Projective Bundle Theorem in MW-Motives

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Motivation

Suppose $0 \le i \le n$, we have:

$$H^i(\mathbb{RP}^n,\mathbb{Z}) = egin{cases} \mathbb{Z} & ext{if } i=0 ext{ or } i=n ext{ and } n ext{ is odd} \ \mathbb{Z}/2\mathbb{Z} & ext{if } i>0 ext{ is even} \ 0 & ext{else}. \end{cases}$$

Theorem (Fasel, 2013)

$$\widetilde{CH}^{i}(\mathbb{P}^{n}) = egin{cases} GW(k) & \textit{if } i = 0 \textit{ or } i = n \textit{ and } n \textit{ is odd} \\ \mathbb{Z} & \textit{if } i > 0 \textit{ is even} \\ 2\mathbb{Z} & \textit{else} \end{cases}$$

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Question

- A motivic explanation?
- How about projective bundles?

Chow Groups

• $CH^n(X) = \mathbb{Z}\{\text{cycles of codimension } n\}/\text{rational equivalence}:$

$$\bigoplus_{y \in X^{(n-1)}} k(y)^* \xrightarrow{div} \bigoplus_{y \in X^{(n)}} \mathbb{Z} \longrightarrow 0.$$

$$H$$

$$CH^n(X)$$

Projective bundle theorem:

$$CH^{n}(\mathbb{P}(E)) = \bigoplus_{i=0}^{rk(E)-1} CH^{n-i}(X) \quad \mathbb{P}(E) = \bigoplus_{i=0}^{rk(E)-1} X(i)[2i].$$

• Chern class:

$$c_i(E) \in CH^i(X)$$
.



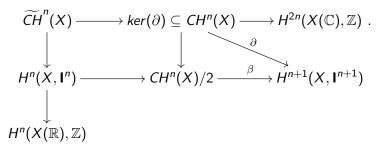
Chow-Witt Groups

Suppose X is smooth and $L \in Pic(X)$. We have the Gersten complex:

$$\bigoplus_{y \in X(n-1)} \mathsf{K}_1^{MW}(k(y), L \otimes \Lambda_y^*) \xrightarrow{\operatorname{div}} \bigoplus_{y \in X(n)} \mathsf{GW}(k(y), L \otimes \Lambda_y^*) \xrightarrow{\operatorname{div}} \bigoplus_{y \in X(n+1)} \mathsf{W}(k(y), L \otimes \Lambda_y^*) \cdot \prod_{i \in X(n-1)} \mathsf{K}_1^{MW}(k(y), L \otimes \Lambda_y^*) \xrightarrow{\operatorname{div}} \bigoplus_{i \in X(n-1)} \mathsf{W}(k(y), L \otimes \Lambda_y^*) \cdot \prod_{i \in X(n-1)} \mathsf{W$$

Chow-Witt Groups

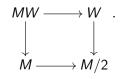
• Suppose *X* is celluar. We have a Cartesian square:



- Pontryagin class
- Bockstein image of Stiefel-Whitney classes
- Orientation class

Four Motivic Theories

• Suppose K = MW, M, W, M/2. We have a homotopy Cartesian:



Definition

Define the category of effective K-motives over S with coefficients in R:

$$DM_{\mathbf{K}}^{eff} = D[(X \times \mathbb{A}^1 \longrightarrow X)^{-1}]$$

where D is the derived category of Nisnevich sheaves with K-transfers.

- $K = MW \Longrightarrow \text{Milnor-Witt Motives}$
- \bullet $K = M \Longrightarrow$ Voevodsky's Motives



Four Motivic Theories

Theorem (BCDFØ, 2020)

For any $X \in Sm/S$ and $n \in \mathbb{N}$, we have

$$[X,\mathbb{Z}(n)[2n]]_{\mathbf{K}} = \widetilde{CH}^{n}(X), CH^{n}(X), CH^{n}(X)/2$$

if $\mathbf{K} = MW, M, M/2$.

