

# PARITY SHEAVES AND THE HECKE CATEGORY

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## Abstract

We survey some applications of parity sheaves and Soergel calculus to representation theory.

*Dedicated to Wolfgang Soergel, in admiration.*

*“...we’re still using your imagination...” (L. A. Murray)*

## Introduction

One of the first theorems of representation theory is Maschke’s theorem: any representation of a finite group over a field of characteristic zero is semi-simple. This theorem is ubiquitous throughout mathematics. (We often use it without realising it; for example, when we write a function of one variable as the sum of an odd and an even function.) The next step is Weyl’s theorem: any finite-dimensional representation of a compact Lie group is semi-simple<sup>1</sup>. It is likewise fundamental: for the circle group Weyl’s theorem is closely tied to the theory of Fourier series.

Beyond the theorems of Maschke and Weyl lies the realm where semi-simplicity fails. Non semi-simple phenomena in representation theory were first encountered when studying the modular (i.e. characteristic  $p$ ) representations of finite groups. This theory is the next step beyond the classical theory of the character table, and is important in understanding the deeper structure of finite groups. A second example (of fundamental importance throughout mathematics from number theory to mathematical physics) occurs when studying the infinite-dimensional representation theory of semi-simple Lie groups and their  $p$ -adic counterparts.

Throughout the history of representation theory, geometric methods have played an important role. Over the last forty years, the theory of intersection cohomology and perverse sheaves has provided powerful new tools. To any complex reductive group is naturally associated several varieties (e.g. unipotent and nilpotent orbits and their closures, the flag variety and its Schubert varieties, the affine Grassmannian and its Schubert varieties ...). In contrast to the group itself, these varieties are often singular. The theory of perverse sheaves provides a collection of constructible complexes of sheaves (intersection cohomology sheaves) on such varieties, and the “IC data” associated to

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<sup>1</sup>Weyl first proved his theorem via integration over the group to produce an invariant Hermitian form. To do this he needed the theory of manifolds. One can view his proof as an early appearance of geometric methods in representation theory.

intersection cohomology sheaves (graded dimensions of stalks, total cohomology, ...) appears throughout Lie theory.

The first example of the power of this theory is the Kazhdan–Lusztig conjecture (a theorem of Beilinson–Bernstein and Brylinski–Kashiwara), which expresses the character of a simple highest weight module over a complex semi-simple Lie algebra in terms of IC data of Schubert varieties in the flag variety. This theorem is an important first step towards understanding the irreducible representations of semi-simple Lie groups. A second example is Lusztig’s theory of character sheaves, which provides a family of conjugation equivariant sheaves on the group which are fundamental to the study of the characters of finite groups of Lie type.<sup>2</sup>

An important aspect of the IC data appearing in representation theory is that it is computable. For example, a key step in the proof of the conjecture of Kazhdan and Lusztig is their theorem that the IC data attached to Schubert varieties in the flag variety is encoded in Kazhdan–Lusztig polynomials, which are given by an explicit combinatorial algorithm involving only the Weyl group. Often this computability of IC data is thanks to the Decomposition Theorem, which asserts the semi-simplicity (with coefficients of characteristic zero) of a direct image sheaf, and implies that one can compute IC data via a resolution of singularities.

One can view the appearance of the Decomposition Theorem throughout representation theory as asserting some form of (perhaps well-hidden) semi-simplicity. A trivial instance of this philosophy is that Maschke’s theorem is equivalent to the Decomposition Theorem for a finite morphism. A less trivial example is the tendency of categories in highest weight representation theory to admit Koszul gradings; indeed, according to Beilinson, Ginzburg, and Soergel [1996], a Koszul ring is “as close to semisimple as a  $\mathbb{Z}$ -graded ring possibly can be”. Since the Kazhdan–Lusztig conjecture and its proof, many character formulae have been discovered resembling the Kazhdan–Lusztig conjecture (e.g. for affine Lie algebras, quantum groups at roots of unity, Hecke algebras at roots of unity, ...) and these are often accompanied by a Koszul grading.

All of the above character formulae involve representations of objects defined over  $\mathbb{C}$ . On the other hand modular representation theory has been dominated since 1979 by conjectures (the Lusztig conjecture Lusztig [1980] on simple representations of reductive algebraic groups and the James conjecture James [1990] on simple representations of symmetric groups) which would imply that characteristic  $p$  representations of algebraic groups and symmetric groups are controlled by related objects over  $\mathbb{C}$  (quantum groups and Hecke algebras at a  $p^{\text{th}}$  root of unity) where character formulae are given by Kazhdan–Lusztig like formulae.

The Decomposition Theorem fails in general with coefficients in a field of characteristic  $p$ , as is already evident from the failure of Maschke’s theorem in characteristic  $p$ . It was pointed out by Soergel [2000] (and extended by Fiebig [2011] and Achar and Riche [2016b]) that, after passage through deep equivalences, the Lusztig conjecture is equivalent to the Decomposition Theorem holding for Bott–Samelson resolutions of certain complex Schubert varieties, with coefficients in a field of characteristic  $p$ . For

<sup>2</sup>The reader is referred to Lusztig’s contribution Lusztig [1991] to these proceedings in 1990 for an impressive list of applications of IC techniques in representation theory.

a fixed morphism, the Decomposition Theorem can only fail in finitely many characteristics, which implies that the Lusztig conjecture holds for large primes<sup>3</sup>. More recently, it was discovered that there are many large characteristics for which the Decomposition Theorem fails for Bott–Samelson resolutions [Williamson \[2017c\]](#). This led to exponentially large counter-examples to the expected bounds in the Lusztig conjecture as well as counter-examples to the James conjecture.

Thus the picture for modular representations is much more complicated than we thought. Recently it has proven useful (see [Soergel \[2000\]](#) and [Juteau, Mautner, and Williamson \[2014\]](#)) to accept the failure of the Decomposition Theorem in characteristic  $p$  and consider indecomposable summands of direct image sheaves as interesting objects in their own right. It was pointed out by Juteau, Mautner and the author that, in examples in representation theory, these summands are often characterised by simple cohomology parity vanishing conditions, and are called parity sheaves.

Most questions in representation theory whose answer involves (or is conjectured to involve) Kazhdan–Lusztig polynomials are controlled by the Hecke category, a categorification of the Hecke algebra of a Coxeter system. Thus it seems that the Hecke category is a fundamental object in representation theory, like a group ring or an enveloping algebra. The goal of this survey is to provide a motivated introduction to the Hecke category in both its geometric (via parity sheaves) and diagrammatic (generators and relations) incarnations.

When we consider the Hecke category in characteristic  $p$  it gives rise to an interesting new Kazhdan–Lusztig-like basis of the Hecke algebra, called the  $p$ -canonical basis. The failure of this basis to agree with the Kazhdan–Lusztig basis measures the failure of the Decomposition Theorem in characteristic  $p$ . Conjecturally (and provably in many cases), this basis leads to character formulae for simple modules for algebraic groups and symmetric groups which are valid for all  $p$ . Its uniform calculation for affine Weyl groups and large  $p$  seems to me to be one of the most interesting problems in representation theory.<sup>4</sup>

If one sees the appearance of the Decomposition Theorem and Koszulity as some form of semi-simplicity, then this semi-simplicity fails in many settings in modular representation theory. However it is tempting to see the appearance of parity sheaves and the  $p$ -canonical basis as a deeper and better hidden layer of semi-simplicity, beyond what we have previously encountered. Some evidence for this is the fact that some form of Koszul duality still holds, although here IC sheaves are replaced by parity sheaves and there are no Koszul rings.

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<sup>3</sup>The first proof of Lusztig’s conjecture for  $p \gg 0$  was obtained as a consequence of works by [Kazhdan and Lusztig \[1993\]](#), [Lusztig \[1994\]](#), [Kashiwara and Tanisaki \[1995\]](#) and [Andersen, Jantzen, and Soergel \[1994\]](#).

<sup>4</sup>See [Lusztig and Williamson \[2018\]](#) for a conjecture in a very special case, which gives some idea of (or at least a lower bound on!) the complexity of this problem.

## Structure of the paper.

1. In §1 we discuss the Decomposition Theorem, parity sheaves and the role of intersection forms. We conclude with examples of parity sheaves and the failure of the Decomposition Theorem with coefficients of characteristic  $p$ .
2. In §2 we introduce the Hecke category. We explain two incarnations of this category (via parity sheaves, and via diagrammatics) and discuss its spherical and anti-spherical modules. We conclude by defining the  $p$ -canonical basis, giving some examples, and discussing several open problems.
3. In §3 we give a bird's eye view of Koszul duality for the Hecke category in its classical, monoidal and modular forms.

Although this work is motivated by representation theory, we only touch on applications in remarks. The reader is referred to [Williamson \[2017a\]](#) for a survey of applications of this material to representation theory.

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## 1 The Decomposition Theorem and Parity Sheaves

The Decomposition Theorem is a beautiful theorem about algebraic maps. However its statement is technical and it takes some effort to understand its geometric content. To motivate the Decomposition Theorem and the definition of parity sheaves, we consider one of the paths that led to its discovery, namely Deligne's proof of the Weil conjectures [Deligne \[1974\]](#). We must necessarily be brief; for more background on the Decomposition Theorem see [Beilinson, Bernstein, and Deligne \[1982\]](#), [de Cataldo and Migliorini \[2009\]](#), and [Williamson \[2017d\]](#).

**1.1 Motivation: The Weil conjectures.** Suppose that  $X$  is a smooth projective variety defined over a finite field  $\mathbb{F}_q$ . On  $X$  one has the Frobenius endomorphism  $\text{Fr} : X \rightarrow X$  and the deepest of the Weil conjectures ("purity") implies that the eigenvalues of  $\text{Fr}$  on the étale cohomology  $H^i(X)^5$  are of a very special form ("Weil numbers of weight  $i$ "). By the Grothendieck–Lefschetz trace formula, we have

$$|X(\mathbb{F}_{q^m})| = \sum_i (-1)^i \text{Tr}((\text{Fr}^*)^m : H^i(X) \rightarrow H^i(X))$$

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<sup>5</sup>In this section only we will use  $H^i(X)$  to denote the étale cohomology group  $H^i(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$  of the extension of scalars of  $X$  to an algebraic closure  $\overline{\mathbb{F}}_q$  of  $\mathbb{F}_q$ , where  $\ell$  is a fixed prime number coprime to  $q$ .

for all  $m \geq 1$ , where  $X(\mathbb{F}_{q^m})$  denotes the (finite) set of  $\mathbb{F}_{q^m}$ -rational points of  $X$ . In this way, the Weil conjectures have remarkable implications for the number of  $\mathbb{F}_{q^m}$ -points of  $X$ .

How should we go about proving purity? We might relate the cohomology of  $X$  to that of other varieties, slowly expanding the world where the Weil conjectures hold. A first attempt along these lines might be to consider long exact sequences associated to open or closed subvarieties of  $X$ . However this is problematic because purity no longer holds if one drops the “smooth” or “proper” assumption.

For any map  $f : X \rightarrow Y$  of varieties we have a push-forward functor  $f_*$  and its derived functor  $Rf_*$  between (derived) categories of sheaves on  $X$  and  $Y$ . (In this paper we will never consider non-derived functors; we will write  $f_*$  instead of  $Rf_*$  from now on.) The cohomology of  $X$  (with its action of Frobenius) is computed by  $p_*\overline{\mathbb{Q}}_{\ell,X}$ , where  $\overline{\mathbb{Q}}_{\ell,X}$  denotes the constant sheaf on  $X$  and  $p : X \rightarrow \text{pt}$  denotes the projection to a point.

This reinterpretation of what cohomology “means” provides a more promising approach to purity. For any map  $f : X \rightarrow Y$  we can use the commutative diagram

$$\begin{array}{ccc} X & & \\ f \downarrow & \searrow p & \\ Y & \xrightarrow{g} & \text{pt} \end{array}$$

and the isomorphism  $p_*\overline{\mathbb{Q}}_{\ell,X} = (g \circ f)_*\overline{\mathbb{Q}}_{\ell,X} = g_*(f_*\overline{\mathbb{Q}}_{\ell,X})$  to factor the calculation of  $H^*(X)$  into two steps: we can first understand  $f_*\overline{\mathbb{Q}}_{\ell,X}$ ; then understand the direct image of this complex to a point. One can think of the complex  $f_*\overline{\mathbb{Q}}_{\ell,X}$  as a linearisation of the map  $f$ . For example, if  $f$  is proper and  $y$  is a (geometric) point of  $Y$  then the stalk at  $y$  is

$$(f_*\overline{\mathbb{Q}}_{\ell,X})_y = H^*(f^{-1}(y)).$$

It turns out that<sup>6</sup>  $f_*\overline{\mathbb{Q}}_{\ell,X}$  splits as a direct sum of simple pieces (this is the Decomposition Theorem). Thus, each summand contributes a piece of the cohomology of  $X$ , and one can try to understand them separately. This approach provides the skeleton of Deligne’s proof of the Weil conjectures: after some harmless modifications to  $X$ , the theory of Lefschetz pencils provides a surjective morphism  $f : X \rightarrow \mathbb{P}^1$ , and one has to show purity for the cohomology of each of the summands of  $f_*\overline{\mathbb{Q}}_{\ell,X}$  (sheaves on  $\mathbb{P}^1$ ). Showing the purity of the cohomology of each summand is the heart of the proof, which we don’t enter into here!

**1.2 The Decomposition Theorem.** We now change setting slightly: from now on we consider complex algebraic varieties equipped with their classical (metric) topology and sheaves of  $\mathbb{k}$ -vector spaces on them, for some field of coefficients  $\mathbb{k}$ . For such a variety  $Y$  and a stratification

$$Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$$

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<sup>6</sup>after passage to  $\overline{\mathbb{F}}_q$

of  $Y$  into finitely many locally-closed, smooth and connected subvarieties we denote by  $D_{\Lambda}^b(Y; \mathbb{k})$  the full subcategory of the derived category of complexes of sheaves of  $\mathbb{k}$ -vector spaces with  $\Lambda$ -constructible<sup>7</sup> cohomology sheaves. We will always assume that our stratification is such that  $D_{\Lambda}^b(Y; \mathbb{k})$  is preserved under Verdier duality (this is the case, for example, if our stratification is given by the orbits of a group). We denote by  $D_c^b(Y; \mathbb{k})$  the constructible derived category: it consists of those complexes which are  $\Lambda$ -constructible for some  $\Lambda$  as above. Both  $D_{\Lambda}^b(Y; \mathbb{k})$  and  $D_c^b(Y; \mathbb{k})$  are triangulated with shift functor  $[1]$ . For any morphism  $f : X \rightarrow Y$  we have functors

$$D_c^b(X; \mathbb{k}) \begin{array}{c} \xrightarrow{f_*, f!} \\ \xleftarrow{f^*, f^!} \end{array} D_c^b(Y; \mathbb{k})$$

satisfying a menagerie of relations (see e.g. [de Cataldo and Migliorini \[2009\]](#)).

