Journal of Computational Mathematics, Vol.26, No.3, 2008, 347–364.

REFLECTION/TRANSMISSION CHARACTERISTICS OF A DISCONTINUOUS GALERKIN METHOD FOR MAXWELL'S EQUATIONS IN DISPERSIVE INHOMOGENEOUS MEDIA*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper, we analyze the transmission and reflection properties of a high order discontinuous Galerkin method for dispersive Maxwell's equations, originally proposed by Lu et al. [J. Comput. Phys. **200** (2004), pp. 549-580]. We study the reflection and transmission properties of the numerical method for up to second-order polynomial elements for one- and two-dimensional Maxwell's equations with rectangular meshes. High order accuracy has been shown for reflection and transmission coefficients near material interfaces.

Mathematics subject classification: 65N50, 65N15, 35A40, 35J05.
Key words: Discontinuous Galerkin Method, Reflection, Transmission.

1. Introduction

Wave propagation in inhomogeneous and dispersive media can be found in many scientific and engineering applications such as evanescent waves in surface plasmons, and ground penetrating radar detection, and optical devices. In those situations, the need for accurate numerical modeling calls for continuing advances in the development of high order numerical algorithms. An important desirable feature is the capability of the numerical methods in predicting accurately the reflection and transmission of waves across material interfaces. For electromagnetic scattering in a dispersive media, the frequency dependent constitutive relation between displacement field D and the electric field E entails a time domain relationship via a time convolution. In [1, 2], the Auxilary Differential Equation (ADE) method is proposed to address this issue in the framework of discontinuous Galerkin methods for dispersive Maxwell's equations, and various applications of the resulting dispersive discontinuous Galerkin method have been conducted for the modeling of ground penetrating radar [3], resonant microcavity waveguide [4], and plasmon coupling of nanowires [2].

In this paper, we will elaborate the transmission and reflection properties of the above mentioned discontinuous Galerkin method for Maxwell's equation in a dispersive media with material interfaces. The reflection/transmssion near a material interface has a great effect on the quality of the simulation of wave propagation in a dispersive inhomogeneous media. Many physical phenomena involves waves near material interfaces, such as evanescent plasmon waves

* Received September 12, 2007 / Revised version received April 25, 2008 / Accepted April 28, 2008 /

near dielectric and metals, which decay exponentially away from interfaces, and diffractive optics with gratings, to just list a few. Numerical modeling of the waves near material interfaces requires high fidelity in replicating the reflection and transmission of waves near material interfaces and the resolution power of approximating exponentially decaying field distributions.

In this paper, we are mainly concerned with the reflection/transmission property of the discontinuous Galerkin method proposed in [1,2] for the dispersive Maxwell's equations. Such an analysis is important for selecting numerical algorithms for studying electromagnetic waves near material interface and in dispersive media such as soil and metals and even in artificial dispersive media of PML for truncating computational domain [5]. The rest of the paper is organized as follows. In Section 2, we give an introduction of dispersive Maxwell's equations. In Section 3, the analysis of reflection/transmission properties of discontinuous Galerkin method for 1-D Maxwell's equations is given. In Section 4, we investigate the reflection/transmission properties of discontinuous Galerkin method for 2-D Maxwell's equations with rectangular meshes. Section 5 contains the conclusion.

2. Dispersive Maxwell's Equations

Maxwell's equations are fundamental equations of electromagnetism, which are of the form

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \tag{2.1}$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{j}, \tag{2.2}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.4}$$

where $\mathbf{E}$ is the electric field and $\mathbf{B}$ the magnetic induction, $\mathbf{D}$ electric displacement, $\mathbf{H}$ the magnetic field, ρ the free charge density and $\mathbf{j}$ the free current density.

Eqs. (2.1)-(2.4) are not closed in themselves; they must be supplemented with constitutive relations

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}, \quad \mathbf{j} = \sigma\mathbf{E}, \tag{2.5}$$

where ϵ is the permittivity, μ is the permeability, and σ is the conductivity. For dispersive media, ϵ in (2.5) is a function of the frequency ω.

For a lossy and dispersive media, a typical single-pole Drude medium [6] has a relative frequency dependent electric permittivity as

$$\hat{\epsilon}_r(\omega) = \hat{\epsilon}_{r,\infty} - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}, \tag{2.6}$$

where ω_p is the plasma frequency, γ is the damping constant, $\hat{\epsilon}_{r,\infty}$ is the relative electric permittivity at infinite frequency. Fourier transforms are needed to get an expression in time domain.

In [1,4], a discontinuous Galerkin method employs ADE for auxiliary variable $\mathbf{J}$, which is introduced to handle the time convolution resulting from the frequency dependent constitutive relation (2.6)

$$\begin{aligned}
&\frac{\partial}{\partial t}\mathbf{E}(x,y,z,t) = \frac{1}{\epsilon_{r,\infty}}[\nabla\times\mathbf{H}(x,y,z,t) - \mathbf{J}(x,y,z,t)],\\
&\frac{\partial}{\partial t}\mathbf{H}(x,y,z,t) = -\nabla\times\mathbf{E}(x,y,z,t),\\
&\frac{\partial}{\partial t}\mathbf{J}(x,y,z,t) = -\gamma\mathbf{J}(x,y,z,t) + \omega_p^2\mathbf{E}(x,y,z,t).
\end{aligned} \tag{2.7}$$

Here we assume $j = 0, \rho = 0$, and the non-dimensionalized variables are introduced in the above equation

$$\frac{x}{L}\to x, \quad \frac{y}{L}\to y, \quad \frac{ct}{L}\to t, \quad Z_0\mathbf{H}\to\mathbf{H}, \quad \mathbf{E}\to\mathbf{E}, \tag{2.8}$$

where L is the reference length associated with a given problem, c is the speed of light in free space, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the free-space impedance.

3. Reflection/Transmission Properties of 1-D Maxwell's Equations

First, we consider an one-dimensional problem where only $E_y(x,t)$,$H_z(x,t)$, $J_y(x,t)$ are sought to satisfy the following equations

$$\frac{\partial E_y}{\partial t} = -\frac{1}{\epsilon_{r,\infty}}(\frac{\partial H_z}{\partial x} + J_y), \tag{3.1}$$

$$\frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}, \tag{3.2}$$

$$\frac{\partial J_y}{\partial t} = -\gamma J_y + \omega_p^2 E_y. \tag{3.3}$$

Assuming a time-harmonic form for the solutions

$$E_y = e(x)\exp(-i\omega t), \quad H_z = h(x)\exp(-i\omega t), \quad J_y = j(x)\exp(-i\omega t). \tag{3.4}$$

Using (3.1)-(3.3), an equation for $e(x)$ is obtained

$$e''(x) = \Big[\frac{-i\omega\omega_p^2}{-i\omega+\gamma} - \omega^2\epsilon_{r,\infty}\Big]e(x) = -\omega^2\epsilon_r(\omega)e(x). \tag{3.5}$$

The plane waves $\exp(ikx)$ is a solution of (3.5) if the wave number k satisfies the following dispersion relation

$$k^2 = -\Big[\frac{-i\omega\omega_p^2}{-i\omega+\gamma} - \omega^2\epsilon_{r,\infty}\Big] = \omega^2\epsilon_r(\omega). \tag{3.6}$$

Consider an interval $I = [-h, h]$, denote $x_L = -h$, $x_R = h$, all solution values at these two points will be given a subscript $_L$ or $_R$. Let $+$ denote the limit values taken from outside and $-$ from inside of I, respectively. For example,

$$f_L^+ = \lim_{\varepsilon\to 0} f(-h-\varepsilon), \quad f_R^- = \lim_{\varepsilon\to 0} f(h-\varepsilon).$$

