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Author(s): Ying Lung-an and Zhang Ping-Wen
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FULLY DISCRETE CONVERGENCE ESTIMATES FOR VORTEX METHODS IN BOUNDED DOMAINS*

YING LUNG-AN† AND ZHANG PING-WEN†

Abstract. In this paper the authors study vortex method for 2-dimensionsal Euler equations of incompressible flow in bounded domains. To approximate the initial vortex field by a sum of vortex blobs with arbitrary high accuracy, this field is extended smoothly to a small neighborhood of the boundary. And the computation is carried out in the extended domain. To construct the velocity field from vorticity, a second-order isoparametric finite element method is applied, and to solve the characteristic equations, the explicit Euler's scheme is considered. Optimal error bounds for this fully discrete scheme are obtained.

Key words. Euler equation, vortex method, convergence, initial-boundary value problem, finite element method, Euler's scheme

AMS subject classifications. 65N15, 35Q35, 76M25, 76C05

1. Introduction.

Vortex methods are efficient numerical techniques for simulating incompressible flow, especially flow with high Reynord's number. The mathematical foundation of the vortex methods has been studied by many authors. By virtue of a viscous splitting approach, the equations are decomposed into Euler equations and pure diffusion equations. Thus the study of vortex methods for Euler equations is an important part in the theory, and the most fruitful results have been obtained in this direction.

The convergence of vortex methods for the initial value problems of Euler equations was first obtained by Hald [6]. Then the results were improved and different proofs were given by several authors [2], [3], [4], [7]. Recently, the first author of this paper considered the convergence problem for two-dimensional bounded domains [10] and obtained optimal error estimates for a semidiscretization scheme, where it was assumed that the equations for stream functions and the system of ordinary differential equations for partical trajectories were solved exactly. In [10] a finite element approximation for the equations for the stream function was also considered and convergence results were given, but constants in the error bounds depended on the vortex blob parameters.

One purpose of this paper is to prove that the rate of convergence of the finite element schemes can be independent of the vortex blob parameters, provided second-order isoparametric finite elements are used instead of linear ones. The other purpose is to give error estimates for fully discretized two-dimensional vortex methods for initial-boundary value problems of Euler equations.

The paper is organized as follows. In §2, we recall a result in [10]. In §3, we prove an error estimate for the combined effect of vortex discretization and finite element approximation. In §4 we prove the error estimate for full discretization problems.

2. A convergence theorem for semidiscretization.

Let $\Omega \subset \mathbb{R}^2$ be a convex and bounded domain, whose boundary $\partial\Omega$ is sufficiently smooth. Denote by $x =$

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†Department of Mathematics, Peking University, Beijing 100871, People's Republic of China (yingla%bepc2@scs.slac.stanford.edu, mathpu%bepc2@scs.slac.stanford.edu).

(x_1, x_2) the points in $\mathbb{R}^2$. We consider the following initial-boundary value problems

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\rho} \nabla \pi = f, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

$$u \cdot n \mid_{x \in \partial\Omega} = 0, \tag{2.3}$$

$$u \mid_{t=0} = u_0, \tag{2.4}$$

where $u = (u_1, u_2)$ stands for velocity, π stands for pressure, $f = (f_1, f_2)$ is the external force, the density ρ is a positive constant, n is the unit outward normal vector along $\partial\Omega$, and the initial data u_0 satisfies

$$\nabla \cdot u_0 = 0, u_0 \cdot n \mid_{\partial\Omega} = 0.$$

If f, u_0 are sufficiently smooth, then the solutions u and π are also sufficiently smooth on the domain $\overline{\Omega} \times [0, T]$, where T is an arbitrary positive constant.

Let $\omega = -\nabla \wedge u, \omega_0 = -\nabla \wedge u_0$, and ψ be the stream function corresponding to u. Then (2.1)–(2.4) is equivalent to

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = -\nabla \wedge f \equiv F, \tag{2.5}$$

$$-\triangle \psi = \omega, u = \nabla \wedge \psi, \tag{2.6}$$

$$\psi \mid_{x \in \partial\Omega} = 0, \tag{2.7}$$

$$\omega \mid_{t=0} = \omega_0. \tag{2.8}$$

We extend functions u_0 and f, still denoted by u_0 and f, such that they are sufficiently smooth on $\mathbb{R}^2$ and $\mathbb{R}^2 \times [0, T]$, respectively, and the supports of them are compact. Let c be any positive constant. We define

$$\Omega_c = \{x, \operatorname{dist}(x, \overline{\Omega}) < c\}.$$

The "blob function" is defined as follows, $\zeta(x)$ is a cutoff function, such that $\zeta \equiv 0$ for $|x| > 1$ and

$$\zeta_\varepsilon(x) = \frac{1}{\varepsilon^2} \zeta\left(\frac{x}{\varepsilon}\right).$$

With that notation, the semidiscretization scheme for (2.5)–(2.8) is

$$\omega^\varepsilon(x, t) = \sum_{j \in J_1} \alpha_j^\varepsilon(t) \zeta_\varepsilon(x - X_j^\varepsilon(t)), \tag{2.9}$$

$$\frac{d\alpha_j^\varepsilon}{dt} = h^2 F(X_j^\varepsilon(t), t), \alpha_j^\varepsilon(0) = \alpha_j, \tag{2.10}$$

$$\frac{dX_j^\varepsilon}{dt} = g^\varepsilon(X_j^\varepsilon(t), t), X_j^\varepsilon(0) = X_j, \tag{2.11}$$

$$-\triangle \psi^\varepsilon = \omega^\varepsilon, \psi^\varepsilon \mid_{x \in \partial\Omega} = 0, \tag{2.12}$$

$$u^\varepsilon = \nabla \wedge \psi^\varepsilon, \tag{2.13}$$

$$g^\varepsilon(x, t) = \sum_{i=1}^{M} a_i u^\varepsilon(x^{(i)}, t), \tag{2.14}$$

where $j = (j_1, j_2), X_j = (j_1 h, j_2 h), \alpha_j = h^2\omega_0(X_j)$, and $J_1 = \{j, X_j \in \Omega_d\}$, if $x \in \overline{\Omega}$, then $x^{(i)} = x$, otherwise

$$x^{(i)} = (i+1)Y - ix,$$

where Y is the nearest point on $\partial\Omega$ to x; the terms a_i are the solutions of the system

$$\sum_{i=1}^{M}(-i)^j a_i = 1 \quad j = 0, \dots, M;$$

and $\varepsilon > 0, h > 0, d > 0$ are mesh parameters. Equations (2.14) makes sense only if $x^{(i)}$ belongs to $\overline{\Omega}$, but it is proved in [10] that this fact is true provided d is small enough. In this scheme the function g^ε plays the role of velocity, which is equal to u^ε in the domain and interpolated to the exterior part of Ω. This is a natural way to deal with blobs near the boundary. Using g^ε and a "slightly larger" domain Ω_d in computation, all blobs move according to a uniform formula (2.11).

Now we state the convergence results. The notation $W^{m,p}(\Omega)$ for conventional Sobolev spaces and $\|\cdot\|_{m,p}$ for the norms of them are applied throughout this paper. Let $X_j(t)$ be characteristic curves that satisfy

$$\frac{dX_j(t)}{dt} = u(X_j(t), t), X_j(0) = X_j.$$

As a rule, we admit the value of u as an extension if $X_j(t)\overline{\in}\overline{\Omega}$. Then set

$$J_2 = \{j; X_j \in \Omega_{C_0\varepsilon} \cap \Omega_d\},$$

$$\|e(t)\|_p = \left(h^2 \sum_{j\in J_2} |X_j(t) - X_j^\varepsilon(t)|^p\right)^{1/p}, \qquad 1 \le p < \infty,$$

where C_0 is a positive constant to be determined. The following theorem is proved in [10].