Consider a proper morphism  $f : X \rightarrow Y$  of complex algebraic varieties with  $X$  smooth. We consider the constant sheaf  $\mathbb{k}_X$  on  $X$  with values in  $\mathbb{k}$  and its (derived) direct image on  $Y$ :

$$f_* \mathbb{k}_X.$$

A fundamental problem (which we tried to motivate in the previous section) is to understand how this complex of sheaves decomposes. The Decomposition Theorem states that, if  $\mathbb{k}$  is a field of characteristic zero, then  $f_* \mathbb{k}_X$  is semi-simple in the sense of perverse sheaves. Roughly speaking, this means that much of the topology of the fibres of  $f$  is “forced” by the nature of the singularities of  $Y$ . More precisely, if we fix a stratification of  $Y$  as above for which  $f_* \mathbb{k}_X$  is constructible, then we have an isomorphism:

$$(1) \quad f_* \mathbb{k}_X \cong \bigoplus H_{\lambda, \mathcal{L}}^* \otimes_{\mathbb{k}} \mathbf{IC}_{\lambda}^{\mathcal{L}}.$$

Here the (finite) sum is over certain pairs  $(\lambda, \mathcal{L})$  where  $\mathcal{L}$  is an irreducible local system on  $Y_{\lambda}$ ,  $H_{\lambda, \mathcal{L}}^*$  is a graded vector space, and  $\mathbf{IC}_{\lambda}^{\mathcal{L}}$  denotes IC extension of  $\mathcal{L}$ . (The complex of sheaves  $\mathbf{IC}_{\lambda}^{\mathcal{L}}$  is supported on  $\overline{Y}_{\lambda}$  and extends  $\mathcal{L}[\dim_{\mathbb{C}} Y_{\lambda}]$  in a “minimal” way, taking into account singularities. For example, if  $\overline{Y}_{\lambda}$  is smooth and  $\mathcal{L}$  extends to a local system  $\overline{\mathcal{L}}$  on  $\overline{Y}_{\lambda}$ , then  $\mathbf{IC}_{\lambda}^{\mathcal{L}} = \overline{\mathcal{L}}[\dim_{\mathbb{C}} Y_{\lambda}]$ .) If  $\mathcal{L} = \mathbb{k}$  is the trivial local system we sometimes write  $\mathbf{IC}_{\lambda}$  instead of  $\mathbf{IC}_{\lambda}^{\mathbb{k}}$ .

Below we will often consider coefficient fields  $\mathbb{k}$  of positive characteristic, where in general (1) does not hold. We will say that the Decomposition Theorem *holds* (resp. *fails*) with  $\mathbb{k}$ -coefficients if an isomorphism of the form (1) holds (resp. fails).

**1.3 Parity Sheaves.** Consider  $f : X \rightarrow Y$ , a proper map between complex algebraic varieties, with  $X$  smooth. Motivated by the considerations that led to the Decomposition Theorem we ask:

**Question 1.1.** *Fix a field of coefficients  $\mathbb{k}$ .*

1. *What can one say about the indecomposable summands of  $f_* \mathbb{k}_X$ ?*

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<sup>7</sup>i.e. those sheaves whose restriction to each  $Y_{\lambda}$  is a local system

2. What about the indecomposable summands of  $f_*\mathcal{L}$ , for  $\mathcal{L}$  a local system of  $\mathbb{k}$ -vector spaces on  $X$ ?

(Recall  $f_*$  means derived direct image.) If  $\mathbb{k}$  is of characteristic zero, then (1) has a beautiful answer: by the Decomposition Theorem, any indecomposable summand is a shift of an IC extension of an irreducible local system. The same is true of (2) if  $\mathcal{L}$  is irreducible. (This is Kashiwara's conjecture, proved by Mochizuki [2011].)

If the characteristic of  $\mathbb{k}$  is positive this question seems difficult. However it has a nice answer (in terms of “parity sheaves”) under restrictions on  $X$ ,  $Y$  and  $f$ .

*Remark 1.2.* It seems unlikely that this question will have a good answer as phrased in general. It is possible that it does have a good answer if one instead works in an appropriate category of motives, perhaps with restrictions on allowable maps  $f$  and local systems  $\mathcal{L}$ .

Assume that  $Y$  admits a stratification  $Y = \bigsqcup_{\lambda \in \Lambda} Y_\lambda$  as above. For  $\lambda \in \Lambda$ , let  $j_\lambda : Y_\lambda \hookrightarrow Y$  denote the inclusion. A complex  $\mathcal{F} \in D_\Lambda^b(Y; \mathbb{k})$  is *even* if

$$(2) \quad \mathcal{H}^i(j_\lambda^* \mathcal{F}) = \mathcal{H}^i(j_\lambda^! \mathcal{F}) = 0 \quad \text{for } i \text{ odd, and all } \lambda \in \Lambda.$$

(Here  $\mathcal{H}^i$  denotes the  $i^{\text{th}}$  cohomology sheaf of a complex of sheaves.) A complex  $\mathcal{F}$  is *odd* if  $\mathcal{F}[1]$  is even; a complex is *parity* if it can be written as a sum  $\mathcal{F}_0 \oplus \mathcal{F}_1$  with  $\mathcal{F}_0$  (resp.  $\mathcal{F}_1$ ) even (resp. odd).

**Example 1.3.** The archetypal example of a parity complex is  $f_*\mathbb{k}_X[\dim_{\mathbb{C}} X]$ , where  $f$  is proper and  $X$  is smooth as above, and  $f$  is in addition *even*:  $f_*\mathbb{k}_X[\dim_{\mathbb{C}} X]$  is  $\Lambda$ -constructible and the cohomology of the fibres of  $f$  with  $\mathbb{k}$ -coefficients vanishes in odd degree. (Indeed, in this case,  $\mathbb{k}_X[\dim_{\mathbb{C}} X]$  is Verdier self-dual, hence so is  $f_*\mathbb{k}_X[\dim_{\mathbb{C}} X]$  (by properness) and the conditions (2) follow from our assumptions on the cohomology of the fibres of  $f$ .)

We make the following (strong) assumptions on each stratum:

- (3)  $Y_\lambda$  is simply connected;  
 (4)  $H^i(Y_\lambda, \mathbb{k}) = 0$  for  $i$  odd.

**Theorem 1.4.** Suppose that  $\mathcal{F}$  is indecomposable and parity:

1. The support of  $\mathcal{F}$  is irreducible, and hence is equal to  $\overline{Y}_\lambda$  for some  $\lambda \in \Lambda$ .
2. The restriction of  $\mathcal{F}$  to  $Y_\lambda$  is isomorphic to a constant sheaf, up to a shift.

Moreover, any two indecomposable parity complexes with equal support are isomorphic, up to a shift.

(The proof of this theorem is not difficult, see Juteau, Mautner, and Williamson [2014, §2.2].) If  $\mathcal{F}$  is an indecomposable parity complex with support  $Y_\lambda$  then there is a unique shift of  $\mathcal{F}$  making it Verdier self-dual. We denote it by  $\mathcal{E}_\lambda^{\mathbb{k}}$  or  $\mathcal{E}_\lambda$  and call it a *parity sheaf*.

*Remark 1.5.* The above theorem is a uniqueness statement. In general, there might be no parity complex with support  $\overline{Y}_\lambda$ . A condition guaranteeing existence of a parity sheaf with support  $\overline{Y}_\lambda$  is that  $\overline{Y}_\lambda$  admit an even resolution. In all settings we consider below parity sheaves exist for all strata, and thus are classified in the same way as IC sheaves.

*Remark 1.6.* In contrast to IC sheaves, parity sheaves are only defined up to non-canonical isomorphism.

Below it will also be important to consider the equivariant setting. We briefly outline the necessary changes. Suppose that a complex algebraic group  $G$  acts on  $Y$  preserving strata. Let  $D_{G,\Lambda}^b(X; \mathbb{k})$  denote the  $\Lambda$ -constructible equivariant derived category [Bernstein and Lunts \[1994\]](#). We have the usual menagerie of functors associated to  $G$ -equivariant maps  $f : X \rightarrow Y$  which commute with the “forget  $G$ -equivariance functor” to  $D_\Lambda^b(X; \mathbb{k})$ . In the equivariant setting the definition of even, odd and parity objects remain unchanged. Also Theorem 1.4 holds, if we require “equivariantly simply connected” in (3) and state (4) with equivariant cohomology.

**1.4 Intersection Forms.** In the previous section we saw that, for any proper even map  $f : X \rightarrow Y$ , the derived direct image  $f_* \mathbb{k}_X$  decomposes into a direct sum of shifts of parity sheaves. In applications it is important to know precisely what form this decomposition takes. It turns out that this is encoded in the ranks of certain intersection forms associated to the strata of  $Y$ , as we now explain.

For each stratum  $Y_\lambda$  and point  $y \in Y_\lambda$  we can choose a normal slice  $N$  to the stratum  $Y_\lambda$  through  $y$ . If we set  $F := f^{-1}(y)$  and  $\tilde{N} := f^{-1}(N)$  then we have a commutative diagram with Cartesian squares:

$$\begin{array}{ccccc} F & \longrightarrow & \tilde{N} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ \{x\} & \longrightarrow & N & \longrightarrow & Y \end{array}$$

Set  $d := \dim_{\mathbb{C}} \tilde{N} = \dim_{\mathbb{C}} N = \operatorname{codim}_{\mathbb{C}}(Y_\lambda \subset X)$ . The inclusion  $F \hookrightarrow \tilde{N}$  equips the integral homology of  $F$  with an intersection form (see [Juteau, Mautner, and Williamson \[2014, §3.1\]](#))

$$IF_\lambda^j : H_{d-j}(F, \mathbb{Z}) \times H_{d+j}(F, \mathbb{Z}) \rightarrow H_0(\tilde{N}, \mathbb{Z}) = \mathbb{Z} \quad \text{for } j \in \mathbb{Z}.$$

*Remark 1.7.* Let us give an intuitive explanation for the intersection form: suppose we wish to pair the classes of submanifolds of real dimension  $d - j$  and  $d + j$  respectively. We regard our manifolds as sitting in  $\tilde{N}$  and move them until they are transverse. Because  $(d - j) + (d + j) = 2d$  (the real dimension of  $\tilde{N}$ ) they will intersect in a finite number of signed points, which we then count to get the result.

*Remark 1.8.* The above intersection form depends only on the stratum  $Y_\lambda$  (up to non-unique isomorphism): given any two points  $y, y' \in Y_\lambda$  and a (homotopy class of) path from  $y$  to  $y'$  we get an isometry between the two intersection forms.



Let us assume that our parity assumptions are in force, and that the homology  $H_*(F, \mathbb{Z})$  is free for all  $\lambda$ . In this case, for any field  $\mathbb{k}$  the intersection form over  $\mathbb{k}$  is obtained via extension of scalars from  $IF_\lambda^j$ . We denote this form by  $IF_\lambda^j \otimes_{\mathbb{Z}} \mathbb{k}$ . The relevance of these forms to the Decomposition Theorem is the following:

**Theorem 1.9** (Juteau, Mautner, and Williamson [ibid., Theorem 3.13]). *The multiplicity of  $\mathcal{E}_\lambda[j]$  as a summand of  $f_*\mathbb{k}_X[\dim_{\mathbb{C}} X]$  is equal to the rank of  $IF_\lambda^j \otimes \mathbb{k}$ . Moreover, the Decomposition Theorem holds if and only if  $IF_\lambda^j \otimes \mathbb{k}$  and  $IF_\lambda^j \otimes \mathbb{Q}$  have the same rank, for all  $\lambda \in \Lambda$  and  $j \in \mathbb{Z}$ .*

**1.5 Examples.** Our goal in this section is to give some examples of intersection cohomology sheaves and parity sheaves. Throughout,  $\mathbb{k}$  denotes our field of coefficients and  $p$  denotes its characteristic. (The strata in some of examples below do not satisfy our parity conditions. In each example this can be remedied by considering equivariant sheaves for an appropriate group action.)

**Example 1.10.** (A nilpotent cone) Consider the singular 2-dimensional quadric cone

$$X = \{(x, y, z) \mid x^2 = -yz\} \subset \mathbb{C}^3.$$

Then  $X$  is isomorphic to the cone of nilpotent matrices inside  $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$ . Let 0 denote the unique singular point of  $X$  and  $X_{\text{reg}} = X \setminus \{0\}$  the smooth locus. Consider the blow-up of  $X$  at 0:

$$f : \widetilde{X} \rightarrow X$$

This is a resolution of singularities which is isomorphic to the Springer resolution under the above isomorphism of  $X$  with the nilpotent cone. It is an isomorphism over  $X_{\text{reg}}$  and has fibre  $\mathbb{P}^1$  over 0. In particular, the stalks of the direct image of the shifted constant sheaf  $f_*\mathbb{k}_{\widetilde{X}}[2]$  are given by:

$$\begin{array}{c|ccc} & -2 & -1 & 0 \\ \hline X_{\text{reg}} & \mathbb{k} & 0 & 0 \\ \{0\} & \mathbb{k} & 0 & \mathbb{k} \end{array}$$

One has an isomorphism of  $\widetilde{X}$  with the total space of the line bundle  $\mathcal{O}(-2)$  on  $\mathbb{P}^1$ . Under such an isomorphism, the zero section corresponds to  $f^{-1}(0)$ . In particular,  $f^{-1}(0)$  has self-intersection  $-2$  inside  $\widetilde{X}$ . It follows that:

$$\begin{aligned} f_*\mathbb{k}_{\widetilde{X}}[2] &\cong \mathbb{k}_X[2] \oplus \mathbb{k}_{\{0\}}, \text{ if } \mathbb{k} \text{ is of characteristic } \neq 2, \\ f_*\mathbb{k}_{\widetilde{X}}[2] &\text{ is indecomposable, if } \mathbb{k} \text{ is of characteristic } 2. \end{aligned}$$

If  $p = 2$ , the complex  $f_*\mathbb{k}_{\widetilde{X}}[2]$  is an archetypal example of a parity sheaf. For further discussion of this example, see Juteau, Mautner, and Williamson [2012, §2.4].

