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THE SMALL DEBORAH NUMBER LIMIT OF THE DOI-ONSAGER EQUATION TO THE ERICKSEN-LESLIE EQUATION

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Abstract. We present a rigorous derivation of the Ericksen-Leslie equation starting from the Doi-Onsager equation. As in the fluid dynamic limit of the Boltzmann equation, we first make the Hilbert expansion for the solution of the Doi-Onsager equation. The existence of the Hilbert expansion is connected to an open question whether the energy of the Ericksen-Leslie equation is dissipated. We show that the energy is dissipated for the Ericksen-Leslie equation derived from the Doi-Onsager equation. The most difficult step is to prove a uniform bound for the remainder in the Hilbert expansion. This question is connected to the spectral stability of the linearized Doi-Onsager operator around a critical point. By introducing two important auxiliary operators, the detailed spectral information is obtained for the linearized operator around all critical points. However, these are not enough to justify the small Deborah number limit for the inhomogeneous Doi-Onsager equation, since the elastic stress in the velocity equation is also strongly singular. For this, we need to establish a precise lower bound for a bilinear form associated with the linearized operator. In the bilinear form, the interactions between the part inside the kernel and the part outside the kernel of the linearized operator are very complicated. We find a coordinate transform and introduce a five dimensional space called the Maier-Saupe space such that the interactions between two parts can been seen explicitly by a delicate argument of completing the square. However, the lower bound is very weak for the part inside the Maier-Saupe space. In order to apply them to the error estimates, we have to analyze the structure of the singular terms and introduce a suitable energy functional.

1. Introduction

1.1. **The Doi-Onsager theory.** Liquid crystals are a state of matter that have properties between those of a conventional liquid and those of a solid crystal. One of the most common liquid crystal phases is the nematic. The nematic liquid crystals are composed of rod-like molecules with the long axes of neighboring molecules aligned approximately to one another. A classic model which predicts isotropic-nematic phase transition is the hard-rod model proposed by Onsager [18]. Onsager introduced the notion of orientational distribution function and considered a mean-field model in which the rod-rod interaction was modeled by the excluded volume effect. Following Onsager, Maier and Saupe [16] proposed a slightly modified interaction potential, now known as the Maier-Saupe potential. Doi and Edwards [4] extended the Onsager theory for describing the behavior of liquid crystal polymer flows.

We use $\mathbf{x} \in \Omega \subseteq \mathbb{R}^3$ to denote the material point and $f(\mathbf{x}, \mathbf{m}, t)$ to represent the number density for the number of molecules whose orientation is parallel to $\mathbf{m}$ at point $\mathbf{x}$ and time t. For the spatially homogeneous liquid crystal flow, the Doi-Onsager equation [4] takes

$$\frac{\partial f}{\partial t} = \frac{1}{De}\mathcal{R} \cdot (\mathcal{R}f + f\mathcal{R}U) - \mathcal{R} \cdot \big(\mathbf{m} \times \kappa \cdot \mathbf{m}f\big), \tag{1.1}$$

where De is the Deborah number, $\mathcal{R}$ is the rotational gradient operator(see Section 3), κ is a constant velocity gradient, and U is the mean-field interaction potential. Onsager [18]

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considered the potential

$$U = \mathcal{U}f(\mathbf{m},t) = \alpha \int_{\mathbb{S}^2} |\mathbf{m}\times\mathbf{m}'| f(\mathbf{m}',t) d\mathbf{m}',$$

where α is a parameter that measures the potential intensity. In this paper, we will use the Maier-Saupe potential [16] defined by

$$U = \alpha \int_{\mathbb{S}^2} |\mathbf{m}\times\mathbf{m}'|^2 f(\mathbf{m}',t) d\mathbf{m}'.$$

This model has a free energy

$$A[f] = \int_{\mathbb{S}^2} \big(f(\mathbf{m},t)\ln f(\mathbf{m},t) + \frac{1}{2} f(\mathbf{m},t)U(\mathbf{m},t)\big)\mathrm{d}\mathbf{m} \tag{1.2}$$

as its Lyapunov functional. The chemical potential is given by

$$\mu = \frac{\delta A}{\delta f} = \ln f + U.$$

The equation (1.1) can be written as

$$\frac{\partial f}{\partial t} = \frac{1}{De}\mathcal{R}\cdot\big(f\mathcal{R}\mu\big) - \mathcal{R}\cdot\big(\mathbf{m}\times\kappa\cdot\mathbf{m}f\big).$$

The stress tensor is given by

$$\sigma^{De} = \frac{1}{2}\mathbf{D} : \langle \mathbf{mmmm}\rangle_f - \frac{1}{De}\langle \mathbf{mm}\times\mathcal{R}\mu\rangle_f, \tag{1.3}$$

where $\mathbf{D} = \frac{1}{2}(\kappa + \kappa^T)$ is the symmetric part of κ, and

$$\langle(\cdot)\rangle_f \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} (\cdot) f(\mathbf{m},t)\mathrm{d}\mathbf{m}.$$

The homogeneous Doi-Onsager equation has been very successful in describing the properties of liquid crystal polymers in a solvent. This model takes into account the effects of hydrodynamic flow, Brownian motion and intermolecular forces on the molecular orientation distribution. However, it does not include effects such as distortional elasticity. Therefore it is valid only in the limit of spatially homogeneous flows.

The inhomogeneous flows were first studied by Marrucci and Greco [17], and subsequently by many people [8, 20]. Instead of using the distribution as the sole order parameter, they used a combination of the tensorial order parameter and the distribution function, and used the spatial gradients of the tensorial order parameter to describe the spatial variations. This is a departure from the original motivation that led us to the kinetic theory. Wang, E, Liu and Zhang [21] set up a formalism in which the interaction between molecules is treated more directly using the position-orientation distribution function via interaction potentials. They extend the free energy (1.2) to include the effects of nonlocal intermolecular interaction through an interaction potential as follows:

$$A[f] = k_BT \int_\Omega \int_{\mathbb{S}^2} f(\mathbf{x},\mathbf{m},t)(\ln f(\mathbf{x},\mathbf{m},t) - 1) + \frac{1}{2k_BT} U(\mathbf{x},\mathbf{m},t) f(\mathbf{x},\mathbf{m},t)\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}, \tag{1.4}$$

where k_B is the Boltzmann constant, T is the absolute temperature, and the mean-field interaction potential U is defined by

$$U(\mathbf{x},\mathbf{m},t) = k_BT \int_\Omega \int_{\mathbb{S}^2} B(\mathbf{x},\mathbf{x}',\mathbf{m},\mathbf{m}') f(\mathbf{x}',\mathbf{m}',t)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}'.$$

Here $B(\mathbf{x},\mathbf{x}';\mathbf{m},\mathbf{m}')$ is the interaction kernel between the two polymers in the configurations $(\mathbf{x},\mathbf{m})$ and $(\mathbf{x}',\mathbf{m}')$. It should be symmetric with respect to the interchange of $\mathbf{m}$ and $\mathbf{m}'$, $\mathbf{x}$ and $\mathbf{x}'$. B is often translation invariant and hence it can be written in the form

$$B(\mathbf{x}-\mathbf{x}';\mathbf{m},\mathbf{m}').$$

In this paper, we take the following form as in [5, 25]:

$$B(\mathbf{x},\mathbf{x}',\mathbf{m},\mathbf{m}') = \alpha|\mathbf{m}\times\mathbf{m}'|^2\frac{1}{L^3}g\big(\frac{\mathbf{x}-\mathbf{x}'}{L}\big),$$

where L is the lenth of the rods, and $g(\mathbf{x})$ is a radial Schwartz function with $\int_{\mathbb{R}^3} g(\mathbf{x})\mathrm{d}\mathbf{x}=1$. This potential neglects the interaction between orientation and position. But is sufficient in many cases. The chemical potential is given by

$$\mu = \frac{\delta A[f]}{\delta f} = k_BT\ln f(\mathbf{x},\mathbf{m},t) + U(\mathbf{x},\mathbf{m},t).$$

The inhomogeneous Doi-Onsager equation takes the form

$$\begin{aligned}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f =& \frac{1}{k_BT}\nabla\cdot\big\{\big(D_\parallel\mathbf{mm} + D_\perp(\mathbf{I}-\mathbf{mm})\big)\cdot(\nabla\mu)f\big\}\\ &+\frac{D_r}{k_BT}\mathcal{R}\cdot(f\mathcal{R}\mu) - \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f),\\ \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} =& -\nabla p + \nabla\cdot\tau + \mathbf{F}^e,\quad \nabla\cdot\mathbf{v}=0.\end{aligned}$$

Here $D_\parallel$ and $D_\perp$ are respectively the translational diffusion coefficients parallel and normal to the orientation of the LCP molecule, $D_r = \frac{k_BT}{\xi_r}$ is the rotary diffusivity, ∇ is the gradient operator with respect to the spatial variable $\mathbf{x}$. The total stress τ is the sum of the viscous stress τ^s and the elastic stress τ^e. There are two contributions to the viscous stress, one from the solvent and the other from the constraint force arising from the rigidity of the rod (derived in [4]),

$$\tau^s = 2\eta_s\mathbf{D} + \frac{1}{2}\xi_r\mathbf{D}:\langle\mathbf{mmmm}\rangle_f,$$

where $\mathbf{D} = \frac{1}{2}\big(\kappa+\kappa^T\big)$, $\kappa = (\nabla\mathbf{v})^T$ is the velocity gradient tensor, η_s is the solvent viscosity. The elastic stress τ^e and body force $\mathbf{F}^e$ are given by

$$\tau^e = -\langle\mathbf{mm}\times\mathcal{R}\mu\rangle_f,\quad \mathbf{F}^e = -\langle\nabla\mu\rangle_f.$$

Let L_0 be the typical size of the flow region, V_0 be the typical velocity scale, $T_0 = \frac{L_0}{V_0}$ be a typical convective time scale. Another important time scale is the relaxational time scale due to orientation diffusion: $T_r = \frac{\xi_r}{k_BT}$. The ratio of these two time scales is an important parameter called the Deborah number

$$De = \frac{T_r}{T_0} = \frac{\xi_rV_0}{k_BTL_0}.$$

Let $\eta_p = \xi_r, \eta = \eta_s+\eta_p, \gamma = \eta_s/\eta$, and $Re = \frac{V_0L_0}{\eta}$ be the Reynolds number. We denote

$$\begin{aligned}U_\varepsilon(\mathbf{x},\mathbf{m},t) &= \int_\Omega\int_{\mathbb{S}^2} B_\varepsilon(\mathbf{x},\mathbf{x}',\mathbf{m},\mathbf{m}')f(\mathbf{x}',\mathbf{m}',t)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}',\\ \tau^e_\varepsilon &= -\langle\mathbf{mm}\times\mathcal{R}\mu_\varepsilon\rangle_f,\quad \mathbf{F}^e_\varepsilon = -\langle\nabla\mu_\varepsilon\rangle_f,\end{aligned}$$

where the small parameter $\sqrt{\varepsilon}=\frac{L}{L_0}$ represents the typical interaction distance and

$$B_\varepsilon(\mathbf{x},\mathbf{x}',\mathbf{m},\mathbf{m}')=\alpha|\mathbf{m}\times\mathbf{m}'|^2\frac{1}{\sqrt{\varepsilon}^3}g\big(\frac{\mathbf{x}-\mathbf{x}'}{\sqrt{\varepsilon}}\big),$$
$$\mu_\varepsilon=\ln f(\mathbf{x},\mathbf{m},t)+U_\varepsilon(\mathbf{x},\mathbf{m},t).$$

We set

$$f'(\mathbf{x},\mathbf{m},t)=f(L_0\mathbf{x},\mathbf{m},T_0t),\quad \mathbf{v}'(\mathbf{x},t)=\mathbf{v}(L_0\mathbf{x},T_0t)/V_0.$$

Then the non-dimensional Doi-Onsager equation takes the following form (drop the prime for the simplicity):

$$\begin{aligned}\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f&=\frac{\varepsilon}{De}\nabla\cdot\big\{\big(\gamma_\parallel\mathbf{mm}+\gamma_\perp(\mathbf{I}-\mathbf{mm})\big)\cdot(\nabla f+f\nabla U_\varepsilon)\big\}\\&\quad+\frac{1}{De}\mathcal{R}\cdot(\mathcal{R}f+f\mathcal{R}U_\varepsilon)-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f),\qquad(1.5)\\\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}&=-\nabla p+\frac{\gamma}{Re}\Delta\mathbf{v}+\frac{1-\gamma}{2Re}\nabla\cdot(\mathbf{D}:\langle\mathbf{mmmm}\rangle_f)+\frac{1-\gamma}{DeRe}(\nabla\cdot\tau_\varepsilon^e+\mathbf{F}_\varepsilon^e),\end{aligned}$$

where

$$\gamma_\parallel=\frac{L_0De}{V_0L^2}D_\parallel,\quad \gamma_\perp=\frac{L_0De}{V_0L^2}D_\perp.$$

The system (1.5) has the following energy dissipation relation:

$$\begin{aligned}&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega\frac{1}{2}|\mathbf{v}|^2\mathrm{d}\mathbf{x}+\frac{1-\gamma}{DeRe}A_\varepsilon[f]\Big)\\&=\int_\Omega\frac{\gamma}{Re}\mathbf{D}:\mathbf{D}+\frac{1-\gamma}{2Re}\langle(\mathbf{mm}:\mathbf{D})^2\rangle+\frac{1-\gamma}{De^2Re}\big\langle\mathcal{R}\mu_\varepsilon\cdot\mathcal{R}\mu_\varepsilon\big\rangle\\&\quad+\frac{\varepsilon}{De^2Re}\big\langle\nabla\mu_\varepsilon\cdot(\gamma_\parallel\mathbf{mm}+\gamma_\perp(\mathbf{I}-\mathbf{mm}))\cdot\nabla\mu_\varepsilon\big\rangle\mathrm{d}\mathbf{x},\qquad(1.6)\end{aligned}$$

where

$$A_\varepsilon[f]=\int_\Omega\int_{\mathbb{S}^2}f(\mathbf{x},\mathbf{m},t)(\ln f(\mathbf{x},\mathbf{m},t)-1)+\frac{1}{2}U_\varepsilon(\mathbf{x},\mathbf{m},t)f(\mathbf{x},\mathbf{m},t)\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}.$$

We refer to [25, 23] for the numerical study and the well-posedness of the system (1.5).

1.2. **The Ericksen-Leslie theory.** Ericksen-Leslie theory [6, 10] is an elastic continuum theory. The liquid crystal material is treated as a continuum and molecular details are entirely ignored, and this theory considers perturbations to a presumed oriented sample. Elastic continuum theory is a very powerful tool for modeling liquid crystal devices.

The configuration of the liquid crystals is described by a director field $\mathbf{n}(\mathbf{x},t)$. The hydrodynamic equation takes the form

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+\frac{\gamma}{Re}\Delta\mathbf{v}+\frac{1-\gamma}{Re}\nabla\cdot\sigma,\qquad(1.7)$$

where the stress σ is modeled by the phenomenological constitutive relation:

$$\sigma=\sigma^L+\sigma^E.$$

Here σ^L is the viscous (Leslie) stress

$$\sigma^L=\alpha_1(\mathbf{nn}\cdot\mathbf{D})\mathbf{nn}+\alpha_2\mathbf{nN}+\alpha_3\mathbf{Nn}+\alpha_4\mathbf{D}+\alpha_5\mathbf{nn}\cdot\mathbf{D}+\alpha_6\mathbf{D}\cdot\mathbf{nn}\qquad(1.8)$$

with

$$\mathbf{N} = \frac{\partial \mathbf{n}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{n} + \mathbf{\Omega} \cdot \mathbf{n}, \quad \mathbf{\Omega} = \frac{1}{2}(\kappa^T - \kappa).$$

The six constants $\alpha_1, \cdots, \alpha_6$ are called the Leslie coefficients. Parodi's relation [19] gives a constraint for Leslie coefficients: $\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5$. While, σ^E is the elastic (Ericksen) stress

$$\sigma^E = -\frac{\partial E_F}{\partial(\nabla \mathbf{n})} \cdot (\nabla \mathbf{n})^T, \tag{1.9}$$

where $E_F = E_F(\mathbf{n}, \nabla \mathbf{n})$ is the Frank energy. The dynamic equation for the director field is given by

$$\mathbf{n} \times \big(\mathbf{h} - \gamma_1 \mathbf{N} - \gamma_2 \mathbf{D} \cdot \mathbf{n}\big) = 0, \tag{1.10}$$

where $\gamma_1 = \alpha_3 - \alpha_2, \gamma_2 = \alpha_6 - \alpha_5$, and $\mathbf{h}$ is the molecular field

$$\mathbf{h} = -\frac{\delta E_F}{\delta \mathbf{n}} = \nabla \cdot \frac{\partial E_F}{\partial(\nabla \mathbf{n})} - \frac{\partial E_F}{\partial \mathbf{n}}.$$

In this paper, we will consider $E_F = \frac{k}{2}\int_\Omega |\nabla \mathbf{n}(\mathbf{x})|^2 \mathrm{d}\mathbf{x}$. In this case, we have

$$\mathbf{h} = k\Delta \mathbf{n}, \quad \sigma^E = -k\nabla \mathbf{n} \odot \nabla \mathbf{n} = -k(\nabla_i n_k \nabla_j n_k)_{3\times 3}. \tag{1.11}$$

The energy dissipation for Ericksen-Leslie equation is given by

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{Re}{2(1-\gamma)}|\mathbf{v}|^2 \mathrm{d}\mathbf{x} + E_F\Big) \\ &= \int_\Omega \Big(\frac{\gamma}{1-\gamma}|\nabla \mathbf{v}|^2 + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1})|\mathbf{D} : \mathbf{n}\mathbf{n}|^2 + \alpha_4 \mathbf{D} : \mathbf{D} \\ &\qquad + (\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1})|\mathbf{D} \cdot \mathbf{n}|^2 + \frac{1}{\gamma_1}|\mathbf{n} \times \mathbf{h}|^2\Big)\mathrm{d}\mathbf{x}. \end{aligned} \tag{1.12}$$

We refer to [5] for a derivation of (1.12). Concerning the mathematical study of the simplified Ericksen-Leslie equation, we refer to [11, 12, 13, 14] and references therein.

1.3. **From the Doi-Onsager theory to the Ericksen-Leslie theory.** Two kinds of theories were put forward to investigate the liquid crystalline polymers from the different points of view. The Ericksen-Leslie theory is phenomenological in nature, and will be become invalid near defects where the director cannot be defined. The Ericksen-Leslie equation contain six unknown parameters called the Leslie coefficients, which are difficult to determine by using experimental results. Especially, whether the energy defined in (1.12) is dissipated remains unknown in Physics. Hence, it is very important to establish the relationship between two theories.

Kuzuu and Doi [9] formally derive the Ericksen-Leslie equation from the Doi-Onsager equation (1.1), and determine the Leslie coefficients. However, the Ericksen stress is missed in the homogeneous case. E and Zhang [5] extend Kuzuu and Doi's formal derivation to the inhomogeneous case. To recover the Ericksen stress, they find that the Deborah number De and the interaction distance $\sqrt{\varepsilon}$ should satisfy $De \sim \varepsilon$.

Roughly speaking, Kuzuu and Doi shows that when the Deborah number is small, the solution f of (1.1) has the formal expansion

$$f = f_0(\mathbf{m} \cdot \mathbf{n}) + \varepsilon f_1 + \cdots,$$

where $f_0(\mathbf{m} \cdot \mathbf{n})$ denotes the equilibrium distribution function satisfying

$$\mathcal{R} \cdot (\mathcal{R} f_0 + f_0 \mathcal{R}\mathcal{U} f_0) = 0,$$

and $\mathbf{n}$ is determined by (2.3). E and Zhang shows that the solution $(f, \mathbf{v})$ of (1.5) has the formal expansion

$$f = f_0(\mathbf{m}\cdot\mathbf{n}) + \varepsilon f_1 + \cdots,$$
$$\mathbf{v} = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \cdots,$$

where $(\mathbf{v}_0, \mathbf{n})$ is determined by (1.7) and (1.10).

The main goal of this paper is to give a rigorous derivation of the Ericksen-Leslie equation from the Doi-Onsager equation. This is a singular small Deborah number limit problem. To justify this limit, we first make the Hilbert expansion for the solution of the Doi-Onsager equation, then show that the error term is small in a suitable Sobolev space. The existence of the Hilbert expansion is connected to the question whether the energy of the Ericksen-Leslie equation is dissipated. We will show that the energy is dissipated for the Ericksen-Leslie equation derived from the Doi-Onsager equation. The error estimates rely heavily on the spectral analysis of the linearized Doi-Onsager operator around the critical point, which includes

1. Give a complete classification for all critical points h of $A[f]$, which satisfies
$$\mathcal{R}\cdot(\mathcal{R}h + h\mathcal{R}\mathcal{U}h) = 0.$$
2. The spectral analysis of the linearized Doi-Onsager operator $\mathcal{G}_h$ around a critical point h defined by
$$\mathcal{G}_h f = \mathcal{R}\cdot\big(\mathcal{R}f + h\mathcal{R}\mathcal{U}f + f\mathcal{R}\mathcal{U}h\big).$$
3. Establish a precise lower bound for the bilinear form $\big\langle \mathcal{G}_h^\varepsilon f, \mathcal{H}_h^\varepsilon f\big\rangle$ with
$$\mathcal{G}_h^\varepsilon f = \mathcal{R}\cdot\big(\mathcal{R}f + h\mathcal{R}\mathcal{U}_\varepsilon f + f\mathcal{R}\mathcal{U}h\big), \quad \mathcal{H}_h^\varepsilon f = \frac{f}{h} + \mathcal{U}_\varepsilon f.$$

The first point has been given by the second author and coworkers [15]. The second point and the third point are completely new. To prove the second point, we introduce two important auxiliary operators $\mathcal{A}_h$ and $\mathcal{H}_h$ defined by

$$\mathcal{A}_h f = -\mathcal{R}\cdot(h\mathcal{R}f), \quad \mathcal{H}_h = \frac{f}{h} + \mathcal{U}f.$$

It is easy to see that $\mathcal{G}_h = -\mathcal{A}_h\mathcal{H}_h$ and $\mathcal{H}_h$ is self-adjoint. Then we reduce the spectral analysis of $\mathcal{G}_h$ to that of $\mathcal{H}_h$. The proof of the third point is very subtle. Since the orthogonal structure is destroyed when $\varepsilon \neq 0$, the interactions between the part inside the kernel of $\mathcal{G}_h^\varepsilon$ and the part outside the kernel become very complicated. To prove a lower bound, we find a coordinate transform and introduce a generalized kernel space of $\mathcal{G}_h^\varepsilon$(this is a five dimensional space called the Maier-Saupe space) such that the interactions between two parts can be seen explicitly by a delicate argument of completing the square.

With the above preparations, it is still not enough to complete the error estimates in the inhomogeneous case. When $\varepsilon \neq 0$, we can only get a strong lower bound of $\big\langle \mathcal{G}_h^\varepsilon f, \mathcal{H}_h^\varepsilon f\big\rangle$ for the part outside the Maier-Saupe space, and a weak lower bound for the part inside the Maier-Saupe space. In order to apply them to the error estimates, we have to analyze the nonlinear interactions between two parts for the singular term like $\frac{1}{\varepsilon}\big\langle f_R, \partial_t(\frac{1}{f_0})f_R\big\rangle$ and introduce a suitable energy functional. Since we have no decay in ε for the part of f_R inside the kernel, the term $\frac{1}{\varepsilon}\big\langle f_R, \partial_t(\frac{1}{f_0})f_R\big\rangle$ seems to have an order $\frac{1}{\varepsilon}$(**Very singular**). Surprisingly, we will show that it is bounded.

We believe that the spectral information of the linearized operator will be very important to study the other problems like the nonlinear stability and instability of the critical points. These will be left to the future work.

2. Presentation of main results

2.1. **The homogeneous case.** We consider the homogeneous Doi-Onsager equation

$$\frac{\partial f^\varepsilon}{\partial t} = \frac{1}{\varepsilon}\mathcal{R}\cdot\big(\mathcal{R}f^\varepsilon + f^\varepsilon\mathcal{R}\mathcal{U}f^\varepsilon\big) - \mathcal{R}\big(\mathbf{m}\times(\mathbf{D}-\mathbf{\Omega})\cdot\mathbf{m}f^\varepsilon\big). \tag{2.1}$$

Here $\mathbf{D} = \frac{1}{2}(\kappa+\kappa^T), \Omega = \frac{1}{2}(\kappa^T-\kappa)$, and ε is the Deborah number. The corresponding stress tensor σ^ε is given by

$$\sigma^\varepsilon = \frac{1}{2}\mathbf{D} : \langle \mathbf{mmmm}\rangle_{f^\varepsilon} - \frac{1}{\varepsilon}\langle \mathbf{m}\mathbf{m}\times\mathcal{R}\mu^\varepsilon\rangle_{f^\varepsilon} \tag{2.2}$$

with $\mu^\varepsilon = \ln f^\varepsilon + \mathcal{U}f^\varepsilon$ and $\mathcal{U}f = \alpha\int_{\mathbb{S}^2}|\mathbf{m}\times\mathbf{m}'|^2 f(\mathbf{m}',t)\mathrm{d}\mathbf{m}'$.

In the homogeneous case, the Ericksen-Leslie equation is reduced to

$$\mathbf{n}\times\big(\frac{\partial\mathbf{n}}{\partial t} + \mathbf{\Omega}\cdot\mathbf{n} - \lambda\mathbf{D}\cdot\mathbf{n}\big) = 0, \tag{2.3}$$

together with the stress σ^L given by

$$\sigma^L = \alpha_1(\mathbf{nn}\cdot\mathbf{D})\mathbf{nn} + \alpha_2\mathbf{nN} + \alpha_3\mathbf{Nn} + \alpha_4\mathbf{D} + \alpha_5\mathbf{nn}\cdot\mathbf{D} + \alpha_6\mathbf{D}\cdot\mathbf{nn}. \tag{2.4}$$

Our main results are stated as follows.

Theorem 2.1. *Let $h_{\eta,\mathbf{n}}$ be a stable critical point of $A[f]$, and $\mathbf{n}(t)$ be a solution of (2.3) with the initial data $\mathbf{n}_0\in\mathbb{S}^2$ and λ given by*

$$\lambda(\alpha) = \frac{\big\langle 3(\mathbf{m}\cdot\mathbf{n})^2 - 1\big\rangle_{h_{\eta,\mathbf{n}}}}{\big\langle g_0\frac{\mathrm{d}u_0}{\mathrm{d}\theta}\big\rangle_{h_{\eta,\mathbf{n}}}}, \quad u_0 = \mathcal{U}h_{\eta,\mathbf{n}}, \tag{2.5}$$

and g_0 is a solution of (4.8). Assume that the initial data $f_0^\varepsilon(\mathbf{m})\in H^1(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} f_0^\varepsilon(\mathbf{m})\mathrm{d}\mathbf{m} = 1$ takes the form

$$f_0^\varepsilon(\mathbf{m}) = h_{\eta,\mathbf{n}_0}(\mathbf{m}) + \sum_{k=1}^{3}\varepsilon^k f_k(\mathbf{m},0) + \varepsilon^2 f_{R,0}^\varepsilon(\mathbf{m}),$$

where $f_k(\mathbf{m},t)(k=1,2,3)$ is determined by Proposition 6.1, and $f_{R,0}^\varepsilon(\mathbf{m})$ satisfies $\|f_{R,0}^\varepsilon\|_{H^{-1}(\mathbb{S}^2)}\le C$. Then for any $T>0$, there exists an $\varepsilon_0>0$ such that for each $0<\varepsilon<\varepsilon_0$, the solution $f^\varepsilon(\mathbf{m},t)$ of (2.1) takes the form

$$f^\varepsilon(\mathbf{m},t) = h_{\eta,\mathbf{n}(t)}(\mathbf{m}) + \sum_{k=1}^{3}\varepsilon^k f_k(\mathbf{m},t) + \varepsilon^2 f_R^\varepsilon(\mathbf{m},t),$$

where $f_R^\varepsilon(\mathbf{m},t)$ satisfies

$$\|f_R^\varepsilon(t)\|_{H^{-1}(\mathbb{S}^2)} \le C \quad \textit{for any} \quad t\in[0,T].$$

Remark 2.1. *The Ericksen-Leslie equation (2.3) is equivalent to*

$$\frac{\partial\mathbf{n}}{\partial t} + \mathbf{\Omega}\cdot\mathbf{n} - \lambda(\mathbf{I}-\mathbf{nn})\mathbf{D}\cdot\mathbf{n} = 0.$$

It is easy to show that it has a unique global solution.

Let $S_2 = \langle P_2(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\eta,\mathbf{n}}}$ and $S_4 = \langle P_4(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\eta,\mathbf{n}}}$, where $P_k(x)$ is the k-th Legendre polynomial. We take the Leslie coefficients $\alpha_1,\cdots,\alpha_6$ in the definition of σ^L as follows

$$\alpha_1 = -\frac{S_4}{2},\quad \alpha_2 = -\frac{1}{2}\big(1+\frac{1}{\lambda}\big)S_2,\quad \alpha_3 = -\frac{1}{2}\big(1-\frac{1}{\lambda}\big)S_2, \tag{2.6}$$

$$\alpha_4 = \frac{4}{15}-\frac{5}{21}S_2-\frac{1}{35}S_4,\quad \alpha_5 = \frac{1}{7}S_4+\frac{6}{7}S_2,\quad \alpha_6 = \frac{1}{7}S_4-\frac{1}{7}S_2. \tag{2.7}$$

Then we have

Theorem 2.2. *Let $p(t) = -\frac{S_2}{14}\mathbf{D}:\mathbf{nn}$. For any $T>0$, there exist an $\varepsilon_0>0$ such that for each $0<\varepsilon<\varepsilon_0$, there holds*

$$|\sigma^\varepsilon(t)-\sigma^L(t)-p(t)\mathbf{I}| \le C\varepsilon \quad \textit{for} \quad t\in[0,T].$$

2.2. **The inhomogeneous case.** In order to derive the Ericksen-Leslie equation with the Ericksen stress, we have to consider the system (1.5) with $De=\varepsilon$. For the simplicity of presentation, we will consider the case when the translational diffusion coefficients vanish. Then the non-dimensional Doi-Onsager equations takes

$$\frac{\partial f^\varepsilon}{\partial t}+\mathbf{v}^\varepsilon\cdot\nabla f^\varepsilon = \frac{1}{\varepsilon}\mathcal{R}\cdot(\mathcal{R}f^\varepsilon + f^\varepsilon\mathcal{R}\mathcal{U}_\varepsilon f^\varepsilon) - \mathcal{R}\cdot(\mathbf{m}\times\kappa^\varepsilon\cdot\mathbf{m}f^\varepsilon), \tag{2.8}$$

$$\begin{aligned}\frac{\partial \mathbf{v}^\varepsilon}{\partial t}+\mathbf{v}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon &= -\nabla p^\varepsilon + \frac{\gamma}{Re}\Delta\mathbf{v}^\varepsilon + \frac{1-\gamma}{2Re}\nabla\cdot\big(\mathbf{D}^\varepsilon:\langle\mathbf{mmmm}\rangle_{f^\varepsilon}\big)\\ &\quad+\frac{1-\gamma}{\varepsilon Re}(\nabla\cdot\tau^e_\varepsilon+\mathbf{F}^e_\varepsilon),\end{aligned} \tag{2.9}$$

where $\kappa^\varepsilon = (\nabla v^\varepsilon)^T, \mathbf{D}^\varepsilon = \frac{1}{2}\big(\kappa^\varepsilon+(\kappa^\varepsilon)^T\big)$, and

$$\tau^e_\varepsilon = -\langle\mathbf{mm}\times\mathcal{R}\mu_\varepsilon\rangle_{f^\varepsilon},\quad \mathbf{F}^e_\varepsilon = -\langle\nabla\mu_\varepsilon\rangle_{f^\varepsilon},\quad \mu_\varepsilon = \ln f^\varepsilon + \mathcal{U}_\varepsilon f,$$

$$\mathcal{U}_\varepsilon f = \int_\Omega\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2\frac{1}{\sqrt{\varepsilon}^3}g\big(\frac{\mathbf{x}-\mathbf{x}'}{\sqrt{\varepsilon}}\big)f(\mathbf{x}',\mathbf{m}',t)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}'.$$

We also require that the Fourier transform of g satisfies

$$0\le\hat{g}(\xi)<1 \quad \text{for} \quad \xi\neq 0,\quad \hat{g}''(0)<0.$$

Now we can derive the full Ericksen-Leslie equation

$$\mathbf{n}\times\big(\mathbf{h}-\gamma_1\mathbf{N}-\gamma_2\mathbf{D}\cdot\mathbf{n}\big)=0, \tag{2.10}$$

$$\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p+\frac{\gamma}{Re}\Delta\mathbf{v}+\frac{1-\gamma}{Re}\nabla\cdot\sigma, \tag{2.11}$$

where $\gamma_1=\alpha_3-\alpha_2$, $\gamma_2=\alpha_6-\alpha_5$, $\mathbf{h}=k\Delta\mathbf{n}$, $\mathbf{N}=\frac{\partial\mathbf{n}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{n}+\mathbf{\Omega}\cdot\mathbf{n}$, $\mathbf{D}=\frac{1}{2}\big(\nabla\mathbf{v}+(\nabla\mathbf{v})^T\big)$, $\mathbf{\Omega}=\frac{1}{2}\big(\nabla\mathbf{v}-(\nabla\mathbf{v})^T\big)$, and $\sigma=\sigma^L+\sigma^E$ with

$$\begin{aligned}\sigma^L &= \alpha_1(\mathbf{nn}\cdot\mathbf{D})\mathbf{nn}+\alpha_2\mathbf{nN}+\alpha_3\mathbf{Nn}+\alpha_4\mathbf{D}+\alpha_5\mathbf{nn}\cdot\mathbf{D}+\alpha_6\mathbf{D}\cdot\mathbf{nn},\\ \sigma^E &= -k\nabla\mathbf{n}\odot\nabla\mathbf{n}.\end{aligned}$$

Our main result is stated as follows.