THEOREM 1. *If we have* $m \ge 1, k \ge 2$, *such that* $\zeta \in W^{m+1,\infty}(\mathbb{R}^2)$ *and*

$$\int_{\mathbb{R}^2} \zeta(x)dx = 1,$$

$$\int_{\mathbb{R}^2} x^\alpha \zeta(x)dx = 0 \qquad \forall\alpha \in N^2 \quad \text{with } 1 \le |\alpha| \le k-1 \tag{2.15}$$

and if there is a constant $\widetilde{C}$, *such that*

$$\widetilde{C}^{-1}\varepsilon^a \le h \le \widetilde{C}\varepsilon^{1+\frac{k-1}{m}}, \tag{2.16}$$

where $a \ge 1 + \frac{k-1}{m}$, *and if the constant in expression* (2.14), $M \ge k$, *then for any* $p \in [1, \infty)$, *there are positive constants* d_0, C_0, C_1, *and* C_2 *such that if* $d \le d_0$, *then the solution of problem* (2.9)–(2.14) *satisfy*

$$|\nabla u^\varepsilon(x,t)| \le C_1, \qquad x \in \overline{\Omega}, \tag{2.17}$$

$$\|u - u^\varepsilon\|_{0,p,\Omega} + \|e(t)\|_p \le C_2\varepsilon^k \tag{2.18}$$

for $t \in [0, T]$.

For our later use, we need the following.

COROLLARY. *Under the assumption of Theorem* 1, *let* $C_3 > 0$ *be given. Then there is a constant* C_4, *such that*

$$\|\omega^\varepsilon(\cdot, t)\|_{k-1,p,\Omega_{C_3\varepsilon}} \le C_4, \qquad t \in [0,T]. \tag{2.19}$$

Proof. It was proved in [10] that the points X_j out of $\Omega_{C_0\varepsilon}$ do not contribute to the value of ω^ε in Ω. Now we replace C_0 by $C_0 + C_3$. Then the points X_j do not contribute to the value of ω^ε in $\Omega_{C_3\varepsilon}$.

By the proof of Lemmas 2, 3, and 4 of [10],

$$\begin{aligned} &|\omega - \omega^\varepsilon|_{l-1,p,\Omega_{C_3\varepsilon}} \\ &\quad\le C\Bigg\{\varepsilon^k + \left(1+\frac{h}{\varepsilon}\right)^{\frac{2}{r}} \frac{h^m}{\varepsilon^{m+l-1}} + \frac{h^N}{\varepsilon^{N+l-1}} \\ &\qquad + \frac{1}{\varepsilon^l}\left(1 + \frac{1}{\varepsilon}\|e(t)\|_{0,\infty,h}\right)^{\frac{2}{q}} \|e(t)\|_p + \frac{1}{\varepsilon^l}\int_0^t \|e(s)\|_p ds\Bigg\}, \end{aligned} \tag{2.20}$$

where $r \in [1,2]$, $\frac{1}{p} + \frac{1}{q} = 1$, $N \ge 3$, C is a positive constant, and

$$\|e(t)\|_\infty = \max_{j\in J_2} |X_j(t) - X_j^\varepsilon(t)|.$$

By (2.31) of [10], we have

$$\|e(t)\|_\infty \le \frac{1}{h^{\frac{2}{p'}}}\|e(t)\|_{p'}$$

for any $p' \in [1, \infty)$. We take $l = k$, $p' \ge 4$, $N \ge m$. Then we get the upper bound of the right-hand side of (2.20). □

3. Further discretization by finite element methods. Let C denote a generic constant independent of mesh parameters. For simplicity we only consider quadratic triangular isoparametric elements of Lagrange type here. Let a triangle $\hat{K}$ be the reference element, and the set of six nodes consists of three vertices and three midpoints of the edges. Denote by F_K the isoparametric mapping from $\hat{K}$ to each element K. If K is an interior one, then we take F_K affine. If K is a "boundary" element, the nodes of which is shown in Fig. 3.1 where $\tilde{a}_{12,K}$ is the midpoint of $\overline{a_{1,K}a_{2,K}}$, the node $a_{12,K}$ belongs to $\partial\Omega$, and $\overline{\tilde{a}_{12,K}a_{12,K}}$ is perpendicular to $\overline{a_{1,K}a_{2,K}}$. The nodes $a_{13,K}$ and $a_{23,K}$ are simply midpoints. It is known that F_K is uniquely determined by the nodes.

Let $\mathcal{T}_\delta = \{K\}$ and $\overline{\Omega}^\delta = \bigcup_{K\in\mathcal{T}_\delta} K$, where δ is the size of the largest diameter of elements. We assume that the partition is regular and quasiuniform. Then we define the finite element space

$$V^\delta = \{v \in H_0^1(\Omega^\delta);\ v|_K \in P_2(\hat{K}) \circ F_K^{-1}\},$$

where $P_2(\hat{K})$ is the space of all polynomials of degree ≤ 2 and Ω^δ is the interior of $\overline{\Omega}^\delta$. We consider in this section the following scheme:

$$\omega^\delta(x,t) = \sum_{j\in J_1} \alpha_j^\delta(t)\zeta_\varepsilon(x - X_j^\delta(t)), \tag{3.1}$$

$$\frac{d\alpha_j^\delta(t)}{dt} = h^2 F(X_j^\delta(t), t), \qquad \alpha_j^\delta(0) = \alpha_j, \tag{3.2}$$

$$\frac{dX_j^\delta(t)}{dt} = g^\delta(X_j^\delta(t), t), \qquad X_j^\delta(0) = X_j, \tag{3.3}$$

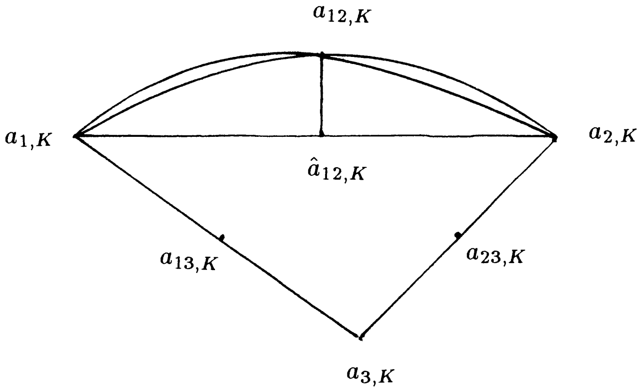

FIG. 3.1

$\psi^\delta \in V^\delta$, and

$$\int_{\Omega^\delta} \nabla\psi^\delta \cdot \nabla v dx = \int_{\Omega^\delta} \omega^\delta v dx \quad \forall v \in V^\delta, \tag{3.4}$$

$$u^\delta = \nabla \wedge \psi^\delta, \tag{3.5}$$

where

$$g^\delta(x,t) = \sum_{i=1}^{M} a_i u^\delta(x_\delta^{(i)}, t).$$

As before $x_\delta^{(i)} = x$ if $x \in \overline{\Omega}^\delta$, and otherwise

$$x_\delta^{(i)} = (i+1)Y_\delta - ix,$$

where Y_δ is the nearest point on $\partial\Omega^\delta$ to x. Y_δ is not necessary unique; we pick up an arbitrary one. Now g^δ is no longer continuous, so (3.3) is satisfied in a generalized sense; see [10]. Here (3.4) and (3.5) are just the definition of a weak finite element solution to (2.12). Since Ω is arbitrary, spacial discretization is needed in solving (2.12). We will prove convergence and get error estimates for it.

