# Well-posedness of Hydrodynamics on the Moving Elastic Surface

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**Abstract**

The dynamics of a membrane is a coupled system comprising a moving elastic surface and an incompressible membrane fluid. We will consider a reduced elastic surface model, which involves the evolution equations of the moving surface, the dynamic equations of the two-dimensional fluid, and the incompressible equation, all of which operate within a curved geometry. In this paper, we prove the local existence and uniqueness of the solution to the reduced elastic surface model by reformulating the model into a new system in the isothermal coordinates. One major difficulty is that of constructing an appropriate iterative scheme such that the limit system is consistent with the original system.

## 1 Introduction

This paper is concerned with the hydrodynamics on the moving surface of bio-membrane, which as the outerwear of living cells and organelles plays an important role in the life process. Consisting of lipids, proteins and carbohydrates, the structures and properties of bio-membrane are very complex. In general, bio-membrane can be viewed as a 2-dimensional fluid surface consisting of a lipid bilayer, as the lipid molecules can move freely on the surface but cannot escape from it. The fluid is viscous and can be viewed as incompressible because it typically has a large tensile module. Moreover, this 2-dimensional fluid surface is bend-resistent. Hence, it tends to minimize the Helfrich energy under the fixed area condition (guaranteed by the incompressible condition)

$$E_H = \int \big(c_1(H-B)^2 - c_2 K\big)\mathrm{d}\sigma, \tag{1.1}$$

where $H$ and $K$ are the mean curvature and the Gaussian curvature, respectively, $B$ is the spontaneous curvature that reflects the initial or intrinsic curvature of the membrane, $c_1$ and $c_2$ are the elastic coefficients, and $\mathrm{d}\sigma$ is the area form of the surface [6]. When $c_2$ is uniform on the membrane, $\int K\mathrm{d}\sigma$ is a constant determined by the topology of the membrane. When $B \equiv 0$, $E_H$ is called the Willmore energy in geometry. A number of studies based on Helfrich's bending energy model explore the mechanics of bio-membrane, for example, see [22, 3, 14].

During the past several decades, membrane dynamics have received considerable attention. Researchers from different fields have developed several models with/without the surrounding fluid to study the behaviors of the membrane. For the models without surrounding fluid, see [25, 26, 19, 23, 3], and for the models with surrounding fluid, see [17, 18, 13, 16].

Waxman [25] may have been the first to study the dynamics of bend-resistant biomembrane using a model without surrounding fluid and in which the incompressibility, bend-resistance, and viscosity effects are all considered. However, Waxman's model does not preserve the energy dissipation law. In [10], Hu-Zhang-E introduced a director field to represent the direction of lipid molecules at every material point and developed an elastic energy model based on the Frank energy of the smectic liquid crystal. When the director is constrained to the normal of the surface, they obtain a reduced elastic surface model, that is very close to Waxman's model, but adds one term to the in-plane stresses whereby the model satisfies a natural energy dissipation law. In the elastic surface model, the dynamics of the membrane involves the evolution equations of the moving surface, the dynamic equations of the two-dimensional fluid, and the incompressible equation, all of which operate within a curved geometry.

For a surface membrane $\Gamma = \mathbf{R}(u_1, u_2, t)$, we denote by $\mathbf{a}_\alpha$ the tangent vectors of $\Gamma$, $\mathbf{n}$ the unit normal vector, $(a_{\alpha\beta})_{1\le\alpha,\beta\le 2}$ the covariant metric tensor, $\Delta_\Gamma$ the Lapalace-Beltrami operator, $K$ the Gaussian curvature, and $H$ the mean curvature. In the simple case, the reduced elastic surface model takes the following form:

$$\begin{cases} \frac{\partial \mathbf{R}}{\partial t} = \mathbf{v}(u_1, u_2, t), \\ \frac{\partial \mathbf{v}}{\partial t} = (-\Pi a^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta} + 2\varepsilon_0 (S^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta} - \frac{1}{2}\big(\Delta_\Gamma H + 2H(H^2-K)\big)\mathbf{n}, \\ \nabla_\Gamma \cdot \mathbf{v} = 0. \end{cases} \tag{1.2}$$

Here, $\mathbf{v}$ is the velocity of the fluid, $\Pi$ is the surface pressure, $S^{\alpha\beta}$ is the rate of the surface strain, and the constant $\varepsilon_0 > 0$ is the shear viscosity. The notation $()_{,\beta}$ denotes the covariant derivative. The first term on the right-hand side of the second equation is induced by the incompressible condition $\nabla_\Gamma \cdot \mathbf{v} = 0$, and the surface pressure $\Pi$ can be viewed as a Lagrangian multiplier; the second term describes the viscosity of the fluid on the surface; the third term is the elastic stress induced by the Helfrich bending energy (1.1) with $B = 0$. Please see Section 2 or [10] for more detail.

When the interaction with bulk fluid is considered, Hu-Zhang-E [10] also derived the incompressible membrane-fluid coupling system in the form

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}, & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0, & \text{in } \Omega, \\ [-p\mathbf{I} + \tau]\cdot\mathbf{n} = \mathbf{F}, & \text{on } \Gamma, \\ [\mathbf{u}] = 0, & \text{on } \Gamma, \\ \nabla_\Gamma\cdot\mathbf{u} = 0, & \text{on } \Gamma, \end{cases} \tag{1.3}$$

where $\tau = \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the stress of the bulk fluid, $\mathbf{F}$ is given by the right-hand side of the second equation of (1.2), $\Omega$ is the fluid domain, $\Gamma$ is the time-dependent surface of the membrane included in $\Omega$, and $[\cdot]$ denotes the jump across the membrane. In a recent review paper [16], a similar model was derived via the direct variational method. Compared with the classical free boundary problem of the Navier-Stokes equations, the main difference is that the system (1.3) contains two unknown pressures: the pressure $p$ of the surrounding fluid and the pressure $\Pi$ of the membrane defined on the surface, where $\Pi$ is determined by the incompressible condition $\nabla_\Gamma \cdot \mathbf{u} = 0$. Due to the coupling between $p$ and $\Pi$, solving the membrane-fluid coupling system (1.3) is still challenge, both mathematically and numerically. In some specific case (e.g., when the velocity of the surrounding fluid is small), the main influence of the bulk fluid is to maintain the enclosed volume of the membrane. In such cases for simplicity, it can be replaced by introducing osmotic pressure. Moreover, although the reduced model (1.2) neglects the fluid interaction, numerical simulation [8] also convinces us that this model can be used to reconstruct some important physical processes, such as exocytosis and endocytosis.

To our knowledge, few mathematical results such as the well-posedness for the fluid bio-membrane dynamics are available. In [4], Cheng-Coutand-Shkoller studied the bulk fluid interacting with a membrane considered a nonlinear elastic bio-fluid shell and modeled by the nonlinear Saint Venant-Kirchhoff constitutive law, where the membrane is compressible and the surface fluid is inviscid. In [9], Hu-Song-Zhang proved the local existence and uniqueness of (1.2) for a simplified case when the membrane is cylindrical. In this case, the membrane is similar to a 1-D incompressible string such that the fluid vanishes. With the introduction of the arc length parameter and the tangent angle of the curve, the system is transformed into a fourth-order wave equation for the tangent angle $\alpha$ coupled with an elliptic equation:

$$\begin{cases} \alpha_{tt} = g_1 + 2T_s\alpha_s + T\alpha_{ss} - (B+\alpha_s)_{sss} + \alpha_s^2(B+\alpha_s)_s, \\ -T_{ss} + \alpha_s^2 T = g_2 + \alpha_t^2 + 2(B+\alpha_s)_{ss}\alpha_s + (B+\alpha_s)_s\alpha_{ss}, \end{cases}$$

where $g_1, g_2$, and $B$ are the given smooth functions.

The purpose of this paper is to prove the local well-posedness of the system (1.2). This is also a key step toward understanding and solving the membrane-fluid coupling system (1.3). Our result is stated as follows.

**Theorem 1.1** *Let $s = 2k$ for some integer $k \geq 3$. Assume that the initial velocity $\mathbf{v}_0 \in H^{s-1}$ and the initial closed surface $\mathbf{R}_0 \in H^{s+1}$. There exists $T > 0$ such that the system (1.2) has a unique solution $(\mathbf{v}(t), \mathbf{R}(t))$ on $[0, T]$ satisfying*

$$\mathbf{v} \in C([0,T]; H^{s-1}), \qquad \mathbf{R} \in C([0,T]; H^{s+1}).$$

**Remark 1.2** *The regularity we imposed on the initial data should not be optimal. To simplify the analysis, we will work in a functional space with high regularity.*

System (1.2) is a coupled system of parabolic, hyperbolic, and elliptic equations. The evolution equations of the tangential velocities are parabolic, the evolution equations of the normal velocity and the mean curvature constitute a hyperbolic system, and the pressure satisfies an elliptic equation, see (3.23)-(3.28). Because the surface is moving, it seems natural to solve (1.2) in the framework of Lagrangian coordinates. However, some essential difficulties will arise. Let us explain it in what follows.

Assume that the initial velocity $\mathbf{v}_0 \in H^{s-1}$ and the initial surface $\mathbf{R}_0 \in H^{s+1}$. Because the tangential velocity $v^\alpha$ satisfies the parabolic equation, and the normal velocity $v^n$ and the mean curvature $H$ together satisfy the hyperbolic system, it seems natural to expect $v^\alpha$ to belong to $L^2(0,T;H^s)$, and $(v^n, H)$ to belong to $L^\infty(0,T;H^{s-1})$. However, these estimates depend on the $H^s$ regularity of the metric of the surface. Hence, we have to recover the

$H^s$ regularity of the metric from $(v^\alpha, v^n, H)$ in order to close the energy estimates. In the Lagrangian coordinates, we have

$$\mathbf{R}_t = \mathbf{v}(u_1, u_2, t),$$

which tells us that $\mathbf{R} \in L^\infty(0, T; H^{s-1})$ by the estimate for the velocity. Hence the metric has only $H^{s-2}$ regularity (a loss of two derivatives). Maybe, one wants to use the regularity of the mean curvature to gain the regularity of $\mathbf{R}$(Note that formally, $H^{s-1}$ regularity of the mean curvature suggests that the free surface has $H^{s+1}$ regularity). However, we cannot expect $\mathbf{R}$ to have more regularity in the Lagrangian coordinates, see the example and argument of Section 5 in [20].

Another way to solve the system is to represent the moving surface locally by $x_3 = g(x_1, x_2, t)$, where $g$ satisfies the following hyperbolic equation

$$\frac{1}{\sqrt{1+|\nabla_\Gamma g|^2}} g_{tt} + \Delta_\Gamma \Big( \mathrm{div}_\Gamma \big( \frac{\nabla_\Gamma g}{\sqrt{1+|\nabla_\Gamma g|^2}} \big) \Big) = \text{lower-order terms}.$$

However, if we make an energy estimate for this equation, the estimate is also not closed, since the lower-order terms contain the third-order derivative of $g$, which cannot be controlled by the main part.

Motivated by [1], we will use the isothermal coordinates to re-parameterize the surface. There are two main advantages adopting the isothermal coordinate: (1) we can gain two more regularities for the surface from the regularity of the mean curvature, and (2) the coefficients of the first fundamental form have the same regularity as the surface. Indeed, there are the following important relations between the surface $\Gamma = \mathbf{R}(u_1, u_2)$, the first fundamental form $E$, and the mean curvature $H$ when $(u_1, u_2)$ is taken as the isothermal coordinate of $\Gamma$:

$$\Delta \mathbf{R} = 2EH\mathbf{n}, \quad \Delta E = 2\big(\partial_{u_1}\partial_{u_2}\mathbf{R} \cdot \partial_{u_1}\partial_{u_2}\mathbf{R} - \partial_{u_1}^2 \mathbf{R} \cdot \partial_{u_2}^2 \mathbf{R}\big). \tag{1.4}$$

Here $\Delta = \partial_{u_1}^2 + \partial_{u_2}^2$, and $\mathbf{n}$ is the unit normal of $\Gamma$.

In general, it is difficult to construct an approximate system preserving the isothermal relation. As the solution of the approximate system does not satisfy the important geometric relation (1.4), there will also be derivative loss once we make the energy estimates for the approximate system. To overcome this difficulty, we incorporate the relation (1.4) into our iterative scheme. However, this produces another very troubling problem—one that arises for the most part from the construction of the iterative scheme and relates to the equivalence of the two systems. The problem is this: we do not know and need to establish whether the limit system is equivalent to the original system, and proving such equivalence involves very complicated geometric calculations.

This paper is organized as follows. In the next section, we review some formulae for the evolving surfaces and introduce the reduced elastic surface model. In Section 3, we derive an equivalent system in the isothermal coordinate by decomposing the velocity into tangential and normal components. Section 4 is devoted to studying the linearized system. In Section 5, we prove our main results, including the construction of the iteration scheme, nonlinear estimates, the convergence of the iteration procedure, and the consistency between the limit system and the original system.

## 2 The elastic model of an incompressible fluid membrane

In this section, we provide a short derivation of the dynamic model of an incompressible elastic fluid membrane in three-dimensional space. We refer to [10] for more details.

### 2.1 Geometric tensors and their evolution equations

For a surface membrane $\Gamma = \mathbf{R}(\vec{u}, t)$ with a curve coordinate $\vec{u} = (u^1, u^2)$, we can get the Frenet coordinate system of the surface. Namely, the tangent vectors $\mathbf{a}_\alpha$ and the unit normal vector $\mathbf{n}$ are given by

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{R}}{\partial u^\alpha}\,(\alpha = 1, 2), \quad \mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|}.$$

The covariant metric tensor $(a_{\alpha\beta})_{1\le\alpha,\beta\le2}$ is defined as

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta.$$

We denote its inverse by $(a^{\alpha\beta})_{1\le\alpha,\beta\le2}$, which can be used to raise or lower the indices of the vectors and tensors. For example,

$$b^\gamma_\beta = a^{\alpha\gamma} b_{\alpha\beta}.$$

The surface Christoffel symbols $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$ and the curvature tensor $b_{\alpha\beta} = b_{\beta\alpha}$ are given by the Gauss-Weingarten-Codazzi equation:

$$\begin{aligned}
&\frac{\partial \mathbf{a}_\alpha}{\partial u^\beta} = \Gamma^\gamma_{\alpha\beta}\mathbf{a}_\gamma + b_{\alpha\beta}\mathbf{n},\\
&\frac{\partial \mathbf{n}}{\partial u^\beta} = -b^\gamma_\beta \mathbf{a}_\gamma = -a^{\alpha\gamma} b_{\alpha\beta}\mathbf{a}_\gamma,\\
&b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}.
\end{aligned}$$

Here we use a comma followed by a lowercase Greek subscript to denote the covariant derivatives based on the metric tensor $a_{\alpha\beta}$, that is,

$$Q^{\cdot\cdot\alpha\cdot}_{\cdot\beta\cdot\cdot,\gamma} = \frac{\partial Q^{\cdot\cdot\alpha\cdot}_{\cdot\beta\cdot\cdot}}{\partial u^\gamma} + \sum \Gamma^\alpha_{\mu\gamma} Q^{\cdot\cdot\mu\cdot}_{\cdot\beta\cdot\cdot} - \sum \Gamma^\mu_{\beta\gamma} Q^{\cdot\cdot\alpha\cdot}_{\cdot\mu\cdot\cdot}. \tag{2.1}$$

For example, we have

$$b_{\alpha\beta,\gamma} = \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \Gamma^\delta_{\alpha\gamma} b_{\delta\beta} - \Gamma^\delta_{\beta\gamma} b_{\alpha\delta}.$$

Thus we can rewrite the Gauss-Weigarten-Codarzzi equation as

$$\mathbf{a}_{\alpha,\beta} = b_{\alpha\beta}\mathbf{n}, \quad \mathbf{n}_{,\alpha} = -b^\beta_\alpha \mathbf{a}_\beta, \quad b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}. \tag{2.2}$$

The mean curvature $H$ and the Gaussian curvature $K$ of the surface are given by

$$H = \frac{1}{2} b^\alpha_\alpha, \quad K = \frac{1}{2}(4H^2 - b^\alpha_\beta b^\beta_\alpha).$$

In the following, let us derive the evolution equations of the geometric tensors. For this purpose, we denote by $\mathbf{v}(\vec{u}, t)$ the velocity of the surface given by

$$\mathbf{v}(\vec{u}, t) = \frac{\partial \mathbf{R}(\vec{u}, t)}{\partial t}, \tag{2.3}$$

and we decompose it into

$$\mathbf{v} = v^\alpha \mathbf{a}_\alpha + v^n \mathbf{n}.$$

Using (2.2), it is easy to find that

$$\begin{aligned} \frac{\partial \mathbf{a}_\alpha}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial \mathbf{R}}{\partial u^\alpha} = \frac{\partial}{\partial u^\alpha}\frac{\partial \mathbf{R}}{\partial t} = \frac{\partial \mathbf{v}}{\partial u^\alpha} = \mathbf{v}_{,\alpha} \\ &= (v^\gamma_{,\alpha} - v^n b^\gamma_\alpha)\mathbf{a}_\gamma + (v^n_{,\alpha} + v^\gamma b_{\alpha\gamma})\mathbf{n}. \end{aligned} \tag{2.4}$$

As $\mathbf{n} \cdot \mathbf{a}_\alpha = 0$, we get by (2.4) that

$$\frac{\partial \mathbf{n}}{\partial t} \cdot \mathbf{a}_\alpha = -\frac{\partial \mathbf{a}_\alpha}{\partial t} \cdot \mathbf{n} = -(v^n_{,\alpha} + v^\gamma b_{\alpha\gamma}),$$

which together with the fact of $\frac{\partial \mathbf{n}}{\partial t} \cdot \mathbf{n} = 0$ implies that

$$\frac{\partial \mathbf{n}}{\partial t} = -(v^\beta b^\alpha_\beta + a^{\alpha\beta} v^n_{,\beta})\mathbf{a}_\alpha. \tag{2.5}$$

The evolution equation of the metric tensor is given by

$$\frac{\partial a_{\alpha\beta}}{\partial t} = \frac{\partial \mathbf{a}_\alpha}{\partial t} \cdot \mathbf{a}_\beta + \frac{\partial \mathbf{a}_\beta}{\partial t} \cdot \mathbf{a}_\alpha = (v_{\alpha,\beta} + v_{\beta,\alpha}) - 2v^n b_{\alpha\beta}. \tag{2.6}$$

Differentiating the identity $a^{\alpha\beta} a_{\beta\gamma} = \delta^\alpha_\gamma$ with respect to $t$, we get by (2.6) that

$$\frac{\partial a^{\alpha\beta}}{\partial t} = -a^{\alpha\gamma} a^{\beta\delta}(v_{\gamma,\delta} + v_{\delta,\gamma}) + 2v^n b^{\alpha\beta}. \tag{2.7}$$

And differentiating $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$, we get by (2.2)–(2.5) that

$$\frac{\partial b_{\alpha\beta}}{\partial t} = (v^n_{,\alpha\beta} - v^n b^\gamma_\alpha b_{\gamma\beta}) + (v^\gamma_{,\beta} b_{\alpha\gamma} + v^\gamma_{,\alpha} b_{\gamma\beta}) + v^\gamma b_{\alpha\beta,\gamma}. \tag{2.8}$$

Due to $2H = a^{\alpha\beta} b_{\alpha\beta}$, we get by (2.7) and (2.8) that

$$2\frac{\partial H}{\partial t} = a^{\alpha\beta} v^n_{,\alpha\beta} + v^n b^\alpha_\beta b^\beta_\alpha + 2v^\alpha H_{,\alpha}. \tag{2.9}$$

### 2.2 The derivation of the elastic surface model

In this subsection, we choose $\vec{u} = (u^1, u^2)$ as the Lagrangian coordinate of the moving fluid surface. In this coordinate system, the velocity of the fluid on the surface is equal to the velocity of the fluid $\mathbf{v}$ given by (2.3). The Helfrich bending elastic energy [6, 7] is

$$E_H = \int_\Gamma C_1^{\alpha\beta\gamma\delta}(B_{\alpha\beta} - b_{\alpha\beta})(B_{\gamma\delta} - b_{\gamma\delta})\mathrm{d}S, \tag{2.10}$$

where $B_{\alpha\beta}$ is the spontaneous curvature tensor, and the fourth-order tensor $C_1^{\alpha\beta\gamma\delta}$ is given by

$$C_1^{\alpha\beta\gamma\delta} = (k_1 - \varepsilon_1)a^{\alpha\beta}a^{\gamma\delta} + \varepsilon_1(a^{\alpha\gamma}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\gamma}),$$

where $k_1$ and $\varepsilon_1$ are positive elastic coefficients and $k_1 \geq \varepsilon_1$.

