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# LOCAL EXISTENCE AND UNIQUENESS OF THE DYNAMICAL EQUATIONS OF AN INCOMPRESSIBLE MEMBRANE IN TWO-DIMENSIONAL SPACE*

DAN HU†, PENG SONG‡, AND PINGWEN ZHANG§

**Abstract.** The dynamics of a membrane is a coupled system of a moving elastic surface and an incompressible membrane fluid. The difficulties in analyzing such a system include the nonlinearity of the curved space (geometric nonlinearity), the nonlinearity of the fluid dynamics (fluid nonlinearity), and the coupling to the surface incompressibility. In the two-dimensional case, the fluid vanishes and the system reduces to a coupling of a wave equation and an elliptic equation. Here we prove the local existence and uniqueness of the solution to the system by constructing a suitable discrete scheme and proving the compactness of the discrete solutions. The risk of blowing up due to the geometric nonlinearity is overcome by the bending elasticity.

**Key words.** Membrane, incompressible, existence, uniqueness, and bending elasticity.

**AMS subject classifications.** 35M13, 65M12, 92C17

## 1. Introduction

The lipid membrane is an important component in cells, which surrounds all living animal cells and their organelles. It helps the cell to maintain the shape and regulate the transport in and out of the cell or subcellular domains. By changing its own shape, the membrane also plays important roles in many vital actions, such as cell division. A membrane consists of lipids, proteins, and carbohydrates [1]. The structures and properties of such a membrane are very complex and refined. In general, the membrane is a surface fluid because the lipids and associated proteins are not allowed to escape from the membrane but are allowed to move freely on it; the fluid can be viewed incompressible since the tensile elastic modulus is very large, and the membrane is bending resistent [7]. In a word, the membrane is basically a two-dimensional incompressible fluid defined on a moving elastic surface.

The equations of the surface's evolution was first considered by Scriven [15] and Waxman [19]. Waxman [19, 20] and Steigmann [18] also derived the dynamic equations of fluid membranes with curvature elasticity [4, 7]. Seifert [16], Pozrikidis [14], and Cai and Lubensky [3] considered the coupled system of elastic membranes and bulk fluids. Recently, Miao, Lomholt, and Hansen [11, 12] took more effects, such as the viscous effect, into account and obtained the membrane-fluid coupled system. Hu, Zhang, and E [9] derived the director model in which a director is endowed to every material point on the membrane and the elastic energy is induced by the directors. They also obtained an elastic surface model, in which the director is constrained to be the normal of the surface. Their elastic surface model is a synthesis of the previous works. In the elastic surface model, the dynamics of the membrane involves the evolution equations of the moving surface, the dynamic equations of the two-dimensional fluid, and the incompressible equation. The difficulties in the analysis of such a system

*Received: November 30, 2008; accepted (in revised version): June 23, 2009.
†LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China (hudan@cims.nyu.edu).
‡LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China (kevinsongpeng@yahoo.com.cn).
§LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China (pzhang@pku.edu.cn).

include the nonlinearity of the curved space (geometric nonlinearity), the nonlinearity of the fluid dynamics (fluid nonlinearity), and the coupling to the incompressible equation.

The aim of this work is to take the first step toward the analysis of these equations and to get a deep understanding of the behavior of the membrane. As the first step, we consider the two dimensional case in this paper, where the membrane can be viewed as a cylinder which is translation invariant along the generatrix. In this case, the membrane fluid vanishes and the behavior of such a membrane is similar to an incompressible elastic string. Therefore, we can focus our attention on the geometrical nonlinearity and the incompressibility. We prove the local existence and uniqueness of such a system and show that the difficulty due to the geometrical nonlinearity is overcome by the bending elasticity. This gives us insights to treat the geometric nonlinearity of the general system.

In the next section, we introduce the elastic surface model. In the latter sections, we prove the local existence and uniqueness of this simple case by introducing a suitable discrete scheme and prove the compactness of the discrete solutions.

## 2. The elastic surface model and the reduced form in two-dimensional space

For a integrate understanding of the background and the properties of the problem, we provide a short derivation of the equations in general case. After that, we reduce the equations to the two dimensional case. Readers who are familiar to the background may directly start from the equation set (2.7) – (2.9), which is the set we consider in this paper.

There are two alternative ways to treat the dynamics of membrane fluid [9]. In one way, the membrane is regarded as an immersed interface in the bulk fluid. It moves along with the bulk fluid and provides surface stresses which cause jumps of the bulk stresses. In the other, the membrane is regarded as a separate surface. The interaction between the membrane and the bulk fluid is taken into account by an external force. In this paper, we consider the membrane using the latter approach. First, we introduce the elastic surface model of a moving membrane. Our tensor notation can be found in classic materials [2, 9, 19].

On a moving surface $\Gamma$, we introduce a Lagrangian coordinate $u^\alpha$ $(\alpha=1,2)$. Let $\vec{R}(u^\alpha,t)$ be the position vector in Euclidean space of the material points on the surface. The Frenet coordinate system of the surface–the tangent vectors $\vec{a}_\alpha$ and the unit normal vector $\vec{n}$ are given by

$$\vec{a}_\alpha=\frac{\partial\vec{R}}{\partial u^\alpha},\vec{n}\cdot\vec{a}_\alpha=0,\quad \vec{n}\cdot\vec{n}=1.$$

The metric tensor $a_{\alpha\beta}$ and the covariant alternating tensor $\varepsilon_{\alpha\beta}$ are calculated as

$$a_{\alpha\beta}=\vec{a}_\alpha\cdot\vec{a}_\beta,\varepsilon_{\alpha\beta}=\vec{a}_\alpha\times\vec{a}_\beta\cdot\vec{n}.$$

The metric tensor $a_{\alpha\beta}$, along with its inverse $a^{\alpha\beta}$, is used to lower or raise the indices of vectors and tensors. For example: $b^\gamma_\beta=a^{\alpha\gamma}b_{\alpha\beta}$. The alternating tensor $\varepsilon_{\alpha\beta}$ takes the values $\varepsilon_{12}=-\varepsilon_{21}=\sqrt{a},\varepsilon_{11}=\varepsilon_{22}=0$, where $a=\det(a_{\alpha\beta})$. The surface Christoffel symbols $\Gamma^\gamma_{\alpha\beta}=\Gamma^\gamma_{\beta\alpha}$ and curvature tensor $b_{\alpha\beta}=b_{\beta\alpha}$ are given by the Gauss-Weingarten-