Since $\Omega^\delta \not\subset \Omega$ we need to extend some functions from Ω to the whole space $\mathbb{R}^2$. Since $\partial\Omega$ is sufficiently smooth, there exists a strong m-extension operator E on Ω, such that [1]

$$\begin{aligned} &\|E\psi\|_{k,p,\mathbb{R}^2} \le C\|\psi\|_{k,p,\Omega}, \\ &\qquad \forall 0 \le k \le m, \quad 1 \le p < \infty, \quad \psi \in W^{m,p}(\Omega). \end{aligned} \tag{3.6}$$

In this section, we take m large enough, and extend the stream function ψ^ε, still denoted by ψ^ε, then set u^ε and ω^ε to be the corresponding velocity and vorticity, all of them now defined on $\mathbb{R}^2 \times [0,T]$.

We are now in a position to estimate $u^\varepsilon - u^\delta$ and $X_j^\varepsilon(t) - X_j^\delta(t)$. We define

$$\|e(t)\|_p = \left(h^2 \sum_{j\in J_1} |X_j^\varepsilon(t) - X_j^\delta(t)|^p \right)^{\frac{1}{p}},$$

$$\|e(t)\|_\infty = \max_{j\in J_1} |X_j^\varepsilon(t) - X_j^\delta(t)|.$$

LEMMA 1. *Under the assumptions of Theorem* 1,

$$\|\psi^\varepsilon(\cdot,t)-\psi^\delta(\cdot,t)\|_{1,p,\Omega^\delta}\le C\delta^2+C\left\{\left(1+\frac{1}{\varepsilon}\|e(t)\|_\infty\right)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t\|e(s)\|_p ds\right\}, \tag{3.7}$$

provided $\delta\le C_3\varepsilon$, *where* $p\ge 2$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We define two operators: If ψ^ε corresponds to ω^ε on the basis of (2.12), then we define $\psi^\varepsilon=\Delta^{-1}\omega^\varepsilon$, likewise (3.4) defines an operator $\psi^\delta=\Delta_\delta^{-1}\omega^\delta$. Then

$$\begin{aligned}\psi^\varepsilon-\psi^\delta&=\Delta^{-1}\Big(\sum_{j\in J_1}\alpha_j^\varepsilon(t)\zeta_\varepsilon(\cdot-X_j^\varepsilon(t))\Big)-\Delta_\delta^{-1}\Big(\sum_{j\in J_1}\alpha_j^\delta(t)\zeta_\varepsilon(\cdot-X_j^\delta(t))\Big)\\&\equiv\phi_1+\phi_2+\phi_3,\end{aligned}$$

on the domain Ω^δ, where

$$\begin{aligned}\phi_1&=\Delta^{-1}\Big(\sum_{j\in J_1}\alpha_j^\delta(t)(\zeta_\varepsilon(\cdot-X_j^\varepsilon(t))-\zeta_\varepsilon(\cdot-X_j^\varepsilon(t))\Big),\\\phi_2&=\Delta^{-1}\Big(\sum_{j\in J_1}(\alpha_j^\varepsilon(t)-\alpha_j^\delta(t))\zeta_\varepsilon(\cdot-X_j^\delta(t))\Big),\\\phi_3&=(\Delta^{-1}-\Delta_\delta^{-1})\sum_{j\in J_1}\alpha_j^\delta(t)\zeta_\varepsilon(\cdot-X_j^\delta(t)).\end{aligned}$$

Proceeding in a manner similar to [10], we have

$$\|\phi_1(\cdot,t)\|_{1,p,\Omega}\le C\left(1+\frac{1}{\varepsilon}\|e(t)\|_\infty\right)^{\frac{2}{q}}\|e(t)\|_p, \tag{3.8}$$

$$\|\phi_2(\cdot,t)\|_{1,p,\Omega}\le C\int_0^t\|e(s)\|_p ds, \tag{3.9}$$

and

$$\|\omega^\varepsilon(\cdot,t)-\omega^\delta(\cdot,t)\|_{1,p,\Omega_{C_3\varepsilon}}\le\frac{C}{\varepsilon^2}\left\{\left(1+\frac{1}{\varepsilon}\|e(t)\|_\infty\right)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t\|e(s)\|_p ds\right\}. \tag{3.10}$$

Inequality (3.10) and the Corollary of Theorem 1 yield

$$\|\omega^\delta(\cdot,t)\|_{1,p,\Omega_{C_3\varepsilon}}\le C+\frac{C}{\varepsilon^2}\left\{\left(1+\frac{1}{\varepsilon}\|e(t)\|_\infty\right)^{\frac{2}{q}}\|e(t)\|_p + \int_0^t\|e(s)\|_p ds\right\}. \tag{3.11}$$

To estimate ϕ_3, we define a function ψ_1 that solves

$$-\Delta\psi_1=\omega^\delta,\quad x\in\Omega,\quad \psi_1\mid_{x\in\partial\Omega}=0.$$

Then function ψ^δ determined by (3.4) is the finite element approximation of ψ_1. An abstract error estimate of the isoparametric finite element method shows (see [5, Thm. 4.4.1])

$$\|\psi_1 - \psi^\delta\|_{1,\Omega^\delta} \le C\Bigg(\inf_{v \in V^\delta} \|\psi_1 - v\|_{1,\Omega^\delta} + \sup_{\substack{v \in V^\delta \\ v = 0}} \frac{|\int_{\Omega^\delta} \nabla\psi_1 \cdot \nabla v dx - \int_{\Omega^\delta} \omega^\delta v dx|}{\|v\|_{1,\Omega^\delta}} \Bigg). \tag{3.12}$$

Integrating by parts and using the Hölder inequality gives

$$\begin{aligned}
&\left| \int_{\Omega^\delta} \nabla\psi_1 \cdot \nabla v dx - \int_{\Omega^\delta} \omega^\delta v dx \right| \\
&= \left| - \int_{\Omega^\delta} (\Delta\psi_1 + \omega^\delta) v dx \right| \\
&= \left| \int_{\Omega^\delta \setminus \Omega} (\Delta\psi_1 + \omega^\delta) v dx \right| \\
&\le \|\Delta\psi_1 + \omega^\delta\|_{0,6,\Omega^\delta \setminus \Omega} \|v\|_{0,6,\Omega^\delta \setminus \Omega} \cdot (\operatorname{meas}(\Omega^\delta \setminus \Omega))^{\frac{2}{3}}.
\end{aligned}$$

It will be shown in the Appendix that

$$\operatorname{meas}(\Omega^\delta \setminus \Omega) \le C\delta^3.$$

This fact and the conclusion of the embedding thorem, $H^1(\Omega^\delta) \longrightarrow L^6(\Omega^\delta)$, shows that the second term of the right-hand side of (3.12) is bounded by

$$C\delta^2(\|\omega^\delta\|_{1,\Omega^\delta} + \|\psi_1\|_{3,\Omega^\delta}) \le C\delta^2 \|\omega^\delta\|_{1,\Omega^\delta \cup \Omega}.$$

An interpolation error estimates theorem [5, Thm. 4.3.4] gives the same bound for the first term; therefore,

$$\|\psi_1 - \psi^\delta\|_{1,\Omega^\delta} \le C\delta^2 \|\omega^\delta\|_{1,\Omega_{C_3\varepsilon}}.$$

