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ON THE CROSSROADS OF ENUMERATIVE GEOMETRY AND GEOMETRIC REPRESENTATION THEORY

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Abstract

The subjects in the title are interwoven in many different and very deep ways. I recently wrote several expository accounts that reflect a certain range of developments, but even in their totality they cannot be taken as a comprehensive survey. In the format of a 30-page contribution aimed at a general mathematical audience, I have decided to illustrate some of the basic ideas in one very interesting example – that of $Hilb(\mathbb{C}^2, n)$, hoping to spark the curiosity of colleagues in those numerous fields of study where one should expect applications.

1 The Hilbert scheme of points in $\mathbb{C}^2$

1.1 Classical geometry. distinct points in the plane $\mathbb{C}^2$ is uniquely specified by the corresponding ideal

$$I_P = \{f(p_1) = \cdots = f(p_n) = 0\} \subset \mathbb{C}[x_1, x_2]$$

in the coordinate ring $\mathbb{C}^2$. The codimension of this ideal, i.e. the dimension of the quotient

$$\mathbb{C}[x_1, x_2]/I_P = \text{functions on } P \stackrel{\text{def}}{=} \mathcal{O}_P\,,$$

clearly equals n.

If the points $\{p_i\}$ merge, the limit of I_P stores more information than just the location of points. For instance, for two points, it remembers the direction along which they came together. One defines

$$\mathrm{Hilb}(\mathbb{C}^2, n) = \{\text{ideals } I \subset \mathbb{C}[x_1, x_2] \text{ of codimension } n\}\,,$$

and with its natural scheme structure Fantechi, Göttsche, Illusie, Kleiman, Nitsure, and Vistoli [2005] and Kollár [1996] this turns out to be a smooth irreducible algebraic variety — a special feature of the Hilbert schemes of surfaces that fails very badly in higher dimensions.

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It is impossible to overestimate the influence of my friends and collaborators M. Aganagic, R. Bezrukavnikov, P. Etingof, I. Loseu, D. Maulik, N. Nekrasov, R. Pandharipande on the thoughts and ideas presented here. I am very grateful to the Simons Foundation Simons Investigator program, the Russian Academic Excellence Project '5-100', and the RSF grant 16-11-10160 for their support of my research.

The map $I_P \mapsto P$ extends to a natural map

$$\pi_{\mathrm{Hilb}} : \mathrm{Hilb}(\mathbb{C}^2, n) \to (\mathbb{C}^2)^n / S(n) \tag{1}$$

which is proper and birational, in other words, a resolution of singularities of $(\mathbb{C}^2)^n/S(n)$. This makes $\mathrm{Hilb}(\mathbb{C}^2, n)$ an instance of an *equivariant symplectic resolution* — a very special class of algebraic varieties Kaledin [2009] and Beauville [2000] that plays a central role in the current development of both enumerative geometry and geometric representation theory. This general notion axiomatizes two key features of π_{Hilb}:

- the source of π is an algebraic symplectic variety (here, with the symplectic form induced from that of $\mathbb{C}^2$),
- the map is equivariant for an action of a torus T that contracts the target to a point (here, T are the diagonal matrices in $GL(2)$ and the special point is the origin in $(\mathbb{C}^2)^n$).

Both enumerative geometry and geometric representation theory really work with algebraic varieties X and correspondences, that is, cycles (or sheaves, etc.) in $X_1 \times X_2$, considered up to a certain equivalence. These one can compose geometrically and they form a nonlinear analog of matrices of linear algebra and classical representation theory. To get to vector spaces and matrices, one considers functors like the equivariant cohomology $H^\bullet_{\mathsf{T}}(X)$, the equivariant K-theory etc., with the induced action of the correspondences.

Working with equivariant cohomology whenever there is a torus action available is highly recommended, in particular, because:

- equivariant cohomology is in many ways simpler than the ordinary, while also more general. E.g. the spectrum of the ring $H^\bullet_{\mathsf{T}}(\mathrm{Hilb}(\mathbb{C}^2, n))$ is a union of explicit essentially linear subvarieties over all partitions of n.
- the base ring $H^\bullet_{\mathsf{T}}(\mathrm{pt}, \mathbb{Q}) = \mathbb{Q}[\mathrm{Lie}\,\mathsf{T}]$ of equivariant cohomology introduces parameters in the theory, on which everything depends in a very rich and informative way,
- equivariance is a way to trade global geometry for local parameters. For instance, all formulas in the classical (that is, not quantum) geometry of Hilbert schemes of points generalize Ellingsrud, Göttsche, and Lehn [2001] to the general surface $\mathcal{S}$ with the substitution

$$c_1(\mathcal{S}) = t_1 + t_2\,, \quad c_2(\mathcal{S}) = t_1 t_2\,,$$

  where $\mathrm{diag}(t_1, t_2) \in \mathrm{Lie}\,\mathsf{T}$.

The fundamental correspondence between Hilbert schemes of points is the Nakajima correspondence

$$\alpha_{-k}(\gamma) \subset \mathrm{Hilb}(\mathcal{S}, n+k) \times \mathrm{Hilb}(\mathcal{S}, n)$$

formed for $k > 0$ by pairs of ideals (I_1, I_2) such that $I_1 \subset I_2$ and the quotient I_2/I_1 is supported at a single point located along a cycle γ in the surface $\mathcal{S}$. This cycle is

Lagrangian if $\gamma \subset \mathcal{S}$ is an algebraic curve. For $\mathcal{S} = \mathbb{C}^2$, there are very few T-equivariant choices for γ and they are all proportional, e.g.

$$[\{x_2 = 0\}] = t_2[\mathbb{C}^2], \quad [0] = t_1 t_2 [\mathbb{C}^2] .$$

One defines $\alpha_k(\gamma)$ for $k > 0$ as $(-1)^k$ times the transposed correspondence (see section 3.1.3 in Maulik and Okounkov [2012] on the author's preferred way to deal with signs in the subject).

The fundamental result of Nakajima [1997] is the following commutation relation[1]

$$[\alpha_n(\gamma), \alpha_m(\gamma')] = -(\gamma, \gamma')\, n\, \delta_{n+m} , \tag{2}$$

where the intersection pairing (γ, γ') for $\mathcal{S} = \mathbb{C}^2$ is $([0], [\mathbb{C}^2]) = 1$. One recognizes in (2) the commutation relation for the Heisenberg Lie algebra $\widehat{\mathfrak{gl}}(1)$ — a central extension of the commutative Lie algebra of Laurent polynomials with values in $\mathfrak{gl}(1)$. The representation theory of this Lie algebra is very simple, yet very constraining, and one deduces the identification

$$\bigoplus_{n \geq 0} H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}, n)) \cong \mathsf{S}^{\bullet}\,(\text{span of } \{\alpha_{-k}(\gamma)\}_{k>0}) \tag{3}$$

with the Fock module generated by the vacuum $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}, 0)) = H_{\mathsf{T}}^{\bullet}(\mathrm{pt})$.

Fock spaces (equivalently, symmetric functions) are everywhere in mathematics and mathematical physics and many remarkable computations and phenomena are naturally expressed in this language. My firm belief is that geometric construction, in particular the DT theory of 3-folds to be discussed below, are the best known way to think about them.

The identification (3) is a good example to illustrate the general idea that the best way to understand an algebraic variety X and, in particular, its equivariant cohomology $H_{\mathsf{T}}^{\bullet}(X)$, is to construct interesting correspondences acting on it.

For a general symplectic resolution

$$\pi : X \to X_0 \tag{4}$$

the irreducible components of the *Steinberg* variety $X \times_{X_0} X$ give important correspondences. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, these will be quadratic in Nakajima correspondences, and hence not as fundamental. In Section 2 we will see one general mechanism into which $\alpha_k(\gamma)$ fit.

Since $\widehat{\mathfrak{gl}}(1)$ acts irreducibly in (3), it is natural to express all other geometrically defined operators in terms of $\alpha_k(\gamma)$. Of special importance in what follows will be the operator of cup product (and also of the quantum product) by Chern classes of the tautological bundle $\mathsf{Taut} = \mathbb{C}[x_1, x_2]/I$ over the Hilbert scheme.

[1] in which the commutator is the supercommutator for odd-dimensional cycles γ, γ'. Similarly, the symmetric algebra in (3) is taken in the $\mathbb{Z}/2$-graded sense.

The operator of multiplication by the divisor $c_1(\mathsf{Taut})$ was computed by Lehn [1999] as follows

$$c_1(\mathsf{Taut})\cup = -\tfrac{1}{2}\sum_{n,m>0}\left(\alpha_{-n}\,\alpha_{-m}\,\alpha_{n+m}+\alpha_{-n-m}\,\alpha_n\,\alpha_m\right)$$
$$+(t_1+t_2)\sum_{n>0}\frac{n-1}{2}\,\alpha_{-n}\,\alpha_n \tag{5}$$

with the following convention[2] about the arguments of the α's. If, say, we have 3 alphas, and hence need 3 arguments, we take the Künneth decomposition of

$$[\text{small diagonal}] = [0]\times[0]\times[\mathbb{C}^2]\in H_{\mathsf{T}}^{\bullet}((\mathbb{C}^2)^3)\,.$$

Similarly, $\alpha_{-n}\,\alpha_n$ is short for $\alpha_{-n}([0])\,\alpha_n([\mathbb{C}^2])$. With this convention, (5) is clearly an operator of cohomological degree 2. There is a systematic way to prove formulas like (5) in the framework of Section 2, see Smirnov [2016a].