**Example 1.11.** (The first singular Schubert variety) Let  $\mathcal{G}r_2^4$  denote the Grassmannian of 2-planes inside  $\mathbb{C}^4$ . Fix a two-dimensional subspace  $\mathbb{C}^2 \subset \mathbb{C}^4$  and let  $X \subset \mathcal{G}r_2^4$  denote the closed subvariety (a Schubert variety)

$$X = \{V \in \mathcal{G}r_2^4 \mid \dim(V \cap \mathbb{C}^2) \geq 1\}.$$

It is of dimension 3, with unique singular point  $V = \mathbb{C}^2 \in \mathcal{G}r_2^4$ . The space

$$\widetilde{X} = \{V \in \mathcal{G}r_2^4, L \subset \mathbb{C}^2 \mid \dim L = 1, L \subset V \cap \mathbb{C}^2\}$$

is smooth, and the map  $f : \widetilde{X} \rightarrow X$  forgetting  $L$  is a resolution of singularities. This morphism is “small” (i.e. the shifted direct image sheaf  $f_* \mathbb{k}_{\widetilde{X}}[3]$  coincides with the intersection cohomology complex for any field  $\mathbb{k}$ ) and even. Thus in this example the parity sheaf and intersection cohomology sheaf coincide in all characteristics.

**Example 1.12.** (Contraction of the zero section) Suppose that  $Y$  is smooth of dimension  $> 0$  and that  $Y \subset T^*Y$  may be contracted to a point (i.e. there exists a map  $f : T^*Y \rightarrow X$  such that  $f(Y) = \{x\}$  and  $f$  is an isomorphism on the complement of  $Y$ ). In this case  $x$  is the unique singular point of  $X$  and the intersection form at  $x$  is  $-\chi(Y)$ , where  $\chi(Y)$  denotes the Euler characteristic of  $Y$ . If  $Y$  has vanishing odd cohomology then  $f$  is even and  $f_* \mathbb{k}_{T^*Y}$  is parity. The Decomposition Theorem holds if and only if  $p$  does not divide  $\chi(Y)$ .

**Example 1.13.** (A non-perverse parity sheaf) For  $n \geq 1$  consider

$$X = \mathbb{C}^{2n}/(\pm 1) = \operatorname{Spec} \mathbb{C}[x_i x_j \mid 1 \leq i, j \leq 2n].$$

If  $\widetilde{X}$  denotes the total space of  $\mathcal{O}(-2)$  on  $\mathbb{P}^{2n-1}$  then we have a resolution

$$f : \widetilde{X} \rightarrow X.$$

It is an isomorphism over  $X_{\text{reg}} = X \setminus \{0\}$  with fibre  $\mathbb{P}^{2n-1}$  over 0. The intersection form

$$IF_0^j : H_{2n-j}(\mathbb{P}^{2n-1}, \mathbb{Z}) \times H_{2n+j}(\mathbb{P}^{2n-1}, \mathbb{Z}) \rightarrow \mathbb{Z}$$

is non-trivial only for  $j = -2n + 2, -2n + 4, \dots, 2n - 2$  in which case it is the  $1 \times 1$  matrix  $(-2)$ . Thus  $f_* \mathbb{k}_{\widetilde{X}}$  is indecomposable if  $p = 2$ . Otherwise we have

$$f_* \mathbb{k}_{\widetilde{X}}[2n] \cong \mathbb{k}_X[2n] \oplus \mathbb{k}_0[2n - 2] \oplus \mathbb{k}_0[2n - 4] \cdots \oplus \mathbb{k}_0[-2n + 2].$$

Because  $f$  is even,  $f_* \mathbb{k}_{\widetilde{X}}[2n]$  is parity. It is indecomposable (and hence is a parity sheaf) if  $p = 2$ . The interest of this example is that  $f_* \mathbb{k}_{\widetilde{X}}[2n]$  has many non-zero perverse cohomology sheaves. (See [Juteau, Mautner, and Williamson \[2012, §3.3\]](#) for more on this example.)

**Example 1.14.** (The generalised Kashiwara–Saito singularity) Fix  $d \geq 2$  and consider the variety of linear maps

$$\begin{array}{ccc} \mathbb{C}^d & \xrightarrow{A} & \mathbb{C}^d \\ D \downarrow & & \downarrow B \\ \mathbb{C}^d & \xrightarrow{C} & \mathbb{C}^d \end{array} \quad \text{satisfying} \quad \begin{array}{l} BA = CD = 0, \\ \operatorname{rank} \begin{pmatrix} A \\ D \end{pmatrix} \leq 1, \\ \operatorname{rank} \begin{pmatrix} B & C \end{pmatrix} \leq 1. \end{array}$$

This is a singular variety of dimension  $6d - 4$ . Let  $0 = (0, 0, 0, 0)$  denote its most singular point. Consider

$$\widetilde{X} = \left\{ (A, B, C, D, H_1, L_2, L_3, L_4) \left| \begin{array}{l} H_1 \in \mathcal{G}r_{d-1}^d, L_i \in \mathcal{G}r_1^d, \\ A \in \text{Hom}(\mathbb{C}^d/H_1, L_2), B \in \text{Hom}(\mathbb{C}^d/L_2, L_4), \\ C \in \text{Hom}(\mathbb{C}^d/L_3, L_4), D \in \text{Hom}(\mathbb{C}^d/H_1, L_3) \end{array} \right. \right\}$$

(where  $\mathcal{G}r_i^d$  denotes the Grassmannian of  $i$ -planes in  $\mathbb{C}^d$ ). The natural map

$$f : \widetilde{X} \rightarrow X$$

is a resolution. We have  $F = f^{-1}(0) \cong (\mathbb{P}^{d-1})^4$ . The intersection form

$$H_{6d-4}(F) \times H_{6d-4}(F) \rightarrow \mathbb{Z}$$

has elementary divisors  $(1, \dots, 1, d)$ . The Decomposition Theorem holds if and only if  $p \nmid d$ .

The  $d = 2$  case yields an 8-dimensional singularity which Kashiwara and Saito showed is smoothly equivalent to a singularity of a Schubert variety in the flag variety of  $\text{SL}_8$  or a quiver variety of type  $A_5$ . It tends to show up as a minimal counterexample to optimistic hopes in representation theory [Kashiwara and Saito \[1997\]](#), [Leclerc \[2003\]](#), and [Williamson \[2014, 2015\]](#). Polo observed (unpublished) that for any  $d$  the above singularities occur in Schubert varieties for  $\text{SL}_{4d}$ . This shows that the Decomposition Theorem can fail for type  $A$  Schubert varieties for arbitrarily large  $p$ .

## 2 The Hecke category

In this section we introduce the Hecke category, a monoidal category whose Grothendieck group is the Hecke algebra. If one thinks of the Hecke algebra as providing Hecke operators which act on representations or function spaces, then the Hecke category consists of an extra layer of “Hecke operators between Hecke operators”.

**2.1 The Hecke algebra.** Let  $G$  denote a split reductive group over  $\mathbb{F}_q$ , and let  $T \subset B \subset G$  denote a maximal torus and Borel subgroup. For example, we could take  $G = \text{GL}_n$ ,  $B =$  upper triangular matrices and  $T =$  diagonal matrices. The set of  $\mathbb{F}_q$ -points,  $G(\mathbb{F}_q)$ , is a finite group (e.g. for  $G = \text{GL}_n$ ,  $G(\mathbb{F}_q)$  is the group of invertible  $n \times n$ -matrices with coefficients in  $\mathbb{F}_q$ ). Many important finite groups, including “most” simple groups, are close relatives of groups of this form.

A basic object in the representation theory of the finite group  $G(\mathbb{F}_q)$  is the Hecke algebra

$$H_{\mathbb{F}_q} := \text{Fun}_{B(\mathbb{F}_q) \times B(\mathbb{F}_q)}(G(\mathbb{F}_q), \mathbb{C})$$

of complex valued functions on  $G(\mathbb{F}_q)$ , invariant under left and right multiplication by  $B(\mathbb{F}_q)$ . This is an algebra under convolution:

$$(f * f')(g) := \frac{1}{|B(\mathbb{F}_q)|} \sum_{h \in G(\mathbb{F}_q)} f(gh^{-1})f'(h).$$

*Remark 2.1.* Instead we could replace  $G(\mathbb{F}_q)$  by  $G(\mathbb{K})$  and  $B(\mathbb{F}_q)$  by an Iwahori subgroup of  $G(\mathbb{K})$  (for a local field  $\mathbb{K}$  with finite residue field), and obtain the affine Hecke algebra (important in the representation theory of  $p$ -adic groups).

Let  $W$  denote the Weyl group,  $S$  its simple reflections,  $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$  the length function of  $W$  with respect to  $S$  and  $\leq$  the Bruhat order. The Bruhat decomposition

$$G(\mathbb{F}_q) = \bigsqcup_{w \in W} B(\mathbb{F}_q) \cdot wB(\mathbb{F}_q)$$

shows that  $H_{\mathbb{F}_q}$  has a basis given by indicator functions  $t_w$  of the subsets  $B(\mathbb{F}_q) \cdot wB(\mathbb{F}_q)$ , for  $w \in W$ .

Iwahori [1964] showed that  $H_{\mathbb{F}_q}$  may also be described as the unital algebra generated by  $t_s$  for  $s \in S$  subject to the relations

$$\begin{aligned} t_s^2 &= (q - 1)t_s + q, \\ \underbrace{t_s t_u \dots}_{m_{su} \text{ factors}} &= \underbrace{t_u t_s \dots}_{m_{su} \text{ factors}} \end{aligned}$$

where  $u \neq s$  in the second relation and  $m_{su}$  denotes the order of  $su$  in  $W$ . These relations depend on  $q$  in a uniform way and make sense for any Coxeter group. Thus it makes sense to use these generators and relations to define a new algebra  $H$  over  $\mathbb{Z}[q^{\pm 1}]$  ( $q$  is now a formal variable); thus  $H$  specialises to the Hecke algebra defined above via  $q \mapsto |\mathbb{F}_q|$ .

For technical reasons it is useful to adjoin a square root of  $q$  and regard  $H$  as defined over  $\mathbb{Z}[q^{\pm 1/2}]$ . We then set  $v := q^{-1/2}$  and  $\delta_s := vt_s$ , so that the defining relations of  $H$  become

$$\begin{aligned} \delta_s^2 &= (v^{-1} - v)\delta_s + 1, \\ \underbrace{\delta_s \delta_u \dots}_{m_{su} \text{ factors}} &= \underbrace{\delta_u \delta_s \dots}_{m_{su} \text{ factors}}. \end{aligned}$$

For any reduced expression  $w = st \dots u$  (i.e. any expression for  $w$  using  $\ell(w)$  simple reflections) we set  $\delta_w := \delta_s \delta_t \dots \delta_u$ . We obtain in this way a well-defined  $\mathbb{Z}[v^{\pm 1}]$ -basis  $\{\delta_x \mid x \in W\}$  for  $H$ , the *standard basis*. (This basis specialises via  $q \mapsto |\mathbb{F}_q|$  to the indicator functions  $t_w$  considered above, up to a power of  $v$ .)

There is an involution  $d : H \rightarrow H$  defined via

$$v \mapsto v^{-1} \quad \text{and} \quad \delta_s \mapsto \delta_s^{-1} = \delta_s + (v - v^{-1}).$$

Kazhdan and Lusztig [1979] (see Soergel [1997] for a simple proof) showed that for all  $x \in W$  there exists a unique element  $b_x$  satisfying

$$\begin{aligned} \text{("self-duality")} \quad & d(b_x) = b_x, \\ \text{("degree bound")} \quad & b_x \in \delta_x + \sum_{y < x} v\mathbb{Z}[v]\delta_y \end{aligned}$$

where  $\leq$  is Bruhat order. For example  $b_s = \delta_s + v$ . The set  $\{b_x \mid x \in W\}$  is the *Kazhdan–Lusztig basis* of  $H$ . The polynomials  $h_{y,x} \in \mathbb{Z}[v]$  defined via  $b_x = \sum h_{y,x}\delta_y$  are (normalisations of) *Kazhdan–Lusztig polynomials*.

**2.2 The Hecke category: geometric incarnation.** Grothendieck’s function-sheaf correspondence (see e.g. [Laumon \[1987, §1\]](#)) tells us how we should categorify the Hecke algebra  $H_{\mathbb{F}_q}$ . Namely, we should consider an appropriate category of  $B \times B$ -equivariant sheaves on  $G$ , with the passage to  $H_{\mathbb{F}_q}$  being given by the trace of Frobenius at the rational points  $G(\mathbb{F}_q)$ .<sup>8</sup> Below we will use the fact that the multiplication action of  $B$  on  $G$  is free, and so instead we can consider  $B$ -invariant functions (resp.  $B$ -equivariant sheaves) on  $G/B$ .

To avoid technical complications, and to ease subsequent discussion, we will change setting slightly. Let us fix a generalised Cartan matrix  $C = (c_{st})_{s,t \in S}$  and let  $(\mathfrak{h}_{\mathbb{Z}}, \{\alpha_s\}_{s \in S}, \{\alpha_s^\vee\}_{s \in S})$  be a Kac–Moody root datum, so that  $\mathfrak{h}_{\mathbb{Z}}$  is a free and finitely generated  $\mathbb{Z}$ -module,  $\alpha_s \in \text{Hom}(\mathfrak{h}_{\mathbb{Z}}, \mathbb{Z})$  are “roots” and  $\alpha_s^\vee \in \mathfrak{h}_{\mathbb{Z}}$  are “coroots” such that  $\langle \alpha_s^\vee, \alpha_t \rangle = c_{st}$ . To this data we may associate a Kac–Moody group  $\mathcal{G}$  (a group ind-scheme over  $\mathbb{C}$ ) together with a canonical Borel subgroup  $\mathcal{B}$  and maximal torus  $\mathcal{T}$ . The reader is welcome to take  $\mathcal{G}$  to be a complex reductive group, as per the following remark. (For applications to representation theory the case of an affine Kac–Moody group is important.)