By the L^∞ estimate [8], we have

$$\|\psi_1 - \psi^\delta\|_{1,\infty,\Omega^\delta} \le C\delta^2 \|\omega^\delta\|_{1,\infty,\Omega_{C_3\varepsilon}}.$$

Then the Stampachia interpolation inequality gives

$$\|\psi_1 - \psi^\delta\|_{1,p,\Omega^\delta} \le C\delta^2 \|\omega^\delta\|_{1,p,\Omega_{C_3\varepsilon}},$$

that is,

$$\|\phi_3\|_{1,p,\Omega^\delta} \le C\delta^2 \|\omega^\delta\|_{1,p,\Omega_{C_3\varepsilon}}. \tag{3.13}$$

Finally (3.8), (3.9), (3.13), and (3.11) yield (3.7). □

LEMMA 2. *Under the assumptions of Theorem* 1 *with* $k \geq 3$,

$$\begin{aligned}\|e(t)\|_p \leq{} & C\delta^2 + C\int_0^t \left(1 + \frac{1}{\delta}\|e(s)\|_\infty + \frac{h}{\delta}\right)^{\frac{2}{p}} \\ & \cdot \left\{\delta^2 + \left(1 + \frac{1}{\varepsilon}\|e(s)\|_\infty\right)^{\frac{2}{q}} \|e(s)\|_p + \int_0^s \|e(t)\|_p dt\right\} ds,\end{aligned} \tag{3.14}$$

provided $\delta \leq C_3\varepsilon$, *and* d *is small enough, where* $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We define

$$g^{\varepsilon\delta}(x,t) = \sum_{i=1}^{M} a_i u^\varepsilon(x_\delta^{(i)}, t).$$

Then taking (2.11) and (3.3) into account, we have

$$\frac{dX_j^\varepsilon(t)}{dt} - \frac{dX_j^\delta(t)}{dt} = I_1 + I_2, \qquad X_j^\varepsilon(0) - X_j^\delta(0) = 0,$$

where

$$\begin{aligned} I_1 &= g^\varepsilon(X_j^\varepsilon(t), t) - g^{\varepsilon\delta}(X_j^\delta(t), t), \\ I_2 &= g^{\varepsilon\delta}(X_j^\varepsilon(t), t) - g^\delta(X_j^\delta(t), t). \end{aligned}$$

By (2.17) and the definitions of functions g^ε and $g^{\varepsilon\delta}$, we obtain

$$\begin{aligned} |I_1| &\leq C|(X_j^\varepsilon(t))^{(i)} - (X_j^\delta(t))_\delta^{(i)}| \\ &\leq C|(X_j^\varepsilon(t))^{(i)} - (X_j^\delta(t))^{(i)}| + C|(X_j^\delta(t))^{(i)} - (X_j^\delta(t))_\delta^{(i)}|. \end{aligned}$$

Since Ω is convex, the first term is bounded by $C|X_j^\varepsilon(t) - X_j^\delta(t)|$,and we will prove in the Appendix that the second term is bounded by $C\delta^2$; therefore,

$$|I_1| \leq C|X_j^\varepsilon(t) - X_j^\delta(t)| + C\delta^2. \tag{3.15}$$

Now we estimate I_2, denoted by $K_j^{(i)}$, the element to which the point $(X_j^\delta(t))_\delta^{(i)}$ belongs. Then

$$\begin{aligned} |I_2| &\leq C\sum_{i=1}^{M} \|u^\varepsilon(\cdot,t) - u^\delta(\cdot,t)\|_{0,\infty,K_j^{(i)}} \\ &= C\sum_{i=1}^{M} |\psi^\varepsilon(\cdot,t) - \psi^\delta(\cdot,t)|_{1,\infty,K_j^{(i)}}. \end{aligned}$$

Let I: $C(K) \longrightarrow P_2(\hat{K})\circ F_K^{-1}$ be the interpolation operator associated with the nodes. We have

$$|\psi^\varepsilon(\cdot,t) - \psi^\delta(\cdot,t)|_{1,\infty,K_j^{(i)}} \leq |\psi^\varepsilon(\cdot,t) - I\psi^\varepsilon(\cdot,t)|_{1,\infty,K_j^{(i)}} + |I\psi^\varepsilon(\cdot,t) - \psi^\delta(\cdot,t)|_{1,\infty,K_j^{(i)}}.$$

By the interpolation estimate for isoparametric elements and the Corollary of Theorem 1, we obtain

$$\begin{aligned}|\psi^\varepsilon(\cdot,t)-I\psi^\varepsilon(\cdot,t)|_{1,\infty,K_j^{(i)}} &\le C\delta^2|\psi^\varepsilon(\cdot,t)|_{3,\infty,K_j^{(i)}}\\ &\le C\delta^2|\psi^\varepsilon(\cdot,t)|_{3,\infty,\Omega}\\ &\le C\delta^2.\end{aligned}$$

Using the inverse inequality gives

$$|I\psi^\varepsilon(\cdot,t)-\psi^\delta(\cdot,t)|_{1,\infty,K_j^{(i)}} \le C\delta^{-\frac{2}{p}}\|I\psi^\varepsilon(\cdot,t)-\psi^\delta(\cdot,t)\|_{1,p,K_j^{(i)}}.$$

Therefore

$$|I_2| \le C\delta^2+\sum_{i=1}^{M}\delta^{-\frac{2}{p}}\|I\psi^\varepsilon(\cdot,t)-\psi^\delta(\cdot)\|_{1,p,k_j^{(i)}}.$$

In conjunction with (3.15), this gives

$$\begin{aligned}\left|\frac{dX_j^\varepsilon(t)}{dt}-\frac{dX_j^\delta(t)}{dt}\right| \le& C\delta^2+C|X_j^\varepsilon(t)-X_j^\delta(t)|\\ &+C\sum_{i=1}^{M}\delta^{-\frac{2}{p}}\|I\psi^\varepsilon(\cdot,t)-\psi^\delta(\cdot,t)\|_{1,p,K_j^{(i)}}.\end{aligned}\tag{3.16}$$

Before summing up (3.16) with respect to j, we should estimate the number of points $(X_j^\delta(t))_\delta^{(i)}$ which lie in one single element K. First of all, let us estimate card $\{j\in J_1; X_j^\delta(t)\in K\}$. We define

$$B_j=\left\{x\in\mathbb{R}^2;\left(j_i-\frac{1}{2}\right)h\le x_i\le\left(j_i+\frac{1}{2}\right)h, i=1,2\right\}.$$

Consider the initial value problem

$$\frac{dy}{dt}=g^\varepsilon(y,t),\qquad y|_{t=0}=y_0\in B_j$$

and define $B_j(t)=\{y(t);$ for all $y_0\in B_j\}$. It is easy to see that

$$\text{meas}B_j\le C\text{meas}B_j^\varepsilon(t),\qquad \text{diam}B_j^\varepsilon(t)\le Ch.$$

Because the distance between $X_j^\delta(t)$ and $X_j^\varepsilon(t)$ is less than $\|e(t)\|_\infty$, if $X_j^\delta(t)\in K$, then $B_j(t)$ lies in a disk with center in K and radius $\delta+\|e(t)\|_\infty+Ch$. Since $B_j(t)$ do not overlap each other, we have

$$\text{card}\{j\in J_1; X_j^\delta(t)\in K\}\le C\frac{\pi(\delta+\|e(t)\|_\infty+Ch)^2}{h^2}.\tag{3.17}$$

