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006

# OPTIMAL $L^1$-RATE OF CONVERGENCE FOR THE VISCOSITY METHOD AND MONOTONE SCHEME TO PIECEWISE CONSTANT SOLUTIONS WITH SHOCKS*

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**Abstract.** We derive optimal error bounds for the viscosity method and monotone difference schemes to an initial-value problem of scalar conservation laws with initial data being a finite number of piecewise constants, subject to the initial discontinuities satisfying the entropy conditions. It is known that the entropy solution of the problem is piecewise constant with a finite number of interacting shocks satisfying the entropy conditions. A rigorous analysis shows that both the viscosity method and monotone schemes to approach the initial-value problem have uniform $L^1$-error bounds of $O(\epsilon)$ and $O(\Delta x)$ for the time $+\infty > t > 0$, respectively, where $\epsilon$ and $\Delta x$ are their corresponding viscosity coefficient and discrete mesh length. The results are improvements over the half-order rates of $L^1$-convergence. Numerical experiments for the Lax–Friedrichs scheme are presented and numerical results justify the theoretical analysis.

**Key words.** monotone-difference scheme, viscosity method, conservation laws, error estimate, convergence rate

**AMS subject classifications.** Primary, 65M25; Secondary, 76D05

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**1. Introduction.** Viscosity methods and monotone-difference schemes play an important role in both theoretical analysis and practical computation for hyperbolic conservation laws. The accuracy and error bound of the two classes of approximation methods are of much concern from the viewpoint of numerical computation. Harten, Hyman, and Lax [5] pointed out that the monotone-difference schemes are of at most first-order accuracy and Kuznetsov [9, 10] showed that their $L^1$-error bound for boundary value (BV) initial data is $O(\sqrt{\Delta x})$ as $\Delta x$ goes to zero, where $\Delta x$ is the size of space. Tang and Teng [16] recently proved that all monotone schemes applied to linear first-order equations with discontinuous initial data is of at most $\sqrt{\Delta x}$ rate of convergence in $L^1$-norm. This means that the $\sqrt{\Delta x}$ rate of convergence in $L^1$-norm is indeed the best possible for the monotone schemes applied to scalar conservation laws if it includes the linear case. But it is widely believed that some approximate methods (such as monotone difference schemes) to approach discontinuous solutions for conservation laws with nonlinear fluxes perform better than those with linear fluxes (see Harten [4]). This property has important implications in numerical calculations [4]. In this paper we justify this observation and derive optimal $L^1$-error estimates for both the viscosity method and monotone difference scheme to piecewise constant solutions with a finite number of shock waves, i.e., solutions of the following initial-value problems:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{1.1a}$$

$$u\big|_{t=0} = u_0(x), \qquad x \in \mathbf{R} \tag{1.1b}$$

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with

$$u_0(x) = \begin{cases} u^{(0)}, & x < x_{1/2}^{(0)}, \\ u^{(k)}, & x_{k-1/2}^{(0)} \le x < x_{k+1/2}^{(0)}, \quad k = 1, \dots, K-1, \\ u^{(K)}, & x \ge x_{K-1/2}^{(0)}, \end{cases} \tag{1.2}$$

where $u^{(k)}$ $(k = 0, 1, \dots, K)$ and $x_{1/2}^{(0)} < \cdots < x_{K-1/2}^{(0)}$ are given constants, $K$ is a positive integer, and the flux $f(u)$ with the initial data (1.2) satisfies Oleinik's condition E (see [15])

$$\frac{f(u) - f(u^{(k)})}{u - u^{(k)}} > D_{k,l} > \frac{f(u) - f(u^{(l)})}{u - u^{(l)}}, \quad k < l \ (k, l = 0, 1, \dots, K) \tag{1.3a}$$

for all $u$ strictly between $u^{(k)}$ and $u^{(l)}$, and satisfies the Lax geometric condition (see [11])

$$f'(u^{(k)}) > D_{k,l} > f'(u^{(l)}), \quad k < l \ (k, l = 0, 1, \dots, K), \tag{1.3b}$$

where

$$D_{k,l} = \frac{f(u^{(k)}) - f(u^{(l)})}{u^{(k)} - u^{(l)}} \tag{1.3c}$$

is shock speed.

*Remark.* If the flux $f(u)$ is strictly convex $f''(u) > 0$, then (1.3) is equivalent to $u^{(0)} > u^{(1)} > \cdots > u^{(K)}$.

It is known [11] that under the conditions (1.3) the entropy solution $u(x,t)$ of (1.1) and (1.2) is piecewise constant with a finite number of interacting shocks. More precisely, the solution is of the following form: for $0 < t \le t^{(1)}$

$$u(x,t) = \begin{cases} u^{(0)}, & x < X_{1/2}(t), \\ u^{(k)}, & X_{k-1/2}(t) \le x < X_{k+1/2}(t), \quad k = 1, \dots, K-1, \\ u^{(K)}, & x \ge X_{K-1/2}(t), \end{cases} \tag{1.4}$$

where

$$X_{k-1/2}(t) = x_{k-1/2}^{(0)} + D_{k-1,k}\, t \tag{1.5}$$

is a shock curve and $t^{(1)}$ is the earliest intersection time of any two neighboring shock curves, i.e.,

$$t^{(1)} = \min_{0 \le k < K-1} t_k, \tag{1.6a}$$

where

$$t_k = \frac{x_{k+1/2}^{(0)} - x_{k-1/2}^{(0)}}{D_{k-1,k} - D_{k,k+1}} \tag{1.6b}$$

is the intersection time of two neighboring shock curves.

At the time $t^{(1)}$, $u(x, t^{(1)})$ is of another piecewise constant with $K-1$ or less discontinuities. The solution $u(x,t)$ can be constructed beyond $t^{(1)}$ as an entropy solution of (1.1) with the initial data $u(x, t^{(1)})$. To repeat the process a finite number of times, $u(x,t)$ can be continued to $t = \infty$.

In this paper we will prove the following main results.

THEOREM 1.1. *Let the entropy conditions* (1.3a) *and* (1.3b) *be satisfied. If* $v_\epsilon$ *and* $w_{\Delta x}$ *are solutions of the viscosity method and monotone scheme, defined by* (1.9) *and* (1.10) *below, respectively, to the initial-value problem* (1.1) *and* (1.2), *then the following uniform error bounds are fulfilled for any* $0 < t < +\infty$:

$$\|v_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le C_K\,\epsilon \tag{1.7}$$

*and*

$$\|w_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \le C_K \Delta x, \tag{1.8}$$

*where* $u$ *is the entropy solution of* (1.1) *and* (1.2), $\epsilon$ *and* $\Delta x$ *are the viscosity coefficient and the discrete mesh length of the two approximate methods, respectively, and* $C_K$ *are constants independent of* $t$, $\epsilon$, *and* $\Delta x$, *but depend on* $K$, *the number of initial discontinuities.*

*Remark.* The error estimates of (1.7) and (1.8) hold not only before but also after interaction of shocks.

Several authors provide convergence rates for various approximate methods to discontinuous solutions of conservation laws (see, e.g., [6, 9, 10, 14, 16, 17, 19, 20]). Although their methods are different, the convergence rates in the $L^1$-norm obtained are half-order in most of the cases. Therefore, Theorem 1.1, having one-order convergence rate, may provide a clue on how to break the half-order barrier to the error estimate.

The proof of Theorem 1.1 consists of two key steps. First, we prove the theorem for a single shock solution, i.e., $K = 1$, by means of Jennings' travelling discrete shock wave [7] and travelling viscosity shock wave. For a finite number of interacting shock waves, we use the matching method to assemble the corresponding travelling viscosity shock waves or travelling discrete shock waves together in each time interval between any two successive shock interactions. We prove that the assembling travelling-waves solution is a good approximation to both the multishock solution and its corresponding viscosity or finite-difference solution within an $O(\epsilon)$ or $O(\Delta x)$ $L^1$-error, respectively.

The analysis used in this paper can be extended to more general cases, such as piecewise-smooth solutions, and a similar first-order $L^1$-convergence rate can be obtained. The result will be reported elsewhere [18].

We would like to mention that the study of this paper is also motivated by the work of Goodman and Xin [3] in which they introduced the matching method to assemble the travelling waves in the theoretical proof of energy estimates for a system of conservation laws and showed that the viscosity method approaching piecewise-smooth solutions with a finite number of noninteracting shocks to a system of conservation laws has a local $\epsilon$ rate of convergence away from shocks. Liu and Xin [13] also used this method to prove the stability of multiple shocks for Lax–Friedrichs schemes to systems of conservation laws. As mentioned by Liu and Xin [12, 13], one would expect a first-order $L^1$-error estimate for the Lax–Friedrichs scheme approximating the Riemann single-shock solution to systems of conservation from their stability estimates. But they also said that this needs some initial-layer estimates for the case of systems.