Theorem 2.3. *Let $\Omega=\mathbb{R}^3$ and $h_{\eta,\mathbf{n}}$ be a stable critical point of $A[f]$, and let $(\mathbf{n},\mathbf{v}_0)\in C\big([0,T];H^{20}(\Omega)\big)$ be a solution of (2.10)-(2.11) on $[0,T]$ for some $T>0$ with the initial data $(\mathbf{n}_0,\mathbf{v}_{0,0})$, λ given by (2.5), and Leslie coefficients defined by (2.6)-(2.7). Assume also that there exist constant vector $\boldsymbol{c}\in\mathbb{S}^2$ and constant $c_0\in(0,1)$ such that*

$$\big|\mathbf{n}(\mathbf{x},t)\times\boldsymbol{c}\big|\ge c_0 \quad \textit{for any} \quad (x,t)\in\Omega\times[0,T]. \tag{2.12}$$

Assume that the initial data $\big(f_0^\varepsilon(\mathbf{x},\mathbf{m}),\mathbf{v}_0^\varepsilon\big)$ *with* $\int_{\mathbb{S}^2} f_0^\varepsilon(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m}=1$ *takes the form*

$$f_0^\varepsilon(\mathbf{x},\mathbf{m}) = h_{\eta,\mathbf{n}_0(\mathbf{x})}(\mathbf{m}) + \sum_{k=1}^{3}\varepsilon^k f_k(\mathbf{x},\mathbf{m},0) + \varepsilon^3 f_{R,0}^\varepsilon(\mathbf{x},\mathbf{m}),$$

$$\mathbf{v}_0^\varepsilon(\mathbf{x}) = \sum_{k=0}^{2}\varepsilon^k \mathbf{v}_{k,0}(\mathbf{x}) + \varepsilon^3 v_{R,0}^\varepsilon(\mathbf{x}),$$

where $\big(f_1,f_2,f_3,\mathbf{v}_1,\mathbf{v}_2\big)$ *is determined by Proposition 7.1, and* $\big(f_{R,0}^\varepsilon(\mathbf{x},\mathbf{m}),\mathbf{v}_{R,0}^\varepsilon(\mathbf{x})\big)$ *satisfies*

$$\|f_{R,0}^\varepsilon\|_{H^2(\Omega\times\mathbb{S}^2)} + \|\mathbf{v}_{R,0}^\varepsilon\|_{H^2(\Omega)} \le C < +\infty, \quad \|f_{R,0}^\varepsilon\|_{L^2(\Omega\times\mathbb{S}^2)} \le C\varepsilon.$$

Then there exist $\varepsilon_0>0$ *such that for each* $0<\varepsilon<\varepsilon_0$*, the system (2.8)- (2.9) has a unique solution* $\big(f^\varepsilon(\mathbf{x},\mathbf{m},t),\mathbf{v}^\varepsilon(\mathbf{x},\mathbf{m},t)\big)$ *on* $[0,T]$ *which takes the form*

$$f^\varepsilon(\mathbf{x},\mathbf{m},t) = h_{\eta,\mathbf{n}(\mathbf{x},t)}(\mathbf{m}) + \sum_{k=1}^{3}\varepsilon^k f_k(\mathbf{x},\mathbf{m},t) + \varepsilon^3 f_R^\varepsilon(\mathbf{x},\mathbf{m},t),$$

$$\mathbf{v}^\varepsilon(\mathbf{x},t) = \sum_{k=0}^{2}\varepsilon^k \mathbf{v}_k(\mathbf{x},t) + \varepsilon^3 \mathbf{v}_R^\varepsilon(\mathbf{x},t),$$

where $\big(f_R^\varepsilon,\mathbf{v}_R^\varepsilon\big)$ *satisfies*

$$\big\|\big(f_R^\varepsilon,\varepsilon^{1/2}\nabla f_R^\varepsilon,\varepsilon^{3/2}\Delta f_R^\varepsilon\big)(t)\big\|_{L^2(\Omega\times\mathbb{S}^2)} + \big\|\big(\mathbf{v}_R^\varepsilon,\varepsilon\nabla\mathbf{v}_R^\varepsilon,\varepsilon^2\Delta\mathbf{v}_R^\varepsilon\big)(t)\big\|_{L^2(\Omega)} \le C$$

for any $t\in[0,T]$.

Remark 2.2. *The non-degenerate assumption (2.12) allows us to construct a global coordinate transformation, which is the key to establish a lower bound of a bilinear form associated with the linearized operator in Section 5.*

Remark 2.3. *We will study the existence of the solution for the full Ericksen-Leslie equation in a separate paper. We refer to* [12] *for the simplified Ericksen-Leslie equation.*

3. Classification and stability of critical points of energy functional

We consider the homogeneous energy functional $A[f]$ defined by

$$A[f] = \int_{\mathbb{S}^2}\big(f(\mathbf{m})\ln f(\mathbf{m}) + \frac{1}{2}f(\mathbf{m})\mathcal{U}f(\mathbf{m})\big)\mathrm{d}\mathbf{m}$$

for $f\in L^2(\mathbb{S}^2)$. We define

$$\mathcal{P}_0(\mathbb{S}^2) = \Big\{\varphi\in L^2(\mathbb{S}^2) : \int_{\mathbb{S}^2}\varphi(\mathbf{m})\mathrm{d}\mathbf{m}=0\Big\}.$$

We are concerned with the local minimizer of $A[f]$. That is, we find all $h\in L^2(\mathbb{S}^2)$ such that

$$A[h+\epsilon\phi]\ge A[h]$$

for all $\phi\in\mathcal{P}_0(\mathbb{S}^2)$ when ϵ is small enough. Taking a formal expansion, we find that

$$A[h+\epsilon\phi] = A[h] + \epsilon\big\langle \ln h + \mathcal{U}h,\phi\big\rangle + \epsilon^2\big\langle\frac{\phi}{h} + \mathcal{U}\phi,\phi\big\rangle + O(\epsilon^3).$$

This motivates us to introduce the following definition.

Definition 3.1. *We say that $h \in L^2(\mathbb{S}^2)$ is a critical point of the energy functional $A[f]$ if*

$$\frac{\delta A[f]}{\delta f}\Big|_{f=h} = \ln h + \mathcal{U}h = \text{const}.$$

A critical point h is said to be stable if for any $\phi \in \mathcal{P}_0(\mathbb{S}^2)$, there holds

$$\Big\langle \frac{\phi}{h} + \mathcal{U}\phi, \phi \Big\rangle \geq 0.$$

It is easy to see that if h is a critical point of $A[f]$, then h is a solution of stationary Doi-Onsager equation

$$\mathcal{R} \cdot \big(\mathcal{R}h + h\mathcal{R}\mathcal{U}h\big) = 0. \tag{3.1}$$

A complete classification for all critical points of $A[f]$ was given by Liu, Zhang and Zhang [15]; see also [3, 7, 24].

Proposition 3.1. *All the critical points of $A[f]$ take the form*

$$h_{\eta,\mathbf{n}}(\mathbf{m}) = \frac{\mathrm{e}^{\eta(\mathbf{m}\cdot\mathbf{n})^2}}{\int_{\mathbb{S}^2} \mathrm{e}^{\eta(\mathbf{m}\cdot\mathbf{n})^2}\mathrm{d}\sigma},$$

where $\mathbf{n}$ is an arbitrary unit vector, and $\eta = \eta(\alpha)$ is determined by the equation

$$\frac{3\mathrm{e}^{\eta}}{\int_0^1 \mathrm{e}^{\eta z^2}\mathrm{d}z} = 3 + 2\eta + \frac{\eta^2}{\alpha}. \tag{3.2}$$

Furthermore, we have

- *For all $\alpha > 0$, $\eta = 0$(i.e. $h = \frac{1}{4\pi}$) is always a solution.*
- *For $\alpha < \alpha^* \approx 6.731393$, $\eta = 0$ is the only solution. While for $\alpha = \alpha^*$, there is another solution $\eta = \eta^*$.*
- *For $\alpha > \alpha^*$, besides $\eta = 0$, there are exactly two solutions $\eta = \eta_1(\alpha), \eta_2(\alpha)$ satisfying*
  - *$\eta_1(\alpha) > \eta^* > \eta_2(\alpha)$, $\lim_{\alpha\to\alpha^*} \eta_1(\alpha) = \lim_{\alpha\to\alpha^*} \eta_2(\alpha) = \eta^*$;*
  - *$\eta_1(\alpha)$ is an increasing function of α, while $\eta_2(\alpha)$ is a decreasing function;*
  - *$\eta_2(7.5) = 0$.*

Except the monotonicity of $\eta(\alpha)$, the others have been proved in [15]. The monotonicity will be proved in Lemma 4.2.

Concerning the stability of the critical point, Zhang and Zhang [22] showed

Proposition 3.2. *$h = \frac{1}{4\pi}$ is a stable critical point of $A[f]$ if and only if $\alpha < 7.5$; If $\alpha > \alpha^*$, $h_{\eta_1,\mathbf{n}}$ is stable, while $h_{\eta_2,\mathbf{n}}$ is unstable.*

Let us conclude this section by collecting some properties of the rotational operator, which will be used throughout the paper. Let $\mathbf{m} \in \mathbb{S}^2$ and $\nabla_{\mathbf{m}}$ be the gradient operator on the unit sphere $\mathbb{S}^2$. The rotational gradient operator $\mathcal{R}$ is defined by

$$\mathcal{R} = \mathbf{m} \times \nabla_{\mathbf{m}}.$$

Let (θ, ϕ) be the sphere coordinate on $\mathbb{S}^2$. Then $\mathcal{R}$ can be written as

$$\begin{aligned}\mathcal{R} =& (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j})\partial_\theta - (\cos\theta\cos\phi\mathbf{i} + \cos\theta\sin\phi\mathbf{j} - \sin\theta\mathbf{k})\frac{1}{\sin\theta}\partial_\phi \\ \stackrel{\text{def}}{=}& \mathbf{i}\mathcal{R}_1 + \mathbf{j}\mathcal{R}_2 + \mathbf{k}\mathcal{R}_3.\end{aligned}$$

The following properties can be easily verified.

1. $\mathcal{R} \cdot \mathcal{R} = \Delta_{\mathbb{S}^2}$;

2. $\mathcal{R}_i m_j = -\epsilon^{ijk} m_k$, where $\mathbf{m} = (m_1, m_2, m_3)$. If $\mathbf{u}$ is a constant vector, then
$$\mathcal{R}(\mathbf{m}\cdot\mathbf{u}) = \mathbf{m}\times\mathbf{u}, \quad \mathcal{R}\cdot(\mathbf{m}\times\mathbf{u}) = -2\mathbf{m}\cdot\mathbf{u};$$
3. $[\mathcal{R}_j,\ \mathcal{R}_k] = \epsilon^{ijk}\mathcal{R}_i$;
4. $\int_{\mathbb{S}^2} \mathcal{R} f_1 f_2 \mathrm{d}\mathbf{m} = -\int_{\mathbb{S}^2} f_1 \mathcal{R} f_2 \mathrm{d}\mathbf{m}$;
5. $[\mathcal{R}, \mathcal{U}] = 0$.

Here $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator on $\mathbb{S}^2$, ϵ^{ijk} is the Levi-Civita symbol.

4. Spectral analysis of the linearized operator

We linearize the Doi-Onsager equation $\mathcal{R}\cdot(\mathcal{R}f + f\mathcal{R}\mathcal{U}f) = 0$ around a critical point h. The linearized Doi-Onsager operator $\mathcal{G}_h$ is given by
$$\mathcal{G}_h f \stackrel{\text{def}}{=} \mathcal{R}\cdot\big(\mathcal{R}f + h\mathcal{R}\mathcal{U}f + f\mathcal{R}\mathcal{U}h\big). \tag{4.1}$$
We denote by $H^k(\mathbb{S}^2)$ the Sobolev space on $\mathbb{S}^2$, and $H_0^k(\mathbb{S}^2) = H^2(\mathbb{S}^2)\cap\mathcal{P}_0$. $\mathcal{G}_h$ is a bounded operator from $H^2(\mathbb{S}^2)$ to $L^2(\mathbb{S}^2)$, and has the discrete spectra.This section is devoted to studying the kernel and spectra of the linearized operator $\mathcal{G}_h$. These information will play a vital role in the study of small Deborah limit, and will be very important in the study of nonlinear stability and instability of the critical point.

When h is a trivial critical point $h_0 = \frac{1}{4\pi}$, the linearized operator $\mathcal{G}_h$ is reduced to
$$\mathcal{G}_{h_0} f = \Delta_{\mathbb{S}^2}\big(f + \frac{1}{4\pi}\mathcal{U}f\big).$$

Proposition 4.1. *The eigenvalues of $\mathcal{G}_{h_0}$ are $\lambda_k = -k(k+1)$ (for $k\neq 2$, $k\geq 1$) and $-6+\frac{4\alpha}{5}$, and the corresponding eigenfunction is the spherical harmonics $Y_{k,\ell}$ of degree k. Specifically, $\mathcal{G}_{h_0}$ has a positive eigenvalue if and only if $\alpha > 7.5$.*

Remark 4.1. *The critical value 7.5 is consistent with that in Proposition 3.2 deduced from the energy stability analysis.*

Proof. Let ψ be an eigenfunction of $\mathcal{G}_{h_0}$ associated with the eigenvalue λ, that is,
$$\mathcal{G}_{h_0}\psi = \lambda\psi.$$
We choose the spherical harmonics $\{Y_{2,\ell}\}_{1\leq\ell\leq 5}$ of degree 2 as
$$Y_1 = (m_1^2 - m_2^2),\ Y_2 = m_3^2 - \frac{1}{3},\ Y_3 = m_1 m_2,\ Y_4 = m_1 m_3,\ Y_5 = m_2 m_3.$$
Then we make a spherical harmonics expansion for ψ:
$$\psi = \sum_{i=1}^{5}\mu_i Y_i + \sum_{k\neq 2,\ell}\mu_{k,\ell}Y_{k,\ell}. \tag{4.2}$$
We have
$$\begin{aligned}\Delta_{\mathbb{S}^2}\mathcal{U}\psi &= -\alpha\mathcal{R}\cdot\mathcal{R}\int_{\mathbb{S}^2}(\mathbf{m}\cdot\mathbf{m}')^2\psi(\mathbf{m}')\mathrm{d}\mathbf{m}' \\ &= -2\alpha\mathcal{R}\cdot\int_{\mathbb{S}^2}(\mathbf{m}\cdot\mathbf{m}')(\mathbf{m}\times\mathbf{m}')\psi(\mathbf{m}')\mathrm{d}\mathbf{m}' \\ &= 6\alpha\int_{\mathbb{S}^2}(\mathbf{m}\cdot\mathbf{m}')^2\psi(\mathbf{m}')\mathrm{d}\mathbf{m}'.\end{aligned}$$

Hence,

$$\begin{aligned}\lambda\psi &= \Delta_{\mathbb{S}^2}\psi + \frac{3\alpha}{2\pi}\int_{\mathbb{S}^2}(\mathbf{m}\cdot\mathbf{m}')^2\psi(\mathbf{m}')\mathrm{d}\mathbf{m}' \\ &= \Delta_{\mathbb{S}^2}\psi + \frac{3\alpha}{2\pi}m_i m_j M_{ij} \end{aligned} \tag{4.3}$$

with $M_{ij} = \int_{\mathbb{S}^2}\mathbf{m}_i\mathbf{m}_j\psi(\mathbf{m})\mathrm{d}\mathbf{m}$. Noting that

$$\Delta_{\mathbb{S}^2}Y_{k,\ell} = -k(k+1)Y_{k,\ell},$$

and plugging (4.2) into (4.3), we find that

$$\lambda\mu_{k,\ell} = -k(k+1)\mu_{k,\ell}.$$

This implies that

$$\lambda = -k(k+1) \quad \text{or} \quad \mu_{k,\ell} = 0.$$

Hence, if $\lambda > 0$, then $\mu_{k,\ell} = 0$ and we have

$$\psi(\mathbf{m}) = \sum_{i=1}^{5}\mu_i Y_i.$$

Then it follows from (4.3) that

$$\begin{aligned}&(\lambda+6)\sum_{i=1}^{5}\mu_i Y_i \\ &= \frac{3\alpha}{2\pi}\int_{\mathbb{S}^2}\big(\frac{1}{2}Y_1Y_1' + \frac{3}{2}Y_2Y_2' + 2Y_3Y_3' + 2Y_4Y_4' + 2Y_5Y_5' + \frac{1}{3}\big)\sum_{i=1}^{5}\mu_i Y_i\mathrm{d}\mathbf{m} \\ &= \frac{3\alpha}{2\pi}\int_{\mathbb{S}^2}\big(\frac{1}{2}\mu_1Y_1Y_1'^2 + \frac{3}{2}\mu_2Y_2Y_2'^2 + 2\mu_3Y_3Y_3'^2 + 2\mu_4Y_4Y_4'^2 + 2\mu_5Y_5Y_5'^2\big)\mathrm{d}\mathbf{m}.\end{aligned}$$

A direct computation shows that

$$\int_{\mathbb{S}^2}Y_1^2\mathrm{d}\mathbf{m} = \frac{16\pi}{15},\ \int_{\mathbb{S}^2}Y_2^2\mathrm{d}\mathbf{m} = \frac{16\pi}{45},\ \int_{\mathbb{S}^2}Y_3^2\mathrm{d}\mathbf{m} = \frac{4\pi}{15},$$

which implies that

$$\big(\lambda + 6 - \frac{4\alpha}{5}\big)\sum_{i=1}^{5}\mu_i Y_i = 0.$$

Hence, $\lambda = \frac{4\alpha}{5} - 6$. Specifically, $\mathcal{G}_{h_0}$ has a positive eigenvalue if and only if $\alpha > 7.5$. □

When $h = h_{\eta_i,\mathbf{n}}(i = 1, 2)$, the problem becomes more complicated. Kuzzu and Doi [9] conjectured that all the eigenvalues of $\mathcal{G}_h$ are non-positive, and Ker $\mathcal{G}_h = \big\{\mathbf{\Theta}\cdot\mathcal{R}h, \mathbf{\Theta}\in\mathbb{R}^3\big\}$. Here we will give a rigorous proof of Kuzzu and Doi's conjecture when h is a stable critical point. Let us introduce an important operator $\mathcal{A}_h$ defined by

$$\mathcal{A}_h\phi \stackrel{\text{def}}{=} -\mathcal{R}\cdot(h\mathcal{R}\phi)$$

The operator $\mathcal{A}_h$ has the following properties:

Lemma 4.1. *The operator $\mathcal{A}_h$ is a one-one mapping from $H_0^2(\mathbb{S}^2)$ to $\mathcal{P}_0(\mathbb{S}^2)$. We denote by $\mathcal{A}_h^{-1}$ its inverse. Then it holds that*

$$\mathcal{A}_h = \mathcal{A}_h^*, \quad \big\langle\mathcal{A}_h\phi,\phi\big\rangle \geq 0, \quad \big\langle\mathcal{A}_h^{-1}\phi,\phi\big\rangle \geq 0.$$

Now we introduce another important operator $\mathcal{H}_h$ defined by

$$\mathcal{H}_h f \stackrel{\text{def}}{=} \frac{f}{h} + \mathcal{U} f.$$

We have the following important relation:

$$\mathcal{G}_h f = -\mathcal{A}_h \mathcal{H}_h f. \tag{4.4}$$

Basically, we can reduce the spectral analysis of $\mathcal{G}_h$ to that of $\mathcal{H}_h$.

Proposition 4.2. *If h is a critical point of $A[f]$, then $\mathcal{G}_h\mathcal{A}_h$ is a symmetric operator and*

$$\big\langle \mathcal{G}_h\psi, \mathcal{A}_h^{-1}\phi\big\rangle = -\big\langle \mathcal{H}_h\psi, \phi\big\rangle = \big\langle \mathcal{G}_h\phi, \mathcal{A}_h^{-1}\psi\big\rangle.$$

Moreover, if h is a stable critical point of $A[f]$, then $\mathcal{G}_h\mathcal{A}_h$ is a non-positive operator, and $\mathcal{G}_h$ has only non-positive eigenvalues.

Proof. The identity follows from (4.4). Since h is a critical point, we have by (3.1) that

$$\mathcal{A}_h\phi = -h\mathcal{R}\cdot\mathcal{R}\phi - \mathcal{R}h\cdot\mathcal{R}\phi = -h\mathcal{R}\cdot\mathcal{R}\phi + h\mathcal{R}(\mathcal{U}h)\cdot\mathcal{R}\phi.$$

Then for any $\psi, \phi \in H^2(\mathbb{S}^2)$, we have

$$\begin{aligned}
\big\langle \mathcal{G}_h\mathcal{A}_h\psi, \phi\big\rangle &= -\big\langle \mathcal{R}\mathcal{A}_h\psi + h\mathcal{R}\mathcal{U}\mathcal{A}_h\psi + \mathcal{A}_h\psi\mathcal{R}\mathcal{U}h, \mathcal{R}\phi\big\rangle \\
&= -\big\langle \mathcal{R}\mathcal{A}_h\psi + \mathcal{A}_h\psi\mathcal{R}\mathcal{U}h, \mathcal{R}\phi\big\rangle + \big\langle \mathcal{U}\mathcal{A}_h\psi, \mathcal{R}\cdot(h\mathcal{R}\phi)\big\rangle \\
&= \big\langle \mathcal{A}_h\psi, \mathcal{R}\cdot\mathcal{R}\phi - \mathcal{R}(\mathcal{U}h)\cdot\mathcal{R}\phi\big\rangle - \big\langle \mathcal{U}\mathcal{A}_h\psi, \mathcal{A}_h\phi\big\rangle \\
&= -\big\langle \mathcal{A}_h\psi, \frac{\mathcal{A}_h\phi}{h}\big\rangle - \big\langle \mathcal{U}\mathcal{A}_h\psi, \mathcal{A}_h\phi\big\rangle.
\end{aligned}$$

Specifically,

$$\big\langle \mathcal{G}_h\mathcal{A}_h\phi, \phi\big\rangle = -\big\langle \mathcal{A}_h\phi, \frac{\mathcal{A}_h\phi}{h} + \mathcal{U}\mathcal{A}_h\phi\big\rangle.$$

This means that $\mathcal{G}_h\mathcal{A}_h$ is a non-positive operator if h is a stable critical point. Furthermore, if ϕ is an eigenfunction of $\mathcal{G}_h$ associated with the eigenvalue λ, then we have

$$0 \geq \big\langle \mathcal{G}_h\mathcal{A}_h\mathcal{A}_h^{-1}\phi, \mathcal{A}_h^{-1}\phi\big\rangle = \big\langle \mathcal{G}_h\phi, \mathcal{A}_h^{-1}\phi\big\rangle = \lambda\big\langle \phi, \mathcal{A}_h^{-1}\phi\big\rangle.$$

Hence, $\lambda \leq 0$. □

Now we establish a lower bound of the operator $\mathcal{H}_h$.

Proposition 4.3. *Let $h_1 = h_{\eta_1,\mathbf{n}}$. For any $f \in \mathcal{P}_0$, there holds*

$$\big\langle \mathcal{H}_{h_1} f, f\big\rangle \geq 0,$$

and the equality holds if and only if $f \in \operatorname{span}\big\{\mathcal{R}_i h_1, i = 1,2,3\big\}$. Moreover, there exists a positive constant c_0 depending only on η_1 such that if f satisfies

$$\int_{\mathbb{S}^2} f(\mathbf{m})\mathcal{A}_{h_1}^{-1}\mathcal{R}h_1 \mathrm{d}\mathbf{m} = 0,$$

then we have a lower bound

$$\big\langle \mathcal{H}_{h_1} f, f\big\rangle \geq c_0\langle f, f\rangle.$$

We need the following key lemma.

Lemma 4.2. *Let $\eta=\eta(\alpha)$ be determined by (3.2), and define $A_k(\eta)=\int_{-1}^{1} z^k \mathrm{e}^{\eta z^2}\mathrm{d}z$. Then there hold*

$$A_{k+2}=\frac{\mathrm{e}^{\eta}}{\eta}-(k+1)\frac{A_k}{2\eta},\quad A_0=\alpha(A_2-A_4).$$

Moreover, $\frac{\partial\alpha(\eta)}{\partial\eta}>0$ when $\eta>\eta^*$; $\frac{\partial\alpha(\eta)}{\partial\eta}<0$ when $\eta<\eta^*$.

Proof. The first equality can be easily verified by integrating by parts. While, the relation (3.2) is equivalent to

$$6\alpha\mathrm{e}^{\eta}-(3+2\eta)\alpha A_0=4\eta^2A_0\Longleftrightarrow A_0=\alpha(A_2-A_4).$$

In order to prove the second statement, it suffices to show that the equation $\frac{\partial\alpha(\eta)}{\partial\eta}=0$ has only one root, since $\frac{\partial\alpha(\eta^*)}{\partial\eta}=0$. We have

$$\begin{aligned}&\frac{\partial}{\partial\eta}\Big(\mathrm{e}^{-\eta}\big(A_0(A_4-A_6)-A_2(A_2-A_4)\big)\Big)\\&=\frac{1}{2}\frac{\partial}{\partial\eta}\int_{-1}^{1}\int_{-1}^{1}(x^2y^4+x^4y^2-x^6-y^6+x^4+y^4-2x^2y^2)\mathrm{e}^{\eta(x^2+y^2-1)}\mathrm{d}x\mathrm{d}y\\&=\frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}-(x^2-y^2)^2(1-x^2-y^2)^2\mathrm{e}^{\eta(x^2+y^2-1)}\mathrm{d}x\mathrm{d}y<0.\end{aligned}$$

Hence, $A_0(A_4-A_6)-A_2(A_2-A_4)=0$ has only one root. Then from the fact that

$$\frac{\partial\alpha(\eta)}{\partial\eta}=\Big(\frac{A_0}{A_2-A_4}\Big)'=\frac{A_2(A_2-A_4)-A_0(A_4-A_6)}{(A_2-A_4)^2},$$

we know that $\frac{\partial\alpha(\eta)}{\partial\eta}=0$ has only one root. □

Proof of Proposition 4.3. Without loss of generality, we may assume $\mathbf{n}=(0,0,1)$. Introduce the sphere coordinates $(\theta,\phi)\in[0,\pi]\times[0,2\pi]$ with $\mathbf{m}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. Hence,

$$h_{\eta,\mathbf{n}}(\mathbf{m})=\frac{\mathrm{e}^{\eta(\cos\theta)^2}}{\int_{\mathbb{S}^2}\mathrm{e}^{\eta(\cos\theta)^2}\mathrm{d}\sigma}.$$

We make a Fourier expansion for f with respect to the variable ϕ:

$$f=a_0(\theta)+\sum_{k\geq1}\big(a_k(\theta)\cos(k\phi)+b_k(\theta)\sin(k\phi)\big).$$

Noting that the area element $\mathrm{d}\mathbf{m}=\sin\theta\mathrm{d}\theta\mathrm{d}\phi$, we make a change of variable $z=\cos\theta$ to get

$$\begin{aligned}\Big\langle\frac{f}{h_{\eta,\mathbf{n}}},f\Big\rangle&=\int_{\mathbb{S}^2}\mathrm{e}^{\eta(\cos\theta)^2}\mathrm{d}\mathbf{m}\int_{\mathbb{S}^2}\mathrm{e}^{-\eta\cos^2\theta}\Big(a_0^2+\frac{1}{2}\sum_{k\geq1}(a_k^2+b_k^2)\Big)\mathrm{d}\mathbf{m}\\&=2\pi^2\Big(\int_{-1}^{1}\mathrm{e}^{\eta z^2}\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}\big\{2a_0^2+\sum_{k\geq1}(a_k^2+b_k^2)\big\}\mathrm{d}z\Big).\end{aligned}$$

Routine computations show that

$$
\begin{aligned}
\langle \mathcal{U}f, f\rangle =& -\alpha \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\mathbf{m}\cdot\mathbf{m}')^2 f(\mathbf{m})f(\mathbf{m}')\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{m}' \\
=& -\alpha \sum_{i,j=1}^{3} \Big(\int_{\mathbb{S}^2} m_i m_j f(\mathbf{m})\mathrm{d}\mathbf{m}\Big)^2 \\
=& -2\alpha\pi^2\Big\{\big(\int_{-1}^{1}(1-z^2)a_0\mathrm{d}z\big)^2 + 2\big(\int_{-1}^{1} z^2 a_0\mathrm{d}z\big)^2 + \big(\int_{-1}^{1} z\sqrt{1-z^2}a_1\mathrm{d}z\big)^2 \\
&+ \big(\int_{-1}^{1} z\sqrt{1-z^2}b_1\mathrm{d}z\big)^2 + \frac{1}{4}\big(\int_{-1}^{1}(1-z^2)a_2\mathrm{d}z\big)^2 + \frac{1}{4}\big(\int_{-1}^{1}(1-z^2)b_2\mathrm{d}z\big)^2\Big\}.
\end{aligned}
$$

We use Lemma 4.2 and Cauchy-Schwartz inequality to get

$$
\begin{aligned}
&2\pi^2\Big(\int_{-1}^{1}\mathrm{e}^{\eta z^2}\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) - 2\alpha\pi^2\big(\int_{-1}^{1} z\sqrt{1-z^2}a_1\mathrm{d}z\big)^2 \\
&= 2\alpha\pi^2\Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}z^2(1-z^2)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) - 2\alpha\pi^2\Big(\int_{-1}^{1} z\sqrt{1-z^2}a_1\mathrm{d}z\Big)^2 \geq 0.
\end{aligned}
$$

Moreover, the equality holds if and only if $a_1(z) = C\mathrm{e}^{\eta z^2}z\sqrt{1-z^2}$ or 0 for some constant C. If $a_1(z)$ satisfies $\int_{-1}^{1}\mathrm{e}^{\eta z^2}z\sqrt{1-z^2}a_1(z)\mathrm{d}z = 0$, then

$$
\begin{aligned}
&\Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}z^2(1-z^2)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) - \Big(\int_{-1}^{1} z\sqrt{1-z^2}a_1\mathrm{d}z\Big)^2 \\
&= \Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}z^2(1-z^2)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) - \Big(\int_{-1}^{1} z\sqrt{1-z^2}(1-\zeta\mathrm{e}^{\eta z^2})a_1\mathrm{d}z\Big)^2 \\
&\geq \Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}\big(z^2(1-z^2) - z^2(1-z^2)(1-\zeta\mathrm{e}^{\eta z^2})^2\big)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) \\
&= \Big(\int_{0}^{1}\mathrm{e}^{2\eta z^2}z^2(1-z^2)\big(2\zeta - \zeta^2\mathrm{e}^{\eta z^2}\big)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_1^2\mathrm{d}z\Big) \\
&\geq c_0(\eta)\int_{-1}^{1} a_1^2\mathrm{d}z, \qquad (4.5)
\end{aligned}
$$

if we take ζ small enough. We denote

$$
W_1(\eta) = \int_{-1}^{1}\mathrm{e}^{\eta z^2}(1-z^2)(5z^2-1)\mathrm{d}z,
$$

which is positive for $\eta > 0$ by noting that

$$
W_1(\eta) = \int_{-1}^{1}(\mathrm{e}^{\eta z^2} - \mathrm{e}^{\eta\frac{1}{5}})(1-z^2)(5z^2-1)\mathrm{d}z > 0.
$$

By Lemma 4.2 and Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&2\pi^2\Big(\int_{-1}^{1}\mathrm{e}^{\eta z^2}\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_2^2\mathrm{d}z\Big)-\frac{1}{2}\alpha\pi^2\Big(\int_{-1}^{1}(1-z^2)a_2\mathrm{d}z\Big)^2\\
&=2\alpha\pi^2\Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}z^2(1-z^2)\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_2^2\mathrm{d}z\Big)-\frac{1}{2}\alpha\pi^2\Big(\int_{-1}^{1}(1-z^2)a_2\mathrm{d}z\Big)^2\\
&=\frac{1}{2}\alpha\pi^2\Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}(1-z^2)^2\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_2^2\mathrm{d}z\Big)-\frac{1}{2}\alpha\pi^2\Big(\int_{-1}^{1}(1-z^2)a_2\mathrm{d}z\Big)^2\\
&\quad+\frac{1}{2}\alpha\pi^2W_1(\eta)\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_2^2\mathrm{d}z\geq c_0(\eta)\int_{-1}^{1}a_2^2\mathrm{d}z
\end{aligned}$$

for some $c_0(\eta)>0$.