Secondly, let us estimate card$\{j\in J_1;(X_j^\delta(t))_\delta^{(i)}\in K, X_j^\delta(t)\notin\overline{\Omega}^\delta\}$. If $(X_j^\delta(t))_\delta^{(i)}\in K$, then similar to (3.15) we have

$$|(X_j^\delta(t))_\delta^{(i)}-(X_j^\varepsilon(t))^{(i)}|\le|X_j^\varepsilon(t)-X_j^\delta(t)|+C\delta^2.$$

Let x_0 be any point on $K \cap \Omega$. Then

$$|(X_j^\varepsilon(t))^{(i)} - x_0| \le \delta + |X_j^\varepsilon(t) - X_j^\delta(t)| + C\delta^2.$$

If d is small enough, then the correspodence $x \longrightarrow x^{(i)}$ is one-to-one. Let $y_0 \notin \overline{\Omega}$ and $x_0 = (y_0)^{(i)}$. Then

$$|X_j^\varepsilon(t) - y_0| \le C(\delta + |X_j^\varepsilon(t) - X_j^\delta(t)| + \delta^2).$$

By the same argument we obtain an analogue of (3.17):

$$\operatorname{card}\{j \in J_1; (X_j^\delta(t))_\delta^{(i)} \in K, X_j^\delta(t) \notin \overline{\Omega}^\delta\} \le C\frac{(\delta + \|e(t)\|_\infty + h)^2}{h^2}. \tag{3.18}$$

Inequalities (3.16)–(3.18) imply that

$$\begin{aligned} \|e(t)\|_p \le{} & C\delta^2 + C\int_0^t \|e(t)\|_p ds + Ch^{\frac{2}{p}} \\ & \cdot \int_0^t \left(\frac{\delta}{h} + \frac{1}{h}\|e(t)\|_\infty + 1\right)^{\frac{2}{p}} \delta^{-\frac{2}{p}} \|I\psi^\varepsilon(\cdot, s) - \psi^\delta(\cdot, s)\|_{1,p,\Omega^\delta} ds. \end{aligned} \tag{3.19}$$

Using the interpolation theorem and Lemma 1, we obtain

$$\begin{aligned} & \|I\psi^\varepsilon(\cdot, s) - \psi^\delta(\cdot, s)\|_{1,p,\Omega^\delta} \\ & \quad \le \|\psi^\varepsilon(\cdot, s) - \psi^\delta(\cdot, s)\|_{1,p,\Omega^\delta} + \|I\psi^\varepsilon(\cdot, s) - \psi^\varepsilon(\cdot, s)\|_{1,p,\Omega^\delta} \\ & \quad \le C\delta^2 + C\left\{\left(1 + \frac{1}{\varepsilon}\|e(s)\|_\infty\right)^{\frac{2}{p}} \|e(s)\|_p + \int_0^s \|e(s)\|_p d\tau\right\} \\ & \qquad + C\delta^2 |\psi^\varepsilon(\cdot, s)|_{3,p,\Omega^\delta}. \end{aligned} \tag{3.20}$$

By the Corollary of Theorem 1, $|\psi^\varepsilon(\cdot, s)|_{3,p,\Omega^\delta}$ is bounded, thus substituting (3.20) into (3.19) gives (3.14). □

THEOREM 2. *If the assumptions of Theorem* 1 *hold with* $k \ge 3$ *and* d_0 *is small enough, and if* $\delta \le C_3\varepsilon$ *and there are constants* $b > 0$ *and* $C_5 > 0$ *such that*

$$C_5^{-1}\delta^b \le h \le C_5\delta,$$

then

$$\|e(t)\|_p + \|u^\varepsilon(\cdot, t) - u^\delta(\cdot, t)\|_{0,p,\Omega^\delta} \le C\delta^2 \tag{3.21}$$

for any $p \in [1, \infty)$.

Proof. It will suffice to prove the conclusion for large number p. If $\|e(t)\|_p \le C\delta^2$, then the factor in (3.14),

$$\begin{aligned} 1 + \frac{1}{\delta}\|e(s)\|_\infty + \frac{h}{\delta} & \le 1 + \frac{\|e(s)\|_p}{\delta h^{\frac{2}{p}}} + \frac{h}{\delta} \\ & \le 1 + C\delta^{1-\frac{2b}{p}} + C_5 \\ & \le C, \end{aligned}$$

provided $1-\frac{2b}{p}>0$. Since $\delta \le C_3\varepsilon$, the factor $(1+\frac{1}{\varepsilon}\|e(t)\|_\infty)^{\frac{2}{q}}$ is bounded too. We have $\|e(0)\|_p=0$. Thus by using Gronwall inequality and a continuous argument it is easy to prove that $\|e(t)\|_p \le C\delta^2$ holds for $t\in[0,T]$. Finally the proof of (3.21) is complete by using Lemma 1. □

4. Full discretization. For simplicity we assume $f=0$ in this section. The forward Euler scheme is applied to the ordinary differential equations (3.3). The full discretization scheme for solving $u^{\triangle t}$, $\omega^{\triangle t}$, and $X_j^{\triangle t}$ is the following:

$$\omega^{\triangle t}(x,n\triangle t)=\sum_{j\in J_1}\alpha_j\zeta_\varepsilon(x-X_j^{\triangle t}(n\triangle t)), \tag{4.1}$$

$$X_j^{\triangle t}((n+1)\triangle t)=X_j^{\triangle t}(n\triangle t)+\triangle t g^{\triangle t}(X_j^{\triangle t}(n\triangle t),n\triangle t), \tag{4.2}$$

$$X_j^{\triangle t}(0)=X_j, \tag{4.3}$$

$\psi^{\triangle t}(n\triangle t)\in V^\delta$, and

$$\int_{\Omega^\delta}\nabla\psi^{\triangle t}\cdot\nabla v dx=\int_{\Omega^\delta}\omega^{\triangle t}v dx \quad \forall v\in V^\delta, \tag{4.4}$$

$$u^{\triangle t}(n\triangle t)=\nabla\wedge\psi^{\triangle t}(n\triangle t), \tag{4.5}$$

where $\triangle t$ is the length of time step, and

$$g^{\triangle t}(x,t)=\sum_{i=1}^{M}a_i u^{\triangle t}(x_\delta^{(i)},t).$$

Now we estimate the error $X_j-X_j^{\triangle t}$ and $u^\varepsilon-u^{\triangle t}$. Let

$$\|e^n\|_p=\left(h^2\sum_{j\in J_3}|X_j^{\triangle t}(n\triangle t)-X_j(n\triangle t)|^p\right)^{\frac{1}{p}},$$

$$\|e^n\|_\infty=\max_{j\in J_3}|X_j^{\triangle t}(n\triangle t)-X_j(n\triangle t)|,$$

where

$$J_3=\left\{j;\quad X_j\in\Omega_d\cap\Omega_{2C_0(\varepsilon+\delta+\triangle t^{\frac{1}{2}})}\right\}.$$

LEMMA 3. *There exists a constant* C_6 *such that*

$$|X((n+1)\triangle t)-X(n\triangle t)-\triangle t\frac{dX}{dt}(n\triangle t)|\le C_6\triangle t^2. \tag{4.6}$$

Proof. Since u is sufficiently smooth, (4.6) obviously follows. □

LEMMA 4. *Under the assumptions of Theorem* 1,

$$\begin{aligned}&\left(h^2\sum_{j\in J_3}|u^\varepsilon((X_j^{\triangle t}(n\triangle t))_\delta^{(i)},n\triangle t)-u((X_j(n\triangle t))_\delta^{(i)},n\triangle t)|^p\right)^{\frac{1}{p}}\\ &\le C(\varepsilon^k+\delta^2+\|e^n\|_p),\end{aligned} \tag{4.7}$$

where $p \geq 2$.