As the membrane is a two-dimensional incompressible fluid, we have

$$\nabla_\Gamma \cdot \mathbf{v} = 0,$$

which is equivalent to

$$v^\alpha_{,\alpha} = 2Hv^n. \tag{2.11}$$

Then by applying the principle of virtual work, we obtain elastic stresses. For isotropic Newtonian membrane fluids, the dynamical equation of the membrane is

$$\varrho \frac{\partial \mathbf{v}}{\partial t} = \mathbf{f} + (T^{\alpha\beta}\mathbf{a}_\beta)_{,\alpha} + (q^\alpha \mathbf{n})_{,\alpha}, \tag{2.12}$$

where $\varrho$ is the membrane fluid density. The in-plane stress tensor $T^{\alpha\beta}$ and transverse shear stress $q^\alpha$ are given by

$$\begin{aligned}
T^{\alpha\beta} &= -\Pi a^{\alpha\beta} + J^{\alpha\beta} + M^{\alpha\mu} b^\beta_\mu, \\
q^\alpha &= M^{\alpha\beta}_{,\beta}, \\
J^{\alpha\beta} &= C^{\alpha\beta\gamma\delta} S_{\gamma\delta}, \\
C^{\alpha\beta\gamma\delta} &= (k_0 - \varepsilon_0) a^{\alpha\beta} a^{\gamma\delta} + \varepsilon_0 (a^{\alpha\gamma} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\gamma}), \\
M^{\alpha\beta} &= C_1^{\alpha\beta\gamma\delta} (B_{\gamma\delta} - b_{\gamma\delta}),
\end{aligned}$$

where $\Pi$ is the surface pressure (tension), and the rate of the surface strain is given by

$$S_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{2} \frac{\partial a_{\alpha\beta}}{\partial t} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) - v^n b_{\alpha\beta}.$$

From (2.11), it is easy to see that $S^\alpha_\alpha = 0$, and thus $J^{\alpha\beta} = 2\varepsilon_0 S^{\alpha\beta}$. Furthermore, the above equations have the following energy dissipation relation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_H + \int_\Gamma \varrho |\mathbf{v}|^2 \mathrm{d}S\Big) = -\int_\Gamma C^{\alpha\beta\gamma\delta} S_{\alpha\beta} S_{\gamma\delta} \mathrm{d}S = -2\varepsilon_0 \int_\Gamma S^{\alpha\beta} S_{\alpha\beta} \mathrm{d}S. \tag{2.13}$$

If the function $B(\vec{u}) = B(\vec{u}) a_{\alpha\beta}$ with $B$ independent of the time $t$, then (2.10) can be reduced to

$$E_H = \int_\Gamma 4k_1 (H - B)^2 + 4\varepsilon_1 (H^2 - K) \mathrm{d}S,$$

and the velocity equation (2.12) can be reduced to the following form

$$\begin{aligned}
\varrho \frac{\partial \mathbf{v}}{\partial t} &= (-\Pi a^{\alpha\beta} \mathbf{a}_\alpha)_{,\beta} + 2\varepsilon_0 (S^{\alpha\beta} \mathbf{a}_\alpha)_{,\beta} - 4k_1 H a^{\alpha\beta} B_{,\beta} \mathbf{a}_\alpha \\
&\quad + 2k_1 (\Delta_\Gamma B - 2BK)\mathbf{n} - 2(k_1 + \varepsilon_1)\Big(\Delta_\Gamma H + 2H(H^2 - K)\Big)\mathbf{n}.
\end{aligned} \tag{2.14}$$

Here $\Delta_\Gamma$ denotes the Lapalace-Beltrami operator on the surface $\Gamma$, and $K$ is the Gaussian curvature. We refer to the appendix for the derivations of (2.13) and (2.14).

In this paper, we only consider the simple case with $B(\vec{u}) \equiv 0$. By the rescaling argument, we can set $\varrho = 1, 4(k_1 + \varepsilon_1) = 1$. Thus, we obtain

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} = (-\Pi a^{\alpha\beta} \mathbf{a}_\alpha)_{,\beta} + 2\varepsilon_0 (S^{\alpha\beta} \mathbf{a}_\alpha)_{,\beta} - \frac{1}{2}\Big(\Delta_\Gamma H + 2H(H^2 - K)\Big)\mathbf{n}, \\ v^\alpha_{,\alpha} = 2Hv^n. \end{cases} \tag{2.15}$$

# 3 New formulation of the system

Motivated by [1], we will reformulate (2.15) into a new system in the isothermal coordinates. That is, we choose a coordinate $\vec{x} = (x^1, x^2)$ such that the metric tensor satisfies

$$a_{11} = a_{22}, \quad a_{12} = a_{21} = 0. \tag{3.1}$$

As the tangential velocity of the surface only serves to reparameterize the surface, in the following we choose them such that

$$\frac{\partial a_{11}}{\partial t} = \frac{\partial a_{22}}{\partial t}, \quad \frac{\partial a_{12}}{\partial t} = 0. \tag{3.2}$$

As a result, if it holds for the initial surface the relation (3.1) will be preserved for any time $t$.

## 3.1 Elliptic system for the tangential velocity of the surface

In the sequel, for convenience, we denote $f_\alpha = \frac{\partial f}{\partial x^1}, f_\beta = \frac{\partial f}{\partial x^2}$ , and whereas $(\cdot)_{,\alpha}$ (or $(\cdot)_{,\beta}$) denotes the covariant derivative with respect to $x^1$ (or $x^2$). The unit tangent vector and the unit normal vector of the surface are given respectively by

$$\mathbf{t}^1 = \frac{\mathbf{R}_\alpha}{|\mathbf{R}_\alpha|}, \quad \mathbf{t}^2 = \frac{\mathbf{R}_\beta}{|\mathbf{R}_\beta|}, \quad \mathbf{n} = \frac{\mathbf{R}_\alpha \times \mathbf{R}_\beta}{|\mathbf{R}_\alpha \times \mathbf{R}_\beta|}. \tag{3.3}$$

We denote $E, F$, and $G$ by the coefficients of the first fundamental form, and $L, M$, and $N$ by the coefficients of the second fundamental form. Namely,

$$\begin{aligned} E = \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha, \quad F = \mathbf{R}_\alpha \cdot \mathbf{R}_\beta, \quad G = \mathbf{R}_\beta \cdot \mathbf{R}_\beta, \\ L = \mathbf{R}_{\alpha\alpha} \cdot \mathbf{n}, \quad M = \mathbf{R}_{\alpha\beta} \cdot \mathbf{n}, \quad N = \mathbf{R}_{\beta\beta} \cdot \mathbf{n}. \end{aligned}$$

In the isothermal coordinates, we have $E = G, F = 0$. The Christoffel symbols can be calculated as follows:

$$\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\Gamma^1_{22} = \tfrac{E_\alpha}{2E}, \\ \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = \tfrac{E_\beta}{2E}. \end{aligned} \tag{3.4}$$

And the following identities can be verified easily:

$$\begin{aligned}
&\mathbf{t}^1_\alpha \cdot \mathbf{t}^2 = -\mathbf{t}^2_\alpha \cdot \mathbf{t}^1 = -\left(\tfrac{\mathbf{R}_\beta}{|\mathbf{R}_\beta|}\right)_\alpha \cdot \tfrac{\mathbf{R}_\alpha}{|\mathbf{R}_\alpha|} = -\tfrac{\mathbf{R}_{\alpha\beta}\cdot\mathbf{R}_\alpha}{E} = -\tfrac{E_\beta}{2E}, \\
&\mathbf{t}^2_\beta \cdot \mathbf{t}^1 = -\mathbf{t}^1_\beta \cdot \mathbf{t}^2 = -\left(\tfrac{\mathbf{R}_\alpha}{|\mathbf{R}_\alpha|}\right)_\beta \cdot \tfrac{\mathbf{R}_\beta}{|\mathbf{R}_\beta|} = -\tfrac{\mathbf{R}_{\alpha\beta}\cdot\mathbf{R}_\beta}{E} = -\tfrac{E_\alpha}{2E}, \\
&\mathbf{t}^1_\alpha \cdot \mathbf{n} = -\mathbf{t}^1 \cdot \mathbf{n}_\alpha = -\tfrac{\mathbf{R}_\alpha}{|\mathbf{R}_\alpha|} \cdot \mathbf{n}_\alpha = \tfrac{L}{\sqrt{E}}, \\
&\mathbf{t}^2_\alpha \cdot \mathbf{n} = -\mathbf{t}^2 \cdot \mathbf{n}_\alpha = -\tfrac{\mathbf{R}_\beta}{|\mathbf{R}_\beta|} \cdot \mathbf{n}_\alpha = \tfrac{M}{\sqrt{E}}, \\
&\mathbf{t}^1_\beta \cdot \mathbf{n} = -\mathbf{t}^1 \cdot \mathbf{n}_\beta = -\tfrac{\mathbf{R}_\alpha}{|\mathbf{R}_\alpha|} \cdot \mathbf{n}_\beta = \tfrac{M}{\sqrt{E}}, \\
&\mathbf{t}^2_\beta \cdot \mathbf{n} = -\mathbf{t}^2 \cdot \mathbf{n}_\beta = -\tfrac{\mathbf{R}_\beta}{|\mathbf{R}_\beta|} \cdot \mathbf{n}_\beta = \tfrac{N}{\sqrt{E}}.
\end{aligned} \tag{3.5}$$

For a given normal velocity $U^n(x^1, x^2, t)$, we assume that the evolution of the surface is determined by

$$\frac{\partial \mathbf{R}(\vec{x}, t)}{\partial t} = U^n \mathbf{n} + W_1 \mathbf{t}^1 + W_2 \mathbf{t}^2 \stackrel{\text{def}}{=} \mathbf{w}. \tag{3.6}$$

Then it follows from (3.5) that

$$\begin{aligned}
\mathbf{R}_{\alpha t} &= (U^n_\alpha + \frac{W_1 L}{\sqrt{E}} + \frac{W_2 M}{\sqrt{E}})\mathbf{n} + (W_{1\alpha} - \frac{U^n L}{\sqrt{E}} + \frac{W_2 E_\beta}{2E})\mathbf{t}^1 \\
&\quad + (W_{2\alpha} - \frac{U^n M}{\sqrt{E}} - \frac{W_1 E_\beta}{2E})\mathbf{t}^2 \\
&\stackrel{\text{def}}{=} A_1\mathbf{n} + A_{01}\mathbf{t}^1 + A_3\mathbf{t}^2, \qquad (3.7) \\
\mathbf{R}_{\beta t} &= (U^n_\beta + \frac{W_1 M}{\sqrt{E}} + \frac{W_2 N}{\sqrt{E}})\mathbf{n} + (W_{1\beta} - \frac{U^n M}{\sqrt{E}} - \frac{W_2 E_\alpha}{2E})\mathbf{t}^1 \\
&\quad + (W_{2\beta} - \frac{U^n N}{\sqrt{E}} + \frac{W_1 E_\alpha}{2E})\mathbf{t}^2 \\
&\stackrel{\text{def}}{=} A_2\mathbf{n} + A_4\mathbf{t}^1 + A_{02}\mathbf{t}^2. \qquad (3.8)
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&E_t = 2\mathbf{R}_{\alpha t}\cdot\mathbf{R}_\alpha = 2\sqrt{E}\mathbf{R}_{\alpha t}\cdot\mathbf{t}^1 = 2\sqrt{E}A_{01}, \\
&G_t = 2\mathbf{R}_{\beta t}\cdot\mathbf{R}_\beta = 2\sqrt{E}\mathbf{R}_{\beta t}\cdot\mathbf{t}^2 = 2\sqrt{E}A_{02}, \\
&F_t = \sqrt{E}(\mathbf{R}_{\alpha t}\cdot\mathbf{t}^2 + \mathbf{R}_{\beta t}\cdot\mathbf{t}^1) = \sqrt{E}(A_3 + A_4).
\end{aligned}$$

Now the relation (3.2) is equivalent to

$$(E-G)_t = 0, \quad F_t = 0,$$

which implies that

$$\begin{cases} \left(\frac{W_1}{\sqrt{E}}\right)_\alpha - \left(\frac{W_2}{\sqrt{E}}\right)_\beta = \frac{U^n(L-N)}{E}, \\ \left(\frac{W_1}{\sqrt{E}}\right)_\beta + \left(\frac{W_2}{\sqrt{E}}\right)_\alpha = \frac{2U^n M}{E}. \end{cases} \qquad (3.9)$$

This is an elliptic system for $(W_1, W_2)$. As mentioned above, if the surface evolves as (3.6) with $(W_1, W_2)$ determined by (3.9), the coordinate will always be isothermal.

**Remark 3.1** *The above system can also be obtained by using (2.6) directly.*

Let us conclude this section by deriving the elliptic equations for $E$ and $\mathbf{R}$. Noticing that $E = \mathbf{R}_\alpha\cdot\mathbf{R}_\alpha = \mathbf{R}_\beta\cdot\mathbf{R}_\beta$ and $\mathbf{R}_\alpha\cdot\mathbf{R}_\beta = 0$, we have

$$\begin{aligned}
\Delta E &= (\mathbf{R}_\alpha\cdot\mathbf{R}_\alpha)_{\beta\beta} + (\mathbf{R}_\beta\cdot\mathbf{R}_\beta)_{\alpha\alpha} - 2(\mathbf{R}_\alpha\cdot\mathbf{R}_\beta)_{\alpha\beta} \\
&= 2(\mathbf{R}_{\alpha\beta}\cdot\mathbf{R}_{\alpha\beta} - \mathbf{R}_{\alpha\alpha}\cdot\mathbf{R}_{\beta\beta}). \qquad (3.10)
\end{aligned}$$

On the other hand, we have

$$\mathbf{R}_\alpha\cdot\Delta\mathbf{R} = 0, \quad \mathbf{n}\cdot\Delta\mathbf{R} = L + N = 2EH,$$

which means that

$$\Delta\mathbf{R} = 2EH\mathbf{n}. \qquad (3.11)$$

**Remark 3.2** *From (3.10) and the standard elliptic estimate, it is easy to find that $E$ has the same regularity as the surface. This fact is noted by S.-S Chern in [5]. Then we can gain two more regularities of* $\mathbf{R}$ *from the regularity of the mean curvature $H$ by using (3.11). Specifically, we will use (3.10) and (3.11) to construct our approximate solutions in Section 5.1.*

### 3.2 The velocity equation in the isothermal coordinate

Assume that $\mathbf{u}(\vec{x},t)$ is the velocity of the fluid in the isothermal coordinate. Hence, $\mathbf{v}(\vec{u},t) = \mathbf{u}(\vec{x}(\vec{u},t),t)$, where $\vec{u}$ is the Lagrangian coordinate. And, we have

$$\begin{aligned}\mathbf{u}(\vec{x}(\vec{u},t),t) &= \mathbf{v}(\vec{u},t) = \frac{\mathrm{d}\mathbf{R}(\vec{x}(\vec{u},t),t)}{\mathrm{d}t} \\ &= \frac{\partial \mathbf{R}}{\partial t}\circ\vec{x} + \frac{\partial x^1(\vec{u},t)}{\partial t}\frac{\partial \mathbf{R}}{\partial x^1}\circ\vec{x} + \frac{\partial x^2(\vec{u},t)}{\partial t}\frac{\partial \mathbf{R}}{\partial x^2}\circ\vec{x} \\ &= \mathbf{w}\circ\vec{x} + x_t^1\mathbf{R}_\alpha\circ\vec{x} + x_t^2\mathbf{R}_\beta\circ\vec{x}.\end{aligned}$$

Hence, we have

$$x_t^1(\vec{u},t) = \big[\frac{1}{\sqrt{E}}(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^1\big]\circ\vec{x}, \quad x_t^2(\vec{u},t) = \big[\frac{1}{\sqrt{E}}(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^2\big]\circ\vec{x}.$$

Consequently,

$$\begin{aligned}\frac{\partial \mathbf{v}(\vec{u},t)}{\partial t} &= \frac{\mathrm{d}\mathbf{u}(\vec{x}(\vec{u},t),t)}{\mathrm{d}t} \\ &= \frac{\partial \mathbf{u}}{\partial t}\circ\vec{x} + \frac{\partial x^1(\vec{u},t)}{\partial t}\frac{\partial \mathbf{u}}{\partial x^1}\circ\vec{x} + \frac{\partial x^2(\vec{u},t)}{\partial t}\frac{\partial \mathbf{u}}{\partial x^2}\circ\vec{x} \\ &= \Big(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\sqrt{E}}\big[(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^1\big]\mathbf{u}_\alpha + \frac{1}{\sqrt{E}}\big[(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^2\big]\mathbf{u}_\beta\Big)\circ\vec{x}.\end{aligned}$$

The above equation can also be derived by Oldroyd's theorem [15].

On the other hand, by (2.2) we have that

$$\begin{aligned}&\mathbf{R}_{\alpha,\alpha} = L\mathbf{n}, \quad \mathbf{R}_{\alpha,\beta} = M\mathbf{n}, \quad \mathbf{R}_{\beta,\beta} = N\mathbf{n}, \\ &H = \frac{1}{2E}(L+N), \quad \Delta_\Gamma H = \frac{1}{E}\Delta H.\end{aligned}$$

Given that the right-hand side of (2.15) is coordinate-invariant, the first equation of (2.15) can be reduced to

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\sqrt{E}}[(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^1]\mathbf{u}_\alpha + \frac{1}{\sqrt{E}}[(\mathbf{u}-\mathbf{w})\cdot\mathbf{t}^2]\mathbf{u}_\beta \\ &= -2H\Pi\mathbf{n} - \frac{\Pi_\alpha}{\sqrt{E}}\mathbf{t}^1 - \frac{\Pi_\beta}{\sqrt{E}}\mathbf{t}^2 + \frac{2\epsilon_0}{E^2}\Big(S_{11}L + 2S_{12}M + S_{22}N\Big)\mathbf{n} \\ &\quad +2\epsilon_0\sqrt{E}(S^{11}_{,\alpha} + S^{12}_{,\beta})\mathbf{t}^1 + 2\epsilon_0\sqrt{E}(S^{12}_{,\alpha} + S^{22}_{,\beta})\mathbf{t}^2 \\ &\quad -\frac{1}{2}\Big(\frac{\Delta H}{E} + \frac{H}{2E^2}\big((L-N)^2 + 4M^2\big)\Big)\mathbf{n}.\end{aligned} \tag{3.12}$$

Here $S_{11} = \mathbf{u}_\alpha\cdot\mathbf{R}_\alpha, S_{12} = \frac{1}{2}(\mathbf{u}_\alpha\cdot\mathbf{R}_\beta + \mathbf{u}_\beta\cdot\mathbf{R}_\alpha), S_{22} = \mathbf{u}_\beta\cdot\mathbf{R}_\beta$. So,

$$\begin{aligned}&S_1^1 = \frac{S_{11}}{E}, \quad S_1^2 = S_2^1 = \frac{S_{12}}{E}, \quad S_2^2 = \frac{S_{22}}{E}, \\ &S^{11} = \frac{S_{11}}{E^2}, \quad S^{12} = S^{21} = \frac{S_{12}}{E^2}, \quad S^{22} = \frac{S_{22}}{E^2}.\end{aligned}$$

### 3.3 New equivalent system

Setting $\mathbf{u}(\vec{x},t) = U^n\mathbf{n} + U_1\mathbf{t}^1 + U_2\mathbf{t}^2$, we infer from (3.5) that

$$\begin{aligned}
\mathbf{u}_\alpha &= (U^n_\alpha + \frac{LU_1 + MU_2}{\sqrt{E}})\mathbf{n} + (U_{1\alpha} - \frac{LU^n}{\sqrt{E}} + \frac{U_2E_\beta}{2E})\mathbf{t}^1 \\
&\quad +(U_{2\alpha} - \frac{MU^n}{\sqrt{E}} - \frac{U_1E_\beta}{2E})\mathbf{t}^2 \\
&\stackrel{\text{def}}{=} f_1\mathbf{n} + g_{11}\mathbf{t}^1 + g_{12}\mathbf{t}^2, \qquad (3.13)\\
\mathbf{u}_\beta &= (U^n_\beta + \frac{MU_1 + NU_2}{\sqrt{E}})\mathbf{n} + (U_{1\beta} - \frac{MU^n}{\sqrt{E}} - \frac{U_2E_\alpha}{2E})\mathbf{t}^1 \\
&\quad +(U_{2\beta} - \frac{NU^n}{\sqrt{E}} + \frac{U_1E_\alpha}{2E})\mathbf{t}^2 \\
&\stackrel{\text{def}}{=} f_2\mathbf{n} + g_{21}\mathbf{t}^1 + g_{22}\mathbf{t}^2. \qquad (3.14)
\end{aligned}$$

Using (3.7) and (3.8), we find that

$$\begin{aligned}
\frac{\partial \mathbf{t}^1}{\partial t} &= \frac{1}{\sqrt{E}}(A_1\mathbf{n} + A_3\mathbf{t}^2),\\
\frac{\partial \mathbf{t}^2}{\partial t} &= \frac{1}{\sqrt{E}}(A_2\mathbf{n} + A_4\mathbf{t}^1),\\
\frac{\partial \mathbf{n}}{\partial t} &= -\frac{1}{\sqrt{E}}(A_1\mathbf{t}^1 + A_2\mathbf{t}^2).
\end{aligned}$$

Thus, the equation (3.12) can be rewritten as

$$\begin{aligned}
&\Big(U^n_t + \frac{1}{\sqrt{E}}(U_1A_1 + U_2A_2) + \frac{U_1 - W_1}{\sqrt{E}}f_1 + \frac{U_2 - W_2}{\sqrt{E}}f_2\Big)\mathbf{n} \\
&\quad +\Big(U_{1t} + \frac{1}{\sqrt{E}}(U_2A_4 - U^nA_1) + \frac{U_1 - W_1}{\sqrt{E}}g_{11} + \frac{U_2 - W_2}{\sqrt{E}}g_{21}\Big)\mathbf{t}^1 \\
&\quad +\Big(U_{2t} + \frac{1}{\sqrt{E}}(U_1A_3 - U^nA_2) + \frac{U_1 - W_1}{\sqrt{E}}g_{12} + \frac{U_2 - W_2}{\sqrt{E}}g_{22}\Big)\mathbf{t}^2 \\
&= -2H\Pi\mathbf{n} - \frac{\Pi_\alpha}{\sqrt{E}}\mathbf{t}^1 - \frac{\Pi_\beta}{\sqrt{E}}\mathbf{t}^2 + \frac{2\epsilon_0}{E^2}\Big(S_{11}L + 2S_{12}M + S_{22}N\Big)\mathbf{n} \\
&\quad +\frac{2\epsilon_0}{E\sqrt{E}}(S_{11,\alpha} + S_{12,\beta})\mathbf{t}^1 + \frac{2\epsilon_0}{E\sqrt{E}}(S_{12,\alpha} + S_{22,\beta})\mathbf{t}^2 \\
&\quad -\frac{1}{2}\Big(\frac{\Delta H}{E} + \frac{H}{2E^2}\big((L-N)^2 + 4M^2\big)\Big)\mathbf{n}. \qquad (3.15)
\end{aligned}$$

Here we used the fact that $E^2\big(S^{11}_{,\alpha} + S^{21}_{,\beta}\big) = S_{11,\alpha} + S_{12,\beta}$.