A remarkable observation, made independently by several people, is that the operator (5) is identical to the second-quantized trigonometric Calogero–Sutherland operator — an classical object in many-body systems and symmetric functions (its eigenfunctions being the Jack symmetric polynomials), see e.g. Costello and Grojnowski [2003] for a comprehensive discussion.

The quantum CS Hamiltonian[3]

$$H_{CS}=\tfrac{1}{2}\sum_{i=1}^{N}\left(w_i\frac{\partial}{\partial w_i}\right)^2+\theta(\theta-1)\sum_{i<j\le N}\frac{1}{|w_i-w_j|^2}\,,\qquad \theta=-\left(t_1/t_2\right)^{\pm 1}\,, \tag{6}$$

describes a system of N identical particles interacting with $|w|^{-2}$-potential on the unit circle $|w|=1$. After conjugation by an eigenfunction $\prod_{i<j}(w_i-w_j)^{\theta}$, it preserves symmetric Laurent polynomials in w_i and stabilizes as $N\to\infty$ to a limit in which the left-movers and right-movers (that is, symmetric polynomials in w_i and polynomials in w_i^{-1}) decouple. This limit is (5) with α_{-k} proportional to multiplication by $\sum w_i^k$.

Another interpretation of the same equation (5) is an integrable quantum version of the Benjamin–Ono equation of 1-dimensional hydrodynamics, see in particular Abanov and Wiegmann [2005]. The BO equation describes waves on a 1-dimensional surface of a fluid of infinite depth, and it involves the Hilbert transform — a nonlocal operation. This nonlocal operation is precisely responsible for the term $\sum_{n>0} n\,\alpha_{-n}\,\alpha_n$ which is present in (5) and looks a bit unconventional when expressed in terms of the field $\boldsymbol{\alpha}(\zeta)=\sum\alpha_{-n}\zeta^n$. Note that other terms in (5) are the normally ordered constant terms in $\boldsymbol{\alpha}(\zeta)^3$ and $\boldsymbol{\alpha}(\zeta)^2$, respectively.

These by now classical connections are only a preview of the kind of connections that exists between enumerative problems and quantum integrable systems in the full unfolding of the theory. Crucial insights into this connection were made in the pioneering work of N. A. Nekrasov and Shatashvili [2009, 2010].

[2]Note that it differs by a sign from the convention used in Maulik and Okounkov [2012].

[3]Note that the well-known, but still remarkable strong/weak duality $\theta\mapsto 1/\theta$ in the CS model becomes simply the permutation of the coordinates in $\mathbb{C}^2$ in the geometric interpretation.

1.2 Counting curves in $\mathrm{Hilb}(\mathbb{C}^2, n)$**.** Enumerative geometry of curves in an algebraic variety Y is a very old subject in mathematics, with the counts like the 27 lines on a smooth cubic surface going as far back as the work of Cayley from 1849. While superficially the subject may be likened to counting points of Y over some field , the actual framework that the geometers have to construct to do the counts looks very different from the number-theoretic constructions. In particular, the counts are defined treating curves in Y as an excess intersection problem, with the result that the counts are invariant under deformation even though there may be no way to deform actual curves.

Also, the subject draws a lot of inspiration from mathematical physics, where various curve counts are interpreted as counts (more precisely, indices) of supersymmetric states in certain gauge or string theories. This leaves a very visible imprint on the field, ranging from how one organizes the enumerative data to what is viewed as an important goal/result in the subject. In particular, any given count, unless it is something as beautiful as 27 lines, is viewed as only an intermediate step in the quest to uncover universal structures that govern a certain totality of the counts.

Depending on what is meant by a "curve" in Y, enumerative theories come in several distinct flavors, with sometimes highly nontrivial interrelations between them. My personal favorite among them is the Donaldson–Thomas theory of 3-folds Donaldson and Thomas [1996] and Thomas [2000], see, in particular, Okounkov [2017c] for a recent set of lecture notes.

The DT theory views a curve $C \subset Y$ as something defined by equations in Y, that is, as a subsheaf $\mathcal{I}_C \subset \mathcal{O}_Y$ of regular functions on Y formed by those functions that vanish on C. The DT moduli space for Y is thus

$$\mathrm{Hilb}(Y, \mathrm{curves}) = \bigsqcup_{\beta, n} \mathrm{Hilb}(Y, \beta, n) \tag{7}$$

where $\beta \in H_2(Y, \mathbb{Z})_{\mathrm{eff}}$ is the degree of the curve and $n = \chi(\mathcal{O}_C)$ is the holomorphic Euler characteristic of $\mathcal{O}_C = \mathcal{O}_Y / \mathcal{I}_C$. In particular, if C is a smooth connected curve of genus g then $n = 1 - g$.

While similar in construction and universal properties to the Hilbert schemes of point in surfaces, Hilbert schemes of 3-folds are, in general, highly singular varieties and nearly nothing is known about their dimension or irreducible components. However, viewed as moduli of sheaves of on Y of the form $\{\mathcal{I}_C\}$, they have a good deformation theory and thus a virtual fundamental cycle of (complex) dimension $\beta \cdot c_1(Y)$ Thomas [2000]. The DT curve counts are defined by pairing this virtual cycle against natural cohomology classes, such as those pulled back from $\mathrm{Hilb}(D, \mathrm{points})$ via a map that assigns to C its pattern of tangency to a fixed smooth divisor $D \subset Y$, see Figure 1 and the discussion in Section 2 of Okounkov [2017c]. Note that the divisor $D = \bigsqcup D_i$ may be disconnected, as in Figure 1, in which case curve counts really define a tensor in a tensor product of several Fock spaces, e.g. an operator from one Fock space to another if there are two components. I believe that this is the geometric source of great many, if not all, interesting tensors in Fock spaces.

There is a broader world of DT counts, in which one counts other 1-dimensional sheaves on 3-folds (a very important example being the Pandharipande–Thomas moduli spaces of stable pairs Pandharipande and Thomas [2009]), and even more broadly stable

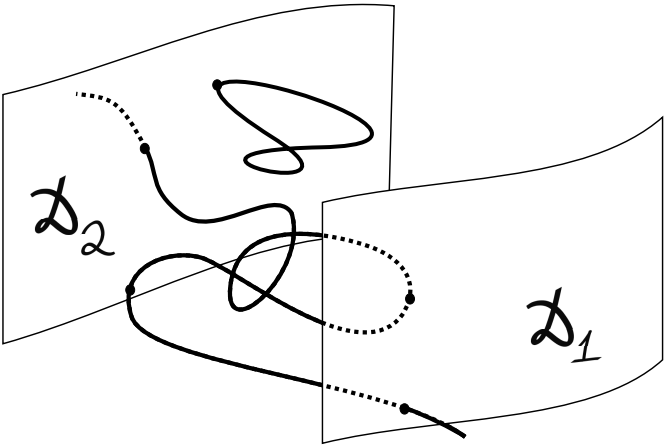

Figure 1: A fundamental object to count in DT theory are algebraic curves, or more precisely, subschemes $C \subset Y$ of given (β, χ) constrained by how they meet a fixed divisor $D \subset Y$.

objects in other categories that look like coherent sheaves on a smooth 3-fold. For all of these, the deformation theory has certain self-duality features that makes K-theoretic and otherwise refined enumerative information a well-behaved and a very interesting object to study.

Since the virtual dimension does not depend on n, it is convenient to organize the DT counts by summing over all n with a weight z^n, where z is a new variable. These are conjectured to be rational functions of z with poles at roots of unity Maulik, N. Nekrasov, Okounkov, and Pandharipande [2006a,b]. This is known in many important cases Maulik, Oblomkov, Okounkov, and Pandharipande [2011], Pandharipande and Pixton [2013], Smirnov [2016b], and Toda [2010] and may be put into a larger conjectural framework as in N. Nekrasov and Okounkov [2016], see Section 3.1.

An algebraic analog of cutting Y into pieces is a degeneration of Y to a transverse union of Y_1 and Y_2 along a smooth divisor D_0 as in Figure 2. A powerful result of Levine and Pandharipande [2009] shows that any smooth projective 3-fold can be linked to a product of projective spaces (or any other basis in algebraic cobordism) by a sequence of such moves. The DT counts satisfy a certain gluing formula for such degeneration Li and Wu [2015] in which the divisor D_0 is added to the divisors from Figure 1. This highlights the importance of understanding the DT counts in certain basic geometries which can serve as building blocks for arbitrary 3-folds.