*Remark 2.2.* If  $\mathcal{G}$  is a complex reductive group and  $\mathcal{T} \subset \mathcal{B} \subset \mathcal{G}$  is a maximal torus and Borel subgroup, then we can consider the corresponding root datum  $(\mathcal{X}, R, \mathcal{X}^\vee, R^\vee)$  (where  $\mathcal{X}$  denotes the characters of  $\mathcal{T}$ ,  $R$  the roots etc.). If  $\{\alpha_s\}_{s \in S} \subset R$  denotes the simple roots determined by  $\mathcal{B}$  then  $(\mathcal{X}^\vee, \{\alpha_s\}, \{\alpha_s^\vee\})$  is a Kac–Moody root datum. The corresponding Kac–Moody group (resp. Borel subgroup and maximal torus) is canonically isomorphic to  $\mathcal{G}$  (resp.  $\mathcal{B}$ ,  $\mathcal{T}$ ).

We denote by  $\mathcal{G}/\mathcal{B}$  the flag variety (a projective variety in the case of a reductive group, and an ind-projective variety in general). As earlier, we denote by  $W$  the Weyl group,  $\ell$  the length function and  $\leq$  the Bruhat order. We have the Bruhat decomposition

$$\mathcal{G}/\mathcal{B} = \bigsqcup_{w \in W} X_w \quad \text{where} \quad X_w := \mathcal{B} \cdot w\mathcal{B}/\mathcal{B}.$$

The  $X_w$  are isomorphic to affine spaces, and are called *Schubert cells*. Their closures  $\overline{X}_w \subset \mathcal{G}/\mathcal{B}$  are projective (and usually singular), and are called *Schubert varieties*.

Fix a field  $\mathbb{k}$  and consider  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$ , the bounded equivariant derived category with coefficients in  $\mathbb{k}$  (see e.g. [Bernstein and Lunts \[1994\]](#)).<sup>9</sup> This is a monoidal category under convolution: given two complexes  $\mathcal{F}, \mathcal{G} \in D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  their convolution is

$$\mathcal{F} * \mathcal{G} := \text{mult}_*(\mathcal{F} \boxtimes_{\mathcal{B}} \mathcal{G}),$$

where:  $\mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B}$  denotes the quotient of  $\mathcal{G} \times \mathcal{G}/\mathcal{B}$  by  $(gb, g'\mathcal{B}) \sim (g, bg'\mathcal{B})$  for all  $g, g' \in \mathcal{G}$  and  $b \in \mathcal{B}$ ;  $\text{mult} : \mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B} \rightarrow \mathcal{G}/\mathcal{B}$  is induced by the multiplication on  $\mathcal{G}$ ; and  $\mathcal{F} \boxtimes_{\mathcal{B}} \mathcal{G} \in D_{\mathcal{B}}^b(\mathcal{G} \times_{\mathcal{B}} \mathcal{G}/\mathcal{B}; \mathbb{k})$  is obtained via descent from  $\mathcal{F} \boxtimes \mathcal{G} \in D_{\mathcal{B}^3}^b(\mathcal{G} \times \mathcal{G}/\mathcal{B}; \mathbb{k})$ .<sup>10</sup> (Note that  $\text{mult}$  is proper, and so  $\text{mult}_* = \text{mult}_!$ .)

<sup>8</sup>As is always the case with Grothendieck’s function-sheaf correspondence, this actually categorifies the Hecke algebras of  $G(\mathbb{F}_{q^m})$  for “all  $m$  at once”.

<sup>9</sup>By definition, any object of  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  is supported on finitely many Schubert cells, and hence has finite-dimensional support.

<sup>10</sup>The reader is referred to [Springer \[1982\]](#) and [Nadler \[2005\]](#) for more detail on this construction.

*Remark 2.3.* If  $\mathcal{G}$  is a reductive group and we work over  $\mathbb{F}_q$  instead of  $\mathbb{C}$ , then this definition categorifies convolution in the Hecke algebra, via the function-sheaf correspondence.

For any  $s \in S$  we can consider the parabolic subgroup

$$\mathcal{P}_s := \overline{\mathcal{B}s\mathcal{B}} = \mathcal{B}s\mathcal{B} \sqcup \mathcal{B} \subset \mathcal{G}.$$

We define the Hecke category (in its geometric incarnation) as follows

$$\mathcal{H}_{\text{geom}}^{\mathbb{k}} := \langle \mathbb{k}_{\mathcal{P}_s/\mathcal{B}} \mid s \in S \rangle_{*, \oplus, [1], \text{Kar}}.$$

That is, we consider the full subcategory of  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  generated by  $\mathbb{k}_{\mathcal{P}_s/\mathcal{B}}$  under convolution  $(*)$ , direct sums  $(\oplus)$ , homological shifts  $([1])$  and direct summands (Kar, for “Karoubi”).

*Remark 2.4.* If we were to work over  $\mathbb{F}_q$ , then  $\mathbb{k}_{\mathcal{P}_s/\mathcal{B}}$  categorifies the indicator function of  $\mathcal{P}_s(\mathbb{F}_q) \subset \mathcal{G}(\mathbb{F}_q)$ . The definition of the Hecke category is imitating the fact that the Hecke algebra is generated by these indicator functions under convolution (as is clear from Iwahori’s presentation).

Let  $[\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$  denote the split Grothendieck group<sup>11</sup> of  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$ . Because  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  is a monoidal category,  $[\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$  is an algebra via  $[\mathcal{F}] \cdot [\mathcal{G}] = [\mathcal{F} * \mathcal{G}]$ . We view  $[\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$  as a  $\mathbb{Z}[v^{\pm 1}]$ -algebra via  $v \cdot [\mathcal{F}] := [\mathcal{F}[1]]$ . Recall the Kazhdan–Lusztig basis element  $b_s = \delta_s + v$  for all  $s \in S$  from earlier. The following theorem explains the name “Hecke category” and is fundamental to all that follows:

**Theorem 2.5.** *The assignment  $b_s \mapsto [\mathbb{k}_{\mathcal{P}_s/\mathcal{B}}[1]]$  for all  $s \in S$  yields an isomorphism of  $\mathbb{Z}[v^{\pm 1}]$ -algebras:*

$$H \xrightarrow{\sim} [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}.$$

(This theorem is easily proved using the theory of parity sheaves, as will be discussed in the next section.) The inverse to the isomorphism in the theorem is given by the *character map*

$$\text{ch} : [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus} \xrightarrow{\sim} H \quad \mathcal{F} \mapsto \sum_{x \in W} \dim_{\mathbb{Z}}(H^*(\mathcal{F}_{x\mathcal{B}/\mathcal{B}}))v^{-\ell(x)}\delta_x$$

where:  $\mathcal{F}_{x\mathcal{B}/\mathcal{B}}$  denotes the stalk of the constructible sheaf on  $\mathcal{G}/\mathcal{B}$  at the point  $x\mathcal{B}/\mathcal{B}$  obtained from  $\mathcal{F}$  by forgetting  $\mathcal{B}$ -equivariance;  $H^*$  denotes cohomology; and

$$\dim_{\mathbb{Z}} H^* := \sum (\dim H^i)v^{-i} \in \mathbb{Z}[v^{\pm 1}]$$

denotes graded dimension.

<sup>11</sup>The split Grothendieck group  $[\mathcal{Q}]_{\oplus}$  of an additive category is the abelian group generated by symbols  $[A]$  for all  $A \in \mathcal{Q}$ , modulo the relations  $[A] = [A'] + [A'']$  whenever  $A \cong A' \oplus A''$ .

**2.3 Role of the Decomposition Theorem.** The category  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$ , and hence also the Hecke category  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$ , is an example of a Krull–Schmidt category: every object admits a decomposition into indecomposable objects; and an object is indecomposable if and only if its endomorphism ring is local.

Recall that the objects of  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  are the direct summands of finite direct sums of shifts of objects of the form

$$\mathcal{E}_{(s,t,\dots,u)} := \mathbb{k}_{\mathcal{P}_s/\mathcal{B}} * \mathbb{k}_{\mathcal{P}_t/\mathcal{B}} * \cdots * \mathbb{k}_{\mathcal{P}_u/\mathcal{B}} \in D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B})$$

for any word  $(s, t, \dots, u)$  in  $S$ . The Krull–Schmidt property implies that any indecomposable object is isomorphic to a direct summand of a single  $\mathcal{E}_{(s,t,\dots,u)}$ . Thus in order to understand the objects of  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  it is enough to understand the summands of  $\mathcal{E}_{(s,t,\dots,u)}$ , for any word as above.

For any such word  $(s, t, \dots, u)$  we can consider a Bott–Samelson space

$$BS_{(s,t,\dots,u)} := \mathcal{P}_s \times_{\mathcal{B}} \mathcal{P}_t \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \mathcal{P}_u / \mathcal{B}$$

and the (projective) morphism  $m : BS_{(s,t,\dots,u)} \rightarrow \mathcal{G}/\mathcal{B}$  induced by multiplication. A straightforward argument (using the proper base change theorem) shows that we have a canonical isomorphism

$$\mathcal{E}_{(s,t,\dots,u)} = m_* \mathbb{k}_{BS_{(s,t,\dots,u)}}.$$

The upshot: in order to understand the indecomposable objects in  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  it is enough to decompose the complexes  $m_* \mathbb{k}_{BS}$ , for all expressions  $(s, t, \dots, u)$  in  $S$ .

*Remark 2.6.* If  $(s, t, \dots, u)$  is a reduced expression for  $w \in W$ , then the map  $m$  provides a resolution of singularities of the Schubert variety  $\overline{X}_w$ . These resolutions are often called Bott–Samelson resolutions, which explains our notation.

If the characteristic of our field is zero then we can appeal to the Decomposition Theorem to deduce that all indecomposable summands of  $m_* \mathbb{k}_{BS}$  are shifts of the intersection cohomology complexes of Schubert varieties. Thus

$$(5) \quad \mathcal{H}_{\text{geom}}^{\mathbb{k}} = \langle \mathbf{IC}_x \mid x \in W \rangle_{\oplus, [1]} \quad \text{if } \mathbb{k} \text{ is of characteristic } 0$$

where  $\mathbf{IC}_x$  denotes the ( $\mathcal{B}$ -equivariant) intersection cohomology sheaf of the Schubert variety  $\overline{X}_x$ . It is also not difficult (see e.g. [Springer \[1982\]](#)) to use (5) to deduce that<sup>12</sup>

$$(6) \quad \text{ch}(\mathbf{IC}_x) = b_x \quad \text{if } \mathbb{k} \text{ is of characteristic } 0.$$

Thus, when the coefficients are of characteristic zero, the intersection cohomology sheaves categorify the Kazhdan–Lusztig basis.

It is known that Bott–Samelson resolutions are even. In particular,  $m_* \mathbb{k}_{BS}$  is a parity complex. Thus, for arbitrary  $\mathbb{k}$  we can appeal to Theorem 1.4 to deduce that all indecomposable summands of  $f_* \mathbb{k}_{BS}$  are shifts of parity sheaves. Thus

$$\mathcal{H}_{\text{geom}}^{\mathbb{k}} = \langle \mathcal{E}_x \mid x \in W \rangle_{\oplus, [1]} \quad \text{for } \mathbb{k} \text{ arbitrary}$$

where  $\mathcal{E}_x$  denotes the ( $\mathcal{B}$ -equivariant) parity sheaf of the Schubert variety  $\overline{X}_x$ .

<sup>12</sup>Roughly speaking, the two conditions (“self-duality” + “degree bound”) characterising the Kazhdan–Lusztig basis mirror the two conditions (“self-duality” + “stalk vanishing”) characterising the IC sheaf.

*Remark 2.7.* Recall that for any map there are only finitely many characteristics in which the Decomposition Theorem fails. Thus, for a fixed  $x$  there will be only finitely many characteristics in which  $\mathfrak{E}_x^{\mathbb{k}} \neq \mathbf{IC}_x^{\mathbb{k}}$ .

**2.4 The Hecke category: generators and relations.** The above geometric definition of the Hecke category is analogous to the original definition of the Hecke algebra as an algebra of  $B$ -biinvariant functions. Now we discuss a description of the Hecke category via generators and relations; this description is analogous to the Iwahori presentation of the Hecke algebra.<sup>13</sup> This description is due to Elias and the author [Elias and Williamson \[2016\]](#), building on work of [Elias and Khovanov \[2010\]](#) and [Elias \[2016\]](#).

*Remark 2.8.* In this section it will be important to keep in mind that monoidal categories are fundamentally two dimensional. While group presentations (and more generally presentations of categories) occur “on a line”, presentations of monoidal categories (and more generally 2-categories) occur “in the plane”. For background on these ideas the reader is referred to e.g. [Street \[1996\]](#) or [Lauda \[2010, §4\]](#).

Recall our generalised Cartan matrix  $C$ , Coxeter system  $(W, S)$  and Kac–Moody root datum from earlier. Given  $s, t \in S$  we denote by  $m_{st}$  the order (possibly  $\infty$ ) of  $st \in W$ . We assume:

$$(7) \quad C \text{ is simply laced, i.e. } m_{st} \in \{2, 3\} \text{ for } s \neq t.$$

(We impose this assumption only to shorten the list of relations below. For the general case the reader is referred to [Elias and Williamson \[2016\]](#).) Recall our “roots” and “coroots”

$$\{\alpha_s\}_{s \in S} \subset \mathfrak{h}_{\mathbb{Z}}^* \quad \text{and} \quad \{\alpha_s^\vee\}_{s \in S} \subset \mathfrak{h}_{\mathbb{Z}}$$

such that  $\langle \alpha_s^\vee, \alpha_t \rangle = c_{st}$  for all  $s, t \in S$ . The formula  $s(v) = v - \langle v, \alpha_s^\vee \rangle \alpha_s$  defines an action of  $W$  on  $\mathfrak{h}_{\mathbb{Z}}^*$ . We also assume that our root datum satisfies that  $\alpha_s : \mathfrak{h}_{\mathbb{Z}} \rightarrow \mathbb{Z}$  and  $\alpha_s^\vee : \mathfrak{h}_{\mathbb{Z}}^* \rightarrow \mathbb{Z}$  are surjective, for all  $s \in S$ . (This condition is called “Demazure surjectivity” in [Elias and Williamson \[ibid.\]](#). We can always find a Kac–Moody root datum satisfying this constraint.)