We now give a more detailed description of the viscosity method and the monotone difference scheme. The viscosity method approaches the scalar conservation laws (1.1) by solving the parabolic equation

$$\frac{\partial v_\epsilon}{\partial t} + \frac{\partial f(v_\epsilon)}{\partial x} = \epsilon \frac{\partial^2 v_\epsilon}{\partial x^2}, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \ \epsilon > 0, \tag{1.9a}$$

$$v_\epsilon\big|_{t=0} = u_0(x), \qquad x \in \mathbf{R} \tag{1.9b}$$

and the entropy solution $u$ of (1.1) can be constructed as the limit of solutions $v_\epsilon$ as $\epsilon \to +0$.

The monotone scheme is a finite-difference approach to (1.1). A $(p+q+1)$-point conservative finite-difference scheme

$$\begin{aligned} w_j^{n+1} &= H(w_{j-p}^n, w_{j-p+1}^n, \dots, w_{j+q}^n) \\ &= w_j^n - \lambda[\overline{f}(w_{j-p+1}^n, \dots, w_{j+q}^n) - \overline{f}(w_{j-p}^n, \dots, w_{j+q-1}^n)] \end{aligned} \tag{1.10a}$$

is said to be monotone if $H$ is a monotone nondecreasing function of each of its arguments, and is said to be consistent with the scalar conservation laws (1.1) if the numerical flux $\overline{f}$ satisfies

$$\overline{f}(w, \dots, w) = f(w), \tag{1.10b}$$

where $\lambda = \Delta t / \Delta x =$ constant, $p$ and $q$ are given nonnegative integers, and

$$w_j^0 = T_{\Delta x}(u_0)(x_j) = \frac{1}{\Delta x} \int_{x_j - \Delta x/2}^{x_j + \Delta x/2} u_0(x)\, dx, \quad x_j = j\Delta x. \tag{1.10c}$$

Extend the lattice function $w_j^n$ to continuous values of $x$ and $t$ by setting

$$w_{\Delta x}(x,t) = w_j^n \quad \text{for} \quad (x,t) \in [x_{j-1/2}, x_{j+1/2}) \times [t_n, t_{n+1}), \tag{1.10d}$$

where $x_{j+1/2} = (j+1/2)\Delta x$, $t_n = n\Delta t$, $n \in \mathbf{N}$, and $j \in \mathbf{Z}$.

Stability, convergence, and an error estimate for the monotone-difference schemes can be found in [1] and [10].

The rest of this paper is organized as follows: in section 2 we will introduce stability lemmas and travelling-wave lemmas for the viscosity method and monotone schemes. In section 3 we prove Theorem 1.1 for the viscosity method, while in section 4 we prove the theorem for monotone schemes. In the last section we present some numerical experiments to verify the theoretical analysis for both noninteracting shock waves and interacting waves. Some lemmas are given in the Appendix.

**2. Some basic lemmas.** First we establish stability lemmas and travelling-wave lemmas for both the viscosity method and the monotone difference scheme. These two kinds of lemmas play central roles in proving Theorem 1.1.

LEMMA 2.1. *If $u_0(x)$ belongs to $L^\infty(\mathbf{R})$, then* (1.9) *has a unique solution $v_\epsilon(x,t)$ that satisfies:*

1. *$v_\epsilon(x,t)$ has continuous derivatives involved in* (1.9) *for $t > 0$,*
2. *$v_\epsilon(x,t)$ is uniformly bounded by*

$$\|v_\epsilon(\cdot,t)\|_{L^\infty(\mathbf{R})} \le \|u_0(\cdot)\|_{L^\infty(\mathbf{R})} \quad \forall t > 0, \tag{2.1}$$

3. *$v_\epsilon(x,t)$ assumes the initial value $u_0(x)$ in the following sense:*

$$\lim_{t \to +0} \|v_\epsilon(\cdot,t) - u_0(\cdot)\|_{L^1([-R,R])} = 0 \quad \forall R > 0. \tag{2.2}$$

The proof of this lemma can be found in [8]. Next we introduce stability results for the viscosity solutions of (1.9) with a nonhomogeneous term.

LEMMA 2.2 (stability). *If $v_\epsilon^{(i)}(x,t)$ for $i = 1,2$ are solutions of* (1.9) *with nonhomogeneous terms $g_i(x,t)$ and initial data $u_0^{(i)}(x) \in L^\infty(\mathbf{R})$, namely,*

$$\frac{\partial v_\epsilon^{(i)}}{\partial t} + \frac{\partial f(v_\epsilon^{(i)})}{\partial x} - \epsilon \frac{\partial^2 v_\epsilon^{(i)}}{\partial x^2} = g_i(x,t), \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{2.3a}$$

$$v_\epsilon^{(i)}\big|_{t=0} = u_0^{(i)}(x), \qquad x \in \mathbf{R}, \tag{2.3b}$$

*then*

$$\|v_\epsilon^{(1)}(\cdot,t) - v_\epsilon^{(2)}(\cdot,t)\|_{L^1(\mathbf{R})} \le \|u_0^{(1)}(\cdot) - u_0^{(2)}(\cdot)\|_{L^1(\mathbf{R})} + \int_0^t \|g_1(\cdot,\tau) - g_2(\cdot,\tau)\|_{L^1(\mathbf{R})} d\tau. \tag{2.4}$$

The lemma can be proved by a technique used by Lax in [11], with some modifications.

A travelling-wave solution is a solution of (1.9a) in the form $v_\epsilon(x,t) = V_\epsilon(x - St)$, where the wave speed $S$ is a constant. We have the following existence lemma for the travelling-wave solution.

LEMMA 2.3 (travelling wave). *Let the assumption* (1.3) *be satisfied. Then for each pair $k < l$ $(k,l = 0,1,\dots,K)$ and each $u_m \in (u^{(k)} \wedge u^{(l)}, u^{(k)} \vee u^{(l)})$ there is a unique travelling-wave solution $V_\epsilon^{(k,l)}(x - D_{k,l}t)$ of* (1.9a) *taking on the values $V_\epsilon^{(k,l)}(0) = u_m$, $V_\epsilon^{(k,l)}(-\infty) = u^{(k)}$, and $V_\epsilon^{(l)}(\infty) = u^{(l)}$, which has the following properties:*

1. *$V_\epsilon^{(k,l)}(x - D_{k,l}t) = V_{k,l}((x - D_{k,l}t)/\epsilon)$, where $V_{k,l}(\xi)$ is defined implicitly by*

$$\xi_{k,l}(V) = \int_{u_m}^{V} \Phi_{k,l}(u)^{-1}\, du, \tag{2.5a}$$

*where*

$$\Phi_{k,l}(u) = f(u) - f(u^{(l)}) - D_{k,l}(u - u^{(l)}); \tag{2.5b}$$

2. *For all $\xi \in \mathbf{R}$*

$$u^{(k)} \wedge u^{(l)} < V_{k,l}(\xi) < u^{(k)} \vee u^{(l)}, \tag{2.6}$$

*where $c \wedge d = \min\{c,d\}$ and $c \vee d = \max\{c,d\}$;*

3. *$V_{k,l}(\xi)$ ($V_\epsilon^{(k,l)}(\xi)$ as well) is a monotone function for $\xi \in \mathbf{R}$ and approaches $u^{(k)}$ and $u^{(l)}$ exponentially in $\xi$ as $\xi \to \mp\infty$; namely, there are positive constants $\gamma_{k,l}$ and $\Gamma_{k,l}$ such that*

$$|V_{k,l}(\xi) - u^{(k)}| \le \Gamma_{k,l} e^{-\gamma_{k,l}|\xi|} \quad \textit{for } \xi \le 0, \tag{2.7a}$$

$$|V_{k,l}(\xi) - u^{(l)}| \le \Gamma_{k,l} e^{-\gamma_{k,l}|\xi|} \quad \textit{for } \xi \ge 0. \tag{2.7b}$$

4. *The following estimates hold:*

$$A_{k,l} \equiv \|H_{k,l}(\cdot) - V_{k,l}(\cdot)\|_{L^1(\mathbf{R})} < \infty \tag{2.8}$$

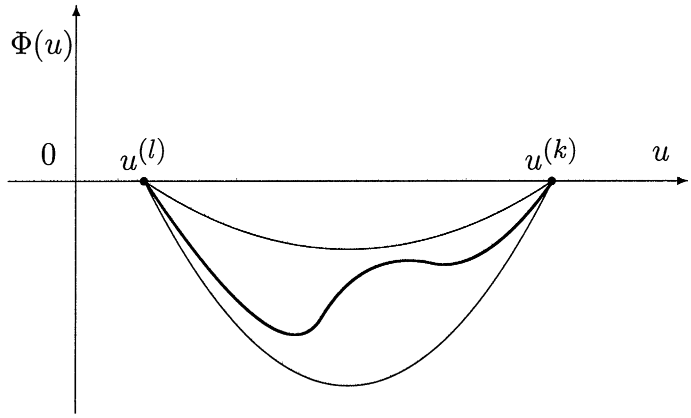


FIG. 1. *Entropy conditions* (1.3a) *and* (1.3b). *Dark line is* $\Phi_{k,l}(u)$, *light lines are* $\alpha_{k,l}(u-u^{(k)})(u-u^{(l)})$ *and* $\beta_{k,l}(u-u^{(k)})(u-u^{(l)})$.

