ICM 2018
RIO DE JANEIRO

Proc. Int. Cong. of Math. – 2018
Rio de Janeiro, Vol. 1 (935–978)

# SYSTEMS OF POINTS WITH COULOMB INTERACTIONS

Sylvia Serfaty

**Abstract**

Large ensembles of points with Coulomb interactions arise in various settings of condensed matter physics, classical and quantum mechanics, statistical mechanics, random matrices and even approximation theory, and give rise to a variety of questions pertaining to calculus of variations, Partial Differential Equations and probability. We will review these as well as "the mean-field limit" results that allow to derive effective models and equations describing the system at the macroscopic scale. We then explain how to analyze the next order beyond the mean-field limit, giving information on the system at the microscopic level. In the setting of statistical mechanics, this allows for instance to observe the effect of the temperature and to connect with crystallization questions.

## 1 General setups

We are interested in large systems of points with Coulomb-type interactions, described through an energy of the form

$$\mathcal{H}_N(x_1,\dots,x_N) = \frac{1}{2}\sum_{i\neq j} \mathsf{g}(x_i-x_j) + N\sum_{i=1}^N V(x_i). \tag{1-1}$$

Here the points $x_i$ belong to the Euclidean space $\mathbb{R}^{\mathsf{d}}$, although it is also interesting to consider points on manifolds. The interaction kernel $\mathsf{g}(x)$ is taken to be

$$\text{(Log2 case)}\quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d}=2, \tag{1-2}$$

$$\text{(Coul case)}\quad \mathsf{g}(x) = \frac{1}{|x|^{\mathsf{d}-2}}, \quad \text{in dimension } \mathsf{d}\geq 3. \tag{1-3}$$

This is (up to a multiplicative constant) the Coulomb kernel in dimension $\mathsf{d}\geq 2$, i.e. the fundamental solution to the Laplace operator, solving

$$-\Delta \mathsf{g} = \mathsf{c}_{\mathsf{d}}\delta_0 \tag{1-4}$$

*MSC2010:* primary 60F05; secondary 60F15, 60K35, 60B30, 82B05, 82C22.
*Keywords:* Coulomb gases, log gases, mean field limits, jellium, large deviations, point processes, crystallization, Gaussian Free Field.

where $\delta_0$ is the Dirac mass at the origin and $\mathsf{c}_\mathsf{d}$ is an explicit constant depending only on the dimension. It is also interesting to broaden the study to the one-dimensional logarithmic case

$$(\mathsf{Log1}\text{ case}) \quad \mathsf{g}(x) = -\log|x|, \quad \text{in dimension } \mathsf{d} = 1, \tag{1-5}$$

which is not Coulombic, and to more general Riesz interaction kernels of the form

$$\mathsf{g}(x) = \frac{1}{|x|^\mathsf{s}} \quad \mathsf{s} > 0. \tag{1-6}$$

The one-dimensional Coulomb interaction with kernel $-|x|$ is also of interest, but we will not consider it as it has been extensively studied and understood, see Lenard [1961], Lenard [1963], and Kunz [1974].

Finally, we have included a possible external field or confining potential $V$, which is assumed to be regular enough and tending to $\infty$ fast enough at $\infty$. The factor $N$ in front of $V$ makes the total confinement energy of the same order as the total repulsion energy, effectively balancing them and confining the system to a subset of $\mathbb{R}^\mathsf{d}$ of fixed size. Other choices of scaling would lead to systems of very large or very small size as $N \to \infty$.

The Coulomb interaction and the Laplace operator are obviously extremely important and ubiquitous in physics as the fundamental interactions of nature (gravitational and electromagnetic) are Coulombic. Coulomb was a French engineer and physicist working in the late 18$^{\text{th}}$ century, who did a lot of work on applied mechanics (such as modeling friction and torsion) and is most famous for his theory of electrostatics and magnetism. He is the first one who postulated that the force exerted by charged particles is proportional to the inverse distance squared, which corresponds in dimension $\mathsf{d} = 3$ to the gradient of the Coulomb potential energy $\mathsf{g}(x)$ as above. More precisely he wrote in Coulomb [1785] " It follows therefore from these three tests, that the repulsive force that the two balls [which were] electrified with the same kind of electricity exert on each other, follows the inverse proportion of the square of the distance." He developed a method based on systematic use of mathematical calculus (with the help of suitable approximations) and mathematical modeling (in contemporary terms) to predict physical behavior, systematically comparing the results with the measurements of the experiments he was designing and conducting himself. As such, he is considered as a pioneer of the "mathematization" of physics and in trusting fully the capacities of mathematics to transcribe physical phenomena Blondel and Wolff [2015].

Here we are more specifically focusing on Coulomb interactions between points, or in physics terms, discrete point charges. There are several mathematical problems that are interesting to study, all in the asymptotics of $N \to \infty$:

**(1)** understand minimizers and possibly critical points of (1-1) ;

**(2)** understand the statistical mechanics of systems with energy $\mathcal{H}_N$ and inverse temperature $\beta > 0$, governed by the so-called Gibbs measure

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N. \tag{1-7}$$

Here, as postulated by statistical mechanics, $\mathbb{P}_{N,\beta}$ is the density of probability of observing the system in the configuration $(x_1, \dots, x_N)$ if the number of particles is fixed to $N$ and the inverse of the temperature is $\beta > 0$. The constant $Z_{N,\beta}$ is called the "partition function" in physics, it is the normalization constant that makes $\mathbb{P}_{N,\beta}$ a probability measure, [1] i.e.

$$Z_{N,\beta} = \int_{(\mathbb{R}^d)^N} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N, \tag{1-8}$$

where the inverse temperature $\beta = \beta_N$ can be taken to depend on $N$, as there are several interesting scalings of $\beta$ relative to $N$;

**(3)** understand dynamic evolutions associated to (1-1), such as the gradient flow of $\mathcal{H}_N$ given by the system of coupled ODEs

$$\dot{x_i} = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N), \tag{1-9}$$

conservative dynamics given by the systems of ODEs

$$\dot{x_i} = \frac{1}{N} \mathbb{J} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) \tag{1-10}$$

where $\mathbb{J}$ is an antisymmetric matrix (for example a rotation by $\pi/2$ in dimension 2), or the Hamiltonian dynamics given by Newton's law

$$\ddot{x_i} = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N); \tag{1-11}$$