We denote by  $R = S(\mathfrak{h}_{\mathbb{Z}}^*)$  the symmetric algebra of  $\mathfrak{h}_{\mathbb{Z}}^*$  over  $\mathbb{Z}$ . We view  $R$  as a graded  $\mathbb{Z}$ -algebra with  $\deg \mathfrak{h}_{\mathbb{Z}}^* = 2$ ;  $W$  acts on  $R$  via graded automorphisms. For any  $s \in S$  we define the *Demazure operator*  $\partial_s : R \rightarrow R[-2]$  by

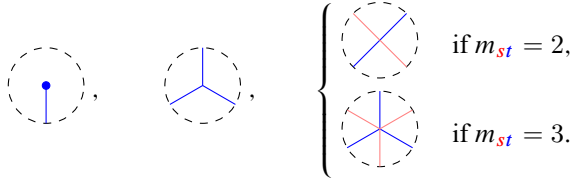
$$(8) \quad \partial_s(f) = \frac{f - sf}{\alpha_s}.$$

An  $S$ -graph is a finite, decorated graph, properly embedded in the planar strip  $\mathbb{R} \times [0, 1]$ , with edges coloured by  $S$ . The vertices of an  $S$ -graph are required to be of the

<sup>13</sup>The Iwahori presentation can be given on two lines. Unfortunately all current presentations of the Hecke category need more than two pages!

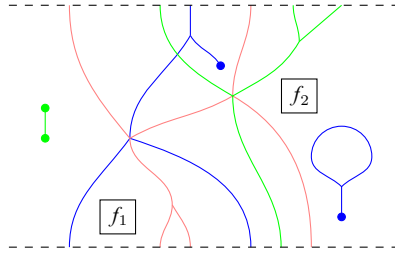


form:



The regions (i.e. connected components of the complement of our  $S$ -graph in  $\mathbb{R} \times [0, 1]$ ) may be decorated by boxes containing homogeneous elements of  $R$ .

**Example 2.9.** An  $S$ -graph (with  $m_{s,t} = 3$ ,  $m_{s,u} = 2$ ,  $m_{t,u} = 3$ , the  $f_i \in R$  are homogeneous polynomials):



The *degree* of an  $S$ -graph is the sum over the degrees of its vertices and boxes, where each box has degree equal to the degree of the corresponding element of  $R$ , and the vertices have degrees given by the following rule: univalent vertices have degree 1, trivalent vertices have degree  $-1$  and  $2m_{st}$ -valent vertices have degree 0. The boundary points of any  $S$ -graph on  $\mathbb{R} \times \{0\}$  and on  $\mathbb{R} \times \{1\}$  give<sup>14</sup> two words in  $S$ , called the *bottom boundary* and *top boundary*.

**Example 2.10.** The  $S$ -graph above has degree  $0 + \deg f_1 + \deg f_2$ . Its bottom boundary is  $(s, t, t, s, u, s)$  and its top boundary is  $(t, u, s, t, u, u)$ .

We are now ready to define a second incarnation of the Hecke category, which we will denote  $\mathcal{H}_{\text{diag}}$ . By definition,  $\mathcal{H}_{\text{diag}}$  is monoidally generated by objects  $B_s$ , for each  $s \in S$ . Thus the objects of  $\mathcal{H}_{\text{diag}}$  are of the form

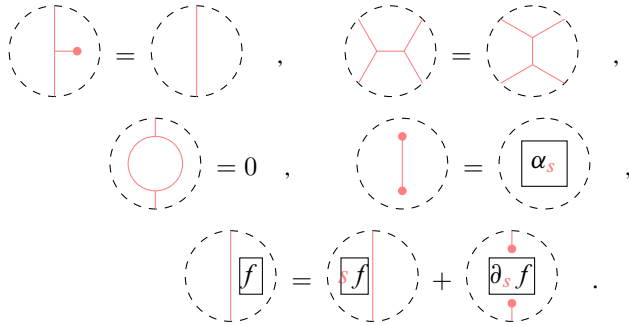
$$B_{(s,t,\dots,u)} := B_s B_t \dots B_u$$

for some word  $(s, t, \dots, u)$  in  $S$ . (We denote the monoidal structure in  $\mathcal{H}_{\text{diag}}$  by concatenation.) Thus  $\mathbb{1} := B_{\emptyset}$  is the monoidal unit. For any two words  $(s, t, \dots, u)$  and  $(s', t', \dots, v')$  in  $S$ ,  $\text{Hom}_{\mathcal{H}_{\text{diag}}}(B_{(s,t,\dots,u)}, B_{(s',t',\dots,v')})$  is defined to be the free  $\mathbb{Z}$ -module generated by isotopy classes<sup>15</sup> of  $S$ -graphs with bottom boundary  $(s, t, \dots, u)$  and top boundary  $(s', t', \dots, v')$ , modulo the local relations below. Composition (resp. monoidal product) of morphisms is induced by vertical (resp. horizontal) concatenation of diagrams.

<sup>14</sup>we read left to right

<sup>15</sup>i.e. two  $S$ -graphs are regarded as the same if one may be obtained from the other by an isotopy of  $\mathbb{R} \times [0, 1]$  which preserves  $\mathbb{R} \times \{0\}$  and  $\mathbb{R} \times \{1\}$

The one colour relations are as follows (see (8) for the definition of  $\partial_s$ ):

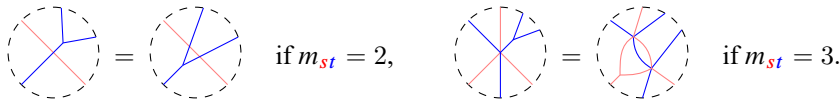


Diagrammatic relations for one colour:

- $\text{A circle with a vertical red line and a red dot on the left} = \text{A circle with a vertical red line}$
- $\text{A circle with three red lines meeting at a central point} = \text{A circle with three red lines meeting at a central point (rotated)}$
- $\text{A circle with a red loop} = 0$
- $\text{A circle with a vertical red line and red dots at both ends} = \boxed{\alpha_s}$
- $\boxed{f} = \boxed{s f} + \boxed{\partial_s f}$

*Remark 2.11.* The first two relations above imply that  $B_s$  is a Frobenius object in  $\mathcal{H}_{\text{diag}}$ , for all  $s \in S$ .

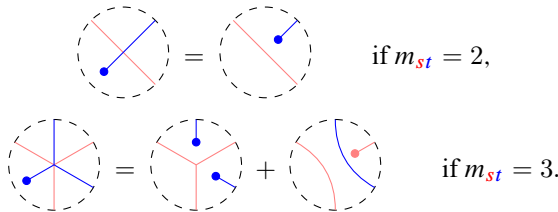
There are two relations involving two colours. The first is a kind of “associativity” (see Elias [2016, (6.12)]):



Associativity relations:

- $\text{A circle with three lines (two blue, one red) meeting at a point} = \text{A circle with three lines (two blue, one red) meeting at a point (rotated)}$  if  $m_{st} = 2$ ,
- $\text{A circle with four lines (two blue, two red) meeting at a point} = \text{A circle with four lines (two blue, two red) meeting at a point (rotated)}$  if  $m_{st} = 3$ .

The second is Elias’ “Jones–Wenzl relation” (see Elias [ibid.]):



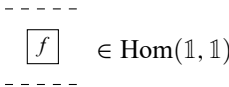
Jones–Wenzl relation:

- $\text{A circle with two crossing lines (one blue, one red) and a blue dot} = \text{A circle with two crossing lines (one blue, one red) and a blue dot (rotated)}$  if  $m_{st} = 2$ ,
- $\text{A circle with three lines (two blue, one red) meeting at a point and a blue dot} = \text{A circle with three lines (two blue, one red) meeting at a point and a blue dot (rotated)} + \text{A circle with three lines (two blue, one red) meeting at a point and a red dot}$  if  $m_{st} = 3$ .

Finally, for each finite standard parabolic subgroup of rank 3 there is a 3-colour “Zamolodchikov relation”, which we don’t draw here (see Elias and Williamson [2016]). This concludes the definition of  $\mathcal{H}_{\text{diag}}$ . (We remind the reader that if we drop the assumption that  $C$  is simply laced there are more complicated relations, see Elias [2016] and Elias and Williamson [2016].)

*Remark 2.12.* Another way of phrasing the above definition is that  $\mathcal{H}_{\text{diag}}$  is the monoidal category with:

- 1. generating objects  $B_s$  for all  $s \in S$ ;
- 2. generating morphisms



$\boxed{f} \in \text{Hom}(\mathbb{1}, \mathbb{1})$

for homogeneous  $f \in R$  (recall  $\mathbb{1}$  denotes the monoidal unit), as well as

$$\begin{array}{cc} \begin{array}{c} \text{---} \text{---} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \in \text{Hom}(B_s, \mathbb{1}), & \begin{array}{c} \text{---} \text{---} \text{---} \\ | \\ \bullet \\ | \\ \text{---} \end{array} \in \text{Hom}(\mathbb{1}, B_s), \\ \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \in \text{Hom}(B_s B_s, B_s), & \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \in \text{Hom}(B_s, B_s B_s) \end{array}$$

for all  $s \in S$  and

$$\begin{array}{cc} \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} \in \text{Hom}(B_s B_t, B_t B_s), & \begin{array}{c} \text{---} \text{---} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} \in \text{Hom}(B_s B_t B_s, B_t B_s B_t), \\ \text{(if } m_{st} = 2\text{)} & \text{(if } m_{st} = 3\text{)} \end{array}$$

for all pairs  $s, t \in S$ ,

subject to the above relations (and additional relations encoding isotopy invariance).

*Remark 2.13.* The above relations are complicated, and perhaps a more efficient presentation is possible. The following is perhaps psychologically helpful. Recall that a *standard parabolic subgroup* is a subgroup of  $W$  generated by a subset  $I \subset S$ , and its *rank* is  $|I|$ . In Iwahori's presentation one has:

$$\begin{array}{l} \text{generators} \leftrightarrow \text{rank } 1, \\ \text{relations} \leftrightarrow \text{ranks } 1, 2. \end{array}$$

In  $\mathcal{H}_{\text{diag}}$  one has:

$$\begin{array}{l} \text{generating objects} \leftrightarrow \text{rank } 1, \\ \text{generating morphisms} \leftrightarrow \text{ranks } 1, 2, \\ \text{relations} \leftrightarrow \text{ranks } 1, 2, 3. \end{array}$$

(More precisely, it is only the *finite* standard parabolic subgroups which contribute at each step.)

All relations defining  $\mathcal{H}_{\text{diag}}$  are homogeneous for the grading on  $S$ -graphs defined above. Thus  $\mathcal{H}_{\text{diag}}$  is enriched in graded  $\mathbb{Z}$ -modules. We denote by  $\mathcal{H}_{\text{diag}}^{\oplus, [1]}$  the additive, graded envelope<sup>16</sup> of  $\mathcal{H}_{\text{diag}}$ . Thus  $\mathcal{H}_{\text{diag}}^{\oplus, [1]}$  is an additive category equipped with a “shift of grading” equivalence [1], and an isomorphism of graded abelian groups

$$\text{Hom}_{\mathcal{H}_{\text{diag}}}(B, B') = \bigoplus_{m \in \mathbb{Z}} \text{Hom}_{\mathcal{H}_{\text{diag}}^{\oplus, [1]}}(B, B'[m]).$$

For any field  $\mathbb{k}$ , we define

$$\mathcal{H}_{\text{diag}}^{\mathbb{k}, \text{Kar}} := (\mathcal{H}_{\text{diag}}^{\oplus, [1]} \otimes_{\mathbb{Z}} \mathbb{k})^{\text{Kar}}$$

where  $(-)^{\text{Kar}}$  denotes Karoubi envelope. In other words,  $\mathcal{H}_{\text{diag}}^{\mathbb{k}, \text{Kar}}$  is obtained as the additive Karoubi envelope of the extension of scalars of  $\mathcal{H}_{\text{diag}}$  to  $\mathbb{k}$ . As for  $\mathcal{H}_{\text{geom}}$ , let us

<sup>16</sup>Objects are formal sums  $F_1[m_1] \oplus F_2[m_2] \oplus \cdots \oplus F_n[m_n]$  where  $F_i$  are objects of  $\mathcal{H}_{\text{diag}}$  and  $m_i \in \mathbb{Z}$ ; and morphisms are matrices, determined by the rule that  $\text{Hom}(F[m], F'[m'])$  is the degree  $m' - m$  part of  $\text{Hom}_{\mathcal{H}_{\text{diag}}}(F, F')$ .

consider the split Grothendieck group  $[\mathcal{H}_{\text{diag}}^{\text{lk}, \text{Kar}}]_{\oplus}$  of  $\mathcal{H}_{\text{diag}}^{\text{lk}, \text{Kar}}$ , which we view as a  $\mathbb{Z}[v^{\pm 1}]$ -algebra in the same way as for  $\mathcal{H}_{\text{geom}}^{\text{lk}}$  earlier. The following is the analogue of Theorem 2.5 in this setting:

**Theorem 2.14** (Elias and Williamson [2016]). *The map  $b_s \mapsto [B_s]$  for all  $s \in S$  induces an isomorphism of  $\mathbb{Z}[v^{\pm 1}]$ -algebras:*

$$H \xrightarrow{\sim} [\mathcal{H}_{\text{diag}}^{\text{lk}, \text{Kar}}]_{\oplus}.$$

The proof is rather complicated diagrammatic algebra, and involves first producing a basis of morphisms between the objects of  $\mathcal{H}_{\text{diag}}$ , in terms of light leaf morphisms Libedinsky [2008]. The following theorem shows that  $\mathcal{H}_{\text{diag}}$  does indeed give a “generators and relations description” of the Hecke category:

**Theorem 2.15** (S. Riche and Williamson [2015, Theorem 10.3.1]). *We have an equivalence of graded monoidal categories:*

$$\mathcal{H}_{\text{diag}}^{\text{lk}, \text{Kar}} \xrightarrow{\sim} \mathcal{H}_{\text{geom}}^{\text{lk}}.$$

*Remark 2.16.* Knowing a presentation of a group or algebra by generators and relations opens the possibility of defining representations by specifying the action of generators and verifying relations. Similarly, studying actions of monoidal categories is sometimes easier when one has a presentation. In principle, the above presentation should allow a detailed study of categories acted on by the Hecke category. For interesting recent classification results, see Mazorchuk and Miemietz [2016] and Mackaay and Tubbenhauer [2016]. One drawback of the theory in its current state is that the above relations (though explicit) can be difficult to check in examples. The parallel theory of representations of categorified quantum groups is much better developed (see e.g. Chuang and Rouquier [2008] and Brundan [2016]).