Now we calculate $S_{11,\alpha} + S_{12,\beta}$. By (3.13) and (3.14), the surface strain rate tensor can be written as

$$\begin{aligned}
S_{11} &= \mathbf{u}_\alpha \cdot \mathbf{R}_\alpha = \sqrt{E}U_{1\alpha} + \frac{U_2E_\beta}{2\sqrt{E}} - LU^n \\
&= (\sqrt{E}U_1)_\alpha + \frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} - LU^n, \qquad (3.16)\\
S_{12} &= S_{21} = \frac{1}{2}\big(\mathbf{u}_\alpha \cdot \mathbf{R}_\beta + \mathbf{u}_\beta \cdot \mathbf{R}_\alpha\big) \\
&= \frac{\sqrt{E}}{2}\big(U_{1\beta} + U_{2\alpha}\big) - \frac{E_\alpha U_2 + E_\beta U_1}{4\sqrt{E}} - MU^n \\
&= \frac{1}{2}(\sqrt{E}U_1)_\beta + \frac{1}{2}(\sqrt{E}U_2)_\alpha - \frac{E_\alpha U_2 + E_\beta U_1}{2\sqrt{E}} - MU^n, \qquad (3.17)\\
S_{22} &= \mathbf{u}_\beta \cdot \mathbf{R}_\beta = \sqrt{E}U_{2\beta} + \frac{E_\alpha U_1}{2\sqrt{E}} - NU^n \\
&= (\sqrt{E}U_2)_\beta + \frac{E_\alpha U_1 - E_\beta U_2}{2\sqrt{E}} - NU^n. \qquad (3.18)
\end{aligned}$$

Since the incompressible condition $\nabla_\Gamma \cdot \mathbf{u} = 0$ can also be written as $S_{11} + S_{22} = 0$, we get by (3.16) and (3.18) that

$$(\sqrt{E}U_1)_\alpha + (\sqrt{E}U_2)_\beta - 2EHU^n = 0. \qquad (3.19)$$

We get by (3.4) and $S_{11} + S_{22} = 0$ that

$$\begin{aligned}
S_{11,\alpha} + S_{12,\beta} &= S_{11\alpha} - 2\Gamma^i_{11}S_{1i} + S_{12\beta} - \Gamma^i_{22}S_{1i} - \Gamma^i_{12}S_{i2} \\
&= S_{11\alpha} + S_{12\beta}.
\end{aligned}$$

Similarly,

$$S_{12,\alpha} + S_{22,\beta} = S_{12\alpha} + S_{22\beta}.$$

Using (3.16)-(3.19), we find that

$$\begin{aligned}
S_{11\alpha} + S_{12\beta} &= \Big((\sqrt{E}U_1)_\alpha + \frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} - LU^n\Big)_\alpha \\
&\quad + \Big(\frac{1}{2}(\sqrt{E}U_1)_\beta + \frac{1}{2}(\sqrt{E}U_2)_\alpha - \frac{E_\alpha U_2 + E_\beta U_1}{2\sqrt{E}} - MU^n\Big)_\beta \\
&= \frac{\Delta(\sqrt{E}U_1)}{2} + \frac{1}{2}\Big((\sqrt{E}U_1)_\alpha + (\sqrt{E}U_2)_\beta\Big)_\alpha \\
&\quad + \Big(\frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} - LU^n\Big)_\alpha - \Big(\frac{E_\alpha U_2 + E_\beta U_1}{2\sqrt{E}} + MU^n\Big)_\beta \\
&= \frac{\Delta(\sqrt{E}U_1)}{2} + \Big(\frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} - LU^n + EHU^n\Big)_\alpha \\
&\quad - \Big(\frac{E_\alpha U_2 + E_\beta U_1}{2\sqrt{E}} + MU^n\Big)_\beta.
\end{aligned}$$

Thus, we obtain from (3.15) the evolution equation for $U_1$:

$$\begin{aligned}
\frac{\partial U_1}{\partial t} \quad = \quad & \frac{\epsilon_0}{E\sqrt{E}}\Delta(\sqrt{E}U_1) - \frac{\Pi_\alpha}{\sqrt{E}} \\
& + \frac{2\epsilon_0}{E\sqrt{E}}\Big(\frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} - LU^n + EHU^n\Big)_\alpha \\
& + \frac{1}{2\sqrt{E}}\Big((U^n)^2\Big)_\alpha - \frac{\epsilon_0}{E\sqrt{E}}\Big(\frac{E_\alpha U_2 + E_\beta U_1}{\sqrt{E}} + 2MU^n\Big)_\beta \\
& - \frac{1}{\sqrt{E}}\Big(U_2 W_{1\beta} + (U_1 - W_1)U_{1\alpha} + (U_2 - W_2)U_{1\beta}\Big) \\
& + \frac{1}{E}(2U_2U^nM + U_1U^nL) - \frac{1}{2E\sqrt{E}}\Big((U_1 - W_1)U_2E_\beta - U_2^2E_\alpha\Big). \qquad (3.20)
\end{aligned}$$

A similar evolution equation for $U_2$ can also be obtained, although we omit the details here. The evolution equation for the normal velocity $U^n$ is

$$\begin{aligned}
\frac{\partial U^n}{\partial t} \quad = \quad & -\frac{1}{2E}\Delta H - 2H\Pi - \frac{1}{\sqrt{E}}\Big((2U_1 - W_1)U_\alpha^n + (2U_2 - W_2)U_\beta^n\Big) \\
& + \frac{2\epsilon_0}{E\sqrt{E}}(LU_{1\alpha} + MU_{1\beta} + MU_{2\alpha} + NU_{2\beta}) \\
& + \frac{\epsilon_0}{E^2\sqrt{E}}(LE_\beta U_2 - ME_\alpha U_2 - ME_\beta U_1 + NE_\alpha U_1) \\
& - \frac{2\epsilon_0 U^n}{E\sqrt{E}}(L^2 + 2M^2 + N^2) \\
& - \frac{H}{4E^2}\Big((L-N)^2 + 4M^2\Big) - \frac{1}{E}(LU_1^2 + 2MU_1U_2 + NU_2^2). \qquad (3.21)
\end{aligned}$$

Due to (2.9), the evolution equation for the mean curvature $H$ is

$$\frac{\partial H}{\partial t} = \frac{1}{2E}\Delta U^n + \frac{W_1}{\sqrt{E}}H_\alpha + \frac{W_2}{\sqrt{E}}H_\beta + \frac{U^n}{2E^2}(L^2 + 2M^2 + N^2). \qquad (3.22)$$

We denote

$$\begin{aligned}
F_1 &= \frac{E_\beta U_2 - E_\alpha U_1}{2\sqrt{E}} + \frac{U^n}{2}(N - L), \quad F_2 = \frac{1}{2}(U^n)^2, \\
F_3 &= \frac{E_\alpha U_2 + E_\beta U_1}{\sqrt{E}} + 2MU^n, \\
F_4 &= -\frac{1}{\sqrt{E}}U_2W_{1\beta} + \frac{1}{E}(2MU_2U^n + LU_1U^n) - \frac{1}{2E\sqrt{E}}\Big((U_1 - W_1)U_2E_\beta - U_2^2E_\alpha\Big), \\
F_5 &= -\frac{1}{\sqrt{E}}U_1W_{2\alpha} + \frac{1}{E}(2MU_1U^n + NU_2U^n) - \frac{1}{2E\sqrt{E}}\Big((U_2 - W_2)U_1E_\alpha - U_1^2E_\beta\Big), \\
F_6 &= \frac{\epsilon_0}{E^2\sqrt{E}}(LE_\beta U_2 - ME_\alpha U_2 - ME_\beta U_1 + NE_\alpha U_1) - \frac{2\epsilon_0 U^n}{E\sqrt{E}}(L^2 + 2M^2 + N^2) \\
&\quad - \frac{H}{4E^2}\Big((L-N)^2 + 4M^2\Big) - \frac{1}{E}(LU_1^2 + 2MU_1U_2 + NU_2^2), \\
F_7 &= \frac{U^n}{2E^2}(L^2 + 2M^2 + N^2),
\end{aligned}$$

and $u_i = \frac{U_i}{\sqrt{E}}$, $w_i = \frac{W_i}{\sqrt{E}}$ for $i = 1, 2$.

From (3.19)-(3.22), we obtain the following equivalent full system:

$$
\begin{aligned}
&\frac{\partial \mathbf{R}}{\partial t} = U^n \mathbf{n} + W_1 \mathbf{t}^1 + W_2 \mathbf{t}^2, && (3.23)\\
&\frac{\partial U_1}{\partial t} = \epsilon_0 \frac{1}{E\sqrt{E}} \Delta(\sqrt{E} U_1) - \frac{1}{\sqrt{E}} \Pi_\alpha + \frac{2\epsilon_0}{E\sqrt{E}} F_{1\alpha} + \frac{1}{2\sqrt{E}} F_{2\alpha} - \frac{\epsilon_0}{E\sqrt{E}} F_{3\beta} \\
&\qquad -(u_1 - w_1) U_{1\alpha} - (u_2 - w_2) U_{1\beta} + F_4, && (3.24)\\
&\frac{\partial U_2}{\partial t} = \epsilon_0 \frac{1}{E\sqrt{E}} \Delta(\sqrt{E} U_2) - \frac{1}{\sqrt{E}} \Pi_\beta - \frac{2\epsilon_0}{E\sqrt{E}} F_{1\beta} + \frac{1}{2\sqrt{E}} F_{2\beta} - \frac{\epsilon_0}{E\sqrt{E}} F_{3\alpha} \\
&\qquad -(u_1 - w_1) U_{2\alpha} - (u_2 - w_2) U_{2\beta} + F_5, && (3.25)\\
&\frac{\partial U^n}{\partial t} = -\frac{1}{2E} \Delta H - 2H\Pi - \Big((2u_1 - w_1) U^n_\alpha + (2u_2 - w_2) U^n_\beta\Big) \\
&\qquad + \frac{2\epsilon_0}{E\sqrt{E}} (L U_{1\alpha} + M U_{1\beta} + M U_{2\alpha} + N U_{2\beta}) + F_6, && (3.26)\\
&\frac{\partial H}{\partial t} = \frac{1}{2E} \Delta U^n + w_1 H_\alpha + w_2 H_\beta + F_7, && (3.27)\\
&(\sqrt{E} U_1)_\alpha + (\sqrt{E} U_2)_\beta - 2EHU^n = 0, && (3.28)
\end{aligned}
$$

where $(W_1, W_2)$ is determined by the elliptic system (3.9).

**Remark 3.3** *Actually, (3.27) is induced by (3.23). However, in order to perform a suitable energy estimate, we add it to the full system.*

## 3.4 The equation of the pressure

Using the incompressible condition $(\sqrt{E} U_1)_\alpha + (\sqrt{E} U_2)_\beta = 2EHU^n$, we find that

$$(\sqrt{E} U_{1t})_\alpha + (\sqrt{E} U_{2t})_\beta - 2EHU^n_t = U^n (2EH)_t - \Big(\frac{E_t}{2\sqrt{E}} U_1\Big)_\alpha - \Big(\frac{E_t}{2\sqrt{E}} U_2\Big)_\beta.$$

We denote the left-hand side of (3.15) by

$$(U^n_t + \frac{K^n}{\sqrt{E}}) \mathbf{n} + (U_{1t} + \frac{K_1}{\sqrt{E}}) \mathbf{t}^1 + (U_{2t} + \frac{K_2}{\sqrt{E}}) \mathbf{t}^2.$$

Noting that $E_t = 2\sqrt{E} A_{01}$, then we have by (3.15) that

$$
\begin{aligned}
&-\Pi_{\alpha\alpha} + 2\epsilon_0 \Big(\frac{S_{11\alpha} + S_{21\beta}}{E}\Big)_\alpha - \Pi_{\beta\beta} + 2\epsilon_0 \Big(\frac{S_{12\alpha} + S_{22\beta}}{E}\Big)_\beta + 4EH^2 \Pi \\
&-K_{1\alpha} - K_{2\beta} + 2\sqrt{E} H K^n - \frac{4\epsilon_0 H}{E} \big(S_{11} L + 2 S_{12} M + S_{22} N\big) \\
&+H\Big(\Delta H + \frac{H}{2E}\big((L - N)^2 + 4M^2\big)\Big) \\
&= U^n (2EH)_t - (A_{01} U_1)_\alpha - (A_{01} U_2)_\beta.
\end{aligned}
$$

By a direct computation, we obtain

$$
\begin{aligned}
&U^n(2EH)_t + (K_1 - A_{01}U_1)_\alpha + (K_2 - A_{01}U_2)_\beta - 2\sqrt{E}HK^n\\
&= \Big(U_1U_{1\alpha} + U_2U_{1\beta} - \frac{U^n}{\sqrt{E}}(2MU_2 - (L-N)U_1) + \frac{U_1U_2E_\beta - U_2^2E_\alpha}{2E}\Big)_\alpha\\
&\quad + \Big(U_1U_{2\alpha} + U_2U_{2\beta} - \frac{U^n}{\sqrt{E}}(2MU_1 - (N-L)U_2) + \frac{U_1U_2E_\alpha - U_1^2E_\beta}{2E}\Big)_\beta\\
&\quad -4H\sqrt{E}(U_1U^n_\alpha + U_2U^n_\beta) - 2H(LU_1^2 + 2MU_2U_2 + NU_2^2) - (U^n_\alpha)^2 - (U^n_\beta)^2\\
&\quad + \frac{(U^n)^2}{E}(L^2 + 2M^2 + N^2 - 4E^2H^2) \overset{\text{def}}{=} \mathcal{G}_1.
\end{aligned}
$$

And using the incompressible condition again, we get

$$
\begin{aligned}
&\Big(\frac{S_{11\alpha} + S_{21\beta}}{E}\Big)_\alpha + \Big(\frac{S_{12\alpha} + S_{22\beta}}{E}\Big)_\beta\\
&= -\frac{E_\alpha}{2E^2}(S_{11\alpha} + S_{21\beta}) - \frac{E_\beta}{2E^2}(S_{12\alpha} + S_{22\beta}) + \frac{1}{E}\Delta(2EHU^n)\\
&\quad + \frac{1}{E}\Big((\sqrt{E})_\beta U_2 - (\sqrt{E})_\alpha U_1 - LU^n\Big)_{\alpha\alpha} - \frac{2}{E}\Big((\sqrt{E})_\alpha U_2 + (\sqrt{E})_\beta U_1 + MU^n\Big)_{\alpha\beta}\\
&\quad + \frac{1}{E}\Big((\sqrt{E})_\alpha U_1 - (\sqrt{E})_\beta U_2 - NU^n\Big)_{\beta\beta} \overset{\text{def}}{=} \mathcal{G}_2.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
-\Delta\Pi + 4EH^2\Pi = {} & H\Big(\Delta H + \frac{H}{2E}\big((L-N)^2 + 4M^2\big)\Big)\\
& + \mathcal{G}_1 - 2\varepsilon_0\mathcal{G}_2 - \frac{4\varepsilon_0 H}{E}\big(S_{11}L + 2S_{12}M + S_{22}N\big) \overset{\text{def}}{=} \mathcal{G}. \qquad (3.29)
\end{aligned}
$$

**Remark 3.4** *In (3.29), $\mathcal{G}$ is a polynomial function of $(\partial^k E, \partial^l U^n, \partial^l U_1, \partial^l U_2, \partial^l L, \partial^l M, \partial^l N, \partial^l H)$, where $0 \le |k| \le 3$ and $0 \le |l| \le 2$.*

**Remark 3.5** *It is reasonable that there is no term involving $\mathbf{w}$ in $\mathcal{G}$. Actually, by differentiating the equation $\mathbf{a}^\alpha \cdot \mathbf{v}_{,\alpha} = 0$, and reformulating the resulting equation in the isothermal coordinate, we can also derive the equation of the pressure.*

# 4 The linearized system

In this section, we study the well-posedness of the linearized system of (3.23)-(3.28). More precisely, we will consider the linear system

$$
\begin{cases}
\frac{\partial U_i}{\partial t} = \frac{1}{E\sqrt{E}}\Delta(\sqrt{E}U_i) + G_i, \quad i = 1, 2,\\
\frac{\partial U^n}{\partial t} = -\frac{1}{2E}\Delta H + B^1 \cdot \nabla U^n + G_3,\\
\frac{\partial H}{\partial t} = \frac{1}{2E}\Delta U^n + B^2 \cdot \nabla H + G_4,
\end{cases} \qquad (4.1)
$$

together with the initial condition

$$
(U_1, U_2, U^n, H)|_{t=0} = (U_1^0, U_2^0, U_0^n, H^0). \qquad (4.2)
$$

Throughout this paper, we assume that $x = (x_1, x_2) \in \mathbf{T}^2$.

**Theorem 4.1** *Let $s = 2(k+1)$ for some integer $k \geq 2$, and let $T > 0$. Suppose that $E \in C([0,T]; H^{s+1}(\mathbf{T}^2)) \cap C^1([0,T]; H^{s-1}(\mathbf{T}^2))$ and $E \geq c_0$ for some $c_0 > 0$. We also assume that $(U_1^0, U_2^0, U_0^n, H^0) \in H^{s-1}(\mathbf{T}^2)$, $(G_1, G_2) \in L^2(0,T; H^{s-2}(\mathbf{T}^2))$, $(G_3, G_4) \in L^2(0,T; H^{s-1}(\mathbf{T}^2))$, and $(B^1, B^2) \in L^2(0,T; H^{s-1}(\mathbf{T}^2))$. Then there exists a unique solution $(U_1, U_2, U^n, H)$ on $[0,T]$ to the linear system (4.1)-(4.2) such that*

$$(U_1, U_2) \in C([0,T]; H^{s-1}(\mathbf{T}^2)) \cap L^2(0,T; H^s(\mathbf{T}^2)),$$
$$(U^n, H) \in C([0,T]; H^{s-1}(\mathbf{T}^2)).$$

*Moreover, for any given $\varepsilon > 0$, it holds that*

$$E_s(t) \leq C(\|E\|_{L_t^\infty H^{s-1}}) \Big[ E_s(0) + \int_0^t \mathcal{F}_\varepsilon(\|E\|_{H^{s+1}}, \|E_t\|_{H^{s-1}})(1 + \|B\|_{H^s}) E_s(\tau) d\tau$$
$$+ \int_0^t \|(G_1, G_2)(\tau)\|_{H^{s-2}}^2 d\tau + \varepsilon \int_0^t \|(G_3, G_4)(\tau)\|_{H^{s-1}}^2 d\tau \Big], \tag{4.3}$$

*where $B = (B^1, B^2)$, $\mathcal{F}_\varepsilon$ is an increasing function, and $E_s(t)$ is defined by*

$$E_s(t) \stackrel{def}{=} \|(U_1, U_2)(t)\|_{H^{s-1}}^2 + \int_0^t \|(U_1, U_2)(\tau)\|_{H^s}^2 d\tau + \|(U^n, H)(t)\|_{H^{s-1}}^2.$$

**Proof.** The existence of $(U_1, U_2)$ is ensured by the classical parabolic theory, whereas $(U^n, H)$ can be obtained by the duality method, see [1] for example. Here we only present the proof of the energy estimate. For this purpose, let us introduce the energy functional $\mathcal{E}$ defined by

$$\mathcal{E} = \mathcal{E}^1 + \mathcal{E}^2,$$

with $\mathcal{E}^1$ and $\mathcal{E}^2$ given, respectively, by

$$\mathcal{E}^1 \stackrel{\text{def}}{=} \|\Lambda^{s-1} U_1\|_{L^2}^2 + \|\Lambda^{s-1} U_2\|_{L^2}^2,$$
$$\mathcal{E}^2 \stackrel{\text{def}}{=} \|\Lambda(\frac{1}{E}\Delta)^k U^n\|_{L^2}^2 + \|\Lambda(\frac{1}{E}\Delta)^k H\|_{L^2}^2, \quad \Lambda = (-\Delta)^{\frac{1}{2}}.$$

**Step 1. Estimate of $\mathcal{E}^1$**

Taking the derivative to $\mathcal{E}^1$ with respect to $t$, we obtain

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}^1 = \langle \Lambda^{s-1} U_1, \Lambda^{s-1} \partial_t U_1 \rangle + \langle \Lambda^{s-1} U_2, \Lambda^{s-1} \partial_t U_2 \rangle.$$

By using the first equation of (4.1), we get that

$$\langle \Lambda^{s-1} U_1, \Lambda^{s-1} \partial_t U_1 \rangle = \langle \Lambda^{s-1} U_1, \Lambda^{s-1} \frac{1}{E\sqrt{E}} \Delta(\sqrt{E} U_1) \rangle + \langle \Lambda^{s-1} U_1, \Lambda^{s-1} G_1 \rangle.$$

By the Cauchy-Schwartz inequality, we have

$$\langle \Lambda^{s-1} U_1, \Lambda^{s-1} G_1 \rangle \leq \|U_1\|_{H^s} \|G_1\|_{H^{s-2}},$$

and we write

$$\langle \Lambda^{s-1} U_1, \Lambda^{s-1} \frac{1}{E\sqrt{E}} \Delta(\sqrt{E} U_1) \rangle$$
$$= \langle \Lambda^{s-1} U_1, \big[\Lambda^{s-1}, \frac{1}{E\sqrt{E}} \Delta \sqrt{E}\big] U_1 \rangle + \langle \Lambda^{s-1} U_1, \frac{1}{E\sqrt{E}} \Delta(\sqrt{E} \Lambda^{s-1} U_1) \rangle.$$