**(4)** understand the previous dynamic evolutions with temperature $\beta^{-1}$ in the form of an added noise (Langevin-type equations) such as

$$dx_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i \tag{1-12}$$

with $W_i$ independent Brownian motions, or

$$dx_i = \frac{1}{N} \mathbb{J} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i \tag{1-13}$$

with $\mathbb{J}$ as above, or

$$dx_i = v_i dt \qquad dv_i = -\frac{1}{N} \nabla_i \mathcal{H}_N(x_1, \dots, x_N) dt + \sqrt{\beta^{-1}} dW_i. \tag{1-14}$$

From a mathematical point of view, the study of such systems touches on the fields of analysis (Partial Differential Equations and calculus of variations, approximation theory) particularly for **(1)-(3)-(4)**, probability (particularly for **(2)-(4)**), mathematical physics, and even geometry (when one considers such systems on manifolds or with

[1] One does not know how to explicitly compute the integrals (1-8) except in the particular case of (1-5) for specific $V$'s where they are called Selberg integrals (cf. Mehta [2004] and Forrester [2010])

curved geometries). Some of the crystallization questions they lead to also overlap with number theory as we will see below.

In the sequel we will mostly focus on the stationary settings **(1)** and **(2)**, while mentioning more briefly some results about **(3)** and **(4)**, for which many questions remain open. Of course these various points are not unrelated, as for instance the Gibbs measure (1-7) can also be seen as an invariant measure for dynamics of the form (1-11) or (1-12).

The plan of the discussion is as follows: in the next section we review various motivations for studying such questions, whether from physics or within mathematics. In Section 3 we turn to the so-called "mean-field" or leading order description of systems **(1)** to **(4)** and review the standard questions and known results. We emphasize that this part can be extended to general interaction kernels $\mathsf{g}$, starting with regular (smooth) interactions which are in fact the easiest to treat. In Section 4, we discuss questions that can be asked and results that can be obtained at the next order level of expansion of the energy. This has only been tackled for problems **(1)** and **(2)**, and the specificity of the Coulomb interaction becomes important then.

## 2 Motivations

It is in fact impossible to list all possible topics in which such systems arise, as they are really numerous. We will attempt to give a short, necessarily biased, list of examples, with possible pointers to the relevant literature.

**2.1 Vortices in condensed matter physics and fluids.** In superconductors with applied magnetic fields, and in rotating superfluids and Bose–Einstein condensates, one observes the occurrence of quantized "vortices" (which are local point defects of superconductivity or superfluidity, surrounded by a current loop). The vortices repel each other, while being confined together by the effect of the magnetic field or rotation, and the result of the competition between these two effects is that, as predicted by Abrikosov [1957], they arrange themselves in a particular *triangular lattice* pattern, called *Abrikosov lattice*, cf. Fig. 1[2]. Superconductors and superfluids are modelled by the celebrated Ginzburg–Landau energy Landau and Ginzburg [1965], which in simplified form [3] can be written

$$\int |\nabla \psi|^2 + \frac{(1-|\psi|^2)^2}{2\varepsilon^2} \tag{2-1}$$

where $\psi$ is a complex-valued unknown function (the "order parameter" in physics) and $\varepsilon$ is a small parameter, and gives rise to the associated Ginzburg–Landau equation

$$\Delta \psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) = 0 \tag{2-2}$$

[2]For more pictures, see www.fys.uio.no/super/vortex/

[3]The complete form for superconductivity contains a gauge-field, but we omit it here for simplicity.

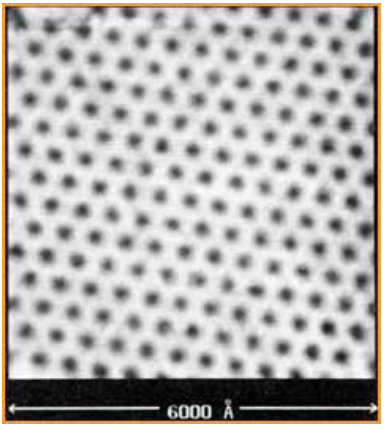


Figure 1: Abrikosov lattice, H. F. Hess et al. Bell Labs *Phys. Rev. Lett.* 62, 214 (1989)

and its dynamical versions, the heat flow

$$\partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2) \tag{2-3}$$

and Schrödinger-type flow (also called the Gross–Pitaevskii equation)

$$i\,\partial_t \psi = \Delta\psi + \frac{1}{\varepsilon^2}\psi(1-|\psi|^2). \tag{2-4}$$

When restricting to a two-dimensional situation, it can be shown rigorously (this was pioneered by Bethuel, Brezis, and Hélein [1994] for (2-1) and extended to the full gauged model Bethuel and Rivière [1995] and Sandier and Serfaty [2007, 2012]) that the minimization of (2-1) can be reduced, in terms of the vortices and as $\varepsilon \to 0$, to the minimization of an energy of the form (1-1) in the case (1-2) (for a formal derivation, see also Serfaty [2015, Chap. 1]) and this naturally leads to the question of understanding the connection between minimizers of (1-1) + (1-2) and the Abrikosov triangular lattice. Similarly, the dynamics of vortices under (2-3) can be formally reduced to (1-9), respectively under (2-4) to (1-10). This was established formally for instance in L. Peres and Rubinstein [1993] and E [1994a] and proven for a fixed number of vortices $N$ and in the limit $\varepsilon \to 0$ in F. H. Lin [1996], Jerrard and Soner [1998], Colliander and Jerrard [1998, 1999], F.-H. Lin and Xin [1999a,b], and Bethuel, Jerrard, and Smets [2008] until the first collision time and in Bethuel, Orlandi, and Smets [2005], Smets, Bethuel, and Orlandi [2007], Bethuel, Orlandi, and Smets [2007], and Serfaty [2007] including after collision.

