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ANALYSIS OF 1+1 DIMENSIONAL STOCHASTIC MODELS OF LIQUID CRYSTAL POLYMER FLOWS*

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Abstract. We consider the stochastic model of concentrated Liquid Crystal Polymers(LCPs) in the plane Couette flow. The dynamic equation for the liquid crystal polymers is described by a nonlinear stochastic differential equation with Maier-Saupe interaction potential. The stress tensor is obtained from an ensemble average of microscopic polymer configurations. We present the local existence and uniqueness theorem for the solution of the coupled fluid-polymer system. We also analyze the error of a fully finite difference-Monte Carlo hybrid numerical scheme by investigating the asymptotic behavior of weakly interacting processes. It is proved that the rate of convergence of the full discretized scheme is $\mathcal{O}(h^2+\delta t+\frac{1}{\sqrt{M}})$.

1. Introduction

The analysis and computation of complex fluids, i.e., non-Newtonian fluids, have attracted much attention in recent years, see, e.g., [29, 30, 12, 23, 8, 9, 16, 17, 24, 34, 4, 5, 20, 25]. Mathematically speaking, the dynamics of a polymer-solvent system may be modeled by a Navier-Stokes-like equation

$$\begin{aligned} \partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \nabla\cdot\boldsymbol{\tau}, \\ \nabla\cdot\boldsymbol{u} &= 0, \end{aligned} \tag{1.1}$$

where $\boldsymbol{u}$ is the velocity field and p is the pressure. The stress is given by $\boldsymbol{\tau}=\boldsymbol{\tau}_s+\boldsymbol{\tau}_p$, where $\boldsymbol{\tau}_s$ is the stress due to the solvent and $\boldsymbol{\tau}_p$ is the polymer contribution.

One typical model is the elastic dumbbell model. The macromolecule is idealized as a coarse-grained dumbbell with two beads joined by an elastic spring. These dumbbells may be transported and stretched by the spring force, frictional force and the thermal force. The polymeric stress $\boldsymbol{\tau}_p$ is derived by the Kramers expression that $\boldsymbol{\tau}_p=\langle \boldsymbol{Q}\otimes\boldsymbol{F}(\boldsymbol{Q})\rangle$[2], where $\boldsymbol{Q}$ is the connector vector of the two beads, $\boldsymbol{F}(\boldsymbol{Q})$ is the spring force vector, and the ensemble average $\langle\,\cdot\,\rangle$ is integrated over the configuration space. Another typical model is the rod-like model for liquid crystal polymers, where each polymer is assumed to be a stiff rod which can translate, rotate and interact with each other in the fluid. The polymeric stress $\boldsymbol{\tau}_p$ is given by Doi theory through virtual work principle [6].

The dumbbell model has been extensively studied in theory, numerics and computations. On the theoretical side, the short time well-posedness of the coupled system has been established both in deterministic version[22, 29, 30] and stochastic version[9] when the spring is infinitely stretched and the force $\boldsymbol{F}(\boldsymbol{Q})$ satisfies the polynomial growth condition at the infinity. For the singular finitely extendable nonlinear elastic (FENE) force $\boldsymbol{F}(\boldsymbol{Q})=\frac{\boldsymbol{Q}}{1-Q^2/\beta^2}$, where β is a prescribed maximal extension for spring, only a short time well-posedness for the shear flow is given in [17]. There the stochastic differential equation for $\boldsymbol{Q}$ is analyzed in detail to guarantee its existence and supply the *a priori* estimate for the shear stress. For the chemical

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engineers, the multi-scale stochastic simulation algorithms such as CONNFFESSIT (Calculation Of Non-Newtonian Flows: Finite Element and Stochastic Simulation Technique), BCF (Brownian Configuration Fields) and LPM (Lagrangian Particle Methods) have been introduced to handle the complex configurations for the dumbbell chains [27, 21, 15, 13]. The convergence of the BCF method for a simple case with linear force law and shear flow is first given by [8, 16]. The special advantage of the simplicity of the model is taken in the analysis. This result is generalized to high dimensional case in [10].

For the LCP model, the Doi-Hess theory is the simplest and the most studied model[14, 6]. Many interesting dynamics were found for the shear flow in [11, 25, 19, 20] via spherical harmonic function expansion and stochastic simulations. But simulation results in more general cases haven't come out yet for its large scale computations. This direction is very promising because these phenomena will be beneficial to understand the defects and patterns in liquid crystal materials from the molecular theory for inhomogeneous systems of rod-like molecules [26, 31]. Mathematically, the structure of the solution for the Smoluchowski equation in the absence of flow is analyzed in [4, 5] recently. The equilibrium structure of the probability density function in 1D case is obtained in [4, 24]. Recently the global existence of solution and the convergence of finite difference scheme for 1+1 case of hydrodynamic coupling Smoluchowski equation is given in [34].

In this work, we are interested in the well-posedness and numerical analysis of the stochastic model of LCPs in the plane Couette flows. We call it "1+1" model because the flow is one dimensional and the configuration variable of the rod is restricted to the circle which is also of dimension one.

For the incompressible plane pressure driven flow, we have

$$\boldsymbol{u} = (u(t,y),0)^T, \qquad \nabla = (0,\partial_y)^T, \qquad \nabla p = (c,0)^T, \tag{1.2}$$

where c is a prescribed pressure gradient. It follows that the deterministic "1+1" model for highly concentrated LCPs reads [34]:

$$\partial_t u = \frac{\gamma}{Re}\partial_{yy} u + \frac{1-\gamma}{ReDe}\partial_y \tau - c, \tag{1.3}$$

$$\partial_t \psi = \frac{1}{De}\partial_{\theta\theta}\psi + \frac{1}{De}\partial_\theta(\psi\,\partial_\theta U) + \partial_y u\,\partial_\theta(\psi \sin^2\theta), \tag{1.4}$$

$$U = \int_0^{2\pi} \sin^2(\theta-\theta')\psi(t,y,\theta')\mathrm{d}\theta', \tag{1.5}$$

$$\tau = 2\langle \sin 2\theta\rangle + \langle \partial_\theta U \cos^2\theta\rangle + \frac{De}{2}\partial_y u \langle \sin^2 2\theta\rangle, \tag{1.6}$$

where De, Re are Deborah and Reynolds numbers, respectively, γ is the viscosity ratio, $\langle g(\theta)\rangle = \int_0^{2\pi} g(\theta)\psi(t,y,\theta)\,\mathrm{d}\theta$, $\psi = \psi(t,y,\theta)$ is the orientational distribution function of the rods at time t and the position y, U is the Maier-Saupe excluded volume interaction potential and τ is the polymeric stress. The space variable y varies in $\Box = (0,1)$.

The initial condition is $u(0,y) = u_0(y)$ and $\psi(0,y,\theta) = \psi_0(y,\theta) \geq 0$ which is non-negative and normalized $\int_0^{2\pi} \psi_0(y,\theta)\,\mathrm{d}\theta = 1$ at each position y. The no-slip boundary condition $u(t,0) = u(t,1) = 0$ for the velocity u and periodic boundary condition for θ are assumed.

We now rewrite (1.3)-(1.6) into another version by replacing the Smoluchowski equation (1.4) with a stochastic differential equation. For the sake of simplicity, we will ignore all constant coefficients later since they do not have a major influence in the proof. We denote by W_t 1-dimension Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, $\mathcal{M}$ the set of probability measures on $\mathbb{R}$, and $\mathcal{L}(\xi)$ the distribution of a random variable ξ under the probability measure $\mathbb{P}$. Then (1.3)-(1.6) are equivalent to (See Remark 1 below):

$$\partial_t u = \partial_{yy} u + \partial_y \tau - c, \tag{1.7}$$

$$\mathrm{d}\Theta_t = -\Big(a(\Theta_t, \mathcal{L}(\Theta_t)) + \partial_y u \sin^2 \Theta_t\Big)\mathrm{d}t + \mathrm{d}W_t, \tag{1.8}$$

$$\tau = \mathbb{E}\left[\sin 2\Theta_t + a(\Theta_t, \mathcal{L}(\Theta_t)) \cos^2 \Theta_t + \partial_y u \sin^2 2\Theta_t\right], \tag{1.9}$$

where

$$a(\theta, \mu) = \int_{-\infty}^{+\infty} \sin 2(\theta - \theta')\mu(\mathrm{d}\theta'), \quad \forall\, \theta \in \mathbb{R}, \mu \in \mathcal{M} \tag{1.10}$$

with initial condition $\Theta(0, y, \omega) = \Theta_0(y, \omega)$. In the above system, $\Theta(t, y, \omega)$ that takes value in $\mathbb{R}$ is a stochastic process presenting at time t, which is the orientational angle of microscopic rod-like polymers at the position y in the flow field.

The nonlinear SDE (1.8) is closely related to the model of the weakly interacting particle system. In [33], some results on chaos propagation for this kind of system are reviewed. Here we have a two-scale coupled system which is very similar to the FENE model studied in [17]. The idea in [17] of eliminating the term $\partial_y u$ with the Girsanov transformation is not applicable directly here since the nonlinear term $a(\Theta_t, \mathcal{L}(\Theta_t))$ involves the distribution of Θ. In this work, we take the fixed point strategy as in [33] by first freezing the nonlinear term and then applying the Girsanov transformation to establish a contraction mapping argument, which enables us to obtain the existence and uniqueness of the SDE. For local time well-posedness of the momentum equation, the standard Galerkin method is applied. The *a priori* estimates are obtained by using the distribution theory in y space because the white noise $\dot{W}_t$ is only in time[9]. Then we study the numerical analysis of a full discretization scheme for the coupled system. We introduce M (i.e. the number of rods per cell) weakly interacting processes to numerically deal with nonlinear term in (1.8) together with Monte-Carlo realization (approximating the probability expectation in τ with the empirical average). The velocity field is discretized by the finite difference method with mesh size h. With the Euler discretization in time with time step δt, we finally show that the full scheme converges to the continuous solution at the order $\mathcal{O}(h^2 + \delta t + \frac{1}{\sqrt{M}})$.

The paper is organized as follows. In section 2, we give the existence and uniqueness result for the SDE (1.8). The local time well-posedness of the system is established in section 3. In section 4 the full discretization scheme will be analyzed theoretically. The conclusions will be given in the final section.