One set of basic geometries is formed by $\mathcal{S}$-bundles over a curve B

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & Y \\ & & \downarrow{\scriptstyle g} \\ & & B\,, \end{array} \tag{8}$$

where $\mathcal{S}$ is a smooth surface[4]. One take $D = \bigcup g^{-1}(b_i)$ for $\{b_i\} \subset B$ and, by degeneration, this defines a TQFT on B with the space of states $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}))$, where $\mathsf{T} \subset \mathrm{Aut}(\mathcal{S})$ is a maximal torus. This TQFT structure is captured by the counts for $B = \mathbb{P}^1 \supset \{b_1, b_2, b_3\}$, which define a new, z- and β-dependent supercommutative multiplication in this Fock space. It is a very interesting question to describe this multiplication explicitly[5].

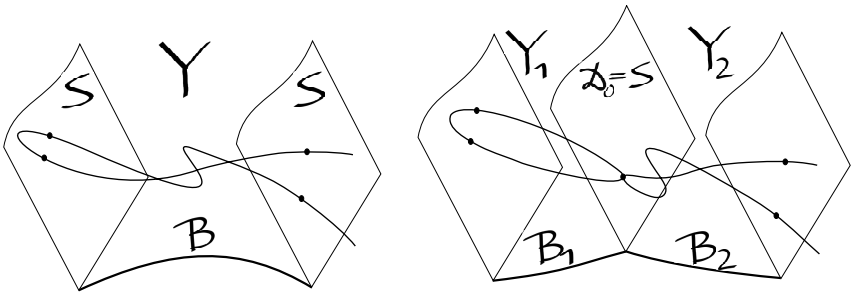

Figure 2: Basic building blocks of DT theory are $\mathcal{S}$ bundles over a curve B as in (8). As B degenerates to a nodal curve $B_1 \cup B_2$, Y degenerates to a transverse union of Y_1 and Y_2 along a smooth divisor $D_0 \cong \mathcal{S}$. DT counts satisfy a gluing formula for such degenerations.

Geometric representation theory provides an answer when $\mathcal{S}$ itself is a symplectic resolution, which concretely means an ADE surface — a minimal resolution of the corresponding surface singularity $\mathcal{S}_0$. In this case, $\mathrm{Hilb}(\mathcal{S}, n)$ is a symplectic resolution of $(\mathcal{S}_0)^n/S(n)$, just like (1), and the multiplication above is nothing but the quantum multiplication in $H_{\mathsf{T}}^{\bullet}(\mathrm{Hilb}(\mathcal{S}))$, see Maulik and Oblomkov [2009] and Okounkov and Pandharipande [2010].

Recall that the quantum product $\star$ is a supercommutative, associative deformation of the classical $\cup$-product in $H_{\mathsf{T}}^{\bullet}(X)$ whose structure constants are the counts of 3-pointed rational curves in X

$$(\alpha \star \beta, \gamma) = \sum_{d \in H_2(X,\mathbb{Z})_{\mathrm{eff}}} z^d \begin{array}{l} \#\text{ of degree } d \text{ rational curves} \\ \text{meeting cycles dual to } \alpha, \beta, \gamma\,. \end{array} \tag{9}$$

This is made mathematically precise using the correspondence

$$\sum_d z^d \, \mathrm{ev}_* \left(\left[\overline{\mathcal{M}}_{0,3}(X, d) \right]_{\mathrm{vir}} \right) \in H_\bullet(X \times X \times X)[[z]] \tag{10}$$

obtained from the virtual fundamental cycle of the moduli space of 3-pointed stable rational maps to X, see Hori, Katz, Klemm, Pandharipande, Thomas, Vafa, Vakil, and Zaslow [2003] for an introduction.

[4]It suffices to take $\mathcal{S} \in \{A_0 = \mathbb{C}^2, A_1, A_2\}$, where A_n is the minimal resolution of the corresponding surface singularity, to generate a basic set of counts.

[5]In particular, there are very interesting results and conjectures for K3 surface fibrations, see Oberdieck [2018]. Note that the dimension counts work out best when $c_1(\mathcal{S}) = 0$.

A closely related structure is the *quantum differential equation*, or Dubrovin connection ∇_X, which is a flat connection on trivial bundle over $H^2(X,\mathbb{C}) \ni \lambda$ with fiber $H^\bullet_\intercal(X,\mathbb{C})$. Its flat sections satisfy

$$\frac{d}{d\lambda}\Psi(z) = \lambda \star \Psi(z)\,, \quad \frac{d}{d\lambda} z^d = (\lambda, d)\, z^d\,, \tag{11}$$

and contain very important enumerative information. For $X = \mathrm{Hilb}(\mathbb{C}^2)$, the quantum multiplication ring is generated by the divisor, so the two structures are really the same.

With the $z \mapsto -z$ substitution, the $\star$-deformation of (5) for $X = \mathrm{Hilb}(\mathbb{C}^2)$ was computed in Okounkov and Pandharipande [2010] as follows

$$c_1(\mathsf{Taut})\star = c_1(\mathsf{Taut}) \cup + (t_1+t_2) \sum_{d>0} d\, \frac{z^d}{1-z^d}\, \alpha_{-d}\, \alpha_d + \dots\,, \tag{12}$$

with the same convention about the arguments of α_n as in (5) and with dots denoting a scalar operator of no importance for us now. Note the simplicity of the purely quantum terms[6].

The largest and the richest class of equivariant symplectic resolutions known to date is formed by the Nakajima quiver varieties Nakajima [1994], of which $\mathrm{Hilb}(\mathbb{C}^2)$ is an example. Formula (12) illustrates many general features of the quantum cohomology of Nakajima quiver varieties proven in Maulik and Okounkov [2012], such as:

- the purely quantum terms are given by a rational function with values in Steinberg correspondences

  $$\text{purely quantum} \in \hbar\, H_{\mathrm{top}}(X \times_{X_0} X) \otimes \mathbb{Q}(z)\,, \tag{13}$$

  where

  $$\hbar = -(t_1+t_2)$$

  is the equivariant weight of the symplectic form.

- the shift $z \mapsto -z$ is an example of the shift by a canonical element of $H^2(X, \mathbb{Z}/2)$, called the theta-characteristic in Maulik and Okounkov [ibid.].

- there is a certain Lie algebra $\mathfrak{g}_Q$ associated to an arbitrary quiver Q in Maulik and Okounkov [ibid.], whose positive roots are represented by effective curve classes $d \in H_2(X,\mathbb{Z})$. Among these, there is a finite set of *Kähler roots* of X such that

  $$\lambda\star = \lambda \cup -\hbar \sum_{d\in\{\text{positive roots}\}} (\lambda, d)\, \frac{z^d}{1-z^d}\, C_d\,, \tag{14}$$

  where $C_d \in \mathfrak{g}_{-d}\, \mathfrak{g}_d$ is the corresponding root components of the Casimir element, that is, the image of the invariant bilinear form on $\mathfrak{g}_{-d} \otimes \mathfrak{g}_d$. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$,

[6] If one interprets (5) as a quantum version of the Benjamin–Ono equation then the new terms deform it to a quantization of the intermediate long wave (ILW) equation. This observation has been rediscovered by many authors.

the quiver is the quiver with one vertex and one loop with $\mathfrak{g}_Q = \widehat{\mathfrak{gl}}(1)$, $C_d = \alpha_{-d}\,\alpha_d$, and

$$\text{positive Kähler roots} = \{1, \dots, n\} \subset \mathbb{Z}\,. \tag{15}$$

- the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_Q$ acts by central elements and by the ranks of the tautological bundles. The operator of cup products by other characteristic classes of the tautological bundles, together with $\mathfrak{g}$, generate a Hopf algebra deformation $\mathcal{Y}(\mathfrak{g}_Q)$ of $\mathcal{U}(\mathfrak{g}_Q[t])$ known as the Yangian.

  The operators of quantum multiplication form a remarkable family of maximal commutative subalgebras of $\mathcal{Y}(\mathfrak{g}_Q)$ known as the Baxter subalgebras in the theory of quantum integrable systems, see Section 2.1. They are parametrized by z and, as $z \to 0$, they become the algebra $\mathcal{Y}(\mathfrak{h}) \subset \mathcal{Y}(\mathfrak{g}_Q)$ of cup products by tautological classes.

  The identification between the $\star$-product ring and Baxter's quantum integrals of motion was predicted by Nekrasov and Shatashvili based on their computation of the spectra of the operators. This served as very important inspiration for Maulik and Okounkov [ibid.].

- The Yangian description identifies the quantum differential equation with the *Casimir connection* for the Lie algebra $\mathfrak{g}_Q$, as studied (in the finite-dimensional case) in Toledano Laredo [2011]. This fits very nicely with the conjecture of Bezrukavnikov and collaborators about the monodromy ∇_X, see below, and was another important inspiration for Maulik and Okounkov [2012], see the historical notes there.

For general symplectic resolutions, there is a definite gap between what is known abstractly, and what can be seen in known examples. I expect that a complete classification of the equivariant symplectic resolutions is within the reach of the current generation of algebraic geometers, and we will see how representative the known examples are. For general symplectic resolutions, the Steinberg correspondence in (13) are constructed in Braverman, Maulik, and Okounkov [2011], while the rationality in z remains abstractly a conjecture that can be checked in all known cases.

