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ON THE NEW MULTISCALE RODLIKE MODEL OF POLYMERIC FLUIDS*

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Abstract. This paper is concerned with the well-posedness for the new rigid rodlike model in a polymeric fluid recently proposed by W.N. E and P.W. Zhang [*Meth. Appl. Anal.*, 13 (2006), pp. 181–198]. The constitutive relations considered in this work are motivated by the kinetic theory. The micro equations involve five independent spatial variables (degrees of freedom): two in the configuration domain and three in the macro flow domain. We obtain the local existence of solutions with large initial data and also global existence of solutions with small Deborah and Reynolds constants in periodic domains.

Key words. polymeric fluid, rodlike model, kinetic theory, global existence

AMS subject classifications. 76B03, 65M12, 35Q35

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1. Introduction.

The Doi kinetic theory for spatially homogeneous flow of rodlike molecules has successfully described the properties of liquid crystal polymers in a solvent [6]. One of the simplest models of polymeric fluids described by the Doi theory is the rigid rodlike model, which takes into account the macro and micro behavior of the dilute or solute polymers—the effects of flow, Brownian motion, and intermolecular forces on the molecular orientation distribution (see [6, 10, 11, 16, 17, 23]). However, it does not include the so-called distortional elasticity. The Doi theory is valid only in the limit of spatial homogeneity. For small molecule liquid crystals, distortional elasticity has been formulated in the limit of weak distribution as Frank elasticity. This is one ingredient of the Leslie–Ericken theory. Several attempts have been made to find a theory which encompasses both the molecular visco-elasticity and the distortional elasticity. Marrucci and Greco [18] made a molecular theory of distortional elasticity. They proposed a nonlocal mean field nematic potential for LCPs (liquid crystal polymers), which accounts for spatial variations of the molecular orientation distribution. Tsuji and Rey [21, 22] add distortional elasticity to the rodlike model of the Doi theory but did not give a stress tensor. Edwards and Beris [2] give an ad hoc generalization of the Frank elasticity in tensorial form. Ericksen [9] allowed the order parameter to be a variable but still required the orientation distribution to be uniaxial. An extension of Kuzuu and Doi [13] theory to flowing systems of nonhomogeneous liquid crystalline polymers is made by Wang [25], in which the author models the LCP molecules as spheroids of equal shape and size. He derives an intermolecular potential which could be considered as an extension of Marrucci–Greco potential.

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All these approaches are phonomenological in nature. They invariably contain a large number of unknown parameters which in general cannot be determined rationally. Another drawback of the phonomenological theories is the lack of consistency with existing theories and among themselves. Then E and Zhang [8] developed a new model for nonhomogeneous flows of liquid crystalline polymers with a few adjustable parameters that could model a variety of configurations of polymeric liquid crystal molecules. This new model is a combination of macroscopic partial differential equations and microscopic Fokker–Planck equations. In this model, the function $\psi(\mathbf{x}, \mathbf{m}, t)$ describes the distribution of an identical rigid rodlike molecule at $(\mathbf{x}, t)$ with the orientation $\mathbf{m}$. Denoting the velocity and pressure of the fluid by $\mathbf{u}$ and p, the new multiscale rodlike model can be expressed as

$$\begin{aligned}\frac{\partial \psi}{\partial t} + \nabla \cdot (\mathbf{u}\psi) = \frac{1}{k_B T} \nabla \cdot \big\{ [D_\| \mathbf{mm} + D_\perp (\mathbf{I} - \mathbf{mm})] \cdot (\psi \nabla \mu) \big\} \\ + \frac{D_r}{k_B T} \mathcal{R} \cdot (\psi \mathcal{R} \mu) - \mathcal{R} \cdot (\mathbf{m} \times \kappa \cdot \mathbf{m} \psi), \quad \mathbf{m} \in \mathbb{S}^2,\end{aligned} \tag{1.1}$$

where k_B is Boltzmann constant and T is the absolute temperature, $D_\| \geq 0$ and $D_\perp \geq 0$ are translational diffusion coefficients parallel and normal to the orientation of the LCP molecule, $D_r = \frac{\xi_r}{k_B T}$ is the rotary diffusivity and ξ_r is the friction coefficient, ∇ is the gradient operator with respect to the spatial variables $\mathbf{x}$, $\mathcal{R} = \mathbf{m} \times \frac{\partial}{\partial \mathbf{m}}$ is the rotational gradient operator, and $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$. The symbol μ denotes the chemical potential

$$\mu = \ln \psi + \bar{U}, \tag{1.2}$$

and $\bar{U}$ represents the excluded-volume potential [6, 11]

$$\bar{U}(\mathbf{x}, \mathbf{m}, t) = k_B T \alpha \int_\Omega \int_{|\mathbf{m}'|=1} B(\mathbf{x}, \mathbf{x}', \mathbf{m}, \mathbf{m}') \psi(\mathbf{x}', \mathbf{m}', t) d\mathbf{m}' d\mathbf{x}'. \tag{1.3}$$

The function B in (1.3) is the interactional factor among rods. Here α denotes the intensity between particles. Now we choose $B(\mathbf{x}, \mathbf{x}', \mathbf{m}, \mathbf{m}') = \frac{1}{\varepsilon^3} \chi(\frac{\mathbf{x}-\mathbf{x}'}{\varepsilon}) |\mathbf{m} \times \mathbf{m}'|^2$, where $\chi(\mathbf{x})$ is the smooth kernel; e.g., $\chi(\mathbf{x}) = C \exp(1/(|\mathbf{x}|^2 - 1))$ as $|\mathbf{x}| < 1$, and $\chi(\mathbf{x}) = 0$ as $|\mathbf{x}| \geq 1$, where C is a constant such that $\int_{|\mathbf{x}| \leq 1} \chi(\mathbf{x}) d\mathbf{x} = 1$. $\kappa = (\nabla \mathbf{u})^T$ is the velocity gradient tensor.

Let L_0 be the typical size of the flow region, V_0 be the typical velocity scale, and $T_0 = \frac{L_0}{V_0}$ be a typical convective time scale. Further De is an important parameter called the Deborah number:

$$De = \frac{\frac{\xi_r}{k_B T}}{\frac{L_0}{V_0}} = \frac{\xi_r V_0}{k_B T L_0}. \tag{1.4}$$

It is the ratio of the orientational diffusion time scale of the rods (which is the relevant relaxation time scale) and the convective time scale of the fluid. Set

$$\varepsilon = \frac{L}{L_0}, \tag{1.5}$$

where L is the length of the rods. Thus the nondimensional kinetic equation is

$$\frac{\partial\psi}{\partial t}+\nabla\cdot(\mathbf{u}\psi)=\frac{\varepsilon^2}{De}\nabla\cdot\left\{\left[D_{\|}^{*}\mathbf{mm}+D_{\perp}^{*}(\mathbf{I}-\mathbf{mm})\right]\cdot(\psi\nabla\mu)\right\}$$
$$+\frac{1}{De}\mathcal{R}\cdot(\psi\mathcal{R}\mu)-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\psi),\quad \mathbf{m}\in\mathbb{S}^2, \tag{1.6}$$

$$\mu=\ln\psi+U, \tag{1.7}$$

$$U(\mathbf{x},\mathbf{m},t)=\alpha\int_{\Omega}\int_{|\mathbf{m}'|=1}B(\mathbf{x},\mathbf{x}',\mathbf{m},\mathbf{m}')\psi(\mathbf{x}',\mathbf{m}',t)d\mathbf{m}'d\mathbf{x}'. \tag{1.8}$$

The velocity field satisfies the Navier–Stokes-like equation

$$\rho(\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u})+\nabla p=\nabla\cdot\tau+F, \tag{1.9}$$

$$\nabla\cdot\mathbf{u}=0. \tag{1.10}$$

In the LCP system, the extra stress τ is given by two parts, the viscous stress τ_s and the elastic stress τ_e, namely

$$\tau=\tau_s+\tau_e. \tag{1.11}$$

The viscous stress comes from two sources, one from the solvent and the other from the constrain of rods derived in [6],

$$\tau_s=2\eta_s\mathbf{D}+\frac{1}{2}\xi_r\mathbf{D}:\langle\mathbf{mmmm}\rangle,$$

where $\mathbf{D}=\frac{1}{2}(\kappa+\kappa^T)=\frac{1}{2}(\nabla\mathbf{u}+(\nabla\mathbf{u})^T)$ is the strain rate tensor, η_s is the solvent viscosity, and $\langle(\cdot)\rangle$ denotes averaging with respect to the distribution ψ; i.e., $\langle(g)\rangle=\int_{|\mathbf{m}|=1}g\psi d\mathbf{m}$. The elastic stress is derived through a generalized virtual work principle [6]. The detail can be found in [8]. Now we cite the result from [8],

$$\tau_e=-\langle(\mathbf{m}\times\mathcal{R}\mu)\mathbf{m}\rangle. \tag{1.12}$$

Meanwhile the body force can also be identified as

$$\mathbf{F}=-\langle\nabla\mu\rangle. \tag{1.13}$$

Now let $\eta_p=\xi_r,\eta=\eta_s+\eta_p,\gamma=\frac{\eta_s}{\eta}$, and Re denotes the Reynolds number. Then the nondimensional Navier–Stokes-like equation

$$\mathbf{u}_t+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\frac{\gamma}{Re}\triangle\mathbf{u}+\frac{1-\gamma}{2Re}\nabla\cdot(\mathbf{D}:\langle\mathbf{mmmm}\rangle)$$
$$+\frac{1-\gamma}{ReDe}(\nabla\cdot\tau_e+\mathbf{F})\text{ for }\mathbf{x}\in\Omega \tag{1.14}$$

$$\nabla\cdot\mathbf{u}=0,\qquad\text{for }\mathbf{x}\in\Omega. \tag{1.15}$$

In this work we mainly investigate the well-posedness of this new multiscale rodlike polymeric model. Moreover, in most cases [6] $D_{\|}^{*}/D_{\perp}^{*}\approx 2$, so we can set $D_{\perp}^{*}=1$ and $D_{\|}^{*}=2$ for simple and without the lost of generality. Then (1.6) can be written as

$$\frac{\partial\psi}{\partial t}+\nabla\cdot(\mathbf{u}\psi)=\frac{\varepsilon^2}{De}\nabla\cdot[(\mathbf{I}+\mathbf{mm})(\psi\nabla\mu)]$$
$$+\frac{1}{De}\mathcal{R}\cdot(\psi\mathcal{R}\mu)-\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\psi),\quad\mathbf{m}\in\mathbb{S}^2. \tag{1.16}$$

We can verify this system (1.14)–(1.16) satisfies the energy identity in the following way.

Multiplying $\mathbf{u}$ to (1.14) and integrating it over Ω yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2 d\mathbf{x} + \int_\Omega \left[\frac{\gamma}{De}|\nabla \mathbf{u}|^2 + \frac{1-\gamma}{2Re}\langle(\mathbf{mm}:\mathbf{D})^2\rangle\right] d\mathbf{x} \\ = \frac{1-\gamma}{ReDe}\int_\Omega [-\tau_e : \nabla\mathbf{u} + \mathbf{F}\cdot\mathbf{u}]\, d\mathbf{x} \\ = \frac{1-\gamma}{ReDe}\int_\Omega [\langle(\mathbf{m}\times\mathcal{R}\mu)\mathbf{m}\rangle : \nabla\mathbf{u} - \langle\nabla\mu\rangle\cdot\mathbf{u}]\, d\mathbf{x} \\ = \frac{1-\gamma}{ReDe}\int_\Omega\int_{|\mathbf{m}|=1} [(\mathbf{m}\times\mathcal{R}\mu)\mathbf{m}\psi : \nabla\mathbf{u} - \psi\nabla\mu\cdot\mathbf{u}]\, d\mathbf{m}d\mathbf{x}.\end{aligned} \tag{1.17}$$

Multiplying μ to (1.16) as well as integrating over in Ω and the unit sphere yields

$$\begin{aligned}\int_\Omega\int_{|\mathbf{m}|=1}\frac{\partial\psi}{\partial t}\mu\, d\mathbf{m}d\mathbf{x} + \int_\Omega\left[\frac{\varepsilon^2}{De}\langle\nabla\mu\cdot(\mathbf{I}+\mathbf{mm})\nabla\mu\rangle + \frac{1}{De}\langle\mathcal{R}\mu\cdot\mathcal{R}\mu\rangle\right] d\mathbf{x} \\ = \int_\Omega\int_{|\mathbf{m}|=1}\mathbf{m}\times\kappa\cdot\mathbf{m}\psi\cdot\mathcal{R}\mu\, d\mathbf{m}d\mathbf{x} + \int_\Omega\int_{|\mathbf{m}|=1}\mathbf{u}\psi\cdot\nabla\mu\, d\mathbf{m}d\mathbf{x}.\end{aligned} \tag{1.18}$$