REMARK 1. *It is observed that the configuration space of the Smoluchowski equation* (1.4) *is essentially* $[0, 2\pi)$ *while the configuration space of the SDE* (1.8) *is* $\mathbb{R}$, *which implies that the two equations are not consistent. However, we can "fold" the configuration space of* Θ_t *to* $[0, 2\pi)$ *and prove the equivalence between* (1.3)-(1.6) *and* (1.7)-(1.9)*. More precisely, we can have the Fokker-Plank equation associated to* (1.8)*. It says that* (1.8)-(1.9) *are equivalent to the following integro-differential system*

$$\partial_t p = \partial_{\theta\theta} p + \partial_\theta \Big(p \int_{\mathbb{R}} \sin 2(\theta - \theta') p(t, y, \theta') \, d\theta' \Big) + \partial_y u \, \partial_\theta (p \sin^2 \theta), \tag{1.11}$$

$$\tau = \int_{\mathbb{R}} \Big\{ \sin 2\theta + \int_{\mathbb{R}} \sin 2(\theta - \theta') p(t, y, \theta') \, d\theta' \cos^2 \theta + \partial_y u \sin^2 2\theta \Big\} p(t, y, \theta) \, d\theta, \tag{1.12}$$

where $p(t, y, \theta)$ *is the probability density function of the random variable* Θ_t*. We define*

$$\tilde{p}(t, y, \theta) = \sum_{k=-\infty}^{+\infty} p(t, y, \theta + 2k\pi) \qquad \textit{for } \theta \in [0, 2\pi) \tag{1.13}$$

and it can be easily verified that $\tilde{p}$ *satisfies* (1.4)-(1.6)*. That is to say* $\tilde{p}$ *and* ψ *are the same (with minor adjustment for neglection of constants in equations) .*

Remark 2. *It may seem strange that the white noise in* (1.8) *is only dependent on time. The validity can be assured from two viewpoints. The first one is that the Fokker-Planck equation* (1.11) *is mathematically equivalent to* (1.8)*, which supplies the correct deterministic version as equation* (1.4)*. Secondly, it can be regarded as a variance reduction technique as BCF. In the BCF algorithm, the Gaussian white noise is also only dependent on time, which introduces correlation between spatial particles and reduces the variance of stress* [28].

2. Existence of a solution to the stochastic differential equation

In this section, we only consider the SDE (1.8) and regard the space variable y as a fixed parameter. The contracting technique to be used in the proof may be referred to [33].

Theorem 2.1 (Existence and uniqueness). *If* $\partial_y u \in L_t^2([0, T])$ *for* $T > 0$*, then there exists a unique solution to the equation* (1.8)*.*

We introduce the Kantorovitch-Rubinstein or Vaserstein metric $D_T(\cdot, \cdot)$ on the set $\mathcal{M}(\mathcal{C})$ of probability measures on $\mathcal{C} = C([0, T], \mathbb{R})$, defined by

$$\begin{aligned} D_T(m^1, m^2) = \inf_m \Big\{ \int_{\mathcal{C} \times \mathcal{C}} \big(|w_T^1 - w_T^2|^* \wedge 1 \big) \, \mathrm{d}m(w^1, w^2), \\ m \in \mathcal{M}(\mathcal{C} \times \mathcal{C}), \ p^1 \circ m = m^1, \text{and } p^2 \circ m = m^2 \Big\}, \end{aligned} \tag{2.1}$$

where $|w_T^1 - w_T^2|^* = \sup_{t \le T} |w_t^1 - w_t^2|$ with w^1, $w^2 \in \mathcal{C}$ and p^1, p^2 being a projection map from $\mathcal{M}(\mathcal{C} \times \mathcal{C})$ to $\mathcal{M}(\mathcal{C})$ with respect to the first and second components, respectively. The definition (2.1) defines a complete metric on $\mathcal{M}(\mathcal{C})$, which gives to $\mathcal{M}(\mathcal{C})$ the topology of weak convergence[7]. Take now $T > 0$, and define a map Υ which associates to $m \in \mathcal{M}(\mathcal{C})$ the law of the solution Θ_t of

$$\Theta_t = \Theta_0 - \int_0^t \int_{\mathcal{C}} \sin 2(\Theta_s - w_s) \, \mathrm{d}m(w) \, \mathrm{d}s - \int_0^t \partial_y u \sin^2 \Theta_s \, \mathrm{d}s + W_t. \tag{2.2}$$

The existence of the strong solution of (2.2) can be obtained by Girsanov and Yamada-Watanabe theorem provided that $\partial_y u \in L_t^2([0,T])$[17, 32].

One easily observes that if $\Theta_t (t \le T)$ is a solution of (1.8) then its law on $\mathcal{C}([0,T],\mathbb{R})$ is a fixed point of Υ. Conversely, if m is a fixed point of Υ, then (2.2) defines a solution of (1.8). Actually, it can be verified that Υ is a contraction map.

LEMMA 2.2. *If $\partial_y u \in L_t^2([0,T])$ for $T>0$, then for $t \le T$,*

$$D_t(\Upsilon(m^1),\Upsilon(m^2)) \le C \int_0^t D_s(m^1,m^2)\, ds, \qquad m^1,m^2 \in \mathcal{M}(\mathcal{C}). \tag{2.3}$$

Proof: By the definition of the map Υ, we have

$$\Theta_t^1 = \Theta_0 - \int_0^t \int_{\mathcal{C}} \sin 2(\Theta_s^1 - w_s^1)\, \mathrm{d}m^1(w^1)\, \mathrm{d}s - \int_0^t \partial_y u \sin^2 \Theta_s^1 \, \mathrm{d}s + W_t,$$

$$\Theta_t^2 = \Theta_0 - \int_0^t \int_{\mathcal{C}} \sin 2(\Theta_s^2 - w_s^2)\, \mathrm{d}m^2(w^2)\, \mathrm{d}s - \int_0^t \partial_y u \sin^2 \Theta_s^2 \, \mathrm{d}s + W_t.$$

For any $m \in \mathcal{M}(\mathcal{C}\times\mathcal{C})$ satisfying $p^1 \circ m = m^1$ and $p^2 \circ m = m^2$, we find that

$$\begin{aligned}
|\Theta_t^1 - \Theta_t^2|^* \le & \int_0^t \int_{\mathcal{C}\times\mathcal{C}} |\sin 2(\Theta_s^1 - w_s^1) - \sin 2(\Theta_s^2 - w_s^2)|\, \mathrm{d}m(w^1,w^2) ds \\
& + \int_0^t |\partial_y u| |\sin^2 \Theta_s^1 - \sin^2 \Theta_s^2| \, \mathrm{d}s \\
\le & \, 2\int_0^t \int_{\mathcal{C}\times\mathcal{C}} (|w_s^1 - w_s^2|^* \wedge 1)\, \mathrm{d}m(w^1,w^2)\, \mathrm{d}s & (2.4) \\
& + 2\int_0^t (1+|\partial_y u|)(|\Theta_s^1 - \Theta_s^2|^* \wedge 1)\, \mathrm{d}s. & (2.5)
\end{aligned}$$

Using Gronwall's lemma and taking infimum for m, we have

$$|\Theta_t^1 - \Theta_t^2|^* \wedge 1 \le 2 \exp\left(2\int_0^T 1 + |\partial_y u|\, \mathrm{d}s\right) \int_0^t D_s(m^1,m^2)\, \mathrm{d}s \tag{2.6}$$

from which the lemma follows. ■

From Lemma 1, we can immediately deduce uniqueness in law for the solution of (1.8) (since $\Upsilon(m^1) = m^1$ and $\Upsilon(m^2) = m^2$ in (2.3), direct application of Gronwall's inequality makes this fact obvious). The existence part also follows now from a standard contraction argument. More precisely, for any $T>0$, and $m \in \mathcal{M}(\mathcal{C}([0,T],\mathbb{R}))$, we can iterate the lemma to obtain:

$$\begin{aligned}
& D_T(\Upsilon^{k+1}(m), \Upsilon^k(m)) \\
\le & \, C \int_0^T D_{t_1}(\Upsilon^k(m), \Upsilon^{k-1}(m))\, \mathrm{d}t_1 & (2.7) \\
\le & \, C^2 \int_0^T \int_0^{t_1} D_{t_2}(\Upsilon^{k-1}(m), \Upsilon^{k-2}(m))\, \mathrm{d}t_2\, \mathrm{d}t_1 & (2.8) \\
\le & \, C^k D_T(\Upsilon(m), m) \int_0^T \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{k-1}} \mathrm{d}t_k\, \mathrm{d}t_{k-1} \cdots \mathrm{d}t_2\, \mathrm{d}t_1 & (2.9) \\
\le & \, C^k \frac{T^k}{k!} D_T(\Upsilon(m), m). & (2.10)
\end{aligned}$$

This indicates that $\Upsilon^k(m)$, $k \geq 1$ is a Cauchy sequence, and converges to a fixed point of Υ.

Now we check the pathwise-uniqueness property of the SDE (1.8). If $\partial_y u \in L_t^2([0,T])$ and assume (1.8) has two solutions Θ^1 and Θ^2 in $\mathcal{C}([0,T],\mathbb{R})$, then

$$\Theta_t^1 - \Theta_t^2 = -\int_0^t a(\Theta_s^1, \mathcal{L}(\Theta_s^1)) - a(\Theta_s^2, \mathcal{L}(\Theta_s^2)) + \partial_y u(s,y)(\sin^2 \Theta_s^1 - \sin^2 \Theta_s^2)\,\mathrm{d}s.$$

We can deduce that

$$\begin{aligned}
&\mathbb{E}\left[|\Theta_t^1 - \Theta_t^2|\right] \\
&\leq \mathbb{E}\left[\int_0^t |a(\Theta_s^1, \mathcal{L}(\Theta_s^1)) - a(\Theta_s^2, \mathcal{L}(\Theta_s^2))| + |\partial_y u(s,y)||\sin^2(\Theta_s^1) - \sin^2(\Theta_s^2)|\mathrm{d}s\right] \\
&\leq 2\mathbb{E}\left[\int_0^t |\Theta_s^1 - \Theta_s^2| + \mathbb{E}\left[|\Theta_s^1 - \Theta_s^2|\right] + |\partial_y u(s,y)||\sin(\Theta_s^1) - \sin(\Theta_s^2)|\mathrm{d}s\right] \\
&\leq \int_0^t 2\big(2 + |\partial_y u(s,y)|\big)\mathbb{E}\left[|\Theta_s^1 - \Theta_s^2)|\right]\mathrm{d}s. \qquad (2.11)
\end{aligned}$$

Since $\partial_y u \in L_t^2([0,T])$, the Gronwall's lemma shows that $\Theta_t^1 = \Theta_t^2$ almost surely (t fixed). This and the continuity of both processes imply that $\Theta_t^1 = \Theta_t^2$ for all $t \leq T$ with probability 1. Therefore, for any $y \in \square$, the SDE (1.8) has a pathwise-unique strong solution in time $[0,T]$, provided that $\partial_y u(\cdot, y) \in L_t^2([0,T])$.