A generalization of these structures appears In enumerative K-theory. There, instead of pairing virtual cycles with cohomology classes, we compute the Euler characteristics of natural sheaves, including the virtual structure sheaf $\mathcal{O}_{\text{vir}}$, on the moduli spaces in question (here, the Hilbert scheme of curves in Y).

From its very beginning, K-theory has been inseparable from the indices of differential operators and related questions in mathematical physics. Equivariant K-theoretic DT counts represent a Hamiltonian approach to supersymmetric indices in a certain physical theory (namely, the theory on a D6 brane), in which the space is Y and the time is periodic[7]. Morally, what one computes is the index of a certain infinite-dimensional Dirac operator as a representation of $\mathrm{Aut}(Y)$ which is additionally graded by (β, n).

[7] Or, more precisely, quasiperiodic, with a twist by an element of $\mathsf{T} \subset \mathrm{Aut}(Y)$ after a full circle of time.

Because this is the index of a Dirac operator, the right analog of the virtual cycle is the symmetrized virtual structure sheaf

$$\widehat{\mathcal{O}}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \mathcal{K}_{\text{vir}}^{1/2} \otimes \dots \tag{16}$$

where $\mathcal{K}_{\text{vir}}^{1/2}$ is a square root of the virtual canonical bundle $\mathcal{K}_{\text{vir}}$, the importance of which was emphasized by N. Nekrasov [2005], and the existence of which is shown in N. Nekrasov and Okounkov [2016]. The dots in (16) denote a certain further twist by a tautological line bundle of lesser importance, see N. Nekrasov and Okounkov [ibid.].

While it is not uncommon for different moduli spaces to give the same or equivalent cohomological counts, the K-theoretic counts really feel every point in the moduli space and are very sensitive to the exact enumerative setup. In particular, for both the existence of (16) and the computations with this sheaf, certain self-duality features of the DT deformation theory are crucial. It remains to be seen whether computations with moduli spaces like $\overline{\mathcal{M}}_{0,3}(X)$, that lack such self-duality, can really reproduce the K-theoretic DT counts.

In K-theory, the best setup for counting curves in $X = \text{Hilb}(\mathbb{C}^2, n)$ is the moduli space of stable *quasimaps* to X, see Ciocan-Fontanine, Kim, and Maulik [2014]. Recall that

$$X = \left\{x_1, x_2 \in \text{End}(\mathbb{C}^n), v \in \mathbb{C}^n \middle| [x_1, x_2] = 0\right\} /\!/ GL(n)\,,$$

where the stability condition in the GIT quotient is equivalent to $\mathbb{C}[x_1, x_2]v$ spanning $\mathbb{C}^n$. By definition Ciocan-Fontanine, Kim, and Maulik [ibid.], a stable quasimap from B to a GIT quotient is a map to the quotient stack[8] that evaluates to a stable point away from a finite set of point in B. In return for allowing such singularities, quasimaps offer many technical advantages.

If Y in (8) is a fibration in $\mathcal{S} = \mathbb{C}^2$, one can consider quasimap sections of the corresponding X-bundle over B, and these are easily seen to be identical to the Pandharipande–Thomas stable pairs for Y. Recall that by definition Pandharipande and Thomas [2009], a stable pair is a complex of the form

$$\mathcal{O}_Y \xrightarrow{s} \mathcal{F}$$

where $\mathcal{F}$ is a pure 1-dimensional sheaf and $\dim \text{Coker}\, s = 0$. For our Y, $g_*\mathcal{F}$ is a vector bundle on B, the section s gives v, while x_1, x_2 come from multiplication by the coordinates in the fiber. If the fiber $\mathcal{S}$ contains curve, the picture becomes modified and the PT spaces for the A_n-fibrations, $n > 0$, are related to quasimaps via a certain sequence of wall crossings.

The K-theoretic quasimap counts to $\text{Hilb}(\mathbb{C}^2)$ and, in fact, to all Nakajima varieties have been computed in Okounkov [2017a] and Okounkov and Smirnov [2016], and their structure is a certain q-difference deformation of what we have seen for the cohomological counts. In particular, the Yangian $\mathcal{Y}(\mathfrak{g})$ is replaced by a quantum loop algebra $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ formed by K-theoretic analogs of the correspondences that define the action of $\mathcal{Y}(\mathfrak{g})$, see Sections 2.2 and 3.1.

[8] i.e. a principal G-bundle on B together with a section of the associated bundle of prequotients, where G is the group by which we quotient. For $G = GL(n)$, a principal G-bundle is the same as a vector bundle on B of rank n.

2 Geometric actions of quantum groups

Geometric representation theory in the sense of making interesting algebras act by correspondences is a mature subject and its exposition in Chriss and Ginzburg [2010] is a classic. Geometric construction of representation of quantum groups has been a very important stimulus in the development of the theory of Nakajima quiver varieties Nakajima [1998, 2001].

Below we discuss a complementary approach of Maulik and Okounkov [2012], which mixes geometry and algebra in a different proportion. It has certain convenient hands-off features, in the sense that it constructs a certain category of representations without, for example, a complete description of the algebra by generators and relations. In algebraic geometry, one certainly prefers having a handle on the category $\mathrm{Coh}(X)$ to a complete list of equation that cut out X inside some ambient variety, so the construction should be of some appeal to algebraic geometers. It also interact very nicely with the enumerative question, as it has certain basic compatibilities built in by design.

2.1 Braiding. If G is a group then the category of G-modules over a field has a tensor product — the usual tensor product $M_1 \otimes M_2$ of vector spaces in which an element $g \in G$ acts by $g \otimes g$. There is also a trivial representation $g \mapsto 1 \in \mathrm{End}()$, which is the identity for $\otimes$. This reflects the existence of a coproduct, that is, of an algebra homomorphism

$$G \ni g \xrightarrow{\Delta} g \otimes g \in G \otimes G\,, \tag{17}$$

where G is the group algebra of G, with the counit

$$G \ni g \xrightarrow{\varepsilon} 1 \in \ .$$

There are also dual module $M^* = \mathrm{Hom}_{(}\, M, \,)$ in which g acts by $(g^{-1})^T$, reflecting the antiautomorphism

$$G \ni g \xrightarrow{\text{antipode}} g^{-1} \in G\,.$$

Just like the inverse in the group, the antipode is unique if it exist, and so it will be outside of our focus in what follows, see Etingof and Schiffmann [2002].

An infinitesimal version of this for a Lie or algebraic group G is to replace G by the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g} = \mathrm{Lie}\, G$, with the coproduct obtained from (17) by Leibniz rule

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi\,, \quad \xi \in \mathfrak{g}\,.$$

The multiplication, comultiplication, unit, counit, and the antipode form an beautiful algebraic structure known as a Hopf algebra, see e.g. Etingof and Schiffmann [ibid.]. Another classical example is the algebra $[G]$ of regular functions on an algebraic group. Remarkably, the axioms of a Hopf algebra are self-dual under taking duals and reversing all arrows. Observe that all of the above examples are either commutative, or cocommutative.

Broadly, a quantum group is a deformation of the above examples in the class of Hopf algebras. Our main interests is in Yangians and quantum loop algebras that are deformations of

$$\mathcal{U}(\mathfrak{g}[t]) \rightsquigarrow \mathcal{Y}(\mathfrak{g}), \quad \mathcal{U}(\mathfrak{g}[t^{\pm 1}]) \rightsquigarrow \mathcal{U}_\hbar(\widehat{\mathfrak{g}}), \tag{18}$$

respectively. Their main feature is the loss of cocommutativity. In other words, the order of tensor factors now matters and

$$M_1 \otimes M_2 \ncong M_2 \otimes M_1,$$

in general, or at least the permutation of the tensor factors is no longer an intertwining operator. While tensor categories are very familiar to algebraic geometers, this may be an unfamiliar feature. But, as the representation-theorists know, a mild noncommutativity of the tensor product makes the theory richer and more constrained.

The Lie algebras $\mathfrak{g}[t]$ and $\mathfrak{g}[t^{\pm 1}]$ in (18) are $\mathfrak{g}$-valued functions on the additive, respectively multiplicative, group of the field and they have natural automorphisms

$$t \mapsto t + a, \quad \text{resp.} \quad t \mapsto at, \quad a \in \mathbb{G},$$

where we use $\mathbb{G}$ as a generic symbol for either an additive or multiplicative group. The action of $\mathbb{G}$ will deform to an automorphism of the quantum group[9] and we denote by $M(a)$ the module M, with the action precomposed by an automorphism from $\mathbb{G}$.