In the following, we take $\eta=\eta_1(\alpha)$ for $\alpha>\alpha^*$. By Lemma 4.2, we have

$$0<\frac{\partial\alpha(\eta)}{\partial\eta}=\frac{A_2(A_2-A_4)-A_0(A_4-A_6)}{(A_2-A_4)^2}=\frac{3A_2^2+2A_0A_2-5A_0A_4}{2\eta(A_2-A_4)^2},$$

which implies that

$$3(A_0A_4-A_2^2)<2A_0(A_2-A_4).$$

Then using the fact $\int_{-1}^{1}a_0\mathrm{d}z=0$ and Cauchy-Schwartz inequality, we infer that

$$\begin{aligned}
&\Big(\int_{-1}^{1}(1-z^2)a_0\mathrm{d}z\Big)^2+2\Big(\int_{-1}^{1}z^2a_0\mathrm{d}z\Big)^2\\
&=3\Big(\int_{-1}^{1}(\frac{A_2}{A_0}-z^2)a_0\mathrm{d}z\Big)^2\\
&\leq 3\Big(\int_{0}^{1}\mathrm{e}^{\eta z^2}(\frac{A_2}{A_0}-z^2)^2\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_0^2\mathrm{d}z\Big)\\
&=3\Big(A_4-\frac{A_2^2}{A_0}\Big)\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_0^2\mathrm{d}z\\
&<2(A_2-A_4)\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_0^2\mathrm{d}z=\frac{2}{\alpha}\Big(\int_{-1}^{1}\mathrm{e}^{\eta z^2}\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_0^2\mathrm{d}z\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{2}{\alpha}\Big(\int_{-1}^{1}\mathrm{e}^{\eta z^2}\mathrm{d}z\Big)\cdot\Big(\int_{-1}^{1}\mathrm{e}^{-\eta z^2}a_0^2\mathrm{d}z\Big)-\Big(\int_{-1}^{1}(1-z^2)a_0\mathrm{d}z\Big)^2-2\Big(\int_{-1}^{1}z^2a_0\mathrm{d}z\Big)^2\\
&\geq c_0(\eta)\int_{-1}^{1}a_0^2\mathrm{d}z\quad\text{ for some } c_0(\eta)>0.
\end{aligned}$$

Summing up all the above estimates, we conclude that if $\eta=\eta_1(\alpha)$, then

$$\big\langle\mathcal{H}_{h_1}f,f\big\rangle=\big\langle\frac{f}{h_{\eta_1,\mathbf{n}}},f\big\rangle+\big\langle\mathcal{U}f,f\big\rangle\geq c_0(\eta)\sum_{k\neq1}\int_{-1}^{1}a_k^2\mathrm{d}z,\tag{4.6}$$

and hence,

$$\big\langle\frac{f}{h_{\eta_1,\mathbf{n}}},f\big\rangle+\big\langle\mathcal{U}f,f\big\rangle=0$$

if and only if f is of the form $C_1 z\sqrt{1-z^2}e^{\eta z^2}\cos\phi + C_2 z\sqrt{1-z^2}e^{\eta z^2}\sin\phi$, which belongs to $\mathrm{span}\big\{\mathcal{R}_i h_{\eta,\mathbf{n}}, i=1,2,3\big\}$, since we have

$$\begin{aligned}\mathcal{R}h_{\eta,\mathbf{n}} =& 2\eta(\mathbf{m}\times\mathbf{n})(\mathbf{m}\cdot\mathbf{n})e^{\eta(\mathbf{m}\cdot\mathbf{n})^2} = 2\eta\big(\sin\theta\sin\phi, -\sin\theta\cos\phi, 0\big)\cos\theta e^{\eta\cos^2\theta}\\ =& 2\eta\big(\sin\phi, -\cos\phi, 0\big) z\sqrt{1-z^2}e^{\eta z^2}.\end{aligned}$$

This proves the first statement of Proposition 4.3. To obtain a lower bound of $\mathcal{H}_{h_1}$, we decompose g into

$$f = f_1 + f_2, \quad f_1 \in \ \mathrm{span}\big\{\mathcal{R}_i h_1, i=1,2,3\big\}.$$

If g satisfy $\int_{\mathbb{S}^2} f(\mathbf{m})\mathcal{A}_{h_1}^{-1}\mathcal{R}h_1 \mathrm{d}\mathbf{m} = 0$, then we have

$$\int_{\mathbb{S}^2} f_1(\mathbf{m})\mathcal{A}_{h_1}^{-1} f_1 \mathrm{d}\mathbf{m} = -\int_{\mathbb{S}^2} f_2(\mathbf{m})\mathcal{A}_{h_1}^{-1} g_1 \mathrm{d}\mathbf{m},$$

which implies that

$$\langle f_1, f_1\rangle \le C\langle f_2, f_2\rangle,$$

since $\langle f_1, f_1\rangle \sim \big\langle f_1, \mathcal{A}_{h_1}^{-1} f_1\big\rangle$. This together with (4.5) and (4.6) gives

$$\big\langle \mathcal{H}_{h_1} f, f\big\rangle = \big\langle \mathcal{H}_{h_1} f_2, f_2\big\rangle \ge c_0\langle f_2, f_2\rangle \ge c_0\langle f, f\rangle.$$

The proof is finished. □

We define $\mathrm{Ker}\mathcal{G}_{h_{\eta,\mathbf{n}}} \overset{\mathrm{def}}{=} \big\{\phi \in H_0^2(\mathbb{S}^2) : \mathcal{G}_{h_{\eta,\mathbf{n}}}\phi = 0\big\}$.

Theorem 4.1. *Let $h_i = h_{\eta_i,\mathbf{n}}, i=1,2$. For $\alpha > \alpha^*$, it holds that*

1. *$\mathcal{G}_{h_1}$ has no positive eigenvalues, while $\mathcal{G}_{h_2}$ has at least one positive eigenvalue;*
2. *$\mathrm{Ker}\mathcal{G}_{h_1} = \big\{\mathbf{\Theta}\cdot\mathcal{R}h_1; \mathbf{\Theta}\in\mathbf{R}^3\big\}$ is a two dimensional space;*
3. *If $\phi \in \mathrm{Ker}\mathcal{G}_{h_1}$, then $\mathcal{H}_{h_1}\phi = 0$;*

Proof. Since h_1 is a stable critical point of $A[f]$, $\mathcal{G}_{h_1}$ has no positive eigenvalues by Proposition 4.2. From the proof of Proposition 4.3, we know that there exists $g \in \mathcal{P}_0(\mathbb{S}^2)$ such that

$$\big\langle \mathcal{H}_{h_2} g, g\big\rangle < 0. \tag{4.7}$$

Assume that all eigenvalues $\{\lambda_k\}$ of $\mathcal{G}_{h_2}$ are non-positive. We denote by E_k the eigen-subspaces of $\mathcal{G}_{h_2}$ corresponding to λ_k. Then for $\psi_k \in E_k, \psi_\ell \in E_\ell (k \ne \ell)$, we have

$$\lambda_k\big\langle \psi_k, \mathcal{A}_{h_2}^{-1}\psi_\ell\big\rangle = \big\langle \mathcal{G}\psi_k, \mathcal{A}_{h_2}^{-1}\psi_\ell\big\rangle = \big\langle \mathcal{G}\psi_\ell, \mathcal{A}_{h_2}^{-1}\psi_k\big\rangle = \lambda_\ell\big\langle \psi_k, \mathcal{A}_{h_2}^{-1}\psi_\ell\big\rangle.$$

Hence, $\langle \psi_k,\ \mathcal{A}_{h_2}^{-1}\psi_\ell\rangle = 0$ for $k \ne \ell$. We write $g = \sum_k \psi_k$ with $\psi_k \in E_k$. Then

$$\big\langle \mathcal{H}_{h_2} g, g\big\rangle = -\big\langle \mathcal{G}g, \mathcal{A}_{h_2}^{-1} g\big\rangle = -\sum_k \lambda_k\big\langle \psi_k, \mathcal{A}_{h_2}^{-1}\psi_k\big\rangle \ge 0,$$

which leads to a contradiction with (4.7). Thus, $\mathcal{G}_{h_2}$ has at least one positive eigenvalue.

If $\phi \in \mathrm{Ker}\mathcal{G}_{h_1}$, then $\mathcal{H}_{h_1}\phi = \text{constant}$. Hence, $\langle \mathcal{H}_{h_1}\phi, \phi\rangle = 0$, and then $\phi \in \mathrm{span}\big\{\mathcal{R}_i h_1, i = 1,2,3\big\}$ by Proposition 4.3. On the other hand, if $\phi = \mathcal{R}h_1$, then we find

$$\mathcal{H}_{h_1}\phi = \mathcal{R}\big(\ln h_1 + \mathcal{U}h_1\big) = 0.$$

This proves $\mathrm{Ker}\mathcal{G}_{h_1} = \big\{\mathbf{\Theta}\cdot\mathcal{R}h_1; \mathbf{\Theta}\in\mathbf{R}^3\big\}$ and the third point. Due to $\mathbf{n}\cdot\mathcal{R}h_1 = 0$, $\mathrm{Ker}\mathcal{G}_{h_1}$ is a two dimensional space. □

Finally let us give a characterization of the functions in $\mathrm{Ker}\mathcal{G}^*_{h_{\eta_1,\mathbf{n}}}$, see also [9].

Proposition 4.4. *If $\psi_0 \in \mathrm{Ker}\mathcal{G}^*_{h_{\eta_1,\mathbf{n}}}$, then ψ_0 takes the form (θ,ϕ)*

$$\psi_0(\theta,\phi) = \mathbf{\Theta}\cdot\mathbf{e}_\phi g_0(\cos\theta),$$

in the spherical coordinate, where $\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0)$ and g_0 satisfies

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\frac{\mathrm{d}g_0}{\mathrm{d}\theta}\Big) - \frac{g_0}{\sin^2\theta} - \frac{\mathrm{d}u_0}{\mathrm{d}\theta}\frac{\mathrm{d}g_0}{\mathrm{d}\theta} = -\frac{\mathrm{d}u_0}{\mathrm{d}\theta}. \tag{4.8}$$

Proof. Note that $\mathrm{Ker}\mathcal{G}^*_{h_{\eta_1,\mathbf{n}}} = \mathcal{A}^{-1}_{h_{\eta_1,\mathbf{n}}}\mathrm{Ker}\mathcal{G}_{h_{\eta_1,\mathbf{n}}}$. Hence, $\psi_0 \in \mathrm{Ker}\mathcal{G}^*_{h_{\eta_1,\mathbf{n}}}$ if and only if there exits a vector $\mathbf{\Theta}$ such that

$$\mathcal{R}\cdot(h_{h_{\eta_1,\mathbf{n}}}\mathcal{R}\psi_0) = \mathbf{\Theta}\cdot\mathcal{R}h_{h_{\eta_1,\mathbf{n}}},$$

which is equivalent to

$$\mathcal{R}\cdot\mathcal{R}\psi_0 - \mathcal{R}u_0\cdot\mathcal{R}\psi_0 = -\mathbf{\Theta}\cdot\mathcal{R}u_0, \tag{4.9}$$

where $u_0 = \mathcal{U}h_{\eta_1,\mathbf{n}}$ is a function of $\mathbf{m}\cdot\mathbf{n}$. We take θ be the angle between $\mathbf{m}$ and $\mathbf{n}$, and rewrite (4.9) in terms of in the spherical coordinate (θ,ϕ) as

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\psi_0}{\partial\theta}\Big) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi_0}{\partial\phi^2} - \frac{\mathrm{d}u_0}{\mathrm{d}\theta}\frac{\partial\psi_0}{\partial\theta} = -\mathbf{\Theta}\cdot\mathbf{e}_\phi\frac{\mathrm{d}u_0}{\mathrm{d}\theta}.$$

We rewrite $\psi_0(\theta,\phi)$ as

$$\psi_0(\theta,\phi) = \mathbf{\Theta}\cdot\mathbf{e}_\phi g_0.$$

Then it easy to find that g_0 satisfies (4.8). □

5. Lower bound of a bilinear form for the linearized operator

In the inhomogeneous case, the linearized operator $\mathcal{G}^\varepsilon_h$ around h is given by

$$\mathcal{G}^\varepsilon_h f = \mathcal{R}\cdot\big(\mathcal{R}f + h\mathcal{R}\mathcal{U}_\varepsilon f + f\mathcal{R}\mathcal{U}h\big).$$

To justify the small Deborah number limit for the inhomogeneous system, the main difficulty is that the elastic stress in the velocity equation is strongly singular(a loss of $\frac{1}{\varepsilon}$). To overcome it, we have to establish a precise lower bound for the following bilinear form:

$$\big\langle\mathcal{G}^\varepsilon_h f, \mathcal{H}^\varepsilon_h f\big\rangle, \quad \mathcal{H}^\varepsilon_h f = \frac{f}{h} + \mathcal{U}_\varepsilon f.$$

When $\varepsilon \neq 0$, the orthogonal structure is destroyed such that the interactions between the part inside the kernel and the part outside the kernel of $\mathcal{G}_h$ become very complicated. We find a coordinate transformation and introduce a generalized kernel space of $\mathcal{G}^\varepsilon_h$ such that the interactions between two parts can been seen explicitly, then a lower bound is obtained by very subtle calculations.

5.1. **New coordinates frame.** At each point $\mathbf{x}$, we choose a right hand cartesian coordinate frame $(\mathbf{k}_1(\mathbf{x}), \mathbf{k}_2(\mathbf{x}), \mathbf{k}_3(\mathbf{x}))$ such that $\mathbf{k}_3(\mathbf{x}) = \mathbf{n}(\mathbf{x})$, and $\mathbf{k}_1(\mathbf{x})$, $\mathbf{k}_2(\mathbf{x})$ depend on $\mathbf{n}(\mathbf{x})$ smoothly. For instance, under the assumption that $|n_1(\mathbf{x})| < 1 - c_0$ for all $\mathbf{x} \in \Omega$, we can take

$$\mathbf{k}_2(\mathbf{x}) = \frac{\mathbf{n}\times(1,0,0)^T}{|\mathbf{n}\times(1,0,0)^T|} = \Big(0, \frac{n_3}{(n_2^2+n_3^2)^{1/2}}, -\frac{n_2}{(n_2^2+n_3^2)^{1/2}}\Big)^T,$$

$$\mathbf{k}_1 = \mathbf{k}_2\times\mathbf{n} = \Big((n_2^2+n_3^2)^{1/2}, -\frac{n_1n_2}{(n_2^2+n_3^2)^{1/2}}, -\frac{n_1n_3}{(n_2^2+n_3^2)^{1/2}}\Big)^T.$$

At each point $\mathbf{x}$, let $(\hat{\theta}, \hat{\varphi})$ be the sphere coordinate on the unit sphere $\mathbb{S}^2$, that is,

$$\begin{aligned}\mathbf{m} &= \sin\hat{\theta}\cos\hat{\varphi}\mathbf{k}_1(\mathbf{x}) + \sin\hat{\theta}\sin\hat{\varphi}\mathbf{k}_2(\mathbf{x}) + \cos\hat{\theta}\mathbf{k}_3(\mathbf{x}) \\ &= \mathbf{A}(\mathbf{x})\cdot(\sin\hat{\theta}\cos\hat{\varphi}, \sin\hat{\theta}\sin\hat{\varphi}, \cos\hat{\theta})^T,\end{aligned}$$

where the matrix $\mathbf{A} = [\mathbf{k}_1\ \mathbf{k}_2\ \mathbf{k}_3]$. We set

$$\mathbf{e}_{\hat{\varphi}} = -\sin\hat{\varphi}\mathbf{k}_1 + \cos\hat{\varphi}\mathbf{k}_2, \quad \mathbf{e}_{\hat{\theta}} = -(\cos\hat{\theta}\cos\hat{\varphi}\mathbf{k}_1 + \cos\hat{\theta}\sin\hat{\varphi}\mathbf{k}_2 - \sin\hat{\theta}\mathbf{k}_3).$$

We denote $\hat{\mathbf{m}} = (\sin\hat{\theta}\cos\hat{\varphi}, \sin\hat{\theta}\sin\hat{\varphi}, \cos\hat{\theta})^T$, hence $\mathbf{m} = \mathbf{A}(\mathbf{x})\cdot\hat{\mathbf{m}}$.

In this coordinate, the rotational gradient operator $\mathcal{R} = \mathbf{m}\times\nabla_{\mathbf{m}}$ can be written as

$$\begin{aligned}\mathcal{R} &= \big(-\sin\hat{\varphi}\mathbf{k}_1 + \cos\hat{\varphi}\mathbf{k}_2\big)\frac{\partial}{\partial\hat{\theta}} - \big(\cos\hat{\theta}\cos\hat{\varphi}\mathbf{k}_1 + \cos\hat{\theta}\sin\hat{\varphi}\mathbf{k}_2 - \sin\hat{\theta}\mathbf{k}_3\big)\frac{1}{\sin\hat{\theta}}\frac{\partial}{\partial\hat{\varphi}} \\ &= \mathbf{A}\cdot\big(-\sin\hat{\varphi}\frac{\partial}{\partial\hat{\theta}} - \cos\hat{\varphi}\frac{\cos\hat{\theta}}{\sin\hat{\theta}}\frac{\partial}{\partial\hat{\varphi}}, \cos\hat{\varphi}\frac{\partial}{\partial\hat{\theta}} - \sin\hat{\varphi}\frac{\cos\hat{\theta}}{\sin\hat{\theta}}\frac{\partial}{\partial\hat{\varphi}}, \frac{\partial}{\partial\hat{\varphi}}\big)^T. \end{aligned} \tag{5.1}$$

We also have

$$\begin{aligned}\mathcal{R}f\cdot\mathcal{R}g &= \big(\mathbf{e}_{\hat{\varphi}}\frac{\partial f}{\partial\hat{\theta}} + \mathbf{e}_{\hat{\theta}}\frac{1}{\sin\hat{\theta}}\frac{\partial f}{\partial\hat{\varphi}}\big)\cdot\big(\mathbf{e}_{\hat{\varphi}}\frac{\partial g}{\partial\hat{\theta}} + \mathbf{e}_{\hat{\theta}}\frac{1}{\sin\hat{\theta}}\frac{\partial g}{\partial\hat{\varphi}}\big) \\ &= \frac{\partial f}{\partial\hat{\theta}}\frac{\partial g}{\partial\hat{\theta}} + \frac{1}{\sin^2\hat{\theta}}\frac{\partial f}{\partial\hat{\varphi}}\frac{\partial g}{\partial\hat{\varphi}}.\end{aligned} \tag{5.2}$$

5.2. **The Maier-Saupe space and the lower bound inequality.** For any $f\in L^2(\Omega\times\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} f(\mathbf{x},\mathbf{m})d\mathbf{m} = 0$, we decompose it as

$$f = a_0(\mathbf{x},\hat{\theta}) + \sum_{k\geq 1}\big(a_k(\mathbf{x},\hat{\theta})\cos k\hat{\varphi} + b_k(\mathbf{x},\hat{\theta})\sin k\hat{\varphi}\big), \tag{5.3}$$

with $a_k(\mathbf{x},0) = a_k(\mathbf{x},\pi) = b_k(\mathbf{x},0) = b_k(\mathbf{x},\pi) = 0$ for $k\geq 1$. We set

$$A_k = \int_0^\pi \mathrm{e}^{\eta\cos^2\theta}(\cos\theta)^k\sin\theta\mathrm{d}\theta, \quad f_0(\hat{\theta}) = \frac{\mathrm{e}^{\eta\cos^2\hat{\theta}}}{2\pi A_0}.$$

We further decompose the coefficients a_0, a_1, a_2, b_1, b_2 as follows

$$\begin{aligned}a_0(\mathbf{x},\hat{\theta}) &= \zeta_0(\mathbf{x})f_0(\hat{\theta})(\cos^2\hat{\theta} - A_2/A_0) + \gamma_0(\mathbf{x},\hat{\theta}), \\ a_1(\mathbf{x},\hat{\theta}) &= \zeta_{a,1}(\mathbf{x})f_0(\hat{\theta})\sin\hat{\theta}\cos\hat{\theta} + \gamma_{a,1}(\mathbf{x},\hat{\theta}), \\ b_1(\mathbf{x},\hat{\theta}) &= \zeta_{b,1}(\mathbf{x})f_0(\hat{\theta})\sin\hat{\theta}\cos\hat{\theta} + \gamma_{b,1}(\mathbf{x},\hat{\theta}), \\ a_2(\mathbf{x},\hat{\theta}) &= \zeta_{a,2}(\mathbf{x})f_0(\hat{\theta})\sin^2\hat{\theta} + \gamma_{a,2}(\mathbf{x},\hat{\theta}), \\ b_2(\mathbf{x},\hat{\theta}) &= \zeta_{b,2}(\mathbf{x})f_0(\hat{\theta})\sin^2\hat{\theta} + \gamma_{b,2}(\mathbf{x},\hat{\theta}),\end{aligned}$$

where the functions $\gamma_0, \gamma_{a,1}, \cdots, \gamma_{b,2}$ satisfy

$$\begin{aligned}&\int_0^\pi \gamma_0(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0, \quad \int_0^\pi (3\cos^2\hat{\theta} - 1)\gamma_0(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0, \\ &\int_0^\pi \sin\hat{\theta}\cos\hat{\theta}\gamma_{a,1}(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0, \quad \int_0^\pi \sin\hat{\theta}\cos\hat{\theta}\gamma_{b,1}(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0, \\ &\int_0^\pi \sin^2\hat{\theta}\gamma_{a,2}(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0, \quad \int_0^\pi \sin^2\hat{\theta}\gamma_{b,2}(\mathbf{x},\hat{\theta})\sin\hat{\theta}\mathrm{d}\hat{\theta} = 0.\end{aligned}$$

Noting that

$$\int_0^\pi f_0(\cos^2\hat\theta - A_2/A_0)(3\cos^2\hat\theta - 1)\sin\hat\theta \mathrm{d}\hat\theta = \frac{3}{A_0}\big(A_4 - \frac{A_2^2}{A_0}\big) > 0,$$

hence ζ_0 is uniquely determined. Obviously, $\zeta_{a,1}, \cdots, \zeta_{b,2}$ are also uniquely determined. Thus, the above decompositions make sense.

The space spanned by the following five functions

$$f_0(\hat\theta)(\cos^2\hat\theta - A_2/A_0), \quad f_0(\hat\theta)\sin\hat\theta\cos\hat\theta\cos\hat\varphi, \quad f_0(\hat\theta)\sin\hat\theta\cos\hat\theta\sin\hat\varphi,$$
$$f_0(\hat\theta)\sin^2\hat\theta\cos(2\hat\varphi), \quad \text{and} \quad f_0(\hat\theta)\sin^2\hat\theta\sin(2\hat\varphi)$$

can be viewed as a generalized kernel of the operator $\mathcal{G}_h^\varepsilon$, and will be called as **the Maier-Saupe space**. The kernel of $\mathcal{G}_h$ is spanned by the second and the third function.

We denote

$$\hat{\mathbf{M}}(\mathbf{x}) = \int_{\mathbb{S}^2} \hat{\mathbf{m}}\hat{\mathbf{m}} f \mathrm{d}\hat{\mathbf{m}}, \quad \mathbf{M}(\mathbf{x}) = \int_{\mathbb{S}^2} \mathbf{m}\mathbf{m} f \mathrm{d}\mathbf{m} = \int_{\mathbb{S}^2} (\mathbf{A}\cdot\hat{\mathbf{m}})(\mathbf{A}\cdot\hat{\mathbf{m}}) f \mathrm{d}\hat{\mathbf{m}},$$
$$N_{ij}(\mathbf{x}) = A_{ki}A_{lj}\big(g_\varepsilon * M_{kl}\big), \quad N_0(\mathbf{x}) = 2N_{33} - N_{11} - N_{22}, \quad N_2(\mathbf{x}) = N_{11} - N_{22}.$$

Proposition 5.1. *(Lower bound inequality) Let $h_{\mathbf{n}(x),\eta}$ be a stable critical point of $A[f]$. Then there exits $c > 0$ such that any $f \in H^1(\Omega\times\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} f(\mathbf{x},\mathbf{m})d\mathbf{m} = 0$, there holds*

$$\big\langle \mathcal{G}^\varepsilon_{h_{\mathbf{n},\eta}} f, \mathcal{H}^\varepsilon_{h_{\mathbf{n},\eta}} f\big\rangle \geq c\Big\{ \int_\Omega\int_0^\pi \big(\gamma_0^2 + \gamma_{a,1}^2 + \gamma_{b,1}^2 + \gamma_{a,2}^2 + \gamma_{b,2}^2 + \sum_{k\geq 3}(a_k^2 + b_k^2)\big)\sin\hat\theta \mathrm{d}\hat\theta\mathrm{d}\mathbf{x}$$
$$+ \int_\Omega \big((2\zeta_0 - \alpha N_0)^2 + \big(\zeta_{a,1} - 2\alpha N_{13}\big)^2 + \big(\zeta_{b,1} - 2\alpha N_{23}\big)^2 + \big(\zeta_{a,2} - \alpha N_2\big)^2 + \big(\zeta_{b,2} - \alpha N_{12}\big)^2\big)\mathrm{d}\mathbf{x}\Big\}.$$

Remark 5.1. *This inequality gives a good bound for the part outside the Maier-Saupe space, and a weak bound for the part inside the Maier-Saupe space.*

Remark 5.2. *By letting ε tend to zero, the lower bound inequality implies that*

$$\big\langle \mathcal{G}_{h_{\mathbf{n},\eta}} f, \mathcal{H}_0 f\big\rangle \geq c\big\langle \mathcal{H}_0 f, f\big\rangle, \quad \mathcal{H}_0 = \mathcal{H}_{h_{\mathbf{n},\eta}},$$

which can also be deduced the following simple argument. Noting that $\mathcal{H}(\mathrm{Ker}\mathcal{G}_{h_{\mathbf{n},\eta}}) = 0$, we may assume that $f \in (\mathrm{Ker}\mathcal{G}^*_{h_{\mathbf{n},\eta}})^\perp$. Then by Poincaré inequality and Proposition 4.3, we have

$$\big\langle \mathcal{H}_0 f, \mathcal{A}\mathcal{H}_0 f\big\rangle \geq \big\langle \mathcal{R}\mathcal{H}_0 f, \mathcal{R}\mathcal{H}_0 f\big\rangle \geq \big\langle \mathcal{H}_0 f - \overline{\mathcal{H}_0 f}, \mathcal{H}_0 f - \overline{\mathcal{H}_0 f}\big\rangle$$
$$\geq \frac{\big\langle \mathcal{H}_0 f - \overline{\mathcal{H}_0 f}, f\big\rangle^2}{\langle f, f\rangle} \geq c\big\langle \mathcal{H}_0 f, f\big\rangle \geq c\langle f, f\rangle,$$

where $\overline{\mathcal{H}_0 f} = \frac{1}{4\pi}\int_{\mathbb{S}^2}\mathcal{H}_0 f \mathrm{d}\mathbf{m}$.

5.3. **Proof of the lower bound inequality.** First of all, we can get by direct calculations that

$$\begin{aligned}
\mathcal{U}_\varepsilon f &= \alpha\int_\Omega\int_{\mathbb{S}^2} g_\varepsilon(\mathbf{x}-\mathbf{x}')(1 - \mathbf{m}\mathbf{m} : \mathbf{m}'\mathbf{m}')f(\mathbf{x}',\mathbf{m}')\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}' \\
&= -\alpha\mathbf{m}\mathbf{m} : \int_\Omega g_\varepsilon(\mathbf{x}-\mathbf{x}')\mathbf{M}(\mathbf{x}')\mathrm{d}\mathbf{x}' \\
&= -\alpha(\mathbf{A}\cdot\hat{\mathbf{m}})(\mathbf{A}\cdot\hat{\mathbf{m}}) : \int_\Omega g_\varepsilon(\mathbf{x}-\mathbf{x}')\mathbf{M}(\mathbf{x}')\mathrm{d}\mathbf{x}' \\
&= -\alpha\hat m_i\hat m_j A_{ki}A_{lj}\big(g_\varepsilon * (A_{ki'}A_{lj'}\hat M_{i'j'})\big).
\end{aligned}$$

For the simplicity, we denote $f_0 = h_{\eta,\mathbf{n}}$ in what follows. Using (5.1)-(5.3), we get by very tedious calculations of competing the square that

$$
\begin{aligned}
&\big\langle \mathcal{G}^{\varepsilon}_{f_0} f, \mathcal{H}^{\varepsilon}_{f_0} f\big\rangle = \big\langle f_0 \mathcal{R}(\frac{f}{f_0} + \mathcal{U}_\varepsilon f), \mathcal{R}(\frac{f}{f_0} + \mathcal{U}_\varepsilon f)\big\rangle \\
&= \big\langle f_0 \mathcal{R}(\frac{f}{f_0} - \alpha \hat{m}_i \hat{m}_j N_{ij}), \mathcal{R}(\frac{f}{f_0} - \alpha \hat{m}_i \hat{m}_j N_{ij})\big\rangle \\
&= \int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{f}{f_0} - \alpha \hat{m}_i \hat{m}_j N_{ij})\big|^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} + \int_\Omega \int_{\mathbb{S}^2} \frac{f_0}{\sin^2\hat\theta} \big|\partial_{\hat\varphi}(\frac{f}{f_0} - \alpha \hat{m}_i \hat{m}_j N_{ij})\big|^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&= \int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{a_0}{f_0}) + \alpha(2N_{33} - N_{11} - N_{22}) \sin\hat\theta\cos\hat\theta\big|^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{a_1}{f_0}) - 2\alpha N_{13}\cos 2\hat\theta\big|^2 + \frac{f_0}{\sin^2\hat\theta}\big(\frac{a_1}{f_0} - 2\alpha N_{13}\sin\hat\theta\cos\hat\theta\big)^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{b_1}{f_0}) - 2\alpha N_{23}\cos 2\hat\theta\big|^2 + \frac{f_0}{\sin^2\hat\theta}\big(\frac{b_1}{f_0} - 2\alpha N_{23}\sin\hat\theta\cos\hat\theta\big)^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{a_2}{f_0}) - \alpha(N_{11} - N_{22})\sin\hat\theta\cos\hat\theta\big|^2 + \frac{f_0}{\sin^2\hat\theta}\big(\frac{2a_2}{f_0} - \alpha(N_{11} - N_{22})\sin^2\hat\theta\big)^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{b_2}{f_0}) - 2\alpha N_{12}\sin\hat\theta\cos\hat\theta\big|^2 + \frac{4f_0}{\sin^2\hat\theta}\big(\frac{b_2}{f_0} - \alpha N_{12}\sin^2\hat\theta\big)^2 \mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\sum_{k\geq 3}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{a_k}{f_0})\big|^2 + f_0\big|\partial_{\hat\theta}(\frac{b_k}{f_0})\big|^2 + \frac{k^2}{\sin^2\hat\theta}\frac{a_k^2 + b_k^2}{f_0}\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Making a change of variable $z = \cos\hat\theta$, we obtain

$$
\begin{aligned}
&\big\langle \mathcal{G}^{\varepsilon}_{f_0} f, \mathcal{H}^{\varepsilon}_{f_0} f\big\rangle \\
&= \int_\Omega \int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{a_0}{f_0}) - \alpha(2N_{33} - N_{11} - N_{22})z\big)^2 \mathrm{d}z\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{-1}^{1} f_0\big|\partial_z(\frac{a_1}{f_0})\sqrt{1-z^2} + 2\alpha N_{13}(2z^2-1)\big|^2 + \frac{f_0}{1-z^2}\big(\frac{a_1}{f_0} - 2\alpha N_{13} z\sqrt{1-z^2}\big)^2 \mathrm{d}z\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{-1}^{1} f_0\big|\partial_z(\frac{b_1}{f_0})\sqrt{1-z^2} + 2\alpha N_{23}(2z^2-1)\big|^2 + \frac{f_0}{1-z^2}\big(\frac{b_1}{f_0} - 2\alpha N_{23} z\sqrt{1-z^2}\big)^2 \mathrm{d}z\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{a_2}{f_0}) + \alpha(N_{11} - N_{22})z\big)^2 + \frac{f_0}{1-z^2}\big(\frac{2a_2}{f_0} - \alpha(N_{11} - N_{22})(1-z^2)\big)^2 \mathrm{d}z\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\int_\Omega \int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{b_2}{f_0}) + 2\alpha N_{12} z\big)^2 + \frac{4f_0}{1-z^2}\big(\frac{b_2}{f_0} - \alpha N_{12}(1-z^2)\big)^2 \mathrm{d}z\mathrm{d}\mathbf{x} \\
&\quad + \frac{1}{2}\sum_{k\geq 3}\int_\Omega \int_{\mathbb{S}^2} f_0 \big|\partial_{\hat\theta}(\frac{a_k}{f_0})\big|^2 + f_0\big|\partial_{\hat\theta}(\frac{b_k}{f_0})\big|^2 + \frac{k^2}{\sin^2\hat\theta}\frac{a_k^2 + b_k^2}{f_0}\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}.
\end{aligned}
$$

- **Lower bound for the term including a_0**

To deal with the cross term, we need to introduce a slightly different decomposition

$$
a_0(\mathbf{x}, z) = \hat\zeta_0(\mathbf{x}) f_0(z)\big(z^2 - \frac{A_2}{A_0}\big) + \hat\gamma_0(\mathbf{x}, z),
$$

where $\hat{\gamma}_0$ satisfies $\int_{-1}^{1}\hat{\gamma}_0(\mathbf{x},z)\big(3z^2-1-2\eta z^2(1-z^2)\big)\mathrm{d}z=0$. From the fact that

$$\begin{aligned}&\int_{-1}^{1} f_0(A_0z^2-A_2)\big(3z^2-1-2\eta z^2(1-z^2)\big)\mathrm{d}z\\&=3(A_0A_4-A_2^2)+2\eta(A_2(A_2-A_4)-A_0(A_4-A_6))>0,\end{aligned}$$

we know that $\hat{\zeta}_0$ are uniquely determined. Then we have

$$\begin{aligned}&\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big|\partial_z(\frac{a_0}{f_0})-\alpha N_0(\mathbf{x})z\big|^2\mathrm{d}z\mathrm{d}\mathbf{x}\\&=\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\hat{\gamma}_0(\mathbf{x},z)}{f_0})\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}+\Big(\int_\Omega\big(2\hat{\zeta}_0-\alpha N_0\big)^2\mathrm{d}\mathbf{x}\Big)\Big(\int_{-1}^{1}(1-z^2)z^2f_0\mathrm{d}z\Big)\\&\quad-2\int_\Omega\big(2A_0\hat{\zeta}_0(\mathbf{x})-\alpha N_0(\mathbf{x})\big)\int_{-1}^{1}\partial_z(\frac{\hat{\gamma}_0(\mathbf{x},z)}{f_0})f_0(z)z(1-z^2)\mathrm{d}z\mathrm{d}\mathbf{x}\\&=\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\hat{\gamma}_0(\mathbf{x},z)}{f_0})\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}+\Big(\int_\Omega\big(2\hat{\zeta}_0-\alpha N_0\big)^2\mathrm{d}\mathbf{x}\Big)\Big(\int_{-1}^{1}(1-z^2)z^2f_0\mathrm{d}z\Big)\\&\quad+2\int_\Omega\big(2A_0\hat{\zeta}_0(\mathbf{x})-\alpha N_0(\mathbf{x})\big)\int_{-1}^{1}\hat{\gamma}_0(\mathbf{x},z)\big(1-3z^2+2\eta z^2(1-z^2)\big)\mathrm{d}z\mathrm{d}\mathbf{x}\\&=\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\hat{\gamma}_0(\mathbf{x},z)}{f_0})\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}+\Big(\int_\Omega\big(2\hat{\zeta}_0-\alpha N_0\big)^2\mathrm{d}\mathbf{x}\Big)\Big(\int_{-1}^{1}(1-z^2)z^2f_0\mathrm{d}z\Big).\end{aligned}$$