*Remark 2.17.* An important historical antecedent to Theorems 2.14 and 2.15 is the theory of Soergel bimodules. We have chosen not to discuss this topic, as there is already a substantial literature on this subject. The above generators and relations were first written down in the context of Soergel bimodules, and Soergel bimodules are used in the proof of Theorem 2.15. We refer the interested reader to the surveys Riche [2017] and Libedinsky [2017] or the papers Soergel [1990a, 1992, 2007].

**2.5 The spherical and anti-spherical module.** In this section we introduce the spherical and anti-spherical modules for the Hecke algebra, as well as their categorifications. They are useful for (at least) two reasons: they are ubiquitous in applications to representation theory; and they often provide smaller worlds in which interesting phenomena become more tractable.

Throughout this section we fix a subset  $I \subset S$  and assume for simplicity that the standard parabolic subgroup  $W_I$  generated by  $I$  is finite. We denote by  $w_I$  its longest element. Let  $H_I$  denote the parabolic subalgebra of  $H$  generated by  $\delta_s$  for  $s \in I$ ; it is canonically isomorphic to the Hecke algebra of  $W_I$ . Consider the induced modules

$$M_I := H \otimes_{H_I} \text{triv}_v \quad \text{and} \quad N_I := H \otimes_{H_I} \text{sgn}_v$$

where  $\text{triv}_v$  (resp.  $\text{sgn}_v$ ) is the rank one  $H_I$ -module with action given by  $\delta_s \mapsto v^{-1}$  (resp.  $\delta_s \mapsto -v$ ). These modules are the *spherical* and *anti-spherical* modules respectively. If  $W^I$  denotes the set of minimal length representatives for the cosets  $W/W_I$  then  $\{\delta_x \otimes 1 \mid x \in W^I\}$  gives a (*standard*) basis for  $M_I$  (resp.  $N_I$ ), which we denote by  $\{\mu_x \mid x \in W^I\}$  (resp.  $\{\nu_x \mid x \in W^I\}$ ). We denote the canonical bases in  $M_I$  (resp.  $N_I$ ) by  $\{c_x \mid x \in W^I\}$  (resp.  $\{d_x \mid x \in W^I\}$ ) (see e.g. [Soergel \[1997\]](#)).

We now describe a categorification of  $M_I$ . To  $I$  is associated a standard parabolic subgroup  $\mathcal{P}_I \subset \mathcal{G}$ , and we may consider the partial flag variety  $\mathcal{G}/\mathcal{P}_I$  (an ind-variety) and its Bruhat decomposition

$$\mathcal{G}/\mathcal{P}_I = \bigsqcup_{x \in W^I} Y_x \quad \text{where} \quad Y_x := \mathcal{B} \cdot x\mathcal{P}_I/\mathcal{P}_I.$$

The closures  $\overline{Y}_x$  are Schubert varieties, and we denote by  $\mathbf{IC}_{x,I}$  (resp.  $\mathcal{E}_{x,I}$ ) the intersection cohomology complex (resp. parity sheaf) supported on  $\overline{Y}_x$ .

Given any  $\mathcal{G}$ -variety or ind-variety  $Z$  the monoidal category  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  acts on  $D_{\mathcal{B}}^b(Z; \mathbb{k})$ . (The definition is analogous to the formula for convolution given earlier.) In particular,  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  acts on  $D_{\mathcal{B}}^b(\mathcal{G}/\mathcal{P}_I; \mathbb{k})$ . One can check that this action preserves

$$\mathfrak{M}_I^{\mathbb{k}} := \langle \mathcal{E}_{x,I} \mid x \in W^I \rangle_{\oplus, [1]}$$

and thus  $\mathfrak{M}_I^{\mathbb{k}}$  is a module over  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$ . We have:

**Theorem 2.18.** *There is a unique isomorphism of  $H = [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$ -modules*

$$M_I \xrightarrow{\sim} [\mathfrak{M}_I^{\mathbb{k}}]_{\oplus}$$

sending  $\mu_{\text{id}} \mapsto [\mathbb{k}_{\mathcal{P}_I/\mathcal{P}_I}]$  (we use the identification  $H = [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$  of Theorem 2.5).

The inverse to the isomorphism in the theorem is given by the *character map*

$$\text{ch} : [\mathfrak{M}_I^{\mathbb{k}}]_{\oplus} \xrightarrow{\sim} M_I \quad \mathcal{F} \mapsto \sum_{x \in W^I} \dim_{\mathbb{Z}}(H^*(\mathcal{F}_{x\mathcal{P}_I/\mathcal{P}_I})) v^{-\ell(x)} \mu_x \in M_I.$$

(The notation is entirely analogous to the previous definition of  $\text{ch}$  in §2.2.)

We now turn to categorifying  $N_I$ . The full additive subcategory

$$\langle \mathcal{E}_x \mid x \notin W^I \rangle \subset \mathcal{H}_{\text{geom}}^{\mathbb{k}}$$

is a left ideal. In particular, if we consider the quotient of additive categories

$$\mathfrak{N}_I^{\mathbb{k}} := \mathcal{H}_{\text{geom}}^{\mathbb{k}} / \langle \mathcal{E}_x \mid x \notin W^I \rangle$$

this is a left  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$ -module. We denote the image of  $\mathcal{F} \in \mathcal{H}_{\text{geom}}^{\mathbb{k}}$  by  $\overline{\mathcal{F}}_I$ . The objects  $\overline{\mathcal{E}}_{x,I}$  for  $x \in W^I$  are precisely the indecomposable objects of  $\mathfrak{N}_I^{\mathbb{k}}$  up to shift and isomorphism. We have:

**Theorem 2.19.** *There is a unique isomorphism of right  $H = [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$ -modules*

$$N_I \xrightarrow{\sim} [\mathfrak{N}_I^{\mathbb{k}}]_{\oplus}$$

sending  $\nu_{\text{id}} \mapsto [\overline{\mathcal{E}}_{\text{id},I}]$  (we use the identification  $H = [\mathcal{H}_{\text{geom}}^{\mathbb{k}}]_{\oplus}$  of Theorem 2.5).

The inverse  $\text{ch} : [\mathfrak{N}_I^{\mathbb{k}}]_{\oplus} \xrightarrow{\sim} N^I$  is more complicated to describe.

*Remark 2.20.* It is also possible to give a geometric description of  $\mathfrak{N}_I^{\mathbb{k}}$  via Iwahori–Whittaker sheaves [S. Riche and Williamson \[2015, Chapter 11\]](#).

**2.6 The  $p$ -canonical basis.** Suppose that  $\mathbb{k}$  is a field of characteristic  $p \geq 0$ . Consider the Hecke category  $\mathcal{H}_{\text{geom}}^{\mathbb{k}}$  with coefficients in  $\mathbb{k}$ . Let us define

$${}^p b_x := \text{ch}(\mathcal{E}_x) \in H.$$

Because  $\mathcal{E}_x$  is supported on  $\overline{X}_x$  and its restriction to  $X_x$  is  $\mathbb{k}_{X_x}[\ell(x)]$ , it follows from the definition of the character map that

$$(9) \quad {}^p b_x = \delta_x + \sum_{y < x} {}^p h_{y,x} \delta_y$$

for certain  ${}^p h_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ . Thus the set  $\{{}^p b_x \mid x \in W\}$  is a basis for  $H$ , the  *$p$ -canonical basis*. The base change coefficients  ${}^p h_{y,x}$  are called  *$p$ -Kazhdan–Lusztig polynomials*, although they are Laurent polynomials in general.

The  $p$ -canonical basis has the following properties (see [Jensen and Williamson \[2017, Proposition 4.2\]](#)):

$$(10) \quad d({}^p b_x) = b_x \quad \text{for all } x \in W;$$

$$(11) \quad \text{if } {}^p b_x = \sum_{y \leq x} {}^p a_{y,x} b_y \text{ then } {}^p a_{y,x} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ and } d({}^p a_{y,x}) = {}^p a_{y,x};$$

$$(12) \quad \text{if } {}^p b_x {}^p b_y = \sum_{z \in W} {}^p \mu_{x,y}^z {}^p b_z \text{ then } {}^p \mu_{x,y}^z \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ and } d({}^p \mu_{x,y}^z) = {}^p \mu_{x,y}^z;$$

$$(13) \quad \text{for fixed } x \in W \text{ we have } {}^p b_x = {}^0 b_x = b_x \text{ for large } p.$$

*Remark 2.21.* Recall that the Kazhdan–Lusztig basis is uniquely determined by the “self-duality” and “degree bound” conditions (see §2.1). The  $p$ -canonical basis satisfies self-duality (10), but there appears to be no analogue of the degree bound condition in general (see Example 2.26 below).

*Remark 2.22.* There is an algorithm to calculate the  $p$ -canonical basis, involving the generators and relations presentation of the Hecke category discussed earlier. This algorithm is described in detail in [Jensen and Williamson \[ibid., §3\]](#).

*Remark 2.23.* The Kazhdan–Lusztig basis only depends on the Weyl group (a fact which is rather surprising from a geometric point of view). The  $p$ -canonical basis depends on the root system. For example, the 2-canonical bases in types  $B_3$  and  $C_3$  are quite different (see [Jensen and Williamson \[ibid., §5.4\]](#)).

*Remark 2.24.* The above properties certainly do not characterise the  $p$ -canonical basis. (For example, for affine Weyl groups the  $p$ -canonical bases are distinct for every prime.) However in certain situations they do appear to constrain the situation quite rigidly. For example, the above conditions are enough to deduce that  ${}^p b_x = b_x$  for all primes  $p$ , if  $\mathcal{G}$  is of types  $A_n$  for  $n < 7$  (see [Williamson and Braden \[2012\]](#)). See [Jensen \[2017\]](#) for further combinatorial constraints on the  $p$ -canonical basis.

We can also define  $p$ -canonical bases in the spherical and anti-spherical module. Let  $I \subset S$  be as in the previous section, and  $\mathbb{k}$  and  $p$  be as above. For  $x \in W^I$ , set

$$\begin{aligned} {}^p c_x &:= \text{ch}(\mathcal{E}_{x,I}) = \sum_{y \in W^I} {}^p m_{y,x} \mu_y \in M_I, \\ {}^p d_x &:= \text{ch}(\overline{\mathcal{E}}_{x,I}) = \sum_{y \in W^I} {}^p n_{y,x} \nu_y \in N_I. \end{aligned}$$

We have  ${}^p m_{y,x} = {}^p n_{y,x} = 0$  unless  $y \leq x$  and  ${}^p m_{x,x} = {}^p n_{x,x} = 1$ . Thus  $\{{}^p c_x\}$  (resp.  $\{{}^p d_x\}$ ) give  $p$ -canonical bases for  $M_I$  (resp.  $N_I$ ). We leave it to the reader to write down the analogues of (10), (11), (12) and (13) that they satisfy.

We define spherical and anti-spherical analogues of the “adjustment” polynomials  ${}^p a_{y,x}$  via:

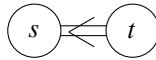
$${}^p c_x = \sum_{y \in W^I} {}^p a_{y,x}^{\text{sph}} c_y \quad \text{and} \quad {}^p d_x = \sum_{y \in W^I} {}^p a_{y,x}^{\text{asph}} d_y.$$

These polynomials give partial information on the  $p$ -canonical basis. For all  $x, y \in W^I$  we have:

$$(14) \quad {}^p a_{yw_I, xw_I} = {}^p a_{y,x}^{\text{sph}} \quad \text{and} \quad {}^p a_{y,x} = {}^p a_{y,x}^{\text{asph}}.$$

We finish this section with a few examples of the  $p$ -canonical basis. These are intended to complement the calculations in §1.5.

**Example 2.25.** Let  $\mathcal{G}$  be of type  $B_2$  with Dynkin diagram:

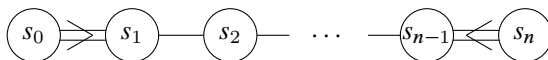


The Schubert variety  $\overline{Y}_{st} \subset \mathcal{G}/\mathcal{P}_s$  has an isolated singularity at  $\mathcal{P}_s/\mathcal{P}_s$ , and a neighbourhood of this singularity is isomorphic to  $X$  from Example 1.10. From this one may deduce that

$${}^2 c_{st} = c_{st} + c_{\text{id}}.$$

For a version of this calculation using diagrams see Jensen and Williamson [2017, §5.1].

**Example 2.26.** Here we explain the implications of Example 1.13 for the  $p$ -canonical basis. The singularity  $\mathbb{C}^{2n}/(\pm 1)$  occurs in the affine Grassmannian for  $\text{Sp}_{2n}$ , which is isomorphic to  $\mathcal{G}/\mathcal{P}_I$ , where  $\mathcal{G}$  is the affine Kac–Moody group of affine type  $C_n$  with Dynkin diagram



and  $I = \{s_1, \dots, s_n\}$  denotes the subset of finite reflections. After some work matching parameters, one may deduce that

$${}^2 c_{w_n w_{n-1} s_0} = c_{w_n w_{n-1} s_0} + (v^{2n-2} + v^{2n-4} + \dots + v^{-2n+2}) \cdot c_{\text{id}}.$$

where  $w_n$  (resp.  $w_{n-1}$ ) denotes the longest element in the standard parabolic subgroup generated by  $\{s_1, \dots, s_n\}$  (resp.  $\{s_2, \dots, s_n\}$ ).

**Example 2.27.** Let  $\mathcal{G} = \mathrm{SL}_8(\mathbb{C})$  with simple reflections:



Let

$$w = s_1 s_3 s_2 s_4 s_3 s_5 s_4 s_3 s_2 s_1 s_6 s_7 s_6 s_5 s_4 s_3$$

and consider  $w_I$  where  $I = \{s_1, s_3, s_4, s_5, s_7\}$ . The singularity of the Schubert variety  $\bar{X}_w$  at  $w_I$  is isomorphic to the Kashiwara–Saito singularity from Example 1.14 (with  $d = 2$ ). It follows that

$${}^2b_w = b_w + b_{w_I}.$$

This is one of the first examples for  $\mathrm{SL}_n$  with  ${}^p b_x \neq b_x$ .