The discontinuous Galerkin solution of (3.1)-(3.3) is to find $H_z, E_y, J_y \in P^k(I)$, such that for any $v_h(x) \in P^k(I)$ (k-th order polynomials defined on I),

$$\int_I \Big(\frac{\partial H_z}{\partial t} v_h - E_y v_h'\Big) dx + F_{H_z}\Big|_{x_R} v(x_R^-) + F_{H_z}\Big|_{x_L} v(x_L^-) = 0, \tag{3.7}$$

$$\int_I \Big(\epsilon_{r,\infty} \frac{\partial E_y}{\partial t} v_h + J_y v_h - H_z v_h'\Big) dx + F_{E_y}\Big|_{x_R} v(x_R^-) + F_{E_y}\Big|_{x_L} v(x_L^-) = 0, \tag{3.8}$$

$$\int_I \Big(\frac{\partial J_y}{\partial t} v_h - \omega_p^2 E_y v_h + \gamma J_y v_h\Big) dx = 0, \tag{3.9}$$

where the numerical flux F has the form [1, 7]

$$F_{H_z} = \frac{(Y n_x E_y + H_z)^- + (Y n_x E_y - H_z)^+}{Y^+ + Y^-},$$
$$F_{E_y} = n_x \frac{(Z H_z + n_x E_y)^- + (Z H_z - n_x E_y)^+}{Z^+ + Z^-}.$$

Here n_x is the outward unit normal to the interval. $Z^\pm$ and $Y^\pm$ are the local impedance and admittance respectively, defined as

$$Z^\pm = 1/Y^\pm = (\mu_r^\pm / \epsilon_r^\pm)^{1/2}.$$

For simplicity, we will assume that $Z = Y \equiv 1$ ($\epsilon_{r,\infty} = 1$) for the rest of this paper.

Now we study the reflection and transmission of discontinuous Galerkin method for a 1-D Drude media. Assuming $x < 0$ is the vacuum with $\epsilon_r = 1$, and $x \geq 0$ is the Drude media with $\epsilon_r(\omega) = 1 - \omega_p^2/(\omega^2 + i\gamma\omega)$.

An incident plane wave $E_y = \exp(i\omega x - i\omega t)$ impinges on $x = 0$ from vacuum, the total electric field can be expressed as

$$E_y = \begin{cases} \exp(i\omega x - i\omega t) + \Gamma \exp(-i\omega x - i\omega t), & x < 0, \\ \tau \exp(ikx - i\omega t), & x \geq 0, \end{cases}$$

where $k = \omega\sqrt{\epsilon_r(\omega)}$. Using (3.2), we can get the magnetic field. Using the boundary conditions of Maxwell's equations at $x = 0$, the reflection coefficient is

$$\Gamma = \frac{\sqrt{\epsilon_r(\omega)} - 1}{\sqrt{\epsilon_r(\omega)} + 1},$$

and transmission coefficient is

$$\tau = \frac{2}{\sqrt{\epsilon_r(\omega)} + 1}.$$

Next, we analyze the reflection and transmission coefficient of the discontinuous Galerkin method, assuming the basic solution of the discontinuous Galerkin method has the form [8]

$$U_j = \lambda^j u, \quad U = E, H. \tag{3.10}$$

Here λ can be obtained similarly as in the dispersion analysis in the appendix, j corresponds to space discretization cell, neighboring cell in the same media differs by a factor of λ. Replacing $\epsilon_r(\omega)$ with 1, we can get the dispersion in vacuum.

The key idea of the reflection/transmission analysis is based on the assumption that the solution has the form (3.10) in 1-D case (or (4.10) in 2-D case). We first write the total wave in different domains with numerical wave number (these are got from dispersion analysis in the appendix). The numerical reflection and transmission coefficients are used in these formulas. Then these expressions are put back into the discontinuous Galerkin scheme to get the error of reflection and transmission coefficients. Different basis will require different form for u. For piecewise constant elements, u is a constant while u will be a linear function for piecewise linear elements and a quadratic function for piecewise quadratic elements, respectively.

3.1. Piecewise constant elements

In this case, u in (3.10) is constant. We consider two cells $[-2h,0)$ and $[0,2h)$ around $x=0$. Then, by (6.6) in Appendix I, λ in the Drude media satisfies

$$2-(\lambda+\lambda^{-1}) = 4\frac{(kh)^2}{-2i\omega h+\omega_p^2 h/(-i\omega+\gamma)+1}, \tag{3.11}$$

while, λ in the vacuum satisfies

$$2-(\lambda+\lambda^{-1}) = 4\frac{(kh)^2}{-2i\omega h+1}. \tag{3.12}$$

Let $(+),(-)$ indicate right travelling wave and left traveling wave, respectively, and l,r indicate left and right side of $x=0$. Eq. (3.12) has two solutions denoted as $\lambda_l^{(\pm)}=\exp(\pm 2hk_l i)$ while the two solutions of (3.11) can be written as $\lambda_r^{(\pm)}=\exp(\pm 2hk_r i)$. Also the error of wave number is $\mathcal{O}(h)$, as will be shown in (6.7).

From the form of the solution (3.10), we can write the numerical solution as (ignoring the time dependence of $\exp(-i\omega t)$)

$$E_y = \begin{cases} \exp^n(-2hk_l i)E_{lr}+\alpha\exp^n(2hk_l i)E_{ll}, \\ \qquad x\in[-2nh,-2(n-1)h),\ n\geq 1, \\ \beta\exp^n(2hk_r i)E_{rr}, \\ \qquad x\in[2(n-1)h,2nh),\ n\geq 1, \end{cases} \tag{3.13}$$

$$H_z = \begin{cases} \exp^n(-2hk_l i)H_{lr}+\alpha\exp^n(2hk_l i)H_{ll}, \\ \qquad x\in[-2nh,-2(n-1)h),\ n\geq 1, \\ \beta\exp^n(2hk_r i)H_{rr}, \\ \qquad x\in[2(n-1)h,2nh),\ n\geq 1, \end{cases} \tag{3.14}$$

where the subscript $_{rr}$ represents the right traveling wave of right side, $_{lr}$ is the right traveling wave of left side, $_{ll}$ is the left traveling wave of left side, corresponding to $\lambda_r^{(+)},\lambda_l^{(+)},\lambda_l^{(-)}$, respectively.

Assuming $E_{lr}=E_{ll}=E_{rr}=1$. The value of H can be solved as (from (6.3) or (6.4) in the appendix)

$$H_{ll}=-1+\mathcal{O}(h),\quad H_{lr}=1+\mathcal{O}(h),\quad H_{rr}=\sqrt{\epsilon_r(\omega)}+\mathcal{O}(h). \tag{3.15}$$

Denote the solution on $[-2h,0)$ as H_0,E_0, on $[-4h,-2h)$ as H_{-1},E_{-1}, on $[0,2h)$ as H_1,E_1. Using (3.7)-(3.9) ($\epsilon_r=1$ in vacuum) on $[-2h,0)$ gives

$$-2i\omega hH_0+\frac{(E_0+H_0)+(E_1-H_1)}{2}-\frac{(E_0-H_0)+(E_{-1}+H_{-1})}{2}=0, \tag{3.16}$$

$$-2i\omega hE_0+\frac{(H_0+E_0)+(H_1-E_1)}{2}-\frac{(H_0-E_0)+(H_{-1}+E_{-1})}{2}=0. \tag{3.17}$$