Codazzi equation,

$$\begin{aligned}\frac{\partial \vec{a}_\alpha}{\partial u^\beta} &= \Gamma^\gamma_{\alpha\beta}\vec{a}_\gamma + b_{\alpha\beta}\vec{n},\\ \frac{\partial \vec{n}}{\partial u^\beta} &= -b^\gamma_\beta \vec{a}_\gamma = -a^{\alpha\gamma} b_{\alpha\beta}\vec{a}_\gamma,\\ b_{\alpha\beta,\gamma} &= b_{\alpha\gamma,\beta},\end{aligned}$$

where we have used a comma followed by a lowercase Greek subscript to denote covariant derivatives based on the metric tensor $a_{\alpha\beta}$:

$$Q^{\cdot\alpha\cdot\cdot}_{\cdot\cdot\beta\cdot,\gamma} = \frac{\partial Q^{\cdot\alpha\cdot\cdot}_{\cdot\cdot\beta\cdot}}{\partial u^\gamma} + \sum Q^{\cdot\mu\cdot\cdot}_{\cdot\cdot\beta\cdot}\Gamma^\alpha_{\mu\gamma} - \sum Q^{\cdot\alpha\cdot\cdot}_{\cdot\cdot\mu\cdot}\Gamma^\mu_{\beta\gamma}.$$

For example, we have

$$b_{\alpha\beta,\gamma} = \frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - b_{\alpha\mu}\Gamma^\mu_{\beta\gamma} - b_{\mu\beta}\Gamma^\mu_{\alpha\gamma}.$$

We may also write the Gauss-Weingarten-Codazzi equation as

$$\vec{a}_{\alpha,\beta} = b_{\alpha\beta}\vec{n}, \vec{n}_{,\beta} = -b^\gamma_\beta \vec{a}_\gamma, b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}.$$

The velocity of the surface is

$$\vec{v}(u^\alpha, t) = \frac{\partial \vec{R}}{\partial t},$$

which can be decomposed as

$$\vec{v} = v^\alpha \vec{a}_\alpha + v^{(n)}\vec{n}.$$

Here, the superscript $\alpha$ and $(n)$ mean the component of $\vec{v}$ along the tangent vector $\vec{a}_\alpha$ and normal vector $\vec{n}$ respectively.

The Helfrich free energy [4, 7] is

$$E_{\mathrm{H}} = \int_\Gamma C_1^{\alpha\beta\mu\delta}(B_{\alpha\beta} - b_{\alpha\beta})(B_{\mu\delta} - b_{\mu\delta})\, dS,$$

where $B_{\alpha\beta} = B(u^\gamma)a_{\alpha\beta}$ is the spontaneous curvature tensor and the fourth order tensor is given by

$$C_1^{\alpha\beta\mu\delta} = (k_1 - \varepsilon_1)a^{\alpha\beta}a^{\gamma\delta} + \varepsilon_1\left(a^{\alpha\mu}a^{\beta\delta} + a^{\alpha\delta}a^{\beta\mu}\right),$$

where $k_1$and $\varepsilon_1$ are positive elastic coefficient and $k_1 \geq \varepsilon_1$. By applying the principle of virtual work, we obtain elastic stresses. For isotropic Newtonian membrane fluids, the dynamical equation set of the membrane is

$$\begin{aligned}\gamma\frac{\partial^2 \vec{R}}{\partial t^2} &= \vec{f} + \left(T^{\alpha\beta}\vec{a}_\beta\right)_{,\alpha} + \left(q^\alpha \vec{n}\right)_{,\alpha},\\ v^\alpha_{,\alpha} - 2Hv^{(n)} &= 0,\end{aligned}$$

where the first equation is the momentum equation and the second one is the incompressible equation. In the above equations, $H$ is the mean curvature, and the in-plane stress tensor $T^{\alpha\beta}$ and transverse shear stress $q^{\alpha}$ are given by

$$\begin{aligned}
T^{\alpha\beta} &= -\Pi a^{\alpha\beta} + J^{\alpha\beta} + M^{\alpha\mu} b^{\beta}_{\mu},\\
q^{\alpha} &= M^{\alpha\beta}_{,\beta}\\
J^{\alpha\beta} &= C^{\alpha\beta\gamma\delta} S_{\gamma\delta},\\
M^{\alpha\beta} &= C_1^{\alpha\beta\mu\delta}(B_{\mu\delta} - b_{\mu\delta}),\\
C^{\alpha\beta\mu\delta} &= (k_0 - \varepsilon_0) a^{\alpha\beta} a^{\gamma\delta} + \varepsilon_0 \left(a^{\alpha\mu} a^{\beta\delta} + a^{\alpha\delta} a^{\beta\mu}\right),
\end{aligned}$$

where $\Pi$ is the surface pressure (tension), and $S_{\alpha\beta} = \frac{1}{2}(v_{\alpha,\beta} + v_{\beta,\alpha}) - v^{(n)} b_{\alpha\beta}$ is the rate of surface strain. The energy dissipation relation of the above equations is

$$\frac{d}{dt}\int \left(\frac{1}{2} C_1^{\alpha\beta\mu\delta}(B_{\alpha\beta} - b_{\alpha\beta})(B_{\mu\delta} - b_{\mu\delta}) + \frac{\gamma}{2}|\vec{v}|^2\right) dS = -\int C^{\alpha\beta\gamma\delta} S_{\alpha\beta} S_{\gamma\delta} dS.$$

For a cylindrical membrane with translation invariance along the generatrix, the dynamics of the membrane is simplified to the dynamics of the transversal curve. Since the velocity component along the direction of the generatrix can be simply described by a diffuse equation and is decoupled to the dynamics in the transversal profile, we only consider the dynamics of a closed elastic curve in a plane and call it "the two dimensional case". Since the curve is incompressible, the Lagrangian coordinate $s$ is an arc length parameter if it is initially set to be. All tensors reduce to a scalar, and particularly, $S_{\alpha\beta}$ vanishes due to the incompressibility. The model equations in the two-dimensional space are

$$\gamma \frac{\partial^2 \vec{R}}{\partial t^2} = \vec{f} + \frac{\partial (T\vec{a})}{\partial s} + \frac{\partial (q\vec{n})}{\partial s}, \tag{2.1}$$