**2.7 Torsion explosion.** In this section we assume that  $\mathcal{G} \cong \mathrm{SL}_n(\mathbb{C})$  and so  $W = S_n$ , the symmetric group. Here the  $p$ -canonical basis is completely known for  $n = 2, 3, \dots, 9$  and difficult to calculate beyond that. The following theorem makes clear some of the difficulties that await us in high rank:

**Theorem 2.28** (Williamson [2017c]). *Let  $\gamma$  be a word of length  $l$  in the generators*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

with product

$$\gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix}.$$

For non-zero  $m \in \{\gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}\}$  and any prime  $p$  dividing  $m$  there exists  $y \in S_{3l+5}$  such that  ${}^p b_y \neq b_y$ .

The moral seems to be that arithmetical issues (“which primes divide entries of this product of elementary matrices?”) are hidden in the question of determining the  $p$ -canonical basis.<sup>17</sup>

We can get some qualitative information out of Theorem 2.28 as follows. Define

$$\Pi_n := \{p \text{ prime} \mid {}^p b_x \neq b_x \text{ for some } x \in S_n\}.$$

Because any Schubert variety in  $\mathrm{SL}_n(\mathbb{C})/\mathcal{B}$  is also a Schubert variety in  $\mathrm{SL}_{n+1}(\mathbb{C})/\mathcal{B}$  we have inclusions  $\Pi_n \subset \Pi_{n+1}$  for all  $n$ . By long calculations by Braden, Polo, Saito and the author, we know the following about  $\Pi_n$  for small  $n$ :

$$\begin{aligned} \Pi_n &= \emptyset \quad \text{for } n \leq 7, \\ \Pi_n &= \{2\} \quad \text{for } n = 8, 9, \\ \{2, 3\} &\subset \Pi_{12}. \end{aligned}$$

<sup>17</sup>Another example of this phenomenon from Williamson [2017c]: for any prime number  $p$  dividing the  $l^{\text{th}}$  Fibonacci number there exists  $y \in S_{3l+5}$  with  ${}^p b_y \neq b_y$ . Understanding the behaviour of primes dividing Fibonacci numbers is a challenging open problem in number theory. It is conjectured, but not known, that infinitely many Fibonacci numbers are prime.



The most interesting values here are  $2 \in \Pi_8$  (discovered by Braden in 2002, see [Williamson and Braden \[2012, Appendix\]](#)) and  $3 \in \Pi_{12}$  (discovered by Polo in 2012). More generally, Polo shows that  $p \in \Pi_{4p}$  for any prime  $p$ , and hence  $\Pi_n$  exhausts all prime numbers as  $n \rightarrow \infty$  (see [Example 1.14](#)).

For applications to representation theory, it is important to know how large the entries of  $\Pi_n$  grow with  $n$ .<sup>18</sup> Some number theory, combined with [Theorem 2.28](#), implies the following:

**Corollary 2.29** ([Williamson \[2017c, Theorem A.1\]](#)). *For  $n$  large,  $n \mapsto \max \Pi_n$  grows at least exponentially in  $n$ . More generally,  $\Pi_n$  contains many exponentially large prime numbers.*

*Remark 2.30.* Let us try to outline how [Theorem 2.28](#) is proved. To  $\underline{\gamma}$  and  $m$  we associate a reduced expression  $\underline{x} = (s_1, \dots, s_n)$  for some particular  $x \in \underline{S}_{3l+5}$ . (There is a precise but complicated combinatorial recipe as to how to do this, which we won't go into here. Let us mention however that the length of  $x$  grows quadratically in  $l$ .) Associated to this reduced expression we have a Bott–Samelson resolution

$$f : BS_{\underline{x}} \rightarrow \overline{X}_x.$$

We calculate the intersection form at a point  $w_I \mathcal{B}/\mathcal{B}$  (corresponding to the maximal element of a standard parabolic subgroup) and discover the  $1 \times 1$ -matrix  $(m)$ . Thus for any  $p$  dividing  $m$  the Decomposition Theorem fails for  $f$  at the point  $w_I \mathcal{B}/\mathcal{B}$ , which is enough to deduce the theorem. The hard part in all of this is finding the appropriate expression  $\underline{x}$  and calculating the intersection form. The intersection form calculation was first done in [Williamson \[ibid.\]](#) using a formula in the nil Hecke ring discovered by [He and Williamson \[2015\]](#). Later a purely geometric argument was found [Williamson \[2017b\]](#).

*Remark 2.31.* Let us keep the notation of the previous remark. In general we do not know whether  ${}^p a_{w_A, x} \neq 0$  for any  $p$  dividing  $m$ , only that there is some  $y$  with  $w_A \leq y \leq x$  and  ${}^p a_{w_A, y} \neq 0$ . Thus, in the statement of [Theorem 2.28](#) we don't know that  ${}^p b_x \neq b_x$ , although this seems likely.

*Remark 2.32.* By a classical theorem of [Zelevinskii \[1983\]](#), Schubert varieties in Grassmannians admit small resolutions, and hence the  $p$ -canonical basis is equal to the canonical basis in the spherical modules for one step flag varieties (we saw a hint of this in [Example 1.11](#)). It is an interesting question (suggested by Joe Chuang) as to how the  $p$ -canonical basis behaves in flag varieties with small numbers of steps and at what point (i.e. at how many steps) the behaviour indicated in [Theorem 2.28](#) begins.

*Remark 2.33.* Any Schubert variety in  $\mathrm{SL}_n(\mathbb{C})/\mathcal{B}$  is isomorphic to a Schubert variety in the flag varieties of types  $B_n$ ,  $C_n$  and  $D_n$ . In particular, the above complexity is present in the  $p$ -canonical bases for all classical finite types.

<sup>18</sup>For example, the Lusztig conjecture would have implied that the entries of  $\Pi_n$  are bounded linearly in  $n$ , and the James conjecture would have implied a quadratic bound in  $n$ , see [Williamson \[ibid.\]](#).

**2.8 Open questions about the  $p$ -canonical basis.** In this section we discuss some interesting open problems about the  $p$ -canonical basis. We also try to outline what is known and point out connections to problems in modular representation theory.

In the following a Kac–Moody root datum is assumed to be fixed throughout. Thus, when we write  ${}^p b_x$ , its dependence on the root datum is implicit. Throughout,  $p$  denotes the characteristic of  $\mathbb{k}$ , our field of coefficients.

**Question 2.34.** *For  $x \in W$  and  $p$  a prime, when is  ${}^p b_x = b_x$ ?*

*Remark 2.35.* This question is equivalent to asking whether  $\mathbf{IC}_x^{\mathbb{k}} \cong \mathcal{E}_x^{\mathbb{k}}$ .

A finer-grained version of this question is:

**Question 2.36.** *For  $x, y \in W$  and  $p$  a prime, when is  ${}^p h_{y,x} = h_{y,x}$ ?*

*Remark 2.37.* If  $h_{y,x} = v^{\ell(x)-\ell(y)}$  then Question 2.36 has a satisfactory answer. In this case  $y\mathcal{B}/\mathcal{B}$  is a rationally smooth point of the Schubert variety  $\overline{X}_x$  and  ${}^p h_{y,x} = h_{y,x}$  if and only if  $\overline{X}_x$  is also  $p$ -smooth at  $y\mathcal{B}/\mathcal{B}$ ; moreover, this holds if and only if a certain combinatorially defined integer (the numerator of the “equivariant multiplicity”) is not divisible by  $p$ , see [Juteau and Williamson \[2014\]](#) and [Dyer \[2001\]](#). (See [Fiebig \[2010\]](#) and [Fiebig and Williamson \[2014\]](#) for related ideas.) It would be very interesting if one could extend such a criterion beyond the rationally smooth case.

In applications the following variants of Question 2.34 and 2.36 (for particular choices of  $Z$ ) are more relevant:

**Question 2.38.** *Fix  $Z \subset W$ . For which  $p$  does there exist  $x \in Z$  with  ${}^p b_x \neq b_x$ ?*

**Question 2.39.** *Fix  $Z \subset W^I$ .*

1. *For which  $p$  is  ${}^p c_x = c_x$  for all  $x \in Z$ ?*

2. *For which  $p$  is  ${}^p d_x = d_x$  for all  $x \in Z$ ?*

*Remark 2.40.* If  $\mathcal{G}$  is finite-dimensional then  ${}^p b_x = b_x$  for all  $x \in W$  if and only if a part of Lusztig’s conjecture holds (see [Soergel \[2000\]](#)). The results of §2.7 give exponentially large counter-examples to the expected bounds in Lusztig’s character formula [Lusztig \[1980\]](#) and [Williamson \[2017c\]](#).

*Remark 2.41.* With  $\mathcal{G}$  as in the previous remark, Xuhua He has suggested that we might have  ${}^p b_x = b_x$  for all  $x$ , if  $p > |W|$ . This seems like a reasonable hope, and it would be wonderful to have a proof.

*Remark 2.42.* Suppose  $\mathcal{G}$  is an affine Kac–Moody group and  $I \subset S$  denotes the “finite” reflections (so that  $W_I = \langle I \rangle$  is the finite Weyl group). Then there exists a finite subset  $Z_1 \subset W^I$  for which Question 2.38(1) is equivalent to determining in which characteristics Lusztig’s character formula holds, see [Achar and Riche \[2016b, §11.6\]](#) and [Williamson \[2017a, §2.6\]](#). Because  $Z_1$  is finite,  ${}^p c_x = c_x$  for all  $x \in Z_1$  for  $p$  large, which translates into the known fact that Lusztig’s conjecture holds in large characteristic.

*Remark 2.43.* Suppose that  $\mathcal{G}$  and  $I$  are as in the previous remark. There exists a subset  $Z(p) \subset W^I$  (depending on  $p$ ) such that if  ${}^p d_x = d_x$  for all  $x \in Z(p)$  then Andersen's conjecture on characters of tilting modules (see [Andersen \[2000, Proposition 4.6\]](#)) holds in characteristic  $p$  (this is a consequence of the character formula proved in [Achar, Makisumi, Riche, and Williamson \[2017b\]](#)). Note that Andersen's conjecture does not give a character formula for the characters of all tilting modules and is not known even for large  $p$ .

The following questions are also interesting:

**Question 2.44.** *For which  $x \in W$  and  $p$  is  ${}^p a_{y,x} \in \mathbb{Z}$  for all  $y \in W$ ?*

*Remark 2.45.* This is equivalent to asking when  $\mathcal{E}_x^{\mathbb{k}}$  is perverse.

**Question 2.46.** *Fix  $? \in \{\text{sph}, \text{asph}\}$ . For which  $x \in W^I$  and  $p$  is  ${}^p a_{y,x}^? \in \mathbb{Z}$  for all  $y \in W^I$ ?*

*Remark 2.47.* Example 2.26 shows that in the affine case  ${}^p a_{y,x}$  can be a polynomial in  $v$  of arbitrarily high degree. An example of Libedinsky and the author [Libedinsky and Williamson \[2017\]](#) shows that there exists  $x, y$  in the symmetric group  $S_{15}$ , for which  ${}^2 a_{y,x} = (v + v^{-1})$ . Recently P. McNamara has proposed new candidate examples, which appear to show that for any  $p$ , the degree of  ${}^p a_{y,x}$  is unbounded in symmetric groups.

*Remark 2.48.* Suppose  $\mathcal{G}$  and  $I$  are as in Remark 2.42. It follows from the the results of [Juteau, Mautner, and Williamson \[2016\]](#) and [Mautner and Riche \[2018\]](#) that  ${}^p a_{y,x} \in \mathbb{Z}$  if  $x$  is maximal in  $W_I x W_I$ . More generally, it seems likely that  ${}^p a_{y,x}^{\text{sph}} \in \mathbb{Z}$  for all  $x, y \in W^I$  and large  $p$  (depending only on the Dynkin diagram of  $\mathcal{G}$ ). This is true for trivial reasons in affine types  $A_1$  and  $A_2$ .

*Remark 2.49.* In contrast, recent conjectures of Lusztig and the author [Lusztig and Williamson \[2018\]](#) imply that, if  $\mathcal{G}$  is of affine type  $\widetilde{A_2}$  then it is never the case (for any  $p \neq 2$ ) that  ${}^p a_{y,x}^{\text{asph}} \in \mathbb{Z}$  for all  $y, x \in W^I$ . In fact, our conjecture implies that

$$\max \{ \deg({}^p a_{y,x}^{\text{asph}}) \mid y, w \in W^I \} = \infty.$$

This contrast in behaviour between the  $p$ -canonical bases in the spherical and anti-spherical modules is rather striking.

### 3 Koszul duality

In this section we discuss Koszul duality for the Hecke category. This is a remarkable derived equivalence relating the Hecke categories of Langlands dual groups. It resembles a Fourier transform. Its modular version involves parity sheaves, and is closely related to certain formality questions. In this section we assume that the reader has some background with perverse sheaves and highest weight categories.

**3.1 Classical Koszul duality.** Let  $C, \mathcal{G}, \mathcal{B}, \mathcal{T}, W, \mathbb{k}$  be as previously. We denote by  $\mathcal{G}^\vee, \mathcal{B}^\vee, \mathcal{T}^\vee$  the Kac–Moody group (resp. Borel subgroup, resp. maximal torus) associated to the dual Kac–Moody root datum. We have a canonical identification of  $W$  with the Weyl group of  $\mathcal{G}^\vee$ .

In this section we assume that  $\mathcal{G}$  is a (finite-dimensional) complex reductive group, i.e. that  $C$  is a Cartan matrix. We denote by  $w_0 \in W$  the longest element. For any  $x \in W$  let  $i_x : X_x = \mathcal{B} \cdot x\mathcal{B}/\mathcal{B} \hookrightarrow \mathcal{G}/\mathcal{B}$  denote the inclusion of the Schubert cell and set

$$\Delta_x := i_{x!} \mathbb{k}_{X_x}[\ell(x)] \quad \text{and} \quad \nabla_x := i_{x*} \mathbb{k}_{X_x}[\ell(x)].$$

Let  $D_{(\mathcal{B})}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  denote the derived category, constructible with respect to  $\mathcal{B}$ -orbits and let  $\mathbf{P}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B}; \mathbb{k}) \subset D_{(\mathcal{B})}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  denote the subcategory of perverse sheaves. The abelian category  $\mathbf{P}_{(\mathcal{B})}(\mathcal{G}/\mathcal{B}; \mathbb{k})$  is highest weight [Beilinson, Ginzburg, and Soergel \[1996\]](#) and [Beilinson, Bezrukavnikov, and Mirković \[2004\]](#) with standard (resp. costandard) objects  $\{\Delta_x\}_{x \in W}$  (resp.  $\{\nabla_x\}_{x \in W}$ ). For  $x \in W$ , we denote by  $\mathcal{P}_x, \mathcal{I}_x$  and  $\mathcal{T}_x$  the corresponding indecomposable projective, injective, and tilting object. The corresponding objects in  $\mathbf{P}_{(\mathcal{B}^\vee)}(\mathcal{G}^\vee/\mathcal{B}^\vee; \mathbb{k})$  are denoted with a check, e.g.  $\mathbf{IC}_x^\vee, \Delta_x^\vee$  etc.