*and*

$$\|H_{k,l}(\cdot) - V_{k,l}(\cdot/\epsilon)\|_{L^1(\mathbf{R})} = A_{k,l}\epsilon, \tag{2.9}$$

*where*

$$H_{k,l}(x) = \begin{cases} u^{(k)}, & x \le 0, \\ u^{(l)}, & x > 0. \end{cases} \tag{2.10}$$

*Proof.* To be specific, we assume $u^{(k)} > u^{(l)}$. The expression (2.5) can be easily derived by substituting $V_\epsilon^{(k,l)}(x - D_{k,l}t) = V_{k,l}((x - D_{k,l}t)/\epsilon)$ with $V_\epsilon^{(k,l)}(0) = u_m$ into (1.9a) and, in view of Oleinik's condition E (1.3a), or equivalently $\Phi_{k,l}(u) < 0$ for $u \in (u^{(l)}, u^{(k)})$ and $\Phi_{k,l}(u^{(k)}) = \Phi_{k,l}(u^{(l)}) = 0$, the inverse function of (2.5a) also satisfies $V_{k,l}(-\infty) = u^{(k)}$ and $V_{k,l}(\infty) = u^{(l)}$.

Property (2) follows immediately from (2.5a) and $V'_{k,l}(\xi) = \Phi_{k,l}(V) < 0$.

In order to prove (3), we show that there exist $0 < \alpha_{k,l} < \beta_{k,l}$ such that

$$\beta_{k,l}(u-u^{(k)})(u-u^{(l)}) < \Phi_{k,l}(u) < \alpha_{k,l}(u-u^{(k)})(u-u^{(l)}) \quad \forall u \in (u^{(l)}, u^{(k)}). \tag{2.11}$$

This is due to the entropy conditions (1.3a) and (1.3b), or equivalently, $\Phi_{k,l}(u) < 0$ for $u \in (u^{(l)}, u^{(k)})$ and $\Phi'_{k,l}(u^{(l)}) < 0$ and $\Phi'_{k,l}(u^{(k)}) > 0$ (see Figure 1). Thus (2.11) and (2.5) imply that

$$\begin{aligned} \alpha_{k,l}{}^{-1}\bar{\xi}(V) < \xi_{k,l}(V) < \beta_{k,l}{}^{-1}\bar{\xi}(V), \qquad V > u_m, \\ \beta_{k,l}{}^{-1}\bar{\xi}(V) < \xi_{k,l}(V) < \alpha_{k,l}{}^{-1}\bar{\xi}(V), \qquad V < u_m, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} \bar{\xi}(V) &= \int_{u_m}^{V} \frac{1}{(u-u^{(k)})(u-u^{(l)})}\,du \\ &= \frac{1}{u^{(k)}-u^{(l)}} \ln\left[\left(\frac{V-u^{(k)}}{V-u^{(l)}}\right)\left(\frac{u_m-u^{(l)}}{u_m-u^{(k)}}\right)\right]. \end{aligned}$$

Since all three functions in (2.12) are decreasing functions for $V \in (u^{(l)}, u^{(k)})$, their inverse functions also satisfy a similar inequality in the same order:

$$\begin{aligned} &\overline{V}(\alpha_{k,l}\xi) < V_{k,l}(\xi) < \overline{V}(\beta_{k,l}\xi), \qquad \xi > 0,\\ &\overline{V}(\beta_{k,l}\xi) < V_{k,l}(\xi) < \overline{V}(\alpha_{k,l}\xi), \qquad \xi < 0, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} \overline{V}(\xi) = &\left(u^{(k)} + u^{(l)}\left(\frac{u^{(k)} - u^{(l)}}{u_m - u^{(l)}}\right)\exp\{\xi(u^{(k)} - u^{(l)})\}\right)\\ &\times\left(1 + \left(\frac{u^{(k)} - u^{(l)}}{u_m - u^{(l)}}\right)\exp\{\xi(u^{(k)} - u^{(l)})\}\right)^{-1}. \end{aligned} \tag{2.14}$$

It is easy to see from (2.14) that $\overline{V}(\xi)$ approaches $u^{(k)}$ and $u^{(l)}$ exponentially in $\xi$ as $\xi \to \mp\infty$, and so does $V_{k,l}(\xi)$. Hence (3) is proved.

Equation (2.8) is an immediate conclusion from property (3). Substituting $V_\epsilon^{(k,l)}(x) = V_{k,l}(x/\epsilon)$ and $H_{k,l}(x) = H_{k,l}(x/\epsilon)$ (because of the special form of (2.10)) into the left-hand side of (2.9) and using integration by substitution, we obtain the equality (2.9). We have now completed the proof of Lemma 2.3. □

*Remark* (i). From the proof we see that the Lax geometric condition (1.3b) is necessary for proving property (3). This condition is also needed for proving a similar property in Lemma 2.5 below.

*Remark* (ii). If the flux is strictly $f''(u) > 0$, then $A_{k,l}$ in (2.9) has an explicit bound

$$A_{k,l} \le 4\left\{\min_{|u|\le\|u_0\|_{L^\infty}} f''(u)\right\}^{-1}.$$

Similarly, we can also establish the stability lemma and the travelling-wave lemma for the monotone scheme.

LEMMA 2.4 (stability). *If $w_{\Delta x}^{(i)}(x,t)$ for $i = 1, 2$ are solutions of* (1.10) *with nonhomogeneous terms $h_i(x,t)$ and initial data $u_0^{(i)}(x) \in L^\infty(\mathbf{R})$, namely, for any $t_n = n\Delta t$ $(n = 1, 2, \dots)$*

$$w_{\Delta x}^{(i)}(x, t_{n+1}) - H(w_{\Delta x}^{(i)}(x - p\Delta x, t_n), \dots, w_{\Delta x}^{(i)}(x + q\Delta x, t_n)) = h_i(x, t_n)\Delta t, \tag{2.15a}$$

$$w_{\Delta x}^{(i)}(x, 0) = u_0^{(i)}(x), \tag{2.15b}$$

*then*

$$\begin{aligned} \|w_{\Delta x}^{(1)}(\cdot, t_n) - w_{\Delta x}^{(2)}(\cdot, t_n)\|_{L^1(\mathbf{R})} \le &\|u_0^{(1)}(\cdot) - u_0^{(2)}(\cdot)\|_{L^1(\mathbf{R})}\\ &+ \sum_{m=0}^{n-1} \|h_1(\cdot, t_m) - h_2(\cdot, t_m)\|_{L^1(\mathbf{R})}\Delta t. \end{aligned} \tag{2.16}$$

The proof can be found in [1].

In what follows we introduce a travelling-wave solution to the monotone scheme (1.10a) by following Jennings' work [7]. A travelling-wave solution of (1.10a) moving with speed $D_{k,l}$ satisfies the difference equation

$$W_{z-D_{k,l}\lambda}^{(k,l)} = H(W_{z-p}^{(k,l)}, \dots, W_{z+q}^{(k,l)}). \tag{2.17}$$

The minimal domain on which (2.17) makes sense consists of functions defined on the linear span over the integers of $\eta = D_{k,l}\lambda$ and 1. Call the closure of the set $\mathcal{L}_\eta$. If $\eta$ is rational, $\mathcal{L}_\eta$ is discrete; if $\eta$ is irrational, then $\mathcal{L}_\eta$ is the entire real line. $L^1(\mathcal{L}_\eta)$ is the space of absolutely integrable functions on $\mathcal{L}_\eta$ with the usual sense.

LEMMA 2.5 (travelling wave). *Let the assumption* (1.3) *be satisfied and* $|\eta| < 1$. *Then for each pair* $k < l$ $(k, l = 0, 1, \ldots, K)$ *and each* $u_m \in (u^{(k)} \wedge u^{(l)}, u^{(k)} \vee u^{(l)})$ *there is a travelling-wave solution* $W^{(k,l)}_{z-\eta}$ *of* (2.17) *continuous on* $\mathcal{L}_\eta$ *taking on the values* $W^{(k,l)}_0 = u_m$, $W^{(k,l)}_{-\infty} = u^{(k)}$, *and* $W^{(k,l)}_{\infty} = u^{(l)}$, *which is a monotone function of* $z$ *and approaches* $u^{(k)}$ *and* $u^{(l)}$ *exponentially in* $z$ *as* $z \to \mp\infty$.

The lemma for the strictly monotone scheme is due to Jennings [7] and it is generalized by Engquist and Osher [2] to any weakly monotone scheme of the form (1.10a) having the property that for any $\epsilon > 0$ sufficiently small there exists a strictly monotone scheme $H^\epsilon(w_{j-p}, \ldots, w_{j+q})$ such that $H^\epsilon(w_{j-p}, \ldots, w_{j+q}) \to H(w_{j-p}, \ldots, w_{j+q})$ as $\epsilon \to 0$.

