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Vanishing Curvature Viscosity for Front Propagation

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In this paper we study the front propagation with constant speed and small curvature viscosity. We first investigate two related problems of conservation laws, one of which is on the nonlinear viscosity methods for the conservation laws, and the other one is on the structure of solutions to conservation laws with L^1 initial data. We show that the nonlinear viscosity methods approaching the piecewise smooth solutions with finitely many discontinuity for convex conservation laws have the first-order rate of L^1-convergence. The solutions of conservation laws with L^1 initial data are shown to be bounded after $t>0$ if all singular points of initial data are from shocks. These results suggest that the front propagation with constant speed and a small curvature viscosity will approach the front movements with a constant speed, as the small parameter goes to zero. After the front breaks down, the cusps will disappear promptly and corners will be formed.

1. INTRODUCTION

Front propagation models with curvature-dependent normal velocities arise in a variety of physical phenomena such as flame propagation, solidification, and phase transition problems. For a front propagating with constant velocity c the level set formulation is a simple Hamilton–Jacobi (H–J) equation

$$\partial_t \phi + c\,|\nabla \phi| = 0. \tag{1.1}$$

The problem is technically simpler than the "general case" in which the Hamiltonian H may depend on $\nabla\phi$

$$\partial_t \phi + H(\nabla \phi) = 0. \tag{1.2}$$

One way of identifying a uniquely existing solution for a class problems which include (1.2) as a special case was given by M. G. Crandall and P. L. Lions [2]. The relevant solutions are called *viscosity solutions*, and they are known to be the solutions of primary interest in many areas of applications.

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In another model, the front moves along its normal vector field with speed $V = c - \varepsilon\kappa$, where κ is the mean curvature,

$$\kappa = \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right)$$

and so the level set formulation can be written as

$$\partial_t \phi^\varepsilon + c\, |\nabla \phi^\varepsilon| = \varepsilon \nabla \cdot \left(\frac{\nabla \phi^\varepsilon}{|\nabla \phi^\varepsilon|} \right). \tag{1.3}$$

Equation (1.3) is nonlinear, degenerate and undefined at points where $\nabla \phi^\varepsilon = 0$. A unique *weak solution* (viscosity solution) exists for the above equation, see Chen, Giga and Goto [1] and Evans and Spruck [5].

The key to the level set approach is the following link [12, 14]. Consider the propagating curve and two solutions: $X^\varepsilon_{curvature}(t)$, obtained by evolving the initial front with $V = c - \varepsilon\kappa$, and $X_{constant}(t)$, obtained with speed $V = c$ and the entropy condition. Then, for any time $t > 0$

$$\lim_{\varepsilon \to 0} X^\varepsilon_{curvature}(t) = X_{constant}(t), \tag{1.4}$$

i.e., the limit of motion with curvature is the entropy solution for the constant speed case. This is known as the *nonlinear viscous limit*. In order to see why nonlinear viscosity is an appropriate name, we turn to consider the link between propagating fronts and hyperbolic conservation laws [12, 13].

We consider a small section of the curve $\phi(x, y, t) = 0$, which, without loss of generality, can be written as $y = \psi(x, t)$. In this case, the front propagation with speed $V = c - \varepsilon\kappa$ is governed by

$$\psi_t + c\sqrt{1 + \psi_x^2} = \varepsilon \frac{\psi_{xx}}{1 + \psi_x^2}. \tag{1.5}$$

Letting $u^\varepsilon(x, t) = \psi_x(x, t)$ gives the conservation law

$$u^\varepsilon_t + (c\sqrt{1 + (u^\varepsilon)^2})_x = \varepsilon \left(\frac{u^\varepsilon_x}{1 + (u^\varepsilon)^2} \right)_x. \tag{1.6}$$

Similarly, the front propagation with speed $V = 1$ is governed by

$$u_t + (c\sqrt{1 + u^2})_x = 0. \tag{1.7}$$

In this paper, we will study the relationship between the solution u^ε to equation (1.6) and the entropy solution u to equation (1.7). In general, the nonlinear parabolic equation (1.6) can be regarded as the nonlinear viscosity approximation to the conservation law (1.7). One of the main results in this work, as stated in Theorem 2.1, is that if solutions to (1.7) is piecewise smooth with finitely many discontinuities then the rate of L^1-convergence for the nonlinear viscosity methods is of first order.

It is clear that the fronts may develop sharp corners and topological changes (merge and break down). The structure of solution to (1.1) and (1.3) are complicated. It is believed that the solution of equation (1.3) approaches the solution of the H–J equation (1.1) when ε goes to zero. In other words,

$$\lim_{\varepsilon \to 0} \Gamma^{\varepsilon}(t) = \Gamma(t), \tag{1.8}$$

where

$$\Gamma(t) = \{(x, y), \phi(x, y, t) = 0\}, \qquad \Gamma^{\varepsilon}(t) = \{(x, y), \phi^{\varepsilon}(x, y, t) = 0\}.$$

This is why the level set method works well for this problem.