Additionally, we can calculate

$$\begin{aligned}\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}\left[\psi\ln\psi + \frac{1}{2}\psi U\right] d\mathbf{m}d\mathbf{x} \\ = \int_\Omega\int_{|\mathbf{m}|=1}\left[\frac{\partial\psi}{\partial t}\ln\psi + \frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial\psi}{\partial t}U + \frac{1}{2}\psi\frac{\partial U}{\partial t}\right] d\mathbf{m}d\mathbf{x} \\ = \int_\Omega\int_{|\mathbf{m}|=1}\left[\frac{\partial\psi}{\partial t}(\ln\psi + U) + \frac{1}{2}\left(\psi\frac{\partial U}{\partial t} - \frac{\partial\psi}{\partial t}U\right)\right] d\mathbf{m}d\mathbf{x} \\ = \int_\Omega\int_{|\mathbf{m}|=1}\frac{\partial\psi}{\partial t}\mu\, d\mathbf{m}d\mathbf{x}.\end{aligned} \tag{1.19}$$

Combining (1.17)–(1.19), we obtain that the system (1.14)–(1.18) satisfies the energy law:

$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{2}\int_\Omega|\mathbf{u}|^2 d\mathbf{x} + \frac{1-\gamma}{ReDe}E(\psi)\right] = -\int_\Omega\left[\frac{\gamma}{De}|\nabla\mathbf{u}|^2 + \frac{1-\gamma}{2Re}\langle(\mathbf{mm}:\mathbf{D})^2\rangle\right] d\mathbf{x} \\ - \frac{1-\gamma}{ReDe}\int_\Omega\left[\frac{\varepsilon^2}{De}\langle\nabla\mu\cdot(\mathbf{I}+\mathbf{mm})\nabla\mu\rangle + \frac{1}{De}\langle\mathcal{R}\mu\cdot\mathcal{R}\mu\rangle\right] d\mathbf{x},\end{aligned} \tag{1.20}$$

where $E(\psi)$ is a nonlocal intermolecular potential given by

$$E(\psi) = \int_\Omega\int_{|\mathbf{m}|=1}\psi(\mathbf{x},\mathbf{m},t)\ln\psi(\mathbf{x},\mathbf{m},t) + \frac{1}{2}U(\mathbf{x},\mathbf{m},t)\psi(\mathbf{x},\mathbf{m},t)d\mathbf{m}d\mathbf{x}. \tag{1.21}$$

By the same way one can see that the original system (1.6)–(1.15) also satisfies the energy law like the form (1.20). Comparing to the Doi model of rodlike polymeric fluid [6], this new model is based on the more rational assumption that the particle distribution function (pdf) ψ is possibly different in the macro variable $\mathbf{x}$. When the pdf is the same at every point $\mathbf{x}$ in the domain Ω, this model is similar to the Doi model. Here the other important difference is the excluded-volume potential (1.3) if we choose $B = |\mathbf{m}\times\mathbf{m}'|^2$ or $B = |\mathbf{m}\times\mathbf{m}'|$ in (1.3), which is the well-known Maier–Saupe or Onsager potential [6]. Now in [8] E and Zhang have proved that the inhomogeneous property reduces to the Ericksen–Leslie theory in the limit of small Debroah number. Recently numerical simulation results [26, 27] have shown that this model can really describe the anisotropic long-range elasticity of polymeric molecules, and the microstructure and defect dynamics of LCP solution. Moreover they have reported in [26] that there are seven in-plane flow modes in plane Couette flow described by this new model. Four of them have also been reported by Rey and Tsuji [22], and the other three modes are new complicated in-plane modes with inner defects. Furthermore, some significant scaling properties were verified in [26], such as the tumbling period is proportional to the inverse of the shear rate. In plane Poiseuille flow, different local states, such as flow-aligning, tumbling, or wagging, arise in different flow region. There are also some related numerical analysis results for special cases of this model of (1.14)–(1.16), e.g., [3] and references therein. These numerical results require a detailed well-posedness analysis for the system (1.14)–(1.16). This is the main objective of the present work. These related problems for the macroscopic nonlinear elasticity and viscoelasticity cases were recently studied by Sideris and Thomases [23] and Lin, Liu, and Zhang [16]. For the micro-macro model with dumbbell type of potential there are lots of works, e.g., [7, 15, 17] and references therein. In [28], we gave the globally classical existence theory and a numerical analysis for the Dirichlet initial boundary problem of the system (1.14)–(1.16) in a simple case, the 1+1-dimensional case, and the pressure-driven channel flow. More precisely, it is assumed that the rodlike particles rotate in shear plane. That work is a first step towards the better understanding for currently more sophisticated models (1.14)–(1.16).

In this paper we consider the system (1.14)–(1.16) with the initial data

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0,\quad \psi(\mathbf{x},\mathbf{m},0)=\psi_0(\mathbf{x},\mathbf{m}). \tag{1.22}$$

We denote the space of functions by $H^i(\Omega), i\in\mathbb{N}$, which are in $H^i_{loc}(\mathbb{R}^3)$ (i.e., $u|_\Omega$ for every open bounded set Ω) and which are periodic with period Ω: $\mathbf{u}(x_j+Le_j)=\mathbf{u}(x_j),\ \psi(x_j+Le_j)=\psi(x_j),\ j=1,2,3$. It is easy to see that ψ is also periodic with respect to the variable $\mathbf{m}$ since $\mathbf{m}\in\mathbb{S}^2$. Denote

$$H^i_d(\Omega)=\text{closure of }\mathcal{V}\text{ in }H^i(\Omega),\ \text{ where }\mathcal{V}=C_0^\infty(\Omega)\cap\{\mathbf{u}:\ \text{div }\mathbf{u}=0\}.$$

We can see that $H^i_d(\Omega)$ is a Sobolev space. Moreover, we define the space

$$H^l(\Omega,\mathcal{X}_k)=\left\{\psi:\left|\sum_{j=0}^{l}\sum_{i=0}^{k}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^j\mathcal{R}^i\psi(\mathbf{x},\mathbf{m},t)|^2d\mathbf{m}d\mathbf{x}<\infty\right.\right\}, \tag{1.23}$$

with the natural topology of a Banach space. Now we state our main results:

Theorem 1.1 (local existence). *Assume the following conditions hold:*
*(**A1**) $\mathbf{u}_0\in H^3_d(\Omega)$, div $\mathbf{u}_0=0$ $\mathbf{u}_0$ is periodic,*
*(**A2**) $\psi_0\in\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j)$ and is periodic in $\mathbf{x}$. Moreover, $\psi_0\ge 0$, and*

$$\int_\Omega\int_{|\mathbf{m}|=1}\psi_0(\mathbf{x},\mathbf{m})d\mathbf{m}d\mathbf{x}=1.$$

If $\|\mathbf{u}_0\|^2_{H^3_d(\Omega)}+\|\psi_0\|^2_{\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j)}$ is bounded, then there exists $T'>0$ such that the problem (1.14)–(1.16) with (1.22) exists a solution $(\mathbf{u},\psi)$, which possesses the regularity

$$\mathbf{u}\in L^\infty([0,T');H^3_d(\Omega))\cap L^2([0,T);H^4_d(\Omega)); \tag{1.24}$$

$$\psi\in L^\infty([0,T');\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j))\cap L^2([0,T');\cap_{i+j=4}H^i(\Omega,\mathcal{X}_j)), \tag{1.25}$$

with redefined a set of measure zero if necessary.

THEOREM 1.2 (global existence). *Assume* (**A1**) *and* (**A2**) *and*

$$\|\mathbf{u}_0\|^2_{H^3_d(\Omega))} + \|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0)\cap H^2(\Omega,\mathcal{X}_1)} \leq B,$$

where B is a positive constant, then problem (1.14)–(1.16) *with* (1.22) *exists a global solution* $(\mathbf{u},\psi)$*, which possesses as the following regularity*

$$\mathbf{u} \in L^\infty([0,\infty); H^3_d(\Omega)), \tag{1.26}$$

$$\psi \in L^\infty([0,\infty); H^3(\Omega,\mathcal{X}_0)\cap H^2(\Omega,\mathcal{X}_1)), \tag{1.27}$$

and

$$\|\mathbf{u}\|^2_{L^\infty([0,\infty);H^3_d(\Omega))} + \|\psi\|^2_{L^\infty([0,\infty);H^3(\Omega,\mathcal{X}_0)\cap H^2(\Omega,\mathcal{X}_1))} \leq B$$

provided that

$$Re < \gamma/C_2, \quad \textit{and} \quad De < \varepsilon^2/C_2;$$

here C_2 is a positive constant depending on n and the domain Ω.

Remark 1.1. From the proof of Theorem 1.2 in section 4 we see that C_2 is a large positive constant. Thus Theorem 1.2 requires the Deborah and Reynolds numbers to be small enough.

Remark 1.2. From the proof of Theorem 1.1 we can also obtain the "global" result for the two-dimensional system. That is, for given $T > 0$, there exists a solution under the conditions of Theorem 1.1,

$$\mathbf{u} \in L^\infty([0,T]; H^3_d(\Omega)) \cap L^2([0,T); H^4_d(\Omega)); \tag{1.28}$$

$$\psi \in L^\infty([0,T]; \cap_{i+j=3} H^i(\Omega,\mathcal{X}_j)) \cap L^2([0,T'); \cap_{i+j=4} H^i(\Omega,\mathcal{X}_j)). \tag{1.29}$$

But the bound of this solution depends on T.

These results are similar to that of the Navier–Stokes of traditional models of complex fluids in the case of the spatially periodic solutions [5, 24] for $n = 3$. However, in contrast to traditional models of complex fluids [5, 24] which express polymer stress τ using empirical constitutive relations; τ in (1.11) expresses the polymer stress in terms of the microscopic conformations of the polymers. So the model considered in this work is closer to the original system for polymeric fluids in kinetic theory of polymers. But, on the other hand, it causes the difficulty of well-posedness analysis and numerical simulation since we have to study the configuration equation (1.16) which involves five spatial freedom variables, two of them are in the configuration domain and the others are in the macro flow domain. We utilized the properties of the Laplace–Bertrami operator on compact Riemannian manifold to obtain the existence and the preservation of the positivity of the solution to the linearized equation of (1.16). Then the regularity of the solution was strengthened by the energy estimates method. Thus, in virtue of the properties of the distribution function ψ, we can obtain the regularity of the stress τ. Then it is finished by the local well-posedness analysis of the rigid rodlike model by utilizing the Galerkin approximation and energy methods. However, we specially point out that the nonlinear stress (1.11) and the nonlinear body force (1.13) concerned with the solution of the micro-scale model (1.16) let us only obtain the global solution for small Deborah and Reynolds numbers in virtue of the method which we choose in this paper.

The paper is organized as follows. In section 2, we give the iterative scheme of the system to obtain the existence of the local solution and the scheme alternates between solving an equation of the same type as encountered in incompressible elasticity and solving a linear diffusion equation. Section 3 is devoted to giving the detailed proof of the main lemmas. We will investigate the global existence of the solution in section 4. In this paper C denotes different constant depending only on γ, De, Re, Ω, and ε if there are no special notations. Some times we denote $H_d^p(\Omega), L^p(\Omega)$ by H^p, L^p for brevity.

2. Local solution. In this section we will construct an iterative scheme of the system (1.14)–(1.16) with (1.22) and with which we can obtain the existence of the local solution.

Motivated by the approach [12, 20], we construct an iterative scheme of the system (1.14)–(1.16) with (1.22). Given an iteration ψ^l we determine $\mathbf{u}^{l+1}$ by solving the equations

$$\mathbf{u}_t^{l+1} + (\mathbf{u}^{l+1}\cdot\nabla)\mathbf{u}^{l+1} + \nabla p^{l+1} = \frac{\gamma}{Re}\Delta\mathbf{u}^{l+1} + \frac{1-\gamma}{2Re}\nabla\cdot(\mathbf{D}:\langle\mathbf{mmmm}\rangle)^{l+1} + \frac{1-\gamma}{ReDe}(\nabla\cdot\tau_e^l + F^l), \tag{2.1}$$

$$\nabla\cdot\mathbf{u}^{l+1} = 0 \tag{2.2}$$

with the initial condition $\mathbf{u}^{l+1}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x})$, where

$$(\mathbf{D}:\langle\mathbf{mmmm}\rangle)^{l+1} = \mathbf{D}_{ks}^{l+1}\langle m_i m_j m_k m_s\rangle^l, \tag{2.3}$$

$$\langle m_i m_j m_k m_s\rangle^l = \int_{|\mathbf{m}|=1} m_i m_j m_k m_s \psi^l(\mathbf{x},\mathbf{m},t)d\mathbf{m}, \tag{2.4}$$

$$(\tau_e^l) = -\langle(\mathbf{m}\times\mathcal{R}\mu^l)\mathbf{m}\rangle^l, \tag{2.5}$$

$$\mu^l = \ln\psi^l + U^l, \tag{2.6}$$

$$U^l = \alpha\int_\Omega\int_{|\mathbf{m}'|=1} B(\mathbf{x},\mathbf{x}';\mathbf{m},\mathbf{m}')\psi^l(\mathbf{x}',\mathbf{m}',t)d\mathbf{m}'d\mathbf{x}', \tag{2.7}$$

$$\kappa^{l+1} = (\nabla\mathbf{u}^{l+1})^T,\ F^l = -\langle\nabla\mu^l\rangle. \tag{2.8}$$

Meanwhile, for given $\mathbf{u}^l$, we determine ψ^l from the following initial value problem:

$$\frac{\partial\psi^l}{\partial t} + \nabla\cdot(\mathbf{u}^l\psi^l) = \frac{\varepsilon^2}{De}[\nabla\cdot(\mathbf{I}+\mathbf{mm})\nabla\psi^l + \nabla\cdot(\mathbf{I}+\mathbf{mm})(\psi^l\nabla U^l)] + \frac{1}{De}[\mathcal{R}\cdot\mathcal{R}\psi^l + \mathcal{R}\cdot(\psi^l\mathcal{R}U^l)] - \mathcal{R}\cdot(\mathbf{m}\times\kappa^l\cdot\mathbf{m}\psi^l), \tag{2.9}$$

$$\psi^l(\mathbf{x},\mathbf{m},0) = \psi_0(\mathbf{x},\mathbf{m}). \tag{2.10}$$

Our eventual task is to show that the mapping $\mathcal{M}:\mathbf{u}^l\mapsto\mathbf{u}^{l+1}$ has a fixed point in an appropriate complete space of functions. The fixed point of the mapping is the solution we seek.