Theorem (Cancellation, BCDFØ, 2020)

Suppose S=pt. For any $A,B\in DM_{\mathbf{K}}^{eff}$, we have

$$[A,B]_{\mathbf{K}} \xrightarrow{\otimes (1)} [A(1),B(1)]_{\mathbf{K}}.$$



Basic Calculations

- \bullet $\mathbb{A}^n = \mathbb{Z}$.
- $\mathbb{G}_m = \mathbb{Z} \oplus \mathbb{Z}(1)[1]$.
- $\bullet \ \mathbb{A}^n \setminus 0 = \mathbb{Z} \oplus \mathbb{Z}(n)[2n-1].$
- $\bullet \mathbb{P}^1 = \mathbb{Z} \oplus \mathbb{Z}(1)[2].$
- $\mathbb{A}^n/(\mathbb{A}^n \setminus 0) = \mathbb{P}^n/(\mathbb{P}^n \setminus pt) = \mathbb{Z}(n)[2n].$
- $E \cong X$ for any \mathbb{A}^n -bundle E over X.

Hopf Map η

Definition

The multiplication map $\mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m$ induces a morphism

$$\mathbb{G}_m \otimes \mathbb{G}_m \longrightarrow \mathbb{G}_m$$
.

It's the suspension of a (unique) morphism $\eta \in [\mathbb{G}_m, \mathbb{1}]$, which is called the Hopf map.

It's also equal, up to a suspension, to the morphism

$$\begin{array}{ccc} \mathbb{A}^2 \setminus 0 & \longrightarrow & \mathbb{P}^1 \\ (x,y) & \longmapsto & [x:y] \end{array}.$$

Remark

The $\eta = 0$ if K = M, M/2, but never zero if K = MW, W!

$$\pi_3(S^2) = \mathbb{Z} \cdot Hopf$$



MW-Motive of \mathbb{P}^n

Theorem (Y)

Suppose $n \in \mathbb{N}$ and $p : \mathbb{P}^n \longrightarrow pt$.

1 If n is odd, there is an isomorphism

$$\mathbb{P}^n \xrightarrow{(\rho,c_n^{2i-1},th_{n+1})} R \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} cone(\eta)(2i-1)[4i-2] \oplus R(n)[2n].$$

If n is even, there is an isomorphism

$$\mathbb{P}^n \xrightarrow{(\rho,c_n^{2i-1})} R \oplus \bigoplus_{i=1}^{\frac{n}{2}} cone(\eta)(2i-1)[4i-2].$$

Here $th_{n+1} = i_*(1)$ for some rational point $i : pt \longrightarrow \mathbb{P}^n$.



$$c_n^{2i-1}: \mathbb{P}^n \longrightarrow cone(\eta)(2i-1)[4i-2]$$

We have $cone(\eta) = \mathbb{Z} \oplus \mathbb{Z}(1)[2]$ in DM_M^{eff} since $\eta = 0$. This implies

$$[\mathbb{P}^n, cone(\eta)(j)[2j]]_M = CH^j(\mathbb{P}^n) \oplus CH^{j+1}(\mathbb{P}^n).$$

We have an adjunction $\gamma^*: DM_{MW}^{eff} \rightleftharpoons DM_{M}^{eff}: \gamma_*$.

Theorem (Y)

Suppose $j = 2i - 1 \le n - 1$. The morphism

$$\gamma^*: [\mathbb{P}^n, cone(\eta)(j)[2j]]_{MW} \longrightarrow [\mathbb{P}^n, cone(\eta)(j)[2j]]_{M} \\
c_n^j \longmapsto (c_1(O(1))^k, c_1(O(1))^{k+1})$$

is injective with coker $(\gamma^*) = \mathbb{Z}/2\mathbb{Z}$.

Splitness in MW-Motives

Definition

We say $X \in Sm/k$ splits in DM_{MW}^{eff} if it's isomorphic to the form

$$\bigoplus_{i} \mathbb{Z}(i)[2i] \oplus \bigoplus_{j} cone(\eta)(j)[2j].$$

Remark

The former (resp. latter) component corresponds to the torsion free (resp. torsion) part of $H^{\bullet}(X(\mathbb{R}), \mathbb{Z})$ if X is cellular.

Remark

We couldn't split $cone(\eta)$ in DM_{MW}^{eff} but we still have

$$cone(\eta)^2 = cone(\eta) \oplus cone(\eta)(1)[2].$$



Goal

Suppose E is a vector bundle. Find out the global definition of c_n^{2i-1} and th_{n+1} on $\mathbb{P}(E)$.