By integration by parts and based on Lemmas 6.1 and 6.2, the second term of the right-hand side is bounded by

$$
\begin{aligned}
&-c\|\Lambda^s U_1\|_{L^2}^2 + C\|\nabla E\|_{L^\infty}^2\|\Lambda^{s-1}U_1\|_{L^2}^2 + C\|\nabla E\|_{L^\infty}\|\Lambda^{s-1}U_1\|_{L^2}\|\Lambda^s U_1\|_{L^2} \\
&\le -\frac{c}{2}\|U_1\|_{H^s}^2 + \mathcal{F}(\|E\|_{H^{s+1}})\|U_1\|_{H^{s-1}}^2.
\end{aligned}
$$

Here the constant $c>0$ depends only on $c_0$. And when Lemma 6.3 and Lemma 6.2 are both used, the first term is bounded by

$$
\mathcal{F}(\|E\|_{H^{s+1}})\|U_1\|_{H^{s-1}}^2 + \frac{c}{4}\|U_1\|_{H^s}^2,
$$

since we can write

$$
\big[\Lambda^{s-1}, \frac{1}{E\sqrt{E}}\Delta\sqrt{E}\big]U_1 = \big[\Lambda^{s-1}, \frac{1}{E\sqrt{E}}\big]\Delta(\sqrt{E}U_1) + \frac{1}{E\sqrt{E}}\Delta\big[\Lambda^{s-1}, \sqrt{E}\big]U_1.
$$

Summing up the above estimates yields that

$$
\langle\Lambda^{s-1}U_1, \Lambda^{s-1}\partial_t U_1\rangle \le -c\|U_1\|_{H^s}^2 + \mathcal{F}(\|E\|_{H^{s+1}})\|U_1\|_{H^{s-1}}^2 + \|G_1\|_{H^{s-2}}^2,
$$

for some $c>0$. Similarly, we have

$$
\langle\Lambda^{s-1}U_2, \Lambda^{s-1}\partial_t U_2\rangle \le -c\|U_2\|_{H^s}^2 + \mathcal{F}(\|E\|_{H^{s+1}})\|U_2\|_{H^{s-1}}^2 + \|G_2\|_{H^{s-2}}^2.
$$

Hence, we obtain

$$
\frac{d}{dt}\mathcal{E}^1 + c\|(U_1,U_2)\|_{H^s}^2 \le \mathcal{F}(\|E\|_{H^{s+1}})\|(U_1,U_2)\|_{H^{s-1}}^2 + \|(G_1,G_2)\|_{H^{s-2}}^2. \tag{4.4}
$$

**Step 2. Estimate of $\mathcal{E}^2$**

Take the derivative to $\mathcal{E}^2$ with respect to $t$ to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathcal{E}^2 &= \langle\Lambda(\frac{1}{E}\Delta)^k\partial_t U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle + \langle\Lambda(\frac{1}{E}\Delta)^k\partial_t H, \Lambda(\frac{1}{E}\Delta)^k H\rangle \\
&\quad + \langle\Lambda\big[\partial_t, (\frac{1}{E}\Delta)^k\big]U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle + \langle\Lambda\big[\partial_t, (\frac{1}{E}\Delta)^k\big]H, \Lambda(\frac{1}{E}\Delta)^k H\rangle \\
&\overset{\text{def}}{=} I + II + III + IV.
\end{aligned}
$$

Based on the last two equations of (4.1), we get

$$
\begin{aligned}
&I + II \\
&= \langle\Lambda(\frac{1}{E}\Delta)^k(B^1\cdot\nabla U^n), \Lambda(\frac{1}{E}\Delta)^k U^n\rangle + \langle\Lambda(\frac{1}{E}\Delta)^k G_3, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle \\
&\quad + \langle\Lambda(\frac{1}{E}\Delta)^k(B^2\cdot\nabla H), \Lambda(\frac{1}{E}\Delta)^k H\rangle + \langle\Lambda(\frac{1}{E}\Delta)^k G_4, \Lambda(\frac{1}{E}\Delta)^k H\rangle \\
&\overset{\text{def}}{=} I_1 + I_2 + II_1 + II_2.
\end{aligned}
$$

Here we use the following fact:

$$
\langle\Lambda(\frac{1}{E}\Delta)^k(-\frac{1}{2E}\Delta)H, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle + \langle\Lambda(\frac{1}{E}\Delta)^k(\frac{1}{2E}\Delta)U^n, \Lambda(\frac{1}{E}\Delta)^k H\rangle = 0.
$$

Using Lemma 6.6 and Lemma 6.4, we get

$$|III| + |IV| \le \mathcal{F}(\|(E,E_t)\|_{H^{s-1}})\big(\|U^n\|_{H^{s-1}}^2 + \|H\|_{H^{s-1}}^2\big),$$

and by Lemma 6.4,

$$|I_2| + |II_2| \le \mathcal{F}_\varepsilon(\|E\|_{H^{s-1}})\big(\|U^n\|_{H^{s-1}}^2 + \|H\|_{H^{s-1}}^2\big) + \varepsilon\|(G_3,G_4)\|_{H^{s-1}}^2.$$

To estimate $I_1$, we write

$$\begin{aligned} I_1 &= \langle \Lambda(\frac{1}{E}\Delta)^k(B^1\cdot\nabla U^n) - B^1\cdot\nabla\Lambda(\frac{1}{E}\Delta)^k U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle \\ &\quad + \langle B^1\cdot\nabla\Lambda(\frac{1}{E}\Delta)^k U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle = I_{11} + I_{12}. \end{aligned}$$

We have by Lemma 6.4 that

$$|I_{12}| \le \mathcal{F}(\|E\|_{H^{s-1}})\|\nabla B^1\|_{L^\infty}\|U^n\|_{H^{s-1}}^2.$$

We further write

$$\begin{aligned} I_{11} &= \langle \Lambda\big[(\frac{1}{E}\Delta)^k, B^1\big]\cdot\nabla U^n + \big[\Lambda, B^1\big]\cdot\nabla(\frac{1}{E}\Delta)^k U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle \\ &\quad + \langle B^1\cdot\Lambda\big[(\frac{1}{E}\Delta)^k,\nabla\big]U^n, \Lambda(\frac{1}{E}\Delta)^k U^n\rangle, \end{aligned}$$

which along with Lemma 6.4 and Lemma 6.6-6.7 implies that

$$|I_{11}| \le \mathcal{F}(\|E\|_{H^s})\|B^1\|_{H^{s-1}}\|U^n\|_{H^{s-1}}^2.$$

On the basis of the above estimates, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathcal{E}^2 &\le \mathcal{F}_\varepsilon(\|E\|_{H^s},\|E_t\|_{H^{s-1}})(1+\|B\|_{H^{s-1}})\big(\|U^n\|_{H^{s-1}}^2 + \|H\|_{H^{s-1}}^2\big) \\ &\quad + \varepsilon\|(G_3,G_4)\|_{H^{s-1}}^2. \end{aligned} \tag{4.5}$$

**Step 3.** $L^2$ **estimate**

Taking the $L^2$ energy estimate for $U_i (i=1,2)$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U_i\|_{L^2}^2 + \|\frac{1}{E}\nabla(\sqrt{E}U_i)\|_{L^2}^2 \\ &\le C(\|E\|_{L^\infty})\|\frac{1}{E}\nabla E\|_{L^\infty}\|\nabla(\sqrt{E}U_i)\|_{L^2}\|U_i\|_{L^2} + \|G_i\|_{L^2}\|U_i\|_{L^2} \\ &\le \mathcal{F}(\|E\|_{H^3})\|U_i\|_{L^2}^2 + \|G_i\|_{L^2}^2 + \|\frac{1}{E}\nabla(\sqrt{E}U_i)\|_{L^2}^2. \end{aligned}$$

Taking the $L^2$ energy estimate for $(U^n, H)$, we get

$$\begin{aligned} &\langle E\partial_t U^n, U^n\rangle + \langle E\partial_t H, H\rangle \\ &= \langle EB^1\cdot\nabla U^n, U^n\rangle + \langle EG_3, U^n\rangle + \langle EB^2\cdot\nabla H, H\rangle + \langle EG_4, H\rangle, \end{aligned}$$

from which, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\sqrt{E}U^n\|_{L^2}^2+\|\sqrt{E}H\|_{L^2}^2\big)\\
&\le \mathcal{F}_\varepsilon(\|(E,E_t)\|_{H^3})(1+\|B\|_{H^3})\big(\|U^n\|_{L^2}^2+\|H\|_{L^2}^2\big)+\varepsilon\|(G_3,G_4)\|_{L^2}^2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_0 \quad &\le \quad \mathcal{F}(\|E\|_{H^3})\|(U_1,U_2)\|_{L^2}^2+\mathcal{F}_\varepsilon(\|(E,E_t)\|_{H^3})(1+\|B\|_{H^3})\|(U^n,H)\|_{L^2}^2\\
&\quad +\|(G_1,G_2)\|_{L^2}^2+\varepsilon\|(G_3,G_4)\|_{L^2}^2. \qquad (4.6)
\end{aligned}$$

Here $\mathcal{E}_0 \stackrel{\text{def}}{=} \|(U_1,U_2)\|_{L^2}^2+\|\sqrt{E}U^n\|_{L^2}^2+\|\sqrt{E}H\|_{L^2}^2$.

Now we are in position to complete the proof. Taken together (4.4)-(4.6) yields that

$$\begin{aligned}
\frac{d}{dt}(\mathcal{E}+\eta\mathcal{E}_0)+c\|(U_1,U_2)\|_{H^s}^2 \quad &\le \quad \mathcal{F}_\varepsilon(\|E\|_{H^{s+1}},\|E_t\|_{H^{s-1}})(1+\|B\|_{H^s})(\mathcal{E}+\eta\mathcal{E}_0)\\
&\quad +\eta\big(\|(G_1,G_2)\|_{H^{s-2}}^2+\varepsilon\|(G_3,G_4)\|_{H^{s-1}}^2\big),
\end{aligned}$$

which implies (4.3) by taking $\eta$ to be bigger than $C(\|E\|_{L_t^\infty H^{s-1}})$, since we have by Lemma 6.5 and an interpolation argument that

$$\mathcal{E}^2 \ge c\|(U^n,H)\|_{H^{s-1}}^2 - C(\|E\|_{H^{s-1}})\|(U^n,H)\|_{L^2}^2.$$

This completes the proof of Theorem 4.1. □

# 5 Nonlinear system

This section is devoted to solving the nonlinear system (3.23)-(3.28).

## 5.1 Iteration scheme

We will construct the solution $(\mathbf{R},U_1,U_2,U^n,H)$ by the iteration method. First of all, we take

$$(\mathbf{R}^{(0)},U_1^{(0)},U_2^{(0)},U^{n,(0)},E^{(0)},H^{(0)})=(\mathbf{R}_0,U_1^0,U_2^0,U_0^n,E_0,H_0);$$

And, $(W_1^{(0)},W_2^{(0)})$ are determined by solving the following elliptic system:

$$\begin{cases}
\Big(\frac{W_1^{(0)}}{\sqrt{E^{(0)}}}\Big)_\alpha-\Big(\frac{W_2^{(0)}}{\sqrt{E^{(0)}}}\Big)_\beta=\frac{U^{n,(0)}(L^{(0)}-N^{(0)})}{E^{(0)}},\\
\Big(\frac{W_1^{(0)}}{\sqrt{E^{(0)}}}\Big)_\beta+\Big(\frac{W_2^{(0)}}{\sqrt{E^{(0)}}}\Big)_\alpha=\frac{2U^{n,(0)}M^{(0)}}{E^{(0)}}.
\end{cases}$$

The pressure $\Pi^{(0)}$ is given by

$$-\Delta\Pi^{(0)}+4E^{(0)}(H^{(0)})^2\Pi^{(0)}=\mathcal{G}^{(0)}$$

with $\mathcal{G}^{(0)}$ determined by $(\mathbf{R}^{(0)},U_1^{(0)},U_2^{(0)},U^{n,(0)})$ $\big($see (3.29)$\big)$.

Assume that $(U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, H^{(\ell)}, E^{(\ell)}, W_1^{(\ell)}, W_2^{(\ell)}, \mathbf{R}^{(\ell)})$ has been constructed. We denote

$$L^{(\ell)} = \mathbf{R}_{\alpha\alpha}^{(\ell)} \cdot \mathbf{n}^{(\ell)}, \quad M^{(\ell)} = \mathbf{R}_{\alpha\beta}^{(\ell)} \cdot \mathbf{n}^{(\ell)}, \quad N^{(\ell)} = \mathbf{R}_{\beta\beta}^{(\ell)} \cdot \mathbf{n}^{(\ell)}, \quad \mathbf{n}^{(\ell)} = \frac{\mathbf{R}_\alpha^{(\ell)} \times \mathbf{R}_\beta^{(\ell)}}{|\mathbf{R}_\alpha^{(\ell)} \times \mathbf{R}_\beta^{(\ell)}|}.$$

Then we construct $(U_1^{(\ell+1)}, U_2^{(\ell+1)}, U^{n,(\ell+1)}, H^{(\ell+1)})$ by solving the following linear system:

$$\begin{cases} \frac{\partial U_1^{(\ell+1)}}{\partial t} = \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}} \Delta(\sqrt{E^{(\ell)}} U_1^{(\ell+1)}) + \mathcal{F}_1^{(\ell)}, \\ \frac{\partial U_2^{(\ell+1)}}{\partial t} = \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}} \Delta(\sqrt{E^{(\ell)}} U_2^{(\ell+1)}) + \mathcal{F}_2^{(\ell)}, \\ \frac{\partial U^{n,(\ell+1)}}{\partial t} = -\frac{1}{2E^{(\ell)}} \Delta H^{(\ell+1)} - (2u_i^{(\ell)} - w_i^{(\ell)}) \partial_i U^{n,(\ell+1)} + \mathcal{F}_3^{(\ell)}, \\ \frac{\partial H^{(\ell+1)}}{\partial t} = \frac{1}{2E^{(\ell)}} \Delta U^{n,(\ell+1)} + w_i^{(\ell)} \partial_i H_\alpha^{(\ell+1)} + F_7^{(\ell)}, \\ (U_1^{(\ell+1)}, U_2^{(\ell+1)}, U^{n,(\ell+1)}, H^{(\ell+1)})|_{t=0} = (U_1^0, U_2^0, U_0^n, H_0), \end{cases} \tag{5.1}$$

where $u_i^{(\ell)} = \frac{U_i^{(\ell)}}{\sqrt{E^{(\ell)}}}$, $w_i^{(\ell)} = \frac{W_i^{(\ell)}}{\sqrt{E^{(\ell)}}}$ for $i = 1, 2$ and

$$\begin{aligned} \mathcal{F}_1^{(\ell)} &= -\frac{1}{\sqrt{E^{(\ell)}}} \Pi_\alpha^{(\ell)} + \frac{2\epsilon_0}{E^{(\ell)}\sqrt{E^{(\ell)}}} F_{1\alpha}^{(\ell)} + \frac{1}{2\sqrt{E^{(\ell)}}} F_{2\alpha}^{(\ell)} \\ &\quad - \frac{\epsilon_0}{E^{(\ell)}\sqrt{E^{(\ell)}}} F_{3\beta}^{(\ell)} - (u_1^{(\ell)} - w_1^{(\ell)}) U_{1\alpha}^{(\ell)} - (u_2^{(\ell)} - w_2^{(\ell)}) U_{1\beta}^{(\ell)} + F_4^{(\ell)}, \\ \mathcal{F}_2^{(\ell)} &= -\frac{1}{\sqrt{E^{(\ell)}}} \Pi_\beta^{(\ell)} + \frac{2\epsilon_0}{E^{(\ell)}\sqrt{E^{(\ell)}}} F_{1\beta}^{(\ell)} + \frac{1}{2\sqrt{E^{(\ell)}}} F_{2\beta}^{(\ell)} \\ &\quad - \frac{\epsilon_0}{E^{(\ell)}\sqrt{E^{(\ell)}}} F_{3\beta}^{(\ell)} - (u_1^{(\ell)} - w_1^{(\ell)}) U_{2\alpha}^{(\ell)} - (u_2^{(\ell)} - w_2^{(\ell)}) U_{2\beta}^{(\ell)} + F_5^{(\ell)}, \\ \mathcal{F}_3^{(\ell)} &= 2H^{(\ell)} \Pi^{(\ell)} + \frac{2\epsilon_0}{E^{(\ell)}\sqrt{E^{(\ell)}}} (L^{(\ell)} U_{1\alpha}^{(\ell)} + M^{(\ell)} U_{1\beta}^{(\ell)} + M^{(\ell)} U_{2\alpha}^{(\ell)} + N^{(\ell)} U_{2\beta}^{(\ell)}) + F_6^{(\ell)}, \end{aligned}$$

with $F_i^{(\ell)} (i = 1, \cdots, 7)$ are given in Section 3.3 where $(U_1, U_2, U^n, W_1, W_2, E, L, M, N)$ are replaced by $(U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, W_1^{(\ell)}, W_2^{(\ell)}, E^{(\ell)}, L^{(\ell)}, M^{(\ell)}, N^{(\ell)})$. Let $\widetilde{\mathbf{R}}_t^{k+1}$ be given by

$$\widetilde{\mathbf{R}}_t^{(\ell+1)} = U^{n,(\ell)} \frac{\mathbf{R}_\alpha^{(\ell)} \times \mathbf{R}_\beta^{(\ell)}}{E^{(\ell)}} + W_1^{(\ell)} \frac{\mathbf{R}_\alpha^{(\ell)}}{\sqrt{E^{(\ell)}}} + W_2^{(\ell)} \frac{\mathbf{R}_\beta^{(\ell)}}{\sqrt{E^{(\ell)}}}, \quad \widetilde{\mathbf{R}}|_{t=0} = \mathbf{R}_0. \tag{5.2}$$

And, $\widehat{\mathbf{R}}^{(\ell+1)}$ is determined by solving

$$\Delta \widehat{\mathbf{R}}^{(\ell+1)} - \widehat{\mathbf{R}}^{(\ell+1)} = 2H^{(\ell)} \widetilde{\mathbf{R}}_\alpha^{(\ell+1)} \times \widetilde{\mathbf{R}}_\beta^{(\ell+1)} - \widetilde{\mathbf{R}}^{(\ell+1)}. \tag{5.3}$$

Then we construct the surface $\mathbf{R}^{(\ell+1)}$ by solving the following elliptic equation:

$$\Delta \mathbf{R}^{(\ell+1)} - \mathbf{R}^{(\ell+1)} = 2H^{(\ell)} \widehat{\mathbf{R}}_\alpha^{(\ell+1)} \times \widehat{\mathbf{R}}_\beta^{(\ell+1)} - \widetilde{\mathbf{R}}^{(\ell+1)}. \tag{5.4}$$

Next we define $E^{(\ell+1)}$ by solving

$$\begin{aligned} \Delta E^{(\ell+1)} - 2E^{(\ell+1)} &= 2(\mathbf{R}_{\alpha\beta}^{(\ell+1)} \cdot \mathbf{R}_{\alpha\beta}^{(\ell+1)} - \mathbf{R}_{\alpha\alpha}^{(\ell+1)} \cdot \mathbf{R}_{\beta\beta}^{(\ell+1)}) \\ &\quad - \big(\mathbf{R}_\alpha^{(\ell+1)} \cdot \mathbf{R}_\alpha^{(\ell+1)} + \mathbf{R}_\beta^{(\ell+1)} \cdot \mathbf{R}_\beta^{(\ell+1)}\big). \end{aligned} \tag{5.5}$$

And, $(W_1^{(\ell+1)}, W_2^{(\ell+1)})$ is determined by solving

$$\begin{cases} \left(\frac{W_1^{(\ell+1)}}{\sqrt{E^{(\ell)}}}\right)_\alpha - \left(\frac{W_2^{(\ell+1)}}{\sqrt{E^{(\ell)}}}\right)_\beta = \frac{U^{n,(\ell)}(L^{(\ell)}-N^{(\ell)})}{E^{(\ell)}}, \\ \left(\frac{W_1^{(\ell+1)}}{\sqrt{E^{(\ell)}}}\right)_\beta + \left(\frac{W_2^{(\ell+1)}}{\sqrt{E^{(\ell)}}}\right)_\alpha = \frac{2U^{n,(\ell)}M^{(\ell)}}{E^{(\ell)}}. \end{cases} \tag{5.6}$$

Finally, we define the pressure $\Pi^{(\ell+1)}$ by solving

$$-\Delta\Pi^{(\ell+1)} + 4E^{(\ell)}(H^{(\ell)})^2\Pi^{(\ell+1)} = \mathcal{G}^{(\ell)}, \tag{5.7}$$

with $\mathcal{G}^{(\ell)}$ determined by $(U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, E^{(\ell)}, L^{(\ell)}, M^{(\ell)}, N^{(\ell)})$, see (3.29).