Recall that we have decomposed $a_0(x,z)$ as

$$a_0(\mathbf{x},z)=\zeta_0(\mathbf{x})f_0(z)(z^2-\frac{A_2}{A_0})+\gamma_0(\mathbf{x},z),$$

thus $(\zeta_0-\hat{\zeta}_0)f_0(z)(z^2-\frac{A_2}{A_0})=\hat{\gamma}_0-\gamma_0$, which implies that

$$\zeta_0(\mathbf{x})-\hat{\zeta}_0(\mathbf{x})=\int_{-1}^{1}\hat{\gamma}_0(3z^2-1)\mathrm{d}z/(3A_0A_4-3A_2^2).$$

Thus we have

$$\begin{aligned}\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\gamma_0}{f_0})\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}&=\int_\Omega\int_{-1}^{1} f_0(1-z^2)\Big(\partial_z(\frac{\hat{\gamma}_0}{f_0})-2A_0(\zeta_0-\hat{\zeta}_0)z\Big)^2\mathrm{d}z\mathrm{d}\mathbf{x}\\&\le C\int_\Omega\int_{-1}^{1} f_0(1-z^2)\Big(\partial_z(\frac{\hat{\gamma}_0}{f_0})\Big)^2+\hat{\gamma}_0^2\mathrm{d}z\mathrm{d}\mathbf{x}.\end{aligned}$$

Then we infer that

$$\begin{aligned}&\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big|\partial_z(\frac{a_0}{f_0})-\alpha N_0(\mathbf{x})z\big|^2\mathrm{d}z\mathrm{d}\mathbf{x}\\&\ge c\Big\{\int_\Omega\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\gamma_0(\mathbf{x},z)}{f_0})\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}+\int_\Omega\big(2\zeta_0-\alpha N_0\big)^2\mathrm{d}\mathbf{x}\int_{-1}^{1}(1-z^2)z^2f_0\mathrm{d}z\Big\}\\&\ge c\Big\{\int_\Omega\int_{-1}^{1}\gamma_0^2+(\partial_z\gamma_0)^2\mathrm{d}z\mathrm{d}\mathbf{x}+\int_\Omega\big(2\zeta_0(\mathbf{x})-\alpha N_0\big)^2\mathrm{d}\mathbf{x}\Big\},\end{aligned}\tag{5.4}$$

where we used the following Poincaré type inequality in the last inequality:

Lemma 5.1. *There exists $c > 0$ such that if $\gamma(z)$ satisfies*

$$\int_{-1}^{1} \gamma \mathrm{d}z = 0, \quad \int_{-1}^{1} (3z^2 - 1)\gamma \mathrm{d}z = 0,$$

then there holds

$$\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\gamma}{f_0})\big)^2 \mathrm{d}z \geq c\int_{-1}^{1} \gamma^2 \mathrm{d}z.$$

Proof. We define $\bar{\gamma}(\mathbf{m}) = \gamma(\mathbf{m}\cdot\mathbf{n})$. By the assumption, we know

$$\int_{\mathbb{S}^2} \mathbf{m}\mathbf{m}\bar{\gamma}(\mathbf{m})\mathrm{d}\mathbf{m} = 0, \quad \mathcal{H}_0\bar{\gamma} = \frac{\bar{\gamma}}{f_0}.$$

Hence, $\bar{\gamma} \in (\mathrm{Ker}\mathcal{G}_{f_0})^{\perp}$ and we have

$$\int_{\mathbb{S}^2} f_0|\mathcal{R}\mathcal{H}_{f_0}\bar{\gamma}|^2 \mathrm{d}\mathbf{m} = 2\pi\int_{-1}^{1} f_0(1-z^2)\big(\partial_z(\frac{\gamma}{f_0})\big)^2 \mathrm{d}z.$$

Set $\bar{C} = \frac{1}{4\pi}\int \mathcal{H}_{f_0}\bar{\gamma}\mathrm{d}\mathbf{m}$. It follows from Poincaré inequality and Proposition 4.3 that

$$\begin{aligned}\int_{\mathbb{S}^2} f_0|\mathcal{R}\mathcal{H}_{f_0}\bar{\gamma}|^2 \mathrm{d}\mathbf{m} &\geq c\int_{\mathbb{S}^2} (\mathcal{H}_{f_0}\bar{\gamma} - \bar{C})^2 \mathrm{d}\mathbf{m} \geq c\frac{\big(\int_{\mathbb{S}^2}(\mathcal{H}_{f_0}\bar{\gamma} - \bar{C})\bar{\gamma}\mathrm{d}\mathbf{m}\big)^2}{\int_{\mathbb{S}^2}\bar{\gamma}^2\mathrm{d}\mathbf{m}} \\ &\geq c\frac{\big(\int_{\mathbb{S}^2}(\mathcal{H}_{f_0}\bar{\gamma})\bar{\gamma}\mathrm{d}\mathbf{m}\big)^2}{\int_{\mathbb{S}^2}\bar{\gamma}^2\mathrm{d}\mathbf{m}} \geq c\int_{\mathbb{S}^2}\bar{\gamma}^2\mathrm{d}\mathbf{m} = c4\pi^2\int_{-1}^{1}\gamma^2\mathrm{d}z,\end{aligned}$$

which completes the proof. □

- **Lower bound for the terms including a_1, b_1**

  Recall that we have a decomposition for $a_1(x,z)$ as

$$a_1(\mathbf{x},z) = \zeta_{a,1}(\mathbf{x})f_0(z)z\sqrt{1-z^2} + \gamma_{a,1}(\mathbf{x},z),$$

where $\int_{-1}^{1} z\sqrt{1-z^2}\gamma_{a,1}\mathrm{d}z = 0$. Thus, we can get

$$\begin{aligned}&\int_{\Omega}\int_{-1}^{1}\frac{f_0}{1-z^2}\big(\frac{a_1}{f_0} - 2\alpha N_{13}z\sqrt{1-z^2}\big)^2\mathrm{d}z\mathrm{d}\mathbf{x} \\ &\geq \int_{\Omega}\int_{-1}^{1} f_0\Big(\frac{\gamma_{a,1}(\mathbf{x},z)}{f_0} + \big(\zeta_{a,1}(\mathbf{x}) - 2\alpha N_{13}(\mathbf{x})\big)z\sqrt{1-z^2}\Big)^2\mathrm{d}z\mathrm{d}\mathbf{x} \\ &= \int_{\Omega}\int_{-1}^{1}\frac{\gamma_{a,1}^2(\mathbf{x},z)}{f_0}\mathrm{d}x\mathrm{d}\mathbf{x} + \int_{\Omega}\big(\zeta_{a,1}(\mathbf{x}) - 2\alpha N_{13}(\mathbf{x})\big)^2\mathrm{d}\mathbf{x}\int_{-1}^{1} f_0 z^2(1-z^2)\mathrm{d}z. \qquad (5.5)\end{aligned}$$

A lower bound for the terms including b_1 can be obtained in the same way.

- **Lower bound for the terms including a_2, b_2**

  We have decomposed $a_2(x,z)$ as

$$a_2(\mathbf{x},z) = \zeta_{a,2}(\mathbf{x})f_0(z)(1-z^2) + \gamma_{a,2}(\mathbf{x},z),$$

where $\int_{-1}^{1}\gamma_{a,2}(\mathbf{x},z)(1-z^2)\mathrm{d}z=0$. Then we have

$$\begin{aligned}&\int_\Omega\int_{-1}^{1}\frac{4f_0}{1-z^2}\big(\frac{a_2}{f_0}-\alpha N_2(1-z^2)\big)^2\mathrm{d}z\mathrm{d}\mathbf{x}\\&\geq\int_\Omega\int_{-1}^{1}4f_0\Big(\frac{\gamma_{a,2}(\mathbf{x},z)}{f_0}+\big(\zeta_{a,2}(\mathbf{x})-\alpha N_2(\mathbf{x})\big)(1-z^2)\Big)^2\mathrm{d}z\mathrm{d}\mathbf{x}\\&=\int_\Omega\int_{-1}^{1}\frac{\gamma_{a,2}^2(\mathbf{x},z)}{f_0}\mathrm{d}z\mathrm{d}\mathbf{x}+\int_\Omega\big(\zeta_{a,2}(\mathbf{x})-\alpha N_2(\mathbf{x})\big)^2\mathrm{d}\mathbf{x}\int_{-1}^{1}f_0(1-z^2)^2\mathrm{d}z.\end{aligned}\tag{5.6}$$

We can obtain a similar bound for the terms including b_2.

Finally, the lower bound inequality follows from (5.4), (5.5) and (5.6). □

6. Small Deborah number limit for the homogeneous system

This section is devoted to justifying the small Deborah number limit for the homogeneous system (2.1). For the simplicity of notations, throughout this section we denote

$$\begin{aligned}&h_{\mathbf{n}}=h_{\eta,\mathbf{n}},\quad \mathcal{G}_{\mathbf{n}}=\mathcal{G}_{h_{\mathbf{n}}},\quad \mathcal{A}_{\mathbf{n}}=\mathcal{A}_{h_{\mathbf{n}}},\\&\|f\|_k=\|f\|_{H^k(\mathbb{S}^2)},\quad |f|_p=\|f\|_{L^p(\mathbb{S}^2)},\quad \big\langle f,g\big\rangle=\int_{\mathbb{S}^2}f(\mathbf{m})g(\mathbf{m})\mathrm{d}\mathbf{m}.\end{aligned}$$

6.1. **Hilbert expansion.** As in the fluid dynamic limit of the Boltzmann equation [1], we make the Hilbert expansion for $f^\varepsilon(\mathbf{m},t)$:

$$f^\varepsilon(\mathbf{m},t)=\sum_{k=0}^{3}\varepsilon^k f_k(\mathbf{m},t)+\varepsilon^2 f_R^\varepsilon(\mathbf{m},t).\tag{6.1}$$

Plugging it into (2.1) and collecting the terms with the same order with respect to ε, we find that $f_k(\mathbf{m},t)(k=0,1,2,3)$ satisfies

$$\mathcal{R}f_0+f_0\mathcal{R}\mathcal{U}f_0=0,\qquad \text{that is}\quad f_0(\mathbf{m},t)=h_{\mathbf{n}(t)}(\mathbf{m}),\tag{6.2}$$

$$\frac{\partial f_0}{\partial t}=\mathcal{G}_{\mathbf{n}(t)}f_1-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_0),\tag{6.3}$$

$$\frac{\partial f_1}{\partial t}=\mathcal{G}_{\mathbf{n}(t)}f_2+\mathcal{R}\cdot(f_1\mathcal{R}\mathcal{U}f_1)-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_1),\tag{6.4}$$

$$\frac{\partial f_2}{\partial t}=\mathcal{G}_{\mathbf{n}(t)}f_3+\mathcal{R}\cdot(f_1\mathcal{R}\mathcal{U}f_2)+\mathcal{R}\cdot(f_2\mathcal{R}\mathcal{U}f_1)-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_2).\tag{6.5}$$

The global in time existence of the Hilbert expansion is nontrivial, since f_1 satisfies a nonlinear equation. However, we find that the part of f_1 inside the kernel of $\mathcal{G}_{\mathbf{n}(t)}$ satisfies a linear equation by Lemma 6.2.

Proposition 6.1. *Let $\mathbf{n}(t)$ be a solution of (2.3) on $[0,T]$ with λ given by (2.5). We can construct smooth functions $f_k(\mathbf{m},t)(k=0,1,2,3)\in\mathcal{P}_0$ defined on $[0,T]$ such that (6.2)-(6.5) hold on $[0,T]$.*

We need the following two lemmas in order to prove it.

Lemma 6.1. [4, 5] *Let $f_0(\mathbf{m},t)=h_{\mathbf{n}(t)}(\mathbf{m})$. Then $f_0(\mathbf{m},t)$ satisfies*

$$\Big\langle\frac{\partial f_0}{\partial t}+\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_0),\psi\Big\rangle=0\tag{6.6}$$

for any $\psi\in\mathrm{Ker}\mathcal{G}^*_{\mathbf{n}(t)}$ if and only if $\mathbf{n}(t)$ is a solution of (2.3) with $\lambda=\lambda(\alpha)$.

Lemma 6.2. *For any $\phi, \tilde{\phi} \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}$, there holds*

$$\phi = -h_{\mathbf{n}}\mathcal{U}\phi, \quad \big\langle (\mathcal{U}\phi)^2, \tilde{\phi}\big\rangle = 0.$$

Proof. For $\phi \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}$, there exists $\mathbf{\Theta} \in \mathbb{R}^3$ such that $\phi = \mathbf{\Theta} \cdot \mathcal{R}h_{\mathbf{n}}$ by Theorem 4.1. Due to $\ln h_{\mathbf{n}} = -\mathcal{U}h_{\mathbf{n}}$, we see that

$$\mathcal{R}h_{\mathbf{n}} = -h_{\mathbf{n}}\mathcal{R}\mathcal{U}h_{\mathbf{n}} = -h_{\mathbf{n}}\mathcal{U}\mathcal{R}h_{\mathbf{n}}.$$

Hence, $\phi = -h_{\mathbf{n}}\mathcal{U}\phi$. To prove the second equality, we choose the sphere coordinates such that $\mathbf{n} = (0,0,1)$ and $\mathbf{m} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Let $\tilde{\phi} = \tilde{\mathbf{\Theta}} \cdot \mathcal{R}h_{\mathbf{n}}$ for some $\tilde{\mathbf{\Theta}} \in \mathbb{R}^3$. Then we have

$$\begin{aligned}\phi &= \eta h_{\mathbf{n}}\cos\theta\sin\theta\big(\Theta_1\sin\varphi - \Theta_2\cos\varphi\big),\\ \tilde{\phi} &= \eta h_{\mathbf{n}}\cos\theta\sin\theta\big(\tilde{\Theta}_1\sin\varphi - \tilde{\Theta}_2\cos\varphi\big).\end{aligned}$$

It is easy to check that

$$\int_0^{2\pi}(\Theta_1\sin\varphi - \Theta_2\cos\varphi)^2(\tilde{\Theta}_1\sin\varphi - \tilde{\Theta}_2\cos\varphi)\mathrm{d}\varphi = 0,$$

which implies that $\int_{\mathbb{S}^2}\frac{\phi^2}{h_{\mathbf{n}}^2}\tilde{\phi}\mathrm{d}\mathbf{m} = 0$, or equivalently $\langle(\mathcal{U}\phi)^2, \tilde{\phi}\rangle = 0$ by $\phi = -h_{\mathbf{n}}\mathcal{U}\phi$. □

Proof of Proposition 6.1. Let us first solve f_1 and write $f_1 = \phi(t) + \phi^{\perp}(t)$, where $\phi \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}, \phi^{\perp} \in (\mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*)^{\perp}$. Then $\phi^{\perp}$ will be determined by (6.3), while ϕ will be determined by (6.4). However, (6.3) has a solution $\phi^{\perp}$ if and only if

$$\Big\langle \frac{\partial f_0}{\partial t} + \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_0), \psi\Big\rangle = 0$$

for any $\psi \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*$. From Lemma 6.1, this is equivalent to require that $\mathbf{n}(t)$ is a solution of (2.3) with $\lambda = \lambda(\alpha)$. Given $\phi^{\perp}$, (6.4) implies that ϕ satisfies

$$\Big\langle \frac{\partial\phi}{\partial t}, \psi\Big\rangle = \big\langle \mathcal{R}\cdot\big((\phi+\phi^{\perp})\mathcal{R}\mathcal{U}(\phi+\phi^{\perp})\big) - \mathcal{R}\cdot\big(\mathbf{m}\times(\kappa\cdot\mathbf{m})(\phi+\phi^{\perp})\big), \psi\big\rangle - \Big\langle\frac{\partial\phi^{\perp}}{\partial t}, \psi\Big\rangle$$

for any $\psi \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*$. Since $\mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*$ is a two dimensional space, we take ψ_1, ψ_2 as a base of $\mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*$ and write $\phi = a_1(t)\psi_1 + a_2(t)\psi_2$. Then we can get an ODE system for $\big(a_1(t), a_2(t)\big)$. For any $\psi \in \mathrm{Ker}\mathcal{G}_{\mathbf{n}}^*$, we write $\psi = \mathcal{A}^{-1}\tilde{\phi}$ with $\tilde{\phi} \in \mathrm{Ker}\mathcal{G}$. Due to Lemma 6.2, we find that

$$\begin{aligned}&\big\langle \mathcal{R}\cdot(\phi\mathcal{R}\mathcal{U}\phi), \psi\big\rangle = \big\langle \mathcal{R}\cdot(\phi\mathcal{R}\mathcal{U}\phi), \mathcal{A}_{\mathbf{n}}^{-1}\tilde{\phi}\big\rangle = -\big\langle \phi\mathcal{R}\mathcal{U}\phi, \mathcal{R}\mathcal{A}_{\mathbf{n}}^{-1}\tilde{\phi}\big\rangle\\ &= \big\langle h_{\mathbf{n}}\mathcal{U}\phi\mathcal{R}\mathcal{U}\phi, \mathcal{R}\mathcal{A}_{\mathbf{n}}^{-1}\tilde{\phi}\big\rangle = \frac{1}{2}\big\langle \mathcal{R}\big[(\mathcal{U}\phi)^2\big], h_{\mathbf{n}}\mathcal{R}\mathcal{A}_{\mathbf{n}}^{-1}\tilde{\phi}\big\rangle = \frac{1}{2}\big\langle(\mathcal{U}\phi)^2, \tilde{\phi}\big\rangle = 0.\end{aligned}$$

This means that $\big(a_1(t), a_2(t)\big)$ satisfies a linear ODE system, hence is global in time.

Once f_1 is determined, we can get f_2 and f_3 by solving (6.4) and (6.5) in a similar way (note that the equation for f_2 is linear). □

6.2. **Error estimates.** This subsection is devoted to proving Theorem 2.1. Due to the weak nonlinearity of the kinetic equation (2.1), given the initial data $f_0^{\varepsilon} \in H^1(\mathbb{S}^2)$, it is easy to show by standard energy method that there exists a unique global solution $f^{\varepsilon}(\mathbf{m}, t)$ to (2.1) such that

$$f^{\varepsilon} \in C([0,+\infty); H^1(\mathbb{S}^2)) \cap L^2(0,T; H^2(\mathbb{S}^2)) \quad \text{for any } T < +\infty.$$

Thanks to Proposition 6.1 and (2.1), it is easy to find that f_R^ε satisfies

$$\begin{aligned}\frac{\partial}{\partial t}f_R^\varepsilon(\mathbf{m},t)=&\frac{1}{\varepsilon}\mathcal{G}_{\mathbf{n}}f_R^\varepsilon-\mathcal{R}\cdot\big(\mathbf{m}\times\kappa\cdot\mathbf{m}f_R^\varepsilon\big)+\varepsilon\mathcal{R}\cdot\big(f_R^\varepsilon\mathcal{R}\mathcal{U}f_R^\varepsilon\big)\\&+\sum_{i=1}^{3}\varepsilon^{i-1}\mathcal{R}\cdot\big(f_i\mathcal{R}\mathcal{U}f_R^\varepsilon+f_R\mathcal{R}\mathcal{U}f_i\big)+\varepsilon A,\end{aligned}\tag{6.7}$$

where A is given by

$$A=\frac{\partial}{\partial t}f_3+\sum_{1\le i,j\le 3,i+j\ge 4}\varepsilon^{i+j-4}\mathcal{R}\big(f_i\mathcal{R}\mathcal{U}f_j\big).$$

To complete the proof, it suffices to prove that

$$\|f_R^\varepsilon(t)\|_{-1}\le C\quad\text{ for all }\quad 0\le t\le T.\tag{6.8}$$

For this purpose, we need the following lemmas.

Lemma 6.3.

$$\big[\frac{\partial}{\partial t},\mathcal{A}_{\mathbf{n}}^{-1}\big]g(\mathbf{m},t)=\mathcal{A}_{\mathbf{n}}^{-1}\mathcal{R}\cdot\Big(\frac{\partial f_0}{\partial t}\mathcal{R}(\mathcal{A}_{\mathbf{n}}^{-1}g)\Big).$$

Proof. Direct calculations give that

$$\begin{aligned}\frac{\partial}{\partial t}(\mathcal{A}_{\mathbf{n}}g)=&-\frac{\partial}{\partial t}\mathcal{R}\cdot(h_{\mathbf{n}}\mathcal{R}g)=-\mathcal{R}\cdot\Big(\frac{\partial h_{\mathbf{n}}}{\partial t}\mathcal{R}g\Big)-\mathcal{R}\cdot\Big(h_{\mathbf{n}}\mathcal{R}\frac{\partial g}{\partial t}\Big)\\=&-\mathcal{R}\cdot\Big(\frac{\partial h_{\mathbf{n}}}{\partial t}\mathcal{R}g\Big)+\mathcal{A}_{\mathbf{n}}\frac{\partial g}{\partial t}.\end{aligned}$$

Replacing g by $\mathcal{A}_{\mathbf{n}}^{-1}g$, the lemma follows. □

Lemma 6.4. *For any vector field $V\in C^1(\mathbb{S}^2)$, there holds*

$$\big\langle\mathcal{R}\cdot(Vf),\mathcal{A}_{\mathbf{n}}^{-1}f\big\rangle\le C\big(|V|_\infty+|\mathcal{R}V|_\infty\big)\big\langle f,\mathcal{A}_{\mathbf{n}}^{-1}f\big\rangle.$$

Proof. Let $V=(V_1,\ V_2,\ V_3)^T$, $\mathcal{R}f=(R_1f,\ R_2f,\ R_3f)^T$, and $g=\mathcal{A}_{\mathbf{n}}^{-1}f$. Recalling $\mathcal{A}_{\mathbf{n}}f=-\mathcal{R}\cdot(h_{\mathbf{n}}\mathcal{R}f)$, we get

$$\begin{aligned}\big\langle\mathcal{R}\cdot(Vf),\mathcal{A}_{\mathbf{n}}^{-1}f\big\rangle=&-\big\langle R_i[V_iR_j(h_{\mathbf{n}}R_jg)],g\big\rangle\\=&-\big\langle h_{\mathbf{n}}R_jg,(R_jV_i)R_ig+V_iR_jR_ig\big\rangle\\=&-\big\langle h_{\mathbf{n}}R_jg,(R_jV_i)R_ig-V_i\epsilon^{kji}R_kg\big\rangle+\big\langle h_{\mathbf{n}}R_jg,V_iR_iR_jg\big\rangle\\=&-\big\langle h_{\mathbf{n}}R_jg,(R_jV_i)R_ig-V_i\epsilon^{kji}R_kg\big\rangle-\frac{1}{2}\big\langle R_i(h_{\mathbf{n}}V_i),(R_jg)^2\big\rangle,\end{aligned}$$

which implies the lemma by using the fact that

$$|\mathcal{R}g|_2\le C\|f\|_{-1}\le C\big\langle f,\mathcal{A}_{\mathbf{n}}^{-1}f\big\rangle.$$

The proof is finished. □

Now we are in position to prove (6.8). With the help of Lemma 6.3, we make the energy estimate for (6.7) to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \\
&= \langle \frac{\partial}{\partial t} f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle + \frac{1}{2}\Big\langle \mathcal{A}_{\mathbf{n}}^{-1}\mathcal{R}\cdot\Big(\frac{\partial f_0}{\partial t}\mathcal{R}(\mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon)\Big), f_R^\varepsilon\Big\rangle \\
&= \frac{1}{\varepsilon}\langle \mathcal{G}_{\mathbf{n}} f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle - \langle \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m} f_R^\varepsilon), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \\
&\quad + \langle \varepsilon\mathcal{R}\cdot(f_R^\varepsilon \mathcal{R}\mathcal{U} f_R^\varepsilon), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle + \sum_{i=1}^{3}\varepsilon^{i-1}\langle \mathcal{R}\cdot(f_i\mathcal{R}\mathcal{U} f_R^\varepsilon + f_R^\varepsilon\mathcal{R}\mathcal{U} f_i), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \\
&\quad + \varepsilon\langle A, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle - \frac{1}{2}\Big\langle \frac{\partial f_0}{\partial t}\mathcal{R}(\mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon), \mathcal{R}\mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\Big\rangle.
\end{aligned}$$

Since $h_{\mathbf{n}}$ is a stable critical point, we know from Proposition 4.3 that

$$\langle \mathcal{G}_{\mathbf{n}} f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle = -\langle \mathcal{H}_{\mathbf{n}} f_R^\varepsilon, f_R^\varepsilon\rangle \le 0.$$

We infer from Lemma 6.4 that

$$\begin{aligned}
&-\langle \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m} f_R^\varepsilon), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \le C\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle, \\
&\langle \varepsilon\mathcal{R}\cdot(f_R^\varepsilon\mathcal{R}\mathcal{U} f_R^\varepsilon), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \le C\varepsilon\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle^{\frac{3}{2}}, \\
&\langle \mathcal{R}\cdot(f_R^\varepsilon\mathcal{R}\mathcal{U} f_i), \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle \le C\|f_i\|_{-1}\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle,
\end{aligned}$$

while the other terms on the right hand side are bounded by

$$\Big(\sum_{i=1}^{3}\varepsilon^{i-1}|f_i|_2 + |\partial_t f_0|_2\Big)\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle + \varepsilon\|A\|_{-1}\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle^{\frac{1}{2}}.$$

Here we use the fact that $|\mathcal{R}^k\mathcal{U} f_R^\varepsilon|_\infty$ for $k\in\mathbf{N}$ can be bounded by $\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle^{\frac{1}{2}}$ and $\langle f_R^\varepsilon, \mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon\rangle^{\frac{1}{2}} \sim \|f_R^\varepsilon\|_{-1}$. In conclusion, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_R^\varepsilon\|_{-1}^2 \le C\big(\|f_R^\varepsilon\|_{-1}^2 + \varepsilon\|f_R^\varepsilon\|_{-1}^3 + \varepsilon\|f_R^\varepsilon\|_{-1}\big).$$

This implies (6.8). □

6.3. **The Lesile stress and coefficients.** This subsection is devoted to proving Theorem 2.2. We introduce the 2-order tensor $\mathbf{Q}_2[f]$ and 4-order tensor $\mathbf{Q}_4[f]$ as follows

$$\begin{aligned}
&\mathbf{Q}_2[f] = \big\langle \mathbf{mm} - \frac{1}{3}\mathbf{I}\big\rangle_f, \\
&\mathbf{Q}_{4\alpha\beta\gamma\mu}[f] = \Big\langle m_\alpha m_\beta m_\gamma m_\mu - \frac{1}{7}(m_\alpha m_\beta\delta_{\gamma\mu} + m_\gamma m_\mu\delta_{\alpha\beta} + m_\alpha m_\gamma\delta_{\beta\mu} + m_\beta m_\mu\delta_{\alpha\gamma} \\
&\qquad + m_\alpha m_\mu\delta_{\beta\gamma} + m_\beta m_\gamma\delta_{\alpha\mu}) + \frac{1}{35}(\delta_{\alpha\beta}\delta_{\gamma\mu} + \delta_{\alpha\gamma}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\gamma})\Big\rangle_f.
\end{aligned}$$

Lemma 6.5. *It holds that*

$$\begin{aligned}
&\mathbf{Q}_2[h_{\mathbf{n}}] = \langle P_2(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\mathbf{n}}}(\mathbf{nn} - \frac{1}{3}), \\
&\mathbf{Q}_{4\alpha\beta\gamma\mu}[h_{\mathbf{n}}] = \langle P_4(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\mathbf{n}}}\Big(n_\alpha n_\beta n_\gamma n_\mu - \frac{1}{7}(n_\alpha n_\beta\delta_{\gamma\mu} + n_\gamma n_\mu\delta_{\alpha\beta} + n_\alpha n_\gamma\delta_{\beta\mu} \\
&\qquad + n_\beta n_\mu\delta_{\alpha\gamma} + n_\alpha n_\mu\delta_{\beta\gamma} + n_\beta n_\gamma\delta_{\alpha\mu}) + \frac{1}{35}(\delta_{\alpha\beta}\delta_{\gamma\mu} + \delta_{\alpha\gamma}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\gamma})\Big).
\end{aligned}$$

Proof. We only prove the first equality, the proof of the second equality is similar but more complicated. Since both sides of the first equality are tensors, they are coordinate-independent. So, we may choose the sphere coordinates such that $\mathbf{n} = (0,0,1)$ and $\mathbf{m} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Since $h_{\mathbf{n}}(\mathbf{m})$ depends only on $\cos\theta$, it is easy to check that

$$\begin{aligned}
&\int_{\mathbb{S}^2} m_i m_j h_{\mathbf{n}}(\mathbf{m})\mathrm{d}\mathbf{m} = 0 \quad \text{for} \quad i \neq j,\\
&\int_{\mathbb{S}^2} (m_3^2 - \frac{1}{3}) h_{\mathbf{n}}(\mathbf{m})\mathrm{d}\mathbf{m} = \frac{2}{3}\int_{\mathbb{S}^2} \frac{1}{2}\big(3\cos^2\theta - 1\big) h_{\mathbf{n}}(\mathbf{m})\mathrm{d}\mathbf{m} = \frac{2}{3}\big\langle P_2(\cos\theta)\big\rangle_{h_{\mathbf{n}}},\\
&\int_{\mathbb{S}^2} (m_1^2 - \frac{1}{3}) h_{\mathbf{n}}(\mathbf{m})\mathrm{d}\mathbf{m} = \int_{\mathbb{S}^2} (m_2^2 - \frac{1}{3}) h_{\mathbf{n}}(\mathbf{m})\mathrm{d}\mathbf{m} = -\frac{1}{3}\big\langle P_2(\cos\theta)\big\rangle_{h_{\mathbf{n}}},
\end{aligned}$$

which give the first equality. □

Lemma 6.6. *Let $f^\varepsilon(t)$ be given by Theorem 2.1 and $P(x)$ be a smooth function on $\mathbf{R}$. Then we have*

$$\big|\big\langle P(\mathbf{m}\cdot\mathbf{n}(t))\big\rangle_{f^\varepsilon(t)} - \big\langle P(\mathbf{m}\cdot\mathbf{n}(t))\big\rangle_{h_{\mathbf{n}(t)}}\big| \leq C\varepsilon.$$

Proof. By the definition, we get

$$\big\langle P(\mathbf{m}\cdot\mathbf{n})\big\rangle_{f^\varepsilon} - \big\langle P(\mathbf{m}\cdot\mathbf{n})\big\rangle_{h_{\mathbf{n}}} = \int_{\mathbb{S}^2} P(\mathbf{m}\cdot\mathbf{n})\Big(\sum_{k=1}^{3}\epsilon^k f_k(\mathbf{m},t) + \epsilon^2 f_R^\varepsilon(\mathbf{m},t)\Big)\mathrm{d}\mathbf{m}.$$

Using the facts that f_1, f_2, f_3 are bounded and

$$\begin{aligned}
\int_{\mathbb{S}^2} P(\mathbf{m}\cdot\mathbf{n}) f_R^\varepsilon(\mathbf{m},t)\mathrm{d}\mathbf{m} &= \int_{\mathbb{S}^2} \mathcal{A}_{\mathbf{n}} P(\mathbf{m}\cdot\mathbf{n})\mathcal{A}_{\mathbf{n}}^{-1} f_R^\varepsilon(\mathbf{m},t)\mathrm{d}\mathbf{m}\\
&\leq \|\mathcal{A}_{\mathbf{n}} P(\mathbf{m}\cdot\mathbf{n})\|_0 \|f_R^\varepsilon\|_{-1},
\end{aligned}$$

the lemma follows. □

Now we are in position to prove Theorem 2.2. Direct computation shows that

$$\begin{aligned}
\frac{d}{dt}\mathbf{Q}_2[f^\varepsilon] =& \frac{1}{\varepsilon}\int_{\mathbb{S}^2} (\mathbf{mm} - \frac{1}{3}\mathbf{I})\big(\mathcal{R}\cdot(f^\varepsilon\mathcal{R}\mu^\varepsilon) - \varepsilon\mathcal{R}(\mathbf{m}\times\kappa\cdot\mathbf{m}f^\varepsilon)\big)\mathrm{d}\mathbf{m}\\
=& \frac{1}{\varepsilon}\big\langle \mathbf{m}\times\mathcal{R}\mu^\varepsilon\mathbf{m} + \mathbf{mm}\times\mathcal{R}\mu^\varepsilon\big\rangle_{f^\varepsilon} - \big(2\mathbf{D} : \langle\mathbf{mmmm}\rangle_{f^\varepsilon}\\
&- \mathbf{D}\cdot\langle\mathbf{mm}\rangle_{f^\varepsilon} + \mathbf{\Omega}\cdot\langle\mathbf{mm}\rangle_{f^\varepsilon} - \langle\mathbf{mm}\rangle_{f^\varepsilon}\cdot(\mathbf{D}+\mathbf{\Omega})\big).
\end{aligned}$$