$$\frac{\partial v^1}{\partial s} - \kappa v^{(n)} = 0, \tag{2.2}$$

where $\kappa$ is the curvature and $q = \mu(B - \kappa)_{,s}$, $\mu = k_1 + \varepsilon_1$, and $B$ is the spontaneous curvature. Since the surface pressure $\Pi$ in the in-plane stresses $T$ can be regarded as a Lagrangian multiplier and is determined by the incompressibility equation, the other components in $T$ can be absorbed by $\Pi$ in the two dimensional case. Thus we simply regard $T$ as the Lagrangian multiplier. Since the curve is closed, $\vec{R}(s,t)$, $\vec{v}(s,t)$, $T(s,t)$ as well as their derivatives satisfy the periodic boundary condition. The initial condition is

$$\vec{R}(0,t) = R^0(s), \quad \vec{v}(0,t) = \vec{v}^0(s).$$

Note that, due to the incompressibility, the fluid vanishes in the two dimensional case. Therefore, the energy of the system preserves

$$\frac{d}{dt}\int_{\Gamma} \left(\frac{\gamma}{2}|\vec{v}|^2 + \mu(B - \kappa)^2\right) ds = 0.$$

Let $L$ be the total arc length. We take the rescaling on the equations

$$\begin{aligned}
&s \to \frac{1}{L} s, \quad t \to t\sqrt{\frac{\mu}{L^4 \gamma}}, \vec{R} \to \frac{1}{L}\vec{R},\\
&T \to \frac{L^2}{\mu} T, \quad B \to LB, \quad \vec{f} \to \frac{L^3}{\mu}\vec{f}.
\end{aligned}$$

Using the subscript $t,s$ to express the time and spatial derivative, the model equations are

$$\vec{R}_{tt}=\vec{f}+(T\vec{a})_s+((B-\kappa)_s\vec{n})_s, \tag{2.3}$$

$$v_s^1-\kappa v^{(n)}=0, s\in[0,1). \tag{2.4}$$

Equation (2.3) is a nonlinear wave equation, which involves the second order time-derivative and the fourth-order spatial derivative of $\vec{R}$. Equation (2.4) is the constraint equation of the membrane incompressibility.

In [8, 10, 13], the arc length parameter and the tangent angle of the curve was introduced to remove the stiffness from interfacial flows. In [17], based on a similar idea, equations (2.3)–(2.4) are changed into the equations of tangent angle and simulated by the tangent angle equations. In this description, the vector equation in equation (2.4) reduces to a single scalar equation, and the incompressible equation is replaced by a second order elliptic equation of the in-plane stress $T$. We set $\vec{a}=\vec{R}_s=(\cos\alpha(s,t),\sin\alpha(s,t))$ and $\vec{n}=(\sin\alpha(s,t),-\cos\alpha(s,t))$, where $\alpha$ is the angle between the curve's tangent vector and the positive direction of the $x$-axis. The tangent angle equation is

$$\alpha_{tt}=g_1+2T_s\alpha_s+T\alpha_{ss}-(B+\alpha_s)_{sss}+\alpha_s^2(B+\alpha_s)_s, \tag{2.5}$$

$$-T_{ss}+\alpha_s^2T=g_2+\alpha_t^2+2(B+\alpha_s)_{ss}\alpha_s+(B+\alpha_s)_s\alpha_{ss}, \tag{2.6}$$

where $g_1=-\vec{f}_s\cdot\vec{n}$ and $g_2=\vec{f}_s\cdot\vec{a}$ are the normal and tangent components of the external force. In return, the position of the curve $\vec{R}(s,t)$ can be determined by the tangent angle and the center of gravity

$$\frac{\partial\vec{R}}{\partial s}=\vec{a},$$

$$\int_\Gamma\vec{R}(s,t)ds=\int_0^t\int_0^{t'}\int_\Gamma\vec{f}(s,\tau)dsd\tau dt'+t\int_\Gamma\vec{v}^0ds+\int_\Gamma\vec{R}^0(s)ds.$$

This suggests that the equation set (2.5)–(2.6) is equivalent to the equation set (2.3)–(2.4). Let $P=T+(\alpha_s)^2$ and $u=\alpha_t$, then (2.5–2.6) can be written as

$$\alpha_t=u, \tag{2.7}$$

$$u_t=2P_s\alpha_s+P\alpha_{ss}-4(\alpha_s)^2\alpha_{ss}-\alpha_{ssss}+h_1, \tag{2.8}$$

$$-P_{ss}+\alpha_s^2P=\alpha_s^4-\alpha_{ss}^2+u^2+h_2, \tag{2.9}$$

where $h_1=g_1-B_{sss}+B_s(\alpha_s)^2$ and $h_2=g_2+2B_{ss}\alpha_s+B_s\alpha_{ss}$. The boundary condition is $\alpha(1,t)=\alpha(0,t)+2\pi$, and $P,\alpha_s$ are periodic. The initial condition is

$$\alpha(s,0)=\alpha^0(s), u(s,0)=u^0(s).$$

Equation (2.8) is a nonlinear wave equation for the angle $\alpha$ and equation (2.9) is an elliptic equation for $P$.

Next, we prove the local existence and uniqueness of the equation set (2.7)–(2.9). First, we design an implicit numerical scheme for these equations. Then we prove that, as the time step tends to zero, the numerical solution converges and the limit is the solution of the equations. In this paper, $L^p(\Omega)$, $W^{\alpha,p}(\Omega)$, and $H^\alpha(\Omega)$ are the usual Sobolev Spaces, and $\|\cdot\|_{L^p}$, $\|\cdot\|_{W^{\alpha,p}}$ and $\|\cdot\|_{H^\alpha}$ are the corresponding norms.

## 3. Basic lemmas

Before we begin to introduce the discrete numerical scheme, we introduce two lemmas, which are the base of the discrete scheme. Lemma 3.1 is an $L^1$ theory of a linear elliptic equation. This may not be included in classic results, so we give a short proof here. For simplicity, we use $C$ to denote the constants which depend only on the spaces but not on the functions. These constants may be different.