Vortices also arise in classical fluids, where in contrast with what happens in superconductors and superfluids, their charge is not quantized. In that context the energy (1-1)+(1-2) is sometimes called the Kirchhoff energy and the system (1-10) with $\mathbb{J}$ taken to be a rotation by $\pi/2$, known as the point-vortex system, corresponds to the dynamics of idealized vortices in an incompressible fluid whose statistical mechanics analysis

was initiated by Onsager, cf. Eyink and Sreenivasan [2006] (one of the motivations for studying (1-13) is precisely to understand fluid turbulence as he conceived). It has thus been quite studied as such, see Marchioro and Pulvirenti [1994] for further reference. The study of evolutions like (1-11) is also motivated by plasma physics in which the interaction between ions is Coulombic, cf. Jabin [2014].

**2.2 Fekete points and approximation theory.** Fekete points arise in interpolation theory as the points minimizing interpolation errors for numerical integration Saff and Totik [1997]. More precisely, if one is looking for $N$ interpolation points $\{x_1, \dots, x_N\}$ in $K$ such that the relation

$$\int_K f(x)dx = \sum_{j=1}^N w_j f(x_j)$$

is exact when $f$ is any polynomial of degree $\leq N-1$, one sees that one needs to compute the coefficients $w_j$ such that $\int_K x^k = \sum_{j=1}^N w_j x_j^k$ for $0 \leq k \leq N-1$, and this computation is easy if one knows to invert the Vandermonde matrix of the $\{x_j\}_{j=1\dots N}$. The numerical stability of this operation is as large as the *condition number* of the matrix, i.e. as the Vandermonde determinant of the $(x_1, \dots, x_N)$. The points that minimize the maximal interpolation error for general functions are easily shown to be the Fekete points, defined as those that maximize

$$\prod_{i\neq j} |x_i - x_j|$$

or equivalently minimize

$$-\sum_{i\neq j} \log|x_i - x_j|.$$

They are often studied on manifolds, such as the $\mathsf{d}$-dimensional sphere. In Euclidean space, one also considers "weighted Fekete points" which maximize

$$\prod_{i<j} |x_i - x_j| e^{-N\sum_i V(x_i)}$$

or equivalently minimize

$$-\frac{1}{2}\sum_{i\neq j} \log|x_i - x_j| + N\sum_{i=1}^N V(x_i)$$

which in dimension 2 corresponds exactly to the minimization of $\mathcal{H}_N$ in the particular case $\mathsf{Log2}$. They also happen to be zeroes of orthogonal polynomials, see Simon [2008].

Since $-\log|x|$ can be obtained as $\lim_{s\to 0} \frac{1}{s}(|x|^{-s} - 1)$, there is also interest in studying "Riesz $\mathsf{s}$-energies", i.e. the minimization of

$$\sum_{i\neq j} \frac{1}{|x_i - x_j|^{\mathsf{s}}} \tag{2-5}$$

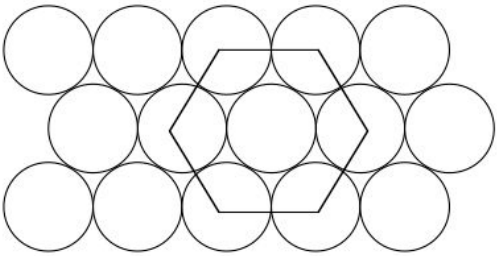

Figure 2: The triangular lattice solves the sphere packing problem in dimension 2

for all possible $\mathsf{s}$, hence a motivation for (1-6). For these aspects, we refer to the the review papers Saff and Kuijlaars [1997] and Brauchart, Hardin, and Saff [2012] and references therein.

Varying $\mathsf{s}$ from 0 to $\infty$ connects Fekete points to the optimal sphere packing problem, which formally corresponds to the minimization of (2-5) with $\mathsf{s} = \infty$.

The optimal sphere packing problem has been solved in 1, 2 and 3 dimensions, as well as in dimensions 8 and 24 in the recent breakthrough Viazovska [2017] and Cohn, Kumar, Miller, Radchenko, and Viazovska [2017] (we refer the reader to the nice presentation in Cohn [2017] and the review Sloane [1998]). The solution in dimension 2 is the triangular lattice Fejes [1940] (i.e. the same as the Abrikosov lattice, see Figure 2), in dimension 3 it is the FCC (face-centered cubic) lattice Hales [2005], in dimension 8 the $E_8$ lattice Viazovska [2017], and in dimension 24 the Leech lattice Cohn, Kumar, Miller, Radchenko, and Viazovska [2017].

In other dimensions, the solution is in general not known and it is expected that in high dimension, where the problem is important for error-correcting codes, it is *not* a lattice (in dimension 10 already, the so-called "Best lattice", a non-lattice competitor, is known to beat the lattices), see Conway and Sloane [1999] for these aspects.

**2.3 Statistical mechanics and quantum mechanics.** The ensemble given by (1-7) in the $\mathsf{Log2}$ case is called in physics a two-dimensional *Coulomb gas* or *one-component plasma* and is a classical ensemble of statistical mechanics (see e.g. Alastuey and Jancovici [1981], Jancovici, Lebowitz, and Manificat [1993], Jancovici [1995], Sari and Merlini [1976], Kiessling [1993], and Kiessling and Spohn [1999]). The Coulomb case with $\mathsf{d} = 3$ can be seen as a toy (classical) model for matter (see e.g. Penrose and Smith [1972], Jancovici, Lebowitz, and Manificat [1993], Lieb and Lebowitz [1972], and Lieb and Narnhofer [1975]). Several additional motivations come from quantum mechanics. Indeed, the Gibbs measure of the two-dimensional Coulomb gas happens to be directly related to the Laughlin wave-function in the fractional quantum Hall effect Girvin [2004] and Stormer [1999]: this is the "plasma analogy", cf. Laughlin [1983], Girvin [1999], and Laughlin [1999], and for recent mathematical progress using this correspondence, cf. Rougerie, Serfaty, and Yngvason [2014], Rougerie and Yngvason [2015], and Lieb, Rougerie, and Yngvason [2018]. For $\beta = 2$ it also arises as the wave-function density of the ground state for the system of $N$ non-interacting fermions

confined to a plane with a perpendicular magnetic field Forrester [2010, Chap. 15]. The 1-dimensional log gas $\mathsf{Log1}$ also arises as the wave-function density in several exactly solvable quantum mechanics systems: examples are the Tonks–Girardeau model of "impenetrable" Bosons Girardeau, Wright, and Triscari [2001] and Forrester, Frankel, Garoni, and Witte [2003], the Calogero–Sutherland quantum many-body Hamiltonian Forrester, Jancovici, and McAnally [2001] and Forrester [2010] and finally the density of the many-body wave function of non-interacting fermions in a harmonic trap Dean, Le Doussal, Majumdar, and Schehr [2016]. It also arises in several non-intersecting paths models from probability, cf. Forrester [2010].