So by Lemma 6.6, the stress σ^ε can be written as

$$\begin{aligned}
\sigma^\varepsilon =& \frac{1}{2}\mathbf{D} : \langle\mathbf{mmmm}\rangle_{f^\varepsilon} - \frac{1}{\varepsilon}\langle\mathbf{mm}\times\mathcal{R}\mu[f^\varepsilon]\rangle_{f^\varepsilon}\\
=& \frac{1}{2}\mathbf{D} : \langle\mathbf{mmmm}\rangle_{f^\varepsilon} - \frac{1}{2}\Big(2\mathbf{D} : \langle\mathbf{mmmm}\rangle_{f^\varepsilon} - \mathbf{D}\cdot\langle\mathbf{mm}\rangle_{f^\varepsilon}\\
&+ \mathbf{\Omega}\cdot\langle\mathbf{mm}\rangle_{f^\varepsilon} - \langle\mathbf{mm}\rangle_{f^\varepsilon}\cdot(\mathbf{D}+\mathbf{\Omega}) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}_2[f^\varepsilon]\Big)\\
=& \frac{1}{2}\mathbf{D} : \langle\mathbf{mmmm}\rangle_{h_{\mathbf{n}}} - \frac{1}{2}\Big(2\mathbf{D} : \langle\mathbf{mmmm}\rangle_{h_{\mathbf{n}}} - \mathbf{D}\cdot\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}\\
&+ \mathbf{\Omega}\cdot\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}} - \langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}\cdot(\mathbf{D}+\mathbf{\Omega}) + \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}_2[h_{\mathbf{n}}]\Big)\\
&- \frac{1}{2}\mathbf{Q}_2\Big[\sum_{k=1}^{3}\epsilon^k\frac{\partial}{\partial t}f_k(\mathbf{m},t) + \epsilon^2\frac{\partial}{\partial t}f_R^\varepsilon(\mathbf{m},t)\Big] + C\varepsilon.
\end{aligned}$$

For any constant vector $U,\ V$, we have

$$\varepsilon^2 U^T\cdot\mathbf{Q}_2\Big[\frac{\partial}{\partial t}f_R^\varepsilon(\mathbf{m},t)\Big]\cdot V=\varepsilon^2\int_{\mathbb{S}^2}(\mathbf{m}\cdot U)(\mathbf{m}\cdot V)\Big[\varepsilon^{-1}\mathcal{G}_{\mathbf{n}}f_R^\varepsilon-\mathcal{R}(\mathbf{m}\times\kappa\cdot\mathbf{m}f_R^\varepsilon)$$
$$+\varepsilon\mathcal{R}\cdot(f_R^\varepsilon\mathcal{R}\mathcal{U}f_R^\varepsilon)+\sum_{i=1}^{3}\varepsilon^{i-1}\mathcal{R}\cdot(f_i^\varepsilon\mathcal{R}\mathcal{U}f_R^\varepsilon+f_R^\varepsilon\mathcal{R}\mathcal{U}f_i^\varepsilon)+\varepsilon A\Big]\mathrm{d}\mathbf{m},$$

which is bounded by $C\varepsilon\|f_R\|_{-1}$ by using the argument of integration by parts. Then we infer that

$$\sigma^\varepsilon=\frac{1}{2}\mathbf{D}:\langle\mathbf{mmmm}\rangle_{h_{\mathbf{n}}}-\frac{1}{2}\Big(2\mathbf{D}:\langle\mathbf{mmmm}\rangle_{h_{\mathbf{n}}}-\mathbf{D}\cdot\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}$$
$$+\mathbf{\Omega}\cdot\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}-\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}\cdot(\mathbf{D}+\mathbf{\Omega})+\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Q}_2[h_{\mathbf{n}}]\Big)+C\varepsilon.$$

We see from the definition of $\mathbf{Q}_4$ that

$$\mathbf{D}:\langle\mathbf{mmmm}\rangle_{h_{\mathbf{n}}}=\mathbf{D}:\mathbf{Q}_4+\frac{1}{7}\mathbf{D}:\langle\mathbf{mm}\rangle_{h_{\mathbf{n}}}\mathbf{I}+\frac{2}{7}(\mathbf{D}:\mathbf{Q}_2+\mathbf{Q}_2:\mathbf{D})+\frac{2}{15}\mathbf{D}.$$

Using Lemma 6.5, we can calculate

$$\sigma^\varepsilon=\frac{1}{2}\Big(-S_4(\mathbf{D}:\mathbf{nn})\mathbf{nn}-\frac{S_2}{7}(\mathbf{D}:\mathbf{nn})\mathbf{I}-S_2(\mathbf{nN}+\mathbf{Nn})$$
$$+(\frac{8}{15}-\frac{10}{21}S_2-\frac{2}{35}S_4)\mathbf{D}+(\frac{5}{7}S_2+\frac{2}{7}S_4)(\mathbf{nn}\cdot\mathbf{D}+\mathbf{D}\cdot\mathbf{nn})\Big)+C\varepsilon.$$

So, we conclude that

$$\sigma^\varepsilon-\sigma^L-p\mathbf{I}=-\frac{S_2}{2}\Big(\mathbf{n}(\frac{1}{\lambda}\mathbf{N}-\mathbf{D}\cdot\mathbf{n})-(\frac{1}{\lambda}\mathbf{N}-\mathbf{D}\cdot\mathbf{n})\mathbf{n}\Big)+C\varepsilon=C\varepsilon,$$

here we used the equation (2.3). This completes the proof of Theorem 2.2. □

7. Small Deborah number limit for the inhomogeneous system

This section is devoted to justifying the small Deborah number limit for the inhomogeneous system (2.8)-(2.9). Throughout this section, we will use the following notations. $\langle,\rangle$ denotes the inner product on $L^2(\Omega\times\mathbb{S}^2)$ or $L^2(\Omega)$. We also denote $\|f\|_{L^p}=\|f\|_{L^p(\Omega\times\mathbb{S}^2)}(\|f\|_{L^p(\Omega)})$ for f defined on $\Omega\times\mathbb{S}^2$(for f defined on Ω), and $\|\cdot\|_{H^{0,k}}=\big\|\|\cdot\|_{H^k(\Omega)}\big\|_{L^2(\mathbb{S}^2)}$.

7.1. **Hilbert expansion.** Due to the choice of g, we have the formal expansion to the operator $\mathcal{U}_\varepsilon$:

$$\begin{aligned}\mathcal{U}_\varepsilon f-\mathcal{U}f&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2g_\varepsilon(\mathbf{x}-\mathbf{x}')(f(\mathbf{x}',\mathbf{m}',t)-f(\mathbf{x},\mathbf{m}',t))\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}'\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2g(\mathbf{y})(f(\mathbf{x}+\sqrt{\varepsilon}\mathbf{y},\mathbf{m}',t)-f(\mathbf{x},\mathbf{m}',t))\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{y}\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2g(\mathbf{y})\Big(\sum_{k\geq1}\frac{\varepsilon^{\frac{k}{2}}}{k!}(\mathbf{y}\cdot\nabla)^kf(\mathbf{x},\mathbf{m}',t)\Big)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{y}\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2g(\mathbf{y})\Big(\sum_{k\geq1}\frac{\varepsilon^{k}}{(2k)!}(\mathbf{y}\cdot\nabla)^{2k}f(\mathbf{x},\mathbf{m}',t)\Big)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{y}.\end{aligned}$$

We denote

$$U_k[f](\mathbf{x},\mathbf{m},t)=\frac{1}{(2k)!}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}\alpha|\mathbf{m}\times\mathbf{m}'|^2g(\mathbf{y})(\mathbf{y}\cdot\nabla)^{2k}f(\mathbf{x},\mathbf{m}',t)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{y}\quad\text{for}\quad k\geq1.$$

Formally, we have

$$\mathcal{U}_\varepsilon f = U_0[f] + \varepsilon U_1[f] + \varepsilon^2 U_2[f] + \cdots, \quad U_0[f] = \mathcal{U} f. \tag{7.1}$$

Then we make a formal expansion for the solution of (2.8)-(2.9):

$$f^\varepsilon(\mathbf{x},\mathbf{m},t) = \sum_{k=0}^{3} \varepsilon^k f_k(\mathbf{x},\mathbf{m},t) + \varepsilon^3 f_R^\varepsilon(\mathbf{x},\mathbf{m},t),$$
$$v^\varepsilon(\mathbf{x},t) = \sum_{k=0}^{2} \varepsilon^k v_k(\mathbf{x},t) + \varepsilon^3 v_R^\varepsilon(\mathbf{x},t).$$

Plugging them into (2.8)-(2.9) and collecting the terms with the same order with respect to ε, we find that

$$\mathcal{R} f_0 + f_0 \mathcal{R}\mathcal{U} f_0 = 0, \quad \text{that is,} \quad f_0 = h_{\eta,\mathbf{n}(t,\mathbf{x})}(\mathbf{m}); \tag{7.2}$$

and for the terms of order $O(1)$, there hold

$$\frac{\partial f_0}{\partial t} + \mathbf{v}_0 \cdot \nabla f_0 = \mathcal{G}_{f_0} f_1 + \mathcal{R}\cdot(f_0 \mathcal{R} U_1 f_0) - \mathcal{R}\cdot(\mathbf{m}\times(\nabla \mathbf{v}_0)^T \cdot \mathbf{m} f_0), \tag{7.3}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 =& -\nabla p_0 + \frac{\gamma}{Re}\Delta \mathbf{v}_0 + \frac{1-\gamma}{2Re}\nabla\cdot(\mathbf{D}_0 : \langle \mathbf{mmmm}\rangle_{f_0}) \\ &- \frac{1-\gamma}{Re}\Big\{\nabla\cdot\big\langle \mathbf{mm}\times(\mathcal{R} f_1 + \sum_{i+j+k=1} f_i U_j f_k)\big\rangle_1 + \langle \nabla U_1 f_0\rangle_{f_0}\Big\};\end{aligned} \tag{7.4}$$

and for the terms of order $O(\varepsilon)$, there hold

$$\begin{aligned}\frac{\partial f_1}{\partial t} + \mathbf{v}_0 \cdot \nabla f_1 =& \mathcal{G}_{f_0} f_2 + \mathcal{R}\cdot\Big(\sum_{i+j+k=2, j\geq 1} f_i \mathcal{R} U_j f_k\Big) - \mathbf{v}_1\cdot\nabla f_0 \\ &- \mathcal{R}\cdot\big(\mathbf{m}\times(\nabla\mathbf{v}_0)^T\cdot\mathbf{m} f_1 + \mathbf{m}\times(\nabla\mathbf{v}_1)^T\cdot\mathbf{m} f_0\big),\end{aligned} \tag{7.5}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_1 =& -\nabla p_1 + \frac{\gamma}{Re}\Delta \mathbf{v}_1 + \frac{1-\gamma}{2Re}\nabla\cdot\Big(\sum_{i+j=1}\mathbf{D}_i : \langle \mathbf{mmmm}\rangle_{f_j}\Big) - \mathbf{v}_1\cdot\nabla\mathbf{v}_0 \\ &- \frac{1-\gamma}{Re}\Big\{\nabla\cdot\big\langle \mathbf{mm}\times(\mathcal{R} f_2 + \sum_{i+j+k=2} f_i \mathcal{R} U_j f_k)\big\rangle_1 + \big\langle \sum_{i+j+k=2} f_i \nabla U_j f_k\big\rangle_1\Big\};\end{aligned} \tag{7.6}$$

and for the terms of $O(\varepsilon^2)$, there hold

$$\begin{aligned}\frac{\partial f_2}{\partial t} + \mathbf{v}_0 \cdot \nabla f_2 =& \mathcal{G}_{f_0} f_3 + \mathcal{R}\cdot\Big(\sum_{i+j+k=3, j\geq 1} f_i \mathcal{R} U_j f_k\Big) \\ &- \mathcal{R}\cdot\Big(\sum_{i+j=2}\mathbf{m}\times\big((\nabla\mathbf{v}_i)^T\cdot\mathbf{m}\big) f_j\Big) - \mathbf{v}_1\cdot\nabla f_1 - \mathbf{v}_2\cdot\nabla f_0,\end{aligned} \tag{7.7}$$

$$\begin{aligned}\frac{\partial \mathbf{v}_2}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_2 =& -\nabla p_2 + \frac{\gamma}{Re}\Delta \mathbf{v}_2 + \frac{1-\gamma}{2Re}\nabla\cdot\Big(\sum_{i+j=2}\mathbf{D}_i : \langle \mathbf{mmmm}\rangle_{f_j}\Big) \\ &- \frac{1-\gamma}{Re}\Big\{\nabla\cdot\big\langle \mathbf{mm}\times(\mathcal{R} f_3 + \sum_{i+j+k=3} f_i \mathcal{R} U_j f_k)\big\rangle_1 + \langle \sum_{i+j+k=3} f_i \nabla U_j f_k\rangle_1\Big\} \\ &- \mathbf{v}_1\cdot\nabla\mathbf{v}_1 - \mathbf{v}_2\cdot\nabla\mathbf{v}_0.\end{aligned} \tag{7.8}$$

Proposition 7.1. *Let $(\mathbf{v}_0, \mathbf{n}) \in C\big([0,T]; H^{20}(\Omega)\big)$ be a solution of (2.10)-(2.11) with λ given by (2.5) and Leslie coefficients defined by (2.6)-(2.7) on $[0,T]$. Then there exist the functions $f_i \in C\big([0,T]; H^{20-4i}(\Omega \times \mathbb{S}^2)\big)(i = 0,1,2,3)$ and $\mathbf{v}_i \in C\big([0,T]; H^{20-4i}(\Omega)\big)(i = 0,1,2)$ such that (7.2)-(7.8) holds on $[0,T]$.*

We need the following lemma.

Lemma 7.1. [5] *Let $f_0 = h_{\eta,\mathbf{n}(\mathbf{x},t)}(\mathbf{m})$. Then $f_0(\mathbf{x},\mathbf{m},t)$ satisfies*

$$\Big\langle \frac{\partial f_0}{\partial t} + \mathbf{v}_0 \cdot \nabla f_0 + \mathcal{R} \cdot (\mathbf{m} \times (\nabla \mathbf{v}_0)^T \cdot \mathbf{m} f_0) + \mathcal{R} \cdot (f_0 \mathcal{R} U_1 f_0), \psi \Big\rangle_{L^2(\mathbb{S}^2)} = 0$$

for any $\psi \in \mathrm{Ker}\mathcal{G}^*_{f_0}$ if and only if $\mathbf{n}(\mathbf{x},t)$ is a solution of (2.10) with $\mathbf{v} = \mathbf{v}_0$.

Proof of Proposition 7.1. We denote by $\mathbb{P}_{\mathrm{in}}$ the projection operator from $\mathcal{P}_0(\mathbb{S}^2)$ to $\mathrm{Ker}\mathcal{G}_{f_0}$, and denote by $\mathbb{P}_{\mathrm{out}}$ the projection operator from $\mathcal{P}_0(\mathbb{S}^2)$ to $(\mathrm{Ker}\mathcal{G}^*_{f_0})^\perp$. Let $\mathbb{P}_{\mathrm{in}} f_1 = \phi_1$, $\mathbb{P}_{\mathrm{out}} f_1 = \psi_1$. Now ψ_1 will be determined by (7.3), whose existence is ensured by Lemma 7.1. Once ψ_1 is determined, it can be proved that the equation (7.4) is equivalent to (2.11), see [5]. Now we solve $(\phi_1, \mathbf{v}_1)$. In what follows, we denote by $L(\phi, \mathbf{v})$ the terms which only depend on $(\phi, \mathbf{v})$(not their derivatives) linearly. Let $\phi_1 = \mathbf{n}^\perp \cdot \mathcal{R} f_0$. We have

$$\big(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\big)\phi_1 = \big(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\big)\mathbf{n}^\perp \cdot \mathcal{R} f_0 + \mathbf{n}^\perp \cdot \mathcal{R}\big((\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla) f_0\big).$$

This means that

$$\mathbb{P}_{\mathrm{out}}\Big((\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla)\phi_1\Big) = \mathbb{P}_{\mathrm{out}}\Big(\mathbf{n}^\perp \cdot \mathcal{R}\big((\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla) f_0\big)\Big) \triangleq L(\phi_1), \tag{7.9}$$

$$\mathbb{P}_{\mathrm{in}}\Big((\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla)\phi_1\Big) = \big(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\big)\phi_1 - L(\phi_1). \tag{7.10}$$

We also have

$$\mathbb{P}_{\mathrm{in}}\big(\mathcal{G}_{f_0} f_2 + \mathcal{R} \cdot (\phi_1 \mathcal{R}\mathcal{U}\phi_1)\big) = 0. \tag{7.11}$$

For a matrix κ, we denote

$$\mathcal{K}(\kappa) = \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot \big(\mathbf{m} \times \kappa \cdot \mathbf{m} f_0\big)\big), \quad \mathcal{L}(\kappa) = \mathbb{P}_{\mathrm{out}}\big(\mathcal{R} \cdot \big(\mathbf{m} \times \kappa \cdot \mathbf{m} f_0\big)\big),$$
$$\mathcal{B}_{in}(\phi_1) = \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot (f_0 \mathcal{R} U_1 \phi_1)\big), \quad \mathcal{B}_{out}(\phi_1) = \mathbb{P}_{\mathrm{out}}\big(\mathcal{R} \cdot (f_0 \mathcal{R} U_1 \phi_1)\big).$$

Taking $\mathbb{P}_{\mathrm{in}}$ for the equation of f_1, we get by (7.10) and (7.11) that

$$\begin{aligned}
\big(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla\big)\phi_1 &= L(\phi_1) - \mathcal{K}((\nabla \mathbf{v}_1)^T) + \mathcal{B}_{in}(\phi_1) \\
&\quad + \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot (\phi_1 \mathcal{R}\mathcal{U}\psi_1 + \psi_1 \mathcal{R}\mathcal{U}\phi_1 + \psi_1 \mathcal{R}\mathcal{U}\psi_1)\big) - \mathbf{v}_1 \cdot \nabla f_0 \\
&\quad - \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot \big(\mathbf{m} \times (\nabla \mathbf{v}_0)^T \cdot \mathbf{m} f_1\big)\big) + \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot (f_0 \mathcal{R} U_1 \psi_1)\big) + \mathbb{P}_{\mathrm{in}}\big(\mathcal{R} \cdot (f_1 \mathcal{R} U_1 f_0)\big) \\
&= -\mathcal{K}((\nabla \mathbf{v}_1)^T) + \mathcal{B}_{in}(\phi_1) + L(\phi_1, \mathbf{v}_1).
\end{aligned} \tag{7.12}$$

Taking $\mathbb{P}_{\mathrm{out}}$ for the equation of f_1, we get by (7.9) that

$$\begin{aligned}
-\big(\mathcal{G}_{f_0} f_2 + \mathcal{R} \cdot (\phi_1 \mathcal{R}\mathcal{U}\phi_1)\big) &= -L(\phi_1) + \mathcal{L}((\nabla \mathbf{v}_1)^T) - \mathcal{B}_{out}(\phi_1) \\
&\quad + \mathbb{P}_{\mathrm{out}}\big(\mathcal{R} \cdot (\phi_1 \mathcal{R}\mathcal{U}\psi_1 + \psi_1 \mathcal{R}\mathcal{U}\phi_1 + \psi_1 \mathcal{R}\mathcal{U}\psi_1)\big) \\
&\quad - \mathbb{P}_{\mathrm{out}}\big(\mathcal{R} \cdot \big(\mathbf{m} \times (\nabla \mathbf{v}_0)^T \cdot \mathbf{m} f_1\big)\big) + \mathbb{P}_{\mathrm{out}}\big(\mathcal{R} \cdot (f_0 \mathcal{R} U_1 \psi_1)\big) + \mathbb{P}_{\mathrm{out}}(\mathcal{R} \cdot (f_1 \mathcal{R} U_1 f_0)\big) \\
&= \mathcal{L}((\nabla \mathbf{v}_1)^T) - \mathcal{B}_{out}(\phi_1) - L(\phi_1, \mathbf{v}_1).
\end{aligned}$$

So, the stress in the equation of $\mathbf{v}_1$ can be rewritten as

$$\begin{aligned}
&-\big\langle \mathbf{mm}\times\big(\mathcal{R}f_2+\sum_{i+j+k=2} f_i\mathcal{R}U_jf_k\big)\big\rangle_1\\
&=-\frac{1}{2}\big\langle(\mathbf{mm}-\frac{1}{3}\mathbf{I})(\mathcal{G}_{f_0}f_2+\mathcal{R}\cdot(\phi_1\mathcal{R}\mathcal{U}\phi_1))\big\rangle_1-\big\langle\mathbf{mm}\times(f_0\mathcal{R}U_1\phi_1)\big\rangle_1+L(\phi_1)\\
&=\frac{1}{2}\big\langle(\mathbf{mm}-\frac{1}{3}\mathbf{I})(\mathcal{L}((\nabla\mathbf{v}_1)^T)-\mathcal{B}_{out}(\phi_1))\big\rangle_1-\big\langle\mathbf{mm}\times(f_0\mathcal{R}U_1\phi_1)\big\rangle_1+L(\phi_1,\mathbf{v}_1)\\
&\triangleq\sigma_1+\sigma_2+L(\phi_1,\mathbf{v}_1).
\end{aligned}$$

Set $\sigma_3=\mathbf{D}_1:\langle\mathbf{mmmm}\rangle_{f_0}$. Then the equation for $\mathbf{v}_1$ can be rewritten as

$$\begin{aligned}
\frac{\partial\mathbf{v}_1}{\partial t}-\frac{\gamma}{Re}\Delta\mathbf{v}_1+\mathbf{v}_0\cdot\nabla\mathbf{v}_1+\nabla p_1&=\frac{1-\gamma}{2Re}\nabla\cdot\big(\sigma_1+\sigma_2+\sigma_3\big)\\
&-\frac{1-\gamma}{Re}\big\langle f_0\nabla U_1\phi_1\big\rangle_1+\nabla L(\phi_1,\mathbf{v}_1)+L(\phi_1,\mathbf{v}_1).
\end{aligned}\tag{7.13}$$

In order to solve (7.12)-(7.13), we introduce the energy functional

$$E(t)=\langle\phi_1,\phi_1\rangle+\langle\phi_1,U_1\phi_1\rangle+\frac{Re}{1-\gamma}\langle\mathbf{v}_1,\mathbf{v}_1\rangle.$$

Due to the choice of g, it is easy to see that

$$\langle\phi_1,\phi_1\rangle+\langle\phi_1,U_1\phi_1\rangle\geq c\big(\langle\phi_1,\phi_1\rangle+\langle\nabla\phi_1,\nabla\phi_1\rangle\big).$$

Notice that $\langle\nabla\cdot\sigma_3,\mathbf{v}_1\rangle=-\langle\mathbf{D}_1:\langle\mathbf{mmmm}\rangle_{f_0},\mathbf{D}_1\rangle\leq 0$ and by Lemma 8.8 and Lemma 8.6,

$$\begin{aligned}
&\big\langle-\mathcal{K}((\nabla\mathbf{v}_1)^T)+\mathcal{B}_{in}(\phi_1),U_1\phi_1\big\rangle-\big\langle\nabla\cdot(\sigma_1+\sigma_2),\mathbf{v}_1\big\rangle\\
&\leq\big\langle-\mathcal{K}((\nabla\mathbf{v}_1)^T)+\mathcal{B}_{in}(\phi_1),-\mathcal{A}_{f_0}^{-1}(\mathcal{B}_{in}(\phi_1)+\mathcal{B}_{out}(\phi_1))\big\rangle\\
&\qquad+\frac{1}{2}\big\langle\mathcal{B}_{out}(\phi_1),\mathbf{m}\cdot\nabla\mathbf{v}_1\cdot\mathbf{m}\big\rangle+\big\langle U_1\phi_1,\mathcal{R}\big(\mathbf{m}\times(\nabla\mathbf{v}_1)^T\cdot\mathbf{m}f_0\big)\big\rangle\\
&=\big\langle-\mathcal{K}((\nabla\mathbf{v}_1)^T)+\mathcal{B}_{in}(\phi_1),-\mathcal{A}_{f_0}^{-1}(\mathcal{B}_{in}(\phi_1)+\mathcal{B}_{out}(\phi_1))\big\rangle+\frac{1}{2}\big\langle\mathcal{B}_{out}(\phi_1),-2\mathcal{A}_{f_0}^{-1}(\mathcal{K}(\mathbf{D}_1)+\mathcal{L}(\mathbf{D}_1))\big\rangle\\
&\qquad+\big\langle-\mathcal{A}_{f_0}^{-1}(\mathcal{B}_{in}(\phi_1)+\mathcal{B}_{out}(\phi_1)),\mathcal{K}((\nabla\mathbf{v}_1)^T)+\mathcal{L}((\nabla\mathbf{v}_1)^T)\big\rangle\\
&=-\big\langle\mathcal{B}_{in}(\phi_1),\mathcal{A}_{f_0}^{-1}\mathcal{B}_{in}(\phi_1)\big\rangle\leq 0.
\end{aligned}$$

Then by a simple energy estimate, we can deduce that

$$\frac{d}{dt}E(t)\leq C\big(1+E(t)\big).$$

The estimate of the higher order derivative for $(\phi_1,\mathbf{v}_1)$ can be obtained by introducing a similar energy functional. Once $(f_1,\mathbf{v}_1)$ is determined, we can get $(f_2,\mathbf{v}_2)$ and f_3 by solving (7.7)-(7.8) in a similar way. Here we omit the details. □

7.2. **The remainder equations.** We denote

$$
\begin{aligned}
&\widetilde{\mathbf{v}} = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \varepsilon^2 \mathbf{v}_2, \quad \widetilde{f} = f_1 + \varepsilon f_2 + \varepsilon^2 f_3, \quad \widetilde{\mathbf{D}} = \mathbf{D}_0 + \varepsilon \mathbf{D}_1 + \varepsilon^2 \mathbf{D}_2,\\
&X_{\mathcal{T}} = (f_1 + \varepsilon f_2 + \varepsilon^2 f_3)\mathcal{T}\mathcal{U}_\varepsilon f_3 + f_3 \mathcal{T}\mathcal{U}_\varepsilon (f_1 + \varepsilon f_2 + \varepsilon^2 f_3) + f_2 \mathcal{T}\mathcal{U}_\varepsilon f_2 + f_3 \mathcal{T}\frac{\mathcal{U}_\varepsilon - U_0}{\varepsilon} f_0\\
&\quad + f_1 \mathcal{T}\Big(\frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1 - \varepsilon^2 U_2}{\varepsilon^3} f_0 + \frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1}{\varepsilon^2} f_1 + \frac{\mathcal{U}_\varepsilon - U_0}{\varepsilon} f_2\Big)\\
&\quad + f_0 \mathcal{T}\Big(\frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1 - \varepsilon^2 U_2 - \varepsilon^3 U_3}{\varepsilon^4} f_0 + \frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1 - \varepsilon^2 U_2}{\varepsilon^3} f_1\\
&\quad + \frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1}{\varepsilon^2} f_2 + \frac{\mathcal{U}_\varepsilon - U_0}{\varepsilon} f_3\Big) + f_2 \mathcal{T}\big(\frac{\mathcal{U}_\varepsilon - U_0 - \varepsilon U_1}{\varepsilon^2} f_0 + \frac{\mathcal{U}_\varepsilon - U_0}{\varepsilon} f_1\big), \quad \text{for } \mathcal{T} = \mathcal{R} \text{ or } \nabla,\\
&L_1 = -\Big(\frac{\partial f_3}{\partial t} + \mathbf{v}_0 \cdot \nabla f_3 + \mathbf{v}_1 \cdot \nabla (f_2 + \varepsilon f_3) + \mathbf{v}_2 \cdot \nabla (f_1 + \varepsilon f_2 + \varepsilon^2 f_3)\Big)\\
&\quad + \mathcal{R} \cdot \Big(X_{\mathcal{R}} - \mathbf{m} \times (\nabla \mathbf{v}_0)^T \cdot \mathbf{m} f_3 - \mathbf{m} \times (\nabla \mathbf{v}_1)^T \cdot \mathbf{m}(f_2 + \varepsilon f_3) - \mathbf{m} \times (\nabla \mathbf{v}_2)^T \cdot \mathbf{m}(f_1 + \varepsilon f_2 + \varepsilon^2 f_3)\Big),\\
&L_2 = \frac{1-\gamma}{2Re} \nabla \cdot \Big\{\mathbf{D}_0 : \langle \mathbf{mmmm} f_3 \rangle + \mathbf{D}_1 : \langle \mathbf{mmmm}(f_2 + \varepsilon f_3)\rangle + \mathbf{D}_2 : \langle \mathbf{mmmm}(f_1 + \varepsilon f_2 + \varepsilon^2 f_3)\rangle\Big\}\\
&\quad - \frac{1-\gamma}{Re}\big\{\nabla \cdot \big\langle \mathbf{mm} \times X_{\mathcal{R}}\big\rangle_1 + \langle X_\nabla \rangle_1\big\} - \mathbf{v}_2 \cdot \nabla \mathbf{v}_1 - \mathbf{v}_1 \cdot \nabla \mathbf{v}_2 - \varepsilon \mathbf{v}_2 \cdot \nabla \mathbf{v}_2.
\end{aligned}
$$

Then we can deduce the equations of $(f_R^\varepsilon, \mathbf{v}_R^\varepsilon)$(drop ε for the simplicity):

$$
\begin{aligned}
&\frac{\partial f_R}{\partial t} + \widetilde{\mathbf{v}} \cdot \nabla f_R + \mathbf{v}_R \cdot \nabla (f_0 + \varepsilon \widetilde{f}) + \frac{1}{\varepsilon}\mathcal{A}_{f_0}\mathcal{H}_{f_0}^\varepsilon f_R\\
&= -\mathcal{R} \cdot \Big(\mathbf{m} \times (\nabla \widetilde{\mathbf{v}})^T \cdot \mathbf{m} f_R + \mathbf{m} \times (\nabla \mathbf{v}_R)^T \cdot \mathbf{m}(f_0 + \varepsilon \widetilde{f}) + \varepsilon^3 \mathbf{m} \times (\nabla \mathbf{v}_R)^T \cdot \mathbf{m} f_R\Big)\\
&\quad - \mathcal{R} \cdot \Big(f_R \mathcal{R}\mathcal{U}_\varepsilon \widetilde{f} + f_R \mathcal{R}\frac{(\mathcal{U}_\varepsilon - \mathcal{U}) f_0}{\varepsilon} + \widetilde{f}\mathcal{R}\mathcal{U}_\varepsilon f_R + \varepsilon^2 f_R \mathcal{R}\mathcal{U}_\varepsilon f_R\Big) + L_1,\\
&\frac{\partial \mathbf{v}_R}{\partial t} + \mathbf{v}_R \cdot \nabla \widetilde{\mathbf{v}} + \widetilde{\mathbf{v}} \cdot \nabla \mathbf{v}_R + \varepsilon^3 \mathbf{v}_R \cdot \nabla \mathbf{v}_R + \nabla p - \frac{\gamma}{Re}\Delta \mathbf{v}_R\\
&= \frac{1-\gamma}{2Re}\nabla \cdot \Big\{\widetilde{\mathbf{D}} : \langle \mathbf{mmmm} f_R \rangle_1 + \mathbf{D}_R : \langle \mathbf{mmmm}(f_0 + \varepsilon \widetilde{f})\rangle_1 + \varepsilon^3 \mathbf{D}_R : \langle \mathbf{mmmm} f_R \rangle_1\Big\}\\
&\quad - \frac{1-\gamma}{Re}\nabla \cdot \Big\langle \mathbf{mm} \times \Big(\frac{1}{\varepsilon} f_0 \mathcal{R}\mathcal{H}_{f_0}^\varepsilon f_R + f_R \mathcal{R}\mathcal{U}_\varepsilon \widetilde{f} + f_R \mathcal{R}\frac{(\mathcal{U}_\varepsilon - \mathcal{U}) f_0}{\varepsilon} + \widetilde{f}\mathcal{R}\mathcal{U}_\varepsilon f_R + \varepsilon^2 f_R \mathcal{R}\mathcal{U}_\varepsilon f_R\Big)\Big\rangle_1\\
&\quad - \frac{1-\gamma}{Re}\Big\langle \Big(\frac{1}{\varepsilon} f_0 \nabla \mathcal{H}_{f_0}^\varepsilon f_R + f_R \nabla \mathcal{U}_\varepsilon \widetilde{f} + f_R \nabla \frac{(\mathcal{U}_\varepsilon - \mathcal{U}) f_0}{\varepsilon} + \widetilde{f}\nabla \mathcal{U}_\varepsilon f_R + \varepsilon^2 f_R \nabla \mathcal{U}_\varepsilon f_R\Big)\Big\rangle_1 + L_2.
\end{aligned}
$$

Here $\mathbf{D}_R = \frac{1}{2}\big(\nabla \mathbf{v}_R + (\nabla \mathbf{v}_R)^T\big)$. We denote $F_R = F_1 + \cdots + F_6$ with

$$
\begin{aligned}
&F_1 = -\mathbf{v}_R \cdot \nabla (f_0 + \varepsilon \widetilde{f}),\\
&F_2 = -\mathcal{R} \cdot \Big(\mathbf{m} \times (\nabla \widetilde{\mathbf{v}})^T \cdot \mathbf{m} f_R + f_R \mathcal{R}\mathcal{U}_\varepsilon \widetilde{f} + f_R \mathcal{R}\frac{(\mathcal{U}_\varepsilon - \mathcal{U}_0) f_0}{\varepsilon} + \widetilde{f}\mathcal{R}\mathcal{U}_\varepsilon f_R\Big),\\
&F_3 = -\varepsilon \mathcal{R} \cdot \big(\mathbf{m} \times (\nabla \mathbf{v}_R)^T \cdot \mathbf{m}\widetilde{f})\big), \quad F_4 = -\varepsilon^3 \mathcal{R} \cdot \big(\mathbf{m} \times (\nabla \mathbf{v}_R)^T \cdot \mathbf{m} f_R\big),\\
&F_5 = -\varepsilon^2 \mathcal{R} \cdot \big(f_R \mathcal{R}\mathcal{U}_\varepsilon f_R\big), \quad F_6 = \varepsilon^3 \mathbf{v}_R \cdot \nabla f_R,
\end{aligned}
$$

and $G_R = G_1 + \cdots + G_8$ with

$$
\begin{aligned}
&G_1 = -\mathbf{v}_R \cdot \nabla \widetilde{\mathbf{v}} - \widetilde{\mathbf{v}} \cdot \nabla \mathbf{v}_R, \quad G_2 = \frac{1-\gamma}{2Re} \nabla \cdot \big(\varepsilon \mathbf{D}_R : \langle \mathbf{mmmm} \widetilde{f} \rangle_1\big),\\
&G_3 = -\frac{1-\gamma}{Re} \nabla \cdot \Big\langle \mathbf{mm} \times \big(\widetilde{\mathbf{D}} : \langle \mathbf{mmmm} f_R \rangle_1 + f_R \mathcal{R}\mathcal{U} \widetilde{f} + f_R \mathcal{R} \frac{(\mathcal{U}_\varepsilon - \mathcal{U}) f_0}{\varepsilon} + \widetilde{f} \mathcal{R}\mathcal{U} f_R\big)\Big\rangle_1,\\
&G_4 = \frac{1-\gamma}{Re} \Big\langle \frac{1}{\varepsilon} f_0 \nabla \mathcal{H}_\varepsilon f_R + f_R \nabla \mathcal{U}_\varepsilon \widetilde{f} + f_R \nabla \frac{(\mathcal{U}_\varepsilon - \mathcal{U}) f_0}{\varepsilon} + \widetilde{f} \nabla \mathcal{U}_\varepsilon f_R \Big\rangle_1,\\
&G_5 = \frac{1-\gamma}{2Re} \varepsilon^3 \nabla \cdot \big(\mathbf{D}_R : \langle \mathbf{mmmm} f_R \rangle_1\big), \quad G_6 = -\varepsilon^3 \mathbf{v}_R \cdot \nabla \mathbf{v}_R,\\
&G_7 = -\varepsilon^2 \frac{1-\gamma}{Re} \nabla \cdot \big\langle \mathbf{mm} \times \big(f_R \mathcal{R}\mathcal{U}_\varepsilon f_R\big)\big\rangle_1, \quad G_8 = -\varepsilon^2 \frac{1-\gamma}{Re} \big\langle f_R \nabla \mathcal{U}_\varepsilon f_R \big\rangle_1.
\end{aligned}
$$

Then we rewrite the equations for $(f_R, \mathbf{v}_R)$ as

$$
\begin{aligned}
&\frac{\partial f_R}{\partial t} + \widetilde{\mathbf{v}} \cdot \nabla f_R + \varepsilon^3 \mathbf{v}_R \cdot \nabla f_R + \frac{1}{\varepsilon} \mathcal{A}_{f_0} \mathcal{H}^\varepsilon_{f_0} f_R\\
&\qquad = -\mathcal{R} \cdot \big(\mathbf{m} \times (\nabla \mathbf{v}_R)^T \cdot \mathbf{m} f_0\big) + F_R + L_1, \qquad (7.14)\\
&\frac{\partial \mathbf{v}_R}{\partial t} - \frac{\gamma}{Re} \Delta \mathbf{v}_R - \frac{1-\gamma}{2Re} \nabla \cdot \big(\mathbf{D}_R : \langle \mathbf{mmmm} f_0 \rangle_1\big) + \nabla p\\
&\qquad = -\frac{1-\gamma}{Re} \nabla \cdot \Big\langle \mathbf{mm} \times \big(\frac{1}{\varepsilon} f_0 \mathcal{R} \mathcal{H}^\varepsilon_{f_0} f_R\big)\Big\rangle_1 + G_R + L_2. \qquad (7.15)
\end{aligned}
$$

7.3. **Some key estimates.** In this subsection, we mainly present a control for a singular term in the error estimates. The proof is based on the lower bound inequality. Since we only have good lower bound for the part outside the Maier-Saupe space, we have to analyze the nonlinear interactions between the part inside the Maier-Saupe space and the part outside the Maier-Saupe space. Throughout this section, we will repeatedly use the notations from Section 5. Due to the assumption (2.12), we can construct a global coordinate transformation so that all results from Section 5 can be applied.