**Remark 5.1** *If $\mathbf{R}^{(l+1)}$ is directly defined by (5.2), then we can only obtain the $H^{s-1}$ regularity of $\mathbf{R}^{(l+1)}$. However, we need the $H^{s+1}$ regularity of $\mathbf{R}^{(l+1)}$ to close the energy estimates. Motivated by (3.11), we determine $\mathbf{R}^{(l+1)}$ by using (5.2)-(5.4) so that the $H^{s+1}$ regularity of $\mathbf{R}^{(l+1)}$ can be obtained by the elliptic estimates.*

## 5.2 Nonlinear estimates

Before presenting the estimates, let us make the following assumptions on the step-$\ell^{th}$ approximate solutions $(\mathbf{R}^{(\ell)}, U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, E^{(\ell)}, H^{(\ell)})$:

$$E^{(\ell)}(x,t) \geq c_0 > 0, \quad \text{for any } (t,x) \in [0,T] \times \mathbf{T}^2, \tag{5.8}$$

$$\int_{\mathbf{T}^2} (H^{(\ell)}(x,t))^2 dx \geq c_1 > 0, \quad \text{for any } t \in [0,T], \tag{5.9}$$

$$|\mathbf{R}_\alpha^{(\ell)}(x,t) \times \mathbf{R}_\beta^{(\ell)}(x,t)| \geq c_0, \quad \text{for any } (t,x) \in [0,T] \times \mathbf{T}^2 \tag{5.10}$$

$$\|(U_1^{(\ell)}, U_2^{(\ell)}\|_{L^\infty(0,T;H^{s-1})} + \|(U_1^{(\ell)}, U_2^{(\ell)})\|_{L^2(0,T;H^s)} \leq C_1, \tag{5.11}$$

$$\|(U^{n,(\ell)}, H^{(\ell)})\|_{L^\infty(0,T;H^{s-1})} \leq C_2, \tag{5.12}$$

$$\|\mathbf{R}^{(\ell)}\|_{C^i([0,T];H^{s+1-2i})} + \|E^{(\ell)}\|_{C^i([0,T];H^{s+1-2i})} \leq C_3, \quad i = 0, 1, \tag{5.13}$$

$$\|\mathbf{R}^{(\ell)}\|_{L^\infty(0,T;H^{s-1})} + \|E^{(\ell)}\|_{L^\infty(0,T;H^{s-1})} \leq C_4. \tag{5.14}$$

Here $T > 0, s = 2(k+1), k \geq 2$, and $C_1, C_2, C_3$, and $C_4$ are some fixed constants to be determined in Section 5.3. Note that the assumptions (5.8)-(5.10) and (5.13)-(5.14) are made so that we can use Theorem 4.1 at each step of the iterations, and the assumptions (5.11)-(5.12) are determined by the energy estimates for the linearized system.

In what follows, we denote $\mathcal{C}$ by an increasing function, which may be different from line to line. From the definition, it is easy to see that

$$\|(L^{(\ell)}, M^{(\ell)}, N^{(\ell)})\|_{L^\infty(0,T;H^{s-1})} \leq \mathcal{C}(C_3). \tag{5.15}$$

Using Lemma 6.8 and Lemma 6.1, we find that

$$\|W_1^{(\ell+1)}\|_{L^\infty(0,T;H^s)} + \|W_2^{(\ell+1)}\|_{L^\infty(0,T;H^s)} \leq \mathcal{C}(C_2, C_3). \tag{5.16}$$

From (5.15), (5.16), and Lemmas 6.1-6.2, we infer that for $i = 1, \cdots, 7$,

$$\|F_i^{(\ell)}\|_{L^\infty(0,T;H^{s-1})} \leq \mathcal{C}(C_1, C_2, C_3). \tag{5.17}$$

Thanks to Remark 3.4, we get by using Lemma 6.1-6.2 that

$$\|\mathcal{G}^{(\ell)}\|_{L^\infty(0,T;H^{s-3})} \le \mathcal{C}(C_1, C_2, C_3).$$

Thus, we infer from Lemma 6.9 that

$$\|\Pi^{(\ell+1)}\|_{L^\infty(0,T;H^{s-1})} \le \mathcal{C}(C_1, C_2, C_3). \tag{5.18}$$

Using (5.15)-(5.18), we obtain

**Proposition 5.2** *The nonlinear terms $\mathcal{F}_1^{(\ell)}, \mathcal{F}_2^{(\ell)}$, and $\mathcal{F}_3^{(\ell)}$ satisfy*

$$\begin{aligned}
&\|\mathcal{F}_i^{(\ell)}\|_{L^\infty(0,T;H^{s-2})} \le \mathcal{C}(C_1, C_2, C_3), \quad i = 1, 2,\\
&\|\mathcal{F}_3^{(\ell)}\|_{L^2(0,T;H^{s-1})} \le \mathcal{C}(C_1, C_2, C_3).
\end{aligned}$$

In order to prove the convergence of the iteration scheme, we need to establish some difference estimates in the lower-order Sobolev spaces. For this, we set

$$\begin{aligned}
&\delta^\ell_{U_i} = U_i^{(\ell)} - U_i^{(\ell-1)} (i = 1, 2), \quad \delta^\ell_{U^n} = U^{n,(\ell)} - U^{n,(\ell-1)}, \quad \delta^\ell_H = H^{(\ell)} - H^{(\ell-1)},\\
&\delta^\ell_{\mathbf{R}} = \mathbf{R}^{(\ell)} - \mathbf{R}^{(\ell-1)}, \quad \delta^\ell_E = E^{(\ell)} - E^{(\ell-1)}.
\end{aligned}$$

First of all, we have

$$\begin{aligned}
&\|(L^{(\ell)}, M^{(\ell)}, N^{(\ell)}) - (L^{(\ell-1)}, M^{(\ell-1)}, N^{(\ell-1)})\|_{H^{s-3}} \le \mathcal{C}(C_3)\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}},\\
&\|W^{(\ell+1)} - W^{(\ell)}\|_{H^{s-2}} \le \mathcal{C}(C_3)(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}} + \|\delta^\ell_{U^n}\|_{H^{s-3}}),
\end{aligned}$$

which imply that for $i = 1, \cdots, 7$,

$$\begin{aligned}
\|F_i^{(\ell)} - F_i^{(\ell-1)}\|_{H^{s-3}} \le\ & \mathcal{C}(C_1, C_2, C_3)\big(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}}\\
&+ \|\delta^\ell_{U_1}\|_{H^{s-3}} + \|\delta^\ell_{U_2}\|_{H^{s-3}} + \|\delta^\ell_{U^n}\|_{H^{s-3}} + \|\delta^\ell_H\|_{H^{s-3}}\big).
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
\|\mathcal{G}^{(\ell)} - \mathcal{G}^{(\ell-1)}\|_{H^{s-5}} \le\ & \mathcal{C}(C_1, C_2, C_3)\big(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}}\\
&+ \|\delta^\ell_{U_1}\|_{H^{s-3}} + \|\delta^\ell_{U_2}\|_{H^{s-3}} + \|\delta^\ell_{U^n}\|_{H^{s-3}} + \|\delta^\ell_H\|_{H^{s-3}}\big).
\end{aligned}$$

Hence, we infer from Lemma 6.9 that

$$\begin{aligned}
\|\Pi^{(\ell+1)} - \Pi^{(\ell)}\|_{H^{s-3}} \le\ & \mathcal{C}(C_1, C_2, C_3)\big(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}}\\
&+ \|\delta^\ell_{U_1}\|_{H^{s-3}} + \|\delta^\ell_{U_2}\|_{H^{s-3}} + \|\delta^\ell_{U^n}\|_{H^{s-3}} + \|\delta^\ell_H\|_{H^{s-3}}\big).
\end{aligned}$$

From the above estimates, we can deduce

**Proposition 5.3** *For $i = 1, 2$, it holds that*

$$\begin{aligned}
\|\mathcal{F}_i^{(\ell)} - \mathcal{F}_i^{(\ell-1)}\|_{H^{s-4}} \le\ & \mathcal{C}(C_1, C_2, C_3)\big(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}}\\
&+ \|\delta^\ell_{U_1}\|_{H^{s-3}} + \|\delta^\ell_{U_2}\|_{H^{s-3}} + \|\delta^\ell_{U^n}\|_{H^{s-3}} + \|\delta^\ell_H\|_{H^{s-3}}\big),\\
\|\mathcal{F}_3^{(\ell)} - \mathcal{F}_3^{(\ell-1)}\|_{H^{s-3}} \le\ & \mathcal{C}(C_1, C_2, C_3)\big(\|\delta^\ell_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^\ell_E\|_{H^{s-1}}\\
&+ \|\delta^\ell_{U_1}\|_{H^{s-2}} + \|\delta^\ell_{U_2}\|_{H^{s-2}} + \|\delta^\ell_{U^n}\|_{H^{s-3}} + \|\delta^\ell_H\|_{H^{s-3}}\big).
\end{aligned}$$

### 5.3 Proof of the main result

To simplify the analysis, we will first prove the well-posedness of the system by assuming that the surface can be globally parameterized by the isothermal coordinates. In Section 5.5, we will indicate how to extend this result to a general closed surface, and thereby conclude the proof of Theorem 1.1.

**Theorem 5.4** *Let $s = 2(k+1)$ for some integer $k \geq 2$. Assume that $(U_1^0, U_2^0, U_0^n) \in H^{s-1}(\mathbf{T}^2)$, and the initial surface $\mathbf{R}_0 \in H^{s+1}$. Moreover, the coefficient of the first fundamental form $E_0$ and the mean curvature $H_0$ satisfy*

$$
\begin{aligned}
&(\sqrt{E_0}U_1^0)_\alpha + (\sqrt{E_0}U_2^0)_\beta - 2E_0H_0U_0^n = 0,\\
&E_0(x) \geq 2c_0, \quad \int_{\mathbf{T}^2} H_0^2(x)dx \geq 2c_1,
\end{aligned}
$$

*for some $c_0 > 0, c_1 > 0$. Then there exists $T > 0$ such that the nonlinear system (3.23)-(3.28) has a unique solution $(\mathbf{R}, U_1, U_2, U^n, H)$ on $[0, T]$ satisfying*

$$
\begin{aligned}
&(U_1, U_2) \in C([0,T]; H^{s-1}) \cap L^2(0,T; H^s),\\
&\mathbf{R} \in C([0,T]; H^{s+1}), \quad (U^n, H) \in C([0,T]; H^{s-1}).
\end{aligned}
$$

**Remark 5.5** *We have chosen the isothermal coordinate for the initial surface. Hence, the conditions*

$$
E_0(x) \geq 2c_0, \quad \textstyle\int_{\mathbf{T}^2} H_0^2(x)dx \geq 2c_1
$$

*are naturally satisfied for any smooth closed surface.*

**Proof.** We split the proof into two steps.

**Step 1. Uniform estimates**

Let us assume that $(\mathbf{R}^{(\ell)}, U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, E^{(\ell)}, H^{(\ell)})$ satisfies (5.8)-(5.14). We will show that $(\mathbf{R}^{(\ell+1)}, U_1^{(\ell+1)}, U_2^{(\ell+1)}, U^{n,(\ell+1)}, E^{(\ell+1)}, H^{(\ell+1)})$ also satisfies the same estimates.

We denote

$$
E_s^{(\ell)}(t) \stackrel{\text{def}}{=} \|(U_1^{(\ell)}, U_2^{(\ell)})(t)\|_{H^{s-1}}^2 + \int_0^t \|(U_1^{(\ell)}, U_2^{(\ell)})(\tau)\|_{H^s}^2 d\tau + \|(U^{n,(\ell)}, H^{(\ell)})(t)\|_{H^{s-1}}^2.
$$

Then we infer from Theorem 4.1 and Proposition 5.2 that

$$
\begin{aligned}
E_s^{(\ell+1)}(t) \quad \leq \quad & \mathcal{C}(C_4)\Big(E_0 + \mathcal{C}_\varepsilon(C_1, C_2, C_3)\int_0^t (1 + \|(U_1^{(\ell)}, U_2^{\ell})\|_{H^s})E_s^{(\ell+1)}(\tau)d\tau\\
&+\mathcal{C}(C_1, C_2, C_3)(t+\varepsilon)\Big).
\end{aligned}
$$

Here $E_0 \stackrel{\text{def}}{=} \|(U_1^0, U_2^0, U_0^n, H^0)\|_{H^{s-1}}$. Then we get by Gronwall's inequality that

$$
E_s^{(\ell+1)}(t) \leq \big(\mathcal{C}(C_4)E_0 + \mathcal{C}(C_1, C_2, C_3, C_4)(t+\varepsilon)\big)\exp(\mathcal{C}(C_1, C_2, C_3, C_4)t).
$$

Taking $T$ and $\varepsilon$ small enough yields that

$$
E_s^{(\ell+1)}(t) \leq 2\mathcal{C}(C_4)E_0 \quad \text{for } t \in [0, T].
$$

This means that if we take $C_1 = C_2 = 2\mathcal{C}(C_4)E_0$, $(U_1^{(\ell+1)}, U_2^{(\ell+1)}, U^{n,(\ell+1)}, H^{(\ell+1)}))$ satisfies (5.11)-(5.12).

Due to (5.2), we find that

$$\|\widetilde{\mathbf{R}}^{(\ell+1)}\|_{L^\infty(0,T;H^{s-1})} \le \|R_0\|_{H^{s-1}} + \mathcal{C}(C_1, C_2, C_3)t.$$

Hence, by taking $T$ to be small enough if necessary, we get

$$\|\widetilde{\mathbf{R}}^{(\ell+1)}\|_{L^\infty(0,T;H^{s-1})} \le 2\|\mathbf{R}_0\|_{H^{s-1}}. \tag{5.19}$$

We also have by (5.2) that

$$\|\partial_t\widetilde{\mathbf{R}}^{(\ell+1)}\|_{L^\infty(0,T;H^{s-2})} \le \mathcal{C}(C_2, C_4). \tag{5.20}$$

We get by the elliptic estimate that

$$\|\widehat{\mathbf{R}}^{(\ell+1)}\|_{L^\infty(0,T;H^{s})} \le \mathcal{C}(\|\mathbf{R}_0\|_{H^{s-1}}, C_2),$$

which along with (5.19) and (5.4) implies that

$$\|\mathbf{R}^{(\ell+1)}\|_{L^\infty(0,T;H^{s+1})} \le \mathcal{C}(\|\mathbf{R}_0\|_{H^{s}}, C_2).$$

Hence, by (5.5) and the elliptic estimate,

$$\|E^{(\ell+1)}\|_{L^\infty(0,T;H^{s+1})} \le \mathcal{C}(\|\mathbf{R}_0\|_{H^{s}}, C_2).$$

Taking the derivative to (5.4) and (5.5) with respect to time, we get by (5.20) that

$$\|\partial_t\mathbf{R}^{(\ell+1)}\|_{L^\infty(0,T;H^{s-1})} + \|\partial_t E^{(\ell+1)}\|_{L^\infty(0,T;H^{s-1})} \le \mathcal{C}(\|\mathbf{R}_0\|_{H^{s}}, C_2, C_4). \tag{5.21}$$

Hence, taking $C_3 = \mathcal{C}(\|\mathbf{R}_0\|_{H^s}, C_2, C_4)$, we see that $(\mathbf{R}^{(\ell+1)}, E^{(\ell+1)})$ satisfies (5.13).

Now taking $C_4 = 2(\|\mathbf{R}_0\|_{H^{s-1}} + \|E_0\|_{H^{s-1}}) \le C\|\mathbf{R}_0\|_{H^s}$, it follows from (5.21) that

$$\|(\mathbf{R}^{(\ell+1)}, E^{(\ell+1)})\|_{L^\infty(0,T;H^{s-1})} \le \|\mathbf{R}_0\|_{H^{s-1}} + \|E_0\|_{H^{s-1}} + \mathcal{C}(\|\mathbf{R}_0\|_{H^s}, C_2, C_4)T,$$

which implies that $(\mathbf{R}^{(\ell+1)}, E^{(\ell+1)})$ satisfies (5.14) when $T$ is taken to be small enough. Similarly, we can show that $(\mathbf{R}^{(\ell+1)}, E^{(\ell+1)}, H^{(\ell+1)})$ also satisfies (5.8)-(5.10).

In conclusion, we prove that there exists a $T > 0$ depending only on $\|(U_1^0, U_2^0, U_0^n)\|_{H^{s-1}}$ and $\|\mathbf{R}_0\|_{H^{s+1}}$ such that (5.8)-(5.14) hold for $(\mathbf{R}^{(\ell+1)}, U_1^{(\ell+1)}, U_2^{(\ell+1)}, U^{n,(\ell+1)}, E^{(\ell+1)}, H^{(\ell+1)})$.

**Step 2. Existence and uniqueness**

It suffices to show that the approximate solution sequence is a Cauchy sequence. For this purpose, we set

$$\delta_{U_i}^{\ell+1} = U_i^{(\ell+1)} - U_i^{(\ell)} (i = 1, 2), \quad \delta_{U^n}^{\ell+1} = U^{n,(\ell+1)} - U^{n,(\ell)}, \quad \delta_H^{\ell+1} = H^{(\ell+1)} - H^{(\ell)},$$
$$\delta_{\mathbf{R}}^{\ell+1} = \mathbf{R}^{(\ell+1)} - \mathbf{R}^{(\ell)}, \quad \delta_E^{\ell+1} = E^{(\ell+1)} - E^{(\ell)}.$$

Then $(\delta^{\ell+1}_{U_1}, \delta^{\ell+1}_{U_2}, \delta^{\ell+1}_{U_n}, \delta^{\ell+1}_{H})$ satisfies the following system:

$$\begin{cases} \frac{\partial \delta^{\ell+1}_{U_1}}{\partial t} = \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}} \Delta(\sqrt{E^{(\ell)}}\delta^{\ell+1}_{U_i}) + \delta\mathcal{F}^{(\ell)}_1, \\ \frac{\partial \delta^{\ell+1}_{U_2}}{\partial t} = \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}} \Delta(\sqrt{E^{(\ell)}}\delta^{\ell+1}_{U_2}) + \delta\mathcal{F}^{(\ell)}_2, \\ \frac{\partial \delta^{\ell+1}_{U^n}}{\partial t} = -\frac{1}{2E^{(\ell)}} \Delta\delta^{\ell+1}_{H} - (2u^{(\ell)}_i - w^{(\ell)}_i)\partial_i \delta^{\ell+1}_{U^n} + \delta\mathcal{F}^{(\ell)}_3, \\ \frac{\partial \delta^{\ell+1}_{H}}{\partial t} = \frac{1}{2E^{(\ell)}} \Delta\delta^{\ell+1}_{U^n} + w^{(\ell)}_i \partial_i \delta^{\ell+1}_{H} + \delta\mathcal{F}^{(\ell)}_4, \\ (\delta^{\ell+1}_{U_1}, \delta^{\ell+1}_{U_2}, \delta^{\ell+1}_{U^n}, \delta^{\ell+1}_{H})|_{t=0} = (0,0,0,0), \end{cases} \tag{5.22}$$

where

$$\begin{aligned} \delta\mathcal{F}^{(\ell)}_1 &= \mathcal{F}^{(\ell)}_1 - \mathcal{F}^{(\ell-1)}_1 + \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}}\Delta(\sqrt{E^{(\ell)}}U^{(\ell)}_1) - \epsilon_0 \frac{1}{E^{(\ell-1)}\sqrt{E^{(\ell-1)}}}\Delta(\sqrt{E^{(\ell-1)}}U^{(\ell)}_1), \\ \delta\mathcal{F}^{(\ell)}_2 &= \mathcal{F}^{(\ell)}_2 - \mathcal{F}^{(\ell-1)}_2 + \epsilon_0 \frac{1}{E^{(\ell)}\sqrt{E^{(\ell)}}}\Delta(\sqrt{E^{(\ell)}}U^{(\ell)}_2) - \epsilon_0 \frac{1}{E^{(\ell-1)}\sqrt{E^{(\ell-1)}}}\Delta(\sqrt{E^{(\ell-1)}}U^{(\ell)}_2), \\ \delta\mathcal{F}^{(\ell)}_3 &= \mathcal{F}^{(\ell)}_3 - \mathcal{F}^{(\ell-1)}_3 - \frac{1}{2E^{(\ell)}}\Delta H^{(\ell)} + \frac{1}{2E^{(\ell-1)}}\Delta H^{(\ell)} - (2\delta^{\ell}_{u_i} - \delta^{\ell}_{w_i})\partial_i U^{n,(\ell)}, \\ \delta\mathcal{F}^{(\ell)}_4 &= F^{(\ell)}_7 - F^{(\ell-1)}_7 + \frac{1}{2E^{(\ell)}}\Delta U^{n,(\ell)} - \frac{1}{2E^{(\ell-1)}}\Delta U^{n,(\ell)} + \delta^{\ell}_{w_i}\partial_i H^{(\ell)}. \end{aligned}$$

From Proposition 5.3, it is easy to see that

$$\begin{aligned} &\|\delta\mathcal{F}^{(\ell)}_i\|_{H^{s-4}} \le C\big(\|\delta^{\ell}_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^{\ell}_{E}\|_{H^{s-1}} + \|(\delta^{\ell}_{U_1}, \delta^{\ell}_{U_2}, \delta^{\ell}_{U^n}, \delta^{\ell}_{H})\|_{H^{s-3}}\big), \quad i = 1,2, \\ &\|\delta\mathcal{F}^{(\ell)}_i\|_{H^{s-3}} \le C\big(\|\delta^{\ell}_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^{\ell}_{E}\|_{H^{s-1}} + \|(\delta^{\ell}_{U_1}, \delta^{\ell}_{U_2})\|_{H^{s-2}} + \|(\delta^{\ell}_{U^n}, \delta^{\ell}_{H})\|_{H^{s-3}}\big), \quad i = 3,4. \end{aligned}$$

Revisiting the proof of Theorem 4.1, we can obtain

$$\begin{aligned} D^{\ell+1}_1(t) &\le C_\varepsilon \int_0^t D^{\ell+1}_1(\tau)d\tau + C\sum_{i=1}^{2}\int_0^t \|\delta\mathcal{F}^{(\ell}_i(\tau)\|^2_{H^{s-4}}d\tau \\ &\quad + C\varepsilon\sum_{i=3}^{4}\int_0^t \|\delta\mathcal{F}^{(\ell}_i(\tau)\|^2_{H^{s-4}}d\tau \\ &\le C_\varepsilon \int_0^t D^{\ell+1}_1(\tau)d\tau + C(t+\varepsilon)\sup_{\tau\in[0,t]} D^{\ell}(\tau), \end{aligned} \tag{5.23}$$

where $D^{\ell}(t)$ is defined by

$$D^{\ell}(t) = D^{\ell}_1(t) + \|\delta^{\ell}_{\mathbf{R}}(t)\|_{H^{s-1}} + \|\delta^{\ell}_{E}\|_{H^{s-1}},$$

with $D^{\ell}_1(t) = \|(\delta^{\ell}_{U_1}, \delta^{\ell}_{U_2}, \delta^{\ell}_{U^n}, \delta^{\ell}_{H})(t)\|_{H^{s-3}} + \int_0^t \|(\delta^{\ell}_{U_1}, \delta^{\ell}_{U_2})(\tau)\|^2_{H^{s-2}}d\tau$.