We will consider the mapping $\mathcal{M}$ in the function space $S(M,T)$, which is defined as a set of all functions $\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^n (n=2,3)$ with the following properties:

$$\mathbf{u}\in L^\infty([0,T];H_d^3(\Omega))\cap L^2([0,T];H_d^4(\Omega)), \tag{2.11}$$

$$\|\mathbf{u}\|^2_{L^\infty([0,T];H_d^3(\Omega))} + \|\mathbf{u}\|^2_{L^2([0,T];H_d^4(\Omega))} \le M. \tag{2.12}$$

On $S(M,T)$, we define the metric

$$d(\mathbf{u}_1,\mathbf{u}_2) = \|\mathbf{u}_1-\mathbf{u}_2\|_{L^\infty([0,T];H_d^3(\Omega))} + \|\mathbf{u}_1-\mathbf{u}_2\|_{L^2([0,T];H_d^4(\Omega))}. \tag{2.13}$$

It is easy to verify that $S(M,T)$ is complete with the associated metric, and also it is nonempty for large M, as evidenced in [19]. The properties of the mapping $\mathcal{M}$ is established by proving the next three lemmas.

LEMMA 2.1. *Assume that the following bounds hold:*

$$\|\psi^l\|_{L^\infty([0,T];H^3(\Omega,\mathcal{X}_0))\cap L^2([0,T];H^4(\Omega,\mathcal{X}_0))} \leq K. \tag{2.14}$$

Then there exists $T'>0$ such that (2.1)–(2.8) *exists a local solution*

$$\mathbf{u}^{l+1} \in L^\infty([0,T'];H^3_d(\Omega)) \cap L^2([0,T'];H^4_d(\Omega)) \tag{2.15}$$

and

$$\|\mathbf{u}^{l+1}\|^2_{L^\infty([0,T'];H^3_d(\Omega))} + \|\mathbf{u}^{l+1}\|^2_{L^2([0,T'];H^4_d(\Omega))} \leq \phi_1(T',K), \tag{2.16}$$

where if setting $f=\frac{1-\gamma}{ReDe}(\nabla\cdot\tau^l_e+F^l)$,

$$\begin{aligned}\phi_1(T,K) = {} & [\|\mathbf{u}_0\|^2_{H^3_d(\Omega)} + C\|f\|^2_{L^2([0,T],H^2(\Omega))} + C(K)]e^{CK^2T+CK^4T}\\ & + \|\mathbf{u}_0\|^2_{H^3_d(\Omega)} + C\|f\|^2_{L^2([0,T],H^2(\Omega))} + C(K)T.\end{aligned} \tag{2.17}$$

LEMMA 2.2.

$$\tau^l_e \in L^2([0,T];H^3(\Omega)), \quad \textit{and} \quad F^l \in L^2([0,T];H^2(\Omega)) \tag{2.18}$$

and

$$\|\tau^l_e\|_{L^2([0,T];H^3(\Omega))} \leq C\|\psi^l\|_{L^2([0,T];H^3(\Omega,\mathcal{X}_0))}, \tag{2.19}$$

$$\|F^l\|_{L^2([0,T];H^2(\Omega))} \leq C\|\psi^l\|_{L^2([0,T];H^3(\Omega,\mathcal{X}_0))} \tag{2.20}$$

provided that $\psi^l\in L^2([0,T];H^3(\Omega,\mathcal{X}_0))$.

LEMMA 2.3. *Given $\mathbf{u}^l \in S(M,T)$, there exists a unique solution of* (2.9)–(2.10) *which has the regularity*

$$\psi^l \in L^\infty([0,T];\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j)) \cap L^2([0,T];\cap_{i+j=4}H^i(\Omega,\mathcal{X}_j)), \tag{2.21}$$

and

$$\|\psi^l\|^2_{L^2([0,T],\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j))} \leq \|\psi_0\|^2_{\cap_{i+j=3}H^i(\Omega,\mathcal{X}_j)}(CT+CTM)\cdot e^{CT+CMT}. \tag{2.22}$$

By combining Lemmas 2.1–2.3, it follows easily that the map $\mathcal{M}: S(M,T') \longrightarrow S(M,T')$ is a compact operator if M is chosen sufficiently large and T' is chosen sufficiently small. In fact, by using (2.22), we know

$$\|\psi\|^2_{L^2([0,T],H^3(\Omega,\mathcal{X}_0))} \leq \|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0))}(CT+CMT)e^{CT+CMT}.$$

Thus

$$\|f\|_{L^2([0,T],H^2(\Omega))} \leq C\|\nabla\cdot\tau^l_e+F^l\|_{L^2([0,T],H^2(\Omega))} \leq C\|\psi\|_{L^2([0,T],H^3(\Omega,\mathcal{X}_0))}.$$

Then, from (2.17), we have

$$\begin{aligned}& \|\mathbf{u}^{l+1}\|^2_{L^\infty([0,T'];H^3_d(\Omega))} + \|\mathbf{u}^{l+1}\|^2_{L^2([0,T'];H^4_d(\Omega))}\\ & \leq [\|\mathbf{u}_0\|^2_{H^3_d(\Omega)} + \|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0)}(CT+CMT)e^{CT+CMT} + C(K)]e^{CK^2T+CK^4T}\\ & + \|\mathbf{u}_0\|^2_{H^3_d(\Omega)} + \|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0)}(CT+CMT)e^{CT+CMT} + C(K)T.\end{aligned}$$

Now we choose

$$M \geq 6\|\mathbf{u}_0\|^2_{H^3(\Omega)} + 2C(K)(2+T_0) + 12CT_0\|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0))}, \tag{2.23}$$

$$T' \leq T_0 \triangleq \min\left\{ \frac{\ln 2}{C(1+M)}, \frac{\ln 2}{C(K^2+K^4)}, \frac{1}{12C\|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0))}} \right\}. \tag{2.24}$$

Then $e^{CT'+CMT'} \leq 2, e^{CK^2T'+CK^4T'} \leq 2$, and

$$\begin{aligned} &C\|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0)} e^{CT'+CMT'} MT'[1+e^{CK^2T'+CK^4T'}] \leq \frac{M}{2}, \\ &\|\mathbf{u}_0\|^2_{H^3(\Omega)}(1+e^{CK^2T'+CK^4T'}) + C(K)[T' + e^{CK^2T'+CK^4T'}] \\ &+ C\|\psi_0\|^2_{H^3(\Omega,\mathcal{X}_0)} e^{CT'+CMT'} MT'[1+e^{CK^2T'+CK^4T'}]T' \leq \frac{M}{2}. \end{aligned}$$

Thus we have (2.12).

Since $S(M,T')$ is clearly a closed, convex subset of $L^2([0,T], H^4_d(\Omega))$ and is also compact, by the fixed point theorem of Leray and Schauder [14], the conclusion of Theorem 1.1 can be obtained.

3. Proof of lemmas.

3.1. Estimates of u. In this section we will give the proof of Lemma 2.1. Let $\mathbf{u}^{l+1} = w$, $q = p^{l+1}$), and the fourth order tensor

$$A(\mathbf{x},t) = (a_{ijks}), \quad \text{where } a_{ijks}(\mathbf{x},t) = \int_{|\mathbf{m}|=1} m_i m_j m_k m_s \psi^l(\mathbf{x},\mathbf{m},t) d\mathbf{m}. \tag{3.1}$$

(2.1) can be rewritten as

$$w_t + (w\cdot\nabla)w + \nabla q = \frac{\gamma}{Re}\triangle w + f + \frac{1-\gamma}{2Re}\nabla\cdot[\mathbf{D} : A(\mathbf{x},t)], \tag{3.2}$$

$$\nabla\cdot w = 0, \tag{3.3}$$

where $f = \frac{1-\gamma}{ReDe}(\nabla\cdot\tau_e^l + F^l)$. In the following we solve this problem to obtain the existence and uniqueness of the solution by using the Galerkin approximation similar to solving the standard Navier–Stokes equation [5, 24]. The difference here is the appearance of the term $\frac{1-\gamma}{2Re}\nabla\cdot[\mathbf{D} : A(\mathbf{x},t)]$. We can see that it is a good term when we give a priori estimates because we will obtain $-\frac{1-\gamma}{2Re}\int_\Omega\langle(\mathbf{D}:\mathbf{mm})^2\rangle d\mathbf{x}$ while multiplying w to (3.2) and integrating it in Ω. The high derivatives estimates are obtained similarly. Thus we will obtain the result in Lemma 2.1 provided that

$$f \in L^2([0,T]; H^2(\Omega)), \tag{3.4}$$

$$A \in L^\infty([0,T], H^3(\Omega)) \cap L^2([0,T], H^4(\Omega)). \tag{3.5}$$

The regularity of f and A can be easily obtained from the estimates of τ_e and F in section 3.2 and ones of ψ in section 3.3, respectively. Why we need conditions (3.4) and (3.5), for the aim of self-contained, will be answered in the outline of the proof to Lemma 2.1 when $n = 2$ and $n = 3$ in the appendix. We also refer the readers to [5, 24] for further details.

3.2. Estimates of τ_e and F.

Proof of Lemma 2.2. From the definition of τ_e, it is straightforward to obtain its estimates from the assumption of ψ. In fact,

$$
\begin{aligned}
\tau_e &= -\int_{|\mathbf{m}|=1} (\mathbf{m}\times\mathcal{R}\mu)\mathbf{m}\psi d\mathbf{m} \\
&= -\int_{|\mathbf{m}|=1} \left[\mathbf{m}\times\left(\frac{1}{\psi}\mathcal{R}\psi+\mathcal{R}U\right)\right]\mathbf{m}\psi d\mathbf{m} \\
&= -\int_{|\mathbf{m}|=1} (\mathbf{m}\times\mathcal{R}\psi)\mathbf{m}d\mathbf{m} - \int_{|\mathbf{m}|=1} (\mathbf{m}\times\mathcal{R}U)\mathbf{m}\psi d\mathbf{m} \\
&= -\mathbf{I} + 3\int_{|\mathbf{m}|=1} \mathbf{m}\mathbf{m}\psi d\mathbf{m} - \int_{|\mathbf{m}|=1} (\mathbf{m}\times\mathcal{R}U)\mathbf{m}\psi d\mathbf{m},
\end{aligned}
$$

where we used the property of operators $\mathcal{R}$ and $\int_{|\mathbf{m}|=1}\cdot\, d\mathbf{m}$ (p. 293 in [6]),

$$
\int_{|\mathbf{m}|=1} G(\mathbf{m})\mathcal{R}[F(\mathbf{m})]d\mathbf{m} = -\int_{|\mathbf{m}|=1} F(\mathbf{m})\mathcal{R}[G(\mathbf{m})]d\mathbf{m}. \tag{3.6}
$$

From the definition (2.7) of U,

$$
\begin{aligned}
U &= \alpha\int_\Omega\int_{|\mathbf{m}'|=1} \frac{1}{\varepsilon^3}\chi\left(\frac{\mathbf{x}-\mathbf{x}'}{\varepsilon}\right)|\mathbf{m}\times\mathbf{m}'|^2\psi^l(\mathbf{x}',\mathbf{m}',t)d\mathbf{m}'d\mathbf{x}' \\
&= \alpha\int_\Omega\int_{|\mathbf{m}'|=1} \frac{1}{\varepsilon^3}\chi\left(\frac{\mathbf{x}-\mathbf{x}'}{\varepsilon}\right)\psi^l(\mathbf{x}',\mathbf{m}',t)d\mathbf{m}'d\mathbf{x}' \\
&\quad - \alpha\mathbf{m}\mathbf{m} : \int_\Omega\int_{|\mathbf{m}'|=1} \frac{1}{\varepsilon^3}\chi\left(\frac{\mathbf{x}-\mathbf{x}'}{\varepsilon}\right)\mathbf{m}'\mathbf{m}'\psi^l(\mathbf{x}',\mathbf{m}',t)d\mathbf{m}'d\mathbf{x}';
\end{aligned} \tag{3.7}
$$

here $\varepsilon > 0$ is in (1.5) and $\chi(\mathbf{x})$ is a smooth kernel, and we can see that $U \in C^\infty(\Omega\times\mathbb{S}^2)$. But the bounds of the derivatives of U with respect to $\mathbf{x}$ and $\mathbf{m}$ depend on ε, denoted by $C(\varepsilon)$.