For rational $\eta$ the set $\mathcal{L}_\eta$ is discrete. So it is of the form $\{z_j\}_{-\infty}^{\infty}$ with $z_j < z_{j+1}$ $\forall j \in \mathbf{Z}$. We extend the discrete travelling-wave solution $W^{(k,l)}_{z_j}$ $(j \in \mathbf{Z})$ to continuous values of $\xi$ $(\xi \in \mathbf{R})$ by setting

$$W_{k,l}(\xi) = W^{(k,l)}_{z_j}, \quad \xi \in [z_j, z_{j+1}), \tag{2.18a}$$

where $j \in \mathbf{Z}$.

For irrational $\eta$ we identify $W_{k,l}(\xi)$ with $W^{(k,l)}_\xi$, i.e.,

$$W_{k,l}(\xi) = W^{(k,l)}_\xi, \quad \xi \in \mathbf{R}. \tag{2.18b}$$

For $W_{k,l}(\xi)$ we have the following lemma.

LEMMA 2.6. *Under the conditions of Lemma* 2.5, $W_{k,l}(\xi)$ *has the following properties:*

1. $u^{(k)} \wedge u^{(l)} < W_{k,l}(\xi) < u^{(k)} \vee u^{(l)}$ *for all* $\xi \in \mathbf{R}$.
2. $W_{k,l}(\xi)$ *is a monotone function for* $\xi \in \mathbf{R}$ *and approaches* $u^{(k)}$ *and* $u^{(l)}$ *exponentially in* $\xi$ *as* $\xi \to \mp\infty$; *namely, there are positive constants* $\delta_{k,l}$ *and* $\Delta_{k,l}$ *such that*

$$|W_{k,l}(\xi) - u^{(k)}| \le \Delta_{k,l} e^{-\delta_{k,l}|\xi|} \quad \textit{for } \xi \le q + q, \tag{2.19a}$$

$$|W_{k,l}(\xi) - u^{(l)}| \le \Delta_{k,l} e^{-\delta_{k,l}|\xi|} \quad \textit{for } \xi \ge -(p + q), \tag{2.19b}$$

*where* $p$ *and* $q$ *are parameters given by* (2.10a).

3. *The following estimates hold:*

$$B_{k,l} \equiv \|H_{k,l}(\cdot) - W_{k,l}(\cdot)\|_{L^1(\mathbf{R})} < \infty \tag{2.20}$$

*and*

$$\|H_{k,l}(\cdot) - W^{(k,l)}_{\Delta x}(\cdot)\|_{L^1(\mathbf{R})} = B_{k,l}\Delta x, \tag{2.21}$$

*where* $H_{k,l}(\xi)$ *is defined by* (2.10) *and*

$$W^{(k,l)}_{\Delta x}(x) = W_{k,l}(x/\Delta x). \tag{2.22}$$

*Proof.* Lemma 2.6 is a direct consequence of Lemma 2.5. □

## 3. Proof of Theorem 1.1 for the viscosity method.

**3.1. Proof of (1.7) for $K = 1$.** Now we first prove the simplest case of Theorem 1.1, i.e., $K = 1$. In this case (1.1)–(1.2) is reduced to a Riemann problem subject to the initial data

$$u_0(x) = H_{0,1}(x - x^{(0)}_{1/2}) = \begin{cases} u^{(0)}, & x < x^{(0)}_{1/2}, \\ u^{(1)}, & x \geq x^{(0)}_{1/2}, \end{cases} \tag{3.1}$$

where $u^{(0)}$ and $u^{(1)}$ satisfy (1.3). From (3.1) and (1.3) we see that the Riemann solution of (1.1) and (3.1) can be expressed as

$$u(x,t) = H_{0,1}(x - X_{1/2}(t)), \tag{3.2}$$

where $H_{0,1}(x)$ is defined by (2.10) and

$$X_{1/2}(t) = x^{(0)}_{1/2} + D_{0,1}t \tag{3.3}$$

is a shock curve. Substituting (3.2) into the right-hand side of (1.7) and using the triangular inequality we obtain

$$\begin{aligned} \|v_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} &= \|v_\epsilon(\cdot,t) - H_{0,1}(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\leq \|v_\epsilon(\cdot,t) - V^{(0,1)}_\epsilon(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\quad + \|V^{(0,1)}_\epsilon(\cdot - X_{1/2}(t)) - H_{0,1}(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &= \|v_\epsilon(\cdot,t) - V^{(0,1)}_\epsilon(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\quad + \|V^{(0,1)}_\epsilon(\cdot) - H_{0,1}(\cdot)\|_{L^1(\mathbf{R})}, \end{aligned} \tag{3.4}$$

where $V^{(0,1)}_\epsilon(x - X_{1/2}(t))$ is a travelling-wave solution of (1.9a) given in Lemma 2.3. Since $V^{(0,1)}_\epsilon(x - X_{1/2}(t))$ is also a solution of (1.9a), (2.4) implies that

$$\|v_\epsilon(\cdot,t) - V^{(0,1)}_\epsilon(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \leq \|H_{0,1}(\cdot) - V^{(0,1)}_\epsilon(\cdot)\|_{L^1(\mathbf{R})}.$$

Therefore substituting this into (3.4) yields

$$\|v_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \leq 2\,\|H_{0,1}(\cdot) - V^{(0,1)}_\epsilon(\cdot)\|_{L^1(\mathbf{R})}$$

and applying (2.9) to the above inequality gives the desired (1.7) with a constant $C_1 = 2A_{0,1}$, where $A_{0,1}$ is defined by (2.8) and is obviously independent of $\epsilon$ and $t$. This has verified (1.7).

**3.2. Proof of (1.7) for general $K$.** We prove the general case by using induction on $K$. For $K = 1$ (1.7) is true, as has been proved in the previous section. We now assume that the theorem is true for $K - 1$ and prove it is true for $K$.

A key step in the proof is to construct a combining approximate solution: for $0 \leq t \leq t^{(1)}$

(3.5a)

$$\overline{v}_\epsilon(x,t) = u(x,t) + \sum_{k=1}^{K} \left[ V_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\epsilon}\right) - H_{k-1,k}\left(x - X_{k-1/2}(t)\right)\right],$$

where $u(x,t)$ is defined by (1.4) and $t^{(1)}$ is given by (1.6). On account of the definitions of $u(x,t)$ and $H_{k-1,k}(\xi)$, the combining solution $\overline{v}_\epsilon(x,t)$ can also be written as

$$\overline{v}_\epsilon(x,t) = -\sum_{k=0}^{K-1} u^{(k)} + \sum_{k=1}^{K} V_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\epsilon}\right) \qquad \text{for } 0 \leq t \leq t^{(1)}. \tag{3.5b}$$

The expression of (3.5b) was first used by Liu and Xin [13] to prove the stability of multiple shocks for Lax–Friedrichs schemes to systems of conservation laws. The next lemma easily follows from the two expressions.

LEMMA 3.1. *$\overline{v}_\epsilon(x,t)$ is a smooth function on $\{(x,t)\,|\,x \in \mathbf{R},\ t \in [0,\, t^{(1)}]\}$ and satisfies*

$$\|u(\cdot,t) - \overline{v}_\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \le C_K\,\epsilon \quad \text{for } 0 \le t \le t^{(1)}, \tag{3.6}$$

*where $C_K$ is some constant.*

*Proof.* Equation (3.5b) indicates the smoothness of $\overline{v}_\epsilon(x,t)$, while (3.5a) and (2.9) give the desired error estimate. □

Next, with the aid of the stability estimate (2.4), we will prove that the $L_1$-error of the difference between $\overline{v}_\epsilon(\cdot,t)$ and $v_\epsilon(\cdot,t)$, the viscosity solution of (1.9), is also bounded by $O(\epsilon)$. We first derive a nonhomogeneous viscosity equation for $\overline{v}_\epsilon(\cdot,t)$. Substituting $\overline{v}_\epsilon(\cdot,t)$ into (2.3) yields the following lemma.