The general Riesz case (1-6) can be seen as a generalization of the Coulomb case, motivations for studying Riesz gases are numerous in the physics literature (in solid state physics, ferrofluids, elasticity), see for instance Mazars [2011], Barré, Bouchet, Dauxois, and Ruffo [2005], Campa, Dauxois, and Ruffo [2009], and Torquato [2016], they can also correspond to systems with Coulomb interaction constrained to a lower-dimensional subspace: for instance in the quantum Hall effect, electrons confined to a two-dimensional plane interact via the three-dimension Coulomb kernel.

In all cases of interactions, the systems governed by the Gibbs measure $\mathbb{P}_{N,\beta}$ are considered as difficult systems of statistical mechanics because the interactions are truly long-range, singular, and the points are not constrained to live on a lattice.

As always in statistical mechanics K. Huang [1987], one would like to understand if there are phase-transitions for particular values of the (inverse) temperature $\beta$ in the large volume limit. For the systems studied here, one may expect, after a suitable blow-up of the system, what physicists call a liquid for small $\beta$, and a crystal for large $\beta$. The meaning of crystal in this instance is not to be taken literally as a lattice, but rather as a system of points whose 2-point correlation function $\rho^{(2)}(x, y)$ defined as the probability to have jointly one point at $x$ and one point at $y$ (see Section 3.1) does not decay too fast as $x - y \to \infty$. A phase-transition at finite $\beta$ has been conjectured in the physics literature for the $\mathsf{Log2}$ case (see e.g. Brush, Sahlin, and Teller [1966], Caillol, Levesque, Weis, and Hansen [1982], and Choquard and Clerouin [1983]) but its precise nature is still unclear (see e.g. Stishov [1998] for a discussion).

**2.4 Two component plasma case.** The two-dimensional "one component plasma", consisting of positively charged particles, has a "two-component" counterpart which consists in $N$ particles $x_1, \dots, x_N$ of charge $+1$ and $N$ particles $y_1, \dots, y_N$ of charge $-1$ interacting logarithmically, with energy

$$\mathcal{H}_N(x_1, \dots, x_N, y_1, \dots, y_N) = -\sum_{i \neq j} \log|x_i - x_j| - \sum_{i \neq j} \log|y_i - y_j| + \sum_{i,j} \log|x_i - y_j|$$

and the Gibbs measure

$$\frac{1}{Z_{N,\beta}} e^{-\beta \mathcal{H}_N(x_1,\dots,x_N,y_1,\dots,y_N)} dx_1 \dots dx_N \, dy_1 \dots dy_N.$$

Although the energy is unbounded below (positive and negative points attract), the Gibbs measure is well defined for $\beta$ small enough, more precisely the partition function converges for $\beta < 2$. The system is then seen to form dipoles of oppositely

charged particles which attract but do not collapse, thanks to the thermal agitation. The two-component plasma is interesting due to its close relation to two important theoretical physics models: the XY model and the sine-Gordon model (cf. the review Spencer [1997]), which exhibit a Berezinski–Kosterlitz–Thouless phase transition Bietenholz and Gerber [2016] consisting in the binding of these "vortex-antivortex" dipoles. For further reference, see Fröhlich [1976], Deutsch and Lavaud [1974], Fröhlich and Spencer [1981], and Gunson and Panta [1977].

**2.5 Random matrix theory.** The study of (1-7) has attracted a lot of attention due to its connection with random matrix theory (we refer to Forrester [2010] for a comprehensive treatment). Random matrix theory (RMT) is a relatively old theory, pioneered by statisticians and physicists such as Wishart, Wigner and Dyson, and originally motivated by the study of sample covariance matrices for the former and the understanding of the spectrum of heavy atoms for the two latter, see Mehta [2004]. For more recent mathematical reference see Anderson, Guionnet, and Zeitouni [2010], Deift [1999], and Forrester [2010]. The main question asked by RMT is : what is the law of the spectrum of a large random matrix ? As first noticed in the foundational papers of Wigner [1955] and Dyson [1962], in the particular cases (1-5) and (1-2) the Gibbs measure (1-7) corresponds in some particular instances to the joint law of the eigenvalues (which can be computed algebraically) of some famous random matrix ensembles:

- for $\mathsf{Log2}$, $\beta = 2$ and $V(x) = |x|^2$, (1-7) is the law of the (complex) eigenvalues of an $N \times N$ matrix where the entries are chosen to be normal Gaussian i.i.d. This is called the Ginibre ensemble Ginibre [1965].
- for $\mathsf{Log1}$, $\beta = 2$ and $V(x) = x^2/2$, (1-7) is the law of the (real) eigenvalues of an $N \times N$ Hermitian matrix with complex normal Gaussian iid entries. This is called the Gaussian Unitary Ensemble.
- for $\mathsf{Log1}$, $\beta = 1$ and $V(x) = x^2/2$, (1-7) is the law of the (real) eigenvalues of an $N \times N$ real symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Orthogonal Ensemble.
- for $\mathsf{Log1}$, $\beta = 4$ and $V(x) = x^2/2$, (1-7) is the law of the eigenvalues of an $N \times N$ quaternionic symmetric matrix with normal Gaussian iid entries. This is called the Gaussian Symplectic Ensemble.
- the general-$\beta$ case of $\mathsf{Log1}$ can also be represented, in a slightly more complicated way, as a random matrix ensemble Dumitriu and Edelman [2002] and Killip and Nenciu [2004].