On the other hand, we revisit the proof of Step 1 to find that

$$\|\delta^{\ell+1}_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^{\ell+1}_{E}\|_{H^{s-1}} \le C_0 D^{\ell}_1(t) + Ct(\|\delta^{\ell}_{\mathbf{R}}\|_{H^{s-1}} + \|\delta^{\ell}_{E}\|_{H^{s-1}}). \tag{5.24}$$

For some small $\delta > 0$ depending only on $C_0$, with $\varepsilon$ and $T$ taken to be small enough, it follows from (5.23) and (5.24) that

$$\sup_{t\in[0,T]} (D^{\ell+1}_1(t) + \delta D^{\ell+1}_2(t)) \le \frac{1}{2}\sup_{t\in[0,T]} (D^{\ell}_1(t) + \delta D^{\ell}_2(t)),$$

with $D_2^{\ell+1} = \|\delta_{\mathbf{R}}^{\ell+1}\|_{H^{s-1}} + \|\delta_E^{\ell+1}\|_{H^{s-1}}$. This implies that

$$(\mathbf{R}^{(\ell)}, U_1^{(\ell)}, U_2^{(\ell)}, U^{n,(\ell)}, E^{(\ell)}, H^{(\ell)}, \Pi^{(\ell)}, \widetilde{\mathbf{R}}^{(\ell)}, \widehat{\mathbf{R}}^{(\ell)})$$

is a Cauchy sequence. More precisely, there exists the limit $(\mathbf{R}, U_1, U_2, U^n, E, H, \Pi, \widetilde{\mathbf{R}}, \widehat{\mathbf{R}})$ such that

$$\begin{aligned}
&\mathbf{R}^{(\ell)} \to \mathbf{R}, \quad E^{(\ell)} \to E \quad \text{in} \quad L^\infty(0,T;H^{s-1});\\
&U_1^{(\ell)} \to U_1, \quad U_2^{(\ell)} \to U_2 \quad \text{in} \quad L^\infty(0,T;H^{s-3}) \cap L^2(0,T;H^{s-2});\\
&U^{n,(\ell)} \to U^n, \quad H^{(\ell)} \to H, \quad \Pi^{(\ell)} \to \Pi \quad \text{in} \quad L^\infty(0,T;H^{s-3});\\
&\widetilde{\mathbf{R}}^{(\ell)} \to \widetilde{\mathbf{R}} \quad \text{in} \quad L^\infty(0,T;H^{s-3}), \quad \widehat{\mathbf{R}}^{(\ell)} \to \widehat{\mathbf{R}} \quad \text{in} \quad L^\infty(0,T;H^{s-2}).
\end{aligned}$$

With the above information, it is easy to prove that $(\mathbf{R}, U_1, U_2, U^n, E, H, \Pi, \widetilde{\mathbf{R}}, \widehat{\mathbf{R}})$ satisfies the system (5.1)-(5.7) without the index $\ell$. In particular, we have

$$\frac{\partial \widetilde{\mathbf{R}}}{\partial t} = \frac{U^n}{E}\mathbf{R}_\alpha \times \mathbf{R}_\beta + \frac{W_1}{\sqrt{E}}\mathbf{R}_\alpha + \frac{W_2}{\sqrt{E}}\mathbf{R}_\beta, \tag{5.25}$$

$$\Delta\widehat{\mathbf{R}} - \widehat{\mathbf{R}} = 2H\widetilde{\mathbf{R}}_\alpha \times \widetilde{\mathbf{R}}_\beta - \widetilde{\mathbf{R}}, \tag{5.26}$$

$$\Delta\mathbf{R} - \mathbf{R} = 2H\widehat{\mathbf{R}}_\alpha \times \widehat{\mathbf{R}}_\beta - \widetilde{\mathbf{R}}, \tag{5.27}$$

$$\Delta E - 2E = 2(\mathbf{R}_{\alpha\beta} \cdot \mathbf{R}_{\alpha\beta} - \mathbf{R}_{\alpha\alpha} \cdot \mathbf{R}_{\beta\beta}) - (\mathbf{R}_\alpha \cdot \mathbf{R}_\alpha + \mathbf{R}_\beta \cdot \mathbf{R}_\beta), \tag{5.28}$$

$$\frac{\partial H}{\partial t} = \frac{1}{2E}\Delta U^n + \frac{W_1}{\sqrt{E}}H_\alpha + \frac{W_2}{\sqrt{E}}H_\beta + \frac{U^n}{2E^2}(L^2 + 2M^2 + N^2) \tag{5.29}$$

$$L = \mathbf{R}_{\alpha\alpha} \cdot \mathbf{n}, \quad M = \mathbf{R}_{\alpha\beta} \cdot \mathbf{n}, \quad N = \mathbf{R}_{\beta\beta} \cdot \mathbf{n}, \quad \mathbf{n} = \frac{\mathbf{R}_\alpha \times \mathbf{R}_\beta}{|\mathbf{R}_\alpha \times \mathbf{R}_\beta|}. \tag{5.30}$$

And, $(W_1, W_2)$ satisfies

$$\begin{cases} \left(\frac{W_1}{\sqrt{E}}\right)_\alpha - \left(\frac{W_2}{\sqrt{E}}\right)_\beta = \frac{U^n(L-N)}{E}, \\ \left(\frac{W_1}{\sqrt{E}}\right)_\beta + \left(\frac{W_2}{\sqrt{E}}\right)_\alpha = \frac{2U^nM}{E}. \end{cases} \tag{5.31}$$

It remains to show that the solution of the limit system is a solution of the original system. For this purpose, it suffices to prove the following relations:

$$\mathbf{R} = \widetilde{\mathbf{R}} = \widehat{\mathbf{R}}, \quad E = \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha = \mathbf{R}_\beta \cdot \mathbf{R}_\beta, \quad \mathbf{R}_\alpha \cdot \mathbf{R}_\beta = 0, \quad H = \frac{L+N}{2E}. \tag{5.32}$$

And the incompressible condition follows easily from (5.32). As the proof is very complicated, it will be given in the following subsection.

## 5.4 Consistency with the original system

This subsection is devoted to proving (5.32). Let us introduce some notations:

$$\begin{aligned}
&a_{11} = \mathbf{R}_\alpha \cdot \mathbf{R}_\alpha, \quad a_{12} = \mathbf{R}_\alpha \cdot \mathbf{R}_\beta, \quad a_{22} = \mathbf{R}_\beta \cdot \mathbf{R}_\beta,\\
&\widetilde{a}_{11} = \widetilde{\mathbf{R}}_\alpha \cdot \widetilde{\mathbf{R}}_\alpha, \quad \widetilde{a}_{12} = \widetilde{\mathbf{R}}_\alpha \cdot \widetilde{\mathbf{R}}_\beta, \quad \widetilde{a}_{22} = \widetilde{\mathbf{R}}_\beta \cdot \widetilde{\mathbf{R}}_\beta,\\
&\mathbf{w} = \frac{U^n}{E}\mathbf{R}_\alpha \times \mathbf{R}_\beta + \frac{W_1}{\sqrt{E}}\mathbf{R}_\alpha + \frac{W_2}{\sqrt{E}}\mathbf{R}_\beta,\\
&\overrightarrow{\mathbf{w}} = (w_1, w_2), \quad w_i = \frac{W_i}{\sqrt{E}}(i = 1, 2).
\end{aligned}$$

We set

$$\delta_R = \mathbf{R} - \widehat{\mathbf{R}}, \quad \delta_{\widetilde{R}} = \widetilde{\mathbf{R}} - \widehat{\mathbf{R}}, \quad \delta_E^1 = E - a_{11}, \quad \delta_E^2 = E - a_{22}, \quad \delta_a = \widetilde{a}_{11} - \widetilde{a}_{22}.$$

In what follows, we denote by $\mathfrak{F}$ some operator bounded in $H^k(\mathbf{T}^2)(0 \le k \le 1)$, which may be different from line to line. For example,

$$\|\mathfrak{F}(u,v)\|_{H^k} \le C(\|u\|_{H^k} + \|v\|_{H^k}).$$

We get by using (5.25) and (5.31) that

$$\begin{aligned}
&(\tilde{a}_{11} - \tilde{a}_{22})_t = \mathbf{w}_\alpha \cdot \widetilde{\mathbf{R}}_\alpha - \mathbf{w}_\beta \cdot \widetilde{\mathbf{R}}_\beta \\
&= \mathbf{w}_\alpha \cdot \mathbf{R}_\alpha - \mathbf{w}_\beta \cdot \mathbf{R}_\beta + \mathfrak{F}\big(\nabla(\mathbf{R} - \widetilde{\mathbf{R}})\big) \\
&= \frac{U^n(L-N)}{2E}(2|\mathbf{R}_\alpha \times \mathbf{R}_\beta| - a_{11} - a_{22}) + \frac{1}{2}(w_{1\alpha} + w_{2\beta})(a_{11} - a_{22}) \\
&\quad + w_1(a_{11} - a_{22})_\alpha + w_2(a_{11} - a_{22})_\beta + (w_{1\alpha} - w_{1\beta})a_{12} + \mathfrak{F}\big(\nabla\delta_R, \nabla\delta_{\widetilde{R}}\big).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\tilde{a}_{12})_t &= \mathbf{w}_\alpha \cdot \widetilde{\mathbf{R}}_\beta + \mathbf{w}_\beta \cdot \widetilde{\mathbf{R}}_\alpha \\
&= \mathbf{w}_\alpha \cdot \mathbf{R}_\beta + \mathbf{w}_\beta \cdot \mathbf{R}_\alpha + \mathfrak{F}\big(\nabla(\mathbf{R} - \widetilde{\mathbf{R}})\big) \\
&= \frac{MU^n}{E}(a_{11} + a_{22} - 2|\mathbf{R}_\alpha \times \mathbf{R}_\beta|) + \frac{1}{2}(w_{1\beta} - w_{2\alpha})(a_{11} - a_{22}) \\
&\quad + w_1 a_{12\alpha} + w_2 a_{12\beta} + (w_{1\alpha} + w_{2\beta})a_{12} + \mathfrak{F}\big(\nabla\delta_R, \nabla\delta_{\widetilde{R}}\big).
\end{aligned}$$

Noting

$$\begin{aligned}
a_{11} + a_{22} - 2|\mathbf{R}_\alpha \times \mathbf{R}_\beta| &= \frac{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2)}{a_{11} + a_{22} + 2|\mathbf{R}_\alpha \times \mathbf{R}_\beta|} \\
&= \frac{(a_{11} - a_{22})^2 - 4a_{12}^2}{a_{11} + a_{22} + 2|\mathbf{R}_\alpha \times \mathbf{R}_\beta|},
\end{aligned}$$

we find that

$$\partial_t \delta_a + \overrightarrow{\mathbf{w}} \cdot \nabla \delta_a = \mathfrak{F}\big(\nabla\delta_R, \nabla\delta_{\widetilde{R}}, \nabla^2\delta_R, \nabla^2\delta_{\widetilde{R}}, \delta_a, \widetilde{a}_{12}\big), \tag{5.33}$$

$$\partial_t \tilde{a}_{12} + \overrightarrow{\mathbf{w}} \cdot \nabla \widetilde{a}_{12} = \mathfrak{F}\big(\nabla\delta_R, \nabla\delta_{\widetilde{R}}, \nabla^2\delta_R, \nabla^2\delta_{\widetilde{R}}, \delta_a, \widetilde{a}_{12}\big). \tag{5.34}$$

By (5.26) and (5.27), we have

$$(\Delta - 1)(\widehat{\mathbf{R}} - \mathbf{R}) = 2H(\widetilde{\mathbf{R}}_\alpha \times \widetilde{\mathbf{R}}_\beta - \widehat{\mathbf{R}}_\alpha \times \widehat{\mathbf{R}}_\beta) = \mathfrak{F}\big(\nabla\delta_{\widetilde{R}}\big). \tag{5.35}$$

And by (5.26) and (5.28),

$$(\Delta - 1)(E - a_{11}) = -2\big(\mathbf{R}_\alpha \cdot (2H\widehat{\mathbf{R}}_\alpha \times \widehat{\mathbf{R}}_\beta)\big)_\alpha + E - a_{22} - 2(\mathbf{R}_\alpha \cdot (\mathbf{R} - \widetilde{\mathbf{R}}))_\alpha,$$

which implies that

$$(\Delta - 2)\delta_E^1 = \mathfrak{F}\big(\nabla\delta_R, \nabla\delta_{\widetilde{R}}, \delta_a\big). \tag{5.36}$$

Similarly, we have

$$\begin{aligned}(\Delta-2)(E-a_{22}) &= -2\big(\mathbf{R}_\beta\cdot(2H\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta)\big)_\beta+a_{22}-a_{11}-2(\mathbf{R}_\beta\cdot(\mathbf{R}-\widetilde{\mathbf{R}}))_\beta\\ &= \mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\delta_a\big), \qquad (5.37)\\ (\Delta-1)a_{12} &= 2H\big((\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_{\alpha\beta})\cdot\mathbf{R}_\beta-(\widehat{\mathbf{R}}_\alpha\times\mathbf{R}_{\alpha\beta})\cdot\widehat{\mathbf{R}}_\beta\big)\\ &\quad+2H\big((\widehat{\mathbf{R}}_{\alpha\beta}\times\widehat{\mathbf{R}}_\beta)\cdot\mathbf{R}_\alpha-(\mathbf{R}_{\alpha\beta}\times\widehat{\mathbf{R}}_\beta)\cdot\widehat{\mathbf{R}}_\alpha\big)\\ &\quad-a_{12}+\mathfrak{F}(\delta_R,\nabla\delta_R,\delta_{\widetilde{R}},\nabla\delta_{\widetilde{R}})\\ &= \sum_{k=0}^{2}\mathfrak{F}\big(\nabla^k\delta_R,\nabla^k\delta_{\widetilde{R}},\widetilde{a}_{12}\big). \qquad (5.38)\end{aligned}$$

The following facts will be used frequently:

$$|\mathbf{R}_\alpha\times\mathbf{R}_\beta|-E=\sqrt{\det(a_{ij})}-E=\mathfrak{F}\big(\delta_E^1,\delta_E^2,a_{12}\big), \qquad (5.39)$$

$$\Gamma_{11}^1,\Gamma_{12}^2,\Gamma_{21}^2,-\Gamma_{22}^1=\frac{E_\alpha}{2E}+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big), \qquad (5.40)$$

$$\Gamma_{12}^1,\Gamma_{21}^1,\Gamma_{22}^2,-\Gamma_{11}^2=\frac{E_\beta}{2E}+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big). \qquad (5.41)$$

$$(\Delta\mathbf{R})_\gamma=(2H\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\gamma+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}}\big),\quad \gamma=\alpha,\beta. \qquad (5.42)$$

Indeed, we have

$$\begin{aligned}\Gamma_{12}^2 &= \frac{1}{|\mathbf{R}_\alpha\times\mathbf{R}_\beta|}\Big(-\frac{a_{12}}{2}\partial_\beta a_{11}+\frac{a_{11}}{2}\partial_\alpha a_{22}\Big)\\ &= \frac{E_\alpha}{2E}+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^2,a_{12}\big),\end{aligned}$$

and the others can be deduced similarly. For the last fact, we have by (5.27) that

$$\begin{aligned}\Delta\mathbf{R} &= 2H\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta-\widetilde{\mathbf{R}}+\mathbf{R}\\ &= 2H\mathbf{R}_\alpha\times\mathbf{R}_\beta+2H(\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta-\mathbf{R}_\alpha\times\mathbf{R}_\beta)-\widetilde{\mathbf{R}}+\mathbf{R}, \qquad (5.43)\end{aligned}$$

thus (5.42) follows easily.

To proceed, we also need the following lemma.

**Lemma 5.6** *For $\gamma=\alpha,\beta$, it holds that*

$$(2EH-L-N)_\gamma=\sum_{k=0}^{2}\sum_{j=0}^{1}\mathfrak{F}\big(\nabla^k\delta_R,\nabla^k\delta_{\widetilde{R}},\nabla^j\delta_E^1,\nabla^j\delta_E^2,\nabla^j a_{12}\big), \qquad (5.44)$$

$$LN-M^2=\frac{1}{2}\Big(-\Delta E+\frac{E_\alpha^2+E_\beta^2}{E}\Big)+\sum_{k=0}^{2}\mathfrak{F}\big(\nabla^k\delta_E^1,\nabla^k\delta_E^2,\nabla^k a_{12}\big), \qquad (5.45)$$

$$N_\alpha=M_\beta+HE_\alpha+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big), \qquad (5.46)$$

$$L_\beta=M_\alpha+HE_\beta+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big). \qquad (5.47)$$

**Proof.** First of all, a direct calculation gives

$$\begin{aligned} 2EH-(L+N) &= \frac{E}{|\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta|^2}\Delta\mathbf{R}\cdot(\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta)-\frac{1}{|\mathbf{R}_\alpha\times\mathbf{R}_\beta|}\Delta\mathbf{R}\cdot(\mathbf{R}_\alpha\times\mathbf{R}_\beta) \\ &= \Delta\mathbf{R}\cdot\Big(\frac{E\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta}{|\widehat{\mathbf{R}}_\alpha\times\widehat{\mathbf{R}}_\beta|^2}-\frac{\mathbf{R}_\alpha\times\mathbf{R}_\beta}{|\mathbf{R}_\alpha\times\mathbf{R}_\beta|}\Big), \end{aligned}$$

which implies (5.44). From the Gauss equation, we infer that

$$\begin{aligned} LN-M^2 &= \frac{1}{a_{11}a_{22}-a_{12}^2}\left(\begin{vmatrix} a_{12\alpha\beta}-\frac{1}{2}a_{11\beta\beta}-\frac{1}{2}a_{22\alpha\alpha} & \frac{1}{2}a_{11\alpha} & a_{12\alpha}-\frac{1}{2}a_{11\beta} \\ a_{12\beta}-\frac{1}{2}a_{22\alpha} & a_{11} & a_{12} \\ \frac{1}{2}a_{22\beta} & a_{12} & a_{22} \end{vmatrix}\right. \\ &\qquad \left.-\begin{vmatrix} 0 & \frac{a_{11\beta}}{2} & \frac{a_{22\alpha}}{2} \\ \frac{a_{11\beta}}{2} & a_{11} & a_{12} \\ \frac{a_{22\alpha}}{2} & a_{12} & a_{22} \end{vmatrix}\right), \end{aligned}$$

which implies (5.45). The Codazzi equation $b_{\alpha\beta,\gamma}=b_{\alpha\gamma,\beta}$ implies that

$$\begin{aligned} &\frac{\partial b_{11}}{\partial x^2}-\Gamma^1_{12}b_{11}-\Gamma^2_{12}b_{12}-\Gamma^1_{12}b_{11}-\Gamma^2_{12}b_{21} \\ &\quad=\frac{\partial b_{12}}{\partial x^1}-\Gamma^1_{21}b_{11}-\Gamma^2_{21}b_{12}-\Gamma^1_{11}b_{12}-\Gamma^2_{11}b_{22}, \\ &\frac{\partial b_{21}}{\partial x^2}-\Gamma^1_{12}b_{21}-\Gamma^2_{12}b_{22}-\Gamma^1_{22}b_{11}-\Gamma^2_{22}b_{21} \\ &\quad=\frac{\partial b_{22}}{\partial x^1}-\Gamma^1_{21}b_{21}-\Gamma^2_{21}b_{22}-\Gamma^1_{21}b_{12}-\Gamma^2_{21}b_{22}, \end{aligned}$$

where $b_{11}=L$, $b_{12}=b_{21}=M$, and $b_{22}=N$. Then (5.46)-(5.47) follow easily from (5.40) and (5.41). The proof is completed. □

In the following, we calculate $\widetilde{\mathbf{R}}-\widehat{\mathbf{R}}$. By (5.25) and (5.26), we have

$$(\Delta-1)(\widetilde{\mathbf{R}}-\widehat{\mathbf{R}})_t=\Delta\mathbf{w}-(2H\widetilde{\mathbf{R}}_\alpha\times\widetilde{\mathbf{R}}_\beta)_t. \tag{5.48}$$