Proposition 7.2. *For any $\delta > 0$, there exists $C = C(\delta)$ such that for any $f \in H^1(\Omega \times \mathbb{S}^2)$ with $\int_{\mathbb{S}^2} f(\mathbf{x}, \mathbf{m}) \mathrm{d}\mathbf{m} = 0$, there holds*

$$
\frac{1}{\varepsilon} \big\langle f, f \partial_t (\frac{1}{f_0}) \big\rangle \le \frac{\delta}{\varepsilon^2} \big\langle f_0 \mathcal{R} \mathcal{H}^\varepsilon_{f_0} f, \mathcal{R} \mathcal{H}^\varepsilon_{f_0} f \big\rangle + C\big(\frac{1}{\varepsilon} \langle \mathcal{H}^\varepsilon_{f_0} f, f \rangle + \langle f, f \rangle\big).
$$

We need the following lemmas.

Lemma 7.2. *It holds that*

$$
\begin{aligned}
&\hat{M}_{11} - \hat{M}_{22} = \zeta_{a,2} \frac{(A_0 - 2A_2 + A_4)}{2A_0},\\
&2\hat{M}_{33} - \hat{M}_{11} - \hat{M}_{22} = 3\zeta_0 \frac{A_0 A_4 - A_2^2}{A_0^2}.
\end{aligned}
$$

Proof. Recall that $\hat{\mathbf{M}}(\mathbf{x}) = \int_{\mathbb{S}^2} \hat{\mathbf{m}}\hat{\mathbf{m}} f \mathrm{d}\hat{\mathbf{m}}$. We get by direct calculations that

$$\begin{aligned}
\hat{M}_{11} =& \frac{1}{2}\int \sin^2\hat{\theta} a_0 \mathrm{d}\hat{\mathbf{m}} + \frac{1}{4}\int \sin^2\hat{\theta} a_2 \mathrm{d}\hat{\mathbf{m}} = -\zeta_0 \frac{A_0A_4 - A_2^2}{2A_0^2} + \zeta_{a,2}\frac{(A_0 - 2A_2 + A_4)}{4A_0},\\
\hat{M}_{22} =& \frac{1}{2}\int \sin^2\hat{\theta} a_0 \mathrm{d}\hat{\mathbf{m}} - \frac{1}{4}\int \sin^2\hat{\theta} a_2 \mathrm{d}\hat{\mathbf{m}} = -\zeta_0 \frac{A_0A_4 - A_2^2}{2A_0^2} - \zeta_{a,2}\frac{(A_0 - 2A_2 + A_4)}{4A_0},\\
\hat{M}_{33} =& \int \cos^2\hat{\theta} a_0 \mathrm{d}\hat{\mathbf{m}} = \zeta_0 \frac{A_0A_4 - A_2^2}{A_0^2},\\
\hat{M}_{12} =& \frac{1}{4}\int \sin^2\hat{\theta} b_2 \mathrm{d}\hat{\mathbf{m}} = \zeta_{b,2}\frac{(A_0 - 2A_2 + A_4)}{4A_0},\\
\hat{M}_{13} =& \frac{1}{2}\int \sin\hat{\theta}\cos\hat{\theta} a_1 \mathrm{d}\hat{\mathbf{m}} = \zeta_{a,1}\frac{A_2 - A_4}{A_0},\\
\hat{M}_{23} =& \frac{1}{2}\int \sin\hat{\theta}\cos\hat{\theta} b_1 \mathrm{d}\hat{\mathbf{m}} = \zeta_{b,1}\frac{A_2 - A_4}{A_0}.
\end{aligned}$$

Then the lemma follows. □

Lemma 7.3. *There exists $c > 0$ such that*

$$\big\langle \mathcal{H}^{\varepsilon}_{f_0} f, f\big\rangle \geq c\langle f^{\perp}, f^{\perp}\rangle + c\big\langle M_{kl}, M_{kl} - g_{\varepsilon} * M_{kl}\big\rangle.$$

Proof. Noting that

$$\begin{aligned}
\big\langle \mathcal{H}^{\varepsilon}_{f_0} f, f\big\rangle =& \langle \mathcal{H}_{f_0} f, f\rangle + \alpha\Big(\int_{\Omega}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\mathbf{m}\cdot\mathbf{m}')^2 f(\mathbf{x},\mathbf{m}) f(\mathbf{x},\mathbf{m}')\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}\\
&- \int_{\Omega}\int_{\Omega}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} (\mathbf{m}\cdot\mathbf{m}')^2 g_{\varepsilon}(\mathbf{x}-\mathbf{x}') f_R(\mathbf{x},\mathbf{m}) f_R(\mathbf{x}',\mathbf{m}')\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}'\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}\Big)\\
=& \big\langle \mathcal{H}_{f_0} f, f\big\rangle + \alpha\int_{\Omega} \mathbf{M}(\mathbf{x}) : \big(\mathbf{M}(\mathbf{x}) - g_{\varepsilon} * \mathbf{M}(\mathbf{x})\big)\mathrm{d}\mathbf{x},
\end{aligned}$$

then the lemma follows from Proposition 4.3. □

Lemma 7.4. *We have*

$$\frac{1}{\varepsilon}\big\langle (f_1 - g_{\varepsilon} * f_1), f_3 f_2\big\rangle \leq C\Big(\frac{1}{\varepsilon}\langle (f_1 - g_{\varepsilon} * f_1), f_1\rangle + \frac{1}{\varepsilon}\langle (f_2 - g_{\varepsilon} * f_2), f_2\rangle + \langle f_2, f_2\rangle\Big).$$

where the constant C depends on $\|f_3\|_{L^\infty}$ and $\|\nabla f_3\|_{L^\infty}$.

Proof. We write $f_1 - g_{\varepsilon} * f_1 = \big(1 - \chi(\sqrt{\varepsilon}D)\big)^2 f_1$, that is $\chi(\xi) = 1 - \sqrt{1 - \hat{g}(\xi)}$. Hence,

$$\begin{aligned}
\frac{1}{\varepsilon}\big\langle (f_1 - g_{\varepsilon} * f_1), f_3 f_2\big\rangle =& \frac{1}{\varepsilon}\big\langle \big(1 - \chi(\sqrt{\varepsilon}D)\big)^2 f_1, f_3 f_2\big\rangle\\
=& \frac{1}{\varepsilon}\big\langle \big(1 - \chi(\sqrt{\varepsilon}D)\big) f_1, f_3(1 - \chi(\sqrt{\varepsilon}D) f_2\big\rangle\\
&+ \frac{1}{\varepsilon}\big\langle \big(1 - \chi(\sqrt{\varepsilon}D)\big) f_1, [f_3, \chi(\sqrt{\varepsilon}D)] f_2\big\rangle.
\end{aligned}$$

Then the lemma follows from the commutator estimate

$$\|[f_3, \chi(\sqrt{\varepsilon}D)] f_2\|_{L^2} \leq C\varepsilon^{\frac{1}{2}}\|\nabla f_3\|_{L^\infty}\|f_2\|_{L^2}.$$

By a scaling argument, it suffices to prove the commutator estimate with $\varepsilon = 1$. Let $K_j(\mathbf{x})$ be the kernel associated with the Fourier multiplier $i(\partial_j\chi)(D)$ (It is easy to show that K_j is

a Calderon-Zygmund kernel). We have

$$\big[f_3,\chi(\sqrt{\varepsilon}D)\big]f_2=\sum_{j=1}^{3}\int_{\mathbb{R}^3}K_j(\mathbf{x}-\mathbf{y})\int_0^1\partial_j f_3(\tau\mathbf{x}+(1-\tau)\mathbf{y})\mathrm{d}\tau f_2(\mathbf{y})\mathrm{d}\mathbf{y},$$

whose L^2 norm is bounded by $\|\nabla f_3\|_{L^\infty}\|f_2\|_{L^2}$; see [2] for example. □

Now we are ready to prove Proposition 7.2. With the notations in section 5, we decompose f as

$$f=a_0(\mathbf{x},\hat{\theta})+\sum_{k\geq 1}\big(a_k(\mathbf{x},\hat{\theta})\cos k\hat{\varphi}+b_k(\mathbf{x},\hat{\theta})\sin k\hat{\varphi}\big).$$

Then we can get

$$\begin{aligned}\frac{1}{\varepsilon}\big\langle f,f\partial_t(\frac{1}{f_0})\big\rangle=&\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\Big(a_0(\mathbf{x},\hat{\theta})+\sum_{k\geq 1}\big(a_k(\mathbf{x},\hat{\theta})\cos k\hat{\varphi}+b_k(\mathbf{x},\hat{\theta})\sin k\hat{\varphi}\big)\Big)^2\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
\leq& C\int_\Omega\int_{\mathbb{S}^2}\frac{1}{\varepsilon^2}\Big(\gamma_0^2+\gamma_1^2+\beta_1^2+\gamma_2^2+\beta_2^2+\sum_{k\geq 3}(a_k^2+b_k^2)\Big)\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
&+C\int_\Omega\int_{\mathbb{S}^2}\big(\zeta_0^2+\zeta_{a,1}^2+\zeta_{a,2}^2+\zeta_{b,1}^2+\zeta_{b,2}^2\big)\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
&+\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\Big(\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})+\sin\hat{\theta}\cos\hat{\theta}(\zeta_{a,1}\cos\hat{\varphi}+\zeta_{b,1}\sin\hat{\varphi})\\
&\qquad+\sin^2\hat{\theta}(\zeta_{a,2}\cos 2\hat{\varphi}+\zeta_{b,2}\sin 2\hat{\varphi})\Big)^2\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}.\end{aligned}$$

As $\partial_t\big(1/f_0\big)=-\partial_t f_0/f_0^2$ and $\partial_t f_0\in\operatorname{Ker}\mathcal{G}_{f_0}$, we may assume that

$$\partial_t(\frac{1}{f_0})=\frac{1}{f_0}(w_1(\mathbf{x})\cos\hat{\varphi}+w_2(\mathbf{x})\sin\hat{\varphi})\sin\hat{\theta}\cos\hat{\theta}.$$

We have

$$\begin{aligned}&\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\Big(\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})+\sin\hat{\theta}\cos\hat{\theta}(\zeta_{a,1}\cos\hat{\varphi}+\zeta_{b,1}\sin\hat{\varphi})\\
&\qquad+\sin^2\hat{\theta}(\zeta_{a,2}\cos 2\hat{\varphi}+\zeta_{b,2}\sin 2\hat{\varphi})\Big)^2\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
=&\frac{2}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\sin\hat{\theta}\cos\hat{\theta}\big(\zeta_{a,1}\cos\hat{\varphi}+\zeta_{b,1}\sin\hat{\varphi}\big)\big(\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})\\
&\qquad+\sin^2\hat{\theta}(\zeta_{a,2}\cos 2\hat{\varphi}+\zeta_{b,2}\sin 2\hat{\varphi})\big)\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
&+\frac{1}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\big(\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})+\sin^2\hat{\theta}(\zeta_{a,2}\cos 2\hat{\varphi}+\zeta_{b,2}\sin 2\hat{\varphi})\big)^2\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}\\
\leq&\frac{2}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t(\frac{1}{f_0})\sin\hat{\theta}\cos\hat{\theta}\big(\zeta_{a,1}\cos\hat{\varphi}+\zeta_{b,1}\sin\hat{\varphi}\big)\big(\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})\\
&\qquad+\sin^2\hat{\theta}(\zeta_{a,2}\cos 2\hat{\varphi}+\zeta_{b,2}\sin 2\hat{\varphi})\big)\mathrm{d}\hat{\mathbf{m}}\mathrm{d}\mathbf{x}+\frac{C}{\varepsilon}\int_\Omega(\zeta_0^2+\zeta_{a,2}^2+\zeta_{b,2}^2)\mathrm{d}\mathbf{x},\end{aligned}\tag{7.16}$$

where we have used the fact

$$\int_{\mathbb{S}^2}\frac{1}{f_0}(w_1(\mathbf{x})\cos\hat{\varphi}+w_2(\mathbf{x})\sin\hat{\varphi})\sin\hat{\theta}\cos\hat{\theta}\big(\sin\hat{\theta}\cos\hat{\theta}(\zeta_{a,1}\cos\hat{\varphi}+\zeta_{b,1}\sin\hat{\varphi})\big)^2\mathrm{d}\hat{\mathbf{m}}=0.$$

We get by Lemma 7.2 that

$$\begin{aligned}&2\zeta_0(\mathbf{x})-\alpha N_0-\alpha(2\hat{M}_{33}-\hat{M}_{22}-\hat{M}_{11}-N_0)\\&=2\zeta_0(\mathbf{x})-3\alpha\zeta_0(\mathbf{x})\frac{A_0A_4-A_2^2}{A_0^2}=\frac{3A_2^2+2A_0A_2-5A_0A_4}{A_0(A_2-A_4)}\zeta_0(\mathbf{x}).\end{aligned}$$

Note that the coefficient is positive in the front of $\zeta_0(\mathbf{x})$ if f_0 is a stable critical point. This implies that

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega w_1(\mathbf{x})\zeta_{a,1}(\mathbf{x})\zeta_0(\mathbf{x})\mathrm{d}\mathbf{x}\le&\int_\Omega\frac{\delta}{\varepsilon^2}\big(2\zeta_0(\mathbf{x})-\alpha N_0(\mathbf{x})\big)^2+C(\delta,\|w_1\|_{L^\infty})\zeta_{a,1}^2(\mathbf{x})\mathrm{d}\mathbf{x}\\&+\frac{C}{\varepsilon}\big\langle 2\hat{M}_{33}(\mathbf{x})-\hat{M}_{11}(\mathbf{x})-\hat{M}_{22}(\mathbf{x})-N_0(\mathbf{x}),\zeta_{a,1}(\mathbf{x})w_1(\mathbf{x})\big\rangle.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\hat{M}_{ij}-N_{ij}=A_{ki}A_{lj}M_{kl}-A_{ki}A_{lj}(g_\varepsilon*M_{kl})=A_{ki}A_{lj}(M_{kl}-g_\varepsilon*M_{kl}),\\&\zeta_{a,1}=\frac{2}{\alpha}\hat{M}_{13}=\frac{2}{\alpha}A_{k'1}A_{l'3}M_{k'l'},\end{aligned}$$

from which, Lemma 7.4 and Lemma 7.3, it follows that

$$\begin{aligned}\frac{1}{\varepsilon}\big\langle\hat{M}_{ij}(\mathbf{x})-N_{ij}(\mathbf{x}),\zeta_{a,1}(\mathbf{x})w_1(\mathbf{x})\big\rangle&=\frac{1}{\varepsilon}\big\langle M_{kl}-g_\varepsilon*M_{kl},\frac{2}{\alpha}A_{ki}A_{lj}A_{k'1}A_{l'3}M_{k'l'}(\mathbf{x})w_1(\mathbf{x})\big\rangle\\&\le C(\mathbf{A},w_1)\big(\frac{1}{\varepsilon}\langle M_{kl}-g_\varepsilon*M_{kl},M_{kl}\rangle+\langle M_{kl},M_{kl}\rangle\big)\\&\le C(\mathbf{A},w_1)\big(\frac{1}{\varepsilon}\langle\mathcal{H}_{f_0}^\varepsilon f,f\rangle+\langle f,f\rangle\big).\end{aligned}$$

Thus, it follows from Proposition 5.1 that

$$\begin{aligned}\frac{1}{\varepsilon}\int_\Omega w_1(\mathbf{x})\zeta_{a,1}(\mathbf{x})\zeta_0(\mathbf{x})\mathrm{d}\mathbf{x}\le&\int_\Omega\frac{\delta}{\varepsilon^2}\big(2\zeta_0(\mathbf{x})-\alpha N_0(\mathbf{x})\big)^2+C(\delta,\|w_1\|_{L^\infty})\zeta_{a,1}^2(\mathbf{x})\mathrm{d}\mathbf{x}\\&+C(\mathbf{A},w_1)\big(\frac{1}{\varepsilon}\langle\mathcal{H}_{f_0}^\varepsilon f,f\rangle+\langle f,f\rangle\big)\\\le&\,C\big(\frac{\delta}{\varepsilon^2}\big\langle f_0\mathcal{R}\mathcal{H}_{f_0}^\varepsilon f,\mathcal{R}\mathcal{H}_{f_0}^\varepsilon f\big\rangle+\frac{1}{\varepsilon}\langle\mathcal{H}_{f_0}^\varepsilon f,f\rangle+\langle f,f\rangle\big).\end{aligned}$$

This gives the desired estimate for the term in (7.16):

$$\frac{2}{\varepsilon}\int_\Omega\int_{\mathbb{S}^2}\partial_t\big(\frac{1}{f_0}\big)\sin\hat{\theta}\cos\hat{\theta}\zeta_{a,1}\cos\hat{\varphi}\zeta_0(\cos^2\hat{\theta}-\frac{A_2}{A_0})\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}.$$

The other terms in (7.16) can be treated similarly. We omit the details. □

The following lemma is used to control the other singular terms like $\frac{1}{\varepsilon}\big\langle\widetilde{\mathbf{v}}\cdot\nabla f_R,\mathcal{H}_{f_0}^\varepsilon f_R\big\rangle$ in the error estimates.

Lemma 7.5. *We have*

$$\|\nabla f\|_{L^2}^2\le C\big(\big\langle\mathcal{H}_{f_0}^\varepsilon\nabla f,\nabla f\big\rangle+\frac{1}{\varepsilon}\big\langle\mathcal{H}_{f_0}^\varepsilon f,f\big\rangle\big).\tag{7.17}$$

Proof. Let us first claim that

$$\|f\|_{L^2}^2\le C\big(\big\langle\mathcal{H}_\varepsilon f,f\big\rangle+\big\langle\mathbf{M}[f],\mathbf{M}[f]\big\rangle\big),\tag{7.18}$$

where $\mathbf{M}[f]=\int_{\mathbb{S}^2}\mathbf{m}\mathbf{m}f\mathrm{d}\mathbf{m}$. Due to the choice of g, we have

$$|\xi\hat{f}(\xi)|^2\le C\big((1-\hat{g}(\varepsilon\xi))|\xi\hat{f}(\xi)|^2+\frac{1}{\varepsilon}(1-\hat{g}(\varepsilon\xi))|\hat{f}(\xi)|^2\big).$$

This implies that

$$\|\nabla f\|_{L^2}^2 \leq C\big(\langle f - g_\varepsilon * \nabla f, \nabla f\rangle + \frac{1}{\varepsilon}\langle f - g_\varepsilon * f, f\rangle\big),$$

which along with (7.18) gives

$$\begin{aligned}\|\nabla f\|_{L^2}^2 \leq& C\big(\big\langle \mathcal{H}_{f_0}^\varepsilon \nabla f, \partial_i f\big\rangle + \langle \nabla \mathbf{M}[f], \nabla \mathbf{M}[f]\rangle\big)\\ \leq& C\big(\big\langle \mathcal{H}_{f_0}^\varepsilon \nabla f, \nabla f\big\rangle + \frac{1}{\varepsilon}\big\langle \mathcal{H}_{f_0}^\varepsilon, f\big\rangle\big),\end{aligned}$$

where we have used

$$\begin{aligned}\big\langle \mathcal{H}_{f_0}^\varepsilon f, f\big\rangle \geq& \alpha \int_\Omega \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (\mathbf{m}\cdot\mathbf{m}')^2 f(\mathbf{x},\mathbf{m})\Big(f(\mathbf{x},\mathbf{m}') - \int_\Omega g_\varepsilon(\mathbf{x}-\mathbf{x}') f(\mathbf{x}',\mathbf{m}')\mathrm{d}\mathbf{x}'\Big)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}\\ =& \alpha\big\langle (1-g_\varepsilon) * \mathbf{M}[f], \mathbf{M}[f]\big\rangle.\end{aligned}$$

To complete the proof, it remains to prove the claim. We write $f = f^\perp + f^\top$ with $f^\top \in \mathrm{Ker}\mathcal{G}_{f_0}$ and $f^\perp \in (\mathrm{Ker}\mathcal{G}_{f_0}^*)^\perp$. By Proposition 4.3, we have

$$\|f^\perp\|_{L^2}^2 \leq C\big\langle \mathcal{H}_{f_0} f, f\big\rangle \leq C\big\langle \mathcal{H}_{f_0}^\varepsilon f, f\big\rangle.$$

While, from the proof of Lemma 7.2, we know that

$$\|f^\top\|_{L^2}^2 \leq C\big(\big\langle \hat{M}_{13}, \hat{M}_{13}\big\rangle + \big\langle \hat{M}_{23}, \hat{M}_{23}\big\rangle\big) \leq C\big\langle \hat{\mathbf{M}}[f^\top], \hat{\mathbf{M}}[f^\top]\big\rangle = C\big\langle \mathbf{M}[f^\top], \mathbf{M}[f^\top]\big\rangle.$$

This implies (7.18). □

In the nonlinear estimates, we will frequently use the following basic lemmas.

Lemma 7.6. *It holds that*

$$\begin{aligned}&\int_\Omega \int_{\mathbb{S}^2} M_1(\mathbf{x}) M_2(\mathbf{x},\mathbf{m}) M_3(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x} \leq C\|M_1\|_{L^2}\|M_2\|_{H^{0,2}}\|M_3\|_{L^2},\\ &\int_\Omega \int_{\mathbb{S}^2} M_1(\mathbf{x}) M_2(\mathbf{x},\mathbf{m}) M_3(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x} \leq C\|M_1\|_{H^1}\|M_2\|_{H^{0,1}}\|M_3\|_{L^2},\\ &\int_\Omega \int_{\mathbb{S}^2} M_1(\mathbf{x}) M_2(\mathbf{x},\mathbf{m}) M_3(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x} \leq C\|\nabla M_1\|_{H^1}\|M_2\|_{L^2}\|M_3\|_{L^2}.\end{aligned}$$

Proof. By Hölder inequality and Sobolev embedding, we get

$$\begin{aligned}&\int_\Omega \int_{\mathbb{S}^2} M_1(\mathbf{x}) M_2(\mathbf{x},\mathbf{m}) M_3(\mathbf{x},\mathbf{m})\mathrm{d}\mathbf{m}\mathrm{d}\mathbf{x}\\ &\leq \|M_1\|_{L^2}\|M_2\|_{L_x^\infty L_m^2}\|M_3(\mathbf{x},\mathbf{m})\|_{L^2} \leq \|M_1\|_{L^2}\|M_2\|_{L_m^2 L_x^\infty}\|M_3\|_{L^2}\\ &\leq C\|M_1\|_{L^2}\|M_2\|_{H^{0,2}}\|M_3\|_{L^2}.\end{aligned}$$

The other two inequalities can be proved similarly. □

The following Bernstein type lemma is a direct consequence of Young's inequality.

Lemma 7.7. *Let $k \geq 0$ be an integer and $p \geq 2$. Then it holds that*

$$\|\nabla^k \mathcal{U}_\varepsilon f\|_{L^p(\Omega\times\mathbb{S}^2)} \leq C\varepsilon^{-3/2(1/2-1/p)-k/2}\|f\|_{L^2(\Omega\times\mathbb{S}^2)}.$$

7.4. **Error estimates.** Let us first explain how to choose a suitable energy functional. It's helpful to look at the following toy model for $(f_R, \mathbf{v}_R)$:

$$\begin{aligned}
&\frac{\partial f_R}{\partial t} + \frac{1}{\varepsilon}\mathcal{A}_{f_0}\mathcal{H}^{\varepsilon}_{f_0} f_R = -\mathcal{R}\cdot\big(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0\big),\\
&\frac{\partial \mathbf{v}_R}{\partial t} - \frac{\gamma}{Re}\Delta\mathbf{v}_R + \nabla p = -\frac{1-\gamma}{Re}\nabla\cdot\big\langle \mathbf{mm}\times\big(\frac{1}{\varepsilon}f_0\mathcal{R}\mathcal{H}^{\varepsilon}_{f_0}f_R\big)\big\rangle_1.
\end{aligned}$$

Compared with the homogeneous case, the new difficulty is caused by the singular term $\frac{1}{\varepsilon}\nabla\cdot\big\langle \mathbf{mm}\times\big(f_0\mathcal{R}\mathcal{H}^{\varepsilon}_{f_0}f_R\big)\big\rangle_1$. To deal with it, it is natural to introduce the energy functional

$$\frac{1}{\varepsilon}\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \frac{Re}{1-\gamma}\langle \mathbf{v}_R, \mathbf{v}_R\rangle,$$

since we have the following important observation:

$$\big\langle \mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0, \mathcal{R}\mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \big\langle\langle \mathbf{mm}\times f_0\mathcal{R}\mathcal{H}^{\varepsilon}_{f_0}f_R\rangle_1, \nabla\mathbf{v}_R\big\rangle = 0.$$

However, $\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle$ does not give a control for the part of f_R inside the kernel. To have a control for the part inside the kernel, we need to introduce another functional $\big\langle f_R, \mathcal{A}^{-1}_{f_0}f_R\big\rangle$ similar to the homogeneous case. So, the suitable energy functional for the toy model should be

$$\big\langle f_R, \mathcal{A}^{-1}_{f_0}f_R\big\rangle + \frac{1}{\varepsilon}\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \frac{Re}{1-\gamma}\langle \mathbf{v}_R, \mathbf{v}_R\rangle.$$

However, if we take $\frac{1}{\varepsilon}\frac{d}{dt}\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle$, a new singular term $\frac{1}{\varepsilon}\big\langle f_R, \partial_t(\frac{1}{f_0})f_R\big\rangle$ will appear. Since there is no any decay in ε for the part of f_R inside the kernel, this term seems to have the order of $O(1/\varepsilon)$(**Very singular!**). Surprisingly, by analyzing the nonlinear interactions for $\big\langle f_R, \partial_t(\frac{1}{f_0})f_R\big\rangle$ and using the lower bound inequality, we find that it is bounded.

In order to control the nonlinear terms, we also need to introduce a higher order analogous of the energy functional, whose choice is also very subtle. In all, our energy functional takes the form

$$\begin{aligned}
\mathfrak{E}_\varepsilon(t) \overset{\text{def}}{=}& \big\langle f_R, \mathcal{A}^{-1}_{f_0}f_R\big\rangle + \frac{1}{\varepsilon}\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \frac{Re}{1-\gamma}\langle \mathbf{v}_R, \mathbf{v}_R\rangle\\
&+C_1\varepsilon\big\langle \nabla f_R, \mathcal{A}^{-1}_{f_0}\nabla f_R\big\rangle + C_2\Big[\varepsilon\big\langle \nabla f_R, \mathcal{H}^{\varepsilon}_{f_0}\nabla f_R\big\rangle + \frac{Re}{1-\gamma}\varepsilon^2\langle \nabla\mathbf{v}_R, \nabla\mathbf{v}_R\rangle\Big]\\
&+\varepsilon^3\big\langle \Delta f_R, \mathcal{A}^{-1}_{f_0}\Delta f_R\big\rangle + C_3\Big[\varepsilon^3\big\langle \Delta f_R, \mathcal{H}^{\varepsilon}_{f_0}\Delta f_R\big\rangle + \frac{Re}{1-\gamma}\varepsilon^4\big\langle \Delta\mathbf{v}_R, \Delta\mathbf{v}_R\big\rangle\Big],\\
\mathfrak{F}_\varepsilon(t) \overset{\text{def}}{=}& \frac{1}{\varepsilon}\big\langle f_R, \mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \frac{1}{\varepsilon^2}\big\langle \mathcal{H}^{\varepsilon}_{f_0}f_R, \mathcal{A}\mathcal{H}^{\varepsilon}_{f_0}f_R\big\rangle + \frac{\gamma}{1-\gamma}\big\langle \nabla\mathbf{v}_R, \nabla\mathbf{v}_R\big\rangle\\
&+ C_1\big\langle \nabla f_R, \mathcal{H}^{\varepsilon}_{f_0}\nabla f_R\big\rangle + C_2\Big[\big\langle \mathcal{H}^{\varepsilon}_{f_0}\nabla f_R, \mathcal{A}\mathcal{H}^{\varepsilon}_{f_0}\nabla f_R\big\rangle + \varepsilon^2\frac{\gamma}{1-\gamma}\big\langle \nabla^2\mathbf{v}_R, \nabla^2\mathbf{v}_R\big\rangle\Big]\\
&+ \varepsilon^2\big\langle \Delta f_R, \mathcal{H}^{\varepsilon}_{f_0}\Delta f_R\big\rangle + C_3\Big[\varepsilon^2\big\langle \mathcal{H}^{\varepsilon}_{f_0}\Delta f_R, \mathcal{A}\mathcal{H}^{\varepsilon}_{f_0}\Delta f_R\big\rangle + \varepsilon^4\frac{\gamma}{1-\gamma}\big\langle \nabla\Delta\mathbf{v}_R, \nabla\Delta\mathbf{v}_R\big\rangle\Big].
\end{aligned}$$

Here the constants C_1, C_2 and C_3 bigger than one will be determined later.

Proposition 7.3. *There exist $c_1 > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, T]$, there holds*

$$\frac{d}{dt}\mathfrak{E}_\varepsilon(t) + c_1\mathfrak{F}_\varepsilon(t) \le C\big(1 + \mathfrak{E}_\varepsilon(t) + \varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2} + \varepsilon\mathfrak{E}_\varepsilon(t)^2\big) + C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t).$$

where the constant C depends on $\|f_i\|_{L^\infty(0,T;H^3(\Omega\times\mathbb{S}^2))}(i=0,1,2,3)$, $\|\mathbf{v}_i\|_{L^\infty(0,T;H^3(\Omega))}(i=0,1,2)$.