Therefore

$$
\|\tau_e\|^2_{L^2(\Omega)} \le C(1+\|\psi\|^2_{L^2(\Omega,\mathcal{X}_0)}) \tag{3.8}
$$

since $\psi \in L^2([0,T],H^3(\Omega,\mathcal{X}_0))$. Higher derivatives can be obtained similarly. Thus (2.19) is obtained. By the same way we can obtain

$$
F = \langle\nabla\mu\rangle = \int_{|\mathbf{m}|=1} (\nabla\psi+\psi\nabla U)d\mathbf{m} \in L^2([0,T];H^2(\Omega)) \tag{3.9}
$$

since $\psi \in L^2([0,T],H^3(\Omega,\mathcal{X}_0))$. Hence (2.20) is obtained by using (3.9).

3.3. Estimates of ψ. In this section we first review some results about the Laplace–Beltrami operator on a compact Riemannian manifold which will be utilized to show the existence of the solution of (2.9)–(2.10). Then we will give the regularity estimates of the solution to (2.9)–(2.10).

3.3.1. Review about the Laplace–Beltrami operator. We recall some known results about the Laplace–Beltrami operator on a compact Riemannian manifold (M_n, g); see (section 4 of Chap. 4) of [1] and [14].

LEMMA 3.1. *Let $v(Q,t)$ be a continuous function on $M_n \times [0, t_0]$. Assume $v \geq 0$ on $M_n \times \{0\}$ and $\partial M_n \times [0, t_0]$. Moreover, it satisfies*

$$\partial v/\partial t \geq \triangle_{M_n} v + b^i(Q,t)\partial_i v + c(Q,t)v \tag{3.10}$$

with the b^i, c bounded. Then we have always $v \geq 0$.

Proof. This lemma can be proved similar to the proof of the maximum principle in p. 130 of [1]. Let $C_m = \max_{M_n \times \mathbb{R}^+} |c(Q,t)|$ and $w = e^{-(C_m+1)t}v$. Then w and v have the same sign. Since

$$\partial w/\partial t = e^{-(C_m+1)t}[\partial v/\partial t - (C_m + 1)v],$$

we have

$$\partial w/\partial t \geq \triangle_{M_n} w + b^i(Q,t)\partial_i w + [c(Q,t) - C_m - 1]w. \tag{3.11}$$

Assume w is negative somewhere and let (Q,t) be a point where w achieves its minimum. Then $\triangle_{M_n} w \geq 0, \partial_i w = 0$, and $\partial w/\partial t \leq 0$. Thus (3.11) implies $w(Q,t) \geq 0$, which yields a contradiction. □

LEMMA 3.2. *For every $g \in L^\infty([0,t_0], L^p(M_n))$, there exists a unique*

$$v \in W^{1,\infty}([0,t_0], W^{2,p}(M_n))$$

satisfying

$$\partial v^i/\partial t = \triangle_{M_n} v^i + a^i_j \partial_i v^j + b^i_j v^j + g^i, \tag{3.12}$$

for $1 \leq \alpha \leq k$ and $v(P,0) \equiv 0$, $P \in M_n$, where $v^i (i = 1,2,\cdots,k)$ are k unknown functions in $M_n \times [0,\infty)$ and $g^i(i = 1,2,\cdots,k)$ are k given functions on $M_n \times [0,\infty)$. The coefficients a^i_j and b^i_j are supposed to be smooth.

Remark 3.1. In fact, the condition on the coefficients a^i_j and b^i_j in the above lemma can be weakened to be bounded in $L^\infty([0,t_0] \times M_n)$. The proof is analogous.

3.3.2. Estimate of ψ.

Proof of Lemma 2.3. In this section we simply denote

$$\phi(\mathbf{x},\mathbf{m},t) = \psi^l(\mathbf{x},\mathbf{m},t) \qquad W = U^l. \tag{3.13}$$

Then (2.9)–(2.10) can be rewritten as

$$\frac{\partial \phi}{\partial t} + \mathbf{u}^l \cdot \nabla\phi = \frac{\varepsilon^2}{De}[\nabla \cdot (\mathbf{I} + \mathbf{mm})\nabla\phi + \nabla \cdot (\mathbf{I} + \mathbf{mm})(\phi\nabla W)]$$
$$+\frac{1}{De}[\mathcal{R}\cdot\mathcal{R}\phi + \mathcal{R}\cdot(\phi\mathcal{R}W)] - \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\phi), \tag{3.14}$$

$$\phi(\mathbf{x},\mathbf{m},0) = \psi_0(\mathbf{x},\mathbf{m}). \tag{3.15}$$

Here we still denote $(\nabla\mathbf{u}^l(\mathbf{x},t))^T$ by κ. Equation (3.14) is a nonlinear differential-integral equation. We first linearize (3.14) replacing W by U^{l-1} and obtaining the solution of this linear equation with the initial data (3.15). If now we transform the Cartesian coordinates $\mathbf{m}$ to the local coordinates (θ,φ) of the unit sphere $\mathbb{S}^2$ in

$\mathbb{R}^3$ by $m_1(\theta,\varphi) = \sin\theta\cos\varphi$, $m_2(\theta,\varphi) = \sin\theta\sin\varphi$, and $m_3 = \cos\varphi$, the operator $\mathcal{R}\cdot\mathcal{R}$ is the Laplace–Beltrami operator on the unit sphere [4]. Thus the operator $\frac{\varepsilon^2}{De}\triangle + \frac{1}{De}\mathcal{R}\cdot\mathcal{R}$ is also the Laplace–Beltrami operator. Moreover, the coefficients of (3.14) are all bounded and smooth when κ is bounded, which is obtained from $\mathbf{u} \in L^\infty([0,T]; H_d^3(\Omega))$. Therefore we can utilize the above lemmas to show the existence and nonnegativeness of the solution to the linearized equation when the initial data is nonnegative. In fact, using Lemma 3.2 and Remark 3.1 for $v = \phi - \psi_0$, we obtain that (3.14)–(3.15) possesses a unique global solution $v \in W^{1,\infty}([0,T], H^2(\Omega,\mathcal{X}_0) \cap H^1(\Omega,\mathcal{X}_1) \cap L^2(\Omega,\mathcal{X}_2))$. Therefore, $\psi^l \in W^{1,\infty}([0,T]; H^2(\Omega,\mathcal{X}_0) \cap H^1(\Omega,\mathcal{X}_1) \cap L^2(\Omega,\mathcal{X}_2))$ for given $T > 0$. Now the coefficient of ϕ is $\frac{\varepsilon^2}{De}\nabla\cdot(\mathbf{I} + \mathbf{mm})\nabla W + \frac{1}{De}\mathcal{R}\cdot\mathcal{R}W - \mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m})$ and it is bounded since $\psi^{l-1} \in L^\infty([0,T]; \cap_{i+j=3} H^i(\Omega,\mathcal{X}_j))$ and $\mathbf{u}^l \in L^\infty([0,T]; H_d^3(\Omega))$. Then it follows that the positivity is preserved by Lemma 3.1, and by integrating both sides of (3.14) we find that

$$\int_\Omega\int_{|\mathbf{m}|=1} \phi(\mathbf{x},\mathbf{m},t)d\mathbf{m}d\mathbf{x} = \int_\Omega\int_{|\mathbf{m}|=1} \phi(\mathbf{x},\mathbf{m},0)d\mathbf{m}d\mathbf{x} = 1 \text{ for all } t.$$

We can obtain the solution of (3.14) with (3.15) from the solution of the linearized equation to construct a suitable Sobolev space and use the Schauder fixed point theorem in the standard argument. So here we omit the detail.

Next we will further prove the regularity of the solution $\psi^l(\mathbf{x},\mathbf{m},t)$ if the initial data is more regular.

I. The estimate of $\boldsymbol{\phi}$. Multiplying ϕ to (3.14) and integrating on $\mathbb{S}^2$ with respect to $\mathbf{m}$ and in Ω with respect to $\mathbf{x}$ yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1} |\phi|^2 d\mathbf{m}d\mathbf{x} + \frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1} (|\nabla\phi|^2 + |\mathbf{m}\cdot\nabla\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad + \frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1} |\mathcal{R}\phi|^2 d\mathbf{m}d\mathbf{x}\\
&= -\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1} (\mathbf{I}+\mathbf{mm})\phi\nabla W\cdot\nabla\phi d\mathbf{m}d\mathbf{x} - \frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1} \phi\mathcal{R}W\cdot\mathcal{R}\phi d\mathbf{m}d\mathbf{x}\\
&\quad + \int_\Omega\int_{|\mathbf{m}|=1} \mathbf{m}\times\kappa\cdot\mathbf{m}\phi\cdot\mathcal{R}\phi d\mathbf{m}d\mathbf{x},
\end{aligned}$$

where we used the periodic condition. Thus we can obtain the estimate,

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1} |\phi|^2 d\mathbf{m}d\mathbf{x} + \frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1} (|\nabla\phi|^2 + |\mathbf{m}\cdot\nabla\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad + \frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1} |\mathcal{R}\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1} |\phi|^2 d\mathbf{m}d\mathbf{x} + C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1} |\phi|^2 d\mathbf{m}d\mathbf{x},
\end{aligned} \tag{3.16}$$

where we used the fact that $|\kappa|$ is also bounded in $\Omega\times[0,T]$ redefined by a set of measure zero from (3.7) since $\mathbf{u}^l \in L^\infty([0,T]; H^3(\Omega))$ and $|\nabla W|, |\mathcal{R}W|$ are bounded by $C(\varepsilon)$. Therefore from (3.16) we obtain

$$\sup_{t\in[0,T]}\int_\Omega\int_{|\mathbf{m}|=1} |\phi|^2 d\mathbf{m}d\mathbf{x} \le \|\psi_0\|^2_{L^2(\Omega,\mathcal{X}_0)} e^{CT+CMT} \triangleq N_1, \tag{3.17}$$

and

$$\frac{\varepsilon^2}{De}\int_0^T\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\phi|^2+|\mathbf{m}\cdot\nabla\phi|^2)d\mathbf{m}d\mathbf{x}dt+\frac{1}{De}\int_0^T\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}dt \leq CN_1T+CMN_1T\triangleq N_2. \tag{3.18}$$

This shows that

$$\phi\in L^\infty([0,T];L^2(\Omega,\mathcal{X}_0))\cap L^2([0,T];L^2(\Omega,\mathcal{X}_1))\cap L^2([0,T];H^1(\Omega,\mathcal{X}_0))$$

provided that $\psi_0\in L^2(\Omega,\mathcal{X}_0)$ and $\mathbf{u}^l\in S(M,T)$.

II. The estimate of $\boldsymbol{\mathcal{R}\phi}$, $\boldsymbol{\nabla\phi}$. Applying the operator $\mathcal{R}$ to (3.14) and multiplying $\mathcal{R}\phi$ to (3.14) and integrating on $\mathbb{S}^2$ with respect to $\mathbf{m}$ and in Ω with respect to $\mathbf{x}$, respectively, yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\mathcal{R}\phi|^2+|\mathbf{m}\cdot\nabla\mathcal{R}\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\cdot\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}\\
&=-\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}\{\mathcal{R}\cdot(\mathbf{I}+\mathbf{mm})\cdot\nabla\phi\cdot\nabla\mathcal{R}\phi-\nabla\cdot[(\mathbf{I}+\mathbf{mm})(\phi\nabla W)]\mathcal{R}\cdot\mathcal{R}\phi\}d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}\mathcal{R}\cdot(\phi\mathcal{R}W)\mathcal{R}\cdot\mathcal{R}\phi d\mathbf{m}d\mathbf{x}+\int_\Omega\int_{|\mathbf{m}|=1}\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\phi)\mathcal{R}\cdot\mathcal{R}\phi d\mathbf{m}d\mathbf{x}.
\end{aligned}$$

Thus, by using $|\nabla W|,|\nabla^2W|$, $|\nabla\mathcal{R}W|$, $|\mathcal{R}W|,|\mathcal{R}^2W|$ are bounded for all $\mathbf{x}\in\Omega$ and $\mathbf{m}\in\mathbb{S}^2$, and $|\kappa|$ is bounded by redefined in a set of measure zero, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\mathcal{R}\phi|^2+|\mathbf{m}\cdot\nabla\mathcal{R}\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\cdot\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}\\
&\leq C\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}\phi|^2+|\nabla_{\mathbf{x}}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}.
\end{aligned}\tag{3.19}$$

Similarly, differentiating (3.14) with respect to $\mathbf{x}$, then multiplying $\nabla\phi$ and integrating it, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla\phi|^2d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\triangle\phi|^2+|\mathbf{m}\cdot\nabla^2\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla\mathcal{R}\phi|^2d\mathbf{m}d\mathbf{x}\\
&\leq C\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\phi|^2+|\mathcal{R}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}.
\end{aligned}\tag{3.20}$$

Here we used that $|\nabla W|, |\nabla^2 W|$, $|\mathcal{R}W|, |\mathcal{R}^2 W|$ and κ are bounded. Combination of (3.16), (3.19), and (3.20) and application of the Grownwall inequality yields

$$\begin{aligned}
&\sup_{t\in[0,T]} \int_\Omega \int_{|\mathbf{m}|=1} (|\nabla\phi|^2 + |\mathcal{R}\phi|^2 + |\phi|^2) d\mathbf{m} d\mathbf{x} \\
&\quad \leq \int_\Omega \int_{|\mathbf{m}|=1} (|\nabla\psi_0|^2 + |\mathcal{R}\psi_0|^2 + |\psi_0|^2) d\mathbf{m} d\mathbf{x}\, e^{CT+CM} \triangleq N_3, \\
&\frac{\varepsilon^2}{De} \int_0^T \int_\Omega \int_{|\mathbf{m}|=1} (|\nabla\phi|^2 + |\triangle\phi|^2 + |\nabla\mathcal{R}\phi|^2) d\mathbf{m} d\mathbf{x} dt \\
&\quad + \frac{1}{De} \int_0^T \int_\Omega \int_{|\mathbf{m}|=1} |\nabla\mathcal{R}\phi|^2 + |\mathcal{R}\cdot\mathcal{R}\phi|^2 d\mathbf{m} d\mathbf{x} dt \leq CN_3 T + CMN_3 T.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\phi \in L^\infty([0,t]; \cap_{i+j=1,i,j\geq 0} H^i(\Omega, \mathcal{X}_j)) \quad \text{and} \\
&\phi \in L^2([0,t]; \cap_{i+j=2,i,j\geq 0} H^i(\Omega, \mathcal{X}_j))
\end{aligned} \tag{3.21}$$

provided that $\psi_0 \in \cap_{i+j=1,i,j\geq 0} H^i(\Omega, \mathcal{X}_j)$.