3. Existence and uniqueness of the solution to the coupled system

We now begin to consider the coupled system (1.7)-(1.9). From now on we suppose that the space variable y varies in $\square = (0,1)$. The notation $L_t^2(L_y^2)$ is a shortcut for $L_t^2([0,T], L_y^2(\square))$.

The aim of this section is to prove the following theorem:

THEOREM 3.1 (Local-in-time existence and uniqueness). *Given $u_0 \in H_{0,y}^1$, $T > 0$, and a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$. If Θ_0^y is an $\mathcal{F}_0$-measurable random variable for a.e. $y \in \square$ satisfying $\Theta_0^y \in H_y^1(L_\omega^2)$, and W_t is an $(\mathcal{F}_t)$ one dimensional standard Brownian motion, then there exits $T' \in (0,T)$ (only depending on data) such that the coupled system admits a unique solution $u(t,y) \in L_t^\infty(H_{0,y}^1) \cap L_t^2(H_y^2)$ and $\Theta_t^y \in C_t(H_y^1(L_\omega^2))$.*

3.1. Formal *a priori* estimates.

We first establish *a priori* estimates on the solution (u,Θ) of the system (1.7)-(1.9). These estimates will be made rigorous at the spatially discrete level in the next subsection. Here and in the following C will be a generic positive constant.

LEMMA 3.2 (Global-in-time first energy estimate). *Let $(u,\Theta)_{t\leq T}$ be the solution of the system (1.8)-(1.9). If $u_0 \in L_y^2$ then we have formally on $[0,T]$*

$$\|u\|_{L_t^\infty(L_y^2)}^2 + \|u\|_{L_t^2(H_y^1)}^2 \leq C, \qquad (3.1)$$

where constant C only depends on $\|u_0\|_{L_y^2}$, c and T. Here c is defined in (1.2).

Proof: One can multiply (1.7) by u and integrate the resulting equation in space and in time $[0,t]$. The main step in the proof is to use the following inequality

$$\begin{aligned}
&\int_0^t\int_\square \partial_y\tau u \,\mathrm{d}y\,\mathrm{d}s = -\int_0^t\int_\square \tau\partial_y u \,\mathrm{d}y\,\mathrm{d}s\\
&\le 2\int_0^t\int_\square |\partial_y u| \,\mathrm{d}y\,\mathrm{d}s - \int_0^t\int_\square |\partial_y u|^2 \mathbb{E}\left[\sin^2 2\Theta_s\right]\,\mathrm{d}y\,\mathrm{d}s\\
&\le \frac{1}{2}\|\partial_y u\|^2_{L^2_t(L^2_y)} + C
\end{aligned}$$

where C only depends on T. The remaining part of the proof is standard. ■

LEMMA 3.3 (Local-in-time second energy estimate). *If $u_0 \in H^1_{0,y}$ then we have the formal second energy estimate on $[0,T']$, with $T' \le T$ only dependent on c and $\|\partial_y u_0\|_{L^2_y}$,*

$$\|u\|_{L^\infty_t(H^1_y)} + \|u\|_{L^2_t(H^2_y)} + \|\partial_t u\|_{L^2_t(L^2_y)} \le C \tag{3.2}$$

where constant C depends only on $\|\partial_y u_0\|_{L^2_y}$ and c.

Proof: Multiplying (1.7) by $-\partial_{yy}u$, and integrating spatially and temporally, we obtain

$$\|\partial_y u\|^2_{L^2_y}(t) + \int_0^t \|\partial_{yy}u\|^2_{L^2_y}(s)\mathrm{d}s \le \|\partial_y u_0\|^2_{L^2_y} + c^2 t - 2\int_0^t\int_\square \partial_y\tau\partial_{yy}u\mathrm{d}y\mathrm{d}s. \tag{3.3}$$

We will focus on the estimate of $-\int_0^t\int_\square \partial_y\tau\partial_{yy}u\mathrm{d}y\mathrm{d}s$. Define $\Phi_t = \partial_y\Theta_t$. From the definition of τ,

$$\tau = \mathbb{E}\left[\sin 2\Theta_t + \cos^2\Theta_t\int_\Omega \sin 2(\Theta_t - \Theta_t(\omega'))\mathbb{P}(\,\mathrm{d}\omega') + \partial_y u \sin^2 2\Theta_t\right], \tag{3.4}$$

we have

$$\partial_y\tau = K_{11} + K_{12} + K_2 + \partial_{yy}u\mathbb{E}\left[\sin^2 2\Theta_t\right], \tag{3.5}$$

where

$$\begin{aligned}
K_{11} &= 2\mathbb{E}\left[\Phi_t\cos 2\Theta_t\right] - \mathbb{E}\left[\Phi_t\sin 2\Theta_t\int_\Omega \sin 2(\Theta_t - \Theta_t(\omega'))\mathbb{P}(\,\mathrm{d}\omega')\right],\\
K_{12} &= 2\partial_y u\mathbb{E}\left[\Phi_t\sin 4\Theta_t\right],\\
K_2 &= 2\mathbb{E}\left[\cos^2\Theta_t\int_\Omega (\Phi_t - \Phi_t(\omega'))\cos 2(\Theta_t - \Theta_t(\omega'))\mathbb{P}(\,\mathrm{d}\omega')\right].
\end{aligned}$$

By using Sobolev's interpolation inequality $\|\partial_y u\|_{L^\infty_y} \le \|\partial_y u\|^{1/2}_{L^2_y}\|\partial_y u\|^{1/2}_{H^1_y}$, we have

$$\begin{aligned}
&\int_0^t\int_\square K_{12}\partial_{yy}u\,\mathrm{d}y\,\mathrm{d}s \le 2\int_0^t \|\partial_y u\|_{L^\infty_y}\|\partial_{yy}u\|_{L^2_y}\|\Phi_s\|_{L^2_y(L^2_\omega)}\,\mathrm{d}s\\
&\le 2\int_0^t \|\partial_y u\|_{L^2_y}\|\partial_{yy}u\|_{L^2_y}\|\Phi_t\|_{L^2_y(L^2_\omega)}\,\mathrm{d}s + 2\int_0^t \|\partial_y u\|^{\frac{1}{2}}_{L^2_y}\|\partial_{yy}u\|^{\frac{3}{2}}_{L^2_y}\|\Phi_t\|_{L^2_y(L^2_\omega)}\,\mathrm{d}s\\
&\le \epsilon\|\partial_{yy}u\|^2_{L^2_tL^2_y} + \frac{2}{\epsilon}\int_0^t \|\partial_y u\|^2_{L^2_y}\|\Phi_t\|^2_{L^2_y(L^2_\omega)}\,\mathrm{d}s + \frac{27}{2\epsilon^3}\int_0^t \|\partial_y u\|^2_{L^2_y}\|\Phi_t\|^4_{L^2_y(L^2_\omega)}\,\mathrm{d}s,
\end{aligned}$$

where ϵ is a fixed small positive number. Energy estimate shows

$$-\int_0^t\int_\square \partial_y\tau\partial_{yy}u \,\mathrm{d}y\,\mathrm{d}s \le \frac{1}{4}\|\partial_{yy}u\|^2_{L^2_tL^2_y} + C\Big(\|K_{11}\|^2_{L^2_tL^2_y} + \|K_2\|^2_{L^2_tL^2_y} + \int_0^t \|\partial_y u\|^2_{L^2_y}\|\Phi_t\|^2_{L^2_y(L^2_\omega)}\,\mathrm{d}s + \int_0^t \|\partial_y u\|^2_{L^2_y}\|\Phi_t\|^4_{L^2_y(L^2_\omega)}\,\mathrm{d}s\Big), \tag{3.6}$$

where C is a fixed positive real number. By the definition of K_{11} and K_2, it is easy to show that $\|K_{11}\|^2_{L^2_y} \le 9\|\Phi_t\|^2_{L^2_y(L^2_\omega)}$ and $\|K_2\|^2_{L^2_y} \le 4\|\Phi_t(\omega)-\Phi_t(\omega')\|^2_{L^2_y(L^2_\omega(L^2_{\omega'}))} \le 16\|\Phi_t\|^2_{L^2_y(L^2_\omega)}$. By (1.8), we have

$$\mathrm{d}\Phi_t = -\Big(\int_\Omega 2\cos 2(\Theta_t - \Theta_t(\omega'))(\Phi_t - \Phi_t(\omega'))\,\mathrm{d}\mathbb{P}(\omega') + \partial_{yy}u\sin^2\Theta_t + \partial_y u\Phi_t\sin 2\Theta_t\Big)\,\mathrm{d}t. \tag{3.7}$$

Applying Ito's formula to Φ_t^2 and integrating over y gives

$$\|\Phi_t\|^2_{L^2_y(L^2_\omega)} \le \|\Phi_0\|^2_{L^2_y(L^2_\omega)} + 4\int_0^t \Big((1+\|\partial_y u\|_{L^\infty_y})\,\|\Phi_s\|^2_{L^2_y(L^2_\omega)} + \|\partial_{yy}u\|^2_{L^2_y}\Big)\,\mathrm{d}s. \tag{3.8}$$

The Gronwall's inequality leads to

$$\|\Phi_t\|^2_{L^2_y(L^2_\omega)} \le \left(\|\Phi_0\|^2_{L^2_y(L^2_\omega)} + 4\int_0^t \|\partial_{yy}u\|^2_{L^2_y}\,\mathrm{d}s\right)\exp\left(4\int_0^t (1+\|\partial_y u\|_{L^\infty_y})\,\mathrm{d}s\right). \tag{3.9}$$