Proof. From the definition of $\mathfrak{E}_\varepsilon(t),\mathfrak{F}_\varepsilon(t)$ and Lemma 7.3, it is easy to see that

$$\begin{aligned}&\|f_R\|_{L^2}^2+C_1\|\varepsilon^{1/2}f_R\|_{H^{0,1}}^2+\|\varepsilon^{3/2}f_R\|_{H^{0,2}}^2\\&\qquad+\|\mathbf{v}_R\|_{L^2}^2+C_2\|\varepsilon\mathbf{v}_R\|_{H^1}^2+C_3\|\varepsilon^2\mathbf{v}_R\|_{H^2}^2\le C\mathfrak{E}_\varepsilon(t),\\&\|\nabla\mathbf{v}_R\|_{L^2}^2+C_2\|\varepsilon\nabla\mathbf{v}_R\|_{H^1}^2+C_3\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}^2\le C\mathfrak{F}_\varepsilon(t).\end{aligned}$$

And it is easy to show that

$$\|L_1\|_{H^{0,2}}+\|L_2\|_{H^2}\le C.$$

These facts will be repeatedly used in the following calculations. For the simplicity of notations, we denote $\mathcal{A}=\mathcal{A}_{f_0}$ and $\mathcal{H}_\varepsilon=\mathcal{H}_{f_0}^\varepsilon$ in what follows.

Step 1. L^2 energy estimate

Making $L^2(\Omega\times\mathbb{S}^2)$ inner product to (7.14) with $\mathcal{A}_{f_0}^{-1}f_R$, we get

$$\begin{aligned}\Big\langle\frac{\partial}{\partial t}f_R,\mathcal{A}^{-1}f_R\Big\rangle+\frac{1}{\varepsilon}\big\langle\mathcal{H}_\varepsilon f_R,f_R\big\rangle&=-\big\langle\widetilde{\mathbf{v}}\cdot\nabla f_R,\mathcal{A}^{-1}f_R\big\rangle\\&\quad+\big\langle\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0,\mathcal{R}\mathcal{A}^{-1}f_R\big\rangle+\big\langle F_R+L_1,\mathcal{A}^{-1}f_R\big\rangle.\end{aligned}$$

By Lemma 7.6 and Lemma 7.7, we have

$$\begin{aligned}&\big\langle F_4,\mathcal{A}^{-1}f_R\big\rangle\le C\varepsilon^{1/2}\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^{1/2}f_R\|_{H^{0,1}}\|f_R\|_{L^2}\le C\varepsilon^{1/2}\mathfrak{E}_\varepsilon^{3/2}(t),\\&\big\langle F_5,\mathcal{A}^{-1}f_R\big\rangle\le C\varepsilon^{5/4}\|f_R\|_{L^2}^3\le C\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)^{3/2},\\&\big\langle F_6,\mathcal{A}^{-1}f_R\big\rangle\le C\varepsilon\|f_R\|_{L^2}^2\|\varepsilon^2\mathbf{v}_R\|_{H^2}\le C\varepsilon\mathfrak{E}_\varepsilon(t)^{3/2},\end{aligned}$$

and the other terms can be estimated as follows

$$\begin{aligned}&\big\langle\widetilde{\mathbf{v}}\cdot\nabla f_R,\mathcal{A}^{-1}f_R\big\rangle+\big\langle F_2,\mathcal{A}^{-1}f_R\big\rangle\le C\|f_R\|_{L^2}^2\le C\mathfrak{E}_\varepsilon(t),\\&\big\langle F_1,\mathcal{A}^{-1}f_R\big\rangle\le C\|\mathbf{v}_R\|_{L^2}\|f_R\|_{L^2}^2\le C\mathfrak{E}_\varepsilon(t),\\&\big\langle F_3,\mathcal{A}^{-1}f_R\big\rangle\le C\varepsilon\|\nabla\mathbf{v}_R\|_{L^2}\|f_R\|_{L^2}\le C\mathfrak{E}_\varepsilon(t),\end{aligned}$$

So, we get

$$\Big\langle\frac{\partial f_R}{\partial t},\mathcal{A}^{-1}f_R\Big\rangle+\frac{1}{\varepsilon}\big\langle\mathcal{H}_\varepsilon f_R,f_R\big\rangle\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)^{3/2}\big)+\delta\mathfrak{F}_\varepsilon(t).\tag{7.19}$$

Make $L^2(\Omega\times\mathbb{S}^2)$ inner product to (7.14) with $\mathcal{H}_\varepsilon f_R$ to obatin

$$\begin{aligned}\frac{1}{\varepsilon}\Big\langle\frac{\partial}{\partial t}f_R,\mathcal{H}_\varepsilon f_R\Big\rangle+\frac{1}{\varepsilon^2}\big\langle\mathcal{R}\mathcal{H}_\varepsilon f_R,f_0\mathcal{R}\mathcal{H}_\varepsilon f_R\big\rangle&=-\frac{1}{\varepsilon}\big\langle\widetilde{\mathbf{v}}\cdot\nabla f_R,\mathcal{H}_\varepsilon f_R\big\rangle\\&\quad+\frac{1}{\varepsilon}\big\langle\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0,\mathcal{R}\mathcal{H}_\varepsilon f_R\big\rangle+\frac{1}{\varepsilon}\big\langle F_R+L_1,\mathcal{H}_\varepsilon f_R\big\rangle.\end{aligned}$$

By Lemma 7.6 and Lemma 7.7, we have

$$\begin{aligned}&\frac{1}{\varepsilon}\big\langle F_4+F_6,\mathcal{H}_\varepsilon f_R\big\rangle\le C\varepsilon^{1/2}\|\varepsilon^{1/2}f_R\|_{H^{0,1}}\|\varepsilon^2\mathbf{v}_R\|_{H^2}\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2}\le C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\&\frac{1}{\varepsilon}\big\langle F_5,\mathcal{H}_\varepsilon f_R\big\rangle\le C\varepsilon^{1/4}\|f_R\|_{L^2}^2\|\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2}\le C\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}.\end{aligned}$$

Noting that

$$\int_{\mathbb{S}^2}\nabla(f_0+\varepsilon\widetilde{f})\mathrm{d}\mathbf{m}=0,\quad\int_{\mathbb{S}^2}N_1\mathrm{d}\mathbf{m}=0,$$

we get by Poincaré inequality that

$$\frac{1}{\varepsilon}\big\langle F_1, \mathcal{H}_\varepsilon f_R\big\rangle \le C\|\mathbf{v}_R\|_{L^2}\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t),$$
$$\frac{1}{\varepsilon}\big\langle N_1, \mathcal{H}_\varepsilon f_R\big\rangle \le C\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2} \le C + \delta\mathfrak{F}_\varepsilon(t).$$

We infer from Lemma 7.5 that

$$\frac{1}{\varepsilon}\big\langle \widetilde{\mathbf{v}}\cdot\nabla f_R, \mathcal{H}_\varepsilon f_R\big\rangle \le C\|\nabla f_R\|_{L^2}\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2} \le \frac{C_0}{\sqrt{C_1}}\mathfrak{F}_\varepsilon(t) + C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t).$$

Here and what follows C_0 denotes a constant independent of δ. The other terms are estimated as follows

$$\frac{1}{\varepsilon}\big\langle F_2, \mathcal{H}_\varepsilon f_R\big\rangle \le C\|f_R\|_{L^2}\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t),$$
$$\frac{1}{\varepsilon}\big\langle F_3, \mathcal{H}_\varepsilon f_R\big\rangle \le C\varepsilon\|\nabla\mathbf{v}_R\|_{L^2}\|\frac{1}{\varepsilon}\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2} \le C\varepsilon\mathfrak{F}_\varepsilon(t),$$

Hence, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\big\langle \frac{\partial f_R}{\partial t}, \mathcal{H}_\varepsilon f_R\big\rangle + \frac{1}{\varepsilon^2}\big\langle f_0\mathcal{R}\mathcal{H}_\varepsilon f_R, \mathcal{R}\mathcal{H}_\varepsilon f_R\big\rangle \\
&\le C\big(1+\mathfrak{E}_\varepsilon(t)\big) + C\varepsilon^{1/2}\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t) + \big(\delta+\varepsilon+\frac{C_0}{\sqrt{C_1}}\big)\mathfrak{F}_\varepsilon(t) \\
&\qquad + \frac{1}{\varepsilon}\big\langle \mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0, \mathcal{R}\mathcal{H}_\varepsilon f_R\big\rangle.
\end{aligned} \tag{7.20}$$

Make $L^2(\Omega)$ inner product to (7.15) with $\mathbf{v}_R$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle \mathbf{v}_R, \mathbf{v}_R\rangle + \frac{\gamma}{Re}\langle\nabla\mathbf{v}_R, \nabla\mathbf{v}_R\rangle + \frac{1-\gamma}{2Re}\big\langle\big(\mathbf{D}_R : \langle\mathbf{mmmm}f_0\rangle_1\big), \nabla\mathbf{v}_R\big\rangle \\
&= \frac{1-\gamma}{Re}\Big\langle \mathbf{mm}\times\big(\frac{1}{\varepsilon}f_0\mathcal{R}\mathcal{H}_\varepsilon f_R\big), \nabla\mathbf{v}_R\Big\rangle + \big\langle G_R + L_2, \mathbf{v}_R\big\rangle.
\end{aligned}$$

Obviously, $\big\langle G_6, \mathbf{v}_R\big\rangle = 0$. We have by Lemma 7.6 and Lemma 7.7 that

$$\begin{aligned}
&\big\langle G_5, \mathbf{v}_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^{1/2}f_R\|_{H^{0,1}}\|\varepsilon^2\mathbf{v}_R\|_{H^2}\|\nabla\mathbf{v}_R\|_{L^2} \le C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}, \\
&\big\langle G_7, \mathbf{v}_R\big\rangle \le C\varepsilon^{1/4}\|f_R\|_{L^2}^2\|\varepsilon\nabla\mathbf{v}_R\|_{L^2} \le C\varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2}, \\
&\big\langle G_8, \mathbf{v}_R\big\rangle \le C\varepsilon^{3/4}\|f_R\|_{L^2}^2\|\mathbf{v}_R\|_{L^2} \le C\varepsilon^{3/4}\mathfrak{E}_\varepsilon(t)^{3/2}.
\end{aligned}$$

While, $\big\langle \mathbf{D}_R : \langle\mathbf{mmmm}f_0\rangle_1, \nabla\mathbf{v}_R\big\rangle = \big\langle \mathbf{D}_R : \langle\mathbf{mmmm}f_0\rangle_1, \mathbf{D}_R\big\rangle \ge 0$. The other terms are estimated as follows

$$\begin{aligned}
&\langle G_1, \mathbf{v}_R\rangle \le C\|\mathbf{v}_R\|_{L^2}^2 \le C\mathfrak{E}_\varepsilon(t), \\
&\big\langle G_2, \mathbf{v}_R\big\rangle \le C\varepsilon\|\nabla\mathbf{v}_R\|_{L^2}^2 \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t), \\
&\big\langle G_3, \mathbf{v}_R\big\rangle \le C\|f_R\|_{L^2}\|\nabla\mathbf{v}_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t), \\
&\big\langle G_4, \mathbf{v}_R\big\rangle \le C\frac{1}{\varepsilon}\|\mathcal{R}\mathcal{H}_\varepsilon f_R\|_{L^2}\|\mathbf{v}_R\|_{L^2} + C\|f_R\|_{L^2}\|\mathbf{v}_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\langle \mathbf{v}_R, \mathbf{v}_R\big\rangle + \frac{\gamma}{Re}\langle \nabla \mathbf{v}_R, \nabla \mathbf{v}_R\rangle\\ &\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2}\big)+C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}+\delta\mathfrak{F}_\varepsilon(t)\\ &\qquad+\big\langle \mathbf{m}\mathbf{m}\times\frac{1}{\varepsilon}f_0\mathcal{R}\mathcal{H}_\varepsilon f_R, \nabla\mathbf{v}_R\big\rangle. \end{aligned}\tag{7.21}$$

Step 2. H^1 energy estimate

Taking the derivative to (7.14) with respect to $\mathbf{x}_i$, then making $L^2(\Omega\times\mathbb{S}^2)$ inner product with $\varepsilon\mathcal{A}^{-1}\partial_i f_R$, we get

$$\begin{aligned}&\varepsilon\big\langle\frac{\partial}{\partial t}\partial_i f_R, \mathcal{A}^{-1}\partial_i f_R\big\rangle + \big\langle \partial_i f_R, \mathcal{H}_\varepsilon\partial_i f_R\big\rangle\\ &=-\big\langle \partial_i f_0\mathcal{R}\mathcal{H}_\varepsilon f_R, \mathcal{R}\mathcal{A}^{-1}\partial_i f_R\big\rangle - \big\langle \partial_i(\frac{1}{f_0})f_R, \partial_i f_R\big\rangle - \varepsilon\big\langle \partial_i(\widetilde{\mathbf{v}}\cdot\nabla f_R), \mathcal{A}^{-1}\partial_i f_R\big\rangle\\ &\quad+\varepsilon\big\langle \partial_i(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0), \mathcal{R}\mathcal{A}^{-1}f_R\big\rangle + \varepsilon\big\langle \partial_i F_R+\partial_i L_1, \mathcal{A}^{-1}\partial_i f_R\big\rangle.\end{aligned}$$

By Lemma 7.6 and Lemma 7.7, we get

$$\begin{aligned}&\varepsilon\big\langle \partial_i F_4, \mathcal{A}^{-1}\partial_i f_R\big\rangle \le C\varepsilon\|\varepsilon\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^{1/2}f_R\|_{H^{0,1}}\|\varepsilon^{3/2}\partial_i f_R\|_{H^{0,1}} \le C\varepsilon\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t),\\ &\varepsilon\big\langle \partial_i F_5, \mathcal{A}^{-1}\partial_i f_R\big\rangle \le C\varepsilon^{5/4}\|f_R\|_{L^2}\|\varepsilon^{1/2}f_R\|^2_{H^{0,1}} \le C\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)^{3/2},\\ &\varepsilon\big\langle \partial_i F_6, \mathcal{A}^{-1}\partial_i f_R\big\rangle \le C\varepsilon\|\varepsilon\nabla\mathbf{v}_R\|_{H^1}\|\varepsilon^{1/2}f_R\|_{H^{0,12}}\|\varepsilon^{3/2}\partial_i f_R\|_{H^{0,1}} \le C\varepsilon\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t).\end{aligned}$$

It follows from Lemma 7.5 that

$$\begin{aligned}&\varepsilon\big\langle \partial_i\big(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0\big), \mathcal{R}\mathcal{A}^{-1}\partial_i f_R\big\rangle + \varepsilon\big\langle \partial_i F_3, \mathcal{A}^{-1}\partial_i f_R\big\rangle\\ &\le C\|\varepsilon\nabla\mathbf{v}_R\|_{H^1}\|\partial_i f_R\|_{L^2} \le \frac{C_0}{\sqrt{C_2}}\mathfrak{F}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t)+C\mathfrak{E}_\varepsilon(t).\end{aligned}$$

The other terms are estimated as follows

$$\begin{aligned}&\big\langle \partial_i f_0\mathcal{R}\mathcal{H}_\varepsilon f_R, \mathcal{R}\mathcal{A}^{-1}\partial_i f_R\big\rangle + \big\langle \partial_i(\frac{1}{f_0})f_R, \partial_i f_R\big\rangle \le C\|f_R\|_{L^2}\|\partial_i f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t),\\ &-\varepsilon\big\langle \partial_i\big(\widetilde{\mathbf{v}}\cdot\nabla f_R\big), \mathcal{A}^{-1}\partial_i f_R\big\rangle + \varepsilon\big\langle \partial_i F_2, \mathcal{A}^{-1}\partial_i f_R\big\rangle \le C\varepsilon||f_R\|^2_{H^{0,1}} \le C\mathfrak{E}_\varepsilon(t),\\ &\varepsilon\big\langle \partial_i F_1, \mathcal{A}^{-1}\partial_i f_R\big\rangle \le C\varepsilon^{1/2}\|\mathbf{v}_R\|_{H^1}\|\varepsilon^{1/2}\partial_i f_R\|_{L^2} \le \delta\mathfrak{F}_\varepsilon(t)+C\mathfrak{E}_\varepsilon(t).\end{aligned}$$

So, we get

$$\begin{aligned}&\big\langle\frac{\partial}{\partial t}\partial_i f_R, \mathcal{A}^{-1}\partial_i f_R\big\rangle + \frac{1}{\varepsilon}\big\langle \mathcal{H}_\varepsilon\partial_i f_R, \partial_i f_R\big\rangle\\ &\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)^{3/2}\big)+C\varepsilon\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t)+\big(\delta+\frac{C_0}{\sqrt{C_2}}\big)\mathfrak{F}_\varepsilon(t).\end{aligned}\tag{7.22}$$

Taking the derivative to (7.14) with respect to $\mathbf{x}_i$, then making $L^2(\Omega\times\mathbb{S}^2)$ inner product with $\varepsilon\mathcal{H}\partial_i f_R$, we get

$$\begin{aligned}&\varepsilon\big\langle\frac{\partial}{\partial t}\partial_i f_R, \mathcal{H}_\varepsilon\partial_i f_R\big\rangle + \big\langle f_0\mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R, \mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R\big\rangle\\ &=-\big\langle \partial_i f_0\mathcal{R}\mathcal{U}_\varepsilon f_R + f_R\mathcal{R}\partial_i(\mathcal{U}_0 f_0), \mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R\big\rangle - \varepsilon\big\langle \partial_i(\widetilde{\mathbf{v}}\cdot\nabla f_R), \mathcal{H}_\varepsilon\partial_i f_R\big\rangle\\ &\quad+\varepsilon\big\langle \partial_i(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0), \mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R\big\rangle + \varepsilon\big\langle \partial_i F_R+\partial_i L_1, \mathcal{H}_\varepsilon\partial_i f_R\big\rangle.\end{aligned}$$

The first term on the right hand side is bounded by

$$\|f_R\|^2_{L^2}\|\mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R\|^2_{L^2} \le C\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t).$$

By Lemma 7.6 and Lemma 7.7, we get

$$\begin{aligned}
&\varepsilon\big\langle \partial_i F_4, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^2 \mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2} f_R\|_{H^{0,2}}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\
&\varepsilon\big\langle \ \partial_i F_5, \mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\varepsilon^{7/4}\|\varepsilon^{1/2} f_R\|_{H^{0,1}}\|f_R\|_{L^2}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\varepsilon^{7/4}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\
&\varepsilon\big\langle \partial_i F_6, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^2 \mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2} f_R\|_{H^{0,2}}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}.
\end{aligned}$$

and by Lemma 7.5,

$$\begin{aligned}
&\varepsilon\big\langle \partial_i\big(\widetilde{\mathbf{v}}\cdot\nabla f_R\big), \mathcal{H}_\varepsilon \partial_i f_R\big\rangle\\
&= \varepsilon\big\langle \partial_i \widetilde{\mathbf{v}}\cdot\nabla f_R, \frac{\partial_i f_R}{f_0}\big\rangle - \varepsilon\big\langle \widetilde{\mathbf{v}}\cdot\nabla(\frac{1}{f_0})\partial_i f_R, \partial_i f_R\big\rangle - \varepsilon\big\langle \widetilde{\mathbf{v}}\cdot\nabla f_R, \partial_i(\mathcal{U}_\varepsilon \partial_i f_R)\big\rangle\\
&\le C\varepsilon^{1/2}\|\nabla f_R\|_{L^2}\|\partial_i f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t).
\end{aligned}$$

The other terms are estimated as follows

$$\begin{aligned}
&\varepsilon\big\langle \partial_i F_1, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\|\varepsilon\nabla \mathbf{v}_R\|_{L^2}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t),\\
&\varepsilon\big\langle \partial_i F_2, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^{\frac{1}{2}} f_R\|_{H^{0,1}}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2}^2 \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t),\\
&\varepsilon\big\langle \partial_i F_3, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\|\varepsilon^2 \mathbf{v}_R\|_{H^2}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t),\\
&\varepsilon\big\langle \mathbf{m}\times(\nabla \mathbf{v}_R)^T\cdot\mathbf{m}\partial_i f_0, \mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\big\rangle \le C\|\varepsilon \mathbf{v}_R\|_{H^1}\|\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\varepsilon\big\langle \frac{\partial}{\partial t}\partial_i f_R, \mathcal{H}_\varepsilon \partial_i f_R\big\rangle + \big\langle f_0\mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R, \mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\big\rangle\\
&\le C\big(1+\mathfrak{E}_\varepsilon(t)\big) + C\varepsilon^{1/2}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2} + \delta\mathfrak{F}_\varepsilon(t)\\
&\qquad + \frac{1}{\varepsilon}\big\langle \mathbf{m}\times(\nabla\partial_i \mathbf{v}_R)^T\cdot\mathbf{m} f_0, \mathcal{R}\mathcal{H}_\varepsilon \partial_i f_R\big\rangle. \qquad (7.23)
\end{aligned}$$

Making $L^2(\Omega)$ inner product to (7.15) with $\varepsilon^2\partial_i^2 \mathbf{v}_R$, we get

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\frac{d}{dt}\big\langle \partial_i \mathbf{v}_R, \partial_i \mathbf{v}_R\big\rangle + \frac{\gamma}{Re}\varepsilon^2\big\langle \nabla\partial_i \mathbf{v}_R, \nabla\partial_i \mathbf{v}_R\big\rangle + \frac{1-\gamma}{2Re}\varepsilon^2\big\langle \partial_i\big(\mathbf{D}_R : \langle \mathbf{mmmm} f_0\rangle_1\big), \nabla\partial_i \mathbf{v}_R\big\rangle\\
&= \frac{1-\gamma}{Re}\varepsilon\big\langle \mathbf{m}\mathbf{m}\times\partial_i\big(f_0\mathcal{R}\mathcal{H}_\varepsilon f_R\big), \nabla\partial_i \mathbf{v}_R\big\rangle + \varepsilon^2\big\langle \partial_i G_R + \partial_i L_2, \partial_i \mathbf{v}_R\big\rangle.
\end{aligned}$$

First of all, we know that $\big\langle \big(\partial_i \mathbf{D}_R : \langle \mathbf{mmmm} f_0\rangle_1\big), \nabla\partial_i \mathbf{v}_R\big\rangle \ge 0$ and

$$\varepsilon^2\big\langle \big(\mathbf{D}_R : \partial_i\langle \mathbf{mmmm} f_0\rangle_1\big), \nabla\partial_i \mathbf{v}_R\big\rangle \le C\|\varepsilon \mathbf{v}_R\|_{H^1}\|\varepsilon\nabla\partial_i \mathbf{v}_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t) + \delta\mathfrak{F}_\varepsilon(t).$$

By Lemma 7.6 and Lemma 7.7, we get

$$\begin{aligned}
&\varepsilon^2\big\langle \partial_i G_5, \partial_i \mathbf{v}_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^{1/2} f_R\|_{H^{0,1}}\|\varepsilon^2 \mathbf{v}_R\|_{H^2}\|\varepsilon^2\nabla \mathbf{v}_R\|_{H^2} \le C\varepsilon^{\frac{1}{2}}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\
&\varepsilon^2\big\langle \partial_i G_6, \partial_i \mathbf{v}_R\big\rangle \le C\varepsilon\|\varepsilon^2 \mathbf{v}_R\|_{H^2}^2\|\nabla \mathbf{v}_R\|_{L^2} \le C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\
&\varepsilon^2\big\langle \partial_i G_7, \partial_i \mathbf{v}_R\big\rangle \le C\varepsilon^{3/4}\|\varepsilon^{1/2} f_R\|_{H^{0,1}}\|f_R\|_{L^2}\|\varepsilon^2 \mathbf{v}_R\|_{H^2} \le C\varepsilon^{3/4}\mathfrak{E}_\varepsilon(t)^{3/2},\\
&\varepsilon^2\big\langle \partial_i G_8, \partial_i \mathbf{v}_R\big\rangle \le C\varepsilon\|f_R\|_{L^2}\|\varepsilon^{1/2} f_R\|_{H^{0,1}}\|\varepsilon^2 \mathbf{v}_R\|_{H^2} \le C\varepsilon^{3/4}\mathfrak{E}_\varepsilon(t)^{3/2}.
\end{aligned}$$

And the other terms are estimates as

$$\begin{aligned}
&\varepsilon^2\big\langle \partial_i G_1, \partial_i \mathbf{v}_R\big\rangle \le C\|\varepsilon \mathbf{v}_R\|_{H^1}^2 \le C\mathfrak{E}_\varepsilon(t),\\
&\varepsilon^2\big\langle \partial_i G_2, \partial_i \mathbf{v}_R\big\rangle \le C\|\varepsilon^2 \mathbf{v}_R\|_{H^2}\|\varepsilon\Delta \mathbf{v}_R\|_{L^2} \le C\varepsilon\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t),,\\
&\varepsilon^2\big\langle \partial_i G_3, \partial_i \mathbf{v}_R\big\rangle \le C\varepsilon^{1/2}\|\varepsilon^{1/2} f_R\|_{H^{0,1}}\|\varepsilon\nabla\partial_i \mathbf{v}_R\|_{L^2} \le C\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t),\\
&\varepsilon^2\big\langle \partial_i G_4, \partial_i \mathbf{v}_R\big\rangle \le C\|f_R\|_{L^2}\|\varepsilon\Delta \mathbf{v}_R\|_{L^2}^2 \le C\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t).
\end{aligned}$$

So, we get

$$\begin{aligned}
&\varepsilon^2\big\langle \frac{\partial}{\partial t}\partial_i \mathbf{v}_R, \partial_i \mathbf{v}_R\big\rangle + \varepsilon^2\frac{Re}{1-\gamma}\langle \nabla\partial_i \mathbf{v}_R, \nabla\partial_i \mathbf{v}_R\rangle\\
&\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{3/4}\mathfrak{E}_\varepsilon(t)^{3/2}\big)+C\varepsilon^{\frac12}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}+\delta\mathfrak{F}_\varepsilon(t)\\
&\qquad+\big\langle \mathbf{m}\mathbf{m}\times\frac{1}{\varepsilon}f_0\mathcal{R}\mathcal{H}_\varepsilon\partial_i f_R, \nabla\partial_i \mathbf{v}_R\big\rangle.
\end{aligned}\tag{7.24}$$

Step 3. H^2 energy estimate

Taking Δ to (7.14), then making $L^2(\Omega\times\mathbb{S}^2)$ inner product with $\varepsilon^3\mathcal{A}^{-1}\Delta f_R$, we get

$$\begin{aligned}
&\varepsilon^3\big\langle \frac{\partial}{\partial t}\Delta f_R, \mathcal{A}^{-1}\Delta f_R\big\rangle + \varepsilon^2\big\langle \Delta f_R, \mathcal{H}_\varepsilon\Delta f_R\big\rangle\\
&=\varepsilon^2\langle f_R\mathcal{R}\Delta\mathcal{U}_0 f_0+\Delta f_0\mathcal{R}\mathcal{U}_\varepsilon f_R+2\partial_i f_0\mathcal{R}\mathcal{U}_\varepsilon\partial_i f_R+2\partial_i f_R\mathcal{R}\mathcal{U}_0\partial_i f_0, \mathcal{R}\mathcal{A}^{-1}\Delta f_R\rangle\\
&\quad-\varepsilon^3\big\langle \Delta(\widetilde{\mathbf{v}}\cdot\nabla f_R), \mathcal{A}^{-1}\Delta f_R\big\rangle+\varepsilon^3\big\langle \Delta(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0), \mathcal{R}\mathcal{A}^{-1}\Delta f_R\big\rangle\\
&\quad+\varepsilon^3\big\langle \Delta F_R+\Delta L_1, \mathcal{A}^{-1}\Delta f_R\big\rangle.
\end{aligned}$$

The first term on the right hand side is bounded by

$$\|\varepsilon^{3/2}\mathcal{R}\mathcal{A}^{-1}\Delta f_R\|_{L^2}^2+\|\varepsilon^{1/2}\partial_i f_R\|_{L^2}^2+\varepsilon\|f_R\|_{L^2}^2\le C\mathfrak{E}_\varepsilon(t).$$

By Lemma 7.6 and Lemma 7.7, we have

$$\begin{aligned}
&\varepsilon^3\big\langle \Delta F_4, \mathcal{A}^{-1}\Delta f_R\big\rangle \le C\varepsilon\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\le C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\
&\varepsilon^3\langle \Delta F_5, \mathcal{A}^{-1}\Delta f_R\rangle \le C\varepsilon^{5/4}\|f_R\|_{L^2}\|f_R\|_{H^{0,2}}^2\le C\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)^{3/2},\\
&\varepsilon^3\big\langle \Delta F_6, \mathcal{A}^{-1}\Delta f_R\big\rangle \le C\varepsilon\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\le C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}.
\end{aligned}$$

And by Lemma 7.5, the term $\varepsilon^3\big\langle \Delta(F_1+F_2+F_3), \mathcal{A}^{-1}\Delta f_R\big\rangle$ is bounded by

$$\begin{aligned}
&\varepsilon^{1/2}\|\varepsilon\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}+\|\varepsilon^{3/2}f_R\|_{H^{0,2}}^2+\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon\Delta f_R\|_{L^2}\\
&\le C\mathfrak{E}_\varepsilon(t)+\big(\frac{C_0}{\sqrt{C_3}}+\delta\big)\mathfrak{F}_\varepsilon(t).
\end{aligned}$$

So, we get

$$\begin{aligned}
&\big\langle \frac{\partial}{\partial t}\Delta f_R, \mathcal{A}^{-1}\Delta f_R\big\rangle+\frac{1}{\varepsilon}\big\langle \mathcal{H}_\varepsilon\Delta f_R, \Delta f_R\big\rangle\\
&\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{5/4}\mathfrak{E}_\varepsilon(t)^{3/2}\big)+C\varepsilon\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t)+\big(\delta+\frac{C_0}{\sqrt{C_3}}\big)\mathfrak{F}_\varepsilon(t).
\end{aligned}\tag{7.25}$$

Taking Δ to (7.14), then making $L^2(\Omega\times\mathbb{S}^2)$ inner product with $\varepsilon^3\mathcal{H}\Delta f_R$, we get

$$\begin{aligned}
&\varepsilon^3\langle\frac{\partial}{\partial t}\partial_i f_R,\mathcal{H}_\varepsilon\Delta f_R\rangle+\varepsilon^2\langle f_0\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R,\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\rangle\\
&=\varepsilon^2\langle f_R\mathcal{R}\Delta\mathcal{U}_0 f_0+\Delta f_0\mathcal{R}\mathcal{U}_\varepsilon f_R+2\partial_i f_0\mathcal{R}\mathcal{U}_\varepsilon\partial_i f_R+2\partial_i f_R\mathcal{R}\mathcal{U}_0\partial_i f_0,\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\rangle\\
&\quad+\varepsilon^3\big\langle\Delta(\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0),\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\big\rangle+\varepsilon^3\big\langle\Delta F_R+\Delta L_1,\mathcal{H}_\varepsilon\partial_i f_R\big\rangle.
\end{aligned}$$

The first term on the right hand side is bounded by

$$\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}\big(\varepsilon^{1/2}\|\varepsilon^{1/2}\partial_i f_R\|_{L^2}+\varepsilon\|f_R\|_{L^2}\big)\le\delta\mathfrak{F}_\varepsilon(t)+C\mathfrak{E}_\varepsilon(t)).$$

By Lemma 7.6 and Lemma 7.7, we get

$$\begin{aligned}
\varepsilon^3\langle\Delta F_4,\mathcal{H}_\varepsilon\Delta f_R\rangle&\le C\varepsilon^{3/2}\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}\le C\varepsilon^{3/2}\mathfrak{F}_\varepsilon(t)\mathfrak{E}_\varepsilon(t)^{1/2},\\
\varepsilon^3\langle\Delta F_5,\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\rangle&\le C\varepsilon^{7/4}\|f_R\|_{L^2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}\le C\varepsilon^{7/4}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},
\end{aligned}$$

and for F_6, we have

$$\begin{aligned}
\varepsilon^3\big\langle\Delta F_6,\mathcal{H}_\varepsilon\Delta f_R\big\rangle&=\varepsilon^6\big\langle\Delta\mathbf{v}_R\cdot\nabla f_R,\frac{\Delta f_R}{f_0}\big\rangle+2\varepsilon^6\big\langle\partial_i\mathbf{v}_R\cdot\nabla\partial_i f_R,\frac{\Delta f_R}{f_0}\big\rangle\\
&\quad+\varepsilon^6\big\langle\mathbf{v}_R\cdot\nabla f_R,\Delta(\mathcal{U}_\varepsilon\Delta f_R)\big\rangle-\frac{\varepsilon^6}{2}\big\langle\mathbf{v}_R\cdot\nabla(\frac{1}{f_0})\Delta f_R,\Delta f_R\big\rangle\\
&\le C\varepsilon\big(\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}\nabla f_R\|_{H^{0,1}}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}\\
&\qquad+\varepsilon^{7/4}\|\mathbf{v}_R\|_{L^2}\|\varepsilon^{1/2}\nabla f_R\|_{L^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}+\|\varepsilon^2\mathbf{v}_R\|_{H^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}^2\big)\\
&\le C\varepsilon\mathfrak{F}_\varepsilon(t)^{1/2}\mathfrak{E}_\varepsilon(t)+C\varepsilon\mathfrak{E}_\varepsilon(t)^{3/2}.
\end{aligned}$$