One thus observes in these ensembles the phenomenon of "repulsion of eigenvalues": they repel each other logarithmically, i.e. like two-dimensional Coulomb particles.

The stochastic evolution (1-12) in the case $\mathsf{Log1}$ is (up to proper scaling) the Dyson Brownian motion, which is of particular importance in random matrices since the GUE process is the invariant measure for this evolution, it has served to prove universality for the statistics of eigenvalues of general Wigner matrices, i.e. those with iid but not

necessarily Gaussian entries, see Erdős and Yau [2017] (and Tao and Vu [2011] for another approach).

For the Log1 and Log2 cases, at the specific temperature $\beta = 2$, the law (1-7) acquires a special algebraic feature: it becomes a *determinantal* process, part of a wider class of processes (see Hough, Krishnapur, Y. Peres, and Virág [2009] and Borodin [2011]) for which the correlation functions are explicitly given by certain determinants. This allows for many explicit algebraic computations, and is part of *integrable probability* on which there is a large literature Borodin and Gorin [2016].

**2.6 Complex geometry and theoretical physics.** Coulomb systems and higher-dimensional analogues involving powers of determinantal densities are also of interest to geometers as a way to construct Kähler–Einstein metrics with negative Ricci curvature on complex manifolds, cf. Berman [2017] and Berman, Boucksom, and Witt Nyström [2011].

Another important motivation is the construction of Laughlin states for the Fractional Quantum Hall effect on complex manifolds, which effectively reduces to the study of a two-dimensional Coulomb gas on a manifold. The coefficients in the expansion of the (logarithm of the) partition function have interpretations as geometric invariants, cf. for instance Klevtsov [2016].

## 3 The mean field limits and macroscopic behavior

**3.1 Questions.** The first question that naturally arises is to understand the limit as $N \to \infty$ of the *empirical measure* defined by [4]

$$\mu_N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \tag{3-1}$$

for configurations of points that minimize the energy (1-1), critical points, solutions of the evolution problems, or typical configurations under the Gibbs measure (1-7), thus hoping to derive effective equations or minimization problems that describe the average or mean-field behavior of the system. The term mean-field refers to the fact that, from the physics perspective, each particle feels the collective field generated by all the other particles, averaged by dividing it by the number of particles. That collective field is $g * \mu_N$, except that it is singular at each particle, so to evaluate it at $x_i$ one first has to remove the contribution of $x_i$ itself.

Another point of view is that of correlation functions. One may denote by

$$\rho_N^{(k)}(x_1, \dots, x_k) \tag{3-2}$$

the $k$-point correlation function, which is the probability density (for each specific problem) of observing a particle at $x_1$, a particle at $x_2$, ..., and a particle at $x_k$ (these functions should of course be symmetric with respect to permutation of the labels). For

[4] Note that the configurations contain $N$ points which also implicitly depend on $N$ themselves, but we do not keep track of this dependence for the sake of lightness of notation.

instance, in the case (1-7), $\rho_N^{(N)}$ is simply $\mathbb{P}_{N,\beta}$ itself, and the $\rho_N^{(k)}$ are its marginals (obtained by integrating $\mathbb{P}_{N,\beta}$ with respect to all its variables but $k$). One then wants to understand the limit as $N \to \infty$ of each $\rho_N^{(k)}$, with fixed $k$. Mean-field results will typically imply that the limiting $\rho^{(k)}$'s have a factorized form

$$\rho^{(k)}(x_1, \dots, x_k) = \mu(x_1) \dots \mu(x_k) \tag{3-3}$$

for the appropriate $\mu$ which is also equal to $\rho^{(1)}$. This is called *molecular chaos* according to the terminology introduced by Boltzmann, and can be interpreted as the particles becoming independent in the limit. When looking at the dynamic evolutions of problems **(3)** and **(4)**, starting from initial data for which $\rho^{(k)}(0,\cdot)$ are in such a factorized form, one asks whether this remains true for $\rho^{(k)}(t,\cdot)$ for $t > 0$, if so this is called *propagation of (molecular) chaos*. It turns out that the convergence of the empirical measure (3-1) to a limit $\mu$ and the fact that each $\rho^{(k)}$ can be put in factorized form are essentially equivalent, see Hauray and Mischler [2014] and Golse [2016] and references therein — ideally, one would also like to find quantitative rates of convergences in $N$, and they will typically deteriorate as $k$ gets large. In the following we will focus on the mean-field convergence approach, via the empirical measure.

In the case of minimizers **(1)**, a major question is to obtain an expansion as $N \to \infty$ for $\min \mathcal{H}_N$. In the setting of manifolds, the coefficients in such an expansion have geometric interpretations. In the same way, in the statistical mechanics setting **(2)**, one searches for expansions as $N \to \infty$ of the so-called *free energy* $-\beta^{-1} \log Z_{N,\beta}$. The free energy encodes a lot of the physical quantities of the system. For instance, points of non-differentiability of $\log Z_{N,\beta}$ as a function of $\beta$ are interpreted as phase-transitions.

We will see below that understanding the mean-field behavior of the system essentially amounts to understanding the leading order term in the large $N$ expansion of the minimal energy or respectively the free energy, while understanding the next order term in the expansion essentially amounts to understanding the next order (or fluctuations) of the system.

**3.2 The equilibrium measure.** The leading order behavior of $\mathcal{H}_N$ is related to the functional

$$\mathcal{I}_V(\mu) := \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y) d\mu(x) d\mu(y) + \int_{\mathbb{R}^d} V(x) d\mu(x) \tag{3-4}$$

defined over the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$ (which may also take the value $+\infty$). This is something one may naturally expect since $\mathcal{I}_V(\mu)$ appears as the continuum version of the discrete energy $\mathcal{H}_N$. From the point of view of statistical mechanics, $\mathcal{I}_V$ is the mean-field limit energy of $\mathcal{H}_N$, while from the point of view of probability, $\mathcal{I}_V$ plays the role of a *rate function*.