LEMMA 3.1. *If $\alpha(s)\in W^{2,1}[0,1]$, $f(s)\in L^1[0,1]$, and $\alpha(1)=\alpha(0)+2\pi$, the equation*

$$-Q_{ss}+(\alpha_s)^2Q=f,$$

*with the boundary condition*

$$\begin{aligned}Q(0)&=Q(1),\\ Q_s(0)&=Q_s(1),\end{aligned}$$

*has an unique solution $Q\in W^{2,1}[0,1]$. Moreover, there exists a constant $C$, such that*

$$\|Q\|_{W^{2,1}}\le C\left(1+\|\alpha\|^2_{W^{2,1}}\right)\|f\|_{L^1}.$$

*If $\alpha(s)\in C^{l+1}(0,1)$, $l\geqslant 0$, $\alpha(1)=\alpha(0)+2\pi$, and $f\in W^{l,1}(0,1)$, we have $\|Q\|_{W^{2+l,1}}\le C\left(1+\|\alpha_s\|^2_{C^{1+l}}\right)\|f\|_{W^{l,1}}$.*

*Proof.* First, we consider the equation

$$\begin{aligned}&-U_{ss}=f-\int_0^1 f(\tau)\,d\tau,\\ &U(0)=U(1),\\ &U_s(0)=U_s(1).\end{aligned}$$

It has a solution

$$U(s)=-\int_0^s\int_0^{s'}f(\tau)\,d\tau ds'+s\int_0^1\int_0^{s'}f(\tau)\,d\tau ds'+\frac{1}{2}s(s-1)\int_0^1 f(\tau)\,d\tau$$

which satisfies $U(0)=U(1)=0$. We have the inequality

$$\|U(s)\|_{W^{2,1}}\le C\|f(s)\|_{L^1}.$$

Let $V=Q-U$; the equation of $V$ is

$$\begin{aligned}&-V_{ss}+(\alpha_s)^2V=-(\alpha_s)^2U+\int_0^1 f(\tau)\,d\tau,\\ &V(0)=V(1),\\ &V_s(0)=V_s(1).\end{aligned}\tag{3.1}$$

By applying the embedding theorem $W^{2,1}(0,1)\hookrightarrow W^{1,4}(0,1)\hookrightarrow L^\infty(0,1)$, we have

$$\begin{aligned}&\left\|-(\alpha_s)^2U+\int_0^1 f(\tau)\,d\tau\right\|_{L^2}\\ &\le C\left(\left\|(\alpha_s)^2\right\|_{L^2}\|U\|_{L^\infty}+\|f\|_{L^1}\right)\\ &\le C\left(\|\alpha\|^2_{W^{1,4}}\|U\|_{W^{2,1}}+\|f\|_{L^1}\right)\\ &\le C\left(\|\alpha\|^2_{W^{2,1}}\|U\|_{W^{2,1}}+\|f\|_{L^1}\right)\\ &\le C\left(1+\|\alpha\|^2_{W^{2,1}}\right)\|f\|_{L^1}.\end{aligned}$$

If the first (smallest) eigenvalue $\lambda$ of equation (3.1) is positive, we can find an unique solution of equation (3.1) by the theory of second order elliptic equations and the embedding theorem $H^2(0,1)\hookrightarrow W^{2,1}(0,1)$ with the following estimation

$$\|V\|_{W^{2,1}}\leq C\|V\|_{H^2}\leq C\left\|-(\alpha_s)^2U+\int_0^1 f(\tau)\,d\tau\right\|_{L^2}\leq C\left(1+\|\alpha\|_{W^{2,1}}^2\right)\|f\|_{L^1}.$$

Thus there exists an unique solution of $Q$ which satisfies

$$\|Q\|_{W^{2,1}}\leq\|V\|_{W^{2,1}}+\|U\|_{W^{2,1}}\leq C\left(1+\|\alpha\|_{W^{2,1}}^2\right)\|f\|_{L^1}.$$

Next we make and estimate the first eigenvalue $\lambda$ of equation (3.1). $\lambda$ is given by

$$\lambda=\inf_{\|V\|_{L^2}=1}\int_0^1\left((V_s)^2+(\alpha_s)^2V^2\right)ds.$$

For $V\in C^0(0,1)$ and $\|V\|_{L^2}=1$, we define $M=\max_{s\in[0,1]}V(s)$, $m=\min_{s\in[0,1]}V(s)$. If $M\geq 0\geq m$, there exists an $s_0\in[0,1]$ such that $V(s_0)=0$. Note that the first eigenvalue of the following problem is $\pi^2$

$$\begin{aligned}-V_{ss}&=\lambda V,\\ V(s_0)=V(s_0+1)&=0.\end{aligned}$$

Therefore, we have

$$\int_0^1\left((V_s)^2+(\alpha_s)^2V^2\right)ds\geq\int_0^1(V_s)^2\,ds\geq\pi^2.$$

If $M\geq m>0$, there exists $s_0\in[0,1]$ such that $V(s_0)-m=0$. Similarly, we estimate

$$\int_0^1(V_s)^2\,ds\geq\pi^2\int_0^1(V-m)^2\,ds.$$

At the same time, the boundary condition $\alpha(1)=\alpha(0)+2\pi$ leads us to the estimation $\int_0^1(\alpha_s)^2\,ds\geq 4\pi^2$. Thus we have

$$\int_0^1(\alpha_s)^2V^2\,ds\geq m^2\int_0^1(\alpha_s)^2\,ds\geq 4\pi^2\int_0^1 m^2\,ds,$$

and

$$\begin{aligned}\int_0^1\left((V_s)^2+(\alpha_s)^2V^2\right)ds&\geq\pi^2\int_0^1\left((V-m)^2+4m^2\right)ds\\ &=\frac{\pi^2}{2}\int_0^1\left(V^2+(V-2m)^2+6m^2\right)ds\\ &\geq\frac{\pi^2}{2}.\end{aligned}$$

Therefore the first (smallest) eigenvalue $\lambda$ of equation (3.1) is always bigger than $\frac{\pi^2}{2}$. $\square$

Lemma 3.2 is an $L^2$ result of a constant-coefficient linear elliptic equation. Since it can be easily obtained by the Fourier transform, we list it without proof.

LEMMA 3.2. *If $f \in H^{-2+l}(0,1)$, there exists an unique solution $\alpha \in H^{2+l}$ of the equation*

$$\alpha + \mu^2 \alpha_{ssss} = f$$

*with periodic boundary condition (or the boundary condition $\alpha(1) = \alpha(0) + 2\pi$, and $\alpha_s$ is periodic).*

## 4. Discrete scheme and discrete solution

In this section, we construct an implicit discrete scheme for (2.7)–(2.9) and show the energy properties of the discrete solution.