It follows from (3.3), (3.6) and (3.9) that

$$\|\partial_y u\|^2_{L^2_y}(t) + \frac{1}{2}\int_0^t \|\partial_{yy}u\|^2_{L^2_y}(s)\mathrm{d}s \le \|\partial_y u_0\|^2_{L^2_y} + c^2 t \tag{3.10}$$
$$+C_1\int_0^t (1+\|\partial_y u\|^2_{L^2_y})\left(1+\int_0^s \|\partial_{yy}u\|^2_{L^2_y}\,\mathrm{d}r)\right)^2\exp\left(8\int_0^s (1+\|\partial_y u\|_{L^\infty_y})\,\mathrm{d}r\right)\,\mathrm{d}s,$$

where C_1 depends on $\|\Phi_0\|^2_{L^2_y(L^2_\omega)}$. Let $f_1(t) = \|\partial_y u\|^2_{L^2_y}(t)$, $f_2(t) = \int_0^t \|\partial_{yy}u\|^2_{L^2_y}(s)\,\mathrm{d}s$, and $A = \|\partial_y u_0\|^2_{L^2_y} + c^2 T$. Then we have

$$f_1(t) + \frac{1}{2}f_2(t) \le A + C_1\int_0^t \Big(1+f_1(s)\Big)\Big(1+f_2(s)\Big)^2\exp\Big(C_2(1+f_2(s))\Big)\,\mathrm{d}s. \tag{3.11}$$

Here the first energy estimate is used for $\int_0^t f_1(s)\,\mathrm{d}s$, thus C_2 may depend on $\|u_0\|^2_{L^2_y}, c$ and T . Denoting by $R(t)$ the right hand side of (3.11) and noticing that $R(t)$ is an increasing function, we have

$$R(t) \le A + C_1\int_0^t \Big(R(s)+1\Big)^3\exp\Big(C_2(R(s)+1)\Big)\,\mathrm{d}s. \tag{3.12}$$

For any given $T>0$, there exist two positive constants γ and C, such that

$$R(t) \le A + \int_0^t C\exp\Big(\gamma R(s)\Big)ds.$$

Therefore, if $t \le e^{-\gamma A}/\gamma C$, then

$$R(t) \le \frac{1}{\gamma}\ln\Big(\frac{1}{e^{-\gamma A}-\gamma C t}\Big).$$

If we let

$$T' = \big(e^{-\gamma A} - e^{-2\gamma A}\big/\,(\gamma C) \le e^{-\gamma A}/(\gamma C), \tag{3.13}$$

then $\forall t \in (0, T')$

$$R(t) \le \frac{1}{\gamma}\ln\Big(\frac{1}{e^{-\gamma A}-\gamma C t}\Big) \le 2A.$$

We have for any $t \in (0,T')$, $\|\partial_y u\|^2_{L^2_y}(t) + \frac{1}{2}\int_0^t \|\partial_{yy}u\|^2_{L^2_y} \le 2\|\partial_y u_0\|^2_{L^2_y} + 2c^2 T$. This fact, together with (3.1), gives

$$\|u\|_{L^\infty_t([0,T'],H^1_y)} + \|u\|_{L^2_t([0,T'],H^2_y)} \le C, \tag{3.14}$$

from which it is also easy to obtain that $\Phi_t \in L^\infty_t(L^2_y(L^2_\omega))$. We now estimate $\partial_t u$. Observe that $\partial_t u = \partial_{yy}u + \partial_y\tau - c$. Thus (3.5) implies

$$\|\partial_y\tau\|_{L^2_t([0,T'],L^2_y)} \le \|K_{11}+K_{12}+K_2\|_{L^2_t([0,T'],L^2_y)} + \|\partial_{yy}u\|_{L^2_t([0,T'],L^2_y)}\,.$$

Only the term K_{12} needs to be considered. We have

$$\begin{aligned}\|K_{12}\|^2_{L^2_t([0,T'],L^2_y)} &\le \int_0^t \|\partial_y u\|^2_{L^\infty_y}\|\Phi_s\|^2_{L^2_y(L^2_\omega)}\,\mathrm{d}s\\ &\le \|\Phi_t\|_{L^\infty_t(L^2_y(L^2_\omega))}\int_0^t \|\partial_y u\|^2_{L^\infty_y}\,\mathrm{d}s,\end{aligned} \tag{3.15}$$

which implies that $K_{12}{\in}L^2_t([0,T'],L^2_y)$. Similarly, we can show that $\partial_y\tau{\in}L^2_t([0,T'],L^2_y)$ and therefore $\partial_t u \in L^2_t([0,T'],L^2_y)$. This completes the proof of this lemma. ■

3.2. Existence and uniqueness. We first prove the uniqueness part in Theorem 3.1. Suppose we have two solutions (u,Θ) and $(\tilde u,\tilde\Theta)$ defined in Theorem 3.1, whose differences are denoted by $w = u - \tilde u$ and $\Gamma = \Theta - \tilde\Theta$, respectively. Then one can easily have for any $t \in [0,T']$

$$\frac{1}{2}\int_\square w^2(t) + \int_0^t\!\!\int_\square (\partial_y w)^2 = -\int_0^t\!\!\int_\square \partial_y w(\tau - \tilde\tau), \tag{3.16}$$

where

$$\tilde\tau = \mathbb{E}\left[\sin 2\tilde\Theta_t + \cos^2\tilde\Theta_t\int_\Omega \sin 2(\tilde\Theta_t - \tilde\Theta_t(\omega'))\,\mathrm{d}\mathbb{P}(\omega') + \partial_y\tilde u\sin^2 2\tilde\Theta_t\right].$$

By writing $\tau - \tilde\tau = \mathbb{E}\,[z] + \partial_y w\sin^2 2\Theta_t$, we have

$$-\int_0^t\!\!\int_\square \partial_y w(\tau-\tilde\tau) \le -\int_0^t\!\!\int_\square \partial_y w\mathbb{E}\,[z] \le \frac{1}{2}\int_0^t\!\!\int_\square |\partial_y w|^2 + \frac{1}{2}\int_0^t\!\!\int_\square \|z\|^2_{L^2_\omega}\,, \tag{3.17}$$

where

$$\begin{aligned} z =& (\sin 2\Theta_t - \sin 2\tilde{\Theta}_t) + \partial_y \tilde{u}(\sin^2 2\Theta_t - \sin^2 2\tilde{\Theta}_t) \\ &+ (\cos^2 \Theta_t - \cos^2 \tilde{\Theta}_t) \int_\Omega (\sin 2(\Theta_t - \Theta_t(\omega'))) \, \mathrm{d}\mathbb{P}(\omega') \\ &+ \cos^2 \tilde{\Theta}_t \int_\Omega \sin 2(\Theta_t - \Theta_t(\omega')) - \sin 2(\tilde{\Theta}_t - \tilde{\Theta}_t(\omega')) \, \mathrm{d}\mathbb{P}(\omega'). \end{aligned}$$

By this, we have

$$\|z\|^2_{L^2_y(L^2_\omega)} \le C(1 + \|\partial_y \tilde{u}\|^2_{L^\infty_y}) \|\Gamma_t\|^2_{L^2_y(L^2_\omega)}. \tag{3.18}$$

Subtracting the stochastic differential equations of Θ and $\tilde{\Theta}$, we find that

$$\begin{aligned} \mathrm{d}\Gamma_t = \Bigg(&-\int_\Omega \Big(\sin 2(\Theta_t - \Theta_t(\omega')) - \sin 2(\tilde{\Theta}_t - \tilde{\Theta}_t(\omega')) \Big) \, \mathrm{d}\mathbb{P}(\omega') \\ &-\partial_y w \sin^2 \Theta_t - \partial_y \tilde{u}(\sin^2 \Theta_t - \sin^2 \tilde{\Theta}_t) \Bigg) \, \mathrm{d}t. \end{aligned} \tag{3.19}$$

Applying Ito's formula to Γ_t^2 and integrating over y leads to

$$\|\Gamma_t\|^2_{L^2_y(L^2_\omega)} \le \int_0^t \|\partial_y w\|^2_{L^2_y} \, \mathrm{d}s + C \int_0^t (1 + \|\partial_y \tilde{u}\|_{L^\infty_y}) \, \|\Gamma_s\|^2_{L^2_y(L^2_\omega)} \, \mathrm{d}s, \tag{3.20}$$

which, together with the Gronwall's lemma gives

$$\|\Gamma_t\|^2_{L^2_y(L^2_\omega)} \le C \exp\left(\int_0^t \|\partial_y \tilde{u}\|_{L^\infty_y} \, \mathrm{d}s \right) \int_0^t \|\partial_y w\|^2_{L^2_y} \, \mathrm{d}s. \tag{3.21}$$

Combing (3.16), (3.17), (3.18) and (3.21) yields

$$\begin{aligned} &\int_\square w^2(t) + \int_0^t \int_\square (\partial_y w)^2 \\ &\le \int_0^t C \Bigg((1 + \|\partial_y \tilde{u}\|^2_{L^\infty_y}) \exp\left(\int_0^s \|\partial_y \tilde{u}\|_{L^\infty_y} \, \mathrm{d}r \right) \int_0^s \|\partial_y w\|^2_{L^2_y} \, \mathrm{d}r \Bigg) \mathrm{d}s. \end{aligned} \tag{3.22}$$

Since $\tilde{u} \in L^\infty_t(H^1_{0,y}) \cap L^2_t(H^2_y)$, the Gronwall's lemma shows that $w = 0$, from which the uniqueness of Θ follows.

Let us show now the existence part of Theorem 3.1 by a Galerkin method. Let $\{v_i\}_{1 \le i \le +\infty} \in \mathcal{C}^\infty(\square) \cap H^1_0(\square)$ is basis in $H^1_0(\square)$. Denote $V^m = \mathrm{Vect}\{v_j, 1 \le j \le m\}$. Then our space Galerkin method reads: Find $U^m \in L^\infty_t(\mathbb{R}^m)$ such that $u^m(t,y) = \sum_{i=1}^m U^m_i(t) v_i(y)$ satisfies, $\forall t \in [0,T]$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\square u^m v = -\int_\square \partial_y u^m \partial_y v - \int_\square \tau^m \partial_y v - c \int_\square v, \qquad \forall v \in V^m \tag{3.23}$$

with initial condition $U^m(t=0) = U_0$ (the vector with components of $u^m_0 = \Pi^m u_0 \in H^1_y$, here Π^m is the projection operator), and

$$\partial_y \tau^m = K^m_{11} + K^m_{12} + K^m_2 + \partial_{yy} u^m \mathbb{E}\left[\sin^2 2\Theta^m_t\right], \tag{3.24}$$

where

$$
\begin{aligned}
K_{11}^m &= 2\mathbb{E}\left[\Phi_t^m \cos 2\Theta_t^m\right] - \mathbb{E}\left[\Phi_t^m \sin 2\Theta_t \int_\Omega \sin 2(\Theta^m - \Theta^m(\omega'))\mathbb{P}(\,\mathrm{d}\omega')\right],\\
K_{12}^m &= 2\partial_y u^m \mathbb{E}\left[\Phi_t^m \sin 4\Theta_t^m\right],\\
K_2^m &= 2\mathbb{E}\left[\cos^2\Theta_t^m \int_\Omega (\Phi_t^m - \Phi_t^m(\omega'))\cos 2(\Theta_t^m - \Theta_t^m(\omega'))\mathbb{P}(\,\mathrm{d}\omega')\right],
\end{aligned}
$$

Θ^m is the solution of

$$
\Theta_t^m = \Theta_0 - \int_0^t \Big(a(\Theta_s^m, \mathcal{L}(\Theta_s^m) + \partial_y u^m \sin^2\Theta_s^m\Big)\mathrm{d}s + W_t \tag{3.25}
$$

and $\Phi^m = \partial_y \Theta^m$.