We note that the following $\mathcal{O}(h)$ approximation expressions of (3.13) and (3.14)

$$E_0 = E_{lr} + \alpha E_{ll} + \mathcal{O}(h), \quad H_0 = H_{lr} + \alpha H_{ll} + \mathcal{O}(h), \tag{3.18}$$

$$E_{-1} = E_{lr} + \alpha E_{ll} + \mathcal{O}(h), \quad H_{-1} = H_{lr} + \alpha H_{ll} + \mathcal{O}(h), \tag{3.19}$$

$$E_1 = \beta E_{rr} + \mathcal{O}(h), \quad H_1 = \beta H_{rr} + \mathcal{O}(h). \tag{3.20}$$

Plugging (3.18)-(3.20) and (3.15) into (3.16)-(3.17), we obtain an equation for α, β

$$-\alpha + \frac{1-\sqrt{\epsilon_r(\omega)}}{2}\beta = 0 + \mathcal{O}(h). \tag{3.21}$$

Repeating the same analysis on cell $[0, 2h)$ and using the discontinuous Galerkin method in the Drude media, we have the second equation of α and β

$$\frac{1+\sqrt{\epsilon_r(\omega)}}{2}\beta - 1 = 0 + \mathcal{O}(h). \tag{3.22}$$

Combining (3.21) with (3.22) yields

$$\alpha = \frac{\sqrt{\epsilon_r(\omega)}-1}{\sqrt{\epsilon_r(\omega)}+1} + \mathcal{O}(h) = \Gamma + \mathcal{O}(h), \tag{3.23}$$

$$\beta = \frac{2}{\sqrt{\epsilon_r(\omega)}+1} + \mathcal{O}(h) = \tau + \mathcal{O}(h). \tag{3.24}$$

Therefore, the reflection coefficient and transmission coefficient is first order accurate.

3.2. Piecewise linear and quadratic elements

Considering two cells $[-2h, 0)$ and $[0, 2h)$ around $x = 0$, u in (3.10) is assumed to be a linear function. From (6.12) in the Appendices, λ satisfies

$$2 - (\lambda + \lambda^{-1}) = \frac{-4CDh^2(9 + 3(C+D)h + CDh^2)}{9 + 3(C+D)h - 2CDh^2 - 2CD(C+D)h^3}. \tag{3.25}$$

Let the two solutions of (3.25) be $\lambda_r^{(\pm)} = \exp(\pm 2hk_r i)$. When $\omega_p = 0$ in (3.25), the two solutions are written as $\lambda_l^{(\pm)} = \exp(\pm 2hk_l i)$ (case of vacuum), $(+), (-)$ and l, r have the same definition as before. It can be shown in (6.14) of the appendix that the wave number error in the piecewise linear element case is of $\mathcal{O}(h^3)$.

In view of (3.10), we can write the solution as

$$E_y = \begin{cases} \dfrac{1}{h}\Big[\exp^n(-2hk_l i)(\exp(2hk_l i)(\dfrac{x}{2} + hn) - \dfrac{x + 2(n-1)h}{2}) \\ \quad +\alpha \exp^n(2hk_l i)(\exp(-2hk_l i)(\dfrac{x}{2} + nh) - \dfrac{x + 2(n-1)h}{2})\Big], \\ \quad x \in [-2nh, -2(n-1)h), \ n \geq 1, \\ \dfrac{1}{h}\Big[\beta \exp^n(2hk_r)(\exp(-2hk_r i)(\dfrac{-x}{2} + nh) - \dfrac{-x + 2(n-1)h}{2})\Big], \\ \quad x \in [2(n-1)h, 2nh), \ n \geq 1. \end{cases}$$

Similar form for H_z can be obtained. We are now ready to solve α and β. Consider the interval $[-2h, 0)$. Assume the solution on this interval as H_0 with left value H_{0L} and right value H_{0R}, solution H_1 on $[0, 2h)$ with H_{1L}, H_{1R}, solution H_{-1} on $[-4h, -2h)$ with H_{-1L}, H_{-1R}. The definition of E is similar. From the expression above, we have

$$
\begin{aligned}
&E_{0L} = \exp(-2hk_l i) + \alpha \exp(2hk_l i), \quad E_{0R} = 1 + \alpha, \\
&H_{0L} = \exp(-2hk_l i) - \alpha \exp(2hk_l i), \quad H_{0R} = 1 - \alpha, \\
&E_{-1L} = \exp(-4hk_l i) + \alpha \exp(4hk_l i), \; E_{-1R} = \exp(-2hk_l i) + \alpha \exp(2hk_l i), \\
&H_{-1L} = \exp(-4hk_l i) - \alpha \exp(4hk_l i), \; H_{-1R} = \exp(-2hk_l i) - \alpha \exp(2hk_l i), \\
&E_{1L} = \beta, \; E_{1R} = \beta \exp(2hk_r i), \; H_{1L} = \beta\sqrt{\epsilon_r(\omega)}, \; H_{1R} = \beta \exp(2hk_r i)\sqrt{\epsilon_r(\omega)},
\end{aligned}
$$

with terms of order $\mathcal{O}(h^2)$ ignored. Substituting above equations into discontinuous Galerkin equations (3.7)-(3.9), we can get a relation between α and β

$$
-\alpha + \frac{1 - \sqrt{\epsilon_r(\omega)}}{2}\beta = 0 + \mathcal{O}(h^2). \tag{3.26}
$$

Repeating the same procedure on $[0, 2h)$, we get

$$
\frac{1 + \sqrt{\epsilon_r(\omega)}}{2}\beta - 1 = 0 + \mathcal{O}(h^2). \tag{3.27}
$$

Combining (3.26) with (3.27) gives

$$
\alpha = \frac{\sqrt{\epsilon_r(\omega)} - 1}{\sqrt{\epsilon_r(\omega)} + 1} + \mathcal{O}(h^2) = \Gamma + \mathcal{O}(h^2), \tag{3.28}
$$

$$
\beta = \frac{2}{\sqrt{\epsilon_r(\omega)} + 1} + \mathcal{O}(h^2) = \tau + \mathcal{O}(h^2). \tag{3.29}
$$

So, the reflection coefficient and transmission coefficient is of second-order accurate.

The process for piecewise quadratic elements is similar though u will be a quadratic function. For example, on the interval $[-2h, 0)$, the value at $-2h$ is $\exp(-2hk_l i)$, at $-h$ is $\exp(-hk_l i)$, at 0 is 1, and the form of incidence wave is

$$
\frac{\exp(-2hk_l i) - 2\exp(-hk_l i) + 1}{2h^2}x^2 + \frac{\exp(-2hk_l i) - 4\exp(-hk_l i) + 3}{2h}x + 1.
$$

We can handle other intervals and wave numbers in the same way, the resulting error of reflection and transmission coefficient is found to be $\mathcal{O}(h^3)$.

We have addressed the case of wave propagating from vacuum to metal, the cases of metal to vacuum, vacuum to PML, metal to PML can be handled, similarly.

4. Reflection/Transmission Properties for 2-D Maxwell's Equations

Assuming a time harmonic form, the non-dimensionalized 2-D Maxwell's equations for the TE wave of Drude media are given as

$$
\begin{cases}
\dfrac{\partial H_z}{\partial y} = -i\omega\epsilon_r(\omega)E_x, \\
-\dfrac{\partial H_z}{\partial x} = -i\omega\epsilon_r(\omega)E_y, \\
-\dfrac{\partial E_y}{\partial x} + \dfrac{\partial E_x}{\partial y} = -i\omega\mu_r H_z.
\end{cases} \tag{4.1}
$$

For plane wave solution of H_z in the form $\exp(i\beta_x x + i\beta_y y)$, the wave numbers β_x, β_y satisfy a dispersion relation

$$\beta_x^2 + \beta_y^2 = \omega^2 \epsilon_r(\omega). \tag{4.2}$$

Electric fields E_x, E_y can then be obtained from the Maxwell's equations.