Let $\Delta t > 0$ be the time step, $t_n = n\Delta t$, $n = 0,1,2,\ldots$. The superscript $n$ means the value of the function on time $t_n$, i.e., $\alpha^n(s) = \alpha(s,t_n)$. At each time step $t_n$, if $\alpha^n$ and $u^n$ are known, we have the value of $h_1^n$, $h_2^n$ and solve $P^n$ by equation (2.9). Using $\alpha^n, u^n$, and $P^n$ to take the place of the nonlinear terms in equation (2.8), we discretize the time derivatives and obtain $\alpha^{n+1}$ and $u^{n+1}$ as the following:

$$-P_{ss}^n + (\alpha_s^n)^2 P^n = (\alpha_s^n)^4 - (\alpha_{ss}^n)^2 + (u^n)^2 + h_2^n, \tag{4.1}$$

$$\alpha^{n+1} = \alpha^n + u^{n+1}\Delta t, \tag{4.2}$$

$$u^{n+1} = u^n + \left(2P_s^n \alpha_s^n + P^n \alpha_{ss}^n - 4(\alpha_s^n)^2 \alpha_{ss}^n - \alpha_{ssss}^{n+1} + h_1^n\right)\Delta t. \tag{4.3}$$

Equations (4.2) and (4.3) are equivalent to a decoupled form

$$\alpha^{n+1} + \alpha_{ssss}^{n+1}\Delta t^2 = \alpha^n + u^n \Delta t + \left(2P_s^n \alpha_s^n + P^n \alpha_{ss}^n - 4(\alpha_s^n)^2 \alpha_{ss}^n + h_1^n\right)\Delta t^2, \tag{4.4}$$

$$u^{n+1} + u_{ssss}^{n+1}\Delta t^2 = u^n + \left(2P_s^n \alpha_s^n + P^n \alpha_{ss}^n - 4(\alpha_s^n)^2 \alpha_{ss}^n - \alpha_{ssss}^n + h_1^n\right)\Delta t. \tag{4.5}$$

Therefore, at each time step, we need only to solve three linear equations (4.1), (4.4)–(4.5). The existence and uniqueness of their solutions can be obtained by Lemma 3.1 and Lemma 3.2. Next, we give the energy estimate of the discrete system.

LEMMA 4.1. *If the spontaneous curvature $B(s) \in W^{2,1}(0,1)$, $B_{sss} \in H^{-2}(0,1)$, the external force $\vec{f}^n(s) \in W^{1,1}(0,1)$, and $\alpha^n \in H^2(0,1)$, $u^n \in L^2(0,1)$, then the functions at the next time step $t_{n+1}$ obtained from (4.1), (4.4)–(4.5) satisfy $\alpha^{n+1} \in H^2(0,1)$, $u^{n+1} \in L^2(0,1)$, $P^n \in W^{2,1}(0,1)$. More accurately, there exists a polynomial $p(\cdot)$ with non-negative coefficients which depend only on the norms of $B(s)$ and $\vec{f}^n(s)$, such that*

$$E_2^{n+1} \le E_2^n + \Delta t p(E_2^n), \tag{4.6}$$

*where*

$$E_2^n = \int_0^1 \left((\alpha_{ss}^n)^2 + (u^n)^2\right) ds. \tag{4.7}$$

*Proof.* From the given conditions, we have $\alpha_s^n \in C^0(0,1)$, $g_1^n, g_2^n \in L^1(0,1) \subset H^{-1}(0,1)$. $h_1^n$ and $h_2^n$ are given as

$$h_1^n = g_1^n - B_{sss} + B_s(\alpha_s^n)^2,$$
$$h_2^n = g_2^n + 2B_{ss}\alpha_s^n + B_s \alpha_{ss}^n,$$

thus we have $h_1^n \in H^{-2}(0,1)$, $h_2^n \in L^1(0,1)$, and $\|h_1^n\|_{H^{-2}} \le c_1 + c_2\|\alpha^n\|_{H^2}^2$, $\|h_2^n\|_{L^1} \le c_3 + c_4\|\alpha^n\|_{H^2}$, where $c_i$ depend on the norms of $B$ and $\vec{f}^n$. Therefore, the right hand side of equation (4.1) belongs to $L^1(0,1)$. From Lemma 3.1, we obtain $P^n \in W^{2,1}(0,1)$ which satisfies

$$\begin{aligned}\|P^n\|_{W^{2,1}} &\le C\left(1+\|\alpha^n\|_{W^{2,1}}^2\right)\left(1+\|\alpha^n\|_{W^{1,4}}^4+\|\alpha^n\|_{H^2}^2+\|u^n\|_{L^2}^2\right)\\ &\le p^{(3)}(E_2^n),\end{aligned}$$

where $p^{(3)}(\cdot)$ is a 3-order polynomial with non-negative coefficient. The embedding theory indicates that $\|P^n\|_{L^\infty} \le C\|P^n\|_{W^{2,1}}$ and $\|P_s^n\|_{L^\infty} \le C\|P^n\|_{W^{2,1}}$. Thus the right hand side of equation (4.4) and (4.5) are $H^{-2}(0,1)$ functions. By Lemma 3.2, there exist solutions $\alpha^{n+1} \in H^4(0,1)$ and $u^{n+1} \in H^2(0,1)$. Moreover, we have the inequality

$$\begin{aligned}E_2^{n+1} &= \int_0^1 \left(\left(\alpha_{ss}^{n+1}\right)^2 + \left(u^{n+1}\right)^2\right) ds\\ &\le \int_0^1 \left[\left(\alpha_{ss}^{n+1}\right)^2 + \left(u^{n+1}\right)^2 + \Delta t^2\left(\left(u_{ss}^{n+1}\right)^2 + \left(\alpha_{ssss}^{n+1}\right)^2\right)\right] ds\\ &= \int_0^1 \left(\left(\alpha_{ss}^{n+1} - u_{ss}^{n+1}\Delta t\right)^2 + \left(u^{n+1} + \alpha_{ssss}^{n+1}\Delta t\right)^2\right) ds\\ &= \int_0^1 \left[(\alpha_{ss}^n)^2 + \left(u^n + \left(2P_s^n\alpha_s^n + P^n\alpha_{ss}^n - 4(\alpha_s^n)^2\alpha_{ss}^n + h_1^n\right)\Delta t\right)^2\right] ds\\ &\le E_2^n + 2\Delta t\int_0^1 \left|u^n\left(2P_s^n\alpha_s^n + P^n\alpha_{ss}^n - 4(\alpha_s^n)^2\alpha_{ss}^n + h_1^n\right)\right| ds\\ &\quad + \Delta t^2\int_0^1 \left|\left(2P_s^n\alpha_s^n + P^n\alpha_{ss} - 4(\alpha_s^n)^2\alpha_{ss}^n + h_1^n\right)\right|^2 ds\\ &\le E_2^n + C\Delta t p(E_2^n),\end{aligned}$$

where $p(\cdot)$ is a polynomial with non-negative coefficient. □

From the proof, we can see that the nonlinear terms are controlled by the fourth-order spatial derivatives, which comes from the bending resistance. We will see that this is sufficient to make the system well-posed locally.