When the front has topology change ($\nabla\phi = 0$), the H–J equation (1.1) is not equivalent to a conservation law, then we can not use the results of conservation laws to study the H–J equations. But after the front breaking down, generally a cusp will appear. How the solution will be beyond the formation of the cusp in the front? Sethian [14] constructed the entropy weak solution by removing the "tail" from the "swallowtail" and discussed the vanishing viscosity approach. We will handle it relating the entropy weak solution of conservation laws. Using the result of Theorem 3.1, we conclude that the cusp will disappear and a corner will be formed. It is also known that the level set method can handle the topology change but the accuracy is low at singular points. We will provide some explanations in Section 3.

Finally, we point out that based on the Crandall and Lions theory, our results are new in two aspects. Firstly, Crandall and Lions theory considers the linear viscosity, while we deal with the nonlinear viscosity in this work. Secondly, a half-order L^1-convergence rate is obtained if we employ the Crandall and Lions theory. However, a first-order rate of convergence is established in this work.

2. FIRST-ORDER L^1-CONVERGENCE FOR NONLINEAR VISCOSITY METHODS

In this section, we will investigate the convergence rate for nonlinear viscosity methods to conservation laws. In particular, we are interested in the optimal L^1-convergence rate.

Consider the scalar hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad -\infty < x < \infty, \quad t > 0, \tag{2.1}$$

subject to initial condition

$$u(x, 0) = u_0(x), \qquad -\infty < x < \infty. \tag{2.2}$$

The nonlinear viscosity method approximating the conservation laws (2.1) and (2.2) is to solve the nonlinear parabolic equations

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon(a(u^\varepsilon)\, u^\varepsilon_x)_x, \tag{2.3}$$

subject to the same initial condition

$$u(x, 0) = u_0(x), \tag{2.4}$$

where $u_0(x) \in BV(\mathbf{R}) \cap L^\infty(\mathbf{R})$. By the maximum principle, we have

$$|u^\varepsilon(x, t)| \leqslant \max_x \; |u_0(x)| \doteq M \tag{2.5}$$

In this section we will establish the L^1-convergence rate of $O(\varepsilon\, |\ln \varepsilon| + \varepsilon)$ for the nonlinear viscosity approximations (2.3) and (2.4) to the entropy solutions of the scalar conservation laws (2.1) and (2.2) under the assumptions that (1): the fluxes are convex and the entropy solutions u of (2.1) and (2): (2.2) are piecewise smooth with finitely many discontinuities. As a matter of fact the piecewise smooth entropy solutions are quite general and practical, which include initial central rarefaction waves, initial shocks, possible spontaneous formation of shocks in a future time and interactions of all these patterns. If neither central rarefaction waves nor spontaneous shocks occur in the piecewise entropy solutions, the rate of L^1-convergence is improved to $O(\varepsilon)$.

In this study we use a matching method, which developed by Goodman and Xin [6]. Goodman and Xin first introduced the matching method to assemble the travelling waves and showed that the viscosity methods to approximate piecewise smooth solutions with a finite number of noninteracting shocks have a local ε rate of convergence away from shocks. Later Teng and Zhang in [19] used a similar technique to prove that both viscosity methods and monotone difference schemes approaching piecewise constant solutions with shocks for convex conservation laws have a first-order rate of L^1-convergence. Tang and Teng [17] showed that the viscosity methods approaching the general class of piecewise smooth solutions have the same first-order L^1-convergence rate.

Let $b(u)=f'(u)$. Before giving the statement of our main theorems we make the following assumptions:

- (A1) $f(u)$ is strictly convex

$$\min_{|u|\leqslant M} f''(u)\geqslant\gamma>0.$$

The function $a(u)>0$ associated with the viscosity is continuously differentiable. Therefore, we have

$$\min_{|u|\leqslant M} a(u)\geqslant A_s>0, \qquad \max_{|u|\leqslant M} a(u)\leqslant A_L.$$

- (A2) $u_0(x)$ is bounded and piecewise C^2-smooth with a finite number of discontinuous points γ_i, $1\leqslant i\leqslant I$; $u_0(\gamma_i\pm 0)$ and $\dot{u}_0(\gamma_i\pm 0)$ exist and are finite; where $\dot{u}_0(x)=(d/dx)\,u_0(x)$.

- (A3) $(d^2/dx^2)(b(u_0))$ changes signs a finite number of times, i.e., $b(u_0(x))$ has a finite number of inflection points.

Under the above assumptions, we can obtain the following estimates.