Proof. We have

$$\begin{aligned}
&|u^\varepsilon((X_j^{\triangle t}(n \triangle t))_\delta^{(i)}, n \triangle t) - u((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)| \\
&\quad \leq |u^\varepsilon((X_j^{\triangle t}(n \triangle t))_\delta^{(i)}, n \triangle t) - u^\varepsilon((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)| \\
&\qquad + |u^\varepsilon((X_j(n \triangle t))_\delta^{(i)}, n \triangle t) - u^\varepsilon((X_j(n \triangle t))^{(i)}, n \triangle t)| \\
&\qquad + |u^\varepsilon((X_j(n \triangle t))^{(i)}, n \triangle t) - u((X_j(n \triangle t))^{(i)}, n \triangle t)| \\
&\qquad + |u((X_j(n \triangle t))^{(i)}, n \triangle t) - u((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)|.
\end{aligned}$$

By (2.17) and the Appendix, we get

$$\begin{aligned}
&|u^\varepsilon((X_j^{\triangle t}(n \triangle t))_\delta^{(i)}, n \triangle t) - u^\varepsilon((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)| \\
&\quad \leq C_1 |(X_j^{\triangle t}(n \triangle t))_\delta^{(i)} - (X_j(n \triangle t))_\delta^{(i)}| \\
&\quad \leq C_1 (|(X_j^{\triangle t}(n \triangle t))_\delta^{(i)} - (X_j^{\triangle t}(n \triangle t))^{(i)}| \\
&\qquad + |(X_j^{\triangle t}(n \triangle t))^{(i)} - (X_j(n \triangle t))^{(i)}| + |(X_j(n \triangle t))^{(i)} - (X_j(n \triangle t))_\delta^{(i)}|) \\
&\quad \leq C\delta^2 + C|X_j^{\triangle t}(n \triangle t) - X_j(n \triangle t)|,
\end{aligned}$$

$$\begin{aligned}
&|u^\varepsilon((X_j(n \triangle t))_\delta^{(i)}, n \triangle t) - u^\varepsilon((X_j(n \triangle t))^{(i)}, n \triangle t)| \\
&\quad \leq C|(X_j(n \triangle t))_\delta^{(i)} - (X_j(n \triangle t))^{(i)}| \\
&\quad \leq C\delta^2,
\end{aligned}$$

and

$$\begin{aligned}
&|u((X_j(n \triangle t))^{(i)}, n \triangle t) - u((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)| \\
&\quad \leq C|(X_j(n \triangle t))^{(i)} - (X_j(n \triangle t))_\delta^{(i)}| \\
&\quad \leq C\delta^2.
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\left(h^2 \sum_{j \in J_3} |u^\varepsilon((X_j^{\triangle t}(n \triangle t))_\delta^{(i)}, n \triangle t) - u((X_j(n \triangle t))_\delta^{(i)}, n \triangle t)|^p\right)^{\frac{1}{p}} \\
&\quad \leq C\delta^2 + \left(h^2 \sum_{j \in J_3} |u^\varepsilon((X_j(n \triangle t))^{(i)}, n \triangle t)\right. \\
&\qquad\qquad \left. - u((X_j(n \triangle t))^{(i)}, n \triangle t)|^p\right)^{\frac{1}{p}} + C\|e^n\|_p.
\end{aligned}$$

By (2.25), (2.26) of [10],

$$\begin{aligned}
&\left(h^2 \sum_{j \in J_3} |u^\varepsilon((X_j(n \triangle t))^{(i)}, n \triangle t) - u((X_j(n \triangle t))^{(i)}, n \triangle t)|^p\right)^{\frac{1}{p}} \\
&\quad \leq C(\|u - u^\varepsilon\|_{0,p,\Omega} + h|u - u^\varepsilon|_{1,p,\Omega}) \\
&\quad \leq C\varepsilon^k.
\end{aligned}$$

Thus (4.7) is obtained. □

We will prove by induction that

$$\|e^l\|_p \le C_7(\varepsilon^k + \delta^2 + \Delta t) \tag{4.8}$$

holds for a suitable C_7 and all l. Since $\|e^0\|_p = 0$, we assume that (4.8) is valid for $0 \le l \le n$.

From now on we assume that all the hypotheses of Theorem 2 are satisfied. Since only large p is needed to be taken into consideration, we may assume the constant a in (2.16) satisfies $a < \frac{p}{2}$. Also we assume

$$\Delta t \le C_8\delta^2, \qquad \varepsilon^{k-1} \le C_8\delta. \tag{4.9}$$

Again by (2.31) of [10] and (2.16) we have

$$\|e^l\|_\infty \le \frac{1}{h^{\frac{2}{p}}}\|e^l\|_p \le \frac{\widetilde{C}^{\frac{2}{p}}}{\varepsilon^{\frac{2a}{p}}}\|e^l\|_p. \tag{4.10}$$

Then (4.8) and $\delta \le C_3\varepsilon$ yield

$$\|e^l\|_\infty \le C(\varepsilon^{k-1} + \delta + \Delta t^{\frac{1}{2}})\varepsilon^{1-\frac{2a}{p}}.$$

Noting $k \ge 3$, we get

$$\|e^l\|_\infty \le \varepsilon + \delta + \Delta t^{\frac{1}{2}} \tag{4.11}$$

for sufficiently small ε.

LEMMA 5. *If the hypotheses of Theorem* 2 *and* (4.9) *hold, and* (4.8) *holds for* $l = n$, *and if* ε *is sufficiently small, then*

$$\|\psi^{\Delta t}(\cdot, n\,\Delta\, t) - \psi^\varepsilon(\cdot, n\,\Delta\, t)\|_{1,p,\Omega^\delta} \le C_9\delta^2$$
$$+ C_9\left(1 + \frac{1}{\varepsilon}\|e^n\|_\infty\right)^{\frac{2}{q}}\left(h^2\sum_{j\in J_3}|X_j^{\Delta t}(n\,\Delta\, t) - X_j^\varepsilon(n\,\Delta\, t)|^p\right)^{\frac{1}{p}},$$

where $p \ge 2$, $\frac{1}{p} + \frac{1}{q} = 1$, *and the constant* C_9 *is independent of* C_7.

Proof. The proof is almost the same as that of Lemma 1. Now

$$\psi^\varepsilon - \psi^\delta = \phi_1 + \phi_2,$$

$$\phi_1 = \Delta^{-1}\left(\sum_{j\in J_1}\alpha_j(\zeta_\varepsilon(\cdot - X_j^\varepsilon(n\,\Delta\, t)) - \zeta_\varepsilon(\cdot - X_j^{\Delta t}(n\,\Delta\, t)))\right),$$
$$\phi_2 = (\Delta^{-1} - \Delta_\delta^{-1})\sum_{j\in J_1}\alpha_j\zeta_\varepsilon(\cdot - X_j^{\Delta t}(n\,\Delta\, t)).$$

Inequality (4.11) implies

$$\phi_1 = \Delta^{-1}\left(\sum_{j\in J_3}\alpha_j(\zeta_\varepsilon(\cdot - X_j^\varepsilon(n\,\Delta\, t)) - \zeta_\varepsilon(\cdot - X_j^{\Delta t}(n\,\Delta\, t)))\right).$$