LEMMA 3.4. *For any fixed positive integer m, and any $T > 0$, the finite dimensional approximation* (3.23) *has a unique solution u^m.*

The proof of this lemma is given in Appendix A. Now, the formal *a priori* estimates of (3.1) and (3.3) can be rigorous on the discretized problem. Therefore, we know that there exists $T' > 0$ such that there exists an uniform bound (independent of m) on u^m in norm $L_t^\infty([0,T'], H_y^1) \cap L_t^2([0,T'], H_y^2)$ and on $\partial_t u^m$ in norm $L_t^2([0,T'], L_y^2)$.

Up to the extraction of a subsequence, we obtain a function $u \in L_t^\infty([0,T'], H_y^1) \cap L_t^2([0,T'], H_y^2)$ such that u^m converges towards u weakly in $L_t^2([0,T'], H_y^2)$, weak-* in $L_t^\infty([0,T'], H_y^1)$, $\partial_t u^m$ converges towards $\partial_t u$ weakly in $L_t^2([0,T'], L_y^2)$, and u^m strongly converges towards u in $L_t^2([0,T'], H_y^1)$. Also one can define Θ for any $y \in \square$ by

$$
\Theta_t = \Theta_0 - \int_0^t \Big(a(\Theta_s, \mathcal{L}(\Theta_s) + \partial_y u \sin^2\Theta_s\Big)\mathrm{d}s + W_t,
$$

whose existence is guaranteed by the regularity of $u \in L_t^2([0,T'], H_y^2)$.

We next verify that (u, Θ) defined above is indeed the solution of the coupled system satisfied by (u^m, Θ^m). The only non-trivial term is $\int_0^{T'}\int_\square \tau^m \partial_y v\, w$, where $w \in C_0^\infty([0,T'])$. From the estimates (3.18), (3.21), and the strong convergence of u^m in $L_t^2([0,T'], H_y^1)$, one can show that

$$
\int_0^{T'}\int_\square \tau^m \partial_y v\, w \to \int_0^{T'}\int_\square \tau \partial_y v\, w \tag{3.26}
$$

is valid. This concludes the proof of the well-posedness of the system (1.7)-(1.9).

4. Numerical scheme and convergence analysis

This section is concerned with the convergence of a finite difference-Monte Carlo hybrid numerical scheme for the coupled system (1.7)-(1.9). We choose a finite difference discretization in space of the velocity and apply the Monte Carlo method to approximate the probabilistic expectation in the expression of stress τ by the statistical average. A key issue in our case is how to numerically treat the nonlinearity term $a(\Theta_t, \mathcal{L}(\Theta_t)$ in the SDE (1.7) arising from the mean field potential. Based on the law of large numbers for the interacting diffusions[33], we introduce M weakly interacting processes $(\bar{\Theta}^j)_{j\le M}$ [27] and show their asymptotic behavior is an approximation to the nonlinear diffusion processes Θ at the convergence rate of $\mathcal{O}(M^{-1/2})$. We begin by stating our numerical scheme and the main convergence result about its convergence. The rigorous proofs will be in the second subsection.

4.1. The full discretization and our main result. We use a staggered grid in the velocity field. The interval $[0,1]$ is divided into a uniform mesh with the grid size $h=1/N$. Denote the numerical solution by $\bar{u}^n_{i+\frac{1}{2}}$ at the position $y_{i+\frac{1}{2}}=(i+\frac{1}{2})h$, and the discrete weakly interacting processes $\bar{\Theta}^n_{i,j}$ at the position $y_i=ih$, for $j=1,2,\cdots,M$. n is the time step number and $t_n=n\delta t$. We use the finite difference method and the backward Euler scheme:

$$\begin{aligned}\frac{\bar{u}^{n+1}_{i+\frac{1}{2}}-\bar{u}^n_{i+\frac{1}{2}}}{\delta t}=&\frac{(D_h\bar{u})^{n+1}_{i+1}-(D_h\bar{u})^{n+1}_i}{h}+\frac{(\bar{\tau}_\alpha)^n_{i+1}-(\bar{\tau}_\alpha)^n_i}{h}+\\&+\frac{(D_h\bar{u})^{n+1}_{i+1}(\bar{\tau}_\beta)^n_{i+1}-(D_h\bar{u})^{n+1}_i(\bar{\tau}_\beta)^n_i}{h}-c\end{aligned}\tag{4.1}$$

with

$$(D_h\bar{u})^n_j=\frac{\bar{u}^n_{j+\frac{1}{2}}-\bar{u}^n_{j-\frac{1}{2}}}{h},\qquad(\bar{\tau}_\beta)^n_i=\frac{1}{M}\sum_{j=1}^M\sin^2 2\bar{\Theta}^n_{i,j},$$

$$(\bar{\tau}_\alpha)^n_i=\frac{1}{M}\sum_{j=1}^M\sin 2\bar{\Theta}^n_{i,j}+\frac{1}{M^2}\sum_{j=1}^M\cos^2\bar{\Theta}^n_{i,j}\sum_{k=1}^M\sin 2(\bar{\Theta}^n_{i,j}-\bar{\Theta}^n_{i,k}).$$

Then, one computes $\bar{\Theta}^{n+1}_{i,j}(j=1,2,\cdots,M)$ using

$$\begin{aligned}&\bar{\Theta}^{n+1}_{i,j}-\bar{\Theta}^n_{i,j}\\=&-\Big(\frac{1}{M}\sum_{k=1}^M\sin 2(\bar{\Theta}^n_{i,j}-\bar{\Theta}^n_{i,k})+(D_h\bar{u})^{n+1}_i\sin^2\bar{\Theta}^n_{i,j}\Big)\delta t+W^j_{t_{n+1}}-W^j_{t_n}.\end{aligned}\tag{4.2}$$

The initial condition and boundary condition are

$$\bar{u}^0_{i+\frac{1}{2}}=u_0(y_{i+\frac{1}{2}}),\qquad\bar{\Theta}^0_{i,j}=\Theta_0(y_j),\quad i=0,\cdots,N;\ j=1,\cdots,M\tag{4.3}$$

$$\bar{u}^n_{-\frac{1}{2}}=-\bar{u}^n_{\frac{1}{2}},\qquad\bar{u}^n_{N+\frac{1}{2}}=-\bar{u}^n_{N-\frac{1}{2}},\qquad n\geq 1.\tag{4.4}$$

Define the discrete norm,

$$\|u_{i+\frac{1}{2}}\|^2_{L^2_h(L^2_\omega)}=h\sum_{i=0}^{N-1}\mathbb{E}|u_{i+\frac{1}{2}}|^2,\quad\|\Theta_i\|^2_{L^2_h(L^2_\omega)}=h\sum_{i=0}^N\mathbb{E}|\Theta_i|^2.$$

We can prove the following convergence result for the full discretization scheme.

THEOREM 4.1 (Convergence analysis). *Given $u(t,y)\in L^\infty_t(H^2_y)\cap L^2_t(H^3_y)$, $u_t(t,y)\in L^2_t(H^2_y)$ and $\Theta^y_t\in C_t(H^1_y(L^2_\omega))$. If the coupled system is discretized by* (4.1)-(4.4)*, then the following error estimate for u and Θ holds:*

$$\begin{aligned}&\|u(t_n,y_{i+\frac{1}{2}})-\bar{u}^n_{i+\frac{1}{2}}\|^2_{L^2_h(L^2_\omega)}+\frac{1}{M}\sum_{j=1}^M\|\Theta_{t_n}(y_i)-\bar{\Theta}^n_{i,j}\|^2_{L^2_h(L^2_\omega)}\\&\leq C\left(\frac{1}{M}+\delta t^2+h^4\right).\end{aligned}\tag{4.5}$$

REMARK 3. *Assume that $u(t,y) \in L_t^\infty(H_y^2) \cap L_t^2(H_y^3)$, $u_t(t,y) \in L_t^2(H_y^2)$ and $\Theta_t^y \in C_t(H_y^1(L_\omega^2))$ in Theorem 3, which is a stronger assumption on u than that in well-posedness analysis. It can be obtained by taking higher order differential to the equation of u, and a similar analysis gives the result. The readers may be referred to* [9] *for details.*