Motivic Stable Homotopy Category $\mathcal{SH}(k)$

- $\{\mathbb{P}^1 \text{spectra of simp. Nis. sheaves}\}/\text{stable }\mathbb{A}^1$ -equivalences.
- E-cohomologies:

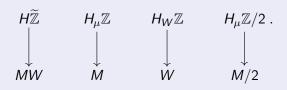
$$[\Sigma^{\infty}X_+, E(q)[p]]_{\mathcal{SH}(k)} = E^{p,q}(X).$$

- $H^n(X, \mathbf{K}_n) = H_{\mathbf{K}}^{2n,n}(X) = CH^n(X), \widetilde{CH}^n(X), \cdots,$ if $E = H_{\mu}\mathbb{Z}, H\widetilde{\mathbb{Z}}, \cdots.$
- $(DM_{MW})_{\mathbb{Q}} = \mathcal{SH}_{\mathbb{Q}}$.

Motivic Cohomology Spectra

Definition

Every motivic theory corresponds to a spectrum in SH(k), namely



The spectrum represents the $cone(\eta)$ (induces the same cohomologies) of, for example, MW-motive is denoted by $H\widetilde{\mathbb{Z}}/\eta$.

$$H\widetilde{\mathbb{Z}}/\eta$$

Theorem (Y)

We have a distinguished triangle

$$\mathbb{P}^1 \wedge H_{\mu}\mathbb{Z} \longrightarrow H\widetilde{\mathbb{Z}}/\eta \longrightarrow H_{\mu}\mathbb{Z} \oplus H_{\mu}\mathbb{Z}/2[2] \longrightarrow \mathbb{P}^1 \wedge H_{\mu}\mathbb{Z}[1].$$

Remark

The triangle doesn't split since applying $\pi_2()_0$ we get an exact sequence of Nisnevich sheaves

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow O^* \longrightarrow 2O^* \longrightarrow 0.$$

$$\eta_{MW}^i(X)$$

Definition

$$\eta_{MW}^{i}(X) := [X, cone(\eta)(i)[2i]]_{MW} = [\Sigma^{\infty}X_{+}, H\widetilde{\mathbb{Z}}/\eta(i)[2i]]_{\mathcal{SH}(k)}.$$

Theorem (Y)

If $R = \mathbb{Z}$ and ${}_{2}CH^{i+1}(X) = 0$, we have a natural isomorphism

$$\theta^i: CH^i(X) \oplus CH^{i+1}(X) \longrightarrow \eta^i_{MW}(X).$$

Corollary

If $R = \mathbb{Z}[\frac{1}{2}]$, we have a natural isomorphism

$$\theta^{i}: CH^{i}(X)[\frac{1}{2}] \oplus CH^{i+1}(X)[\frac{1}{2}] \longrightarrow \eta^{i}_{MW}(X)$$

for any $X \in Sm/k$.

 a^k, b^k

Definition

Suppose $n \ge k + 1$ and k is odd. Define $a^k, b^k \in \mathbb{Z}$ by

$$\begin{array}{ccc} \textit{CH}^k(\mathbb{P}^n) \oplus \textit{CH}^{k+1}(\mathbb{P}^n) & \xrightarrow{\theta^k} & [\mathbb{P}^n, \textit{cone}(\eta)(k)[2k]]_{\textit{MW}} \\ (\textit{a}^k\textit{c}_1(\textit{O}(1))^k, \textit{b}^k\textit{c}_1(\textit{O}(1))^{k+1}) & \longmapsto & \textit{c}^k_n \end{array} .$$

They are independent of n.

$$c(E)^k: \mathbb{P}(E) \longrightarrow cone(\eta)(k)[2k]$$

Definition

Suppose *E* is a vector bundle of rank *n* over *X*, $R = \mathbb{Z}$, ${}_{2}CH^{*}(X) = 0$ and $k \leq n-2$ is odd. Define $c(E)^{k}$ by

$$\begin{array}{ccc} \mathit{CH}^k(\mathbb{P}(E)) \oplus \mathit{CH}^{k+1}(\mathbb{P}(E)) & \stackrel{\theta^k}{\longrightarrow} & [\mathbb{P}(E), \mathit{cone}(\eta)(k)[2k]]_{\mathit{MW}} \\ (\mathit{a}^k c_1(O(1))^k, \mathit{b}^k c_1(O(1))^{k+1}) & \longmapsto & \mathit{c}(E)^k \end{array} .$$

If $R = \mathbb{Z}[\frac{1}{2}]$, $c(E)^k$ is defined for all $X \in Sm/k$.