Using the same argument we can estimate ϕ_1 and ϕ_2. □

LEMMA 6. *If the hypotheses of Theorem* 2 *and* (4.9) *hold, and if* ε *is sufficiently small, then*

$$\begin{aligned}(4.12)\qquad &\left(h^2 \sum_{j\in J_3} |g^{\varepsilon\delta}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t) - g^{\Delta t}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t)|^p\right)^{\frac{1}{p}}\\ &\quad\le C\left(1+\frac{1}{\delta}\|e^n\|_\infty\right)(\delta^2 + \|\psi^\varepsilon(\cdot, n\,\Delta\, t) - \psi^{\Delta t}(\cdot, n\,\Delta\, t)\|_{1,p,\Omega^\delta}).\end{aligned}$$

Proof. Like the proof of Lemma 2, we can get

$$(4.13)\qquad \|I\psi^\varepsilon(\cdot,t) - \psi^\varepsilon(\cdot,t)\|_{1,\infty,\Omega^\delta} \le C\delta^2|\psi^\varepsilon(\cdot,t)|_{3,\infty,\Omega} \le C\delta^2,$$

$$(4.14)\qquad \|I\psi^\varepsilon(\cdot,t) - \psi^{\Delta t}(\cdot,t)\|_{1,\infty,K_j^{(i)}} \le C\delta^{-\frac{2}{p}}\|I\psi^\varepsilon(\cdot,t) - \psi^{\Delta t}(\cdot,t)\|_{1,p,K_j^{(i)}}.$$

Then

$$\begin{aligned}&|g^{\Delta t}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t) - g^{\varepsilon\delta}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t)|\\ &\quad\le \sum_{i=1}^{M} |a_i|\cdot\|u^\varepsilon - u^{\Delta t}\|_{0,\infty,K_j^{(i)}}\\ &\quad= \sum_{i=1}^{M} |a_i|\cdot\|\psi^\varepsilon - \psi^{\Delta t}\|_{1,\infty,K_j^{(i)}}\\ &\quad\le C\delta^2 + C\delta^{-\frac{2}{p}}\sum_{i=1}^{M}|a_i|\cdot\|I\psi^\varepsilon(\cdot, n\,\Delta\, t) - \psi^{\Delta t}(\cdot, n\,\Delta\, t)\|_{1,p,K_j^{(i)}}.\end{aligned}$$

For any $K\in\mathcal{T}_\delta$, we also have

$$\operatorname{card}\{j\in J_1, X_j^{\Delta t}(n\,\Delta\, t)\in K\} \le C\frac{(\delta + \|e^n\|_\infty + h)^2}{h^2},$$

and therefore

$$\begin{aligned}&\left(h^2\sum_{j\in J_3}|g^{\Delta t}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t) - g^{\varepsilon\delta}(X_j^{\Delta t}(n\,\Delta\, t), n\,\Delta\, t)|^p\right)^{\frac{1}{p}}\\ &\quad\le C\delta^2 + C\left(1+\frac{1}{\delta}\|e^n\|_\infty\right)^{\frac{2}{p}}\|I\psi^\varepsilon(\cdot, n\,\Delta\, t) - \psi^{\Delta t}(\cdot, n\,\Delta\, t)\|_{1,p,\Omega^\delta}.\end{aligned}$$

In view of (4.13) we obtain (4.12). □

LEMMA 7. *Under the hypotheses of Lemma* 5,

$$\begin{aligned}(4.15)\qquad \|e^{n+1}\|_p \le&\|e^n\|_p + C_{10}\,\Delta\, t\left(1+\frac{1}{\varepsilon}\|e^n\|_\infty\right)^{\frac{2}{q}}\left(1+\frac{1}{\delta}\|e^n\|_\infty\right)^{\frac{1}{p}}\\ &\cdot(\varepsilon^k + \delta^2 + \Delta t + \|e^n\|_p),\end{aligned}$$

where the constant C_{10} *is independent of* C_7.

Proof. By Lemma 3 and (4.2) we obtain

$$\begin{aligned}
\|e^{n+1}\|_p &= \left(h^2 \sum_{j\in J_3} |X_j^{\triangle t}((n+1)\triangle t) - X_j((n+1)\triangle t)|^p \right)^{\frac{1}{p}} \\
&\le \|e^n\|_p + C\triangle t^2 + \triangle t \Bigg(h^2 \sum_{j\in J_3} |g^{\triangle t}(X_j^{\triangle t}(n\triangle t), n\triangle t) \\
&\qquad - u(X_j(n\triangle t), n\triangle t)|^p \Bigg)^{\frac{1}{p}}. \qquad (4.16)
\end{aligned}$$

We have

$$\left(h^2 \sum_{j\in J_3} |g^{\triangle t}(X_j^{\triangle t}(n\triangle t), n\triangle t) - u(X_j(n\triangle t), n\triangle t)|^p \right)^{\frac{1}{p}} \le I_1 + I_2,$$

where

$$I_1 = \left(h^2 \sum_{j\in J_3} |g^{\triangle t}(X_j^{\triangle t}(n\triangle t), n\triangle t) - g^{\varepsilon\delta}(X_j^{\triangle t}(n\triangle t), n\triangle t)|^p \right)^{\frac{1}{p}},$$

$$I_2 = \left(h^2 \sum_{j\in J_3} |g^{\varepsilon\delta}(X_j^{\triangle t}(n\triangle t), n\triangle t) - u(X_j(n\triangle t), n\triangle t)|^p \right)^{\frac{1}{p}},$$

$$\begin{aligned}
I_2 &= \left(h^2 \sum_{j\in J_3} \left| \sum_{i=1}^M a_i (u^\varepsilon((X_j^{\triangle t}(n\triangle t))_\delta^{(i)}, n\triangle t) - u(X_j(n\triangle t), n\triangle t)) \right|^p \right)^{\frac{1}{p}} \\
&\le \left(h^2 \sum_{j\in J_3} \left| \sum_{i=1}^M a_i (u^\varepsilon((X_j^{\triangle t}(n\triangle t))_\delta^{(i)}, n\triangle t) - u((X_j(n\triangle t))_\delta^{(i)}, n\triangle t)) \right|^p \right)^{\frac{1}{p}} \\
&\quad + \left(h^2 \sum_{j\in J_3} \left| \sum_{i=1}^M a_i (u((X_j(n\triangle t))_\delta^{(i)}, n\triangle t) - u((X_j(n\triangle t))^{(i)}, n\triangle t)) \right|^p \right)^{\frac{1}{p}} \\
&\quad + \left(h^2 \sum_{j\in J_3} \left| \sum_{i=1}^M a_i (u((X_j(n\triangle t))^{(i)}, n\triangle t) - u((X_j(n\triangle t)), n\triangle t)) \right|^p \right)^{\frac{1}{p}} \\
&\le \sum_{i=1}^M |a_i| \left(h^2 \sum_{j\in J_3} |u^\varepsilon((X_j^{\triangle t}(n\triangle t))_\delta^{(i)}, n\triangle t) - u((X_j(n\triangle t))_\delta^{(i)}, n\triangle t)|^p \right)^{\frac{1}{p}} \\
&\quad + \sum_{i=1}^M |a_i| \left(h^2 \sum_{j\in J_3} |u((X_j(n\triangle t))_\delta^{(i)}, n\triangle t) - u((X_j(n\triangle t))^{(i)}, n\triangle t)|^p \right)^{\frac{1}{p}} \\
&\quad + \left(h^2 \sum_{j\in J_3} \left| \sum_{i=1}^M a_i (u((X_j(n\triangle t))^{(i)}, n\triangle t) - u(X_j(n\triangle t), n\triangle t)) \right|^p \right)^{\frac{1}{p}}
\end{aligned}$$