The discontinuous Galerkin method has the following form

$$\begin{cases} -i\omega\epsilon_r(\omega)(E_x, v) + (H_z, v_y') + \int_{\partial K} F_{E_x} v dS = 0, \\ -i\omega\epsilon_r(\omega)(E_y, v) - (H_z, v_x') + \int_{\partial K} F_{E_y} v dS = 0, \\ -i\omega\mu_r(H_z, v) + (E_x, v_y') - (E_y, v_x') + \int_{\partial K} F_{H_z} v dS = 0, \end{cases} \tag{4.3}$$

where K is a cell of domain discretization. $(,)$ denotes inner product over K, ∂K is the boundary of K, v is the test function, and F is the numerical flux [4] given by

$$\begin{cases} F_{E_x} = -n_y \dfrac{[ZH_z + (n_x E_y - n_y E_x)]^- + [ZH_z - (n_x E_y - n_y E_x)]^+}{Z^- + Z^+}, \\ F_{E_y} = n_x \dfrac{[ZH_z + (n_x E_y - n_y E_x)]^- + [ZH_z - (n_x E_y - n_y E_x)]^+}{Z^- + Z^+}, \\ F_{H_z} = \dfrac{[Y(n_x E_y - n_y E_x) + H_z]^- + [Y(n_x E_y - n_y E_x) - H_z]^+}{Y^- + Y^+}, \end{cases} \tag{4.4}$$

where $\hat{\mathbf{n}}_K = (n_x, n_y)$ is the outward unit normal to ∂K, $Z^\pm$ and $Y^\pm$ are the local impedance and admittance, respectively. Similarly to 1-D case, the definition of $+, -$ is shown in Fig. 6.2.

In the x-y plane, assuming that the half plane of $x < 0$ is the vacuum, and $x > 0$ is the Drude media with

$$\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2 + i\gamma\omega}.$$

Time harmonic wave with time-dependence $\exp(-i\omega t)$ is assumed.

Take TE (Transverse Electric) wave as an example. In this case, the total magnetic field wave in $x < 0$ (incidence plus reflection wave) can be written as

$$H_z = H_0(1 + \Gamma \exp(-2i\beta_{1x}x)) \exp(i\beta_{1x}x + i\beta_{1y}y). \tag{4.5}$$

The transmitted magnetic field in $x > 0$ is

$$H_z = H_0 \tau \exp(i\beta_{2x}x + i\beta_{2y}y). \tag{4.6}$$

Here, Γ is the reflection coefficient and τ is transmission coefficient, electric field can be gotten by magnetic field through Maxwell's equations. Now we let

$$\beta_{1x} = k_1 \cos\theta, \quad \beta_{1y} = k_1 \sin\theta, \quad k_1 = \omega, \tag{4.7}$$

$$\beta_{2x} = \sqrt{k_2^2 - \beta_{2y}^2}, \quad k_2 = \omega\sqrt{\epsilon_r(\omega)}, \tag{4.8}$$

where θ is incidence angle. Using boundary conditions of Maxwell's equations that E_y and H_z are continuous at the interface, we get

$$\Gamma = \Big(\frac{\beta_{1x}}{\omega} - \frac{\beta_{2x}}{\omega\epsilon_r(\omega)}\Big) \Big/ \Big(\frac{\beta_{1x}}{\omega} + \frac{\beta_{2x}}{\omega\epsilon_r(\omega)}\Big), \quad \tau = \Big(2\frac{\beta_{1x}}{\omega}\Big) \Big/ \Big(\frac{\beta_{1x}}{\omega} + \frac{\beta_{2x}}{\omega\epsilon_r(\omega)}\Big). \tag{4.9}$$

From the Snell's theory [9], we have $\beta_{2y} = \beta_{1y} = k_1 \sin\theta$.

Next, we discuss the reflection and transmission coefficients for the discontinuous Galerkin solution. Assume the solution has the form (see [8])

$$U_{jl} = \lambda_x^j \lambda_y^l u, \quad U = H_z, E_x, E_y, \tag{4.10}$$

where λ_x, λ_y can be obtained as in the dispersion analysis in the appendix, j, l are the indices of space discretization, neighboring cell in the same media differs a factor of λ_x in x-direction and λ_y in y-direction. Replacing $\epsilon_r(\omega)$ with 1, we can have the dispersion relation in the vacuum.

4.1. Piecewise constant elements

In this case u in (4.10) is assumed to be constant. We consider cells $[-2h, 0) \times [0, 2h)$ and $[0, 2h) \times [0, 2h)$ along interface $x = 0$. Cell $[0, 2h) \times [0, 2h)$ lies in the Drude media, $\lambda_{2x}, \lambda_{2y}$ in (4.10) satisfy (6.16), i.e.

$$\lambda_{2x} + \lambda_{2x}^{-1} - 2 = \frac{(D^2h^2 + 4Dh + CDh^2)(\lambda_{2y} + \lambda_{2y}^{-1} - 2) - CD^2h^3}{(8 + 2Dh + Ch)(\lambda_{2y} + \lambda_{2y}^{-1} - 2) - (D^2h^2 + 4Dh + CDh^2)}, \tag{4.11}$$

where $C = -i4\omega\mu, D = -i4\omega\epsilon_r(\omega)$.

As $[-2h, 0) \times [0, 2h)$ lies in vacuum, $\lambda_{1x}, \lambda_{1y}$ satisfy

$$\lambda_{1x} + \lambda_{1x}^{-1} - 2 = \frac{(D'^2h^2 + 4D'h + CD'h^2)(\lambda_{1y} + \lambda_{1y}^{-1} - 2) - CD'^2h^3}{(8 + 2D'h + Ch)(\lambda_{1y} + \lambda_{1y}^{-1} - 2) - (D'^2h^2 + 4D'h + CD'h^2)}, \tag{4.12}$$

and $D' = -i4\omega$. Let

$$\lambda_{1x} = \exp(2ih\beta_{h1x}), \quad \lambda_{1y} = \exp(2ih\beta_{h1y}), \quad \lambda_{2x} = \exp(2ih\beta_{h2x}), \quad \lambda_{2y} = \exp(2ih\beta_{h2y}),$$

where $\beta_{h1x}, \beta_{h1y}, \beta_{h2x}, \beta_{h2y}$ are numerical wave numbers, which are given by

$$\beta_{h2x} = k_{2h}\cos\theta, \quad \beta_{h2y} = k_{2h}\sin\theta, \quad \beta_{h1x} = k_{1h}\cos\theta, \quad \beta_{h1y} = k_{1h}\sin\theta. \tag{4.13}$$

From the dispersion analysis in appendix, errors between k_{1h} and k_1, and k_{2h} and k_2 are of the order $\mathcal{O}(h)$.

Assume (4.12) has two solutions

$$\lambda_{xl}^{(\pm)} = \exp(\pm 2hik_{1h}\cos\theta), \quad \lambda_{yl}^{(\pm)} = \exp(\pm 2hik_{1h}\sin\theta),$$

while (4.11) has two solutions

$$\lambda_{xr}^{(\pm)} = \exp(\pm 2hik_{2h}\cos\theta), \quad \lambda_{yr}^{(\pm)} = \exp(\pm 2hik_{2h}\sin\theta),$$

l, r present left side and right side of $x = 0$, respectively. As material has no discontinuity in y direction, we only need to consider the case of $\lambda_{yl}^{(-)}, \lambda_{yr}^{(-)}$.