4.2. Numerical analysis of the full discretization scheme. It is easy to verify that $u(t_n, y_{j+\frac{1}{2}})$ satisfies the difference equation (4.1) with the truncation error $\mathcal{O}(\delta t + h^2)$. Let $e^n_{j+\frac{1}{2}} = \bar{u}^n_{j+\frac{1}{2}} - u(t_n, y_{j+\frac{1}{2}})$. Then we have

$$
\begin{aligned}
\frac{e^{n+1}_{i+\frac{1}{2}} - e^n_{i+\frac{1}{2}}}{\delta t} =& \frac{(D_h e)^{n+1}_{i+1} - (D_h e)^{n+1}_i}{h} + \frac{(\bar{\tau}_\alpha - \tau_\alpha)^n_{i+1} - (\bar{\tau}_\alpha - \tau_\alpha)^n_i}{h} \\
&+ \frac{(D_h e)^{n+1}_{i+1}(\bar{\tau}_\beta)^n_{i+1} - (D_h e)^{n+1}_i(\bar{\tau}_\beta)^n_i}{h} \\
&+ \frac{(D_h u)^{n+1}_{i+1}(\bar{\tau}_\beta - \tau_\beta)^n_{i+1} - (D_h u)^{n+1}_i(\bar{\tau}_\beta - \tau_\beta)^n_i}{h} \\
&+ C(\delta t + h^2),
\end{aligned}
\tag{4.6}
$$

where

$$
(D_h u)^n_i = \frac{u(t_n, y_{i+\frac{1}{2}}) - u(t_n, y_{i-\frac{1}{2}})}{h}
$$

and $(\tau_\alpha)^n_i$ and $(\tau_\beta)^n_i$ are defined in a similar manner. Multiplying (4.6) by $e^{n+1}_{i+\frac{1}{2}}$, summing over i and using summation by parts, yield

$$
\begin{aligned}
\frac{h}{2\delta t}\sum_{i=0}^N |e^{n+1}_{i+\frac{1}{2}}|^2 + \frac{h}{2}\sum_{i=0}^N |(D_h e)^{n+1}_i|^2 \le& \frac{h}{2\delta t}\sum_{i=0}^N |e^n_{i+\frac{1}{2}}|^2 \\
&+ Ch\sum_{i=0}^N |(\bar{\tau}_\alpha)^n_i - (\tau_\alpha)^n_i|^2 + Ch\sum_{i=0}^N |(\bar{\tau}_\beta)^n_i - (\tau_\beta)^n_i|^2 + C(\delta t + h^2)^2,
\end{aligned}
\tag{4.7}
$$

where the positivity of the term $(\bar{\tau}_\beta)^n_i$ and the bound of $D_h u^n_i$ are used. Note that (the computation for $|(\bar{\tau}_\alpha)^n_i - (\tau_\alpha)^n_i|$ is similar) for $i = 1, \cdots, N$

$$
\begin{aligned}
&\mathbb{E}|(\bar{\tau}_\beta)_i^n - (\tau_\beta)_i^n|^2 \\
&= \left\| \mathbb{E}\left[\sin 2\Theta_{t_n}(y_i)\right] - \frac{1}{M}\sum_{j=1}^{M} \sin 2\bar{\Theta}_{i,j}^n \right\|_{L^2_\omega}^2 \\
&\le 4\left\| \mathbb{E}\left[\sin 2\Theta_{t_n}(y_i)\right] - \frac{1}{M}\sum_{j=1}^{M} \sin 2\tilde{\Theta}_{t_n}^j(y_i) \right\|_{L^2_\omega}^2 \\
&\quad +4\left\| \frac{1}{M}\sum_{j=1}^{M} (\sin 2\tilde{\Theta}_{t_n}^j(y_i) - \sin 2\Theta_{t_n}^j(y_i)) \right\|_{L^2_\omega}^2 \\
&\quad +4\left\| \frac{1}{M}\sum_{j=1}^{M} (\sin 2\Theta_{t_n}^j(y_i) - \sin 2\Theta_{i,j}^n) \right\|_{L^2_\omega}^2 + 4\left\| \frac{1}{M}\sum_{j=1}^{M} (\sin 2\Theta_{i,j}^n - \sin 2\bar{\Theta}_{i,j}^n) \right\|_{L^2_\omega}^2 \\
&\le 4\left\| \mathbb{E}\left[\sin 2\Theta_{t_n}(y_i)\right] - \frac{1}{M}\sum_{j=1}^{M} \sin 2\tilde{\Theta}_{t_n}^j(y_i) \right\|_{L^2_\omega}^2 + \frac{C}{M}\sum_{j=1}^{M} \|\tilde{\Theta}_{t_n}^j(y_i) - \Theta_{t_n}^j(y_i)\|_{L^2_\omega}^2 \\
&\quad +\frac{C}{M}\sum_{j=1}^{M} \|\Theta_{t_n}^j(y_i) - \Theta_{i,j}^n\|_{L^2_\omega}^2 + \frac{C}{M}\sum_{j=1}^{M} \|\Theta_{i,j}^n - \bar{\Theta}_{i,j}^n\|_{L^2_\omega}^2,
\end{aligned}
\tag{4.8}
$$

where M i.i.d. processes $\tilde{\Theta}_t^j$ (copies of Θ_t) and M weakly interacting processes Θ_t^j for the nonlinear diffusion processes Θ_t are introduced:

$$
\tilde{\Theta}_t^j = \Theta_0 - \int_0^t \Big(a(\tilde{\Theta}_s^j, \mathcal{L}(\tilde{\Theta}_s^j)) + \partial_y u \sin^2 \tilde{\Theta}_s^j \Big) \mathrm{d}s + W_t^j, \quad j = 1, \cdots, M, \tag{4.9}
$$

$$
\Theta_t^j = \Theta_0 - \int_0^t \Big(\frac{1}{M}\sum_{k=1}^{M} \sin 2(\Theta_s^j - \Theta_s^k) + \partial_y u \sin^2 \Theta_s^j \Big) \mathrm{d}s + W_t^j, j=1, \cdots, M. \tag{4.10}
$$

And $\Theta_{i,j}^n$ is the time-discretized solution of Θ_t^j at the time t_n and the spatial position y_i:

$$
\begin{aligned}
\Theta_{i,j}^{n+1} - \Theta_{i,j}^n = &- \Big(\frac{1}{M}\sum_{k=1}^{M} \sin 2(\Theta_{i,j}^n - \Theta_{i,k}^n) + \partial_y u(t_n, y_i) \sin^2 \Theta_{i,j}^n \Big) \delta t \\
&+ W_{t_{n+1}}^j - W_{t_n}^j, \qquad j = 1, \cdots, M.
\end{aligned}
\tag{4.11}
$$

We are about to treat the four terms one by one on the right hand side of (4.8). For the first term, the classical Law of Large Numbers[3] shows

$$
\left\| \mathbb{E}\left[\sin 2\Theta_{t_n}(y_i)\right] - \frac{1}{M}\sum_{j=1}^{M} \sin 2\tilde{\Theta}_{t_n}^j(y_i) \right\|_{L^2_\omega}^2 \le \frac{C}{M}. \tag{4.12}
$$

By the analysis of the numerical scheme of (4.11), we prove the convergence of the third term when δt tends to zero in Appendix B:

$$
\|\Theta_{t_n}^j(y_i) - \Theta_{i,j}^n\|_{L^2_\omega}^2 \le C\delta t^2. \tag{4.13}
$$

The asymptotic behavior of the weak interaction processes $\Theta^j_{t_n}$ in the second term of (4.8) is summarized as the following lemma.

LEMMA 4.2.

$$\frac{1}{M}\sum_{j=1}^{M}\|\Theta^j_t-\tilde{\Theta}^j_t\|^2_{L^2_\omega}\le\frac{C}{M},\qquad \textit{for any } y\in\square \tag{4.14}$$

where C only depends on T, u_0, Θ_0 and c.

Proof: Observe

$$\begin{aligned}
&|\Theta^j_t-\tilde{\Theta}^j_t|^2\\
=&\Big|\int_0^t\frac{1}{M}\sum_{k=1}^{M}\sin 2(\Theta^j_s-\Theta^k_s)-a(\tilde{\Theta}^j_s,\mathcal{L}(\tilde{\Theta}^j_s))+\partial_y u(\sin^2\Theta^j_s-\sin^2\tilde{\Theta}^j_s)\,\mathrm{d}s\Big|^2\\
\le&\, 2t\int_0^t\Big|\frac{1}{M}\sum_{k=1}^{M}\sin 2(\Theta^j_s-\Theta^k_s)-a(\tilde{\Theta}^j_s,\mathcal{L}(\Theta^j_s))\Big|^2+8t\int_0^t|\partial_y u|^2|\Theta^j_s-\tilde{\Theta}^j_s|^2.
\end{aligned}\tag{4.15}$$

Now look at the first term of the right hand side. We find:

$$\begin{aligned}
&\sin 2(\Theta^j_t-\Theta^k_t)-a(\tilde{\Theta}^j_t,\mathcal{L}(\tilde{\Theta}^j_t))\\
=&\Big(\sin 2(\Theta^j_t-\Theta^k_t)-\sin 2(\tilde{\Theta}^j_t-\Theta^k_t)\Big)+\Big(\sin 2(\tilde{\Theta}^j_t-\Theta^k_t)-\sin 2(\tilde{\Theta}^j_t-\tilde{\Theta}^k_t)\Big)+b_{jk},
\end{aligned}$$

where $b_{jk}=\sin 2(\tilde{\Theta}^j_t-\tilde{\Theta}^k_t)-a(\tilde{\Theta}^j_t,\mathcal{L}(\tilde{\Theta}^j_t))$. Insert this equality into (4.15), use Lipschitz property of sine function, perform summations over j and take expectation, then we have

$$\begin{aligned}
&\frac{1}{M}\sum_{j=1}^{M}\|\Theta^j_t-\tilde{\Theta}^j_t\|^2_{L^2_\omega}\\
\le&\, 24t\int_0^t\frac{1}{M}\sum_{j=1}^{M}\Big(\|\Theta^j_s-\tilde{\Theta}^j_s\|^2_{L^2_\omega}+\frac{1}{M}\sum_{k=1}^{M}\|\Theta^k_s-\tilde{\Theta}^k_s\|^2_{L^2_\omega}+\|\frac{1}{M}\sum_{k=1}^{M}b_{jk}\|^2_{L^2_\omega}\Big)\\
&+8t\int_0^t\frac{1}{M}\sum_{j=1}^{M}|\partial_y u|^2\|\Theta^j_s-\tilde{\Theta}^j_s\|^2_{L^2_\omega}\\
\le&\, 48t\int_0^t(1+|\partial_y u|^2)\frac{1}{M}\sum_{j=1}^{M}\|\Theta^j_s-\tilde{\Theta}^j_s\|^2_{L^2_\omega}+24t\int_0^t\frac{1}{M^3}\sum_{j=1}^{M}\|\sum_{k=1}^{M}b_{jk}\|^2_{L^2_\omega}.
\end{aligned}$$

Applying Gronwall's lemma and taking advantage of $u\in L^2_tH^2_y$, we find

$$\begin{aligned}
\frac{1}{M}\sum_{j=1}^{M}\|\Theta^j_t-\tilde{\Theta}^j_t\|^2_{L^2_\omega}&\le C\int_0^t\frac{1}{M^3}\sum_{j=1}^{M}\|\sum_{k=1}^{M}b_{jk}\|^2_{L^2_\omega}\\
&=C\int_0^t\frac{1}{M^3}\sum_{j=1}^{M}\mathbb{E}[\sum_{k,l=1}^{M}b_{jk}b_{jl}].
\end{aligned}$$