Using Lemma 4, the fact that u is smooth enough, and Taylor's formula, we get

$$I_2 \leq C(\varepsilon^k + \delta^2 + \|e^n\|_p).$$

The term in (4.16) with respect to I_1 can be estimated by using Lemma 6. Then we get

$$\begin{aligned} \|e^{n+1}\|_p \leq {} & \|e^n\|_p + C \triangle t^2 + C \triangle t \left(1 + \frac{1}{\delta}\|e^n\|_\infty\right)^{\frac{2}{p}} \\ & \cdot (\delta^2 + \|\psi^\varepsilon(\cdot, n \triangle t) - \psi^{\Delta t}(\cdot, n \triangle t)\|_{1,p,\Omega^\delta}) \\ & + C \triangle t(\varepsilon^k + \delta^2 + \|e^n\|_p). \end{aligned} \tag{4.17}$$

By Theorem 1

$$\left(h^2 \sum_{j \in J_3} |X_j(t) - X_j^\varepsilon(t)|^p\right)^{\frac{1}{p}} \leq C_2(2(\varepsilon + \delta + \Delta t^{\frac{1}{2}}))^k \leq C\varepsilon^k.$$

Hence Lemma 5 implies

$$\|\psi^{\Delta t}(\cdot, n \triangle t) - \psi^\varepsilon(\cdot, n \triangle t)\|_{1,p,\Omega^\delta} \leq C\delta^2 + C\left(1 + \frac{1}{\varepsilon}\|e^n\|_\infty\right)^{\frac{2}{q}} (\varepsilon^k + \|e^n\|_p).$$

Substituting this into (4.17) gives (4.15). □

THEOREM 3. *If the hypotheses of Theorem* 2 *and* (4.9) *hold, then*

$$\|e^n\|_p + \|u^\varepsilon(\cdot, n \triangle t) - u^{\Delta t}(\cdot, n \triangle t)\|_{0,p,\Omega^\delta} \leq C(\varepsilon^k + \delta^2 + \Delta t), \tag{4.18}$$

where $n \triangle t \leq T$.

Proof. It will suffice to prove (4.18) for large p and small ε. We have assumed that (4.8) is valid for $0 \leq l \leq n$. From (4.10) and the relations among ε, δ, and Δt, we get

$$\frac{1}{\delta}\|e^n\|_\infty \leq C_7 \widetilde{C}^{\frac{2}{p}} (\varepsilon^k + \delta^2 + \Delta t)\varepsilon^{\frac{-2a}{p}} \delta^{-1} \leq C_{11},$$

for $\varepsilon \leq \varepsilon_0$, where ε_0 depends on C_7 but C_{11} is independent of C_7. Analogously we can estimate $\frac{1}{\varepsilon}\|e^n\|_\infty$. Then (4.15) becomes

$$\|e^{n+1}\|_p \leq \|e^n\|_p + C_{12} \triangle t(\varepsilon^k + \delta^2 + \Delta t + \|e^n\|_p),$$

for $\varepsilon \leq \varepsilon_0$, where C_{12} is independent of C_7.

We set $C_7 = C_{12}Te^{C_{12}T}$. Then we determine ε_0 according to C_7. It is easy to verify that

$$\|e^n\|_p \leq C_{12}e^{C_{12}n\Delta t}(\varepsilon^k + \delta^2 + \Delta t)n \triangle t$$

for $\varepsilon \leq \varepsilon_0$ and all $n, n \triangle t \leq T$. Thus the estimate for $\|e^n\|_p$ is obtained, and the estimate for $u^\varepsilon - u^{\Delta t}$ follows from Lemma 5. □

Remark. The extension of convergence proof to higher order schemes for time stepping is straightforward. A numerical example was given to show the accuracy of this method [11].

Appendix. Let domains Ω and Ω^δ be the above. We prove

$$\text{meas}(\Omega^\delta \setminus \Omega) \le C\delta^3 \tag{A.1}$$

and

$$|x^{(i)} - x_\delta^{(i)}| \le C\delta^2 \quad \forall x \in \Omega_d, \tag{A.2}$$

provided d is small enough.

Introducing local coordinates, $\partial\Omega^\delta$ is the quadratic interpolation of $\partial\Omega$. Thus we have (Chapter 2 of [9], for example)

$$\sup_{x \in \partial\Omega^\delta} \inf_{y \in \partial\Omega} |x - y| \le C\delta^3,$$

from which (A.1) follows.

Let us consider (A.2). If $x \in \overline{\Omega}$, then (A.2) is trivial, so we assume $x \notin \overline{\Omega}$. From Fig. 3.1 it is clear that

$$|a_{12,K} - \tilde{a}_{12,K}| \le C\delta^2.$$

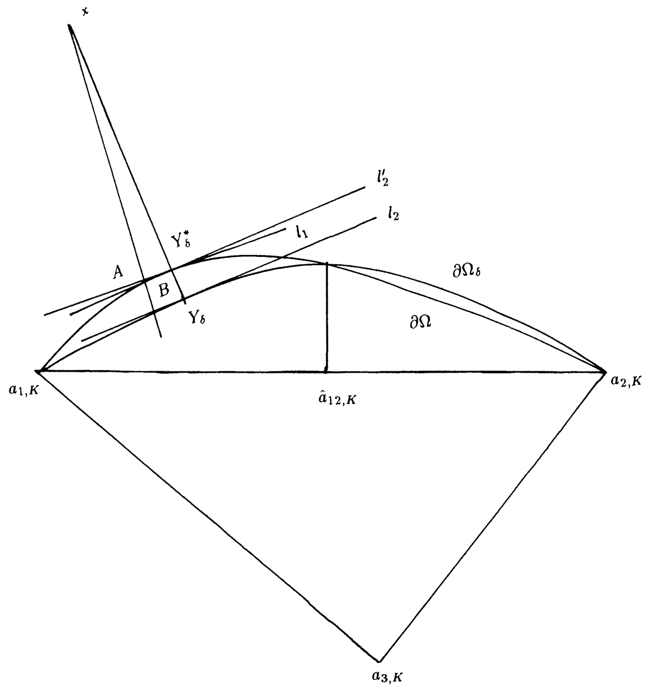

Fig. A.1

Let Y_δ^* be the intersecting point of line $\overline{xY_\delta}$ and $\partial\Omega$, l_1 be the tangent line of $\partial\Omega$ through Y_δ^*, l_2 be the tangent line of $\partial\Omega^\delta$ through Y_δ, and l_2' be the parallel line of l_2 through Y_δ^* (Fig. A.1). The angle between l_1 and l_2' is less than $C\delta^2$. We draw the vertical line of l_1 through point x. Let A be the foot of perpendicular and B be the intersecting point of $\overline{xA}$ with l_2'. Then Y lies in the triangle Y_δ^*AB. The angle $\angle Y_\delta^*xB$ is equal to the angle between l_1 and l_2'. Now $x \in \Omega_d$, hence $|xY_\delta^*| \leq Cd$. Then

$$|Y_\delta^*Y| \leq |Y_\delta^*B| \leq Cd \cdot C\delta^2 \leq C\delta^2.$$

Consequently,

$$|Y_\delta Y| \leq |Y_\delta Y_\delta^*| + |Y_\delta^*Y| \leq C\delta^3 + C\delta^2 \leq C\delta^2.$$

By definition,

$$|x_\delta^{(i)} - x^{(i)}| = (i+1)|Y_\delta Y| \leq C\delta^2,$$

which proves (A.2) if Y_δ is not a node. Conversely, if Y_δ is just a node, say $a_{1,K}$, then we consider the two triangles containing $a_{1,K}$. The argument is similar.

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