III. The estimate of $\mathcal{R}^2\phi, \nabla^2\phi$, and $\nabla\mathcal{R}\phi$. Similar to that in section II, we can obtain the following estimates:

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_{|\mathbf{m}|=1} |\nabla\mathcal{R}\phi|^2 d\mathbf{m} d\mathbf{x} + \frac{\varepsilon^2}{De} \int_\Omega \int_{|\mathbf{m}|=1} (|\triangle\mathcal{R}\phi|^2 + |\mathbf{m}\cdot\nabla^2\mathcal{R}\phi|^2) d\mathbf{m} d\mathbf{x} \\
&\quad + \frac{1}{De} \int_\Omega \int_{|\mathbf{m}|=1} |\nabla\mathcal{R}\cdot\mathcal{R}\phi|^2 d\mathbf{m} d\mathbf{x} \\
&\leq C \int_\Omega \int_{|\mathbf{m}|=1} (|\nabla\mathcal{R}\phi|^2 + |\nabla^2_{\mathbf{x}}\phi|^2 + |\nabla\phi|^2 + |\mathcal{R}\phi|^2 + |\phi|^2) d\mathbf{m} d\mathbf{x} \\
&\quad + C|\kappa| \int_\Omega \int_{|\mathbf{m}|=1} (|\mathcal{R}^2\phi|^2 + |\mathcal{R}\phi|^2 + |\phi|^2) d\mathbf{m} d\mathbf{x},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \int_{|\mathbf{m}|=1} |\mathcal{R}^2\phi|^2 d\mathbf{m} d\mathbf{x} + \frac{\varepsilon^2}{De} \int_\Omega \int_{|\mathbf{m}|=1} (|\nabla\mathcal{R}\cdot\mathcal{R}\phi|^2 + |\mathbf{m}\cdot\nabla\mathcal{R}^2\phi|^2) d\mathbf{m} d\mathbf{x} \\
&\quad + \frac{1}{De} \int_\Omega \int_{|\mathbf{m}|=1} |\mathcal{R}(\mathcal{R}\cdot\mathcal{R}\phi)|^2 d\mathbf{m} d\mathbf{x} \\
&\leq C \int_\Omega \int_{|\mathbf{m}|=1} (|\mathcal{R}^2\phi|^2 + |\mathcal{R}\phi|^2 + |\nabla\phi|^2 + |\nabla\mathcal{R}\phi|^2 + |\phi|^2) d\mathbf{m} d\mathbf{x} \\
&\quad + C|\kappa| \int_\Omega \int_{|\mathbf{m}|=1} (|\mathcal{R}^2\phi|^2 + |\mathcal{R}\phi|^2 + |\phi|^2) d\mathbf{m} d\mathbf{x},
\end{aligned} \tag{3.23}$$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\phi|^2 d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\triangle\phi|^2+|\mathbf{m}\cdot\nabla^3\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}\triangle_{\mathbf{x}}\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^2\phi|^2+|\nabla\phi|^2+|\mathcal{R}\phi|^2+|\nabla\mathcal{R}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\nabla\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}\phi|^2+|\nabla\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\mathcal{R}\phi|^2+|\nabla\phi|^2)d\mathbf{m}d\mathbf{x}.
\end{aligned}\tag{3.24}
$$

Combining the above estimates and applying the Gronwall inequality with $|\nabla\kappa|\in L^2([0,T],H^4(\Omega))$, we can obtain

$$
\begin{aligned}
\|\phi\|^2_{L^\infty([0,t];\cap_{i+j=2,i,j\ge0}H^i(\Omega,\mathcal{X}_j))}\le\|\psi_0\|^2_{\cap_{i+j=2,i,j\ge0}H^i(\Omega,\mathcal{X}_j)}e^{CT+CMT}\triangleq N_4,\\
\|\phi\|^2_{L^2([0,t];\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j))}\le CN_4T+CMN_4T.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\phi\in L^\infty([0,t];\cap_{i+j=2,i,j\ge0}H^i(\Omega,\mathcal{X}_j))\quad\text{and}\\
&\phi\in L^2([0,t];\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j))
\end{aligned}\tag{3.25}
$$

provided that $\psi_0\in\cap_{i+j=2,i,j\ge0}H^i(\Omega,\mathcal{X}_j)$.

IV. The estimate of $\mathcal{R}^i\nabla^j\phi(i+j=3)$. Analogously to that in section III, we can obtain the following estimates:

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}^3\phi|^2 d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\mathcal{R}^3\phi|^2+|\mathbf{m}\cdot\nabla\mathcal{R}^3\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}^4\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}^3\phi|^2+|\mathcal{R}^2\phi|^2+|\mathcal{R}\phi|^2+|\phi|^2\\
&\quad+|\nabla\mathcal{R}^2\phi|^2+|\nabla\mathcal{R}\phi|^2+|\nabla\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}^3\phi|^2+|\mathcal{R}^2\phi|^2+|\mathcal{R}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x},
\end{aligned}\tag{3.26}
$$

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla\mathcal{R}^2\phi|^2 d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^2\mathcal{R}^2\phi|^2+|\mathbf{m}\cdot\nabla^2\mathcal{R}^2\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\mathcal{R}^3\nabla\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^2\mathcal{R}\phi|^2+|\nabla^2\phi|^2+|\nabla\mathcal{R}^2\phi|^2+|\nabla\mathcal{R}\phi|^2\\
&\quad+|\nabla\phi|^2+|\mathcal{R}^2\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}|\nabla\phi|^2 d\mathbf{m}d\mathbf{x}+C|\nabla\kappa|\int_\Omega\int_{|\mathbf{m}|=1}|\phi|^2 d\mathbf{m}d\mathbf{x},
\end{aligned}\tag{3.27}
$$

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\mathcal{R}\phi|^2 d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^3\mathcal{R}\phi|^2+|\mathbf{m}\cdot\nabla^3\mathcal{R}\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\mathcal{R}^2\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^3\phi|^2+|\nabla^2\mathcal{R}\phi|^2+|\nabla\mathcal{R}\phi|^2+|\mathcal{R}\phi|^2\\
&\quad+|\nabla^2\phi|^2+|\nabla\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\phi|^2+|\nabla\mathcal{R}\phi|^2+|\nabla^2\mathcal{R}\phi|^2+|\nabla\mathcal{R}^2\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\nabla\kappa|\int_\Omega\int_{|\mathbf{m}|=1}(|\mathcal{R}^2\phi|^2+|\nabla\mathcal{R}\phi|^2+|\mathcal{R}\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x},
\end{aligned}\tag{3.28}$$

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^3\phi|^2 d\mathbf{m}d\mathbf{x}+\frac{\varepsilon^2}{De}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^4\phi|^2+|\mathbf{m}\cdot\nabla^4\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+\frac{1}{De}\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^3\mathcal{R}\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\le C\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^3\phi|^2+|\nabla^2\phi|^2+|\nabla\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C|\kappa|\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^3\phi|^2 d\mathbf{m}d\mathbf{x}+C|\nabla\kappa|\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\phi|^2 d\mathbf{m}d\mathbf{x}\\
&\quad+C\|\phi\|^2_{\mathcal{X}_0}\int_\Omega|\nabla^3\kappa|^2 d\mathbf{x}+C\|\nabla^3\kappa\|^2_{L^2}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla^2\phi|^2+|\nabla\phi|^2)d\mathbf{m}d\mathbf{x}\\
&\quad+C\|\nabla^2\kappa\|^2_{L^2}\int_\Omega\int_{|\mathbf{m}|=1}(|\nabla\phi|^2+|\phi|^2)d\mathbf{m}d\mathbf{x}.
\end{aligned}\tag{3.29}$$

In the last estimate we used the following estimates:

$$\begin{aligned}
&\int_\Omega\left[|\nabla^2\kappa|\int_{|\mathbf{m}|=1}|\nabla\phi||\nabla^3\mathcal{R}\phi|d\mathbf{m}\right]d\mathbf{x}\\
&\le\int_\Omega|\nabla^2\kappa|\|\nabla\phi\|_{L^2(\mathbb{S}^2)}\|\nabla^3\mathcal{R}\phi\|_{L^2(\mathbb{S}^2)}d\mathbf{x}\\
&\le\|\nabla^2\kappa\|_{L^3(\Omega)}\|\|\nabla\phi\|_{L^2(\mathbb{S}^2)}\|_{L^6(\Omega)}\|\|\nabla^3\mathcal{R}\phi\|_{L^2(\mathbb{S}^2)}\|_{L^2(\Omega)}\\
&\le\|\nabla^2\kappa\|_{H^{\frac12}(\Omega)}\|\|\nabla\phi\|_{L^2(\mathbb{S}^2)}\|_{H^1(\Omega)}\|\|\nabla^3\mathcal{R}\phi\|_{L^2(\mathbb{S}^2)}\|_{L^2(\Omega)}\\
&\le\|\nabla^2\kappa\|_{H^4(\Omega)}\|\phi\|_{H^2(\Omega,\mathcal{X}_0)}\|\|\nabla^3\mathcal{R}\phi\|_{L^2(\mathbb{S}^2)}\|_{L^2(\Omega)}\\
&\le\frac{1}{De}\|\nabla^3\mathcal{R}\phi\|^2_{L^2(\Omega,\mathcal{X}_0)}+C(De)\|\mathbf{u}\|^2_{H^4(\Omega)}\|\phi\|^2_{H^2(\Omega,\mathcal{X}_0)}.
\end{aligned}\tag{3.30}$$

Combining the above estimates and applying the Gronwall inequality by using $\mathbf{u}\in L^2([0,T],H^4(\Omega))$, we can obtain

$$\|\phi\|_{L^\infty([0,t];\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j))}\le\|\psi_0\|_{\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j)}e^{CT+CMT}\triangleq N_6,$$
$$\|\phi\|_{L^2([0,t];\cap_{i+j=4,i,j\ge0}H^i(\Omega,\mathcal{X}_j))}\le CN_6T+CMN_6T.$$

This implies that

$$\phi\in L^\infty([0,t];\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j))\quad\text{and}$$
$$\phi\in L^2([0,t];\cap_{i+j=4,i,j\ge0}H^i(\Omega,\mathcal{X}_j))\tag{3.31}$$

provided that $\psi_0\in\cap_{i+j=3,i,j\ge0}H^i(\Omega,\mathcal{X}_j)$.

Up to now, we have completed the proof of Lemma 2.3.

4. Global solution. In this section, we will show that the local solution obtained in Theorem 1.1 is actually defined for $t \in \mathbb{R}^+$ if the Deborah and Reynolds numbers are small enough. To this end we derive some a priori bounds, satisfied by that solution. Notably, C_1 in this section denotes different constants depending only on n and Ω.

4.1. Some a priori estimates. Recall that the local solution $(\mathbf{u}, \psi)$ obtained in Theorem 1.1 satisfies (1.24) and (1.25) together with (1.14) and (1.16). Now we give the detail of a priori estimates for the case $n = 3$. For $n = 2$, it can be similarly obtained.