But when $k \neq l$, using Fubini theorem, noticing that $\mathcal{L}(\tilde{\Theta}_t^j) = \mathcal{L}(\tilde{\Theta}_t^k) = \mathcal{L}(\tilde{\Theta}_t^l)$

$$\begin{aligned}
&\mathbb{E}[b_{jk}b_{jl}] \\
= \ & \mathbb{E}_{(\tilde{\Theta}_t^j,\tilde{\Theta}_t^k,\tilde{\Theta}_t^l)}\Big[\Big(\sin 2(\tilde{\Theta}_t^j - \tilde{\Theta}_t^k) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\Big(\sin 2(\tilde{\Theta}_t^j - \tilde{\Theta}_t^l) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\Big] \\
= \ & \mathbb{E}_{\tilde{\Theta}_t^j}\Big[\mathbb{E}_{\tilde{\Theta}_t^k}\Big(\sin 2(\tilde{\Theta}_t^j - \tilde{\Theta}_t^k) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\mathbb{E}_{\tilde{\Theta}_t^l}\Big(\sin 2(\tilde{\Theta}_t^j - \tilde{\Theta}_t^l) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\Big] \\
= \ & \mathbb{E}_{\tilde{\Theta}_t^j}\Big[\Big(a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^k)) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\Big(a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^l)) - a(\tilde{\Theta}_t^j, \mathcal{L}(\tilde{\Theta}_t^j))\Big)\Big] \\
= \ & \mathbb{E}_{\tilde{\Theta}_t^j}[0] = 0.
\end{aligned}$$

Using this result and the boundedness of b_{jk}, we have

$$\frac{1}{M}\sum_{j=1}^{M} \|\Theta_t^j - \tilde{\Theta}_t^j\|_{L_\omega^2}^2 \leq \frac{C}{M}.$$

This completes the proof of Lemma 4.2. ■

Finally, we consider the last term on the right hand side of (4.8). We define $f_{i,j}^n = \bar{\Theta}_{i,j}^n - \Theta_{i,j}^n$ and $|F_i^n|^2 = \frac{1}{M}\sum_{j=1}^{M} \mathbb{E}\left[|f_{i,j}^n|^2\right]$. By (4.6), (4.8), (4.12), (4.13) and (4.14) we have

$$\begin{aligned}
&\|e_{i+\frac{1}{2}}^{n+1}\|_{L_h^2(L_\omega^2)}^2 + \delta t\|(D_h e)_i^{n+1}\|_{L_h^2(L_\omega^2)}^2 \\
\leq &\|e_{i+\frac{1}{2}}^{n}\|_{L_h^2(L_\omega^2)}^2 + \frac{C}{M}\delta t + Ch\delta t\sum_{i=0}^{N}|F_i^n|^2 + C\delta t^3 + Ch^4\delta t.
\end{aligned} \tag{4.16}$$

What remains is to estimate the error F_i^n that arises from the difference between the continuous term $\partial_y u$ in (4.11) and its numerical discretization $D_h\bar{u}$ in (4.2). We easily have by subtracting (4.11) from (4.2) to obtain

$$|f_{i,j}^{n+1}| \leq (1 + 2C_1\delta t)|f_{i,j}^n| + 2\delta t\left(\frac{1}{M}\sum_{k=1}^{M}|f_{i,k}^n| + g_i^n\right), \tag{4.17}$$

where $C_1 = \|\partial_y u\|_{L_t^\infty(L_y^\infty)} + 1$. It can be verified that

$$\begin{aligned}
g_i^n &= |\partial_y u(t_n, y_i) - (D_h u)_i^{n+1}| \\
&\leq |(D_h e)_i^{n+1}| + \int_{t_n}^{t_{n+1}} |\partial_t\partial_y u|\,\mathrm{d}t + |\partial_y u(t_{n+1}, y_i) - (D_h u)_i^{n+1}|) \\
&\leq |(D_h e)_i^{n+1}| + C(\delta t + h^2),
\end{aligned} \tag{4.18}$$

where $\partial_t\partial_y u \in L_t^2 L_y^\infty$ is applied. Squaring the both sides of (4.17) yields

$$\begin{aligned}
|f_{i,j}^{n+1}|^2 &\leq (1 + C\delta t)|f_{i,j}^n|^2 + C\delta t^2\left(|g_i^n|^2 + \frac{1}{M}\sum_{k=1}^{M}|f_{i,k}^n|^2\right) + C\delta t|f_{i,j}^n|\left(g_i^n + \frac{1}{M}\sum_{k=1}^{M}|f_{i,k}^n|\right) \\
&\leq |f_{i,j}^n|^2\left(1 + C_\varepsilon\delta t\right) + \varepsilon\delta t|g_i^n|^2 + C\delta t\left(\frac{1}{M}\sum_{k=1}^{M}|f_{i,k}^n|^2\right) \\
&\leq |f_{i,j}^n|^2(1 + C_\varepsilon\delta t) + \varepsilon\delta t|(D_h e)_i^{n+1}|^2 + \varepsilon C\delta t^3 + \varepsilon h^4\delta t + C\delta t\left(\frac{1}{M}\sum_{k=1}^{M}|f_{i,k}^n|^2\right)
\end{aligned}$$

where $C_\varepsilon = C + C/\varepsilon$ and ε is a sufficiently small positive number. Summing over $j = 1, \cdots, M$ and $i = 0, \cdots, N$ leads to

$$\begin{aligned} & h\sum_{i=0}^{N} |F_i^{n+1}|^2 \\ \le & (1 + C_\varepsilon \delta t)h \sum_{i=0}^{N} |F_i^n|^2 + \varepsilon\delta t \|(D_h e)_i^{n+1}\|^2_{L_h^2(L_\omega^2)} + \varepsilon C\delta t^3 + \varepsilon h^4 \delta t. \end{aligned} \tag{4.19}$$

Choose the positive number $\varepsilon < 1$. Adding (4.16) and (4.19) gives

$$\begin{aligned} & \|e_{i+\frac{1}{2}}^{n+1}\|^2_{L_h^2(L_\omega^2)} + h\sum_{i=0}^{N} |F_i^{n+1}|^2 \\ \le & \|e_{i+\frac{1}{2}}^{n}\|^2_{L_h^2(L_\omega^2)} + (1 + C_\varepsilon \delta t)h\sum_{i=0}^{N} |F_i^n|^2 + \frac{C}{M}\delta t + C\delta t^3 + Ch^4\delta t, \end{aligned} \tag{4.20}$$

from which our main result follows

$$\begin{aligned} & \|u(t_n, y_{i+\frac{1}{2}}) - \bar{u}_{i+\frac{1}{2}}^n\|^2_{L_h^2(L_\omega^2)} + \frac{1}{M}\sum_{j=1}^{M} \|\Theta_{t_n}(y_i) - \bar{\Theta}_{i,j}^n\|^2_{L_h^2 L_\omega^2} \\ \le & C\left(\frac{1}{M} + \delta t^2 + h^4\right). \end{aligned} \tag{4.21}$$

5. Conclusion

In this paper, the 1D stochastic model of LCPs with Maier-Saupe potential is investigated. The local time well-posedness is studied, and a numerical scheme is proposed and analyzed. The existence of the solution to the nonlinear SDE is established by using a contraction argument. The well-posedness of the solution to the coupled system is obtained under suitable *a priori* estimates for the velocity u and the stress τ. In our finite difference-Monte Carlo numerical scheme, a staggered grid in space for u and τ is applied. The novelty of the numerical approximation is the convergence of the weakly interacting processes to the nonlinear process by the asymptotic theory of the weakly interacting particle system in stochastic analysis. In particular, the optimal convergence rate $\mathcal{O}(\delta t + h^2 + \frac{1}{\sqrt{M}})$ is derived.

Appendix A. The proof of Lemma 3.4.

We first construct a fixed point U of the following mapping $F(X)$, $F\colon \mathcal{C}([0,T], \mathbb{R}^m) \to \mathcal{C}([0,T], \mathbb{R}^m)$

$$F(X)(t) = U_0 - A^{-1}\int_0^t \Big(BX(s) + \int_\square \mathcal{T}(X)_s \partial_y \Xi + c\int_\square \Xi\Big)\mathrm{d}s, \tag{A.1}$$

where $X \in \mathcal{C}([0,T], \mathbb{R}^m)$, matrices $A_{i,j} = \int_\square v_i v_j$, $B_{i,j} = \int_\square \partial_y v_i \partial_y v_j$, and $\Xi = (v_1, v_2, \cdots, v_m)$. The map $\mathcal{T}(X)$ is defined by

$$\begin{aligned} \mathcal{T}(X)_s = \mathbb{E}\Big[& \sin 2\Theta_s + \sum_{i=1}^{m} X_i(s)\partial_y v_i \sin^2 2\Theta_s \\ & + \cos^2\Theta_s \int_\Omega \sin 2(\Theta_s - \Theta_s(\omega'))\,\mathrm{d}\mathbb{P}(\omega')\Big] \end{aligned} \tag{A.2}$$

where Θ associates to X by the following SDE :

$$\mathrm{d}\Theta_s = -a(\Theta_s, \mathcal{L}(\Theta_s))\,\mathrm{d}s - \sum_{i=1}^{m} X_i(s)\partial_y v_i \sin^2 2\Theta_s\,\mathrm{d}s + \mathrm{d}W_s \tag{A.3}$$

with the same initial condition as before.

Proof: We are going to show by the Picard fixed point theorem that the function F in (A.1) has a fixed point when restricted on $\mathcal{C}([0,\alpha_0], B(U_0,1))$ endowed with the uniform convergence topology, for some small $\alpha_0 \in [0,T]$, and continue to construct the solution up to time T. These two parts will complete our proof.