Projective Orientability

Recall SL^c -bundles are vector bundles E over X such that

$$det(E) \in 2Pic(X)$$
.

Definition

Let E be an SL^c -bundle with even rank n over X. It's said to be projective orientable if there is an element $th(E) \in \widetilde{CH}^{n-1}(\mathbb{P}(E))$ such that for any $x \in X$, there is a neighbourhood U of x such that $E|_U$ is trivial and

$$th(E)|_{U}=p^{*}th_{n},$$

where $p: \mathbb{P}^{n-1} \times U \longrightarrow \mathbb{P}^{n-1}$.

Projective Orientability

- In Chow rings, we can always let $th(E) = c_1(O_{\mathbb{P}(E)}(1))^{n-1}$. But this doesn't work for Chow Witt rings!
- If E has a quotient line bundle, it's projective orientable.
- If E has a quotient bundle being projective orientable, it's projective orientable.
- Further characterization?

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X. Suppose $_2CH^*(X)=0$ and X admits an open covering $\{U_i\}$ such that $CH^j(U_i)=0$ for all j>0 and i. Denote by $p:\mathbb{P}(E)\longrightarrow X$.

• If n is even and E is projective orientable, the morphism $(p, p \boxtimes c(E)^{2i-1}, p \boxtimes th(E))$

$$\mathbb{P}(E) \longrightarrow X \oplus \bigoplus_{i=1}^{\frac{n}{2}-1} X \otimes cone(\eta)(2i-1)[4i-2] \oplus X(n-1)[2n-2]$$

is an isomorphism.

2 If n is odd, there is an isomorphism

$$\mathbb{P}(E) \xrightarrow{(p,p\boxtimes c(E)^{2i-1})} X \oplus \bigoplus_{i=1}^{\frac{n-1}{2}} X \otimes cone(\eta)(2i-1)[4i-2].$$

Projective Bundle Theorem

Corollary

Let E is a vector bundle of odd rank n over X. If X is quasi-projective, we have

$$\mathbb{P}(E)\cong X\oplus igoplus_{i=1}^{rac{n-1}{2}}X\otimes cone(\eta)(2i-1)[4i-2].$$

In particular, we have $(k = min\{\lfloor \frac{i+1}{2} \rfloor, \frac{n-1}{2}\})$

$$\widetilde{CH}^{i}(\mathbb{P}(E)) = \widetilde{CH}^{i}(X) \oplus \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(X \times \mathbb{P}^{2}) / \widetilde{CH}^{i-2j+2}(X).$$

Projective Bundle Theorem

Theorem (Y)

Let E be a vector bundle of rank n over X. Suppose $2 \in R^{\times}$. Denote by $p : \mathbb{P}(E) \longrightarrow X$. If n is even and E is projective orientable, the morphism $(p, p \boxtimes c(E)^{2i-1}, p \boxtimes th(E))$

$$\mathbb{P}(E) \longrightarrow X \oplus \bigoplus_{i=1}^{\frac{n}{2}-1} X \otimes cone(\eta)(2i-1)[4i-2] \oplus X(n-1)[2n-2]$$

is an isomorphism.

In particular, we have (k = min{ $\lfloor \frac{i+1}{2} \rfloor, \frac{n}{2} - 1$ })

$$\widetilde{CH}^{i}(\mathbb{P}(E)) = \widetilde{CH}^{i}(X) \oplus \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(X \times \mathbb{P}^{2}) / \widetilde{CH}^{i-2j+2}(X) \oplus \widetilde{CH}^{i-n+1}(X)$$

after inverting 2.

Blow-ups

Theorem (Y)

Suppose Z is smooth and closed in X, $n := codim_X(Z)$ is odd and Z is quasi-projective. We have

$$Bl_{Z}(X)\cong X\oplus igoplus_{i=1}^{rac{n-1}{2}}Z\otimes cone(\eta)(2i-1)[4i-2].$$

In particular, we have $(k = min\{\lfloor \frac{i+1}{2} \rfloor, \frac{n-1}{2}\})$

$$\widetilde{CH}^{i}(Bl_{Z}(X)) = \widetilde{CH}^{i}(X) \oplus \bigoplus_{j=1}^{k} \widetilde{CH}^{i-2j+2}(Z \times \mathbb{P}^{2})/\widetilde{CH}^{i-2j+2}(Z).$$

Thank you!