Using the form given in (4.10), solution can be written as (ignoring the time dependence

$\exp(-i\omega t)$)

$$H_z = \begin{cases} ((\lambda_{xl}^{(-)})^n H_{zl^{(-)}} + \alpha(\lambda_{xl}^{(+)})^n H_{zl^{(+)}})(\lambda_{yl}^{(-)})^m, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [-2nh, -2(n-1)h), \quad n \geq 1, \\ \beta(\lambda_{xr}^{(-)})^n (\lambda_{yr}^{(-)})^m H_{zr^{(-)}}, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [2(n-1)h, 2nh), \quad n \geq 1, \end{cases} \tag{4.14}$$

$$E_x = \begin{cases} ((\lambda_{xl}^{(-)})^n E_{xl^{(-)}} + \alpha(\lambda_{xl}^{(+)})^n E_{xl^{(+)}})(\lambda_{yl}^{(-)})^m, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [-2nh, -2(n-1)h), \quad n \geq 1, \\ \beta(\lambda_{xr}^{(-)})^n (\lambda_{yr}^{(-)})^m E_{xr^{(-)}}, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [2(n-1)h, 2nh), \quad n \geq 1, \end{cases} \tag{4.15}$$

$$E_y = \begin{cases} ((\lambda_{xl}^{(-)})^n E_{yl^{(-)}} + \alpha(\lambda_{xl}^{(+)})^n E_{yl^{(+)}})(\lambda_{yl}^{(-)})^m, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [-2nh, -2(n-1)h), \quad n \geq 1, \\ \beta(\lambda_{xr}^{(-)})^n (\lambda_{yr}^{(-)})^m E_{yr^{(-)}}, \\ \quad y \in [2mh, 2m+2h) \quad \& \quad x \in [2(n-1)h, 2nh), \quad n \geq 1. \end{cases} \tag{4.16}$$

Assuming $H_{zl^{(+)}} = H_{zl^{(+)}} = H_{zr^{(-)}} = 1$, every component of E can be solved by (6.15) with different component using different λ_x, λ_y. In vacuum ϵ_r is 1, $\epsilon_r = \epsilon_r(\omega)$ in the Drude media.

We get

$$\begin{aligned} E_{xl^{(-)}} &= \frac{\beta_{h1y}}{\omega}, \quad E_{xl^{(+)}} = \frac{\beta_{h1y}}{\omega}, \quad E_{xr^{(-)}} = \frac{\beta_{h2y}}{\omega\epsilon_r(\omega)}, \\ E_{yl^{(-)}} &= -\frac{\beta_{h1x}}{\omega}, \quad E_{yl^{(+)}} = \frac{\beta_{h1x}}{\omega}, \quad E_{yr^{(-)}} = -\frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}, \end{aligned} \tag{4.17}$$

where terms of order $\mathcal{O}(h)$ are ignored.

Denote solution on $[-2h, 0) \times [0, 2h)$ as H_{0z}, E_{0x}, E_{0y}, on $[-4h, -2h) \times [0, 2h)$ as H_{-1z}, E_{-1x}, E_{-1y}, on $[0, 2h) \times [0, 2h)$ as H_{1z}, E_{1x}, E_{1y}, on $[-2h, 0) \times [-2h, 0)$ as H_{-2z}, E_{-2x}, E_{-2y}, on $[-2h, 0) \times [2h, 4h)$ as H_{2z}, E_{2x}, E_{2y}. Using the discontinuous Galerkin method (4.3) on $[-2h, 0) \times [0, 2h)$ gives

$$(-i4\omega h + 2)E_{0x} - E_{-2x} - E_{2x} + H_{-2z} - H_{2z} = 0, \tag{4.18}$$

$$(-i4\omega h + 2)E_{0y} - E_{-1y} - E_{1y} + H_{1z} - H_{-1z} = 0, \tag{4.19}$$

$$\begin{aligned} &(-i4\omega\mu h + 4)H_{0z} - H_{-2z} - H_{2z} - H_{1z} - H_{-1z} \\ &\quad + E_{-2x} - E_{2x} + E_{1y} - E_{-1y} = 0. \end{aligned} \tag{4.20}$$

Approximating H_z, E_x, E_y (4.14)–(4.16) within $\mathcal{O}(h)$ errors, we have the expressions for $H_{kz}, E_{kx}, E_{ky}, k = \pm 1, 0, \pm 2$. Putting them and (4.17) into (4.18)-(4.20), we have an equation for α and β

$$-\frac{\beta_{h1x}}{\omega} + \alpha\frac{\beta_{h1x}}{\omega} + \beta\frac{\beta_{h2x}}{\omega\epsilon_r(\omega)} + \beta - 1 - \alpha = 0. \tag{4.21}$$

Next, repeating the same procedure as that on $[0, 2h) \times [0, 2h)$, we get the second equation for α and β

$$\frac{\beta_{h1x}}{\omega} - \alpha\frac{\beta_{h1x}}{\omega} - \beta\frac{\beta_{h2x}}{\omega\epsilon_r(\omega)} + \beta - 1 - \alpha = 0. \tag{4.22}$$

Combining (4.21) with (4.22), we get

$$\begin{aligned}\alpha &= \Big(\frac{\beta_{h1x}}{\omega} - \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big)\Big/\Big(\frac{\beta_{h1x}}{\omega} + \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big) + \mathcal{O}(h),\\ \beta &= \Big(2\frac{\beta_{h1x}}{\omega}\Big)\Big/\Big(\frac{\beta_{h1x}}{\omega} + \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big) + \mathcal{O}(h).\end{aligned} \tag{4.23}$$

Comparing with (4.9), we conclude that the reflection coefficient and transmission coefficient are both first-order accurate.

4.2. Piecewise linear and quadratic elements

Now u in (4.10) is a linear function. We still consider cells $[-2h,0)\times[0,2h)$ and $[0,2h)\times[0,2h)$. λ_x,λ_y in (4.10) can be found by the dispersive/dispersion analysis. Assume $\lambda_{2x} = \exp(2ih\beta_{h2x})$,$\lambda_{2y} = \exp(2ih\beta_{h2y})$ in the Drude media, $\lambda_{1x} = \exp(2ih\beta_{h1x}), \lambda_{1y} = \exp(2ih\beta_{h1y})$ in the vacuum.

The numerical wave numbers in the Drude media, β_{h2x} and β_{h2y}, can be written as

$$\beta_{h2x} = \pm k_{2h}\cos\theta, \quad \beta_{h2y} = \pm k_{2h}\sin\theta, \tag{4.24}$$

while β_{h1x},β_{h1y} are numerical wave numbers in the vacuum

$$\beta_{h1x} = \pm k_{1h}\cos\theta, \quad \beta_{h1y} = \pm k_{1h}\sin\theta. \tag{4.25}$$

Using the results of the last section, the error between k_{1h} and the wave number k_1 is $\mathcal{O}(h^3)$, between k_{2h} and k_2 is $\mathcal{O}(h^3)$. In view of the form given in (4.10), we can write the solution as

$$H_z = \begin{cases} \exp(2ih\beta_{h1x})^n \exp(-2ih\beta_{h1y})^m \cdot \Big(\dfrac{1-\exp(2ih\beta_{h1x})}{2h}(x+2nh) \\ +\dfrac{\exp(-2ih\beta_{h1y})-1}{2h}(y-2mh)+1\Big) + \alpha\exp(-2ih\beta_{h1x})^n\exp(-2ih\beta_{h1y})^m\cdot \\ \Big(\dfrac{1-\exp(-2ih\beta_{h1x})}{2h}(x+2nh) + \dfrac{\exp(-2ih\beta_{h1y})-1}{2h}(y-2mh)+1\Big), \\ \quad y\in[2mh,2m+2h) \quad \& \quad x\in[-2(n+1)h,-2nh), n\ge 0, \\ \beta\exp(-2ih\beta_{h2x})^n\exp(-2ih\beta_{h1y})^m\cdot \\ \Big(\dfrac{\exp(-2ih\beta_{h2x})-1}{2h}(x-2nh) + \dfrac{\exp(-2ih\beta_{h2y})-1}{2h}(y-2mh)+1\Big), \\ \quad y\in[2mh,2m+2h) \quad \& \quad x\in[2nh,2(n+1)h),\ n\ge 0. \end{cases} \tag{4.26}$$