The main feature of the theory is the existence of intertwiner (known as the braiding, or the R-matrix)

$$R^\vee(a_1 - a_2) : M_1(a_1) \otimes M_2(a_2) \to M_2(a_2) \otimes M_1(a_1) \tag{19}$$

which is invertible as a rational function of $a_1 - a_2 \in \mathbb{G}$ and develops a kernel and cokernel for those values of the parameters where the two tensor products are really not isomorphic. One often works with the operator $R = (12) \circ R^\vee$ that intertwines two different actions on the same vector space

$$R(a_1 - a_2) : M_1(a_1) \otimes M_2(a_2) \to M_1(a_1) \otimes_{\text{opp}} M_2(a_2). \tag{20}$$

As the word *braiding* suggest, there is a constraint on the R-matrices coming from two different ways to put three tensor factors in the opposite order. In our situation, the corresponding products of intertwiners will be simply be equal, corresponding to the Yang–Baxter equation

$$\begin{aligned} R_{M_1,M_2}(a_1 - a_2)\, R_{M_1,M_3}(a_1 - a_3)\, R_{M_2,M_3}(a_2 - a_3) = \\ = R_{M_2,M_3}(a_2 - a_3)\, R_{M_1,M_3}(a_1 - a_3)\, R_{M_1,M_2}(a_1 - a_2), \end{aligned} \tag{21}$$

satisfied by the R-matrices.

There exists general reconstruction theorem that describe tensor categories of a certain shape and equipped with a fiber functor to vector spaces as representation categories

[9] In fact, for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, it is natural to view this *loop rotation* automorphism as part of the Cartan torus.

of quantum groups, see Etingof, Gelaki, Nikshych, and Ostrik [2015] and Etingof and Schiffmann [2002]. For practical purposes, however, one may be satisfied by the following simple-minded approach that may be traced back to the work of the Leningrad school of quantum integrable systems N. Y. Reshetikhin [1989].

Let $\{M_i\}$ be a collection of vector spaces over a field and

$$R_{M_i,M_j}(a) \in GL(M_1 \otimes M_2, (a))$$

a collection of operators satisfying the YB equation (21). From this data, one constructs a certain category $\mathcal{C}$ of representation of a Hopf algebra $\mathcal{Y}$ as follows. We first extend the R-matrices to tensor products by the rule

$$R_{M_1(a_1),M_2(a_2)\otimes M_3(a_3)} = R_{M_1,M_3}(a_1 - a_3)\, R_{M_1,M_2}(a_1 - a_2)\,, \quad \text{etc.}$$

It is clear that these also satisfy the YB equation. One further extends R-matrices to dual vector spaces using inversions and transpositions, see N. Y. Reshetikhin [ibid.]. Note that in the noncocommutative situation, one has to distinguish between left and right dual modules. For simplicity, we may assume that the set $\{M_i\}$ is already closed under duals.

The objects of the category $\mathcal{C}$ are thus $M = \bigotimes M_i(a_i)$ and we define the quantum group operators in M as the as the matrix coefficients of the R-matrices. Concretely, we have

$$\mathsf{T}_{M_0,m_0}(u) \stackrel{\text{def}}{=} \operatorname{tr}_{M_0}(m_0 \otimes 1) R_{M_0(u),M} \in \operatorname{End}(M) \otimes (u) \tag{22}$$

for any operator m_0 of finite rank in an auxiliary space $M_0 \in \operatorname{Ob}(\mathcal{C})$. The coefficients of u in (22) give us a supply of operators

$$\mathcal{Y} \subset \prod_M \operatorname{End}(M)\,.$$

The YB equation (21) can now be read in two different ways, depending on whether we designate one or two factors as auxiliary. With these two interpretations, it gives either:

- a commutation relation between the generators (22) of $\mathcal{Y}$, or
- a braiding of two $\mathcal{Y}$ modules.

Further, any morphism in $\mathcal{C}$, that is, any operator that commutes with $\mathcal{Y}$ gives us relations in $\mathcal{Y}$ when used in the auxiliary space. This is a generalization of the following classical fact: if $G \subset \prod GL(M_i)$ is a reductive algebraic group, then to know the equations of G is equivalent to knowing how $\otimes M_{k_i}$ decompose as G-modules.

The construction explained below gives geometric R-matrices, that is, geometric solutions of the YB equation acting in spaces like (3) in the Yangian situation, or in the corresponding equivariant K-theories for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. This gives a quantum group $\mathcal{Y}$ which is precisely of the right size for the enumerative application. With certain care, see Maulik and Okounkov [2012], the above construction works over a ring like $= H_{\mathsf{T}}^\bullet(\mathrm{pt})$.

The construction of R-matrices uses *stable envelopes*, which is a certain technical notion that continues to find applications in both in enumerative and representation-theoretic contexts.

Let z be an operator in each M_i such that

$$[z \otimes z, R_{M_1,M_2}] = 0\,.$$

A supply of such is provided by the Cartan torus $\exp(\mathfrak{h})$ where, in geometric situations, $\mathfrak{h}$ acts by the ranks of the universal bundles. A classical observation of Baxter then implies that

$$\left[\mathsf{T}_{M_0,z}(u), \mathsf{T}_{M_0',z}(u')\right] = 0$$

for fixed z and any $M_0(u)$ and $M_0'(u')$. In general, these depend rationally on the entries of z as it is not, usually, an operator of finite rank. The corresponding commutative subalgebras of $\mathcal{Y}$ are known as the Baxter, or Bethe subalgebras. They have a direct geometric interpretation as the operators of quantum multiplication, see below.

2.2 Stable envelopes. There are two ways in which a symplectic resolution $X \xrightarrow{\pi} X_0$ may break up into simpler symplectic resolutions. One of them is deformation. There is a stratification of the deformation space $\mathrm{Def}(X,\omega) = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ by the different singularities that occur, the open stratum corresponding to smooth X_0 or, equivalently, affine X, see Kaledin [2009]. In codimension 1, one sees the simplest singularities into which X can break, and this is related to the decompositions (13) and (14), see Braverman, Maulik, and Okounkov [2011], and so to the notion of the Kähler roots of X introduced above. In the context of Section 3.2, the hyperplanes of the quantum dynamical Weyl group may be interpreted as a further refinement of this stratification that records singular *noncommutative* deformations of X_0.

A different way to break up X into simpler pieces is to consider the fixed points X^{A} of a symplectic torus $\mathsf{A} \subset \mathrm{Aut}(X,\omega)$. The first order information about the geometry of X around X^{A} is given by the A-weights in the normal bundle $N_{X/X^{\mathsf{A}}}$. These are called the equivariant roots for the action of A, or just the equivariant roots of X if A is a maximal torus in $\mathrm{Aut}(X,\omega)$.

There is a certain deep Langlands-like (partial) duality for equivariant symplectic resolutions that interchanges the roles of equivariant and Kähler variables. The origin of this duality, sometimes called the 3-dimensional mirror symmetry, is in 3-dimensional supersymmetric gauge theories, see Intriligator and Seiberg [1996] and de Boer, Hori, Ooguri, and Oz [1997] and also e.g. Bullimore, Dimofte, and Gaiotto [2017], Bullimore, Dimofte, Gaiotto, and Hilburn [2016], and Braden, Licata, Proudfoot, and Webster [2010] for a thin sample of references. As this duality interchanges K-theoretic enumerative information, the quantum difference equation for X becomes the shift operators for its dual $X^\vee$, see Okounkov [2017a] for an introduction to these notions. Among other things, the Kähler and equivariant roots control the poles in these difference equations, which makes it clear that they should be exchanges by the duality.

For the development of the theory, it is very important to be able to both break X into simpler piece and also to find X as such a piece in a more complex ambient geometry.

For instance for $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, we can take the maximal torus $\mathsf{A} \subset SL(2)$, in which case X^{A} is a finite set of monomial ideals

$$I_\lambda \subset \mathbb{C}[x_1, x_2]\,, \quad |\lambda| = n\,,$$

indexed by partitions λ of the number n. At I_λ, the normal weights are $\{\pm\text{hook}(\square)\}_{\square\in\lambda}$, with the result that

$$\text{equivariant roots of } \operatorname{Hilb}(\mathbb{C}^2,n) = \{\pm 1,\dots,\pm n\}\,. \tag{23}$$

Notice the parallel with (15). It just happens that $\operatorname{Hilb}(\mathbb{C}^2,n)$ is self-dual, we will see other manifestations of this below.

More importantly, products $\prod_{i=1}^r \operatorname{Hilb}(\mathbb{C}^2,n_i)$ may be realized in a very nontrivial way as fixed loci of a certain torus A on an ambient variety $\mathcal{M}(r,\sum n_i)$. Here $\mathcal{M}(r,n)$ is the moduli space of framed torsion free sheaves $\mathcal{F}$ on $\mathbb{P}^2$ of rank r and $c_2(\mathcal{F}) = n$. A framing is a choice of an isomorphism

$$\phi : \mathcal{F}\big|_L \xrightarrow{\sim} \mathcal{O}_L^{\oplus r}\,, \quad L = \mathbb{P}^2\setminus\mathbb{C}^2\,,$$

on which the automorphism group $GL(r)$ acts by postcomposition. The spaces $\mathcal{M}(r,n)$ are the general Nakajima varieties associated to the quiver with one vertex and one loop. They play a central role in supersymmetric gauge theories as symplectic resolutions of the Uhlenbeck spaces of framed instantons, see Nakajima [1999]. It is easy to see that $\mathcal{M}(1,n) = \operatorname{Hilb}(\mathbb{C}^2,n)$ and

$$\mathcal{M}(r,n)^{\mathsf{A}} = \bigsqcup_{\sum n_i = n} \prod_{i=1}^r \operatorname{Hilb}(\mathbb{C}^2,n_i)\,,$$

where $\mathsf{A}\subset GL(r)$ is the maximal torus. For general Nakajima varieties, there is a similar decomposition for the maximal torus A of framing automorphisms.