For (1.14), the inequality (A.36) in the appendix implies that there exists constant C such that

(4.1)
$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{H^3_d} + \left(\frac{\gamma}{Re} - 5\epsilon - \frac{6\epsilon}{11}\right)\|\mathbf{u}\|^2_{H^4} \le \frac{5\epsilon}{11}\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \frac{C_1}{\epsilon}[\|\mathbf{u}\|^{14}_{H^1} + \|\mathbf{u}\|^4_{H^1} + \|\mathbf{u}\|^{24}_{H^2} + \|\psi\|^{20}_{H^1(\Omega,\mathcal{X}_0)} + \|\psi\|^{24}_{H^2(\Omega,\mathcal{X}_0)} + \|\psi\|^8_{H^3(\Omega,\mathcal{X}_0)} + \|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}], \quad (n = 3).$$

Here in the left side of the above inequality we omit the term $\frac{1-\gamma}{2Re}\int_\Omega \langle|\mathbf{mm} : \nabla^3 D|^2\rangle d\mathbf{x}$ since it is positive. The main difficulty to obtain (4.1) is to estimate

$$\int_\Omega \left|\nabla^3\left(\mathbf{D} : \int_{|\mathbf{m}|=1} \mathbf{mmmm}\nabla\psi d\mathbf{m}\right)\nabla^4\mathbf{u}\right| d\mathbf{x}.$$

Now we will use the following inequalities:

$$\begin{aligned}\int_\Omega |\nabla^2\mathbf{D}\int_{|\mathbf{m}|=1} &\mathbf{mmmm}\nabla\psi d\mathbf{m}\nabla^4\mathbf{u}|d\mathbf{x} \\ &\le C_1\|\nabla^4\mathbf{u}\|_{L^2}\|\nabla^2\mathbf{D}\|_{L^3}\|\nabla\psi\|_{L^6(\Omega,\mathcal{X}_0)} \\ &\le C_1\|\mathbf{u}\|_{H^4}\|\nabla^2\mathbf{D}\|_{H^{\frac{1}{2}}}\|\nabla\psi\|_{H^1(\Omega,\mathcal{X}_0)} \\ &\le C_1\|\mathbf{u}\|_{H^4}\|\mathbf{u}\|_{H^{3+\frac{1}{2}}}\|\psi\|_{H^2(\Omega,\mathcal{X}_0)} \\ &\le C_1\|\mathbf{u}\|_{H^4}\|\mathbf{u}\|^{\frac{5}{6}}_{H^4}\|\mathbf{u}\|^{\frac{1}{6}}_{H^1}\|\psi\|_{H^2(\Omega,\mathcal{X}_0)} \\ &\le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^1}\|\psi\|^{12}_{H^2(\Omega,\mathcal{X}_0)},\end{aligned} \tag{4.2}$$

where ϵ is a positive constant and will be chosen later. Here we used the Hölder inequality ($\int_\Omega |abc|d\mathbf{x} \le \|a\|_{L^2}\|a\|_{L^3}\|c\|_{L^6}$), Sobolev embedding theorems ($H^1(\Omega) \subset L^6(\Omega), H^{1/2}(\Omega) \subset L^3(\Omega)$ for $n = 3$), the interpolation inequality ($\|\mathbf{u}\|_{H^{3+\frac{1}{2}}} \le C_1\|\mathbf{u}\|^{\frac{5}{6}}_{H^4} \|\mathbf{u}\|^{\frac{1}{6}}_{H^1}$), and Young's inequality ($ab \le \epsilon a^{12/11} + \frac{1}{\epsilon}b^{12}$), respectively. Similarly, we have

$$\begin{aligned}\int_\Omega |\nabla\mathbf{D}\int_{|\mathbf{m}|=1} &\mathbf{mmmm}\nabla^2\psi d\mathbf{m}\nabla^4\mathbf{u}|d\mathbf{x} \\ &\le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^1}\|\psi\|^4_{H^3(\Omega,\mathcal{X}_0)}.\end{aligned}$$

Further, a different estimate from the above two is

$$
\begin{aligned}
&\int_\Omega |\mathbf{D}\int_{|\mathbf{m}|=1} \mathbf{mmmm}\nabla^3\psi d\mathbf{m}\nabla^4\mathbf{u}|d\mathbf{x} \\
&\quad\le C_1\|\nabla^4\mathbf{u}\|_{L^2}\|\mathbf{D}\|_{L^6}\|\nabla^3\psi\|_{L^3(\Omega,\mathcal{X}_0)} \\
&\quad\le C_1\|\mathbf{u}\|_{H^4}\|\mathbf{D}\|_{H^1}\|\nabla^3\psi\|_{H^{\frac{1}{2}}(\Omega,\mathcal{X}_0)} \\
&\quad\le C_1\|\mathbf{u}\|_{H^4}\|\mathbf{u}\|_{H^2}\|\psi\|^{\frac{1}{6}}_{H^1(\Omega,\mathcal{X}_0)}\|\psi\|^{\frac{5}{6}}_{H^4(\Omega,\mathcal{X}_0)} \\
&\quad\le \epsilon\left[\|\mathbf{u}\|_{H^4}\|\psi\|^{\frac{5}{6}}_{H^4(\Omega,\mathcal{X}_0)}\right]^{\frac{12}{11}} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^{12}_{H^2}\|\psi\|^{10}_{H^1(\Omega,\mathcal{X}_0)} \\
&\quad\le \epsilon\left[\|\mathbf{u}\|^{\frac{12}{11}}_{H^4}\|\psi\|^{\frac{10}{11}}_{H^4(\Omega,\mathcal{X}_0)}\right] + \frac{C_1}{\epsilon}\|\mathbf{u}\|^{12}_{H^2}\|\psi\|^{10}_{H^1(\Omega,\mathcal{X}_0)} \\
&\quad\le \frac{6\epsilon}{11}\|\mathbf{u}\|^2_{H^4} + \frac{5\epsilon}{11}\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^{12}_{H^2}\|\psi\|^{10}_{H^1(\Omega,\mathcal{X}_0)}. \qquad (4.3)
\end{aligned}
$$

For the convect term we used the same estimate in [24]

$$\int |(\mathbf{u}\cdot\nabla)\mathbf{u}\triangle^3\mathbf{u}|dx \le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^{14}_{H^1},$$

and by using the results of Lemma 2.2 we have

$$
\begin{aligned}
&\int_\Omega |\nabla^3\tau_e\nabla^4\mathbf{u}|d\mathbf{x} \le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\tau_e\|^2_{H^3} \le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}, \\
&\int_\Omega |\nabla^2 F\nabla^4\mathbf{u}|d\mathbf{x} \le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|F\|^2_{H^2} \le \epsilon\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}.
\end{aligned}
$$

Combination of all yields (4.1). For the equation of ψ, we can obtain the following estimates:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)} + \left(\frac{\varepsilon^2}{De} - 5\epsilon - \frac{\epsilon}{11}\right)\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \left(\frac{1}{De} - \frac{5\epsilon}{11}\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)} \\
&\le \frac{5\epsilon}{11}\|\mathbf{u}\|^2_{H^4} + \frac{C_1}{\epsilon}[\|\mathbf{u}\|^4_{H^3} + \|\mathbf{u}\|^{24}_{H^2} + \|\mathbf{u}\|^4_{H^2} + \|\psi\|^4_{H^3(\Omega,\mathcal{X}_0)} + \|\psi\|^4_{H^1(\Omega,\mathcal{X}_0)} \\
&\quad + \|\psi\|^{24}_{H^1(\Omega,\mathcal{X}_1)} + \|\psi\|^4_{H^2(\Omega,\mathcal{X}_1)} + \|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}]. \qquad (4.4)
\end{aligned}
$$

Here the difficulties are the estimates to the nonlinear terms $\mathbf{u}\cdot\nabla\psi$ and $\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\psi)$ in the equation. To overcome them we use the following inequalities similar to (4.2):

$$
\begin{aligned}
&\int_\Omega\int_{|\mathbf{m}|=1} |\nabla^2\mathbf{u}\nabla\psi\nabla^4\psi|d\mathbf{m}d\mathbf{x} \le \epsilon\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^3}\|\psi\|^2_{H^2(\Omega,\mathcal{X}_0)}, \\
&\int_\Omega\int_{|\mathbf{m}|=1} |\nabla\mathbf{u}\nabla^2\psi\nabla^4\psi d\mathbf{m}d\mathbf{x} \le \epsilon\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^2}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}, \\
&\int_\Omega\int_{|\mathbf{m}|=1} |\mathbf{u}\nabla^3\psi\nabla^4\psi d\mathbf{m}d\mathbf{x} \le \epsilon\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \frac{C_1}{\epsilon}\|\mathbf{u}\|^{12}_{H^1}\|\psi\|^2_{H^1(\Omega,\mathcal{X}_0)}.
\end{aligned}
$$

For the other term $\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\mathcal{R}\cdot(\mathbf{m}\times\kappa\cdot\mathbf{m}\psi)\nabla^4\psi|d\mathbf{m}d\mathbf{x}$, we used the estimates obtained similar to (4.3):

$$\int_\Omega\int_{|\mathbf{m}|=1}|\nabla^2\kappa\mathcal{R}\psi\nabla^4\psi|d\mathbf{m}d\mathbf{x} \le \frac{6\epsilon}{11}\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\frac{5\epsilon}{11}\|\mathbf{u}\|^2_{H^4}+\frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^1}\|\psi\|^{12}_{H^1(\Omega,\mathcal{X}_1)},$$

$$\int_\Omega\int_{|\mathbf{m}|=1}|\kappa\nabla^2\mathcal{R}\psi\nabla^4\psi|d\mathbf{m}d\mathbf{x} \le \frac{6\epsilon}{11}\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\frac{5\epsilon}{11}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}+\frac{C_1}{\epsilon}\|\mathbf{u}\|^{12}_{H^2}\|\psi\|^{2}_{H^1(\Omega,\mathcal{X}_1)},$$

$$\int_\Omega\int_{|\mathbf{m}|=1}|\nabla\kappa\nabla\mathcal{R}\psi\nabla^4\psi|d\mathbf{m}d\mathbf{x}\le\epsilon\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\frac{C_1}{\epsilon}\|\mathbf{u}\|^2_{H^3}\|\psi\|^2_{H^2(\Omega,\mathcal{X}_1)}.$$

Now from (4.4), we see that it is necessary to estimate $\|\psi\|^2_{H^2(\Omega,\mathcal{X}_1)}$. But it is not difficult to obtain the following inequality in the similar way with the estimate (4.4).

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2_{H^2(\Omega,\mathcal{X}_1)}+\left(\frac{\varepsilon^2}{De}-4\epsilon\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}+\frac{1}{De}\|\psi\|^2_{H^2(\Omega,\mathcal{X}_2)}$$
$$\le\frac{C_1}{\epsilon}\left[\|\mathbf{u}\|^4_{H^3}+\|\mathbf{u}\|^{\frac{16}{3}}_{H^2}+\|\mathbf{u}\|^{16}_{H^1}+\|\psi\|^4_{H^1(\Omega,\mathcal{X}_1)}+\|\psi\|^8_{H^1(\Omega,\mathcal{X}_1)}+\|\psi\|^2_{H^2(\Omega,\mathcal{X}_1)}\right]. \tag{4.5}$$

For the case $n=2$, we can obtain a priori estimates by using similar approaches, and we omit the details and give the results directly as follows:

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{H^3_d}+\left(\frac{\gamma}{Re}-5\epsilon-\frac{9\epsilon}{16}\right)\|\mathbf{u}\|^2_{H^4}\le\frac{7\epsilon}{16}\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\frac{C_1}{\epsilon}[\|\mathbf{u}\|^4_{H^1}+\|\mathbf{u}\|^{18}_{H^2}$$
$$+\|\psi\|^{18}_{H^2(\Omega,\mathcal{X}_0)}+\|\psi\|^{\frac{9}{2}}_{H^3(\Omega,\mathcal{X}_0)}],\quad (n=2). \tag{4.6}$$

The a priori estimates for ψ are

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}+\left(\frac{\varepsilon^2}{De}-5\epsilon-\frac{9\epsilon}{16}\right)\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\left(\frac{1}{De}-\frac{4\epsilon}{13}\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}$$
$$\le\frac{7\epsilon}{16}\|\mathbf{u}\|^2_{H^4}+\frac{C_1}{\epsilon}[\|\mathbf{u}\|^4_{H^3}+\|\mathbf{u}\|^4_{H^1}+\|\mathbf{u}\|^8_{H^1}+\|\mathbf{u}\|^{18}_{H^1}$$
$$+\|\psi\|^4_{H^1(\Omega,\mathcal{X}_1)}+\|\psi\|^{18}_{H^1(\Omega,\mathcal{X}_1)}+\|\psi\|^4_{H^3(\Omega,\mathcal{X}_0)}]. \tag{4.7}$$

4.2. Global existence. In this subsection we will give the proof of Theorem 1.2. Let

$$Y(t)=\|\mathbf{u}\|^2_{H^3(\Omega)}+\|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)}+\|\psi\|^2_{H^2(\Omega,\mathcal{X}_1)}. \tag{4.8}$$

Combining the above estimates (4.1), (4.4), and (4.5) we have

$$\frac{1}{2}\frac{d}{dt}Y(t)+\left(\frac{\gamma}{Re}-6\epsilon\right)\|\mathbf{u}\|^2_{H^4}$$
$$+\left(\frac{\varepsilon^2}{De}-5\epsilon-\frac{6\epsilon}{11}\right)\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)}+\left(\frac{1}{De}-\frac{5\epsilon}{11}\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}$$
$$+\left(\frac{\varepsilon^2}{De}-4\epsilon\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}+\frac{1}{De}\|\psi\|^2_{H^2(\Omega,\mathcal{X}_2)}$$
$$\le\frac{C_1}{\epsilon}(Y^{12}+Y^{10}+Y^7+Y^4+Y^2+Y). \tag{4.9}$$

Now we choose

$$\epsilon = \min\left\{\frac{\gamma}{7Re}, \frac{\varepsilon^2}{7De}\right\}, \quad \text{and} \quad \beta = \frac{C_1}{\epsilon}. \tag{4.10}$$

Since $Y(t) \le \|\mathbf{u}\|^2_{H^4} + \|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)}$, (4.9) implies

$$Y'(t) + 2\epsilon Y(t) \le 2\beta(Y^{12} + Y^{10} + Y^7 + Y^4 + Y^2 + Y). \tag{4.11}$$

LEMMA 4.1. *Let Y be a nonnegative, absolutely continuous function satisfying inequality* (4.11). *Let B be a positive constant and $0 < B < B_0$, where B_0 is the unique positive solution of*

$$B^{11} + B^9 + B^6 + B^3 + B + 1 - \frac{\epsilon}{2\beta} = 0. \tag{4.12}$$

If $Y(0) \le B$, then $Y(t)$ is bounded by B for all $t \ge 0$.