And the term $\varepsilon^3\big\langle\Delta(F_1+F_2+F_3),\mathcal{H}_\varepsilon\Delta f_R\big\rangle$ is bounded by

$$\begin{aligned}
&\|\varepsilon^2\mathbf{v}_R\|_{H^2}\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}+\varepsilon^{1/2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}\\
&\quad+\varepsilon\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\|\varepsilon\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\|_{L^2}\le C\mathfrak{E}_\varepsilon(t)+(\varepsilon+\delta)\mathfrak{F}_\varepsilon(t),
\end{aligned}$$

and by Lemma 7.7,

$$\begin{aligned}
\varepsilon^3\big\langle\Delta\big(\widetilde{\mathbf{v}}\cdot\nabla f_R\big),\mathcal{H}_\varepsilon\Delta f_R\big\rangle&=\varepsilon^3\big\langle\Delta\widetilde{\mathbf{v}}\cdot\nabla f_R,\frac{\Delta f_R}{f_0}\big\rangle+2\varepsilon^3\big\langle\partial_i\widetilde{\mathbf{v}}\cdot\nabla\partial_i f_R,\frac{\Delta f_R}{f_0}\big\rangle\\
&\quad+\varepsilon^3\big\langle\widetilde{\mathbf{v}}\cdot\nabla f_R,\Delta(\mathcal{U}_\varepsilon\Delta f_R)\big\rangle-\frac{\varepsilon^3}{2}\big\langle\widetilde{\mathbf{v}}\cdot\nabla(\frac{1}{f_0})\Delta f_R,\Delta f_R\big\rangle\\
&\le C\|\varepsilon^{1/2}\nabla f_R\|_{L^2}\|\varepsilon^{3/2}\Delta f_R\|_{L^2}^2+C\|\varepsilon^{3/2}\Delta f_R\|_{L^2}^2\le C\mathfrak{E}_\varepsilon(t).
\end{aligned}$$

So, we get

$$\begin{aligned}
&\varepsilon^3\big\langle\frac{\partial}{\partial t}\Delta f_R,\mathcal{H}_\varepsilon\Delta f_R\big\rangle+\varepsilon^2\big\langle\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R,f_0\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R\big\rangle\\
&\le C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon\mathfrak{E}_\varepsilon(t)^{3/2}\big)+C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}+C\big(\delta+\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\big)\mathfrak{F}_\varepsilon(t)\\
&\qquad+\varepsilon^3\big\langle\mathbf{m}\times(\nabla\Delta\mathbf{v}_R)^T\cdot\mathbf{m}f_0,\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R)\big\rangle.
\end{aligned}\tag{7.26}$$

Making $L^2(\Omega)$ inner product to (7.15) with $\varepsilon^4\Delta^2\mathbf{v}_R$, we get

$$\begin{aligned}&\frac{\varepsilon^4}{2}\frac{d}{dt}\big\langle\Delta\mathbf{v}_R,\Delta\mathbf{v}_R\big\rangle+\frac{\gamma}{Re}\varepsilon^4\big\langle\nabla\Delta\mathbf{v}_R,\nabla\Delta\mathbf{v}_R\big\rangle\\&=-\frac{1-\gamma}{2Re}\varepsilon^4\big\langle\Delta\big(\mathbf{D}_R:\langle\mathbf{mmmm}f_0\rangle_1\big),\nabla\Delta\mathbf{v}_R\big\rangle\\&\quad+\frac{1-\gamma}{Re}\varepsilon^3\big\langle\mathbf{mm}\times\Delta\big(f_0\mathcal{R}\mathcal{H}_\varepsilon f_R\big),\nabla\Delta\mathbf{v}_R\big\rangle+\varepsilon^4\big\langle\Delta G_R+\Delta L_2,\Delta\mathbf{v}_R\big\rangle.\end{aligned}$$

Again, $\big\langle\big(\Delta\mathbf{D}_R:\langle\mathbf{mmmm}f_0\rangle_1\big),\nabla\Delta\mathbf{v}_R\big\rangle\geq 0$. The other part of the first term is bounded by

$$\|\varepsilon^2\mathbf{v}_R\|_{H^2}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\leq C\mathfrak{E}_\varepsilon(t)+\delta\mathfrak{F}_\varepsilon(t).$$

By Lemma 7.6 and Lemma 7.7, we get

$$\begin{aligned}&\varepsilon^4\big\langle\Delta G_5,\Delta\mathbf{v}_R\big\rangle\leq C\varepsilon^{3/2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}\leq C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t),\\&\varepsilon^4\big\langle\Delta G_6,\Delta\mathbf{v}_R\big\rangle\leq C\varepsilon\|\varepsilon^2\mathbf{v}_R\|_{H^2}^2\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\leq C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\&\varepsilon^4\big\langle\Delta G_7,\Delta\mathbf{v}_R\big\rangle\leq C\varepsilon^{7/4}\|f_R\|_{L^2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\leq C\varepsilon^{7/4}\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2},\\&\varepsilon^4\big\langle\Delta G_8,\Delta\mathbf{v}_R\big\rangle\leq C\varepsilon^{9/4}\|\varepsilon^{1/2}f_R\|_{H^{0,1}}\|f_R\|_{L^2}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\leq\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}.\end{aligned}$$

And the term $\varepsilon^4\big\langle\Delta(G_1+G_2+G_3+G_4),\Delta\mathbf{v}_R\big\rangle$ is bounded by

$$\|\varepsilon^2\mathbf{v}_R\|_{H^2}^2+\varepsilon\|\varepsilon^2\nabla\mathbf{v}_R\|_{H^2}^2+\varepsilon^{1/2}\|\varepsilon^{3/2}f_R\|_{H^{0,2}}\|\varepsilon^2\nabla\Delta\mathbf{v}_R\|_{L^2}\leq C\mathfrak{E}_\varepsilon(t)+(\delta+\varepsilon)\mathfrak{F}_\varepsilon(t).$$

Then we get

$$\begin{aligned}&\varepsilon^4\langle\frac{\partial\Delta\mathbf{v}_R}{\partial t},\Delta\mathbf{v}_R\rangle+\frac{\gamma}{Re}\varepsilon^4\langle\nabla\Delta\mathbf{v}_R,\nabla\Delta\mathbf{v}_R\rangle\\&\leq C\big(1+\mathfrak{E}_\varepsilon(t)\big)+(\delta+\varepsilon)\mathfrak{F}_\varepsilon(t)+C\varepsilon\mathfrak{E}_\varepsilon(t)\mathfrak{F}_\varepsilon(t)^{1/2}\\&\qquad+C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t)+\big\langle\mathbf{mm}\times\frac{1}{\varepsilon}f_0\mathcal{R}\mathcal{H}_\varepsilon\Delta f_R,\nabla\Delta\mathbf{v}_R\big\rangle.\end{aligned}\tag{7.27}$$

Step 4. The closing of the energy estimates

Noting that

$$\frac{1}{\varepsilon}\big\langle\mathbf{m}\times(\nabla\mathbf{v}_R)^T\cdot\mathbf{m}f_0,\mathcal{R}\mathcal{H}_\varepsilon f_R\big\rangle+\frac{1}{\varepsilon}\big\langle\langle\mathbf{mm}\times f_0\mathcal{R}\mathcal{H}_\varepsilon f_R\rangle_1,\nabla\mathbf{v}_R\big\rangle=0,$$

and then summing up (7.19)-(7.27), and taking C_1 big enough, and then C_2,C_3 big enough, and finally taking δ small enough, we infer that there exist $\varepsilon>0$ and $c_1>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, there holds

$$\begin{aligned}&\big\langle\frac{\partial}{\partial t}f_R,\mathcal{A}^{-1}f_R\big\rangle+\frac{1}{\varepsilon}\big\langle\frac{\partial}{\partial t}f_R,\mathcal{H}_\varepsilon f_R\big\rangle+\frac{Re}{1-\gamma}\big\langle\frac{\partial}{\partial t}\mathbf{v}_R,\mathbf{v}_R\big\rangle\\&+C_1\varepsilon\big\langle\frac{\partial}{\partial t}\nabla f_R,\mathcal{A}^{-1}\nabla f_R\big\rangle+C_2\varepsilon\big\langle\frac{\partial}{\partial t}\nabla f_R,\mathcal{H}_\varepsilon\nabla f_R\big\rangle+C_2\varepsilon^2\frac{Re}{1-\gamma}\big\langle\frac{\partial}{\partial t}\nabla\mathbf{v}_R,\nabla\mathbf{v}_R\big\rangle\\&+\varepsilon^3\big\langle\frac{\partial}{\partial t}\Delta f_R,\mathcal{A}^{-1}\Delta f_R\big\rangle+C_3\varepsilon^3\big\langle\frac{\partial}{\partial t}\Delta f_R,\mathcal{H}_\varepsilon\Delta f_R\big\rangle+C_3\varepsilon^4\frac{Re}{1-\gamma}\big\langle\frac{\partial}{\partial t}\Delta\mathbf{v}_R,\Delta\mathbf{v}_R\big\rangle+c_1\mathfrak{F}_\varepsilon(t)\\&\leq C\big(1+\mathfrak{E}_\varepsilon(t)+\varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2}+\varepsilon\mathfrak{E}_\varepsilon(t)^2\big)+C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t).\end{aligned}$$

Now Proposition 7.2 implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\langle f_R, \mathcal{H}_\varepsilon f_R\rangle \le \langle \frac{\partial f_R}{\partial t}, \mathcal{H}_\varepsilon f_R\rangle + \delta\mathfrak{F}_\varepsilon(t) + C\mathfrak{E}_\varepsilon(t),\\
&\frac{1}{2}\frac{d}{dt}\langle \partial_i f_R, \mathcal{H}_\varepsilon \partial_i f_R\rangle \le \langle \frac{\partial}{\partial t}\partial_i f_R, \mathcal{H}_\varepsilon \partial_i f_R\rangle + \delta\mathfrak{F}_\varepsilon(t) + C\mathfrak{E}_\varepsilon(t),\\
&\frac{1}{2}\frac{d}{dt}\langle \Delta f_R, \mathcal{H}_\varepsilon \Delta f_R\rangle \le \langle \frac{\partial \Delta f_R}{\partial t}, \mathcal{H}_\varepsilon \Delta f_R\rangle + \delta\mathfrak{F}_\varepsilon(t) + C\mathfrak{E}_\varepsilon(t),
\end{aligned}$$

and we have the trivial inequality

$$\frac{1}{2}\frac{d}{dt}\langle f, \mathcal{A}^{-1} f\rangle \le \langle \frac{\partial}{\partial t} f, \mathcal{A}^{-1} f\rangle + C\|\mathcal{R}\mathcal{A}^{-1} f\|_{L^2}^2.$$

Thus, we can deduce that

$$\frac{d}{dt}\mathfrak{E}_\varepsilon(t) + c_1\mathfrak{F}_\varepsilon(t) \le C\big(1 + \mathfrak{E}_\varepsilon(t) + \varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2} + \varepsilon\mathfrak{E}_\varepsilon(t)^2\big) + C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t).$$

This completes the proof of Proposition 7.3. □

Now we are ready to prove Theorem 2.3. Given the initial data $\big(f_0^\varepsilon, \mathbf{v}_0^\varepsilon\big)$, we can show by the energy method [23] that there exists $T_\varepsilon > 0$ and a unique solution $\big(f^\varepsilon(\mathbf{x}, \mathbf{m}, t), \mathbf{v}^\varepsilon(\mathbf{x}, t)\big)$ on $[0, T_\varepsilon]$ to (2.8)-(2.9) such that

$$f^\varepsilon(t) - 1 \in C\big([0, T_\varepsilon]; H^2(\Omega \times \mathbb{S}^2)\big), \quad \mathbf{v}^\varepsilon(t) \in C\big([0, T_\varepsilon]; H^2(\Omega)\big) \cap L^2(0, T_\varepsilon; H^3(\Omega)).$$

While, Proposition 7.3 tells us that

$$\frac{d}{dt}\mathfrak{E}_\varepsilon(t) + c_1\mathfrak{F}_\varepsilon(t) \le C\big(1 + \mathfrak{E}_\varepsilon(t) + \varepsilon^{1/4}\mathfrak{E}_\varepsilon(t)^{3/2} + \varepsilon\mathfrak{E}_\varepsilon(t)^2\big) + C\varepsilon^{3/2}\mathfrak{E}_\varepsilon(t)^{1/2}\mathfrak{F}_\varepsilon(t),$$

for any $t \in [0, T_\varepsilon]$. Due to the assumptions of Theorem 2.3, we know that $\mathfrak{E}_\varepsilon(0) \le C$. Thus, there exist $\varepsilon_0 > 0$ depending on T such that for any $\varepsilon \in (0, \varepsilon_0)$ and $t \in [0, \min(T, T_\varepsilon)]$, there holds

$$\mathfrak{E}_\varepsilon(t) + c_1\int_0^t \mathfrak{F}_\varepsilon(s)\mathrm{d}s \le C.$$

This in turn implies $T_\varepsilon \ge T$ by a continuous argument. Then Theorem 2.3 follows. □

8. The dissipation of the Ericksen-Leslie energy

Recall that the Ericksen-Leslie equation has the following energy law

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{Re}{2(1-\gamma)}|\mathbf{v}|^2\mathrm{d}\mathbf{x} + E_F\Big)\\
&= \int_\Omega \Big(\frac{\gamma}{1-\gamma}|\nabla\mathbf{v}|^2 + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2 + \alpha_4\mathbf{D}:\mathbf{D}\\
&\qquad + (\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1})|\mathbf{D}\cdot\mathbf{n}|^2 + \frac{1}{\gamma_1}|\mathbf{n}\times\mathbf{h}|^2\Big)\mathrm{d}\mathbf{x}.
\end{aligned} \tag{8.1}$$

Because the relations between six Leslie coefficients are unclear in Physics, whether the energy is dissipated remains open. In [12], Lin and Liu present some constrains on the Leslie coefficients to ensure that the energy is dissipated. We will show that the energy (8.1) is dissipated for the Ericksen-Leslie equation derived from the Doi-Onsager equation. More precisely,

Theorem 8.1. *If the Leslie coefficients are determined by (2.6) and (2.7), then there holds*

$$(\alpha_1+\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2+\alpha_4\mathbf{D}:\mathbf{D}+\big(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}\big)|\mathbf{D}\cdot\mathbf{n}|^2\geq 0$$

for any symmetric matrix $\mathbf{D}$ *and* $\mathbf{n}\in\mathbb{S}^2$.

Remark 8.1. *Recall that* $\gamma_1=S_2/\lambda$. *By taking* $\mathbf{u}=\mathbf{u}'$ *in (8.3), we see that* $\lambda>0$, *thus* $\gamma_1>0$.

Throughout this section, we denote by $f_0=h_{\eta,\mathbf{n}}$ a critical point of $A[f]$.

8.1. **Some useful identities.** Recall that $S_2=\langle P_2(\mathbf{m}\cdot\mathbf{n})\rangle_{f_0}$ and $S_4=\langle P_4(\mathbf{m}\cdot\mathbf{n})\rangle_{f_0}$, where $P_k(x)$ is the k-th Legendre polynomial. We define

$$\mathbf{M}^{(2)}=\langle\mathbf{mm}\rangle_{f_0},\quad \mathbf{M}^{(4)}=\langle\mathbf{mmmm}\rangle_{f_0}.$$

Lemma 8.1. *It holds that*

$$\begin{aligned}\mathbf{M}^{(2)}&=S_2\mathbf{nn}+\frac{1-S_2}{3}\mathbf{I},\\ \mathbf{M}^{(4)}_{\alpha\beta\gamma\mu}&=S_4n_\alpha n_\beta n_\gamma n_\mu+\frac{S_2-S_4}{7}\big(n_\alpha n_\beta\delta_{\gamma\mu}+n_\gamma n_\mu\delta_{\alpha\beta}+n_\alpha n_\gamma\delta_{\beta\mu}+n_\beta n_\mu\delta_{\alpha\gamma}\big)\\ &\quad+n_\alpha n_\mu\delta_{\beta\gamma}+n_\beta n_\gamma\delta_{\alpha\mu})+\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\big(\delta_{\alpha\beta}\delta_{\gamma\mu}+\delta_{\alpha\gamma}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\gamma}\big).\end{aligned}$$

The lemma is a direct consequence of Lemma 6.5. Especially, the lemma implies that

Lemma 8.2. *For any symmetric matrix* $\mathbf{D}$, *there hold*

$$\begin{aligned}\mathbf{M}^{(2)}\cdot\mathbf{D}=&S_2\mathbf{n}(\mathbf{D}\cdot\mathbf{n}),\quad \mathbf{D}\cdot\mathbf{M}^{(2)}=S_2(\mathbf{D}\cdot\mathbf{n})\mathbf{n};\\ \mathbf{M}^{(4)}:\mathbf{D}=&S_4\mathbf{nn}(\mathbf{D}:\mathbf{nn})+\frac{2(S_2-S_4)}{7}\big((\mathbf{D}\cdot\mathbf{n})\mathbf{n}+\mathbf{n}(\mathbf{D}\cdot\mathbf{n})\big)\\ &+2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}+\frac{S_2-S_4}{7}\mathbf{I}(\mathbf{D}:\mathbf{nn}).\end{aligned}$$

Lemma 8.3. *For any symmetric constant matrix* $\mathbf{D}$, *there holds*

$$\big\langle\mathcal{R}\cdot\big(\mathbf{m}\times\mathbf{D}\cdot\mathbf{m}f_0\big),f\big\rangle=\frac{1}{2}\mathbf{D}:\int_{\mathbb{S}^2}(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{R}\cdot(f_0\mathcal{R}f)\mathrm{d}\mathbf{m}.$$

Proof. It is easy to show that for any vector field $\mathbf{v}$ defined on $\mathbb{S}^2$,

$$\langle(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{R}\cdot(f\mathbf{v})\rangle_1=\langle(\mathbf{m}\times\mathbf{v})\mathbf{m}+\mathbf{m}(\mathbf{m}\times\mathbf{v})\rangle_f.$$

Applying it with $\mathbf{v}=\mathcal{R}g$ and $\mathbf{v}=\mathbf{m}\times(\kappa\cdot\mathbf{m})$, we deduce that

$$\int_{\mathbb{S}^2}(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{R}\cdot(f\mathcal{R}g)\mathrm{d}\mathbf{m}=\langle\mathbf{m}\times\mathcal{R}g\mathbf{m}+\mathbf{mm}\times\mathcal{R}g\rangle_f.$$

Thus, we have

$$\begin{aligned}&\big\langle\mathcal{R}\cdot\big(\mathbf{m}\times\mathbf{D}\cdot\mathbf{m}f_0\big),f\big\rangle=\mathbf{D}:\langle\mathbf{m}(\mathbf{m}\times\mathcal{R}f)\rangle_{f_0}\\ &=\frac{1}{2}\mathbf{D}:\big(\langle\mathbf{m}(\mathbf{m}\times\mathcal{R}f)\rangle_{f_0}+\langle(\mathbf{m}\times\mathcal{R}f)\mathbf{m}\rangle_{f_0}\big)\\ &=\frac{1}{2}\mathbf{D}:\int_{\mathbb{S}^2}(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{R}\cdot(f_0\mathcal{R}f)\mathrm{d}\mathbf{m}.\end{aligned}$$

The lemma follows. □

Lemma 8.4. *For any antisymmetric constant matrix $\mathbf{\Omega}$, we have*

$$\mathcal{R}\cdot\big(\mathbf{m}\times(\mathbf{\Omega}\cdot\mathbf{m})f_0\big)-(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n}))\cdot\mathcal{R}f_0=0,$$

Proof. The lemma is a direct consequence of the following identities

$$\begin{aligned}&\mathcal{R}\cdot\big(\mathbf{m}\times(\mathbf{\Omega}\cdot\mathbf{m})\big)=\mathcal{R}_i(\epsilon^{ijk}m_j\Omega_{kl}m_l)=(\mathbf{I}-3\mathbf{mm}):\mathbf{\Omega}=0,\\&(\mathbf{m}\times(\mathbf{\Omega}\cdot\mathbf{m}))\cdot\mathcal{R}f_0=(\mathbf{m}\times(\mathbf{\Omega}\cdot\mathbf{m}))\cdot(\mathbf{m}\times\mathbf{n})f_0'\\&\quad=(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n}))\cdot(\mathbf{m}\times\mathbf{n})f_0'=(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n}))\cdot\mathcal{R}f_0.\end{aligned}$$

The proof is finished. □

8.2. **Projection operator and properties.** We denote by $\mathbb{P}_{\rm in}$ the projection operator from $\mathcal{P}_0(\mathbb{S}^2)$ to $\mathrm{Ker}\mathcal{G}_{f_0}$, and denote by $\mathbb{P}_{\rm out}$ the projection operator from $\mathcal{P}_0(\mathbb{S}^2)$ to $(\mathrm{Ker}\mathcal{G}^*_{f_0})^\perp$. Since $\mathrm{Ker}\mathcal{G}_{f_0}$ is orthogonal to $(\mathrm{Ker}\mathcal{G}^*_{f_0})^\perp$ under the inner product $\langle\cdot,\mathcal{A}_{f_0}^{-1}(\cdot)\rangle$, we have

$$\langle f,\mathcal{A}_{f_0}^{-1}f\rangle=\langle\mathbb{P}_{\rm in}f,\mathcal{A}_{f_0}^{-1}\mathbb{P}_{\rm in}f\rangle+\langle\mathbb{P}_{\rm out}f,\mathcal{A}_{f_0}^{-1}\mathbb{P}_{\rm out}f\rangle.$$

For any constant matrix κ, we define

$$\mathcal{K}(\kappa)=\mathbb{P}_{\rm in}\big[\mathcal{R}\cdot\big(\mathbf{m}\times(\kappa\cdot\mathbf{m})f_0\big)\big],\quad\mathcal{L}(\kappa)=\mathbb{P}_{\rm out}\big[\mathcal{R}\cdot\big(\mathbf{m}\times(\kappa\cdot\mathbf{m})f_0\big)\big].$$

Lemma 8.5. *It holds that*

$$\mathcal{K}(\kappa)=\big(\mathbf{n}\times(\lambda\mathbf{D}\cdot\mathbf{n}-\mathbf{\Omega}\cdot\mathbf{n})\big)\cdot\mathcal{R}f_0.$$

Here $\mathbf{D}=\frac{1}{2}(\kappa+\kappa^T),\mathbf{\Omega}=\frac{1}{2}(\kappa^T-\kappa)$.

Proof. By Theorem 4.1, we may assume that

$$\mathbb{P}_{\rm in}[\mathcal{R}\cdot(\mathbf{m}\times(\kappa\cdot\mathbf{m})f_0)]=\mathbf{w}\cdot\mathcal{R}f_0$$

for some vector $\mathbf{w}$ with $\mathbf{w}\perp\mathbf{n}$. Thus for all $\mathbf{\Theta}\cdot\mathcal{R}f_0\in\mathrm{Ker}\mathcal{G}_{f_0}$,

$$\big\langle\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_0),\mathcal{A}_{f_0}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\big\rangle=\big\langle\mathbf{w}\cdot\mathcal{R}f_0,\mathcal{A}_{f_0}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\big\rangle.$$

First we claim that

$$\langle\mathbf{w}\cdot\mathcal{R}f_0,\mathcal{A}_{f_0}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\rangle=\mathbf{\Theta}\cdot\big(\mathbf{n}\times\big(S_2\mathbf{D}\cdot\mathbf{n}-\frac{S_2}{\lambda}\mathbf{\Omega}\cdot\mathbf{n}\big)\big).\tag{8.2}$$

Let $\mathbf{u}$ and $\mathbf{u}'$ be any vectors. By Proposition 4.4, we may write

$$\mathcal{A}_{f_0}^{-1}(\mathbf{u}'\cdot\mathcal{R}f_0)=(u_1'\sin\phi-u_2'\cos\phi)g_0(\theta).$$

Then we get by a direct computation that

$$\begin{aligned}&\big\langle\mathbf{u}\cdot\mathcal{R}f_0,\mathcal{A}_{f_0}^{-1}(\mathbf{u}'\cdot\mathcal{R}f_0)\big\rangle\\&=\int_{\mathbb{S}^2}2\eta\sin\theta\cos\theta f_0(u_1\sin\phi-u_2\cos\phi)(u_1'\sin\phi-u_2'\cos\phi)g(\theta)\mathrm{d}\mathbf{m}\\&=\frac{1}{2}(\mathbf{u}\times\mathbf{n})(\mathbf{u}'\times\mathbf{n})\int_{\mathbb{S}^2}f_0\frac{\mathrm{d}u_0}{\mathrm{d}\theta}g(\theta)\mathrm{d}\mathbf{m}=\frac{S_2}{\lambda}\big(\mathbf{u}-(\mathbf{u}\cdot\mathbf{n})\mathbf{n}\big)\cdot\mathbf{u}'.\end{aligned}\tag{8.3}$$

Therefore, $\mathbf{w}=\mathbf{n}\times(\lambda\mathbf{D}\cdot\mathbf{n}-\mathbf{\Omega}\cdot\mathbf{n})$.

Now, we prove (8.2). By Lemma 8.3, Lemma 8.4 and (8.3), we have

$$\begin{aligned}
&\big\langle \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}f_0), \mathcal{A}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\big\rangle\\
&=\big\langle \mathcal{R}\cdot(\mathbf{m}\times\mathbf{D}\cdot\mathbf{m}f_0), \mathcal{A}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\big\rangle-\big\langle \mathcal{R}\cdot(\mathbf{m}\times\mathbf{\Omega}\cdot\mathbf{m}f_0), \mathcal{A}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\big\rangle\\
&=-\frac{1}{2}\langle(\mathbf{mm}:\mathbf{D}),\mathbf{\Theta}\cdot\mathcal{R}f_0\rangle-\langle(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n}))\cdot\mathcal{R}f_0,\mathcal{A}^{-1}(\mathbf{\Theta}\cdot\mathcal{R}f_0)\rangle\\
&=\langle\mathbf{\Theta}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m}))\rangle_{f_0}-\frac{S_2}{\lambda}\mathbf{\Theta}\cdot(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n}))\\
&=S_2\mathbf{\Theta}\cdot(\mathbf{n}\times(\mathbf{D}\cdot\mathbf{n}))-\frac{S_2}{\lambda}\mathbf{\Theta}\cdot(\mathbf{n}\times(\mathbf{\Omega}\cdot\mathbf{n})).
\end{aligned}$$

The claim follows. □

Lemma 8.6. $\mathcal{L}(\mathbf{\Omega})=0$ *for any antisymmetric matrix* $\mathbf{\Omega}$.

Proof This is equivalent to prove $\mathcal{K}(\mathbf{\Omega})=\mathcal{R}\cdot\big(\mathbf{m}\times(\mathbf{\Omega}\cdot\mathbf{m})f_0\big)$, which is a consequence of Lemma 8.4 and Lemma 8.5. □

Lemma 8.7. *For any symmetric matrix* $\mathbf{D}$*, there holds*

$$\begin{aligned}
\int_{\mathbb{S}^2}\mathcal{L}(\mathbf{D})\mathcal{A}_{f_0}^{-1}\mathcal{L}(\mathbf{D})\mathrm{d}\mathbf{m}=&\big(\frac{3S_2+4S_4}{7}-\lambda S_2\big)|\mathbf{D}\cdot\mathbf{n}|^2\\
&-2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}:\mathbf{D}+(\lambda S_2-S_4)(\mathbf{D}:\mathbf{nn})^2.
\end{aligned}$$

Proof. Applying Lemma 8.3 with $f=\mathcal{A}_{f_0}^{-1}\big(\mathcal{R}\cdot(\mathbf{m}\times\mathbf{D}\cdot\mathbf{m}f_0)\big)$ and Lemma 8.2, we get

$$\begin{aligned}
&\big\langle \mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0), \mathcal{A}_{f_0}^{-1}\mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0)\big\rangle\\
&=-\frac{1}{2}\mathbf{D}:\int_{\mathbb{S}^2}(\mathbf{mm}-\frac{1}{3})\mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0)\mathrm{d}\mathbf{m}\\
&=-\frac{1}{2}\mathbf{D}:\big(2\mathbf{D}:\langle\mathbf{mmmm}\rangle_{f_0}-\mathbf{D}\cdot\langle\mathbf{mm}\rangle_{f_0}-\langle\mathbf{mm}\rangle_{f_0}\cdot\mathbf{D}\big)\\
&=-\mathbf{D}:\Big(S_4\mathbf{nn}(\mathbf{D}:\mathbf{nn})+\frac{2(S_2-S_4)}{7}\big((\mathbf{D}\cdot\mathbf{n})\mathbf{n}+\mathbf{n}(\mathbf{D}\cdot\mathbf{n})\big)\\
&\quad+2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}+\frac{S_2-S_4}{7}\mathbf{I}(\mathbf{D}:\mathbf{nn})\Big)+S_2(\mathbf{D}\cdot\mathbf{n})^2\\
&=\frac{3S_2+4S_4}{7}(\mathbf{D}\cdot\mathbf{n})^2-2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}:\mathbf{D}-S_4(\mathbf{D}:\mathbf{nn})^2,
\end{aligned}$$

which along with Lemma 8.5 gives

$$\begin{aligned}
&\langle\mathcal{L}(\mathbf{D}),\mathcal{A}_{f_0}^{-1}\mathcal{L}(\mathbf{D})\rangle=\big\langle\mathbb{P}_{\mathrm{out}}\big(\mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0)\big),\mathcal{A}_{f_0}^{-1}\mathbb{P}_{\mathrm{out}}\big(\mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0)\big)\big\rangle\\
&=\big\langle \mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0), \mathcal{A}_{f_0}^{-1}\mathcal{R}\cdot(\mathbf{m}\times(\mathbf{D}\cdot\mathbf{m})f_0)\big\rangle-\lambda S_2|\mathbf{n}\times(\mathbf{D}:\mathbf{n})|^2\\
&=\big(\frac{3S_2+4S_4}{7}-\lambda S_2\big)(\mathbf{D}\cdot\mathbf{n})^2-2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}:\mathbf{D}+(\lambda S_2-S_4)(\mathbf{D}:\mathbf{nn})^2.
\end{aligned}$$

The proof is finished. □

8.3. **Proof of Theorem 8.1 and application.** Let us first prove Theorem 8.1. By (2.6)-(2.7) and Lemma 8.7, we find that

$$
\begin{aligned}
&(\alpha_1+\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2+\alpha_4\mathbf{D}:\mathbf{D}+(\gamma_3-\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}\cdot\mathbf{n}|^2\\
&=(-\frac{S_4}{2}+\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2+(-\frac{S_4}{35}-\frac{5S_2}{21}+\frac{4}{15})\mathbf{D}:\mathbf{D}+(\frac{5S_2+2S_4}{7}-\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}\cdot\mathbf{n}|^2\\
&=\big(\frac{3S_2+4S_4}{7}-\frac{\gamma_2^2}{\gamma_1}\big)(\mathbf{D}\cdot\mathbf{n})^2-2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}:\mathbf{D}+\big(-S_4+\frac{\gamma_2^2}{\gamma_1}\big)(\mathbf{D}:\mathbf{nn})^2\\
&\quad+\frac{2(S_2-S_4)}{7}(\mathbf{D}\cdot\mathbf{n})^2+\frac{S_4}{2}(\mathbf{D}:\mathbf{nn})^2+\big(\frac{S_4}{35}-\frac{3S_2}{7}+\frac{2}{5}\big)\mathbf{D}:\mathbf{D} \qquad (8.4)\\
&\geq\big(\frac{3S_2+4S_4}{7}-\frac{\gamma_2^2}{\gamma_1}\big)(\mathbf{D}\cdot\mathbf{n})^2-2\big(\frac{S_4}{35}-\frac{2S_2}{21}+\frac{1}{15}\big)\mathbf{D}^2+\big(-S_4+\frac{\gamma_2^2}{\gamma_1}\big)(\mathbf{D}:\mathbf{nn})^2\geq 0,
\end{aligned}
$$

since all the coefficients in the line (8.4) are positive. Indeed, we have

$$
\begin{aligned}
S_2&=\langle\frac{1}{2}(3(\mathbf{m}\cdot\mathbf{n})^2-1)\rangle_{f_0}=\frac{3A_2-A_0}{2A_0},\\
S_4&=\big\langle\frac{1}{8}(35(\mathbf{m}\cdot\mathbf{n})^4-30(\mathbf{m}\cdot\mathbf{n})^2+3)\big\rangle_{f_0}=\frac{35A_4-30A_2+3A_0}{8A_0}\\
&=\frac{1}{8A_0(2\eta)^2}(A_8-2A_6+A_4)>0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&S_2-S_4=\frac{7}{8A_0}(6A_2-5A_4-A_0)=\frac{7}{16A_0\eta}(A_6-2A_4+A_2)>0.\\
&\frac{S_4}{35}-\frac{3S_2}{7}+\frac{2}{5}=\frac{1}{8A_0}(5A_0-6A_2+A_4)>0.
\end{aligned}
$$

This complete the proof of Theorem 8.1. □

As a byproduct, we get the following dissipation law, which has been used in the existence of the Hilbert expansion.

Lemma 8.8. *For any matrix κ, there holds*

$$\big\langle(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{L}(\kappa)\big\rangle_1:\kappa\leq 0.$$

Proof. Lemma 8.3 implies that

$$\mathcal{A}_{f_0}\big(\mathbf{D}:(\mathbf{mm}-\frac{1}{3}\mathbf{I})\big)=-2\mathcal{R}\cdot\big(\mathbf{m}\times\mathbf{D}\cdot\mathbf{m}f_0\big)=-2(\mathcal{K}(\mathbf{D})+\mathcal{L}(\mathbf{D})).$$

Here $\mathbf{D}=\frac{1}{2}(\kappa+\kappa^T)$. From Lemma 8.6, we know that $\mathcal{L}(\kappa)=\mathcal{L}(\mathbf{D})$. Hence,

$$
\begin{aligned}
\big\langle(\mathbf{mm}-\frac{1}{3}\mathbf{I})\mathcal{L}(\kappa)\big\rangle_1:\kappa&=\int_{\mathbb{S}^2}(\mathbf{mm}-\frac{1}{3}\mathbf{I}):\mathbf{D}\mathcal{L}(\mathbf{D})\mathrm{d}\mathbf{m}\\
&=-2\big\langle\mathcal{A}_{f_0}^{-1}\mathcal{L}(\mathbf{D}),\ \mathcal{K}(\mathbf{D})+\mathcal{L}(\mathbf{D})\big\rangle=-2\big\langle\mathcal{A}_{f_0}^{-1}\mathcal{L}(\mathbf{D}),\ \mathcal{L}(\mathbf{D})\big\rangle\leq 0.
\end{aligned}
$$

The proof is finished. □

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