THEOREM 2.1. *Assume* (A1)–(A3). *Let* u^ε *and* u *be the solutions of the Cauchy problem of* (2.3) *and* (2.1) *for the same initial data* u_0. *Then the following estimate holds for all* $T>0$

$$\sup_{0\leqslant t\leqslant T} \|u^\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R})}\leqslant C(T)\,\varepsilon\,|\ln\varepsilon|. \tag{2.6}$$

If there is no initial central rarefaction wave and no new formed shock in u, *then the error bound is improved to*

$$\sup_{0\leqslant t\leqslant T} \|u^\varepsilon(\cdot,t)-u(\cdot,t)\|_{L^1(\mathbf{R})}\leqslant C(T)\,\varepsilon. \quad \blacksquare \tag{2.7}$$

In order to provide a rigorous proof for the above theorem, we need to study the stability for nonhomogeneous nonlinear viscous equations and some properties for travelling wave. The following stability can be proved by a technique used by Tang and Teng [17], with some modifications.

LEMMA 2.1 ([17]). *Let* $v^{(i)}(x,t)$, $i=1,2$ *be continuous and piecewise smooth solutions of the following equations*:

$$(v^{(i)})_t+(f(v^{(i)}))_x-\varepsilon(a(v^{(i)})\,v_x^{(i)})_x=g_i(x,t), \qquad t\leqslant s\leqslant 0, \quad i=1,2.$$

We assume that the above equation holds for all values of x *except on some curves* $X_m(t)$, $1 \leqslant m \leqslant M$, *where* $v_x^{(i)}$ *may not exist. If* $w \doteq v^{(1)} - v^{(2)} \to 0$ *as* $x \to \infty$, *then*

$$\begin{aligned}\|w(\cdot, t)\|_{L^1} \leqslant \|w(\cdot, s)\|_{L^1} + \int_s^t \|g_1(\cdot, \tau) - g_2(\cdot, \tau)\|_{L^1}\, d\tau \\ + \varepsilon A_L \sum_{m=1}^{M} \int_s^t |w_x(X_m(\tau)+0, \tau) - w_x(X_m(\tau)-0, \tau)|\, d\tau. \quad \blacksquare\end{aligned} \tag{2.8}$$

Travelling wave solutions of (2.3) are of the form

$$u^\varepsilon(x, t) = U\left(\frac{x - St}{\varepsilon}\right), \tag{2.9}$$

which is subject to the following boundary condition at $x = \pm\infty$:

$$\lim_{\xi \to \pm\infty} U(\xi) = U_\pm, \tag{2.10}$$

where $U_\pm$ are constant states. The existence conditions for the travelling waves are the following: the wave speed S and the boundary conditions $U_\pm$ satisfy the Rankine–Hugoniot condition

$$S = S(U_-, U_+) = \frac{f(U_+) - f(U_-)}{U_+ - U_-}, \tag{2.11}$$

and the entropy condition

$$U_- > U_+. \tag{2.12}$$

It is easy to show that the travelling wave solution $U(\xi)$ satisfy the following ordinary differential equation

$$U' = \frac{1}{a(U)}\,(f(U) - f(U_\pm) - S(U - U_\pm)).$$

For the given value

$$U(0; U_-, U_+) = (f')^{-1}(S)$$

the travelling wave solution $U(\eta; U_-, U_+)$ is expressed implicitly by

$$\eta = \int_{(f')^{-1}(S)}^{U} \frac{a(u)\,du}{f(u) - f(U_\pm) - S(u - U_\pm)}. \tag{2.13}$$

Tang and Teng [17] studied the travelling wave behaviour for linear viscosity equations. Here we will prove a parallel lemma for the travelling solution (2.13). In what follows we will use F_2 to denote a constant

$$F_2 = F_2(U_-, U_+) = \max_{u \in [U_+, U_-]} |f''(u)|,$$

A_L, A_s to denote

$$A_L = A_L(U_-, U_+) = \max_{u \in [U_+, U_-]} a(u),$$

$$A_s = A_s(U_-, U_+) = \min_{u \in [U_+, U_-]} a(u)$$

and $C(\gamma, A_s, A_L, F_2)$ to denote some constants which depend only on γ, A_s, A_L and F_2.

LEMMA 2.2. *Assume* (2.11) *and* (2.12). *Let* $U(\eta; U_-, U_+)$ *be defined by* (2.13). *Then the following properties hold*:

(1) $U(\eta; U_-, U_+)$ *is a decreasing function satisfying*

$$U'(\eta; U_-, U_+) < 0, \qquad U(0; U_-, U_+) = (f')^{-1}(S), \tag{2.14}$$

(2) *U approach $U_\pm$ with exponential rate decay as $\eta \to \pm\infty$*

$$|U(\eta; U_-, U_+) - H(\eta; U_-, U_+)| \leqslant \frac{F_2}{\gamma}(U_- U_+) \exp\{-\gamma(u_- U_+)\,|\eta|/(2A_L)\}, \tag{2.15}$$

where H is the so-called Heaviside function defined by

$$H(\eta; U_-, U_+) = \begin{cases} U_-, & \eta < 0 \\ U_+, & \eta \geqslant 0. \end{cases}$$