E_x, E_y can be written as similarly. To solve for α,β, we assume solutions H_{0z},E_{0x},E_{0y} on $[-2h,0)\times[0,2h)$, H_{-1z},E_{-1x},E_{-1y} on $[-4h,-2h)\times[0,2h)$, H_{1z},E_{1x},E_{1y} on $[0,2h)\times[0,2h)$, H_{-2z},E_{-2x},E_{-2y} on $[-2h,0)\times[-2h,0)$, H_{2z},E_{2x},E_{2y} on $[-2h,0)\times[2h,4h)$. Using the discontinuous Galerkin method (4.3)-(4.4) on $[-2h,0)\times[-h,h)$, putting H_z,E_z,E_y of the corresponding cell into the discontinuous Galerkin scheme, we get the following equation of α and β within error of $\mathcal{O}(h^2)$

$$-\frac{\beta_{h1x}}{\omega} + \alpha\frac{\beta_{h1x}}{\omega} + \beta\frac{\beta_{h2x}}{\omega\epsilon_r(\omega)} + \beta - 1 - \alpha = 0. \tag{4.27}$$

Repeating on $[0,2h)\times[0,2h)$, we get another equation for α and β

$$\frac{\beta_{h1x}}{\omega} - \alpha\frac{\beta_{h1x}}{\omega} - \beta\frac{\beta_{h2x}}{\omega\epsilon_r(\omega)} + \beta - 1 - \alpha = 0. \tag{4.28}$$

Combining (4.27) and (4.28), we get

$$\begin{aligned}\alpha &= \Big(\frac{\beta_{h1x}}{\omega} - \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big)\Big/\Big(\frac{\beta_{h1x}}{\omega} + \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big) + \mathcal{O}(h^2),\\ \beta &= \Big(2\frac{\beta_{h1x}}{\omega}\Big)\Big/\Big(\frac{\beta_{h1x}}{\omega} + \frac{\beta_{h2x}}{\omega\epsilon_r(\omega)}\Big) + \mathcal{O}(h^2).\end{aligned} \tag{4.29}$$

Comparing with (4.9), we conclude that the reflection coefficient and transmission coefficient are both second order accurate.

The process for piecewise quadratic elements is similar while u in (4.10) will be a quadratic function. The resulting error of reflection and transmission coefficient can be found to be $\mathcal{O}(h^3)$.

5. Conclusions

In this paper, we have studied reflection and transmission properties of a discontinuous Galerkin method for Maxwell's equations in dispersive Drude media. The reflection/transmission error is summarized in Table 5.1. Fig. 5.1 gives the convergence plot for 2-D piecewise quadratic elements; Fig. 5.1(a) uses $\omega = 1, \omega_p = 2, \gamma = 2, \epsilon_{r,\infty} = 1$, and Fig. 5.1(b) uses $\omega = 1.5, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$.

Table 5.1: Summary of dispersive/dissipative error and reflection/transmission error.

	const elements	linear elements	quadratic elements
1-D reflection/transmission error	$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^3)$
2-D reflection/transmission error	$\mathcal{O}(h)$	$\mathcal{O}(h^2)$	$\mathcal{O}(h^3)$

Based on the results for the piecewise constant, linear and quadratic elements, we can reasonably predict that reflection and transmission coefficients error convergence order should be $\mathcal{O}(h^{n+1})$ for n-degree polynomial elements on rectangle meshes. As the analysis is based on the plane wave Fourier analysis, dispersion and transmission/reflection results will be difficult to extend to general meshes. However, our numerical implementations of the concerned discontinuous Galerkin method are always carried out on general finite element type meshes (triangles or quadrilaterals) — all with high accuracy in wave dispersions and accurate field solutions near material interfaces.

Acknowledgments. The authors appreciate useful discussions with Dr. Tiao Lu during this work. Wei Cai would like to thank the support of the National Science Foundation (grant numbers DMS-0408309, CCF-0513179) and Department of Energy (Grant # DEFG 0205-ER25678) for the work reported in this paper. P. Zhang acknowledges support from the National Science Foundation of China (grant number 90207009) and for Distinguished Young Scholars (grant number 10225103).

6. Appendix

In the appendix, we include the analysis of dispersion error for the discontinuous Galerkin method. Similar results can be found in [10].

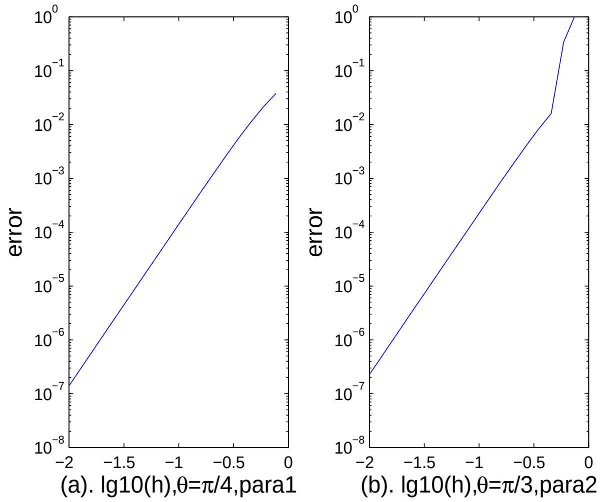

Fig. 5.1. Error of reflection coefficient. (a) $\omega = 1, \omega_p = 2, \gamma = 2, \epsilon_{r,\infty} = 1$; and (b) $\omega = 1.5, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$.

6.1. Dispersion errors for 1-D Maxwell's equations

Our goal is to find a similar dispersion relation for the discontinuous Galerkin method [1] and evaluate the dispersion error. Assuming a time-harmonic form, the numerical solutions are

$$E_y = e(x)\exp(-i\omega t), \quad H_z = h(x)\exp(-i\omega t), \quad J_y = j(x)\exp(-i\omega t).$$

As we seek plane wave solutions

$$U_R^+ = \exp(i2hk_h)U_L^- = \lambda U_L^-, \quad U_L^+ = \exp(-i2hk_h)U_R^- = \lambda^{-1}U_R^-, \quad U = e, h, j,$$

with k_h being the numerical wave number. Combining (3.8) and (3.9), we get

$$\begin{aligned} &-i\omega(h(x), v_h) - (e(x), v_h') + \frac{(e_R + h_R)^- + \lambda(e_L - h_L)^-}{2} v_R^- \\ &\quad + \frac{(-e_L + h_L)^- + \lambda^{-1}(-e_R - h_R)^-}{2} v_L^- = 0, \end{aligned} \tag{6.1}$$

$$\begin{aligned} &(-i\omega\epsilon_{r,\infty} + \frac{\omega_p^2}{-i\omega + \gamma})(e(x), v_h) - (h(x), v_h') + \frac{(h_R + e_R)^- + \lambda(h_L - e_L)^-}{2} v_R^- \\ &\quad - \frac{(h_L - e_L)^- + \lambda^{-1}(h_R + e_R)^-}{2} v_L^- = 0. \end{aligned} \tag{6.2}$$