The cohomological stable envelope is a certain Lagrangian correspondence

$$\operatorname{Stab}\subset X\times X^{\mathsf{A}}\,, \tag{24}$$

which may be seen as an improved version of the attracting manifold

$$\operatorname{Attr} = \Big\{(x,y)\Big|\lim_{a\to 0} a\cdot x = y\Big\}\subset X\times X^{\mathsf{A}}\,.$$

The support of (24) is the full attracting set $\operatorname{Attr}_f\subset X\times X^{\mathsf{A}}$, which is the smallest closed subset that contains the diagonal and is closed under taking $\operatorname{Attr}(\cdot)$.

To define attracting and repelling manifolds, we need to separate the roots for A-weights into positive and negative, that is, we need to choose a chamber $\mathfrak{C}\subset \operatorname{Lie}\mathsf{A}$ in the complement of the root hyperplanes. Note this gives an ordering on the set of components $\bigsqcup F_i = X^{\mathsf{A}}$ of the fixed locus: $F_1 > F_2$ if $\operatorname{Attr}_f(F_1)$ meets F_2.

For

$$\operatorname{Lie}\mathsf{A} = \{\operatorname{diag}(a_1,\dots,a_r)\}\subset \mathfrak{gl}(r) \tag{25}$$

as in Section 2.2, the roots are $\{a_i - a_j\}$, and so a choice of $\mathfrak{C}$ is the usual choice of a Weyl chamber. As we will see, it will correspond to an ordering of tensor factors as in Section 2.1.

The stable envelope is characterized by:

- it is supported on Attr_f,
- it equals[10] $\pm\,\mathrm{Attr}$ near the diagonal in $X^{\mathsf{A}} \times X^{\mathsf{A}} \subset X \times X^{\mathsf{A}}$,
- for an off-diagonal component $F_2 \times F_1$ of $X^{\mathsf{A}} \times X^{\mathsf{A}}$, we have

$$\deg_{\mathrm{Lie}(\mathsf{A})} \mathrm{Stab}\Big|_{F_2\times F_1} < \tfrac{1}{2}\operatorname{codim} F_2 = \deg_{\mathrm{Lie}(\mathsf{A})} \mathrm{Attr}\Big|_{F_2\times F_2}, \tag{26}$$

where the degree is the usual degree of polynomials for

$$H^\bullet_{\mathsf{A}}(X^{\mathsf{A}}, \mathbb{Z}) \cong H^\bullet(X^{\mathsf{A}}) \otimes \mathbb{Z}[\mathrm{Lie}\,\mathsf{A}].$$

Condition (26) is a way to quantify the idea that $\mathrm{Stab}\big|_{F_2\times F_1}$ is smaller than $\mathrm{Attr}(F_2)$. This makes the stable envelope a canonical representative of $\overline{\mathrm{Attr}(F_1)}$ modulo cycles supported on the lower strata of Attr_f.

The existence and uniqueness of stable envelopes are proven, under very general assumptions on X in Maulik and Okounkov [2012]. As these correspondences are canonical, they are invariant under the centralizer of A and, in particular, act in T-equivariant cohomology for any ambient torus T.

To put ourselves in the situation of Section 2.1, we define a category in which the objects are

$$\mathsf{F}(a_1, \dots, a_r) = \bigoplus_{n\geq 0} H^\bullet_{\mathsf{T}}(\mathcal{M}(r,n)) \tag{27}$$

and the maps defined by stable envelopes, like

$$\mathsf{F}(a_1) \otimes \mathsf{F}(a_2) \underset{\mathrm{Stab}_-}{\overset{\mathrm{Stab}_+}{\rightrightarrows}} \mathsf{F}(a_1, a_2)\ , \tag{28}$$

where $\mathfrak{C}_\pm = \{a_1 \gtrless a_2\}$, are declared to be morphisms. Since both maps in (28) are isomorphisms after A-equivariant localization, we get a rational matrix

$$R(a_1 - a_2) = \mathrm{Stab}_-^{-1} \circ \mathrm{Stab}_+ \in \mathrm{End}(\mathsf{F}(a_1) \otimes \mathsf{F}(a_2)) \otimes \mathbb{Q}(a_1 - a_2)\,. \tag{29}$$

Its basic properties are summarized in the following

Theorem 1 (Maulik and Okounkov [ibid.]). *The R-matrix (29) satisfies the YB equation and defines, as in Section 2.1, an action of $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ in equivariant cohomology of $\mathcal{M}(r,n)$. The Baxter subalgebras in $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ are the algebras of operators of quantum multiplication. In particular, the vacuum-vacuum elements of the R-matrix are the operators of classical multiplication in $\mathcal{M}(r,n)$.*

[10] The $\pm$ a choice here, reflecting a choice of a *polarization* of X, which is a certain auxiliary piece of data that one needs to fix in the full development of the theory.

Here z in the Cartan torus of $\widehat{\mathfrak{gl}}(1)$ acts by z^n in the nth term of (27), which clearly commutes with R-matrices. The vacuum in (27) is the $n = 0$ term.

For a general Nakajima variety, it is proven in Maulik and Okounkov [ibid.] that the corresponding R-matrices define an action of $\mathcal{Y}(\mathfrak{g})$ for a certain Borcherds–Kac–Moody Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \,,$$

with finite-dimensional root spaces $\mathfrak{g}_\alpha$. This Lie algebra is additionally graded by the cohomological degree and it has been conjectured in Okounkov [n.d.] that graded dimensions of $\mathfrak{g}_\alpha$ are given by the Kac polynomial for the dimension vector α. A slightly weaker version of this conjecture is proven in Schiffmann [2008a,b].

Again, the operators of classical multiplication are given by the vacuum-vacuum matrix elements of the R-matrix, while for the quantum multiplication we have the formula already announced in Section 1.2

Theorem 2 (Maulik and Okounkov [2012]). *For a general Nakajima variety, quantum multiplication by divisors is given by the formula* (14) *and hence the quantum differential equation is the Casimir connection for* $\mathcal{Y}(\mathfrak{g})$.

Back to the Hilbert scheme case, the R-matrix (29) acting in the tensor product of two Fock spaces is a very important object for which various formulas and descriptions are available. The following description was obtained in Maulik and Okounkov [ibid.].

The operators

$$\alpha_n^{\pm} = \alpha_n \otimes 1 \pm 1 \otimes \alpha_n$$

act in the tensor product of two Fock spaces, and form two commuting Heisenberg subalgebras. They are analogous to the center of mass and separation coordinates in a system of two bosons, and we can similarly decompose

$$\mathsf{F}(a_1) \otimes \mathsf{F}(a_2) = \mathsf{F}_+(a_1 + a_2) \otimes \mathsf{F}_-(a_1 - a_2) \,, \tag{30}$$

where to justify the labels we introduce the zero modes $\alpha_0(\gamma)$ that act on $\mathsf{F}(a_i)$ by $-a_i \int \gamma$. Here integral denotes the equivariant integration of $\gamma \in H_{\mathsf{T}}^\bullet(\mathbb{C}^2)$. Consider the operators L_n defined by

$$\sum L_n \zeta^{-n} = \frac{1}{4} :\boldsymbol{\alpha}_-(\zeta)^2: \pm \frac{1}{2} \hbar \, \partial \boldsymbol{\alpha}_-(\zeta) - \frac{1}{4} \int \hbar^2 \,, \tag{31}$$

where $:\boldsymbol{\alpha}_-(\zeta)^2:$ is the normally ordered square of the operator $\boldsymbol{\alpha}_-(\zeta) = \sum \alpha_n^- \zeta^n$, which now has a constant term in ζ, and ∂ stands for $\zeta \frac{\partial}{\partial \zeta}$. For the cohomology arguments of α_n^-, we use the conventions of Section 1.1. For either choice of sign, (31) form the Virasoro algebra

$$[L_n, L_m] = (m-n) L_{n+m} + \frac{(n^3 - n)}{12} \left(1 + 6 \int \hbar^2\right) \delta_{n+m} \,, \tag{32}$$

in a particular free field realization that is very familiar from the work of B. Feigin and Fuchs [1990] and from CFT. Note that, for generic $a_1 - a_2$, $\mathsf{F}_-(a_1 - a_2)$ is irreducible

with highest weight

$$L_0 \,|\rangle = \frac{1}{4} \int ((a_1 - a_2)^2 - \hbar^2)\, |\rangle\,,$$

which, like (32) is invariant under $a_1 \leftrightarrow a_2$ and $\hbar \mapsto -\hbar$

Theorem 3 (Maulik and Okounkov [2012]). *The R-matrix is the unique operator in* F_- *in* (30) *that preserves the vacuum* $|\rangle$ *and interchanges the two signs in* (31).