Direct calculations yield that

$$\begin{aligned} \Delta\mathbf{w} &= \Delta(\frac{U^n}{E}\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1\mathbf{R}_\alpha+w_2\mathbf{R}_\beta) \\ &= \Delta\Big(\frac{U^n}{E}\Big)\mathbf{R}_\alpha\times\mathbf{R}_\beta+\frac{U^n}{E}(\Delta\mathbf{R})_\alpha\times\mathbf{R}_\beta+\frac{U^n}{E}\mathbf{R}_\alpha\times(\Delta\mathbf{R})_\beta \\ &\quad+\Delta w_1\mathbf{R}_\alpha+\Delta w_2\mathbf{R}_\beta+w_1(\Delta\mathbf{R})_\alpha+w_2(\Delta\mathbf{R})_\beta+2\frac{U^n}{E}(\mathbf{R}_{\alpha\alpha}-\mathbf{R}_{\beta\beta})\times\mathbf{R}_{\alpha\beta} \\ &\quad+2\Big(\frac{U^n}{E}\Big)_\alpha\mathbf{R}_{\alpha\alpha}\times\mathbf{R}_\beta+2\Big(\frac{U^n}{E}\Big)_\alpha\mathbf{R}_\alpha\times\mathbf{R}_{\alpha\beta} \\ &\quad+2\Big(\frac{U^n}{E}\Big)_\beta\mathbf{R}_{\alpha\beta}\times\mathbf{R}_\beta+2\Big(\frac{U^n}{E}\Big)_\beta\mathbf{R}_\alpha\times\mathbf{R}_{\beta\beta} \\ &\quad+2w_{1\alpha}\mathbf{R}_{\alpha\alpha}+2w_{2\beta}\mathbf{R}_{\beta\beta}+2(w_{1\beta}+w_{2\alpha})\mathbf{R}_{\alpha\beta}, \end{aligned} \tag{5.49}$$

and by (5.25),

$$
\begin{aligned}
&\Big(2H(\widetilde{\mathbf{R}}_\alpha\times\widetilde{\mathbf{R}}_\beta)\Big)_t\\
&=\Big(\frac{\Delta U^n}{E}+2w_1H_\alpha+2w_2H_\beta+\frac{U^n}{E^2}(L^2+2M^2+N^2)\Big)\mathbf{R}_\alpha\times\mathbf{R}_\beta\\
&\quad+2H\Big(\frac{U^n}{E}\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1\mathbf{R}_\alpha+w_2\mathbf{R}_\beta\Big)_\alpha\times\widetilde{\mathbf{R}}_\beta\\
&\quad+2H\widetilde{\mathbf{R}}_\alpha\times\Big(\frac{U^n}{E}\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1\mathbf{R}_\alpha+w_2\mathbf{R}_\beta\Big)_\beta\\
&=\Big(\frac{\Delta U^n}{E}+2w_1H_\alpha+2w_2H_\beta+\frac{U^n}{E^2}(L^2+2M^2+N^2)\Big)\mathbf{R}_\alpha\times\mathbf{R}_\beta\\
&\quad+2H\Big(\frac{U^n}{E}\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1\mathbf{R}_\alpha+w_2\mathbf{R}_\beta\Big)_\alpha\times\mathbf{R}_\beta\\
&\quad+2H\mathbf{R}_\alpha\times\Big(\frac{U^n}{E}\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1\mathbf{R}_\alpha+w_2\mathbf{R}_\beta\Big)_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}}\big)\\
&=\Big(\frac{\Delta U^n}{E}+\frac{U^n}{E^2}(L^2+2M^2+N^2)\Big)\mathbf{R}_\alpha\times\mathbf{R}_\beta+w_1(2H\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\alpha\\
&\quad+w_2(2H\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\beta+2H(w_{1\alpha}+w_{2\beta})\mathbf{R}_\alpha\times\mathbf{R}_\beta\\
&\quad+2H\Big(\frac{U^n}{E}\Big)_\alpha(\mathbf{R}_\alpha\times\mathbf{R}_\beta)\times\mathbf{R}_\beta+2H\frac{U^n}{E}\big(\mathbf{R}_\alpha\times\mathbf{R}_\beta\big)_\alpha\times\mathbf{R}_\beta\\
&\quad+2H\Big(\frac{U^n}{E}\Big)_\beta\mathbf{R}_\alpha\times(\mathbf{R}_\alpha\times\mathbf{R}_\beta)+2H\frac{U^n}{E}\mathbf{R}_\alpha\times\big(\mathbf{R}_\alpha\times\mathbf{R}_\beta\big)_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}}\big).
\end{aligned}
\tag{5.50}
$$

From (5.40)-(5.41), it follows that

$$
\begin{aligned}
\mathbf{R}_{\alpha\alpha} &= \Gamma^1_{11}\mathbf{R}_\alpha+\Gamma^2_{11}\mathbf{R}_\beta+L\mathbf{n}\\
&= \frac{E_\alpha}{2E}\mathbf{R}_\alpha-\frac{E_\beta}{2E}\mathbf{R}_\beta+L\mathbf{n}+\mathfrak{F}\big(\delta^1_E,\delta^2_E,\nabla\delta^1_E,\nabla\delta^2_E,a_{12}\big), && (5.51)\\
\mathbf{R}_{\alpha\beta} &= \Gamma^1_{12}\mathbf{R}_\alpha+\Gamma^2_{12}\mathbf{R}_\beta+M\mathbf{n}\\
&= \frac{E_\beta}{2E}\mathbf{R}_\alpha+\frac{E_\alpha}{2E}\mathbf{R}_\beta+M\mathbf{n}+\mathfrak{F}\big(\delta^1_E,\delta^2_E,\nabla\delta^1_E,\nabla\delta^2_E,a_{12}\big), && (5.52)\\
\mathbf{R}_{\beta\beta} &= \Gamma^1_{22}\mathbf{R}_\alpha+\Gamma^2_{22}\mathbf{R}_\beta+N\mathbf{n}\\
&= -\frac{E_\alpha}{2E}\mathbf{R}_\alpha+\frac{E_\beta}{2E}\mathbf{R}_\beta+N\mathbf{n}+\mathfrak{F}\big(\delta^1_E,\delta^2_E,\nabla\delta^1_E,\nabla\delta^2_E,a_{12}\big). && (5.53)
\end{aligned}
$$

And by (5.42), it follows that

$$
\begin{aligned}
&(\Delta\mathbf{R})_\alpha\times\mathbf{R}_\beta\\
&=2H_\alpha(\mathbf{R}_\alpha\times\mathbf{R}_\beta)\times\mathbf{R}_\beta+2H(\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\alpha\times\mathbf{R}_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}}\big)\\
&=-2EH_\alpha\mathbf{R}_\alpha+2H(\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\alpha\times\mathbf{R}_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}},\delta_E,a_{12}\big), && (5.54)\\
&\mathbf{R}_\alpha\times(\Delta\mathbf{R})_\beta\\
&=2H_\beta\mathbf{R}_\alpha\times(\mathbf{R}_\alpha\times\mathbf{R}_\beta)+2H\mathbf{R}_\alpha\times(\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}}\big)\\
&=-2EH_\beta\mathbf{R}_\beta+2H\mathbf{R}_\alpha\times(\mathbf{R}_\alpha\times\mathbf{R}_\beta)_\beta+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}},\delta_E,a_{12}\big). && (5.55)
\end{aligned}
$$

Summing up (5.48)-(5.55), we obtain

$$
\begin{aligned}
&(\Delta-1)(\widetilde{\mathbf{R}}-\widehat{\mathbf{R}})_t=\Delta\mathbf{w}-(2H\widetilde{\mathbf{R}}_\alpha\times\widetilde{\mathbf{R}}_\beta)_t\\
&=\Delta\Big(\frac{U^n}{E}\Big)E\mathbf{n}-2U^n(H_\alpha\mathbf{R}_\alpha+H_\beta\mathbf{R}_\beta)+\Delta w_1\mathbf{R}_\alpha+\Delta w_2\mathbf{R}_\beta\\
&\quad+2\Big(\frac{U^n}{E}\Big)_\alpha(E_\alpha\mathbf{n}-L\mathbf{R}_\alpha-M\mathbf{R}_\beta)+2\Big(\frac{U^n}{E}\Big)_\beta(E_\beta\mathbf{n}-M\mathbf{R}_\alpha-N\mathbf{R}_\beta)\\
&\quad+2(w_{1\alpha}\mathbf{R}_{\alpha\alpha}+w_{2\beta}\mathbf{R}_{\beta\beta})-(w_{1\alpha}+w_{2\beta})(2H\mathbf{R}_\alpha\times\mathbf{R}_\beta)\\
&\quad+2\frac{U^n}{E}(\mathbf{R}_{\alpha\alpha}-\mathbf{R}_{\beta\beta})\times\mathbf{R}_{\alpha\beta}+\frac{4MU^n}{E}\mathbf{R}_{\alpha\beta}-\Big(\Delta U^n+\frac{U^n}{E}(L^2+2M^2+N^2)\Big)\mathbf{n}\\
&\quad+2EH\Big(\frac{U^n}{E}\Big)_\alpha\mathbf{R}_\alpha+2EH\Big(\frac{U^n}{E}\Big)_\beta\mathbf{R}_\beta\\
&\quad+\mathfrak{F}\big(\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}},\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big).
\end{aligned}
$$

On the other hand, we can get by (5.51)-(5.53) that

$$
\begin{aligned}
&(\mathbf{R}_{\alpha\alpha}-\mathbf{R}_{\beta\beta})\times\mathbf{R}_{\alpha\beta}\\
&=\frac{E_\alpha^2+E_\beta^2}{2E}\mathbf{n}-\frac{ME_\alpha}{E}\mathbf{R}_\beta-\frac{ME_\beta}{E}\mathbf{R}_\alpha+\frac{(L-N)E_\beta}{2E}\mathbf{R}_\beta-\frac{(L-N)E_\alpha}{2E}\mathbf{R}_\alpha\\
&\quad+\mathfrak{F}\big(\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big),
\end{aligned}
$$

and by (5.43) and (5.31),

$$
\begin{aligned}
&2(w_{1\alpha}\mathbf{R}_{\alpha\alpha}+w_{2\beta}\mathbf{R}_{\beta\beta})-(w_{1\alpha}+w_{2\beta})(2H\mathbf{R}_\alpha\times\mathbf{R}_\beta)\\
&=(w_{1\alpha}-w_{2\beta})(\mathbf{R}_{\alpha\alpha}-\mathbf{R}_{\beta\beta})+\mathfrak{F}\big(\delta_R,\delta_{\widetilde{R}},\nabla\delta_R,\nabla\delta_{\widetilde{R}}\big)\\
&=\frac{U^n(L-N)}{E}(\mathbf{R}_{\alpha\alpha}-\mathbf{R}_{\beta\beta})+\mathfrak{F}\big(\delta_R,\delta_{\widetilde{R}},\nabla\delta_R,\nabla\delta_{\widetilde{R}}\big)\\
&=\frac{U^n(L-N)}{E}\Big(\frac{E_\alpha}{E}\mathbf{R}_\alpha-\frac{E_\beta}{E}\mathbf{R}_\beta+(L-N)\mathbf{n}\Big)\\
&\quad+\mathfrak{F}\big(\delta_R,\delta_{\widetilde{R}},\nabla\delta_R,\nabla\delta_{\widetilde{R}},\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big).
\end{aligned}
$$

Thus, we arrive at

$$
\begin{aligned}
&(\Delta-1)(\widetilde{\mathbf{R}}-\widehat{\mathbf{R}})_t\\
&=\Delta\Big(\frac{U^n}{E}\Big)E\mathbf{n}-2U^n(H_\alpha\mathbf{R}_\alpha+H_\beta\mathbf{R}_\beta)+\Delta w_1\mathbf{R}_\alpha+\Delta w_2\mathbf{R}_\beta\\
&\quad+2\Big(\frac{U^n}{E}\Big)_\alpha(E_\alpha\mathbf{n}-L\mathbf{R}_\alpha-M\mathbf{R}_\beta)+2\Big(\frac{U^n}{E}\Big)_\beta(E_\beta\mathbf{n}-M\mathbf{R}_\alpha-N\mathbf{R}_\beta)\\
&\quad+\frac{U^n}{E}\Big(\frac{E_\alpha^2+E_\beta^2}{E}\mathbf{n}-\frac{2ME_\alpha}{E}\mathbf{R}_\beta-\frac{2ME_\beta}{E}\mathbf{R}_\alpha+\frac{(L-N)E_\beta}{E}\mathbf{R}_\beta-\frac{(L-N)E_\alpha}{E}\mathbf{R}_\alpha\Big)\\
&\quad+\frac{U^n(L-N)}{E}\Big(\frac{E_\alpha}{E}\mathbf{R}_\alpha-\frac{E_\beta}{E}\mathbf{R}_\beta+(L-N)\mathbf{n}\Big)+\frac{4MU^n}{E}\Big(\frac{E_\beta}{2E}\mathbf{R}_\alpha+\frac{E_\alpha}{2E}\mathbf{R}_\beta+M\mathbf{n}\Big)\\
&\quad-\Big(\Delta U^n+\frac{U^n}{E}(L^2+2M^2+N^2)\Big)\mathbf{n}+2EH\Big(\frac{U^n}{E}\Big)_\alpha\mathbf{R}_\alpha+2EH\Big(\frac{U^n}{E}\Big)_\beta\mathbf{R}_\beta\\
&\quad+\mathfrak{F}\big(\delta_R,\delta_{\widetilde{R}},\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}},\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big)\\
&=\frac{U^n}{E}\Big(2(M^2-LN)-\Delta E+\frac{E_\alpha^2+E_\beta^2}{E}\Big)\mathbf{n}\\
&\quad+\Big(-2U^nH_\alpha+\Delta w_1-(L-N)\big(\frac{U^n}{E}\big)_\alpha-2M\big(\frac{U^n}{E}\big)_\beta\Big)\mathbf{R}_\alpha\\
&\quad+\Big(-2U^nH_\beta+\Delta w_2-2M\big(\frac{U^n}{E}\big)_\alpha+(L-N)\big(\frac{U^n}{E}\big)_\beta\Big)\mathbf{R}_\beta\\
&\quad+\mathfrak{F}\big(\delta_R,\delta_{\widetilde{R}},\nabla\delta_R,\nabla\delta_{\widetilde{R}},\nabla^2\delta_R,\nabla^2\delta_{\widetilde{R}},\delta_E^1,\delta_E^2,\nabla\delta_E^1,\nabla\delta_E^2,a_{12}\big).
\end{aligned}
$$

And thanks to (5.31), we have

$$
\begin{aligned}
\Delta w_1&=\Big(\frac{U^n(L-N)}{E}\Big)_\alpha+\Big(\frac{2MU^n}{E}\Big)_\beta,\\
\Delta w_2&=\Big(\frac{2MU^n}{E}\Big)_\alpha-\Big(\frac{U^n(L-N)}{E}\Big)_\beta,
\end{aligned}
$$

which together with Lemma 5.6 implies that

$$
\begin{aligned}
&(\Delta-1)(\widetilde{\mathbf{R}}-\widehat{\mathbf{R}})_t\\
&=\frac{U^n}{E}\Big(2(M^2-LN)-\Delta E+\frac{E_\alpha^2+E_\beta^2}{E}\Big)\mathbf{n}\\
&\quad+\frac{U^n}{E}\big(L+N-2EH\big)_\alpha\mathbf{R}_\alpha+\frac{U^n}{E}\big(L+N-2EH\big)_\beta\mathbf{R}_\beta\\
&\quad+\sum_{k=0}^{2}\mathfrak{F}\big(\nabla^k\delta_R,\nabla^k\delta_{\widetilde{R}},\nabla^k\delta_E^1,\nabla^k\delta_E^2,\nabla^k a_{12}\big)\\
&=\sum_{k=0}^{2}\mathfrak{F}\big(\nabla^k\delta_R,\nabla^k\delta_{\widetilde{R}},\nabla^k\delta_E^1,\nabla^k\delta_E^2,\nabla^k a_{12}\big). \qquad (5.56)
\end{aligned}
$$

Now we are position to prove (5.32). Firstly, by (5.2) and the fact that the initial surface is parameterized by the isothermal coordinates and (3.11) holds for $t=0$, we know that all the relations in (5.32) hold for $t=0$. Hence,

$$\delta_a(0)=0,\quad \widetilde{a}_{12}(0)=0,\quad \delta_{\widetilde{R}}(0)=0.$$

Taking the $L^2$ energy estimate to (5.33) and (5.34), we obtain

$$\|\delta_a(t)\|_{L^2} \le C\int_0^t \big(\|\delta_a\|_{L^2} + \|\widetilde{a}_{12}\|_{L^2} + \|\delta_R\|_{H^2} + \|\delta_{\widetilde{R}}\|_{H^2}\big)d\tau,$$
$$\|\widetilde{a}_{12}(t)\|_{L^2} \le C\int_0^t \big(\|\delta_a\|_{L^2} + \|\widetilde{a}_{12}\|_{L^2} + \|\delta_R\|_{H^2} + \|\delta_{\widetilde{R}}\|_{H^2}\big)d\tau.$$

Using the elliptic estimate, we deduce from (5.35)-(5.38) that

$$\|\delta_R\|_{H^2} \le C\|\delta_{\widetilde{R}}\|_{H^1},$$
$$\|(\delta_E^1, \delta_E^2, a_{12})\|_{H^2} \le C\big(\|\widetilde{a}_{12}\|_{L^2} + \|\delta_R\|_{H^1} + \|\delta_{\widetilde{R}}\|_{H^1}\big),$$

and from (5.56), it follows that

$$\|\delta_{\widetilde{R}}(t)\|_{H^2} \le C\int_0^t \big(\|\delta_a\|_{L^2} + \|\widetilde{a}_{12}\|_{L^2} + \|\delta_R\|_{H^2} + \|\delta_{\widetilde{R}}\|_{H^2} + \|(\delta_E^1, \delta_E^2, a_{12})\|_{H^2}\big)d\tau.$$

Thus, we obtain

$$\begin{aligned}&\|\delta_a(t)\|_{L^2} + \|\widetilde{a}_{12}(t)\|_{L^2} + \|\delta_{\widetilde{R}}(t)\|_{H^2}\\ &\le C\int_0^t \big(\|\delta_a(\tau)\|_{L^2} + \|\widetilde{a}_{12}(\tau)\|_{L^2} + \|\delta_{\widetilde{R}}(\tau)\|_{H^2}\big)d\tau,\end{aligned}$$

which implies (5.32) by Gronwall's inequality.

### 5.5 Remark on the general case

In this subsection, we describe how to adapt our method to deal with the case in which the surface is parameterized by a finite number of isothermal coordinates. Assume that we need $N$ local chart to parameterize the initial surface $S_0 = \cup_{i=1}^N S_0^i$ where each $S_0^i$ is open and parameterized by isothermal coordinates:

$$\mathbf{R}_0^i(x_1, x_2) : \Omega^i \longrightarrow S_0^i, \quad 1 \le i \le N.$$

Let $\{\psi^i\}_{1\le i\le N}$ be a partition of the unit subordinate to $\{S_0^i\}_{1\le i\le N}$; that is,

$$\sum_{i=1}^N \psi^i = 1, \quad \mathrm{supp}\psi^i \subset S_0^i.$$

At each local chart, $\mathbf{R}^i$ is defined by

$$\frac{\partial \mathbf{R}^i}{\partial t} = v^n\mathbf{n}_i + W_1^i\mathbf{t}_i^1 + W_2^i\mathbf{t}_i^2,$$

where $(W_1^i, W_2^i)$ is defined by

$$\begin{cases} \Big(\frac{W_1^i}{\sqrt{E_i}}\Big)_\alpha - \Big(\frac{W_2^i}{\sqrt{E_i}}\Big)_\beta = \frac{U^n(L_i - N_i)}{E_i}, \\ \Big(\frac{W_1^i}{\sqrt{E_i}}\Big)_\beta + \Big(\frac{W_2^i}{\sqrt{E_i}}\Big)_\alpha = \frac{2U^nM_i}{E_i}. \end{cases}$$

While, $(v, H, \Pi)$ is determined by solving the following system:

$$\begin{aligned}
&\frac{\partial \mathbf{v}}{\partial t} = (-\Pi a^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta} + 2\varepsilon_0(S^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta} - \frac{1}{2}\Big(\Delta_\Gamma H + H(b^\alpha_\beta b^\beta_\alpha - 2H^2)\Big)\mathbf{n},\\
&v^\alpha_{,\alpha} = 2Hv^n,\\
&2\frac{\partial H}{\partial t} = a^{\alpha\beta}v^n_{,\alpha\beta} + v^n b^\alpha_\beta b^\beta_\alpha + 2v^\alpha H_{,\alpha},
\end{aligned}$$

see Section 2 for some notations. As the above equations are coordinate-invariant, $(v, H, \Pi)$ does not depend on the choice of coordinates. In this case, the energy functional is given by

$$\mathcal{E}(t) = \|(-\Delta_{S_t})^k v^T\|^2_{L^2(S_t)} + \|(-\Delta_{S_t})^k v^n\|^2_{L^2(S_t)} + \|(-\Delta_{S_t})^k H\|^2_{L^2(S_t)},$$

where $v^T$ is the tangential component of the velocity, and $\Delta_{S_t}$ is the Laplace-Beltrami operator on the surface $S_t$ at time $t$. In the isothermal coordinates, $\Delta_{S_t} = \frac{1}{E}\Delta$. Then, as in section 4, we can obtain a uniform estimate for $\mathcal{E}(t)$. Let $\{\phi^i(t, x_1, x_2)\}_{1\le i\le N}$ be a partition of the unit on $S_t$ given by

$$\phi^i(t, x_1, x_2) = \frac{\psi^i(R^i_0(x_1, x_2))}{\sum_{j=1}^N \widetilde{\psi}^j(t, R^i(t, x_1, x_2))}, \quad \widetilde{\psi}^i(t, X) = \psi^i(R^i_0 \circ (R^i(t))^{-1}(X)).$$

Indeed, we have

$$\begin{aligned}
\frac{d}{dt}\|(-\Delta_{S_t})^k v^n\|^2_{L^2(S_t)} &= \sum_{i=1}^N \partial_t \int \phi^i\big|(-\frac{1}{E_i}\Delta)^k v^n\big|^2 E_i dx_1 dx_2\\
&= -\sum_{i=1}^N \int \phi^i(-\frac{1}{E_i}\Delta)^k v^n(-\frac{1}{E_i}\Delta)^{k+1} H E_i dx_1 dx_2 + L.W.T.\\
&= -\int_{S_t} (-\Delta_{S_t})^k v^n (-\Delta_{S_t})^{k+1} H dS_t + L.W.T..
\end{aligned}$$

And, similarly,

$$\begin{aligned}
\frac{d}{dt}\|(-\Delta_{S_t})^k H\|^2_{L^2(S_t)} &= \sum_{i=1}^N \int \phi^i(-\frac{1}{E_i}\Delta)^k H(-\frac{1}{E_i}\Delta)^{k+1} v^n E_i dx_1 dx_2 + L.W.T.\\
&= \int_{S_t} (-\Delta_{S_t})^k H(-\Delta_{S_t})^{k+1} v^n dS_t + L.W.T.,
\end{aligned}$$

where $L.W.T.$ denotes the lower-order terms. Thus, we have

$$\frac{d}{dt}\big(\|(-\Delta_{S_t})^k v^n\|^2_{L^2(S_t)} + \|(-\Delta_{S_t})^k H\|^2_{L^2(S_t)}\big) = L.W.T.$$

# 6 Appendix

## 6.1 Derivations of the equation (2.14) and the energy law

In this subsection, we give the derivations of the equation (2.14) and the energy law in the case $B_{\alpha\beta} = Ba_{\alpha\beta}$. The reader can also find a short version of the derivation in [10].