(3) *If $U_\pm$ are time dependent function, i.e. $u_\pm = U_\pm(t)$, and $X(t)$ satisfies the Rankine–Hugoniot condition* (2.11)

$$\dot{X}(t) = S(U_-(t), U_+(t)),$$

then $U = U(x - X(t); U_-, U_+)$ *satisfies*

$$\partial_t U + \partial_x f(U) = (a(U)\, U_x)_x + (U)_{U_-}\, \dot{U}_- + (U)_{U_+}\, \dot{U}_+, \tag{2.16}$$

$$\begin{aligned} &\| U_{U_+}(\cdot\,; U_-, U_+)\, \dot{U}_+ + U_{U_-}(\cdot\,; U_-, U_+)\, \dot{U}_- - H(\cdot\,; \dot{U}_-, \dot{U}_+) \|_{L^1(\mathbf{R})} \\ &\qquad \leqslant C(\gamma, A_s, A_L, F_2)\, \frac{|\dot{U}_+(t)| + |\dot{U}_-(t)|}{U_-(t) - U_+(t)}, \end{aligned} \tag{2.17}$$

where $U_{U_\pm}(\eta; U_-, U_+) = \partial_{U_\pm} U(\eta; U_-, U_+)$ *and* $\dot{U}_\pm = (d/dt)\, U_\pm(t)$. ∎

Proof. It is easy to prove (2.14) by differentiating (2.13) with respect η with the aid of the entropy condition (2.12) and by substituting $\eta = 0$ in to (2.13).

We observe that

$$\Phi(u; U_-, U_+) \doteq \frac{f(u) - f(U_\pm) - S(u - U_\pm)}{a(u)} \tag{2.18}$$

$$= \frac{1}{a(u)} \int_0^1 f''(u^*)\, \theta\, d\theta (u - U_+)(u - U_-). \tag{2.19}$$

where u^* is some intermediate value between U_- and U_+. The assumption $f'' \geqslant \gamma > 0$, $A_s \leqslant a(u) \leqslant A_L$ and the entropy condition $U_- > U_+$ gives that for any u between U_+ and U_- the following inequalities hold

$$\frac{F_2}{2A_s} (u - U_-)(u - U_+) < \Phi(u; U_-, U_+) < \frac{\gamma}{2A_L} (u - U_-)(u - U_+). \tag{2.20}$$

We also observe that

$$\begin{aligned} S - f'(U_\pm) &= \int_0^1 f'(\theta U_+ + (1-\theta)\, U_-)\, d\theta - \int_0^1 f'(U_\pm)\, d\theta \\ &= \pm \int_0^1 f''(\tilde{u}_\pm)\, \theta\, d\theta (U_- U_+) (f')^{-1}(S) - U_\pm \\ &= (f')^{-1}(S) - (f')^{-1}(f'(U_\pm)) = \frac{1}{f''(\hat{u}_\pm)} (S - f'(U_\pm)), \end{aligned}$$

where $\tilde{u}_\pm$ and $\hat{u}_\pm$ are some intermediate values between U_- and U_+. Therefore from the above two equations we obtain

$$\frac{\gamma}{2}(U_-U_+) \leqslant |S-f'(U_\pm)| \leqslant \frac{F_2}{2}(U_-U_+)$$

$$\frac{\gamma}{2F_2}(U_-U_+) \leqslant |(f')^{-1}(S)-U_\pm| \leqslant \frac{F_2}{2\gamma}(U_-U_+). \tag{2.21}$$

The second inequality of (2.20) and the definition U of (2.13) indicate that

$$\begin{cases} U(\eta;U_-,U_+)-U_+ \leqslant ((f')^{-1}(S)-U_+)\exp\{\gamma\eta(U_+-U_-)/(2A_L)\} \\ \qquad \text{for} \quad \eta \leqslant 0. \\ U_-U(\eta;U_-,U_+) \leqslant (U_-(f')^{-1}(S))\exp\{\gamma\eta(U_--U_+)/(2A_L)\} \\ \qquad \text{for} \quad \eta \leqslant 0. \end{cases}$$

It follows from the above results and the equality (2.21) that

$$\begin{cases} |U(\eta;U_-,U_+)-U_+| \leqslant \dfrac{F_2}{2\gamma}(U_-U_+)\exp\{\gamma\eta(U_+-U_-)/(2A_L)\} \\ \qquad \text{for} \quad \eta \geqslant 0, \\ |U(\eta;U_-,U_+)-U_-| \leqslant \dfrac{F_2}{2\gamma}(U_-U_+)\exp\{\gamma\eta(U_-U_+)/(2A_L)\} \\ \qquad \text{for} \quad \eta \leqslant 0. \end{cases}$$

which is equivalent to (2.15). Direct calculation on $U(x-X(t);U_-,U_+)$ with the aid of (2.13) gives the result (2.16).