Assuming some lower semi-continuity of $V$ and that it grows faster than $g$ at $\infty$, it was shown in Frostman [1935] that the minimum of $\mathcal{I}_V$ over $\mathcal{P}(\mathbb{R}^d)$ exists, is finite and is achieved by a unique $\mu_V$ (unique by strict convexity of $\mathcal{I}_V$), which has compact support and a density, and is uniquely characterized by the fact that there exists a constant

$c$ such that

$$\begin{cases} h^{\mu_V} + V \geq c & \text{in } \mathbb{R}^{\mathsf{d}} \\ h^{\mu_V} + V = c & \text{in the support of } \mu_V \end{cases} \tag{3-5}$$

where

$$h^{\mu_V}(x) := \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y) d\mu_V(y) = \mathsf{g} * \mu_V \tag{3-6}$$

is the "electrostatic" potential generated by $\mu_V$.

This measure $\mu_V$ is called the (Frostman) *equilibrium measure*, and the result is true for more general repulsive kernels than Coulomb, for instance for all regular kernels or inverse powers of the distance which are integrable.

**Example 3.1.** When $\mathsf{g}$ is the Coulomb kernel, applying the Laplacian on both sides of (3-5) gives that, in the interior of the support of the equilibrium measure, if $V \in C^2$,

$$\mathsf{c}_{\mathsf{d}} \mu_V = \Delta V \tag{3-7}$$

i.e. the density of the measure on the interior of its support is given by $\frac{\Delta V}{\mathsf{c}_{\mathsf{d}}}$. For example if $V$ is quadratic, this density is constant on the interior of its support. If $V(x) = |x|^2$ then by symmetry $\mu_V$ is the indicator function of a ball (up to a multiplicative factor), this is known as the *circle law* for the Ginibre ensemble in the context of Random Matrix Theory. An illustration of the convergence to this circle law can be found in Figure 3. In dimension $\mathsf{d} = 1$, with $\mathsf{g} = -\log|\cdot|$ and $V(x) = x^2$, the equilibrium measure is $\mu_V(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{|x|\leq 2}$, which corresponds in the context of RMT (GUE and GOE ensembles) to the famous *Wigner semi-circle law*, cf. Wigner [1955] and Mehta [2004].

In the Coulomb case, the equilibrium measure $\mu_V$ can also be interpreted in terms of the solution to a classical *obstacle problem* (and in the Riesz case (1-6) with $\mathsf{d} - 2 \leq \mathsf{s} < \mathsf{d}$ a "fractional obstacle problem"), which is essentially dual to the minimization of $\mathfrak{J}_V$, and better studied from the PDE point of view (in particular the regularity of $\mu_V$ and of the boundary of its support). For this aspect, see Serfaty [2015, Chap. 2] and references therein.

Frostman's theorem is the basic result of potential theory. The relations (3-5) can be seen as the Euler–Lagrange equations associated to the minimization of $\mathfrak{J}_V$. They state that in the static situation, the total potential, sum of the potential generated by $\mu_V$ and the external potential $V$ must be constant in the support of $\mu_V$, i.e. in the set where the "charges" are present.

More generally $\nabla(h^\mu + V)$ can be seen as the total mean-field force acting on charges with density $\mu$ (i.e. each particle feels the average collective force generated by the other particles), and for the particle to be at rest one needs that force to vanish. Thus $\nabla(h^\mu + V)$ should vanish on the support of $\mu$, in fact the stationarity condition that formally emerges as the limit for critical points of $\mathcal{H}_N$ is

$$\mu \nabla(h^\mu + V) = 0. \tag{3-8}$$

The problem with this relation is that the product $\mu \nabla h^\mu$ does not always make sense, since a priori $\mu$ is only a probability measure and $\nabla h^\mu$ is not necessarily continuous, however, in dimension 2, one can give a weak form of the equation which always makes sense, inspired by Delort's work in fluid mechanics Delort [1991], cf. Sandier and Serfaty [2007, Chap. 13].

### 3.3 Convergence of minimizers.

**Theorem 1.** *We have*

$$\lim_{N\to\infty} \frac{\min \mathcal{H}_N}{N^2} = \min \mathcal{L}_V = \mathcal{L}_V(\mu_V) \tag{3-9}$$

*and if* $(x_1, \dots, x_N)$ *minimize* $\mathcal{H}_N$ *then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \rightharpoonup \mu_V \tag{3-10}$$

*in the weak sense of probability measures.*

This result is usually attributed to Choquet [1958], one may see the proof in Saff and Totik [1997] for the logarithmic cases, the general case can be treated exactly in the same way Serfaty [2015, Chap. 2], and is valid for very general interactions $g$ (for instance radial decreasing and integrable near 0). In modern language it can be phrased as a $\Gamma$-convergence result. It can also easily be expressed in terms of convergence of marginals, as a molecular chaos result.

**3.4 Parallel results for Ginzburg–Landau vortices.** The analogue mean field result and leading order asymptotic expansion of the minimal energy has also been obtained for the two-dimensional Ginzburg–Landau functional of superconductivity (2-1), see Sandier and Serfaty [2007, Chap. 7]. It is phrased as the convergence of the vorticity $\nabla \times \langle i\psi, \nabla\psi\rangle$, normalized by the proper number of vortices, to an equilibrium measure, or the solution to an obstacle problem. The analogue of (3-8) is also derived for critical points in Sandier and Serfaty [ibid., Chap. 13].

**3.5 Deterministic dynamics results - problems (3).** For general reference on problems of the form **(3)** and **(4)**, we refer to Spohn [2004]. In view of the above discussion, in the dynamical cases (1-9) or (1-10), one expects as analogue results the convergences of the (time-dependent) empirical measures $\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ to probability densities $\mu$ that satisfy the limiting mean-field evolutions

$$\partial_t \mu = \operatorname{div}(\nabla(h^\mu + V)\mu) \tag{3-11}$$

respectively

$$\partial_t \mu = -\operatorname{div}(\mathbb{J}\nabla(h^\mu + V)\mu) \tag{3-12}$$

where again $h^\mu = \mathsf{g} * \mu$ as in (3-6). These are nonlocal transport equations where the density $\mu$ is transported along the velocity field $-\nabla(h^\mu + V)$ (respectively $\mathbb{J}\nabla(h^\mu + V)$) i.e. advected by the mean-field force that the distribution generates.