Proof. We will prove it by contradiction. Suppose that there exists a t such that $Y(t) > B$, and define $t^* = \inf\{t \in \mathbb{R}^+, Y(t) > B\}$, then $Y(t^*) = B$ and $Y'(t^*) \ge 0$. However from (4.11) and the hypothesis made on B we deduce

$$\begin{aligned} Y'(t^*) &\le -2\epsilon Y(t^*) + 2\beta[Y^{11}(t^*) + Y^9(t^*) + Y^7(t^*) + Y^4(t^*) + Y^2(t^*) + Y(t^*)] \\ &\le -2\epsilon B + 2B\beta\frac{\epsilon}{2\beta} = -\epsilon B < 0, \end{aligned}$$

which contradicts the above statement. Therefore $Y(t) \le B$ for all $t \in \mathbb{R}^+$. □

Proof of Theorem 1.2. We have seen in subsection 4.1 that a specific norm of the local solution obtained in Theorem 1.1 satisfies an inequality with the form (4.11), where C_1 depends on the domain Ω and n while ϵ depends on $\gamma, Re, \varepsilon^2, De$ from (4.10). Lemma 4.1 shows that there exists a constant B_0, depending on initial data such that

$$Y(t) \le B, \text{ for } t \in \mathbb{R}^+, \quad \text{if} \quad Y(0) \le B < B_0.$$

But B_0 is the unique positive solution of (4.12). From (4.12), we can easily see that (4.12) possesses a unique positive solution if and only if

$$1 - \frac{\epsilon}{2\beta} < 0, \tag{4.13}$$

which implies from (4.10) that

$$Re < \frac{\gamma}{7\sqrt{2C_1}} \quad \text{and} \quad De < \frac{\varepsilon^2}{7\sqrt{2C_1}}. \tag{4.14}$$

This completes the proof of Theorem 1.2 for $n = 3$. □

Denoting

$$Z(t) = \|\mathbf{u}\|^2_{H^3(\Omega)} + \|\psi\|^2_{H^3(\Omega,\mathcal{X}_0)},$$

the addition of (4.6) and (4.7) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}Z(t) + \left(\frac{\gamma}{Re} - 6\epsilon\right)\|\mathbf{u}\|^2_{H^4} \\ &+ \left(\frac{\varepsilon^2}{De} - 6\epsilon\right)\|\psi\|^2_{H^4(\Omega,\mathcal{X}_0)} + \left(\frac{1}{De} - \frac{4\epsilon}{13}\right)\|\psi\|^2_{H^3(\Omega,\mathcal{X}_1)} \\ &\le \frac{C_1}{\epsilon}(Z^9 + Y^4 + Y^{9/4} + Y^2) \quad \text{for } n = 2. \end{aligned} \tag{4.15}$$

By the same way the result of Theorem 1.2 for $n = 2$ can be obtained.

Appendix A. Proof of Lemma 2.1. Now we solve (3.2)–(3.3) with the initial value $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0$ by using the Galerkin approximation. Let V be the space of all divergence-free vector in $H^1(\Omega)$, and let $\{\omega^i|i\in\mathbb{N}\}$ be a basis for V. We seek an approximation to w of the form

$$w^N(\mathbf{x},t)=\Sigma_{n=1}^N\rho^n(t)\omega^n(\mathbf{x}). \tag{A.1}$$

The function w^N satisfies, instead of (3.2) and (3.3),

$$\begin{aligned} w_t^N+(w^N\cdot\nabla)w^N &= \frac{\gamma}{Re}\triangle w^N+f\\ &+\frac{1-\gamma}{2Re}\nabla\cdot\left[\mathbf{D}^N:A(\mathbf{x},t)\right], \end{aligned} \tag{A.2}$$

$$w^N(\mathbf{x},0)=P_N\mathbf{u}_0(\mathbf{x}), \tag{A.3}$$

where P_N is the orthogonal projector in H_d^1 onto $W_N=Span\{\omega^i,i=1,\cdots,N\}$. The existence and uniqueness of a solution w_N to (A.2)–(A.3) with periodic boundary conditions defined on some interval $(0,T_N),T_N>0$, is clear; in fact, the following estimate shows that $T_N=T$ for $n=2$ and T_N suitably small for $n=3$.

Multiplying w^N to (A.2) and integrating it, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w^N\|_{L^2}^2+\frac{\gamma}{Re}\|\nabla w^N\|_{L^2}^2+\frac{1-\gamma}{2Re}\int_\Omega\langle|\mathbf{D}^N:\mathbf{mm}|^2\rangle d\mathbf{x}\\ &\le\frac{\gamma}{2Re}\|w^N\|_{L^2}^2+\frac{Re}{2\gamma}\|f\|_{H^{-1}}^2. \end{aligned} \tag{A.4}$$

This implies that

$$\frac{d}{dt}\|w^N\|_{L^2}^2+\frac{\gamma}{Re}\|\nabla w^N\|_{L^2}^2\le\frac{Re}{\gamma}\|f\|_{H^{-1}}^2. \tag{A.5}$$

It shows that

$$\int_0^T\|\nabla w^N(t)\|_{L^2}^2dt\le K_1, \tag{A.6}$$

where

$$K_1=K_1\left(\mathbf{u}_0,f,\frac{\gamma}{Re},T\right)=\frac{Re}{\gamma}\left(\|\mathbf{u}_0\|_{L^2}^2+\frac{Re}{\gamma}\int_0^T\|f(t)\|_{H^{-1}}^2dt\right). \tag{A.7}$$

For $0<s<T$, by (A.5),

$$\|w^N(s)\|_{L^2}^2\le K_2, \tag{A.8}$$

where

$$K_2=K_2\left(\mathbf{u}_0,f,\frac{\gamma}{Re},T\right)=\frac{\gamma}{Re}K_1. \tag{A.9}$$

Multiplying $\triangle w^N$ by (A.2) and integrating it, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla w^N\|_{L^2}^2+\frac{\gamma}{Re}\|\nabla^2 w^N\|_{L^2}^2+\frac{1-\gamma}{2Re}\int_\Omega\langle|\nabla\mathbf{D}^N:\mathbf{mm}|^2\rangle d\mathbf{x}\\ &\le\frac{1}{4}\frac{\gamma}{Re}\|\nabla w^N\|_{L^2}^2+\frac{Re}{\gamma}\|f\|_{L^2}^2\\ &\quad+\int|(w^N\cdot\nabla)w^N\triangle w^N|d\mathbf{x}+\frac{1-\gamma}{2Re}\int|\mathbf{D}^N\nabla A\triangle w^N|d\mathbf{x}. \end{aligned} \tag{A.10}$$

Now the following a priori of $\int |(w^N \cdot \nabla) w^N \triangle w^N| dx$ are different, depending on the dimension. We use the relation [24]

$$\int |(w\cdot\nabla)w\triangle w|dx \le C\|w\|_{L^2}^{1/2}\|\triangle w\|_{L^2}^{3/2}\|\nabla w\|_{L^2}, \qquad (n=2), \tag{A.11}$$

$$\int |(w\cdot\nabla)w\triangle w|dx \le C\|w\|_{L^2}^{1/4}\|\triangle w\|_{L^2}^{7/4}\|\nabla w\|_{L^2}, \qquad (n=3), \tag{A.12}$$

and the estimates

$$\int |\mathbf{D}^N\nabla A\triangle w^N|d\mathbf{x} \le C\|\triangle w^N\|_{L^2}^{4/3}\|\nabla^2 A\|_{L^2}\|\nabla w^N\|_{L^2}^{2/3}, \quad (n=2), \tag{A.13}$$

$$\int |\mathbf{D}^N\nabla A\triangle w^N|d\mathbf{x} \le C\|\triangle w^N\|_{L^2}^{3/2}\|\nabla^2 A\|_{L^2}\|\nabla w^N\|_{L^2}^{1/2}, \quad (n=3). \tag{A.14}$$

By Young's inequality (A.10) implies

$$\begin{aligned}&\frac{d}{dt}\|\nabla w^N\|_{L^2}^2 + \frac{1}{2}\frac{\gamma}{Re}\|\Delta w^N\|_{L^2}^2 \le \frac{2Re}{\gamma}\|f\|_{L^2}^2\\ &\qquad + C\|w^N\|_{L^2}^2\|\nabla w^N\|_{L^2}^4 + C\|\nabla^2 A\|_{L^2}^4\|\nabla w^N\|_{L^2}^2, \quad (n=2),\end{aligned} \tag{A.15}$$

$$\begin{aligned}&\frac{d}{dt}\|\nabla w^N\|_{L^2}^2 + \frac{3}{2}\frac{\gamma}{Re}\|\Delta w^N\|_{L^2}^2 \le \frac{2Re}{\gamma}\|f\|_{L^2}^2\\ &\qquad + C\|w^N\|_{L^2}^2\|\nabla w^N\|_{L^2}^8 + C\|\nabla^2 A\|_{L^2}^4\|\nabla w^N\|_{L^2}^2, \quad (n=3).\end{aligned} \tag{A.16}$$

Since $\psi^l \in L^\infty([0,T], H^3(\Omega, \mathcal{X}_0))$, we can obtain

$$|\nabla A(x,t)| \in L^\infty([0,T]; H^2(\Omega)). \tag{A.17}$$

Then by the same way as the a priori estimate of Theorem 3.2 in [24] we can get for $n=2$ in virtue of (2.14) and (3.1)

$$\sup_{t\in[0,T]} \|\nabla w^N\|_{L^2}^2 \le K_3, \tag{A.18}$$

where

$$K_3 = K_3\left(\mathbf{u}_0, f, \frac{\gamma}{Re}, T\right) = \left(\|\mathbf{u}_0\|_{H^1}^2 + C\int_0^T \|f(t)\|_{L^2}^2 dt\right)\exp(CK_1K_2 + CK^3T).$$

And

$$\int_0^T \|\triangle w^N\|_{L^2}^2 dt \le K_4, \tag{A.19}$$

where

$$K_4 = K_4(\mathbf{u}_0, f, \frac{\gamma}{Re}, T) = C\gamma\left(\|\mathbf{u}_0\|_{H^1}^2 + C\int_0^T \|f(t)\|_{L^2}^2 dt + CK_2K_3^2T + CK^3K_3T\right).$$

For $n=3$, we will estimate in the following from (A.16). Let $y(t) = \|\nabla w^N\|_{L^2}^2 + 1, a(t) = C\|w^N\|_{L^2}^2, b(t) = C\|\nabla^2 A\|_{L^2}^4$, and $c(t) = \frac{2Re}{\gamma}\|f\|_{L^2}^2$. Then from (A.16), we have the inequality

$$\frac{dy}{dt} \le [a(t) + b(t) + c(t)]y^4, \tag{A.20}$$

and

$$\int_0^T [a(t) + b(t) + c(t)]dt \le T(CK_2 + CK^6 + CK^2), \tag{A.21}$$

where we used the condition of (2.14) and the estimate (3.8) and above estimates. Then they imply that for $t \in [0,T]$ and $T < 1/[(CK_2 + CK^4 + \frac{Re}{\gamma}K^2)y^3(0)] \triangleq T_0$

$$y(t) \le \frac{y(0)}{\sqrt[3]{1 - (CK_2 + CK^4 + \frac{Re}{\gamma}K^2)y^3(0)T}} \triangleq K_5. \tag{A.22}$$

Therefore, when $T < T_0$, we have

$$\sup_{s\in[0,T]} \|\nabla w^N\|_{L^2}^2 \le K_5 \tag{A.23}$$

and

$$\int_0^T \|\triangle w^N\|_{L^2}^2 dt \le K_6, \tag{A.24}$$

where

$$K_6 = K_6(\mathbf{u}_0, f, \frac{\gamma}{Re}, T) = C\left(\|\mathbf{u}_0\|_{H^1}^2 + C\int_0^T \|f(t)\|_{L^2}^2 dt + CK_1K_5^4T + CK^4K_5T\right).$$

Multiplying $\triangle^2 w^N$ by (A.2) and integrating it, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla^2 w^N\|_{L^2}^2 + \frac{\gamma}{Re}\|\nabla^3 w^N\|_{L^2}^2 + \frac{1-\gamma}{4Re}\int_\Omega \langle|\nabla^2\mathbf{D}^N : \mathbf{mm}|^2\rangle d\mathbf{x} \\ &\quad \le \frac{1}{4}\frac{\gamma}{Re}\|\nabla\triangle w^N\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2 \\ &\qquad + \frac{1-\gamma}{2Re}\int [|\mathbf{D}^N\nabla^2 A\nabla\triangle w^N| + |\nabla\mathbf{D}^N\nabla A\nabla\triangle w^N|]dx \\ &\qquad + \int |(w^N\cdot\nabla)w^N\triangle^2 w^N|dx. \end{aligned} \tag{A.25}$$