LEMMA 3.2. *$\overline{v}_\epsilon(x,t)$ satisfies*

$$\frac{\partial \overline{v}_\epsilon}{\partial t} + \frac{\partial f(\overline{v}_\epsilon)}{\partial x} - \epsilon\frac{\partial^2 \overline{v}_\epsilon}{\partial x^2} = \overline{g}(x,t) \quad \text{for } 0 \le t \le t^{(1)}, \tag{3.7a}$$

$$\overline{v}_\epsilon\big|_{t=0} = \overline{u}_0(x), \tag{3.7b}$$

*where*

$$|\overline{g}(x,t)| \le C_K \sum_{k=1}^{K}\left[\sum_{s<k}\left|V_{s-1,s}\left(\frac{x - X_{s-1/2}(t)}{\epsilon}\right) - u^{(s)}\right|\right.$$

$$\left. + \sum_{s>k}\left|V_{s-1,s}\left(\frac{x - X_{s-1/2}(t)}{\epsilon}\right) - u^{(s-1)}\right|\right]\left|V_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\epsilon}\right)_x\right| \tag{3.8a}$$

*and*

$$\overline{u}_0(x) = u_0(x) + \sum_{k=1}^{K}\left[V_{k-1,k}\left(\frac{x - x^{(0)}_{k-1/2}}{\epsilon}\right) - H_{k-1,k}\left(x - x^{(0)}_{k-1/2}\right)\right] \tag{3.8b}$$

*with $u_0(x)$ being defined by (1.2) and some constant $C_K$.*

*Proof.* Substituting (3.5b) into the left-hand side of (3.7a), on account of $V_{k-1,k}((x - X_{k-1/2}(t))/\epsilon)$ being a solution of (1.9a), gives

$$\begin{aligned}
\overline{g}(x,t) &= f(\overline{v}_\epsilon)_x - \sum_{k=1}^{K} f(V_{k-1,k})_x \\
&= \sum_{k=1}^{K}\left(f'(\overline{v}_\epsilon) - f'(V_{k-1,k})\right)\left(V_{k-1,k}\right)_x \\
&= \sum_{k=1}^{K}\int_0^1 f''(\theta\overline{v}_\epsilon + (1-\theta)V_{k-1,k})\,d\theta\left[\sum_{s<k}\left(V_{s-1,s}\left(\frac{x - X_{s-1/2}(t)}{\epsilon}\right) - u^{(s)}\right)\right. \\
&\qquad \left. + \sum_{s>k}\left(V_{s-1,s}\left(\frac{x - X_{s-1/2}(t)}{\epsilon}\right) - u^{(s-1)}\right)\right]V_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\epsilon}\right)_x.
\end{aligned}$$

Since $\overline{v}_\epsilon$ and $V_{k-1,k}$ are uniformly bounded, so is the integration in the above expression, and thus we arrive at the conclusion (3.8a). Equation (3.8b) is obvious. □

It is easy to see from (3.8a), (A.1), and (A.2), where (A.1) and (A.2) are given in the Appendix, that

$$\int_0^{t^{(1)}} \|\overline{g}(\cdot,\tau)\|_{L^1(\mathbf{R})} \leq C_K\,\epsilon, \tag{3.9}$$

and from (3.8b) and (2.9) that

$$\|\overline{u}_0(\cdot) - u_0(\cdot)\|_{L^1(\mathbf{R})} \leq C_K\,\epsilon. \tag{3.10}$$

Applying Lemma 2.2 to the combining solution $\overline{v}_\epsilon(x,t)$ and the viscosity solution $v_\epsilon(x,t)$ we conclude the following lemma.

LEMMA 3.3. *For* $0 \leq t \leq t^{(1)}$

$$\|\overline{v}_\epsilon(\cdot,t) - v_\epsilon(\cdot,t)\|_{L^1(\mathbf{R})} \leq C_K\,\epsilon,$$

*where* $C_K$ *is some constant that is independent of* $t \in [0, t^{(1)}]$.

*Proof.* Since $\overline{v}_\epsilon(x,t)$ and $v_\epsilon(x,t)$ satisfy (3.7) and (1.9), respectively, so Lemma 2.2, with the aid of the estimates (3.9) and (3.10), gives the desired result. □

Lemmas 3.1 and 3.3 indicate that

$$\|v_\epsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \leq C_K\,\epsilon \quad \text{for } 0 \leq t \leq t^{(1)}. \tag{3.11}$$

We are now in a position to complete the induction proof. From the definition (1.6) we know that at $t = t^{(1)}$ at least two neighboring shock curves join; therefore one piece of the constant, located between the neighboring shock curves, disappears at $t^{(1)}$. That means $u(x,t^{(1)})$ consists of at most $K-1$ piecewise constants. So according to the induction hypothesis, Theorem 1.1 is valid for the initial-value problem with the initial data

$$u_0(x) = u(x, t^{(1)}). \tag{3.12}$$

We denote the viscosity solution to the initial-value problem (1.9a) and (3.12) by $v_\epsilon^{(1)}(x,t)$ and Theorem 1.1 shows that

$$\|u(\cdot, t+t^{(1)}) - v_\epsilon^{(1)}(\cdot,t)\|_{L^1(\mathbf{R})} \leq C_{K-1}\,\epsilon \quad \text{for } 0 \leq t < +\infty. \tag{3.13}$$

On the other hand, Lemma 2.2 indicates that

$$\|v_\epsilon(\cdot, t+t^{(1)}) - v_\epsilon^{(1)}(\cdot,t)\|_{L^1(\mathbf{R})} \leq \|v_\epsilon(\cdot,t^{(1)}) - u(\cdot,t^{(1)})\|_{L^1(\mathbf{R})} \quad \text{for } 0 \leq t < +\infty. \tag{3.14}$$

Combining (3.13) and (3.14), with the aid of (3.11), verifies that (1.7) is true for $K$, so the induction proof is finished. We have proved the theorem for the viscosity method.

## 4. Proof of Theorem 1.1 for monotone schemes.

**4.1. Proof of (1.8) for $K = 1$.** We now turn to Theorem 1.1 for the finite-difference monotone schemes. The proof of (1.8) is similar to that of the viscosity method.

Let

$$W_{\Delta x}^{(0,1)}(x) = W_{0,1}\left(\frac{x}{\Delta x}\right), \tag{4.1}$$

where $W_{0,1}(\xi)$ is defined by (2.18). Since $W_{0,1}(\xi)$ is a travelling-wave solution of (1.10a), it satisfies

$$W_{0,1}(\xi - D_{0,1}\lambda) = H(W_{0,1}(\xi - p), \dots, W_{0,1}(\xi + q)), \quad \xi \in \mathbf{R}.$$

From this we see that $W_{\Delta x}^{(0,1)}(x - X_{1/2}(t))$ is also a solution of (1.10a); therefore (2.16) implies that

$$\begin{aligned} \|w_{\Delta x}(\cdot,t) - W_{\Delta x}^{(0,1)}(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\le \|w_{\Delta x}(\cdot,0) - W_{\Delta x}^{(0,1)}(\cdot - x_{1/2}^{(0)})\|_{L^1(\mathbf{R})} \\ &= \|H_{0,1}(\cdot - x_{1/2}^{(0)}) - W_{0,1}((\cdot - x_{1/2}^{(0)})/\Delta x)\|_{L^1(\mathbf{R})} \\ &= \|H_{0,1}(\cdot) - W_{0,1}(\cdot)\|_{L^1(\mathbf{R})}\Delta x = B_{0,1}\Delta x, \end{aligned} \tag{4.2}$$

where the last two equalities follow from $H_{0,1}(x) = H_{0,1}(x/\Delta x)$ and (2.20), respectively. By using (4.1), (4.2), and a similar argument as in (3.4), we have

$$\begin{aligned} \|w_{\Delta x}(\cdot,t) - u(\cdot,t)\|_{L^1(\mathbf{R})} \\ &\le \|w_{\Delta x}(\cdot,t) - W_{\Delta x}^{(0,1)}(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\quad + \|H_{0,1}(\cdot - X_{1/2}(t)) - W_{\Delta x}^{(0,1)}(\cdot - X_{1/2}(t))\|_{L^1(\mathbf{R})} \\ &\le B_{0,1}\Delta x + \|H_{0,1}(\cdot - X_{1/2}(t)) - W_{0,1}((\cdot - X_{1/2}(t))/\Delta x)\|_{L^1(\mathbf{R})} \\ &= 2B_{0,1}\Delta x. \end{aligned}$$

So we arrive at the conclusion (1.8) with a constant $C_1 = 2B_{0,1}$, where $B_{0,1}$ is defined by (2.20) and is independent of $\Delta x$ and $t$. This has finished the proof of (1.8) for the simplest case.

**4.2. Proof of (1.8) for general $K$.** We first construct a combining approximate solution for the monotone scheme: for $0 \le t \le t^{(1)}$,

(4.3a)

$$\overline{w}_{\Delta x}(x,t) = u(x,t) + \sum_{k=1}^{K}\left[W_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\Delta x}\right) - H_{k-1,k}\left(x - X_{k-1/2}(t)\right)\right],$$

where $u(x,t)$ is defined by (1.4) and $t^{(1)}$ is given by (1.6). On account of the definitions of $u(x,t)$ and $H_{k-1,k}(\xi)$, the combining solution $\overline{w}_{\Delta x}(x,t)$ can also be written as

$$\overline{w}_{\Delta x}(x,t) = -\sum_{k=0}^{K-1} u^{(k)} + \sum_{k=1}^{K} W_{k-1,k}\left(\frac{x - X_{k-1/2}(t)}{\Delta x}\right) \qquad \text{for } 0 \le t \le t^{(1)}. \tag{4.3b}$$

The next lemma easily follows from the two expressions.