In the two-dimensional Coulomb case (1-2) with $V = 0$, (3-12) with $\mathbb{J}$ chosen as the rotation by $\pi/2$ is also well-known as the vorticity form of the incompressible Euler equation, describing the evolution of the vorticity in an ideal fluid, with velocity given by the Biot–Savart law. As such, this equation is well-studied in this context, and the convergence of solutions of (1-10) to (3-12), also known as the point-vortex approximation to Euler, has been rigorously proven, see Schochet [1996] and Goodman, Hou, and Lowengrub [1990].

As for (3-11), it is a dissipative equation, that can be seen as a gradient flow on the space of probability measures equipped with the so-called Wasserstein $W_2$ (or Monge–Kantorovitch) metric. In the dimension 2 logarithmic case, it was first introduced by Chapman, Rubinstein, and Schatzman [1996] and E [1994b] as a formal model for superconductivity, and in that setting the gradient flow description has been made rigorous (see Ambrosio and Serfaty [2008]) using the theory of gradient flows in metric spaces of Otto [2001] and Ambrosio, Gigli, and Savaré [2005]. The equation can also be studied by PDE methods F. Lin and Zhang [2000] and Serfaty and Vázquez [2014], which generalize to the Coulomb interaction in any dimension. The derivation of this gradient flow equation (3-11) from (2-3) can be guessed by variational arguments, i.e. "Γ-convergence of gradient flows", see Serfaty [2011]. In the non Coulombic case, i.e. for (1-6), (3-11) is a "fractional porous medium equation", analyzed in Caffarelli and Vazquez [2011], Caffarelli, Soria, and Vázquez [2013], and Zhou and Xiao [2017].

We have the following result which states in slightly informal terms that the desired convergence holds provided the limiting solution is regular enough.

**Theorem 2** (Serfaty and Duerinckx [2018]). *For any* $\mathsf{d}$, *any case* (1-2), (1-5) *or* (1-6) *with* $\mathsf{d}-2 \le s < \mathsf{d}$, *let* $\{x_i\}$ *solve* (1-9), *respectively* (1-10) *with initial data* $x_i(0) = x_i^0$. *Then if the limit* $\mu^0$ *of the initial empirical measure* $\frac{1}{N}\sum_{i=1}^N \delta_{x_i^0}$ *is regular enough so that the solution* $\mu_t$ *of* (3-11), *resp.* (3-12), *with initial data* $\mu_0$ *exists until time* $T > 0$ *and is regular enough, and if the initial condition is well-prepared in the sense that*

$$\lim_{N\to\infty} \frac{1}{N^2}\mathcal{H}_N(\{x_i^0\}) = \iint_{\mathbb{R}^\mathsf{d}\times\mathbb{R}^\mathsf{d}} \mathsf{g}(x-y)d\mu^0(x)d\mu^0(y),$$

*then we have that for all* $t \in [0,T)$,

$$\frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)} \to \mu_t \quad \text{as } N\to\infty.$$

Note that the existence of regular enough solutions that exist for all time, provided the initial data is regular enough, is known to hold for all Coulomb cases and all Riesz cases with $\mathsf{s} < \mathsf{d}-1$ Zhou and Xiao [2017].

The difficulty in proving this convergence result is due to the singularity of the Coulomb interaction combined with the nonlinear character of the product $h^\mu\mu$ (and its discrete analogue) which prevents from directly taking limits in the equation.

Prior results existed for less singular interactions Hauray [2009], Carrillo, Choi, and Hauray [2014], and Jabin and Wang [2017] or in dimension 1 Berman and Onnheim [2016]. Theorem 2 was first proven in the dissipative case in dimensions 1 and 2 in Duerinckx [2016], then in all dimensions and in the conservative case in Serfaty and Duerinckx [2018]. Both proofs rely on a "modulated energy" approach inspired from Serfaty [2017]. It consists in considering a *Coulomb-based (or Riesz-based) distance* between probability densities, more precisely the distance defined by

$$d_{\mathsf{g}}(\mu,\nu)^2 = \iint_{\mathbb{R}^{\mathsf{d}}\times\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y)d(\mu-\nu)(x)d(\mu-\nu)(y)$$

which is a good metric thanks to the particular properties of the Coulomb and Riesz kernels. One can prove a "weak-strong" stability result for the limiting equations (3-11), (3-12) in that metric: if $\mu$ is a smooth enough solution to (3-11), resp. (3-12), and if $\nu$ is any solution to the same equation, then

$$d_{\mathsf{g}}(\mu(t),\nu(t)) \leq e^{Ct} d_{\mathsf{g}}(\mu(0),\nu(0)), \tag{3-13}$$

which is proved by showing a Gronwall inequality. One may then exploit this stability property by taking $\mu$ to be the smooth enough expected limiting solution, and $\nu$ to be the empirical measure of the solution to the discrete evolution (1-9) or (1-10), after giving an appropriate *renormalized* meaning to the Coulomb distance (which is otherwise infinite) in that setting. We are able to prove that a relation similar to (3-13) holds, thus proving the desired convergence.

The analogue of the rigorous passage from (1-9) or (1-10) to (3-11) or (3-12) was accomplished at the level of the full parabolic and Schrödinger Ginzburg–Landau PDEs (2-3) and (2-4) Kurzke and Spirn [2014], Jerrard and Spirn [2015], and Serfaty [2017]. The method in Serfaty [2017] relies as above on a modulated energy argument which consists in finding a suitable energy, modelled on the Ginzburg–Landau energy, which measures the distance to the desired limiting solution, and for which a Gronwall inequality can be shown to hold.