Differentiating (2.13) with respect to the parameter U_+ gives

$$0 = \frac{(U)_{U_+}}{\Phi(U;U_-,U_+)} - \frac{S_{U_+}/f''(S)}{\Phi((f')^{-1}(S);U_-,U_+)} - \int_{(f')^{-1}(S)}^{U} \frac{S_{U_+}(u-U_-)\,du}{a(u)\,\Phi(u;U_-,U_+)^2}, \tag{2.22}$$

where $S=S(U_-,U_+)$ is defined by (2.11) and

$$S_{U_+} \doteq \partial_{U_+} S(U_-,U_+) = \frac{f'(U_+)(U_+-U_-)-(f(U_+)-f(U_-))}{(U_+-U_-)^2}.$$

It follows from the above equation that

$$F_2 \geqslant S_{U_+} \geqslant \gamma,$$

we obtain from the equation (2.24) that

$$(U)_{U_+} - 1 = \frac{\Phi(U; U_-, U_+)(S_{U_+}/f''(S) - 1)}{\Phi((f')^{-1}(S); U_-, U_+)} + \Phi(U; U_-, U_+)$$
$$\times \int_{(f')^{-1}(S)}^{U} \frac{\Phi(u; U_-, U_+)_{U_+} + \Phi(u; U_-, U_+)_u}{\Phi(u; U_-, U_+)^2}\, du. \qquad (2.23)$$

Direct calculation on (2.19) gives

$$\Phi_{U_+} + \Phi_u = \frac{1}{a(u)} \left\{ -\frac{f'(U_+)(U_+ - U_-) - (f(U_+) - f(U_-))}{(U_+ - U_-)^2} (u - U_-) \right.$$
$$\left. + f'(u) - \frac{f(U_+) - f(U_-)}{U_+ - U_-} - a'(u)\, \Phi(u; U_-, U_+) \right\}$$
$$= \frac{1}{a(u)} \left\{ -\frac{f'(U_+)(U_+ - U_-) - (f(U_+) - f(U_-))}{(U_+ - U_-)^2} (u - U_+) \right.$$
$$\left. + f'(u) - f'(U_+) - a'(u)\, \Phi(u; U_-, U_+) \right\}$$
$$= \frac{1}{a(u)} \left\{ -\frac{f''(u^*)}{2} (u - U_+) + f''(u^{**})(u - U_+) \right.$$
$$\left. - a'(u) \int_0^1 f''(u^{***})\, \theta\, d\theta (u - U_-)(u - U_+) \right\}$$

where $u^*, u^{**}, u^{***} \in (U_+, U_-)$. Therefore, we obtain that

$$|\Phi_{U_+} + \Phi_u| \leqslant C(A_s, A_L, F_2)\, |u - U_+|. \qquad (2.24)$$

The inequalities of (2.21) and (2.20) show that

$$\frac{\gamma^3}{8F_2^2 A_L} (U_- U_+)^2 \leqslant |\Phi((f')^{-1}(S); U_-, U_+)| \leqslant \frac{F_2^3}{8\gamma^2 A_s} (U_- U_+)^2.$$

for $\eta > 0$, we have $U_+ \leqslant U \leqslant (f')^{-1}(S)$ and hence from (2.24), (2.20) and (2.23) we obtain

$$|(U)_{U_+} - 1| \leqslant C(A_s, A_L, \gamma, F_2) \frac{|U - U_+|}{|U_- U_+|} \left(1 + \int_{(f')^{-1}(S)}^{U} \frac{1}{|u - U_+|}\, du \right)$$
$$\leqslant C(A_s, A_L, \gamma, F_2) \frac{|U - U_+|}{|U_- U_+|} \left(1 + \ln\left(\frac{(f')^{-1}(S) - U_+}{U - U_+} \right) \right).$$

On account of (2.15) and (2.21) we obtain that

$$\int_0^\infty |U_{U_+}(\eta; U_-, U_+) - 1| \, d\eta$$

$$\leqslant C(A_s, A_L, \gamma, F_2) \int_0^\infty \exp\{-\gamma(U_- U_+)\, \eta/(4A_L)\} \, d\eta$$

$$\leqslant \frac{C(A_s, A_l, \gamma, F_2)}{U_- U_+}.$$

In a similar way we obtain that

$$(U)_{U_-} = \frac{\Phi(U; U_-, U_+)(S_{U_-}/f''(S) - 1)}{\Phi((f')^{-1}(S); U_-, U_+)} + \Phi(U; U_-, U_+) \times \int_{(f')^{-1}(S)}^{U} \frac{\Phi(u; U_-, U_+)_{U_-}}{\Phi(u; U_-, U_+)^2} \, du. \tag{2.25}$$

which gives

$$\int_0^\infty |U_{U_-}(\eta; U_-, U_+)| \, d\eta \leqslant \frac{C(A_s, A_l, \gamma, F_2)}{U_- U_+}.$$

Combining the above two inequalities yields

$$\int_0^\infty |U_{U_+}(\eta; U_-, U_+)\, \dot{U}_+ + U_{U_-}(\eta; U_-, U_+)\, \dot{U}_- - H(\eta; \dot{U}_-, \dot{U}_+)| \, d\eta$$

$$\leqslant C(A_s, A_L, \gamma, F_2)\, \frac{|\dot{U}_-| + |\dot{U}_+|}{U_- U_+}.$$

Similar estimate holds for the integral with same integrand over $(-\infty, 0)$. This completes the proof of the Lemma. ▮

As a consequence of (2.11), (2.16) and (2.17) we can easily obtain the following corollary by using rescaling of integration variables in (2.26) and (2.27).