LEMMA 4.2. *$\{a^n\}_{n=0}^\infty$ is a non-negative sequence, which satisfies*

$$a^n \le a^{n-1} + \Delta t p(a^{n-1}), \tag{4.8}$$

*where $\Delta t > 0$ and $p(\cdot)$ is a polynomial with non-negative coefficients. Then there exists a constant $T^*$ which is independent of $\Delta t$ such that $a^n$ has a uniform bound for all $n\Delta t < T^*$.*

*Proof.* We prove this lemma by mathematical induction. Assume $p(x) = xq(x) + c$, where $q(x)$ is also a polynomial with non-negative coefficient and $c \ge 0$ is a constant. By the inequality (4.8), we have

$$a^n \le a^{n-1}(1 + \Delta t q(a^{n-1})) + c\Delta t.$$

Let $M > a^0$. Assume $a^k \le M$ for all $k < n$ ($n\Delta t < T^*$), where

$$T^* = \frac{1}{q(M)}\log\left(\frac{M + c/q(M)}{a^0 + c/q(M)}\right),$$

then we have

$$a^n \le a^{n-1}(1+\Delta t q(M)) + c\Delta t,$$

which is equivalent to

$$a^n + \frac{c}{q(M)} \le \left(a^{n-1} + \frac{c}{q(M)}\right)(1+\Delta t q(M)).$$

Thus we obtain

$$\begin{aligned} a^n + \frac{c}{q(M)} &\le \left(a^0 + \frac{c}{q(M)}\right)(1+\Delta t q(M))^n \\ &\le \left(a^0 + \frac{c}{q(M)}\right)(1+\Delta t q(M))^{\frac{T^*}{\Delta t}} \\ &\le \left(a^0 + \frac{c}{q(M)}\right)\exp\{T^* q(M)\}, \end{aligned}$$

and

$$a^n \le \left(a^0 + \frac{c}{q(M)}\right)\exp\{T^* q(M)\} - \frac{c}{q(M)} = M.$$

Therefore, $a^n$ has a uniform bound for all $n\Delta t < T^*$. □

Based on Lemma 4.1 and Lemma 4.2, we obtain the following lemma.

LEMMA 4.3. *If the conditions of Lemma 4.1 are satisfied, the initial data $\alpha^0 \in H^2(0,1)$ and $u^0 \in L^2(0,1)$, then there exists a constant $T^*$ which is independent on $\Delta t$, such that $E_2^n$ has a uniform bound for all $n\Delta t < T^*$.*

Using the discrete solutions at $t_n$, we define the functions $\alpha_{\Delta t}(s,t)$, $u_{\Delta t}(s,t)$, and $P_{\Delta t}(s,t)$ in $[0,1]\times[0,T^*]$ as

$$\begin{aligned} &\alpha_{\Delta t}(s,t) = \lambda\alpha^n(s) + (1-\lambda)\alpha^{n+1}(s),\ t\in[n\Delta t,(n+1)\Delta t), \\ &u_{\Delta t}(s,t) = \lambda u^n(s) + (1-\lambda)u^{n+1}(s),\ t\in[n\Delta t,(n+1)\Delta t), \\ &P_{\Delta t}(s,t) = \lambda P^n(s) + (1-\lambda)P^{n+1}(s),\ t\in[n\Delta t,(n+1)\Delta t), \\ &\qquad \lambda = \frac{(n+1)\Delta t - t}{\Delta t}. \end{aligned}$$

Obviously, we have $\alpha_{\Delta t} \in L^\infty\left([0,T^*],H^2(0,1)\right)$, $u_{\Delta t} \in L^\infty\left([0,T^*],L^2(0,1)\right)$, and $P_{\Delta t} \in L^\infty\left([0,T^*],W^{2,1}(0,1)\right)$.

Similar to Lemma 4.3, if there are higher-order regularities for the conditions we have the following lemma.

LEMMA 4.4. *Define $E_k^n$ as*

$$E_k^n = \int_0^1 \left[\left((\alpha^n)^{(k)}\right)^2 + \left((u^n)^{(k)}\right)^2\right] ds. \tag{4.9}$$

*Supposing the spontaneous curvature $B(s) \in W^{k,1}(0,1)$, $B_{sss} \in H^{k-4}(0,1)$, the external force $\vec{f}^n(s) \in W^{k-1,1}(0,1)$, and $\alpha^n \in H^k(0,1)$, $u^n \in H^{k-2}(0,1)$, for $k \ge 2$, then there exists a polynomial $p_k(\cdot)$ with non-negative coefficients such that*

$$E_k^{n+1} \le E_k^n + \Delta t p_k(E_k^n). \tag{4.10}$$

*Moreover, if the initial data* $\alpha^0 \in H^k(0,1)$ *and* $u^0 \in H^{k-2}(0,1)$*, there exists a constant* $T^{*k}$ *such that* $\alpha_{\Delta t}(s,t) \in L^\infty\left([0,T^{*k}],H^k(0,1)\right)$*,* $u_{\Delta t} \in L^\infty\left([0,T^{*k}],H^{k-2}(0,1)\right)$*, and* $P_{\Delta t} \in L^\infty\left([0,T^{*k}],W^{k,1}(0,1)\right)$*.*

To sum up, we have gotten the energy estimates for a series of energies $E_k^n$. All the energy can be uniformly bounded for small $t$. These properties are intrinsically based on the bending resistance which controls the nonlinear terms in the equation.

## 5. Local existence and uniqueness of the solution

Based on the above Lemmas, we are able to prove the local existence of the solution of (2.7)–(2.9).