This is the technical basis of numerous fruitful connection between $\mathcal{Y}(\widehat{\mathfrak{gl}}(1))$ and CFT. For instance, the operator $R^\vee = (12) \circ R$ is related to the reflection operator in Liouville CFT, see Zamolodchikov and Zamolodchikov [1996], and the YB equation satisfied by R reveals new unexpected features of affine Toda field theories.

3 Some further directions

3.1 K-theoretic counts. K-theoretic counts require a definite technical investment to be done properly, but offer an ample return producing deeper and more symmetric theories. For example, the dualities already mentioned in Section 2.2 reveal their full power only in equivariant K-theory.

Focusing on $\mathrm{Hilb}(\mathbb{C}^2, n)$, its self-duality may be put into an even deeper and, not surprisingly, widely conjectural framework of M-theory, which is a certain unique 11-dimensional supergravity theory with the power to unify many plots in modern theoretical physics as well as pure mathematics. Its basic actors are membranes with a 3-dimensional world volume that may appear as strings or even point particles to a low-resolution observer, see Bagger, Lambert, Mukhi, and Papageorgakis [2013] for a recent review. The supersymmetric index for membranes of the form

$$C \times S^1 \subset Z \times S^1\,,$$

where C is a complex curve in a complex Calabi–Yau 5-fold Z, should be a virtual representation of the automorphisms $\mathrm{Aut}(Z, \Omega_Z^5)$ that preserve the 5-form, so a certain K-theoretic curve count in Z. See N. Nekrasov and Okounkov [2016] for what it might look like and a conjectural equivalence with K-theoretic DT counts for

$$Y = Z^{\mathbb{C}_z^\times}\,,$$

for any $\mathbb{C}_z^\times \subset \mathrm{Aut}(Z, \Omega_Z^5)$ with a purely 3-dimensional fixed locus. The most striking feature of this equivalence is the interpretation of the variable z

$$\begin{aligned} z &= \text{degree-counting variable in (12)} \\ &= \text{the } \chi(\mathcal{O}_C)\text{-counting variable in DT theory, see Section 1.2} \\ &\overset{!}{=} \text{equivariant variable } z \in \mathbb{C}_z^\times \text{ in M-theory}\,. \end{aligned} \tag{33}$$

Note that in cohomology equivariant variables take values in a Lie algebra, and so cannot be literally on the same footing as Kähler variables.

As a local model, one can take the total space

$$Z = \begin{array}{c} \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \mathcal{L}_4 \\ \downarrow \\ B \end{array}, \quad \bigotimes_{i=1}^{4} \mathcal{L}_i = \mathcal{K}_B\,, \tag{34}$$

of 4 line bundles over a smooth curve B. In this geometry, one can designate any two $\mathcal{L}_i$ to form Y by making z scale the other two line bundles with opposite weights. Thus the counting of Section 1.2 with $\mathcal{S} = \mathbb{C}^2$, when properly set up in K-theory, has a conjectural $S(4)$-symmetry that permutes the weights

$$(t_1, t_2, \frac{z}{\sqrt{q t_1 t_2}}, \frac{1}{z\sqrt{q t_1 t_2}})\,, \quad \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in \mathrm{Aut}(\mathbb{C}^2)\,,$$

where q^{-1} is the Chern root of $\mathcal{K}_B$.

This theory is described by certain q-difference equation that describes the change of the counts as the degrees of $\{\mathcal{L}_i\}$ change, and correspondingly the variables t_1, t_2, z are shifted by powers of q. In fact, similar q-difference equations can be defined for any Nakajima variety including $\mathcal{M}(r,n)$ from Section 2.2. For those, there are also difference equations in the framing equivariant variables a_i.

Theorem 4 (Okounkov [2017a] and Okounkov and Smirnov [2016]). *The q-difference equations in variables a_i are the quantum Knizhnik–Zamolodchikov (qKZ) equations for $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$ and the q-difference equation in z is the lattice part of the dynamical Weyl group of this quantum group. Same is true for a general Nakajima variety and the corresponding quantum loop algebra $\mathcal{U}_\hbar(\mathfrak{g})$.*

The qKZ equations, introduced in Frenkel and N. Reshetikhin [1992], have the form

$$\Psi(q a_1, a_2, \dots, a_n) = (z \otimes 1 \otimes \cdots \otimes 1)\, R_{1,n}(a_1/a_n) \dots R_{1,2}(a_1/a_2)\, \Psi \tag{35}$$

with similar equations in other variables a_i, where z is as in (25). For the R-matrices of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, where $\dim \mathfrak{g} < \infty$, these play the same role in integrable 2-dimensional lattice models as the classical KZ equations play for their conformal limits, see e.g. Jimbo and Miwa [1995].

For $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$, the R-matrix is a generalization of the R-matrix of Theorem 3 constructed using the K-theoretic stable envelopes. The algebra $\mathcal{U}_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$ is constructed from this R-matrix using the general procedure of Section 2.1. For this particular geometry, it also coincides with the algebras constructed by many authors by different means, including explicit presentations see e.g. Burban and Schiffmann [2012], B. Feigin, E. Feigin, Jimbo, Miwa, and Mukhin [2011], Negut [2015], and Schiffmann and Vasserot [2013] for a sample of references where the same algebra appears.

It takes a certain development of the theory to define and make concrete the operators from the quantum dynamical Weyl group of an algebra like $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. They crucially depend on certain features of K-theoretic stable envelopes that are not visible in cohomology.

In the search for the right K-theoretic generalization of (26), one should keep in mind that the correct notion of the degree of a multivariate Laurent polynomial is its Newton polytope, that is, the convex hull of its exponents, considered *up to translation*. Translations correspond to multiplication by monomials, which are invertible functions. The right generalization of (26) is then

$$\text{Newton}\left(\text{Stab}\Big|_{F_2\times F_1}\right) \subset \text{Newton}\left(\text{Attr}\Big|_{F_2\times F_2}\right) + \text{shift}_{F_1,F_2}\,, \tag{36}$$

for a certain collection of shifts in $\mathsf{A}^\vee \otimes \mathbb{R}$, where $\mathsf{A}^\vee$ is the character lattice of the torus A. Shifts come from weights of fractional line bundles $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$ at the fixed points, see Okounkov [2017a] for a survey. This fractional line bundle, called the *slope* of the stable envelope is a new parameter, the dependence on which is locally constant and quasiperiodic — if the slopes differ by integral line bundle $\mathcal{L}$, then the corresponding stable envelopes differ by a twist by $\mathcal{L}$.

Stable envelopes change across certain hyperplanes in $\text{Pic}(X)\otimes\mathbb{R}$ that form a $\text{Pic}(X)$-periodic hyperplane arrangement closely related to the Kähler roots of X. For example, for $\widehat{\mathfrak{gl}}(1)$ the roots are $\{\alpha\} = \mathbb{N}\setminus\{0\}$ and so the affine root hyperplanes are all rationals

$$\{x \,|\, \exists \alpha\ \langle \alpha, x\rangle \in \mathbb{Z}\} = \mathbb{Q} \subset \mathbb{R}\,. \tag{37}$$

The dynamical Weyl group element $\mathsf{B}_{a/b}$ in (39) corresponding to a wall $\frac{a}{b} \in \mathbb{Q}$ acts trivially in $K_\mathsf{T}(\text{Hilb}(\mathbb{C}^2, n)$ if $b > n$, reflecting the fact that the Kähler roots of $\text{Hilb}(\mathbb{C}^2, n)$ form a finite subset $\{\pm 1, \dots, \pm n\}$ of the roots of $\widehat{\mathfrak{gl}}(1)$.

The change across a particular wall is recorded by a certain wall R-matrix R_{wall} and the R-matrices corresponding to a change of chamber $\mathfrak{C}$ as in (29) factor into an infinite product of those, see Okounkov [2017a] and Okounkov and Smirnov [2016]. Each term in such factorization corresponds to a certain root subalgebra

$$\mathcal{U}_\hbar(\mathfrak{g}_{\text{wall}}) \subset \mathcal{U}_\hbar(\widehat{\mathfrak{g}})$$

stable under the action of

$$\widehat{z} \in \text{Cartan torus of } \mathfrak{g} \times \mathbb{C}_q^\times\,, \tag{38}$$

where $\mathbb{C}_q^\times$ acts by loop rotation automorphisms of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$. One defines Okounkov and Smirnov [2016] the dynamical Weyl group by taking certain specific (z, q)-dependent elements

$$\mathsf{B}_{\text{wall}} \in \mathcal{U}_\hbar(\mathfrak{g}_{\text{wall}})\,. \tag{39}$$

For finite-dimensional $\mathfrak{g}$, we have $\mathfrak{g}_{\text{wall}} = \mathfrak{sl}(2)$ for every wall and the construction specializes to the classical construction of Etingof and Varchenko [2002]. The operators B_{wall} satisfy the braid relations of the wall arrangement. Because each B_{wall} depends on (38) through the equation of the corresponding wall, these relations look like the Yang–Baxter equations (21), in which each term depends on $a = (a_1, a_2, a_3)$ through the equation $a_i - a_j = 0$ of the hyperplane being crossed in Lie A.