6.1.1. Piecewise constant elements

In this case, we set $v_h = 1$ in (6.1)-(6.2), $e(x)$ and $h(x)$ are constants denoted as E and H, respectively. Consequently, all derivative terms are zero. So, (6.1) and (6.2) become

$$-2i\omega hH + \frac{(E + H)^- + \lambda(E - H)^-}{2} + \frac{(-E + H)^- + \lambda^{-1}(-E - H)^-}{2} = 0, \tag{6.3}$$

$$2\Big(-i\omega\epsilon_{r,\infty} + \frac{\omega_p^2}{-i\omega + \gamma}\Big)hE + \frac{(H + E)^- + \lambda(H - E)^-}{2} - \frac{(H - E)^- + \lambda^{-1}(H + E)^-}{2} = 0. \tag{6.4}$$

Rewritting in matrix-vector form, (6.3)-(6.4) read

$$AU = 0, \tag{6.5}$$

where $U = (H, E)^T$ and

$$A = \begin{bmatrix} -i2\omega h + 1 - (\lambda^{-1}+\lambda)/2 & (\lambda - \lambda^{-1})/2 \\ (\lambda-\lambda^{-1})/2 & 2\Big(-i\omega\epsilon_{r,\infty} + \frac{\omega_p^2}{-i\omega+\gamma}\Big)h + 1 - (\lambda^{-1}+\lambda)/2 \end{bmatrix}.$$

A non-trivial solution to (6.5) requires $\det(A) = 0$, resulting in the following characteristic equation

$$\begin{aligned}(2 - (\lambda + \lambda^{-1})) &= 4\Big(\frac{i\omega\omega_p^2}{-i\omega+\gamma} + \omega^2\Big)h^2 \Big/ \Big(-2i\omega h + \frac{\omega_p^2}{-i\omega+\gamma}h + 1\Big) \\ &= \frac{4(kh)^2}{-2i\omega h + \frac{\omega_p^2}{-i\omega+\gamma}h + 1},\end{aligned} \tag{6.6}$$

where k is the wave number in (3.6). Let $-2i\omega + \omega_p^2(-i\omega+\gamma)^{-1} = -C$ and $\lambda = \exp(i2hk_h)$. For small hk_h , using Taylor expansion gives

$$k_h - k = \frac{Ck^2}{k_h + k}h + \mathcal{O}(h^2). \tag{6.7}$$

From (6.7), we conclude that the dispersive (real part of $(k_h - k)$) and dissipative (imaginary part of $(k_h - k)$) are both first order accurate.

6.1.2. Piecewise linear elements

In this case, test functions are chosen as $v_{h1} = (h-x)/2$ and $v_{h2} = (h+x)/2$ while the numerical solution is assumed to be of the form

$$U_L(h-x)/2 + U_R(h+x)/2, \quad U = h, e.$$

Putting v_{h1} and v_{h2} into (6.1) and (6.2), we get

$$-i\omega\Big(\frac{2h_L}{3} + \frac{h_R}{3}\Big)h^2 + \frac{(e_L+e_R)h}{2} + \frac{(-e_L+h_L) - \lambda^{-1}(e_R+h_R)}{2}h = 0, \tag{6.8}$$

$$\Big(-i\omega + \frac{\omega_p^2}{-i\omega+\gamma}\Big)\Big(\frac{2e_L}{3} + \frac{e_R}{3}\Big)h^2 + \frac{(h_L+h_R)h}{2} - \frac{(h_L-e_L) + \lambda^{-1}(h_R+e_R)}{2}h = 0, \tag{6.9}$$

$$-i\omega\Big(\frac{2h_R}{3} + \frac{h_L}{3}\Big)h^2 - \frac{(e_L+e_R)h}{2} + \frac{(e_R+h_R) + \lambda(e_L-h_L)}{2}h = 0, \tag{6.10}$$

$$\Big(-i\omega + \frac{\omega_p^2}{-i\omega+\gamma}\Big)\Big(\frac{2e_R}{3} + \frac{e_L}{3}\Big)h^2 - \frac{(h_L+h_R)h}{2} + \frac{(h_R+e_R) + \lambda(h_L-e_L)}{2}h = 0. \tag{6.11}$$

Rewritten again as $AU = 0$, where $U = (h_L, h_R, e_L, e_R)$. A nontrivial solution requires that $\det(A) = 0$, giving

$$2 - \Big(\lambda + \frac{1}{\lambda}\Big) = \frac{-4CDh^2(9 + 3(C+D)h + CDh^2)}{9 + 3(C+D)h - 2CDh^2 - 2CD(C+D)h^3}, \tag{6.12}$$

where

$$C = -i\omega + \frac{\omega_p^2}{-i\omega+\gamma}, \quad D = -i\omega, \quad CD = -k^2, \quad \lambda = \exp(2ihk_h).$$

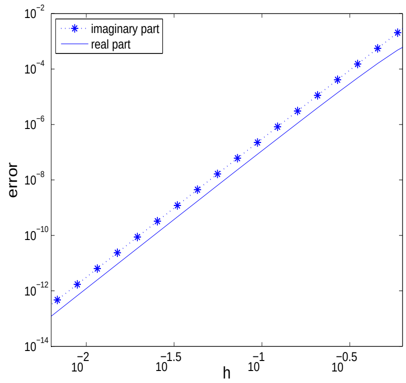

Fig. 6.1. Error of 1-D piecewise quadratic elements, x-axis is $\log(h)$.

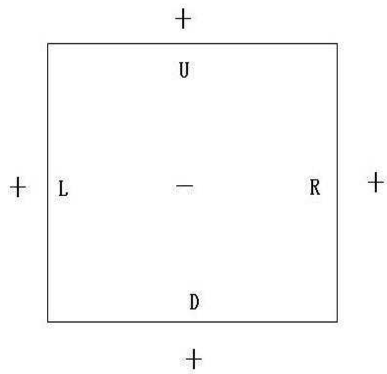

Fig. 6.2. Structure of 2-D cell.

Using Taylor expansion, we have

$$\left(k_h^2 h^2 - \frac{3}{2}\right)^2 - \left(k^2 h^2 - \frac{3}{2}\right)^2 = \mathcal{O}(h^5), \tag{6.13}$$

which gives that

$$k_h - k = \mathcal{O}(h^3). \tag{6.14}$$

Therefore, we have a third-order accuracy in the dispersion and dissipation for piecewise linear element.

6.1.3. Piecewise quadratic elements

Following a similar procedure as above, we write the numerical solution as

$$U_L(h-x)/2 + U_R(h+x)/2 + U_C(h^2 - x^2), \quad U = h, e.$$

By setting test functions $v_{h1} = (h-x)/2$, $v_{h2} = (h+x)/2$ and $v_{h3} = h^2 - x^2$ in (6.1) and (6.2), we get six functions. A nontrivial solution requirement gets

$$k_h - k = \mathcal{O}(h^5),$$

namely, a 5th-order accuracy in the dispersion and dissipation for second order elements.

Fig. 6.1 plots the error for $\omega = 1, \omega_p = 2, \gamma = 2, \epsilon_{r,\infty} = 1$, which indicates that the order of accuracy for the real part is 4.9, and for the imaginary part is 5.0.

6.2. Dispersion errors for 2-D Maxwell's equations

Assuming K is $[-h,h] \times [-h,h]$, as in Fig. 6.2, we define the value on four boundaries as $U_L(y), U_R(y), U_U(x), U_D(x)$, respectively; the limit from inside labels as $-$, outside as $+$. We seek plane wave solutions and, hence, assume $U_L^+ = U_R^-/\lambda_x, U_D^+ = U_U^-/\lambda_y, U_R^+ = U_L^-\lambda_x, U_U^+ = U_D^-\lambda_y$.