In the following we will estimate (A.25). In virtue of the inequality (p. 31 [24])

$$\int |(w^N\cdot\nabla)w^N\triangle^r w^N|dx \le \frac{1}{4}\frac{\gamma}{Re}\|w^N\|_{H^{r+1}}^2 + C\|w^N\|_{H^1}^{2r} \quad (n=2), \tag{A.26}$$

$$\int |(w^N\cdot\nabla)w^N\triangle^r w^N|dx \le \frac{1}{4}\frac{\gamma}{Re}\|w^N\|_{H^{r+1}}^2 + C\|w^N\|_{H^1}^{4r+2} \quad (n=3). \tag{A.27}$$

And the estimates

$$\int |\mathbf{D}^N\nabla^2 A\nabla^3 w^N|d\mathbf{x} \le C\|\nabla^3 w^N\|_{L^2}\|\nabla A\|_{L^2}^{1/3}\|\nabla^3 A\|_{L^2}^{2/3}\|\nabla^2 w^N\|_{L^2}, \quad (n=2),$$

$$\int |\mathbf{D}^N\nabla^2 A\nabla^3 w^N|d\mathbf{x} \le C\|\nabla^3 w^N\|_{L^2}\|\nabla A\|_{L^2}^{1/4}\|\nabla^3 A\|_{L^2}^{3/4}\|\nabla^2 w^N\|_{L^2}, \quad (n=3),$$

$$\int |\nabla\mathbf{D}^N\nabla A\nabla^3 w^N|d\mathbf{x} \le C\|\nabla^3 w^N\|_{L^2}^{5/3}\|\nabla^2 A\|_{L^2}\|\nabla w^N\|_{L^2}^{1/3}, \quad (n=2),$$

$$\int |\nabla\mathbf{D}^N\nabla A\nabla^3 w^N|d\mathbf{x} \le C\|\nabla^3 w^N\|_{L^2}^{7/4}\|\nabla^2 A\|_{L^2}\|\nabla w^N\|_{L^2}^{1/4}. \quad (n=3).$$

Thus, we have

$$\frac{d}{dt}\|w^N\|_{H^2}^2 + \frac{1}{2}\frac{\gamma}{Re}\|w^N\|_{H^3}^2 \le C\|w^N\|_{H^1}^4 + \frac{Re}{\gamma}\|f\|_{H^1}^2 + C\|A\|_{H^1}^{2/3}\|A\|_{H^3}^{4/3}\|w^N\|_{H^2}^2 + C\|A\|_{H^2}^6\|w^N\|_{H^2}^2, \quad (n=2), \tag{A.28}$$

$$\frac{d}{dt}\|w^N\|_{H^2}^2 + \frac{1}{2}\frac{\gamma}{Re}\|w^N\|_{H^3}^2 \le C\|w^N\|_{H^1}^{10} + \frac{Re}{\gamma}\|f\|_{H^1}^2 + C\|A\|_{H^1}^{1/2}\|A\|_{H^3}^{3/2}\|w^N\|_{H^2}^{3/2} + C\|A\|_{H^2}^8\|w^N\|_{H^1}^2, \quad (n=3). \tag{A.29}$$

Similarly we can obtain the following estimates: for $n = 2$,

$$\sup_{t\in[0,T]}\|w^N(t)\|_{H^2}^2 \le [CK_2^2T + CKK_2T + C\|f\|_{L^2(0,T;H^1)}^2 + \|\mathbf{u}_0\|_{H^2}^2]e^{CK^2T+CK^6T} \triangleq K_7, \tag{A.30}$$

and

$$\int_0^T \|w^N(t)\|_{H^3}^2 dt \le K_8, \tag{A.31}$$

where

$$K_8 = C\left(CK_3^2T + C\|f\|_{L^2(0,T;H^1)}^2 + \|\mathbf{u}_0\|_{H^2}^2 + CK^2K_7K_1T + CK^6K_7T\right).$$

While for $n = 3$,

$$\sup_{t\in[0,T]}\|w^N(t)\|_{H^2}^2 \le K_9, \tag{A.32}$$

where

$$K_9 = \left[CK_5^5T + CK^8K_1 + \frac{Re}{\gamma}\|f\|_{L^2(0,T;H^1)}^2 + \|\mathbf{u}_0\|_{H^2}^2\right]e^{CK^2T}.$$

Further, we have

$$\int_0^T \|w^N(t)\|_{H^3}^2 dt \le K_{10}, \tag{A.33}$$

where

$$K_{10} = C\left(CK_5^5T + C\|f\|_{L^2(0,T;H^1)}^2 + \|\mathbf{u}_0\|_{H^2}^2 + CK^8K_1T + CK^2K_9T\right).$$

This shows for T given,

$$w^N \in L^\infty([0,T]; H^2(\Omega)) \cap L^2([0,T]; H^3(\Omega)) \tag{A.34}$$

provided that $f \in L^2([0,T]; H^1(\Omega))$ and $A \in L^\infty([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega))$.

Similarly we can obtain the estimates

$$\frac{d}{dt}\|w^N\|_{H^3}^2 + \frac{1}{2}\frac{\gamma}{Re}\|w^N\|_{H^4}^2 \le C\|w^N\|_{H^1}^6 + \frac{Re}{\gamma}\|f\|_{H^2}^2 + C\|A\|_{H^2}^9\|w^N\|_{H^1}^2 + C\|A\|_{H^3}^4\|w^N\|_{H^1}^2 + C\|A\|_{H^1}^{4/9}\|A\|_{H^4}^{14/9}\|w^N\|_{H^2}^2, \quad (n=2), \tag{A.35}$$

$$\frac{d}{dt}\|w^N\|_{H^3}^2 + \frac{1}{2}\frac{\gamma}{Re}\|w^N\|_{H^4}^2 \le C\|w^N\|_{H^1}^{14} + \frac{Re}{\gamma}\|f\|_{H^2}^2 + C\|A\|_{H^2}^{12}\|w^N\|_{H^1}^2 + C\|A\|_{H^3}^4\|w^N\|_{H^1}^2 + C\|A\|_{H^1}^{1/3}\|A\|_{H^4}^{5/3}\|w^N\|_{H^2}^2, \quad (n=3). \tag{A.36}$$

From (A.35), for $n = 2$ we have

$$\sup_{t\in[0,T]} \|w^N(t)\|_{H^3}^2 \leq K_{11}, \tag{A.37}$$

where

$$K_{11} = \left[\|u_0\|_{H^3}^2 + \frac{Re}{\gamma}\|f\|_{L^2(0,T;H^2)}^2 + CK_3^3T + CK^4K_1 + CK^9K_1\right]e^{CK^4T}.$$

Furthermore, we have

$$\int_0^T \|w^N(t)\|_{H^4}^2 dt \leq K_{12}, \tag{A.38}$$

where

$$K_{12} = C\left(\|u_0\|_{H^3}^2 + C\|f\|_{L^2(0,T;H^2)}^2 + CK_3^3T + CK^4K_1T + CK^2K_{11}T + CK^2K_7T\right).$$

Thus we know that $\phi_1(K,T) = K_{12} + K_{11}$ for $n = 2$.

From (A.36), for $n = 3$ we have

$$\sup_{t\in[0,T]} \|w^N(t)\|_{H^3}^2 \leq K_{13}, \tag{A.39}$$

where

$$K_{13} = [\|u_0\|_{H^3}^2 + C\|f\|_{L^2(0,T;H^2)}^2 + CK_3^7T + CK^{12}K_3T + CK^4K_3T]e^{CK^2T}.$$

Moreover,

$$\int_0^T \|w^N(t)\|_{H^4}^2 dt \leq K_{14}, \tag{A.40}$$

where

$$K_{14} = C\Big(\|u_0\|_{H^3}^2 + C\|f\|_{L^2(0,T;H^2)}^2 + CK_3^7T + CK^{12}K_3T + CK^4K_3T + CK^2K_{13}T\Big).$$

Thus we know that $\phi_1(K,T) = K_{14} + K_{13}$ for $n = 3$.

Therefore, this implies that for T given,

$$w^N \in L^\infty([0,T]; H^3(\Omega)) \cap L^2([0,T]; H^4(\Omega)) \tag{A.41}$$

provided that $f \in L^2([0,T]; H^2(\Omega))$ and $A \in L^\infty([0,T], H^3(\Omega)) \cap L^2([0,T], H^4(\Omega))$.

Passing $N \to \infty$, we can obtain the estimate (2.16).

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REFERENCES

[1] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer, New York, 1998.

[2] A.N. Beris and B.J. Edwards, *Thermodynamics of Flowing Systems with Internal Microstructure*, Oxford Science, New York, 1994.

[3] C.V. Chaubal, A. Srinivasan, Ö. Eğecioğlu, and L.G. Leal, *Smoothed particle hydrodynamics techniques for the solution of kinitic theory problems*, Part 1: Method, J. Non-Newtonian Fluid Mesh., 70 (1997), pp. 125–154.

[4] P. Constantin, I.G. Kevrekids, and E.S. Titi, *Asmptotic states of a smoluchowski equation*, Discrete Continuous Dynamical Systems, 11 (2004), pp. 101–112.

[5] P. Constantin and C. Foias, *Navier-Stokes Equations*, The University of Chicago Press, Chicago and London, 1988.

[6] M. Doi and S.F. Edwards, *The Theory of Polymer Dynamics*, Oxford University Press, Oxford, 1986.

[7] Q. Du, C. Liu, and P. Yu, *FENE dumbbell model and its several linear and nonlinear closure approximations*, Multiscale Model. Simul., 4 (2005), pp. 709–731.

[8] W.N. E and P.W. Zhang, *A molecular kinetic theory of inhomogeneous liquid crystal flow and the small Deborah number limit*, Meth. Appl. Anal., 13 (2006), pp. 181–198.

[9] J. Ericksen, *Liquid crystals with variable degree of orientation*, Arch. Rat. Mech. Anal., 113 (1991), pp. 97–120.

[10] V. Faraoni, M. Grosso, S. Crescitelli, and P.L. Maffettone, *The rigid-rod model for nematic polymers: An analysis of the shear flow problem*, J. Rheol., 43 (1999), pp. 829–843.

[11] P.G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, 2nd ed., Oxford Science Publications, New York, 1993.

[12] C. Guillope and J.C. Saut, *Existence results for the flow of viscoelastic fluids with a differential constitutive law*, Nonl. Anal. TMA, 15 (1990), pp. 849–869.

[13] N. Kuzuu and M. Doi, *Constitutive equation for nematic liquid crystals under weak velocity gradient derived from a molecular kinetic equation*, J. Phys. Soc. Japan, 52 (1983), pp. 3486–3494.

[14] O. Ladyzenskaja, V. Solonnikov, and N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type, Translations of Mathematical Monographs*, Am. Math. Soc., Providence, RI, 1968.

[15] T.J. Li, H. Zhang and P.W. Zhang, *Local existence for the dumbbell model of polymeric fluids*, Comm. Partial Differential Equations, 29 (2004), pp. 903–923.

[16] F.H. Lin, C. Liu, and P. Zhang, *On hydrodynamics of viscoelastic fluids*, Comm. Pure Appl. Math., 58 (2005), pp. 1–35.

[17] F.H. Lin, C. Liu, and P. Zhang, *On a micro-macro model for polymeric fluids near equilibrium*, Comm. Pure Appl. Math., 60 (2006), pp. 838–866.

[18] G. Marrucci and F. Greco, *The elastic constants of Maier-Saupe rodlike molecule nematics*, Mol. Cryst. Liq. Cryst., 206 (1991), pp. 17–30.

[19] M. Renardy, *Local existence of solutions of the Dirichlet initial boundary problem for incompressible hypoelastic materials*, SIAM J. Math. Anal., 21 (1990), pp. 1369–1385.

[20] M. Renardy, *An existence theorem for model equations resulting from kinetic theories*, SIAM J. Math. Anal., 22 (1991), pp. 313–327.

[21] A.D. Rey and T. Tsuji, *Orientation mode selection mechanisms for sheared nematic liquid crystalline materials*, Phys. Rev. E, 57 (1998), pp. 5610–5625.

[22] A.D. Rey and T. Tsuji, *Recent advances in theoretical liquid crystal rheology*, Macromol. Theory Simul., 7 (1998), pp. 623–639.

[23] T.C. Sideris and B. Thomases, *Global existence for 3D incompressible isotropic elastodynamics via the incompressible limit*, Comm. Pure Appl. Math., 58 (2005), pp. 750–788.

[24] R. Teman, *Navier-Stokes Equations and Nonlinear Functional Analysis*, SIAM, Philadelphia, 1995.

[25] Q. Wang, *A hydrodynamic theory for solutions of nonhomogeneous nematic liquid crystalline polymers of different configurations*, J. Chem. Phys., 116 (2002), pp. 9120–9136.

[26] H.J. Yu and P.W. Zhang, *A kinetic-hydrodynamic simulation of microstructure of liquid crystal polymers in plane shear flow*, J. Non-Newtonian Fluid Mech., 141 (2007), pp. 116–127.

[27] H.J. Yu, G.H. Ji, and P.W. Zhang, *A nonhomogeneous kinetic model of liquid crystal polymers and its thermodynamic closure approximation*, preprint.

[28] H. Zhang and P.W. Zhang, *A theoretical and numerical study for the rod-like model of a polymeric fluid*, J. Comp. Math., 22 (2004), pp. 319–330.