3.2 Monodromy and derived equivalences. As a special $\widehat{z}$-independent case, the dynamical Weyl group contains the so-called:

- **(w)** quantum Weyl group of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$, which plays many roles, including
- **(m)** this is the monodromy group of the quantum differential equation (11), and
- **(p)** this group describes the action on $K_T(X)$ of the derived automorphisms of X constructed by Bezrukavnikov and Kaledin using quantizations $\widehat{X}_c$ of X in characteristic $p \gg 0$, see Bezrukavnikov and Finkelberg [2014], Bezrukavnikov and Kaledin [2008], Bezrukavnikov and Losev [2013], Kaledin [2008], and Losev [2014, 2017a, 2016, 2017b]. It thus plays the same role in modular representation theory of $\widehat{X}_c$ as the Hecke algebra plays in the classical Kazhdan–Lusztig theory.

Theorem 5 (Bezrukavnikov and Okounkov [n.d.])**.** *We have* **(w)** $=$ **(m)** $=$ **(p)** *for all Nakajima varieties.*

Other known infinite series of equivariant symplectic resolutions are also considered in Bezrukavnikov and Okounkov [ibid.]. For finite-dimensional $\mathfrak{g}$, the description of the monodromy of the Casimir connection via the quantum Weyl group is a conjecture of V. Toledano Laredo [2011]. The equality **(m)** $=$ **(p)** for all equivariant symplectic resolutions is a conjecture of Bezrukavnikov and the author, see the discussion in Anno, Bezrukavnikov, and Mirković [2015] and Okounkov [2017b].

Here $\widehat{X}_c$ is an associative algebra deformation of the Poisson algebra of functions on X or X_0. While it may be studied abstractly, quantizations of Nakajima varieties may be described concretely as quantum Hamiltonian reductions, see Etingof [2007] and Gan and Ginzburg [2006]. For $X = \mathrm{Hilb}(\mathbb{C}^2, n)$, $\widehat{X}_c$ is the algebra generated by symmetric polynomials in $\{w_1, \dots, w_n\}$, and the operators of the rational Calogero quantum integrable systems — a commutative algebra of differential operators that includes the following rational analog of (6)

$$H_{\mathrm{C,rat}} = \tfrac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial w_i^2} - c(c+1) \sum_{i<j\le N} \frac{1}{(w_i - w_j)^2}\,. \tag{40}$$

The kinship between (6) and (40) is closer than normal because of the self-duality of $\mathrm{Hilb}(\mathbb{C}^2, n)$. Recall that duality swaps equivariant and Kähler variables and instead of an equivariant variable θ in (6) we have a Kähler variable c in (40). It parametrizes deformations of $\widehat{X}_c$ in the same way as $\mathrm{Pic}(X)\otimes_{\mathbb{Z}}$ parametrizes deformations of X over a field .

For $p \gg 0$, the BK theory produces derived equivalences

$$\mathrm{D^b\,Coh}\, X^{(1)} \xleftarrow{\sim} \mathrm{D^b}\, \widehat{X}_c\text{-mod} \xleftarrow{\sim} \mathrm{D^b\,Coh}\, X^{(1)}_{\mathrm{flop}} \tag{41}$$

for every nonsingular value of $c \in \mathbb{Z}$. A shift $c \mapsto c + p$ twists (41) by $\mathcal{O}(1)$. Here $X^{(1)}$ denotes the Frobenius twist of X and X_{flop} refers to a change of stability condition in the GIT construction of X.

In the $(\mathsf{m}) = (\mathsf{p})$ interpretation, the composed equivalence in (41) becomes the transport of the QDE from the point $z = 0$, that corresponds to X, to the point $z = \infty$, that corresponds to X_{flop}, along the ray with

$$\arg z = -2\pi \frac{c}{p}.$$

In particular, this identifies the singularities of the QDE, given by the Kähler roots of X, with the limit $\lim_{p\to\infty} \frac{1}{p}\{c_{\text{sing}}\}$ of the singular parameters of the quantization. See Bezrukavnikov and Okounkov [n.d.] for details.

Monodromy of a flat connection with regular singularities is an analytic map between algebraic varieties that may be seen as a generalization of the exponential map of a Lie group. There is a long tradition, going back to at least the work of Kohno and Drinfeld of computing the monodromy of connections of representation-theoretic origin in terms of closely related algebraic structures. Just like for the exponential map, there is a certain progression in this, as one goes from additive variables to multiplicative, and also from multiplicative — to elliptic. E.g. in the case at hand the QDE (= the Casimir connection) is defined for modules M over the Yangian $\mathcal{Y}(\mathfrak{g})$, and is computed in terms of the action of $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ in a closely related representation[11], see in particular Gautam and Toledano Laredo [2016].

A key step in capturing the monodromy algebraically is typically a certain compatibility constraint between the monodromy for $M = M_1 \otimes M_2$ and the monodromy for the tensor factors. The framework introduced above gives a very conceptual and powerful way to prove such statements. Recall that, geometrically, $\otimes$ arises as a special correspondence between X^{A} and X, where A is a torus that acts on X preserving the symplectic form. Therefore, it is natural to ask, more generally, for a compatibility between the monodromy of the QDE for X and X^{A}.

In fact, one can ask a more general question about the compatibility of the corresponding q-difference equations as in Theorem 4. Let

$$\mathsf{Z} = \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}^\times$$

be the Kähler torus of X and $\overline{\mathsf{Z}}$ be the toric compactification of Z corresponding to the fan of ample cones of flops of X. Its torus-fixed points $0_{X_{\text{flop}}} \in \overline{\mathsf{Z}}$ correspond to all possible flops of X. A regular q-difference connection on a smooth toric variety is an action of the the cocharacter lattice

$$\operatorname{Pic}(X) \ni \lambda \to q^\lambda \in \mathsf{Z}, \quad q \in \mathbb{C}^\times,$$

on a (trivial) vector bundle over $\overline{\mathsf{Z}}$. Shift operators define a commuting regular q-difference connections in the variables $a \in \mathsf{A} \subset \overline{\mathsf{A}}$, where $\overline{\mathsf{A}}$ is the toric variety given by the fan of the chambers $\mathfrak{C}$. The q-difference connection for X^{A} sits overs the torus fixed points $0_{\mathfrak{C}} \in \overline{\mathsf{A}}$.

[11] geometrically, the right relation between $\boldsymbol{H}_{\mathsf{T}}(X)$, where the Yangian acts, and $\boldsymbol{K}_{\mathsf{T}}(X)$ of $\boldsymbol{X}$, where $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ acts, is given by a certain Γ-function analog of the Mukai vector, as in the work of Iritani [2009]

The most interesting analytic feature here is that the connections in z and a, while compatible and separately regular, are *not* regular jointly. This can never happen for differential equations, see Deligne [1970], but is commonplace for q-difference equations as illustrated by the system:

$$f(qz,a) = af(z,a)\,, \quad f(z,qa) = zf(z,a)\,.$$

As a result, near any point $(0_X, 0_{\mathfrak{C}}) \in \bar{\mathsf{Z}} \times \bar{\mathsf{A}}$, we get two kinds of solutions. Those naturally arising enumeratively are holomorphic in z in a punctured neighborhood of 0_X and meromorphic in a with poles accumulating to $0_{\mathfrak{C}}$. These may be called the z-solutions. For a-solutions, the roles of z and a are exchanged. These naturally appear in the Langlands dual setup and the initial conditions at $a = 0_{\mathfrak{C}}$ for them are the z-solutions for X^{A}.

Transition matrices between the a-solutions and the z-solutions, which is by construction elliptic, intertwine the monodromy for X^{A} and X, and vice versa. Note these transition matrices may, in principle, be computed from the series expansions near $(0_X, 0_{\mathfrak{C}})$, which differentiates them from more analytic objects like monodromy or Stokes matrices.

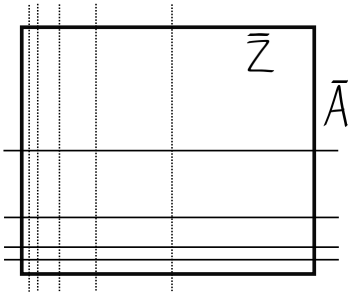

Figure 3: z-solutions are convergent power series in z with coefficients in $\mathbb{Q}(a)$, and the poles (solid lines in the picture) of these coefficients accumulate to $a = 0_{\mathfrak{C}}$. The poles of a-solutions, the dashed lines in the figure, similarly accumulate to $z = 0_X$.

Theorem 6 (Aganagic and Okounkov [2016])**.** *The transformation from the a-solutions to z-solutions is given by elliptic stable envelopes, a certain elliptic analog of* Stab *constructed in Aganagic and Okounkov [ibid.].*

To complete the picture, one can give Mellin–Barnes-type integral solutions to the quantum q-difference equations, and so, in particular to the qKZ and dynamical equations for $\mathcal{U}_\hbar(\widehat{\mathfrak{g}})$ in tensor products of evaluation representations Aganagic and Okounkov [2017]. It is well-known that the stationary phase $q \to 1$ limit in such integrals diagonalizes the Baxter subalgebra and hence generalizes the classical ideas of Bethe Ansatz.

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