Let us assume that  $\mathbb{k} = \mathbb{Q}$ . Motivic considerations, together with the Kazhdan–Lusztig inversion formula (see [Kazhdan and Lusztig \[1979\]](#))

$$(15) \quad \sum_{z \in W} (-1)^{\ell(x) + \ell(y)} h_{y,x} h_{yw_0, zw_0} = \delta_{x,z},$$

led [Beilinson and Ginsburg \[1986\]](#) to the following conjecture<sup>19</sup>:

1. There exists a triangulated category  $D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  equipped with an action of the integers  $\mathcal{F} \mapsto \mathcal{F}\langle m \rangle$  for  $m \in \mathbb{Z}$  (“Tate twist”) and a “forgetting the mixed structure” functor

$$\phi : D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q}) \rightarrow D_{(\mathcal{B})}^b(\mathcal{G}/\mathcal{B}; \mathbb{Q}),$$

such that

$$\mathrm{Hom}(\phi(\mathcal{F}), \phi(\mathcal{G})) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathcal{F}, \mathcal{G}\langle n \rangle)$$

for all  $\mathcal{F}, \mathcal{G} \in D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$ . Furthermore, “canonical” objects (e.g. simple, standard, projective etc. objects) admit lifts<sup>20</sup> to  $D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$ .

2. There is an equivalence of triangulated categories

$$(16) \quad \kappa : D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q}) \xrightarrow{\sim} D_{(\mathcal{B}^\vee)}^{mix}(\mathcal{G}^\vee/\mathcal{B}^\vee; \mathbb{Q})$$

such that  $\kappa \circ \langle -1 \rangle[1] \cong \langle 1 \rangle \circ \kappa$ , and such that  $\kappa$  acts on standard, simple and projective objects (for an appropriate choice of lift) as follows:

$$\Delta_x \mapsto \nabla_{x^{-1}w_0}^\vee, \quad \mathbf{IC}_x \mapsto \mathcal{I}_{x^{-1}w_0}^\vee, \quad \mathcal{P}_x \mapsto \mathbf{IC}_{x^{-1}w_0}^\vee.$$

<sup>19</sup>to simplify the exposition we have modified the statement of their original conjecture slightly (they worked with Lie algebra representations and sought a contravariant equivalence).

<sup>20</sup> $\mathcal{F} \in D_{(\mathcal{B})}^b(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  admits a lift, if there exists  $\tilde{\mathcal{F}} \in D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  such that  $\mathcal{F} \cong \phi(\tilde{\mathcal{F}})$ .

*Remark 3.1.* To understand why the extra grading (provided by the mixed structure) as well as the relation  $\kappa \circ \langle -1 \rangle [1] \cong \langle 1 \rangle \circ \kappa$  is necessary, one only needs to ask oneself where the grading on extensions between simple modules should go under this equivalence.

*Remark 3.2.* One can deduce from (15) and the Kazhdan–Lusztig conjecture that the assignment  $\Delta_x \mapsto \nabla_{x^{-1}w_0}$  on mixed categories forces  $\mathbf{IC}_x \mapsto \mathcal{I}_{x^{-1}w_0}$  and  $\mathcal{P}_x \mapsto \mathbf{IC}_{x^{-1}w_0}$  on the level of Grothendieck groups.

This conjecture was proved by Beilinson, Ginzburg and Soergel in the seminal paper [Beilinson, Ginzburg, and Soergel \[1996\]](#), where they interpreted  $\kappa$  in the framework of Koszul duality for graded algebras. The authors give two constructions of the mixed derived category: one involving mixed étale sheaves (here it is necessary to consider the flag variety for the split group defined over a finite field), and one involving mixed Hodge modules.

*Remark 3.3.* Both constructions of the mixed derived category in [Beilinson, Ginzburg, and Soergel \[ibid.\]](#) involve some non-geometric “cooking” to get the right result. Recently Soergel and Wendt have used various flavours of mixed Tate motives to give a purely geometric construction of these mixed derived categories [Soergel \[1990b\]](#).

After the fact, it is not difficult to see that the mixed derived category admits a simple definition. Indeed, the results of [Beilinson, Ginzburg, and Soergel \[1996\]](#) imply that one has an equivalence

$$(17) \quad \sigma : K^b(\text{Semis}(\mathcal{G}/\mathcal{B}; \mathbb{Q})) \xrightarrow{\sim} D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q}).$$

Here  $\text{Semis}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  denotes the full additive subcategory of  $D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  consisting of direct sums of shifts of intersection cohomology complexes (“semi-simple complexes”), and  $K^b(\text{Semis}(\mathcal{G}/\mathcal{B}; \mathbb{Q}))$  denotes its homotopy category. Note that there are two shift functors on  $\text{Semis}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$ : one coming from its structure as a homotopy category (which we denote  $[1]$ ); and one induced from the shift functor on  $D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q})$  (which we rename  $\langle 1 \rangle$ ). Under the equivalence  $\sigma$ , Tate twist  $\langle 1 \rangle$  corresponds to  $[1](-1)$ .

Now, if  $\mathcal{H}^{\mathbb{Q}}$  denotes the Hecke category we have an equivalence

$$\mathbb{Q} \otimes_R \mathcal{H}_{\text{diag}}^{\mathbb{Q}} \xrightarrow{\sim} \text{Semis}(\mathcal{G}/\mathcal{B}; \mathbb{Q}).$$

Moreover, the left hand side can be described by generators and relations. In particular, Koszul duality can be formulated entirely algebraically as an equivalence

$$\kappa : K^b(\mathbb{Q} \otimes_R \mathcal{H}_{\text{diag}}^{\mathbb{Q}}) \xrightarrow{\sim} K^b(\mathbb{Q} \otimes_R \mathcal{H}_{\text{diag}}^{\vee, \mathbb{Q}}).$$

The existence of such an equivalence (valid more generally for any finite real reflection group, with  $\mathbb{Q}$  replaced by  $\mathbb{R}$ ) has recently been established by [Makisumi \[2017\]](#). (The case of a dihedral group was worked out by [Sauerwein \[2018\]](#).)

**3.2 Monoidal Koszul duality.** The above results raise the following questions:

1. How does Koszul duality interact with the monoidal structure?
2. Can Koszul duality be generalised to the setting of Kac–Moody groups?

The first question was addressed by [Beilinson and Ginzburg \[1999\]](#). They noticed that if one composes Koszul duality  $\kappa$  with the Radon transform and inversion, one obtains a derived equivalence

$$(18) \quad \tilde{\kappa} : D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q}) \xrightarrow{\sim} D_{(\mathcal{B}^\vee)}^{mix}(\mathcal{B}^\vee \setminus \mathcal{G}^\vee; \mathbb{Q})$$

with  $\tilde{\kappa} \circ \langle -1 \rangle [1] \cong \langle 1 \rangle \circ \tilde{\kappa}$  as previously, however now

$$(19) \quad \mathbf{IC}_x \mapsto \mathcal{T}_x^\vee, \quad \Delta_x \mapsto \Delta_x^\vee, \quad \nabla_x \mapsto \nabla_x^\vee, \quad \mathcal{T}_x \mapsto \mathbf{IC}_x^\vee.$$

The new equivalence  $\tilde{\kappa}$  is visibly more symmetric than  $\kappa$ . It also has the advantage that it does not involve the longest element  $w_0$ , and hence makes sense for Kac–Moody groups. Moreover, Beilinson and Ginzburg conjectured that  $\tilde{\kappa}$  can be promoted to a monoidal equivalence (suitably interpreted).

*Remark 3.4.* It has been a stumbling block for some time that (18) cannot be upgraded to a monoidal equivalence in a straightforward way. This is already evident for  $\mathrm{SL}_2$ : the “big” tilting sheaf  $\mathcal{T}_s \in D_{(\mathcal{B})}^b(\mathbb{P}^1; \mathbb{Q})$  does not admit a  $\mathcal{B}$ -equivariant structure.

Subsequently, [Bezrukavnikov and Yun \[2013\]](#) established a monoidal equivalence

$$(20) \quad \tilde{\kappa} : (D_{\mathcal{B}}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{Q}), *) \xrightarrow{\sim} (\widehat{D}^{mix}(\mathcal{B}^\vee \setminus \mathcal{G}^\vee / \mathcal{B}^\vee; \mathbb{Q}), *)$$

which induces the Koszul duality equivalence above after killing the deformations, and is valid for any Kac–Moody group.<sup>21</sup> Here  $\widehat{D}^{mix}(\mathcal{B}^\vee \setminus \mathcal{G}^\vee / \mathcal{B}^\vee; \mathbb{Q})$  denotes a suitable (“free monodromic”) completion of the full subcategory of mixed  $U^\vee$ -constructible complexes on  $\mathcal{G}^\vee / U^\vee$  which have unipotent monodromy along the fibres of the map  $\mathcal{G}^\vee / U^\vee \rightarrow \mathcal{G}^\vee / \mathcal{B}^\vee$ . The construction of this completion involves considerable technical difficulties. The proof involves relating both sides to a suitable category of Soergel bimodules (and thus is by “generators and relations”).

**3.3 Modular Koszul duality.** We now discuss the question of how to generalise (18) to coefficients  $\mathbb{k}$  of positive characteristic. A first difficulty is how to make sense of the mixed derived category. A naïve attempt (carried out in [Riche, Soergel, and Williamson \[2014\]](#)) is to consider a flag variety over a finite field together with the Frobenius endomorphism and its weights, however here one runs into problems because one obtains gradings by a finite cyclic group rather than  $\mathbb{Z}$ . Achar and Riche took the surprising step of simply defining

$$D_{(\mathcal{B})}^{mix}(\mathcal{G}/\mathcal{B}; \mathbb{k}) := K^b(\mathrm{Par}(\mathcal{G}/\mathcal{B}; \mathbb{k}))$$

where  $\mathrm{Par}(\mathcal{G}/\mathcal{B}; \mathbb{k})$  denotes the additive category of  $\mathcal{B}$ -constructible parity complexes on  $\mathcal{G}/\mathcal{B}$ , and the shift  $[1]$  and twist  $(1)$  functors are defined as in the paragraph following (17). (The discussion there shows that this definition is consistent when  $\mathbb{k} = \mathbb{Q}$ .) In doing so one obtains a triangulated category with most of the favourable properties one expects from the mixed derived category. In this setting Koszul duality takes the form:

<sup>21</sup>Actually, Bezrukavnikov and Yun use mixed  $\ell$ -adic sheaves, and no non-geometric “cooking” is necessary.

**Theorem 3.5.** *There is an equivalence of triangulated categories*

$$\kappa : D_{(\mathcal{B})}^{\text{mix}}(\mathcal{G}/\mathcal{B}; \mathbb{k}) \xrightarrow{\sim} D_{(\mathcal{B}^\vee)}^{\text{mix}}(\mathcal{B}^\vee \backslash \mathcal{G}^\vee; \mathbb{k})$$

which satisfies  $\kappa \circ \langle -1 \rangle [1] \cong \langle 1 \rangle \circ \kappa$  and

$$\kappa(\Delta_w) \cong \Delta_w^\vee, \quad \kappa(\nabla_w) \cong \nabla_w^\vee, \quad \kappa(\mathcal{E}_w) \cong \mathcal{T}_w^\vee, \quad \kappa(\mathcal{T}_w) \cong \mathcal{E}_w^\vee.$$

*Remark 3.6.* The important difference in the modular case is that tilting sheaves correspond to parity sheaves (rather than IC sheaves).

*Remark 3.7.* For finite-dimensional  $\mathcal{G}$  this theorem was proved in [Achar and Riche \[2016a\]](#) (in good characteristic). For general  $\mathcal{G}$  this theorem is proved in [Achar, Makisumi, Riche, and Williamson \[2017a,b\]](#), as a corollary of a monoidal modular Koszul duality equivalence, inspired by [Bezrukavnikov and Yun \[2013\]](#).

*Remark 3.8.* The appearance of the Langlands dual group was missing from the original conjectures of [Beilinson and Ginzburg \[1986\]](#) and only appeared in [Beilinson, Ginzburg, and Soergel \[1996\]](#). However in the settings considered there ( $\mathbb{k} = \mathbb{Q}$ ), the Hecke categories associated to dual groups are equivalent. This is no longer the case with modular coefficients, and examples (e.g.  $B_3$  and  $C_3$  in characteristic 2) show that the analogue of Theorem 3.5 is false if one ignores the dual group.

*Remark 3.9.* A major motivation for [Achar, Makisumi, Riche, and Williamson \[2017a,b\]](#) was a conjecture of Riche and the author [S. Riche and Williamson \[2015, §1.4\]](#) giving characters for tilting modules for reductive algebraic groups in terms of  $p$ -Kazhdan–Lusztig polynomials. In fact, a recent theorem of [Achar and Riche \[2016b\]](#) (generalising a theorem of [Arkhipov, Bezrukavnikov, and Ginzburg \[2004\]](#)) combined with (a variant of) the above Koszul duality theorem leads to a solution of this conjecture. We expect that modular Koszul duality will have other applications in modular representation theory.

*Remark 3.10.* One issue with the above definition of the mixed derived category is the absence of a “forget the mixed structure” functor  $\phi : D_{(\mathcal{B})}^{\text{mix}}(\mathcal{G}/\mathcal{B}; \mathbb{k}) \rightarrow D_{(\mathcal{B})}^b(\mathcal{G}/\mathcal{B}; \mathbb{k})$  in general. For finite-dimensional  $\mathcal{G}$  its existence is established in [Achar and Riche \[2016a\]](#). Its existence for affine Weyl groups would imply an important conjecture of [Finkelberg and Mirković \[1999\]](#), see also [Achar and Riche \[2016b\]](#).

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