COROLLARY 2.1.

$$\| U_\varepsilon(\cdot; U_-, U_+) - H(\cdot, U_-, U_+) \|_{L^1(\mathbf{R})} \leqslant C(A_s, A_L, \gamma, F_2)\, \varepsilon \tag{2.26}$$

$$\| U_\varepsilon(\cdot; U_-, U_+)_{U_-} \dot{U}_- + U_\varepsilon(\cdot; U_-, U_+)_{U_+} \dot{U}_+ - H(\cdot; \dot{U}_-, \dot{U}_+) \|_{L^1(\mathbf{R})} \leqslant C(A_s, A_L, \gamma, F_2)\, \frac{|\dot{U}_-| + |\dot{U}_+|}{U_- U_+}\, \varepsilon. \tag{2.27}$$

Proof of Theorem 2.1. We have proved the lemmas on L^1-stability and the behaviour of travelling wave solution for the equation of nonlinear viscosity. Using the structure of solutions to the scalar conservation laws [16, 17, 18], we can prove the main theorem with a little modification to the linear viscosity method [17].

3. NONLINEAR CONSERVATION LAWS WITH L^1 INITIAL DATA

We divide this section into three subsections.

3.1. *Existence and uniqueness.* We consider a scalar conservation law

$$u_t + f(u)_x = 0, \tag{3.1}$$

subject to initial value

$$u(x, 0) = u_0(x), \tag{3.2}$$

We suppose that the flux f is convex

$$f''(u) > 0\,. \tag{3.3}$$

If $u_0(x) \in BV(\mathbf{R}) \cap L^\infty(\mathbf{R})$, then the existence and uniqueness results of entropy solutions can be found anywhere [9, 15].

In this section we suppose that

$$|f'(u)| \leqslant A, \qquad u_0(x) \in L^1(\mathbf{R}). \tag{3.4}$$

where A is a constant. Lax [9] introduced the integrated function $\varphi(x, t)$ defined as follows

$$\varphi(x, t) = \int_{-\infty}^{x} u(y, t)\, dy;$$

then

$$\varphi_x = u.$$

Integrating (3.1) from $-\infty$ to x one obtains

$$\varphi_t + f(\varphi_x) = 0, \tag{3.5}$$

where f has been adjusted so that $f(0)=0$. Lax [9] proved that the existence theorem of the equation (3.5), so we obtain the existence of the solution to (3.1) (3.2) for L^1 initial value. We notice that (3.5) is a Hamilton–Jacobian equation, Crandall and Lions [2] proved that the existence and uniqueness of viscosity solutions. As we know [10, 11], the viscosity solution of (3.5) is equivalent to the entropy solution of (3.1) for one-dimensional problem, so we obtain the existence and uniqueness of entropy solution for L^1 initial value of (3.1).

3.2. *Structure of solution*

THEOREM 3.1. *We assume that* $f''(u)>0$, $|f'(u)|\leqslant A$. *If* $u_0\in L^1(R)$, *and if*

$$\overline{\lim_{x\to x_0+0}}\, u_0(x)<\infty,\quad \underline{\lim_{x\to x_0-0}}\, u_0(x)>-\infty,\qquad \forall x_0\in R, \tag{3.6}$$

then

$$u(\cdot,t)\in L^\infty(R),\qquad \forall t>0.$$

Proof. We prove this theorem by using the following four steps.

Step 1. We denote the norm in $L^p(R)$ by $|\cdot|_p$. Let $|u_0|_1\leqslant C$, then $|u(\cdot,t)|_1\leqslant C$ for all $t>0$. We take $M_1>0$ and fix $t_0>0$, then we define a set $E_{M_1}=\{x;\,|u(x,t_0)|>M_1\}$. Clearly, $\operatorname{meas}(E_{M_1})<C/M_1$. There exists an open set $U\supset E_{M_1}$, such that $\operatorname{meas}(U)<C/M_1$. Let $U=\bigcup_i I_i$, where I_i are open intervals. We take another constant $M_2>M_1$, and define a subset S of $\{I_i\}$, such that $I_i\in S$ if and only if $\sup_{x\in I_i}|u(x,t_0)|>M_2$.