THEOREM 5.1. *If* $B(s) \in W^{k,1}(0,1)$*,* $B_{sss} \in H^{k-4}(0,1)$*,* $\vec{f}(s,t) \in L^\infty\left([0,T],W^{k-1,1}(0,1)\right)$*,* $\alpha^0 \in H^k(0,1)$*, and* $u^0 \in H^{k-2}(0,1)\,(k \geq 2)$*, there exists a constant* $T^{*k}$*,* $0 < T^{*k} < T$*, such that (2.7)–(2.9) has a set of solutions* $(\alpha^*,u^*,P^*)$*, and* $\alpha^* \in L^\infty\left([0,T^{*k}],H^k(0,1)\right)$*,* $u^* \in L^\infty\left([0,T^{*k}],H^{k-2}(0,1)\right)$*,* $P^* \in L^\infty\left([0,T^{*k}],W^{k,1}(0,1)\right)$*.*

*Proof.* Since the conditions of Lemma 4.4 are satisfied, there exists a constant $T^{*k}$ independent on $\Delta t$ and $0 < T^{*k} < T$, such that $\alpha_{\Delta t}(s,t) \in L^\infty\left([0,T^{*k}],H^k(0,1)\right)$ and $u_{\Delta t} \in L^\infty\left([0,T^{*k}],H^{k-2}(0,1)\right)$.

On the other hand, we also have

$$\left(\alpha^{n+1}-\alpha^n\right) = \left(\alpha^n-\alpha^{n-1}\right) + \left(u^{n+1}-u^n\right)\Delta t, \tag{5.1}$$

$$\begin{aligned}\left(u^{n+1}-u^n\right) &= \left(u^n-u^{n-1}\right) - \left(\alpha_{ssss}^{n+1}-\alpha_{ssss}^n\right)\Delta t + 4\Delta t\left(\left(\alpha_s^n\right)^2\alpha_{ss}^n - \left(\alpha_s^{n-1}\right)^2\alpha_{ss}^{n-1}\right)\\ &\quad + \Delta t\left(2P_s^n\alpha_s^n + P^n\alpha_{ss}^n - 2P_s^{n-1}\alpha_s^{n-1} - P^{n-1}\alpha_{ss}^{n-1}\right),\end{aligned} \tag{5.2}$$

$$\begin{aligned}&-\left(P^n-P^{n-1}\right)_{ss} + \left(\alpha_s^n\right)^2\left(P^n-P^{n-1}\right)\\ &\quad = -P^{n-1}\left(\left(\alpha_s^n\right)^2 - \left(\alpha_s^{n-1}\right)^2\right) + \left(\left(\alpha_s^n\right)^4 - \left(\alpha_s^{n-1}\right)^4\right)\\ &\qquad - \left(\left(\alpha_{ss}^n\right)^2 - \left(\alpha_{ss}^{n-1}\right)^2\right) + \left(\left(u^n\right)^2 - \left(u^{n-1}\right)^2\right).\end{aligned} \tag{5.3}$$

It is a set of linear equation $\left(\alpha^{n+1}-\alpha^n\right)$, $\left(u^{n+1}-u^n\right)$, and $\left(P^n-P^{n-1}\right)$. Similar to the above analysis, we can obtain the inequality

$$D_k^n \leq D_k^{n-1} + C\Delta t D_k^{n-1},$$

where $C$ depends on $B(s)$, $\vec{f}^n(s)$, and $E_k^n$, and

$$\begin{aligned}D_k^n &= \int_0^1 \left[\left(\frac{\left(\alpha^{n+1}\right)^{(k)} - \left(\alpha^n\right)^{(k)}}{\Delta t}\right)^2 + \left(\frac{\left(u^{n+1}\right)^{(k-2)} - \left(u^n\right)^{(k-2)}}{\Delta t}\right)^2\right] ds\\ &= \int_0^1 \left[\left(\frac{\partial \alpha_{\Delta t}^{(k)}}{\partial t}\right)^2 + \left(\frac{\partial u_{\Delta t}^{(k-2)}}{\partial t}\right)^2\right] ds.\end{aligned}$$

Thus there is a uniform bound of $D_k^n$ for all $n\Delta t < T^{*k}$, this indicates that $\alpha_{\Delta t} \in W^{1,\infty}\left([0,T^{*k}],H^k(0,1)\right)$, $u_{\Delta t} \in W^{1,\infty}\left([0,T^{*k}],H^{k-2}(0,1)\right)$ and the norms of $\alpha_{\Delta t}$ and $u_{\Delta t}$ depend on $B(s)$, $f(s)$, and the initial conditions continuously.

By the compactness theory, there exists a subsequence $\{\Delta t_i\}_i$ and the functions $\alpha^*(s,t)\in L^\infty\left([0,T^{*k}],H^k(0,1)\right)$, $u^*\in L^\infty\left([0,T^{*k}],H^{k-2}(0,1)\right)$, $P^*\in L^\infty\left([0,T^{*k}],W^{k,1}(0,1)\right)$ such that

$$\lim_{i\to\infty}\Delta t_i=0+,\qquad \lim_{i\to\infty}\alpha_{\Delta t_i}=\alpha^*,$$
$$\lim_{i\to\infty}u_{\Delta t_i}=u^*,\qquad \lim_{i\to\infty}P_{\Delta t_i}=P^*,$$

and $(\alpha^*,u^*,P^*)$ satisfies (2.7)–(2.9). □

THEOREM 5.2. *If the conditions in Theorem 5.1 are satisfied, the solution to (2.7)–(2.9) is unique.*

*Proof.* Suppose there are two solutions $(\alpha_1,u_1,P_1)$ and $(\alpha_2,u_2,P_2)$ of (2.7)–(2.9). Define $\beta=\alpha_1-\alpha_2$, $w=u_1-u_2$, and $R=P_1-P_2$. The equations of $(\beta,w,R)$ are

$$\beta_t=w, \tag{5.4}$$
$$\begin{aligned} w_t=&2P_{1s}\beta_s+2\alpha_{2s}R_s+P_1\beta_{ss}+\alpha_{2ss}R-4(\alpha_{1s}\beta_{ss}+\alpha_{2ss}(\alpha_{1s}+\alpha_{2s})\beta_s)\\ &-\beta_{ssss}-B_s(\alpha_{1s}+\alpha_{2s})\beta_s+f_1\beta, \end{aligned} \tag{5.5}$$
$$\begin{aligned} -R_{ss}+\alpha_{1s}^2R=&-P_2(\alpha_{1s}+\alpha_{2s})\beta_s+\left(\alpha_{1s}^3+\alpha_{1s}^2\alpha_{2s}+\alpha_{1s}\alpha_{2s}^2+\alpha_{2s}^3\right)\beta_s\\ &-(\alpha_{1ss}+\alpha_{2ss})\beta_{ss}-(u_1+u_2)w-2B_{ss}\beta_s-B_s\beta_{ss}+f_2\beta, \end{aligned} \tag{5.6}$$