LEMMA 4.1. *$\overline{w}_{\Delta x}(x,t)$ satisfies*

$$\|u(\cdot,t) - \overline{w}_{\Delta x}(\cdot,t)\|_{L^1(\mathbf{R})} \le C_K\,\Delta x \quad \textit{for } 0 \le t \le t^{(1)}, \tag{4.4}$$

*where $C_K$ is some constant.*

*Proof.* (4.3a) and (2.21) give the desired error estimate. □

Next, with the aid of the stability estimate (2.16), we will prove that the $L_1$-error of the difference between $\overline{w}_{\Delta x}(\cdot,t)$ and $w_{\Delta x}(\cdot,t)$, the monotone-scheme solution of (1.10), is also bounded by $O(\Delta x)$. We first derive a nonhomogeneous monotone-difference scheme for $\overline{w}_{\Delta x}(\cdot,t)$. Substituting $\overline{w}_{\Delta x}(\cdot,t)$ into (2.15) yields the following lemma.

LEMMA 4.2. *$\overline{w}_{\Delta x}(x,t)$ satisfies for $0<t_n\le t^{(1)}$*

$$\overline{w}_{\Delta x}(x,t_{n+1})-H(\overline{w}_{\Delta x}(x-p\Delta x,t_n),\dots,\overline{w}_{\Delta x}(x+q\Delta x,t_n))=\overline{h}(x,t_n)\Delta t, \tag{4.5a}$$

$$\overline{w}_{\Delta x}(x,0)=\overline{u}_0(x), \tag{4.5b}$$

*where*

$$\begin{aligned}|\overline{h}(x,t)|\le C_K\sum_{i,j=-p}^{q}\sum_{k=1}^{K}\Bigg\{\Bigg[&\sum_{s<k}\left|W_{s-1,s}\left(\frac{x+j\Delta x-X_{s-1/2}(t)}{\Delta x}\right)-u^{(s)}\right|\\&+\sum_{s>k}\left|W_{s-1,s}\left(\frac{x+j\Delta x-X_{s-1/2}(t)}{\Delta x}\right)-u^{(s-1)}\right|\Bigg]\\&\times\left|W_{k-1,k}\left(\frac{x+i\Delta x-X_{k-1/2}(t)}{\Delta x}\right)\right.\\&\left.-W_{k-1,k}\left(\frac{x+(i-1)\Delta x-X_{k-1/2}(t)}{\Delta x}\right)\right|\frac{1}{\Delta x}\Bigg\}\end{aligned} \tag{4.6a}$$

*and*

$$\overline{u}_0(x)=u_0(x)+\sum_{k=1}^{K}\left[W_{k-1,k}\left(\frac{x-x^{(0)}_{k-1/2}}{\Delta x}\right)-H_{k-1,k}\left(x-x^{(0)}_{k-1/2}\right)\right] \tag{4.6b}$$

*with $u_0(x)$ being defined by* (1.2) *and some constant $C_K$.*

*Proof.* We first introduce some simplified notations as follows:

$$\mathbf{w}(x+\Delta x/2)=(w(x-(p-1)\Delta x),\dots,w(x+q\Delta x));$$

$$\mathbf{w}(x-\Delta x/2)=(w(x-p\Delta x),\dots,w(x+(q-1)\Delta x));$$

$$\Delta\mathbf{w}(x)=\mathbf{w}(x+\Delta x/2)-\mathbf{w}(x-\Delta x/2);$$

$$\widetilde{\mathbf{w}}(\theta)=\theta\mathbf{w}(x+\Delta x/2)+(1-\theta)\mathbf{w}(x-\Delta x/2);$$

$$\mathbf{D}\overline{f}=(\dots,\overline{f}_j,\dots);\quad \mathbf{D}^2\overline{f}=(\overline{f}_{i,j}),$$

where $\overline{f}_j=\partial\overline{f}/\partial w_j$ and $\overline{f}_{ij}=\partial^2\overline{f}/\partial w_i\partial w_j$, $(i,j=-(p-1),\dots,q)$.

Substituting equation (4.3b) into the left-hand side of (4.5a) (on account of $W_{k-1,k}((x-X_{k-1/2}(t))/\Delta x)$ being a solution of (1.10a)) and using the above notations gives

$$
\begin{aligned}
\overline{h}(x,t)\Delta t &= \lambda\left(\overline{f}\left(\overline{\mathbf{w}}_{\Delta x}(x+\Delta x/2)\right)-\overline{f}\left(\overline{\mathbf{w}}_{\Delta x}(x-\Delta x/2)\right)\right)\\
&\quad -\lambda\sum_{k=1}^{K}\left(\overline{f}\left(\mathbf{W}_{k-1,k}(x+\Delta x/2)\right)-\overline{f}\left(\mathbf{W}_{k-1,k}(x-\Delta x/2)\right)\right)\\
&= \lambda\int_0^1 \mathbf{D}\overline{f}\left(\widetilde{\overline{\mathbf{w}}}_{\Delta x}(\theta)\right)\,d\theta\Delta\overline{\mathbf{w}}_{\Delta x}(x)\\
&\quad -\lambda\sum_{k=1}^{K}\int_0^1 \mathbf{D}\overline{f}\left(\widetilde{\mathbf{W}}_{k-1,k}(\theta)\right)\,d\theta\Delta\mathbf{W}_{k-1,k}(x)\\
&= \lambda\sum_{k=1}^{K}\int_0^1\left(\mathbf{D}\overline{f}\left(\widetilde{\overline{\mathbf{w}}}_{\Delta x}(\theta)\right)-\mathbf{D}\overline{f}\left(\widetilde{\mathbf{W}}_{k-1,k}(\theta)\right)\,d\theta\right)\Delta\mathbf{W}_{k-1,k}(x)\\
&= \lambda\sum_{k=1}^{K}\int_0^1\int_0^1\left(\widetilde{\overline{\mathbf{w}}}_{\Delta x}(\theta)-\widetilde{\mathbf{W}}_{k-1,k}(\theta)\right)\mathbf{D}^2\overline{f}\left(\zeta\widetilde{\overline{\mathbf{w}}}_{\Delta x}(\theta)+(1-\zeta)\widetilde{\mathbf{W}}_{k-1,k}(\theta)\right)\\
&\qquad\times\Delta\mathbf{W}_{k-1,k}(x)\,d\zeta\,d\theta,
\end{aligned}
\tag{4.7}
$$

where the variable $t$ is omitted from the above expressions. Here some terms in (4.7) are expressed more explicitly as

$$
\begin{aligned}
&\widetilde{\overline{\mathbf{w}}}_{\Delta x}(\theta)-\widetilde{\mathbf{W}}_{k-1,k}(\theta)\\
&\quad=\sum_{s<k}\left[\theta\left(\mathbf{W}_{s-1,s}(x+\Delta x/2)-\mathbf{u}^{(s)}\right)+(1-\theta)\left(\mathbf{W}_{s-1,s}(x-\Delta x/2)-\mathbf{u}^{(s)}\right)\right]\\
&\quad\quad+\sum_{s>k}\left[\theta\left(\mathbf{W}_{s-1,s}(x+\Delta x/2)-\mathbf{u}^{(s-1)}\right)\right.\\
&\quad\quad\quad\left.+(1-\theta)\left(\mathbf{W}_{s-1,s}(x-\Delta x/2)-\mathbf{u}^{(s-1)}\right)\right]
\end{aligned}
$$

and

$$
\Delta\mathbf{W}_{k-1,k}(x)=\mathbf{W}_{k-1,k}(x+\Delta x/2)-\mathbf{W}_{k-1,k}(x-\Delta x/2),
$$

where

$$
\mathbf{W}_{k-1,k}(x+\Delta x/2)=\left(\dots,W_{k-1,k}\left(\frac{x+j\Delta x-X_{k-1/2}(t)}{\Delta x}\right),\dots\right),
$$
$$
(j=-(p-1),\dots,q)
$$

and

$$
\mathbf{u}^{(s)}=(\dots,u^{(s)},\dots).
$$

Since $\overline{w}_{\Delta x}$ and $W_{k-1,k}$ are uniformly bounded, so are the components of the matrix $\mathbf{D}^2\overline{f}$ in (4.7), and thus from the above expressions we arrive at the conclusion (4.6a). Equation (4.6b) is obvious. $\square$

It is easy to see from (4.6a), (A.3), and (A.4), where (A.3) and (A.4) are given in the Appendix, that

$$
\sum_{m=0}^{n^{(1)}-1}\|\overline{h}(\cdot,t_m)\|_{L^1(\mathbf{R})}\Delta t\le C_K\,\Delta x,
\tag{4.8}
$$

and from (4.6b) and (2.21) that

$$\|\overline{u}_0(\cdot) - u_0(\cdot)\|_{L^1(\mathbf{R})} \le C_K\,\Delta x, \tag{4.9}$$

where $n^{(1)} = [t^{(1)}/\Delta t]$. Applying Lemma 2.4 to the combining solution $\overline{w}_{\Delta x}(x,t)$ and the finite-difference solution $w_{\Delta x}(x,t)$ we conclude the following lemma.