A.1. Fixed point result at local time. First, we show that for any $X \in \mathcal{C}([0,\alpha], B(U_0,1))$,we have $F(X) \in \mathcal{C}([0,\alpha], B(U_0,1))$ provided that α is small enough. The ball $B(U_0,1)$ is defined by

$$B(U_0,1) = \{K \in \mathbb{R}^m : \|K - U_0\| \le 1\}.$$

Since $X \in \mathcal{C}([0,\alpha], B(U_0,1))$, it follows from (A.2) that

$$\|\mathcal{T}(X)\|_{L^\infty([0,\alpha],L_y^\infty)} \le C'(1 + \|X\|_{L_t^\infty[0,\alpha]}\|\partial_y \Xi\|_{L_y^\infty}) \le C,$$

for any $\alpha > 0$. This, together with (A.1), gives

$$\begin{aligned}&\|F(X) - U_0\|_{L_t^\infty[0,\alpha]}\\ &\le |||A^{-1}|||\alpha\Big(|||B|||(1+\|U_0\|) + \|\mathcal{T}(X)\|_{L_t^\infty([0,\alpha],L_y^\infty)}\|\partial_y\Xi\|_{L_y^\infty} + c\|\Xi\|_{L_y^\infty}\Big),\end{aligned}$$

which yields

$$\|F(X) - U_0\|_{L_t^\infty[0,\alpha]} \le C\alpha \le 1, \qquad \text{(if we take } \alpha \le 1/C.) \tag{A.4}$$

provided that α is sufficiently small, where C depends only $c, V^m, \|u_0\|_{L_y^2}, T$.

We next show that the function F restricted on $\mathcal{C}([0,\alpha], B(U_0,1))$ is contracting as long as α is sufficiently small. Let $X^1, X^2 \in \mathcal{C}([0,\alpha], B(U_0,1))$. Then Θ^1 and Θ^2 are given by (A.3). It follows from (A.1) that

$$\begin{aligned}&\|F(X^1) - F(X^2)\|_{L_t^\infty[0,\alpha]}\\ &\le \alpha C(\|X^1 - X^2\|_{L_t^\infty[0,\alpha]} + \|\mathcal{T}(X^1) - \mathcal{T}(X^2)\|_{L_t^\infty([0,\alpha],L_y^\infty)}).\end{aligned} \tag{A.5}$$

Moreover, by the definition (A.2) of $\mathcal{T}(X)$, one can easily obtain for any time $t \le T$ and $y \in \Box$

$$\begin{aligned}&|\mathcal{T}(X^1)_t - \mathcal{T}(X^2)_t|\\ &\le \mathbb{E}\left[|\Theta_t^1 - \Theta_t^2|\right]\left(1 + \sum_{i=1}^m |X_i^1(t)||\partial_y v_i|\right) + \sum_{i=1}^m |X_i^1(t) - X_i^2(t)||\partial_y v_i|.\end{aligned} \tag{A.6}$$

Using (A.3) and the same argument as that in deriving (3.21), we obtain

$$\begin{aligned}&\mathbb{E}\left[|\Theta_t^1 - \Theta_t^2|\right]\\ &\le C\exp\left(\int_0^t \sum_{i=1}^m |X_i^1(s)||\partial_y v_i|\,\mathrm{d}s\right)\int_0^t \sum_{i=1}^m |X_i^1(s) - X_i^2(s)||\partial_y v_i|\,\mathrm{d}s.\end{aligned} \tag{A.7}$$

Using the above two results gives

$$\|\mathcal{T}(X^1) - \mathcal{T}(X^2)\|_{L_t^\infty([0,\alpha],L_y^\infty)} \leq C\|X^1 - X^2\|_{L_t^\infty[0,\alpha]}, \tag{A.8}$$

which, together with (A.5), yields

$$\|F(X^1) - F(X^2)\|_{L_t^\infty[0,\alpha]} \leq \alpha C\|X^1 - X^2\|_{L_t^\infty[0,\alpha]}. \tag{A.9}$$

The above result indicates that if $\alpha C < 1$, then the function F is contracting on $\mathcal{C}([0,\alpha], B(U_0, 1))$. If we choose $\alpha_0 \leq 1/C$ in both (A.9) and (A.4), then for any initial condition U_0 the discrete problem has a solution $U \in \mathcal{C}([0,\alpha_0], \mathbb{R}^m)$ on the time interval $[0,\alpha_0]$.

A.2. Construct the solution at global time. We can now start again the construction of a solution to (3.23) from the final point $U(\alpha_0)$ at the time interval $t \in [\alpha_0, \alpha_0 + \alpha_1]$ with α_1 small enough and only dependent on T, V^m, c. Furthermore, using the same argument as before, we consider the mapping $F^{\alpha_0} : \mathcal{C}([\alpha_0, T], \mathbb{R}^m) \to \mathcal{C}([\alpha_0, T], \mathbb{R}^m)$

$$F^{\alpha_0}(X)(t) = U(\alpha_0) - A^{-1} \int_{\alpha_0}^{t} \Big(BX(s) + \int_{\Box} \mathcal{T}(X)_s \partial_y \Xi + c\Xi \Big) \mathrm{d}s. \tag{A.10}$$

Going through the same argument as before, we can choose α_1 only dependent on $V^m, T, \Box, c$, such that $F^{\alpha_0}(X)$ has a fixed point on $\mathcal{C}([\alpha_0, \alpha_0 + \alpha_1], \mathbb{R}^m)$. That means that we can extend the solution to the time interval $[0, \alpha_0 + \alpha_1]$. By a continuation argument, we can build a solution of the discrete problem (3.23) up to any finite time T. ■

Appendix B. The proof of $\left\|\Theta_{t_n}^j(y_i) - \Theta_{i,j}^n\right\|_{L_\omega^2} \leq C\delta t$.

We now consider the SDE (4.10) and its numerical scheme (4.11). Our aim is to prove the estimate of $\left\|\Theta_{t_n}^j - \Theta_{i,j}^n\right\|_{L_\omega^2} \leq C\delta t$ under the condition $\partial_y u \in L_t^\infty(L_y^\infty)$ and $\partial_t\partial_y u \in L_t^2(L_y^\infty)$. Let us consider a more general setting of our problem. The basic framework may be refered to [18].

Let $T > 0$. Assume $b_1(x), b_2(x) \in \mathcal{C}^\infty(\mathbb{R}^d)$ which have bounded derivatives up to the second order and assume $g(t,y) : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies $g \in L_t^\infty(L_y^\infty)$ and $\partial_t g \in L_t^2(L_y^\infty)$. We consider the SDE

$$\mathrm{d}X_t = \big(b_1(X_t) + g(t,y)b_2(X_t)\big)\,\mathrm{d}t + \,\mathrm{d}W_t \tag{B.1}$$

where $y \in \mathbb{R}$ is a fixed parameter and $g(t,y)$ is a known function. W_t is the d-dimensional Wiener process. We use Euler method to approximate (B.1)

$$\bar{X}_{n+1} = \bar{X}_n + \big(b_1(\bar{X}_n) + g(t_n,y)b_2(\bar{X}_n)\big)\delta t + \Delta W_n \tag{B.2}$$

with $t_n = n\delta t$ and $\bar{X}_0 = X_0$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. The Ito-Taylor expansion gives that

$$X_{t_{n+1}} = X_{t_n} + \big(b_1(X_{t_n}) + g(t_n,y)b_2(X_{t_n})\big)\delta t + \Delta W_n + R_{11} + R_{12} + R_2 + R_{31} + R_{32}, \tag{B.3}$$

where

$$R_{11} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \nabla b_1(X_s)\cdot\big(b_1(X_s)+g(s,y)b_2(X_s)\big)+\frac{1}{2}\triangle b_1(X_s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$R_{12} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \nabla b_1(X_s)\,\mathrm{d}W_s\,\mathrm{d}t, \qquad R_2 = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} \partial_t g(s,y) b_2(X_s)\,\mathrm{d}s\,\mathrm{d}t,$$
$$R_{31} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} g(s,y)\Big(\nabla b_2(X_s)\cdot\big(b_1(X_s)+g(s,y)b_2(X_s)\big)+\frac{1}{2}\triangle b_2(X_s)\Big)\,\mathrm{d}s\,\mathrm{d}t,$$
$$R_{32} = \int_{t_n}^{t_{n+1}}\int_{t_n}^{t} g(s,y)\nabla b_2(X_s)\,\mathrm{d}W_s\,\mathrm{d}t.$$

We set $E_n = X_{t_n} - \bar{X}_n$. Subtracting (B.2) from (B.3) and noticing the Lipschitz property of b_1, b_2 and the facts $\mathbb{E}\left[E_n R_{12}\right] = \mathbb{E}\left[E_n R_{32}\right] = 0$, we can obtain

$$\begin{aligned}\mathbb{E}\left[|E_{n+1}|^2\right] \le \mathbb{E}\left[|E_n|^2\right]\Big(1 + C\delta t\big(1+\|g\|_{L_t^\infty(L_y^\infty)})|\big) + C\delta t^2\big(1+\|g\|^2_{L_t^\infty(L_y^\infty)}\big)\Big) \\ + C\frac{1}{\delta t}\Big(\mathbb{E}\left[|R_{11}|^2+|R_2|^2+|R_{31}|^2\right]\Big) + C\mathbb{E}\left[|R_{12}|^2+|R_{32}|^2\right]\end{aligned} \tag{B.4}$$

where $|x| = \sqrt{\sum_{i=1}^d x_i^2}$ for $x \in \mathbb{R}^d$. It can be verified that

$$\begin{aligned}&\mathbb{E}\left[|R_{12}|^2\right] \le C\delta t^3, \qquad \mathbb{E}\left[|R_{32}|^2\right] \le C\delta t\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} |g(s,y)|^2\,\mathrm{d}s\,\mathrm{d}t,\\ &\mathbb{E}\left[|R_{11}|^2\right] + \mathbb{E}\left[|R_2|^2\right] + \mathbb{E}\left[|R_{31}|^2\right]\\ \le\ & C\delta t^4 + C\delta t^2\int_{t_n}^{t_{n+1}}\int_{t_n}^{t} |g(s,y)|^2 + |\partial_t g(s,y)|^2 + |g(s,y)|^4\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

These results, together with (B.4) and the fact that $\|g\|_{L_t^\infty(L_y^\infty)} \le C$, give

$$\mathbb{E}\left[|E_n|^2\right] \le C\delta t^2 + C\delta t^2\int_0^T \|\partial_t g(t,y)\|^2_{L_y^\infty}\ \mathrm{d}t. \tag{B.5}$$

Thus, we obtain

$$\sum_{j=1}^d \left\|X_{t_n}^j - \bar{X}_n^j\right\|^2_{L_\omega^2} \le C\delta t^2$$

where C depends only on T, $\|g\|_{L_t^\infty(L_y^\infty)}$, $\|\partial_t g\|_{L_t^2(L_y^\infty)}$ and $\|b_i\|_{L^\infty}$, $\|\nabla b_i\|_{L^\infty}$, $\|\triangle b_i\|_{L^\infty}$ $(i = 1, 2)$.

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