Step 2. We claim that S is a finite set. To prove it, we use the expressions [9]

$$u(x,t)=b\left(\frac{x-y_0(x,t)}{t}\right), \tag{3.7}$$

where

$$b=a^{-1},\qquad a=f',$$

and $y_0(x,t)$ is the unique minimum point of the function

$$v(\xi;x,t)=\int_0^\xi u_0(\eta)\,d\eta+g\left(\frac{x-\xi}{t}\right)t,$$

where

$$g(y)=\int_0^y b(s)\,ds.$$

It is known that [9] [4] $y_0(x,t)$ makes sense for all t and almost all x.

Let $I_i\in S$, $x_0\in I_i$ such that $|u(x_0,t_0)|>M_2$. We consider the case of $u(x_0,t_0)>M_2$ first. If $x<x_0$ and if $y_0(x,t_0)$ makes sense, then by $y_0(x,t_0)\leqslant y_0(x_0,t_0)$ we have

$$u(x,t_0)=b\left(\frac{x-y_0(x,t_0)}{t_0}\right)\geqslant b\left(\frac{x-y_0(x_0,t_0)}{t_0}\right).$$

Since $b(x_0-y_0(x_0,t_0)/t_0)>M_2$, there exists $\tilde{x}<x_0$, such that $b(\tilde{x}-y_0(x_0,t_0)/t_0)=M_1$. Then

$$a(M_1)=\frac{\tilde{x}-y_0(x_0,t_0)}{t_0},\qquad a(M_2)<\frac{x_0-y_0(x_0,t_0)}{t_0},$$

hence

$$x_0-\tilde{x}>t_0(a(M_2)-a(M_1)).$$

If $x\in(\tilde{x},x_0)$, then $u(x,t_0)>M_1$, which implies $(\tilde{x},x_0)\subset I_i$, so $\text{meas}(I_i)\geqslant x_0-\tilde{x}>t_0(a(M_2)-a(M_1))$. If $u(x_0,t_0)<-M_2$, by the same way we can prove $(x_0,\tilde{x})\subset I_i$, and $\tilde{x}-x_0>t_0(a(-M_1)-a(-M_2))$. On the other hand we have $\sum_i \text{meas}(I_i)\leqslant C/M_1$, therefore S is a finite set.

Step 3. We prove by contradiction that $\sup_{x\in I_i}|u(x,t_0)|<+\infty$.

If $\{x_j\}$ is a sequence in I_i such that $\lim_{j\to\infty}|u(x_j,t_0)|=\infty$, by the assumption (3.6) $\{x_j\}$ can be monotonic increasing or decreasing, depending on the limit is $+\infty$ or $-\infty$. For definiteness we suppose $\lim_{j\to\infty}u(x_j,t_0)=+\infty$ and $\{x_j\}$ is monotonic increasing. Let $y=\lim_{j\to\infty}y_0(x_j,t_0)$ and $u_r=\overline{\lim}_{x\to y+0}u_0(x)$. We take an arbitrary $\delta>0$, then $u_0(x)<u_r+\delta$ on one interval $(y,y_1]$.

Without loss of generality we assume that $y=0$. Let $a_1=a(+\infty)$. We take $t_1\in(0,t_0]$ and let $x_0=a_1t_1$. If $\min_\xi v(\xi;x_0,t_1)=v(y_2;x_0,t_1)$, then $y_2\geqslant 0$. Moreover we have $y_2>0$, which is because the derivative of $g((x_0-\xi)/t_1)$ at $\xi=0$ is infinity and u_0 is bounded, $\xi=0$ is not a minimum point. We require that $y_2\leqslant y_1$, otherwise we can reduce t_1 to achieve it. Since $(x_0-y_2)/t_1<a(u_r+\delta)$, we have $y_2>x_0-a(u_r+\delta)\,t_1$. Let

$$\varepsilon=g(a_1)-\frac{1}{t_1}\int_0^{y_2}u_0(\eta)\,d\eta+g\left(\frac{x_0-y_2}{t_1}\right)>0.$$

We take $\xi_0 < 0$ such that

$$\int_0^{\xi} u_0(\eta)\, d\eta > -\frac{\varepsilon}{3}\, t_1 \qquad \text{for all} \quad \xi \in [\xi_0, 0).$$

We take $a' < a_1$ such that

$$g\left(\frac{x}{t_1}\right) > g(a_1) - \frac{\varepsilon}{3} \qquad \text{for all} \quad x \in (a' t_1, a_1 t_1).$$

Besides we take $a'' < a_1$, such that

$$g\left(\frac{x - y_2}{t_1}\right) - g\left(\frac{x_0 - y_2}{t_1}\right) < \frac{\varepsilon}{3} \qquad \text{for all} \quad x \in (a'' t_1, a_1 t_1).$$