where

$$f_1=\begin{cases}\vec{f}\cdot\frac{1}{\beta}(\sin\alpha_1-\sin\alpha_2,\cos\alpha_2-\cos\alpha_1), & \beta\neq0,\\ \vec{f}\cdot(\cos\alpha_1,\sin\alpha_1), & \beta=0,\end{cases}$$
$$f_2=\begin{cases}\vec{f}\cdot\frac{1}{\beta}(\cos\alpha_1-\cos\alpha_2,\sin\alpha_1-\sin\alpha_2), & \beta\neq0,\\ \vec{f}\cdot(-\sin\alpha_1,\cos\alpha_1), & \beta=0,\end{cases}$$

with the periodic boundary condition and the initial condition

$$\beta(s,0)=0,\ w(s,0)=0.$$

These equations are linear for $(\beta,w,R)$. From equation (5.6) and Lemma 3.2, we have the estimation of $R$

$$\|R\|_{H^2}\le C(\|\beta\|_{H^2}+\|w\|_{L^2}),$$

where $C$ depends on $(\alpha_1,u_1,P_1)$ and $(\alpha_2,u_2,P_2)$. Since equations (5.4) and (5.5) are also linear for $(\beta,w,R)$, we obtain

$$\frac{d}{dt}E(t)\le CE(t),\ \text{where } E=\|\beta\|_{H^2}+\|w\|_{L^2}.$$

By the Gronwall Inequality, we obtain

$$E(t)\le E(0)e^{Ct}.$$

Since initially we have $E(0)=0$, $E(t)$ is always zero, which means $\beta(s,t)\equiv0$ and $w\equiv0$. Since the right hand side of equation (5.6) also vanishes, we have $R(s,t)\equiv0$. Therefore, we have $(\alpha_1,u_1,P_1)=(\alpha_2,u_2,P_2)$. □

## 6. Conclusions

A detailed well-posedness analysis for the elastic surface model of the incompressible fluid membrane in a two-dimensional space is carried out in this paper. In general, the model equations contain three parts: the evolution of the geometric quantities, the fluid on the membrane, and the constraint of the incompressibility. In a two-dimensional space, the membrane fluid vanishes and the coordinate system can be simply selected to be the arc length parameter. Using the tangent angle, we reduce the dynamic equations to a wave equation and an elliptic equation. By constructing a suitable discrete scheme for the two equations, we proved the local existence and uniqueness of the solutions.

Generally, nonlinearities may result in blow-up of dynamical systems. For a general fluid membrane, the nonlinearity includes the geometric nonlinearity due to the bending of the surface and the fluid nonlinearity due to the convection of the membrane fluid. In a two dimensional space, the dependence of the geometry on the membrane fluid vanishes, which allows us to consider the geometric nonlinearity separately. From the proof, we can see that the risk of blowing up due to the geometric nonlinearity is overcome by the bending elasticity. This should also be the case in a three-dimensional space. If the fluid nonlinearity can be treated similar to the 2-dimensional Navier-Stokes equation, we may obtain the well-posedness for the dynamic equations of a general fluid membrane. For such a system, there are still more difficulties: 1. the two nonlinearities is coupled; 2. we don't have an efficient coordinate system; 3. we have to treat more equations.

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REFERENCES

[1] B. Alberts, etc. *Essential Cell Biology*, 2nd Edition, Garland Science/Taylor & Francis Group, 2003.
[2] R. Aris, *Vectors, Tensors and the Basic Equations of Fluid Mechanics*, Courier Dover Publications, 1990.
[3] W. Cai and T. Lubensky, *Hydrodynamics and dynamic fluctuations of fluid membranes*, Phys. Rev. E, 4251, 1995.
[4] H. Deuling and W. Helfrich, *The curvature elasticity of fluid membranes: a catalogue of vesicle shapes*, Journal de Physique. Tome, 37, 1335–1345, 1976.
[5] Y. Fu, and R. Ogden, etc. *Nonlinear Elasticity: Theory and Applications*, Cambridge University Press, 2001.
[6] J.H. Heinbockel, *Introduction to Tensor Calculus and Continuum Mechanics*, Trafford Publishing, 2001.
[7] W. Helfrich, *Elastic properties of lipid bilayers: theory and possible experiments*, Z. Naturforsch. Teil C, 28, 693–703, 1973.
[8] T. Hou, J. Lowengrub and M. Shelley, *Removing the stiffness from interfacial flows with surface tension*, J. Comput. Phys., 114, 312–338, 1994.
[9] D. Hu, P. Zhang and W. E, *Continuum theory of a moving membrane*, Phys. Rev. E, 75, 4, 2007.
[10] X. Li and Q. Nie, *Surface diffusion on stressed solid surface*, Commun. Comput. Phys., 2(1), 73–86, 2007.
[11] M. Lomholt, P.L. Hansen and L. Miao, *A general theory of non-equilibrium dynamics of lipid-protein fluid membranes*, European Physical Journal E, 16, 439, 2005.
[12] L. Miao, M. Lomholt and J. Kleis, *Dynamics of shape fluctuations of quas-spherical vesicles revisited*, European Physical Journal E, 9, 143, 2002.
[13] Q. Nie, *The nonlinear evolution of vortex sheets with surface tension in axi-symmetric flows*, J. Comput. Phys., 174, 438–459, 2001.
[14] C. Pozrikidis, *Interfacial dynamics for Stokes flow*, J. Comput. Phys., 169, 250–301, 2001.
[15] L. Scriven, *Dynamics of a fluid interface*, Chem. Eng. Sci., 12–98, 1960.

[16] U. Seifert, *Fluid membranes in hydrodynamic flow fields: formalism and an application to fluctuating quasispherical vesicles in shear flow*, European Physical Journal B, 8, 305, 1999.

[17] P. Song, D. Hu and P. Zhang, *Numerical simulation of fluid membranes in two-dimensional space*, Commun. Comput. Phys., 3, 794–821, 2008.

[18] D. Steigmann, *Fluid films with curvature elasticity*, Arch. Rat. Mech. Anal., 150, 127–152, 1999.

[19] A. Waxman, *Dynamics of a couple-stress fluid membrane*, Studies in Applied Mathematics, 70, 63–86, 1984.

[20] A. Waxman, *A corotational time derivative for surface tensors, constitutive relations and a new measure of bending strain*, J. Non-Newtonian Fluid Mech., 9–235, 1981.