Firstly, we have

$$M^{\alpha\beta}=C_1^{\alpha\beta\gamma\delta}(Ba_{\gamma\delta}-b_{\gamma\delta})=2(k_1B-(k_1-\varepsilon_1)H)a^{\alpha\beta}+2\varepsilon_1 b^{\alpha\beta}.$$

We get by (2.2) that

$$\begin{aligned}(M^{\alpha\mu}b_\mu^\beta\mathbf{a}_\beta)_{,\alpha}+(q^\alpha\mathbf{n})_\alpha &= (M^{\alpha\mu}b_\mu^\beta)_{,\alpha}\mathbf{a}_\beta+M^{\alpha\mu}b_\mu^\beta b_{\alpha\beta}\mathbf{n}+M^{\alpha\beta}_{,\alpha\beta}\mathbf{n}-M^{\alpha\gamma}_\gamma b_\alpha^\beta\mathbf{a}_\beta\\ &= M^{\alpha\mu}b^\beta_{\mu,\alpha}\mathbf{a}_\beta+(M^{\alpha\mu}b_\mu^\beta b_{\alpha\beta}+M^{\alpha\beta}_{,\alpha\beta})\mathbf{n}.\end{aligned}$$

Using $b_{\alpha\beta,\gamma}=b_{\alpha\gamma,\beta}$, we infer that

$$\begin{aligned}&a^{\alpha\mu}b^\beta_{\mu,\alpha}=b^{\alpha\beta}_{,\alpha}=a^{\beta\mu}b^\alpha_{\mu,\alpha}=a^{\beta\mu}b^\alpha_{\alpha,\mu}=2a^{\alpha\beta}H_{,\alpha},\\ &\frac{1}{2}a^{\alpha\beta}(b^{\gamma\delta}b_{\gamma\delta})_{,\alpha}=a^{\alpha\beta}b^{\gamma\delta}b_{\gamma\delta,\alpha}=a^{\alpha\beta}b^{\gamma\delta}b_{\gamma\alpha,\delta}=b^{\gamma\delta}b^\beta_{\gamma,\delta}=b^{\alpha\mu}b^\beta_{\mu,\alpha},\end{aligned}$$

from which, we can deduce that

$$\begin{aligned}M^{\alpha\mu}b^\beta_{\mu,\alpha} &= 2(k_1B-(k_1-\varepsilon_1)H)b^{\alpha\beta}_{,\beta}+2\varepsilon_1 b^{\alpha\mu}b^\beta_{\mu,\alpha}\\ &= -4k_1Ha^{\alpha\beta}B_{,\alpha}-a^{\alpha\beta}\big(2(k_1-\varepsilon_1)H^2+\varepsilon_1 b^{\gamma\delta}b_{\gamma\delta}-4k_1HB\big)_{,\alpha},\\ M^{\alpha\beta}_{,\alpha\beta} &= 2k_1a^{\alpha\beta}B_{,\alpha\beta}-2(k_1+\varepsilon_1)a^{\alpha\beta}H_{,\alpha\beta}.\end{aligned}$$

Let $K$ be the Gaussian curvature. It is easy to see that

$$b^{\alpha\beta}b_{\alpha\beta}=4H^2-2K,\quad b^{\alpha\beta}b^\gamma_\beta b_{\alpha\gamma}=2H(4H^2-3K),$$

which implies that

$$M^{\alpha\mu}b^\beta_\mu b_{\alpha\beta}=4k_1(B-H)(2H^2-K)-8\varepsilon_1H(H^2-K).$$

Let $P\stackrel{\text{def}}{=}\Pi+2(k_1-\varepsilon_1)H^2+2\varepsilon_1(2H^2-K)-4k_1HB$. Then we obtain

$$\begin{aligned}&(T^{\alpha\beta}a_\beta)_{,\alpha}+(q^\alpha\mathbf{n})_{,\alpha}\\ &=-(\Pi a^{\alpha\beta}\mathbf{a}_\beta)_{,\alpha}+2\varepsilon_0(S^{\alpha\beta}\mathbf{a}_\beta)_{,\alpha}+M^{\alpha\mu}b^\beta_{\mu,\alpha}\mathbf{a}_\beta+(M^{\alpha\mu}b^\beta_\mu b_{\alpha\beta}+M^{\alpha\beta}_{,\alpha\beta})\mathbf{n}\\ &=(Pa^{\alpha\beta}\mathbf{a}_\beta)_{,\alpha}+2\varepsilon_0(S^{\alpha\beta}\mathbf{a}_\beta)_{,\alpha}-4k_1Ha^{\alpha\beta}B_{,\alpha}\mathbf{a}_\beta\\ &\quad+\Big(2k_1(\Delta_\Gamma B-2KB)-2(k_1+\varepsilon_1)(\Delta_\Gamma H+2H(H^2-K))\Big)\mathbf{n}.\end{aligned}$$

We still use $\Pi$ to denote $P$. Thus, (2.14) follows easily.

Now we derive the energy law of (2.14). We infer from (2.14) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\frac{1}{2}\varrho|\mathbf{v}|^2\mathrm{d}S=\int\varrho\mathbf{v}\cdot\frac{\partial\mathbf{v}}{\partial t}\mathrm{d}S\\ &=\int\mathbf{v}\cdot\Big(-(\Pi a^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta}+2\varepsilon_0(S^{\alpha\beta}\mathbf{a}_\alpha)_{,\beta}-4k_1Ha^{\alpha\beta}B_{,\beta}\mathbf{a}_\alpha\\ &\quad+2k_1\big(\Delta_\Gamma B+B(b^\alpha_\beta b^\beta_\alpha-4H^2)\big)\mathbf{n}-2\mu_1\big(\Delta_\Gamma H+H(b^\alpha_\beta b^\beta_\alpha-2H^2)\big)\mathbf{n}\Big)\mathrm{d}S\\ &=\int\Pi\mathbf{a}^\alpha\cdot\mathbf{v}_{,\alpha}-2\varepsilon_0S^{\alpha\beta}\mathbf{a}_\alpha\cdot\mathbf{v}_{,\beta}-4k_1Ha^{\alpha\beta}B_{,\beta}v_\alpha\\ &\quad+2k_1\big(\Delta_\Gamma B+B(b^\alpha_\beta b^\beta_\alpha-4H^2)\big)v^n-2\mu_1\big(\Delta_\Gamma H+H(b^\alpha_\beta b^\beta_\alpha-2H^2)\big)v^n\mathrm{d}S\\ &=\int-2\varepsilon_0S^{\alpha\beta}S_{\alpha\beta}+2k_1B\Big(2(Hv^\alpha)_{,\alpha}+(b^\alpha_\beta b^\beta_\alpha-4H^2)v^n+\Delta_\Gamma v^n\Big)\\ &\quad-2\mu_1H\Big(\Delta_\Gamma v^n+(b^\alpha_\beta b^\beta_\alpha-2H^2)v^n\Big)\mathrm{d}S,\quad\mu_1=k_1+\varepsilon_1.\end{aligned}\tag{6.1}$$

Here we used $v^\alpha_{,\alpha} - 2Hv^n = 0$ such that

$$\mathbf{a}^\alpha \cdot \mathbf{v}_{,\alpha} = v^\alpha_{,\alpha} - v^n b^\alpha_\alpha = v^\alpha_{,\alpha} - 2Hv^n = 0.$$

On the other hand, the Helfrich energy can be simplified as

$$E_H = \int_\Gamma 4k_1(H-B)^2 + 4\varepsilon_1(H^2 - K)\mathrm{d}S,$$

where $K = \frac{1}{2}(4H^2 - b^\alpha_\beta b^\beta_\alpha)$ is the Gaussian curvature. As $\int K\mathrm{d}S$ is a constant independent of the time, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_H &= \int_\Gamma \big(8k_1(H-B) + 8\varepsilon_1 H\big)\frac{\partial H}{\partial t}\mathrm{d}S \\ &= 4\int (\mu_1 H - k_1 B)(\Delta_\Gamma v^n + v^n b^\alpha_\beta b^\beta_\alpha + 2v^\alpha H_{,\alpha}). \end{aligned} \tag{6.2}$$

Adding up (6.1) and (6.2), and using $v^\alpha_{,\alpha} - 2Hv^n = 0$ again, we obtain the following energy dissipation law:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_H + \int_\Gamma \varrho|\mathbf{v}|^2\mathrm{d}S\Big) = -2\varepsilon_0\int_\Gamma S^{\alpha\beta}S_{\alpha\beta}\mathrm{d}S.$$

## 6.2 Some basic estimates in Sobolev spaces

Let us first recall some product estimates and commutator estimates.

**Lemma 6.1** *Let $s \geq 0$. Then for any multi-index $\alpha, \beta$, it holds that*

$$\|\partial^\alpha f \partial^\beta g\|_{H^s} \leq C\big(\|f\|_{L^\infty}\|g\|_{H^{s+|\alpha|+|\beta|}} + \|g\|_{L^\infty}\|f\|_{H^{s+|\alpha|+|\beta|}}\big).$$

*In particular, we have*

$$\|fg\|_{H^s} \leq C\big(\|f\|_{L^\infty}\|g\|_{H^s} + \|g\|_{L^\infty}\|f\|_{H^s}\big).$$

**Lemma 6.2** *Let $s \geq 0$ and $F(\cdot) \in C^\infty(\mathbf{R}^+)$ with $F(0) = 0$. Then*

$$\|F(f)\|_{H^s} \leq C(\|f\|_{L^\infty})\|f\|_{H^s}.$$

**Lemma 6.3** *Let $s > 0$. It holds that*

$$\|\big[\Lambda^s, g\big]f\|_{L^2} \leq C\big(\|\nabla g\|_{L^\infty}\|f\|_{H^{s-1}} + \|g\|_{H^s}\|f\|_{L^\infty}\big).$$

*Here $\Lambda = (-\Delta)^{\frac{1}{2}}$.*

Lemmas 6.1-6.3 are well-known, see [11, 24] for example.

**Lemma 6.4** *Let $s \geq 0$ and $k \geq 1$ be an integer. Then it holds that*

$$\|(a\Delta)^k f\|_{H^s} \leq C(\|a\|_{H^2})\big(\|f\|_{H^{s+2k}} + \|a\|_{H^{s+2k}}\|f\|_{H^2}\big).$$

**Proof.** We will prove it by induction on $k$. For $k=1$, using Lemma 6.1 and the Sobolev inequality, we get

$$\begin{aligned}\|a\Delta f\|_{H^s} &\le C\big(\|a\|_{L^\infty}\|f\|_{H^{s+2}}+\|a\|_{H^{s+2}}\|f\|_{L^\infty}\big)\\ &\le C\big(\|a\|_{H^2}\|f\|_{H^{s+2}}+\|a\|_{H^{s+2}}\|f\|_{H^2}\big).\end{aligned}$$

Assume Lemma 6.4 holds for $k-1$. Then using the induction assumption, we have

$$\begin{aligned}\|(a\Delta)^k f\|_{L^2} &= \|(a\Delta)^{k-1}(a\Delta f)\|_{L^2}\\ &\le C(\|a\|_{H^2})\big(\|a\Delta f\|_{H^{s+2k-2}}+\|a\|_{H^{s+2k-2}}\|a\Delta f\|_{H^2}\big).\end{aligned}$$

We get by Lemma 6.1 that

$$\begin{aligned}\|a\Delta f\|_{H^{s+2k-2}} &\le C\big(\|a\|_{H^2}\|f\|_{H^{s+2k}}+\|a\|_{H^{s+2k}}\|f\|_{H^2}\big),\\ \|a\|_{H^{s+2k-2}}\|a\Delta f\|_{H^2} &\le C\|a\|_{H^{s+2k-2}}\big(\|a\|_{H^4}\|f\|_{H^2}+\|a\|_{H^2}\|f\|_{H^4}\big)\\ &\le C(\|a\|_{H^2})\big(\|f\|_{H^{s+2k}}+\|a\|_{H^{s+2k}}\|f\|_{H^2}\big).\end{aligned}$$

Here we used the following interpolation inequality in the last inequality:

$$\|a\|_{H^4}\le\|a\|_{H^2}^{\theta}\|a\|_{H^{s+2k}}^{1-\theta},\quad \|a\|_{H^{s+2k-2}}\le\|a\|_{H^2}^{1-\theta}\|a\|_{H^{s+2k}}^{\theta},$$

with $\theta=(s+2k-4)/(s+2k-2)$. Thus, we get

$$\|(a\Delta)^k f\|_{H^s}\le C(\|a\|_{H^2})\big(\|f\|_{H^{s+2k}}+\|a\|_{H^{s+2k}}\|f\|_{H^2}\big).$$

The proof is completed. $\square$

**Lemma 6.5** *Let $s\ge 0$ and $s_0\in(1,2)$. Assume that $a\ge c_0$ for some positive constant $c_0$. Then we have*

$$\|(a\Delta)^k f\|_{H^s}\ge c\|f\|_{H^{s+2k}}-C(\|a\|_{H^{s_0+1}})\|a\|_{H^{s+2k}}\|f\|_{H^{s_0}}.$$

**Proof.** We prove the lemma based on the induction assumption on $k$. For $k=1$, we have

$$\|(a\Delta)^k f\|_{H^s}\ge c_0\||\Lambda^{s+2}\|_{L^2}-\|\big[\Lambda^s,a\big]\Delta f\|_{L^2}.$$

We write

$$\big[\Lambda^s,a\big]\Delta f=\big[\Lambda^s\Delta,a\big]f-2\Lambda^s(\nabla a\cdot\nabla f)-\Lambda^s(\Delta af),$$

which along with Lemma 6.1 and Lemma 6.3 implies that

$$\|\big[\Lambda^s,a\big]\Delta f\|_{L^2}\le C\big(\|a\|_{H^{s_0+1}}\|f\|_{H^{s+1}}+\|a\|_{H^{s+2}}\|f\|_{H^{s_0}}\big).$$

This yields the case of $k=1$ by an interpolation argument.

Now let us assume that Lemma 6.5 holds for $k-1$. Using the induction assumption, we have

$$\begin{aligned}\|\big(a\Delta\big)^k f\|_{L^2} &= \|(a\Delta)^{k-1}(a\Delta f)\|_{L^2}\\ &\ge c\|a\Delta f\|_{H^{s+2k-2}}-C(\|a\|_{H^{s_0+1}})\|a\|_{H^{s+2(k-1)}}\|a\Delta f\|_{H^s}.\end{aligned}$$

Using the case of $k = 1$, we get

$$\|a\Delta f\|_{H^{s+2k-2}} \geq c\|f\|_{H^{2k+s}} - C(\|a\|_{H^{s_0+1}})\|a\|_{H^{s+2k}}\|f\|_{H^{s_0}},$$

and by Lemma 6.1,

$$\begin{aligned}
&\|a\|_{H^{s+2(k-1)}}\|a\Delta f\|_{H^s} \\
&\leq C\|a\|_{H^{s+2(k-1)}}\big(\|a\|_{H^{s_0+2}}\|f\|_{H^{s_0}} + \|a\|_{H^{s_0}}\|f\|_{H^{s_0+2}}\big) \\
&\leq \varepsilon\|f\|_{H^{s+2k}} + C(\|a\|_{H^{s_0+1}})\|a\|_{H^{s+2k}}\|f\|_{H^{s_0}}.
\end{aligned}$$

Here we used the following interpolation inequality in the last inequality:

$$\|a\|_{H^{s_0+2}} \leq \|a\|_{H^{s_0}}^{\theta}\|a\|_{H^{s+2k}}^{1-\theta}, \quad \|a\|_{H^{s+2k-2}} \leq \|a\|_{H^{s_0}}^{1-\theta}\|a\|_{H^{s+2k}}^{\theta},$$

with $\theta = 2/(s+2k-s_0)$. Taking $\varepsilon$ to be small enough, we obtain

$$\|(a\Delta)^k f\|_{H^s} \geq c\|f\|_{H^{s+2k}} - C(\|a\|_{H^{s_0+1}})\|a\|_{H^{s+2k}}\|f\|_{H^{s_0}}.$$

The proof is completed. □

**Lemma 6.6** *Let $s \geq 0$ and $k \geq 1$ be an integer. Then we have*

$$\begin{aligned}
&\|\big[\partial_t, (a\Delta)^k\big]f\|_{H^s} \leq C(\|(a,\partial_t a)\|_{H^2})\big(\|f\|_{H^{s+2k}} + \|(a,\partial_t a)\|_{H^{s+2k}}\|f\|_{H^2}\big), \\
&\|\big[\nabla, (a\Delta)^k\big]f\|_{H^s} \leq C(\|a\|_{H^3})\big(\|f\|_{H^{s+2k}} + \|a\|_{H^{s+2k+1}}\|f\|_{H^2}\big).
\end{aligned}$$

**Proof.** We write

$$\begin{aligned}
\big[\partial_t, (a\Delta)^k\big]f &= \sum_{\ell=1}^{k-1}(a\Delta)^\ell\big[\partial_t, a\Delta\big](a\Delta)^{k-\ell-1}f \\
&= \sum_{\ell=1}^{k-1}(a\Delta)^\ell(\partial_t a\Delta)(a\Delta)^{k-\ell-1}f.
\end{aligned}$$

Then the first inequality can be seen easily from the proof of Lemma 6.4. The proof of the second inequality is similar. □

**Lemma 6.7** *Let $s \geq 0$ and $k \geq 1$ be an integer. Then we have*

$$\|\big[(a\Delta)^k, g\big]f\|_{H^s} \leq C(\|a\|_{H^2}, \|g\|_{H^3})\big(\|f\|_{H^{s+2k-1}} + \|(a,g)\|_{H^{s+2k}}\|f\|_{H^2}\big).$$

**Proof.** As in Lemma 6.5, this lemma can be proved by the induction argument, however, we omit the details here. □

### 6.3 Elliptic estimates

We consider the following elliptic system:

$$\begin{cases} \left(\frac{W_1}{\sqrt{E}}\right)_\alpha - \left(\frac{W_2}{\sqrt{E}}\right)_\beta = f_1, \\ \left(\frac{W_1}{\sqrt{E}}\right)_\beta + \left(\frac{W_2}{\sqrt{E}}\right)_\alpha = f_2. \end{cases} \tag{6.3}$$

We write

$$\frac{W_1}{\sqrt{E}} = \partial_\alpha \phi + \partial_\beta \psi, \quad \frac{W_2}{\sqrt{E}} = -\partial_\beta \phi + \partial_\alpha \psi.$$

Then (6.3) is reduced to solve the following Poisson equations:

$$\Delta \phi = f_1, \quad \Delta \psi = f_2.$$

Thus, we have

**Lemma 6.8** *Let $s \geq 1$. If $E \in H^s(\mathbf{T}^2), f_1, f_2 \in H^{s-1}(\mathbf{T}^2)$, then the system (6.3) has a solution $(W_1, W_2)$ satisfying*

$$\|W_1\|_{H^{s-1}} + \|W_2\|_{H^{s-1}} \leq C(\|E\|_{H^s})\big(\|f_1\|_{H^{s-1}} + \|f_2\|_{H^{s-1}}\big).$$

Next we consider the elliptic equation:

$$-\Delta U + aU = f. \tag{6.4}$$

**Lemma 6.9** *Let $s \geq 0$. Assume that $f \in H^s(\mathbf{T}^2)$, and $a \in H^s \cap L^\infty(\mathbf{T}^2)$ with*

$$a(x) \geq 0, \quad \int_{\mathbf{T}^2} a(x)dx \geq a_0 > 0. \tag{6.5}$$

*Then there exists a unique solution $U \in H^{s+2}(\mathbf{T}^2)$ to (6.4) satisfying*

$$\|U\|_{H^{s+2}} \leq C\|f\|_{H^s}.$$

*Here $C$ is a constant depending only on $a_0$ and $\|a\|_{H^s \cap L^\infty}$.*

**Proof.** The proof of existence part is standard. Here we only prove the estimate. Taking the $L^2$ inner estimate gives

$$\|\nabla U\|_{L^2}^2 + \int_{\mathbf{T}^2} a|U|^2 dx = \int_{\mathbf{T}^2} fU dx.$$

Let $\bar{U} = \frac{1}{4\pi^2}\int_{\mathbf{T}^2} U dx$. Therefore, we have

$$\|U\|_{L^2}^2 \leq 2\|U - \bar{U}\|_{L^2}^2 + 2\|\bar{U}\|_{L^2}^2 \leq 2\|\nabla U\|_{L^2} + 8\pi^2|\bar{U}|.$$

On the other hand, we have by (6.5) that

$$a_0|\bar{U}|^2 \le \int_{\mathbf{T}^2} a|\bar{U}|^2 dx \le 2\int_{\mathbf{T}^2} a|U|^2 dx + \int_{\mathbf{T}^2} a|U-\bar{U}|^2 dx$$
$$\le 2\int_{\mathbf{T}^2} a|U|^2 dx + \|a\|_{L^\infty}\|\nabla U\|_{L^2}^2.$$

This yields that

$$\|U\|_{L^2} \le C\big(\|\nabla U\|_{L^2}^2 + \int_{\mathbf{T}^2} a|U|^2 dx\big). \tag{6.6}$$

Using the elliptic estimate in $H^s$, we obtain

$$\|U\|_{H^{s+2}} \le C\big(\|aU\|_{H^s} + \|f\|_{H^s}\big) \le C\big(\|a\|_{L^\infty}\|U\|_{H^s} + \|a\|_{H^s}\|U\|_{L^\infty} + \|f\|_{H^s}\big),$$

from which and (6.6), the desired estimate follows from an interpolation argument. □

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