If $\max(x_0 + \xi_0, a' t_1, a'' t_1) < x < x_0$, $\xi \in [x - x_0, 0]$, then

$$\begin{aligned} v(\xi; x, t_1) &= \int_0^{\xi} u_0(\eta)\, d\eta + g\left(\frac{x - \xi}{t_1}\right) t_1 \\ &> -\frac{\varepsilon}{3}\, t_1 + g(a_1)\, t_1 - \frac{\varepsilon}{3}\, t_1 \\ &= \frac{\varepsilon}{3}\, t_1 + \int_0^{y_2} u_0(\eta)\, d\eta + g\left(\frac{x_0 - y_2}{t_1}\right) t_1 \\ &> \int_0^{y_2} u_0(\eta)\, d\eta + g\left(\frac{x - y_2}{t_1}\right) t_1 \\ &= v(y_2; x, t_1). \end{aligned}$$

Therefore ξ is not a minimum point. Consequently $u(x, t_1) \leqslant u_r + \delta$. We have

$$u(x_j, t_0) = u(x_j - a(u(x_j, t_0))(t_0 - t_1), t_1),$$

and

$$\lim_{j \to \infty} (x_j - a(u(x_j, t_0))(t_0 - t_1)) = x_0,$$

so $u(x_j, t_0) \leqslant u_r + \delta$ for large j, which leads to a contradiction.

Following the same lines the case of $\lim_{j \to \infty} u(x_j, t_0) = -\infty$ can also be studied.

Step 4. We have either $\sup_{x \in R} u(x, t_0) \leqslant M_2$, or $|u|_\infty = \sup_{I_j \in S} \sup_{x \in I_j} |u(x, t_0)| < +\infty$. The proof is thus complete. ∎

COROLLARY 3.1. *Under the assumptions of Theorem* 3.1 *total variation of* $u(\cdot, t)$ *is locally bounded for all* $t > 0$.

Proof. Let $M = |u(\cdot, t)|_\infty$, $B = \max_{|u| \leqslant M}(1/|a'(u)|)$. It follows from (3.7) that

$$\operatorname{var}(u(\cdot, t); [-X, X]) \leqslant B \operatorname{var}\left(\frac{\cdot - y_0(\cdot, t)}{t}, [-X, X]\right) < +\infty. \quad \blacksquare$$

Remark. Counterexample can be constructed to show that the condition (3.6) is essential.

The above result explains why after a front breaks down the cusp disappears promptly, and a corner is formed. Let us consider the equation (1.7) and let the initial data be the value of the cusp. If x_0 is the singular point, then $\lim_{x \to x_0 \pm 0} u_0(x) = \mp \infty$. By Theorem 3.6 $u(\cdot, t) \in L^\infty(\mathbf{R})$ for $t > 0$, which means it becomes a corner because the slope of the curve is finite.

3.3. L^1-*convergence of nonlinear viscosity method*

The nonlinear viscosity method approximating the equation (3.1) is to solve the nonlinear parabolic equation

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon(a(u^\varepsilon)\, u_x^\varepsilon)_x, \tag{3.8}$$

subject to the initial condition

$$u(x, 0) = u_0(x), \tag{3.9}$$

where $u_0(x) \in L^1$.

As we know, $BV(\mathbf{R})$ is compact in $L^1(\mathbf{R})$, i.e. $\forall \delta > 0$, it always exists $u_0^\delta(x)$, such that

$$\|u_0^\delta(\cdot) - u_0(\cdot)\|_{L^1} \leqslant C\delta. \tag{3.10}$$

We denote the solution of conservation law (3.1) with initial value $u_0^\delta(x)$ by $u^\delta(x, t)$. By virtue of L^1-stability [9], we have

$$\|u^\delta(\cdot, t) - u(\cdot, t)\|_{L^1} \leqslant \|u_0^\delta(\cdot) - u_0(\cdot)\|_{L^1} \leqslant C\delta. \tag{3.11}$$

We denote the solution of nonlinear parabolic equation (3.8) with initial value $u_0^\delta(x)$ by $u^{\varepsilon\delta}(x, t)$. By virtue of L^1-stability [9], we obtain

$$\|u^{\varepsilon\delta}(\cdot, t) - u^\varepsilon(\cdot, t)\|_{L^1} \leqslant \|u_0^\delta(\cdot) - u_0(\cdot)\|_{L^1} \leqslant C\delta, \tag{3.12}$$

where constant C is independent of ε and δ.

Using the triangle inequality, by the Theorem 2.1, (3.11) and (3.12), we have

$$\|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^1} \leqslant 2\delta + C_\delta \varepsilon, \qquad \forall \delta > 0, \tag{3.13}$$

where C_δ depends on BV norm of u_0^δ.

We summarize what we have shown by stating the following:

THEOREM 3.3. *Assume* (A1)–(A3). *Let* u^ε *and* u *be the solutions of the Cauchy problem of* (3.8) *and* (3.1) *with the same* L^1 *initial data* u_0. *Then the following result holds*

$$\sup_{0 \leqslant t \leqslant T} \|u^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^1} \to 0 \qquad \varepsilon \to 0. \tag{3.14}$$

Remark. Theorem 3.2 suggests that the solutions of nonlinear viscosity methods converge to the solution of conservation laws with L^1 initial data. However, we can not obtain any convergence order since it is not clear how C_δ depends on δ.

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