LEMMA 4.3. *For* $0 \le t_n \le t^{(1)}$

$$\|\overline{w}_{\Delta x}(\cdot, t_n) - w_{\Delta x}(\cdot, t_n)\|_{L^1(\mathbf{R})} \le C_K\,\Delta x,$$

*where* $C_K$ *is some constant which is independent of* $0 < n \le n^{(1)}$.

*Proof.* Since $\overline{w}_{\Delta x}(x,t)$ and $w_{\Delta x}(x,t)$ satisfy (4.5) and (1.10), respectively, Lemma 2.4, with the aid of the estimates (4.8) and (4.9), gives the desired result. □

Lemmas 4.1 and 4.3 indicate that

$$\|w_{\Delta x}(\cdot, t_n) - u(\cdot, t_n)\|_{L^1(\mathbf{R})} \le C_K\,\Delta x \quad \text{for } 0 \le n \le n^{(1)}. \tag{4.10}$$

We are now in a position to complete the induction proof. As shown before, $u(x, t^{(1)})$ consists of at most $K-1$ piecewise constants. So according to the induction hypothesis, Theorem 1.1 is valid for the initial-value problem with the initial data

$$u_0(x) = u(x, t^{(1)}). \tag{4.11}$$

We denote the finite-difference solution to the initial-value problem (1.9a) and (4.11) by $w^{(1)}_{\Delta x}(x,t)$ and Theorem 1.1 shows that

$$\|u(\cdot, t + t^{(1)}) - w^{(1)}_{\Delta x}(\cdot, t)\|_{L^1(\mathbf{R})} \le C_{K-1}\,\Delta x \quad \text{for } 0 \le t < +\infty. \tag{4.12}$$

On the other hand, Lemma 2.2 indicates that for $0 \le t < +\infty$

$$\|w_{\Delta x}(\cdot, t + t^{(1)}) - w^{(1)}_{\Delta x}(\cdot, t)\|_{L^1(\mathbf{R})} \le \|w_{\Delta x}(\cdot, t^{(1)}) - u(\cdot, t^{(1)})\|_{L^1(\mathbf{R})}. \tag{4.13}$$

From the definition (1.10d) we know $w_{\Delta x}(x, t^{(1)}) = w_{\Delta x}(x, [t^{(1)}/\Delta t]\Delta t)$. Therefore, it follows from (4.10) and the $L^1$-continuity of $u(x,t)$ with respect to $t$ (see [1], [10]) that

(4.14)

$$\begin{aligned}\|w_{\Delta x}(\cdot, t^{(1)}) - u(\cdot, t^{(1)})\|_{L^1(\mathbf{R})} &\le \|w_{\Delta x}(\cdot, [t^{(1)}/\Delta t]\Delta t) - u(\cdot, [t^{(1)}/\Delta t]\Delta t)\|_{L^1(\mathbf{R})} \\ &\quad + \|u(\cdot, [t^{(1)}/\Delta t]\Delta t) - u(\cdot, t^{(1)})\|_{L^1(\mathbf{R})} \\ &\le C_K \Delta x + C\|u_0(\cdot)\|_{BV(\mathbf{R})}\Delta t.\end{aligned}$$

Combining (4.12) and (4.13), with the aid of (4.14), verifies that (1.8) is true for $K$, so the induction proof is finished. We have proved the theorem for the finite-difference monotone scheme.

**5. Numerical experiments.** In this section we will present some numerical experiments to verify the theoretical analysis, i.e., the convergence rates of monotone schemes to multishock wave solutions are of one order.

The initial-value problem we consider is of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial u} = 0, \tag{5.1a}$$

$$u|_{t=0} = \begin{cases} 0.0 & x < 0, \\ 0.8 & 1 > x \ge 0, \\ -2.0 & x \ge 1 \end{cases} \tag{5.1b}$$

with

$$f(u) = -\frac{u^3}{3}. \tag{5.2}$$

It is easy to show that the initial data (5.1b) with the nonlinear fluxes (5.2) satisfies the entropy conditions (1.3) and the solution of (5.1) is of the following form: for $0 < t \le 1.25$

$$u(x,t) = \begin{cases} 0.0, & x < -\frac{0.64}{3}t, \\ 0.8, & -\frac{0.64}{3}t \le x < 1 - \frac{3.04}{3}t, \\ -2.0, & x \ge 1 - \frac{3.04}{3}t \end{cases} \tag{5.3a}$$

and for $t > 1.25$

$$u(x,t) = \begin{cases} 0.0, & x < -\frac{0.8}{3} - \frac{4}{3}(t-1.25), \\ -2.0, & x \ge -\frac{0.8}{3} - \frac{4}{3}(t-1.25), \end{cases} \tag{5.3b}$$

where $t^{(1)} = 1.25$ is the intersection time of the two shock curves.

The monotone difference scheme we consider here is the Lax–Friedrichs scheme, which is defined below.

$$w_j^{n+1} = \frac{1}{2}(w_{j-1}^n + w_{j+1}^n) - \frac{\lambda}{2}(f(w_{j+1}^n) - f(w_{j-1}^n)).$$

A simple analysis reveals that the scheme is monotone provided that the Courant–Friedrichs–Lewy (CFL) condition

$$\lambda \max_{|u| \le \sup |w_j^0|} |f'(u)| \le 1$$

holds.

Let us recall some of the notations defined earlier in this paper:

$$\lambda = \frac{\Delta t}{\Delta x} = \text{constant}, \quad w_j^n \approx u(x_j, t_n), \quad x_j = j\Delta x, \quad t_n = n\Delta t.$$

Numerical results for the initial value problems (5.1) and (5.2) are given in Table 1, where $t = 1$ is a time before shocks interact while $t = 1.5$ is after shocks interact. Here errors mean $\|u - w_{\Delta x}\|_{L^1}$ with different steps $\Delta x$ and $\lambda = 1/6$. The numerical results clearly indicate that for the Lax–Friedrichs monotone-difference scheme, the convergence rates for both noninteracting and interacting shocks are of one order, which complies well with the theoretical analysis of Theorem 1.1.

**Appendix.** In order to prove Theorem 1.1 for the more general case we need the following lemmas concerning the travelling-wave solutions.

LEMMA A.1. *For each pair* $0 \le s < k \le K-1$,

(A.1)

$$\int_0^{t^{(1)}} \left\| \left[ V_{s-1,s}\left(\frac{\cdot - X_{s-1/2}(\tau)}{\epsilon}\right) - u^{(s)} \right] V_{k-1,k}\left(\frac{\cdot - X_{k-1/2}(\tau)}{\epsilon}\right)_x \right\|_{L^1(\mathbf{R})} d\tau \le C_{s,k}\,\epsilon$$

*and for each pair* $0 \le k < s \le K-1$,

(A.2)

$$\int_0^{t^{(1)}} \left\| \left[ V_{s-1,s}\left(\frac{\cdot - X_{s-1/2}(\tau)}{\epsilon}\right) - u^{(s-1)} \right] V_{k-1,k}\left(\frac{\cdot - X_{k-1/2}(\tau)}{\epsilon}\right)_x \right\|_{L^1(\mathbf{R})} d\tau \le C_{s,k}\,\epsilon,$$

*where* $t^{(1)}$ *is defined by* (1.6) *and* $C_{s,k}$ *are some constants.*

TABLE 1
*$L^1$-errors and convergence rates for the Lax–Friedrichs scheme to* (5.1) *and* (5.2).

| $\lambda = 1/6$ | $t=1$ | | $t=1.5$ | |
|---|---|---|---|---|
| $\Delta x$ | $L^1$-error | $L^1$-rate | $L^1$-error | $L^1$-rate |
| $2^{-1}$ | 1.654991 | | 1.760314 | |
| $2^{-2}$ | 0.9165800 | 0.8524904 | 0.9282734 | 0.9232112 |
| $2^{-3}$ | 0.6044413 | 0.6006584 | 0.4752705 | 0.9658009 |
| $2^{-4}$ | 0.4495423 | 0.4271453 | 0.2445131 | 0.9588373 |
| $2^{-5}$ | 0.3305978 | 0.4433799 | 0.1270373 | 0.9446595 |
| $2^{-6}$ | 0.2260204 | 0.5486241 | 0.0635880 | 0.9984254 |
| $2^{-7}$ | 0.1393246 | 0.6980032 | 0.0316562 | 1.0062660 |
| $2^{-8}$ | 0.0764073 | 0.8666656 | 0.0160277 | 0.9819129 |
| $2^{-9}$ | 0.0384515 | 0.9906718 | 0.0082601 | 0.9563378 |
| $2^{-10}$ | 0.0194208 | 0.9854328 | 0.0040531 | 1.0271060 |
| $2^{-11}$ | 0.0098752 | 0.9757114 | 0.0019994 | 1.0194890 |