6.2.1. Piecewise constant elements

Assuming test function $v = 1$, E_x, E_y, H_z are constants and all of their derivatives are thus zero. Under the assumption of $Z = Y = 1$, the discontinuous Galerkin solution (4.4) can be simplified as

$$\begin{cases} \Big(-i4\omega\epsilon_r(\omega)h + 2 - \dfrac{1}{\lambda_y} - \lambda_y\Big)E_x + \Big(\dfrac{1}{\lambda_y} - \lambda_y\Big)H_z = 0, \\ \Big(-i4\omega\epsilon_r(\omega)h + 2 - \dfrac{1}{\lambda_x} - \lambda_x\Big)E_y + \Big(\lambda_x - \dfrac{1}{\lambda_x}\Big)H_z = 0, \\ \Big(-i4\omega\mu h + 4 - \dfrac{1}{\lambda_y} - \lambda_y - \dfrac{1}{\lambda_x} - \lambda_x\Big)H_z \\ \qquad +\Big(\dfrac{1}{\lambda_y} - \lambda_y\Big)E_x + \Big(\lambda_x - \dfrac{1}{\lambda_x}\Big)E_y = 0. \end{cases} \tag{6.15}$$

Non-trivial solution demands the determinant of coefficient matrix to be zero, resulting in

$$\lambda_x + \lambda_x^{-1} - 2 = \frac{(D^2h^2 + 4Dh + CDh^2)(\lambda_y + \frac{1}{\lambda_y} - 2) - CD^2h^3}{(8 + 2Dh + Ch)(\lambda_y + \frac{1}{\lambda_y} - 2) - (D^2h^2 + 4Dh + CDh^2)}, \tag{6.16}$$

where

$$C = -i4\omega\mu, \quad D = -i4\omega\epsilon_r(\omega), \quad \lambda_x = \exp(2ih\beta_{hx}), \quad \lambda_y = \exp(2ih\beta_{hy}),$$

β_{hx} and β_{hy} are numerical wave numbers. Next, we will compare the errors between β_{hx} and β_x, β_{hy} and β_y. We assume in (4.2) $\beta_x = k\cos\theta$, $\beta_y = k\sin\theta$, $k^2 = \omega^2\epsilon_r(\omega)$, and, similarly, in (6.16)

$$\beta_{hx} = k_h\cos\theta, \quad \beta_{hy} = k_h\sin\theta. \tag{6.17}$$

Denoting $\lambda_x = \exp(2ih\beta_{hx}), \lambda_y = \exp(2ih\beta_{hy})$, and putting (6.17) into (6.16), we obtain

$$k_h^2 = \frac{-CD}{4(4 + (C + D)h)} + \mathcal{O}(h) = k^2 + \mathcal{O}(h).$$

Consequently, we have

$$k_h - k = \mathcal{O}(h), \tag{6.18}$$

which implies that the dispersion error is first-order accurate for all incident angle θ.

6.2.2. Piecewise linear elements

Assume that the numerical solution has the form

$$U = U_L v_{h1} + U_R v_{h2} + U_C v_{h1}, \quad U = H_z, E_x, E_y. \tag{6.19}$$

Set three basis functions $v_{h1} = x, v_{h2} = y, v_{h3} = h$ in (4.3), respectively. This yields nine equations. Again, we assume

$$\beta_{hx} = k_h\cos\theta, \quad \beta_{hy} = k_h\sin\theta, \quad \lambda_x = \exp(2ih\beta_{hx}), \quad \lambda_y = \exp(2ih\beta_{hy}). \tag{6.20}$$

Putting (6.20) into those nine equations, non-trivial solution demands the determinant of coefficient matrix to be zero. As the determinant is beyond analytical solution, we solve it numerically for a few typical cases.

Fig. 6.3 is the dispersion and dissipation error convergence result of $\omega = 1, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$, incidence angle θ takes $0, \pi/6, \pi/4, \pi/3$, respectively.

Table 6.1 gives the third-order $\mathcal{O}(h^3)$ convergence for three cases of parameters. The first case is $\omega = 1, \omega_p = 2, \gamma = 2, \epsilon_{r,\infty} = 1$; the second case is $\omega = 1, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$, and the third case is $\omega = 1.5, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$.

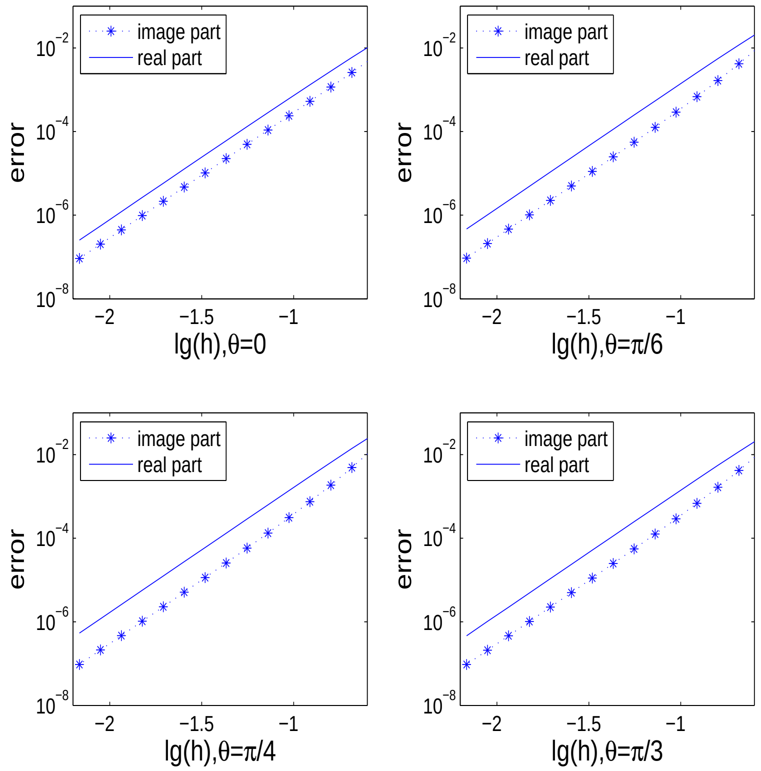

Fig. 6.3. Error of piecewise linear elements with $\omega = 1, \omega_p = 2.5, \gamma = 1.5, \epsilon_{r,\infty} = 1$, at various values of the incidence angle θ.

Table 6.1: The convergence order of piecewise linear elements.

parameter		$\theta = 0$	$\theta = \pi/6$	$\theta = \pi/4$	$\theta = \pi/3$
case 1	Real part	2.96	2.97	2.95	3.00
case 1	Imaginary part	2.92	3.13	3.13	3.13
case 2	Real part	2.74	2.99	3.05	2.99
case 2	Imaginary part	3.50	3.77	3.88	3.77
case 3	Real part	2.84	2.98	2.95	2.98
case 3	Imaginary part	3.40	3.48	3.62	3.48

6.2.3. Piecewise quadratic elements

The numerical solution has the following form

$$U = U_1v_1 + U_2v_2 + U_3v_3 + U_4v_4 + U_5v_5 + U_6v_6, \quad U = H_z, E_x, E_y. \tag{6.21}$$

Setting six test functions $v_1 = h, v_2 = x, v_3 = y, v_4 = x^2, v_5 = y^2, v_6 = xy$ into (4.3), we get eighteen equations. Following the similar numerical process for piecewise linear elements, we can obtain the $\mathcal{O}(h^5)$ convergence results.

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