*Proof.* From (2) and (3) of Lemma 2.3, we know that $V_{s,s+1}(\xi)-u^{(s)}$ and $V'_{k-1,k}(\xi)$ will not change their signs for $\xi \in \mathbf{R}$. Thus the integrand of (A.1), denoted by $I(\tau)$, can be written as

$$\begin{aligned} I(\tau) &= \pm \int_{-\infty}^{+\infty} \left[ V_{s-1,s}\left( \frac{x + (X_{k-1/2}(\tau) - X_{s-1/2}(\tau))}{\epsilon} \right) - u^{(s)} \right] V_{k-1,k}\left(\frac{x}{\epsilon}\right)_x \, dx \\ &= I_1 + I_2, \end{aligned}$$

where $I_1$ is the integral over $\{|x| \le (X_{k-1/2}(\tau) - X_{s-1/2}(\tau))/2\}$ and $I_2$ is over $\{|x| \ge (X_{k-1/2}(\tau) - X_{s-1/2}(\tau))/2\}$. Since $X_{k-1/2}(\tau) - X_{s-1/2}(\tau) > 0$, due to $k > s$, we can use the estimate of (2.7b) to $I_1$ and obtain

$$\begin{aligned} I_1 &\le \pm \Gamma_{s-1,s} \exp\left\{ -\gamma_{s-1,s} \frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right\} \\ &\quad \times \left[ V_{k-1,k}\left( \frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right) - V_{k-1,k}\left( -\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right) \right] \\ &\le |u^{(k)} - u^{(k-1)}| \Gamma_{s-1,s} \exp\left\{ -\gamma_{s-1,s} \frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right\}. \end{aligned}$$

Using the estimates of (2.6) and (2.7) to $I_2$ gives

$$\begin{aligned} I_2 &\le \pm |u^{(s-1)} - u^{(s)}| \left[ u^{(k)} - V_{k-1,k}\left( \frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right) \right. \\ &\qquad \left. + V_{k-1,k}\left( -\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right) - u^{(k-1)} \right] \\ &\le 2|u^{(s-1)} - u^{(s)}| \, \Gamma_{k-1,k} \exp\left\{ -\gamma_{k-1,k} \frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\epsilon} \right\}. \end{aligned}$$

Substituting the above estimates of $I_1$ and $I_2$ into $I$ and integrating the resulting inequality with respect to $\tau$ gives

$$\begin{aligned} \int_0^{t^{(1)}} I(\tau)\, d\tau &\le \int_0^{t_{s,k}} I(\tau)\, d\tau \\ &\le \epsilon \frac{\left( 2|u^{(k)} - u^{(k-1)}| \frac{\Gamma_{s-1,s}}{\gamma_{s-1,s}} + 4|u^{(s)} - u^{(s-1)}| \frac{\Gamma_{k-1,k}}{\gamma_{k-1,k}} \right)}{D_{s-1,s} - D_{k-1,k}} \\ &= C_{k,s}\, \epsilon, \end{aligned}$$

where

$$t_{s,k} = \frac{x^{(0)}_{k-1/2} - x^{(0)}_{s-1/2}}{D_{s-1,s} - D_{k-1,k}}$$

is the intersection time of $X_{k-1/2}(\tau)$ and $X_{s-1/2}(\tau)$, which is greater than or equal to $t^{(1)}$ due to the definition of (1.6). This has completed the proof of (A.1). The proof of (A.2) is similar and is omitted. □

LEMMA A.2. *For each pair $0 \le s < k \le K-1$ and for any $-p \le i, j \le q$*

$$\begin{aligned}\sum_{m=0}^{n^{(1)}-1} \Bigg\| \left[W_{s-1,s}\left(\frac{\cdot + j\Delta x - X_{s-1/2}(t_m)}{\Delta x}\right) - u^{(s)}\right]\left[W_{k-1,k}\left(\frac{\cdot + i\Delta x - X_{k-1/2}(t_m)}{\Delta x}\right)\right.\\ \left. - W_{k-1,k}\left(\frac{\cdot + (i-1)\Delta x - X_{k-1/2}(t_m)}{\Delta x}\right)\right]\frac{1}{\Delta x}\Bigg\|_{L^1(\mathbf{R})} \Delta t \le C_{s,k}\,\Delta x \qquad \text{(A.3)}\end{aligned}$$

*and for each pair $0 \le k < s \le K-1$ and for any $-p \le i, j \le q$*

$$\begin{aligned}\sum_{m=0}^{n^{(1)}-1} \Bigg\| \left[W_{s-1,s}\left(\frac{\cdot + j\Delta x - X_{s-1/2}(t_m)}{\Delta x}\right) - u^{(s-1)}\right]\left[W_{k-1,k}\left(\frac{\cdot + i\Delta x - X_{k-1/2}(t_m)}{\Delta x}\right)\right.\\ \left. - W_{k-1,k}\left(\frac{\cdot + (i-1)\Delta x - X_{k-1/2}(t_m)}{\Delta x}\right)\right]\frac{1}{\Delta x}\Bigg\|_{L^1(\mathbf{R})} \Delta t \le C_{s,k}\,\Delta x, \qquad \text{(A.4)}\end{aligned}$$

*where $n^{(1)} = [t^{(1)}/\Delta t]$ and $C_{s,k}$ are some constants.*

*Proof.* The proof is similar to that of Lemma A.1. From (1) and (2) of Lemma 2.6, we know that $W_{s,s+1}(\xi) - u^{(s)}$ and $W_{k-1,k}(\xi) - W_{k-1,k}(\xi - \eta)$ will not change their signs for $\xi \in \mathbf{R}$ and $\eta$ fixed. Thus the summand in (A.3), denoted by $I(\tau)$, can be written as

$$\begin{aligned} I(\tau) = \pm \int_{-\infty}^{+\infty} &\left[W_{s-1,s}\left(\frac{x + (j-i)\Delta x + X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{\Delta x}\right) - u^{(s)}\right] \\ &\times \left[W_{k-1,k}\left(\frac{x}{\Delta x}\right) - W_{k-1,k}\left(\frac{x - \Delta x}{\Delta x}\right)\right]\frac{1}{\Delta x}\,dx \\ = I_1 + I_2, \end{aligned}$$

where $I_1$ is the integral over $\{|x| \le (X_{k-1/2}(\tau) - X_{s-1/2}(\tau))/2\}$ and $I_2$ is over $\{|x| \ge (X_{k-1/2}(\tau) - X_{s-1/2}(\tau))/2\}$. Since $X_{k-1/2}(\tau) - X_{s-1/2}(\tau) > 0$, due to $k > s$, we can use the estimate of (2.19b) to $I_1$ and obtain

$$\begin{aligned} I_1 &\le \pm \Delta_{s-1,s} \exp\left\{-\delta_{s-1,s}\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\Delta x}\right\} \\ &\quad \cdot (u^{(k)} - u^{(k-1)})\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{\Delta x} \\ &\le 1.5|u^{(k)} - u^{(k-1)}|\Delta_{s-1,s} \exp\left\{-\delta_{s-1,s}\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{4\Delta x}\right\}. \end{aligned}$$

Using the estimates of (2.19) to $I_2$ gives

$$I_2 \le \pm|u^{(s-1)} - u^{(s)}| \Bigg[ u^{(k)} - W_{k-1,k}\left(\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\Delta x} + \widetilde{\theta}_1\right) + W_{k-1,k}\left(-\frac{X_{k-1/2}(\tau) - X_{k-1/2}(\tau)}{2\Delta x} + \widetilde{\theta}_2\right) - u^{(k-1)}\Bigg]$$

$$\le 2|u^{(s-1)} - u^{(s)}|\,\Delta_{k-1,k} \exp\left\{-\delta_{k-1,k}\frac{X_{k-1/2}(\tau) - X_{s-1/2}(\tau)}{2\Delta x}\right\},$$

where $-1 < \widetilde{\theta}_i < 1$ (i=1,2). Substituting the above estimates of $I_1$ and $I_2$ into $I$ and integrating the resulting inequality with respect to $\tau$ gives

$$\sum_{m=0}^{n^{(1)}-1} I(t_m)\Delta t \le \int_0^{t^{(1)}} I(\tau)\,d\tau \le \int_0^{t_{s,k}} I(\tau)\,d\tau$$

$$\le \Delta x \frac{\left(6|u^{(k)} - u^{(k-1)}|\frac{\Delta_{s-1,s}}{\delta_{s-1,s}} + 4|u^{(s)} - u^{(s-1)}|\frac{\Delta_{k-1,k}}{\delta_{k-1,k}}\right)}{D_{s-1,s} - D_{k-1,k}}$$

$$= C_{k,s}\,\Delta x.$$

In the second inequality we have used the fact that

$$t^{(1)} \le t_{s,k} = \frac{x^{(0)}_{k-1/2} - x^{(0)}_{s-1/2}}{D_{s-1,s} - D_{k-1,k}}.$$

This is because of the definition (2.6). This has completed the proof of (A.3). The proof of (A.4) is similar and is omitted. □

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