As far as (1-11) is concerned, the limiting equation is formally found to be the Vlasov–Poisson equation

$$\partial_t \rho + v\cdot\nabla_x\rho + \nabla(h^\mu + V)\cdot\nabla_v\rho = 0 \tag{3-14}$$

where $\rho(t,x,v)$ is the density of particles at time $t$ with position $x$ and velocity $v$, and $\mu(t,x) = \int \rho(t,x,v)dv$ is the density of particles. The rigorous convergence of (1-11) to (3-14) and propagation of chaos are not proven in all generality (i.e. for all initial data) but it has been established in a statistical sense (i.e. randomizing the initial condition) and often truncating the interactions, see Kiessling [2014], Boers and Pickl [2016], Hauray and Jabin [2015], Lazarovici [2016], Lazarovici and Pickl [2017], and Jabin and Wang [2016] and also the reviews on the topic Jabin [2014] and Golse [2016].

**3.6 Noisy dynamics - problems (4).** The noise terms in these equations gives rise to an additive Laplacian term in the limiting equations. For instance the limiting equation

for (1-12) is expected to be the McKean equation

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \operatorname{div}\left(\nabla(h^\mu + V)\mu\right) \tag{3-15}$$

and the convergence is known for regular interactions since the seminal work of McKean [1967], see also the reviews Sznitman [1991] and Jabin [2014].

For singular interactions, the situation has been understood for the one-dimensional logarithmic case Cépa and Lépingle [1997], then for all Riesz interactions (1-6) Berman and Onnheim [2018]. Higher dimensions with singular interactions is largely open, but recent progress of Jabin and Wang [2017] allows to treat possibly rough but bounded interactions, as well as some Coulomb interactions, and prove convergence in an appropriate statistical sense.

For the conservative case (1-13) the limiting equation is a viscous conservative equation of the form

$$\partial_t \mu = \frac{1}{\beta}\Delta\mu - \operatorname{div}\left(\nabla^\perp(h^\mu + V)\mu\right) \tag{3-16}$$

which in the two-dimensional logarithmic case (1-2) is the Navier–Stokes equation in vorticity form. The convergence in that particular case was established in Fournier, Hauray, and Mischler [2014], while the most general available result is that of Jabin and Wang [2017].

For the case of (1-14), the limiting equation is the McKean–Vlasov equation

$$\partial_t \rho + v \cdot \nabla_x \rho + \nabla(h^\mu + V) \cdot \nabla_v \rho - \frac{1}{\beta}\Delta\rho = 0 \tag{3-17}$$

with the same notation as for (3-14), and convergence in the case of bounded-gradient kernels is proven in Jabin and Wang [2016], see also references therein.

**3.7 With temperature: statistical mechanics.** Let us now turn to problem **(2)** and consider the situation with temperature as described via the Gibbs measure (1-7). One can determine that two temperature scaling choices are interesting: the first is taking $\beta$ independent of $N$, the second is taking $\beta_N = \frac{\beta}{N}$ with some fixed $\beta$. In the former, which can be considered a "low temperature" regime, the behavior of the system is still governed by the equilibrium measure $\mu_V$. The result can be phrased using the language of Large Deviations Principles (LDP), cf. Dembo and Zeitouni [2010] for definitions and reference.

**Theorem 3.** *The sequence $\{\mathbb{P}_{N,\beta}\}_N$ of probability measures on $\mathcal{P}(\mathbb{R}^d)$ satisfies a large deviations principle at speed $N^2$ with good rate function $\beta\hat{\mathfrak{l}}_V$ where $\hat{\mathfrak{l}}_V = \mathfrak{l}_V - \min_{\mathcal{P}(\mathbb{R}^d)} \mathfrak{l}_V = \mathfrak{l}_V - \mathfrak{l}_V(\mu_V)$. Moreover*

$$\lim_{N\to+\infty} \frac{1}{N^2} \log Z_{N,\beta} = -\beta\mathfrak{l}_V(\mu_V) = -\beta \min_{\mathcal{P}(\mathbb{R}^d)} \mathfrak{l}_V. \tag{3-18}$$

The concrete meaning of the LDP is that if $E$ is a subset of the space of probability measures $\mathcal{P}(\mathbb{R}^d)$, after identifying configurations $(x_1, \dots, x_N)$ in $(\mathbb{R}^d)^N$ with their

empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$, we may write

$$\mathbb{P}_{N,\beta}(E) \approx e^{-\beta N^2(\min_E \mathcal{W}_V - \min \mathcal{W}_V)}, \tag{3-19}$$

which in view of the uniqueness of the minimizer of $\mathcal{W}_V$ implies that configurations whose empirical measure does not converge to $\mu_V$ as $N \to \infty$ have exponentially decaying probability. In other words the Gibbs measure concentrates as $N \to \infty$ on configurations for which the empirical measure is very close to $\mu_V$, i.e. the temperature has no effect on the mean-field behavior.

This result was proven in the logarithmic cases in Petz and Hiai [1998] (in dimension 2), Ben Arous and Guionnet [1997] (in dimension 1) and Ben Arous and Zeitouni [1998] (in dimension 2) for the particular case of a quadratic potential (and $\beta = 2$), see also Berman [2014] with results for general powers of the determinant in the setting of multidimensional complex manifolds, or Chafaï, Gozlan, and Zitt [2014] which recently treated more general singular $g$'s and $V$'s. This result is actually valid in any dimension, and is not at all specific to the Coulomb interaction (the proof works as well for more general interaction potentials, see Serfaty [2015]).

In the high-temperature regime $\beta_N = \frac{\beta}{N}$, the temperature is felt at leading order and brings an entropy term. More precisely there is a temperature-dependent equilibrium measure $\mu_{V,\beta}$ which is the unique minimizer of

$$I_{V,\beta}(\mu) = \beta \mathcal{W}_V(\mu) + \int \mu \log \mu. \tag{3-20}$$

Contrarily to the equilibrium measure, $\mu_{V,\beta}$ is not compactly supported, but decays exponentially fast at infinity. This mean-field behavior and convergence of marginals was first established for logarithmic interactions Kiessling [1993] and Caglioti, Lions, Marchioro, and Pulvirenti [1992] (see Messer and Spohn [1982] for the case of regular interactions) using an approach based on de Finetti's theorem. In the language of Large Deviations, the same LDP as above then holds with rate function $I_{V,\beta} - \min I_{V,\beta}$, and the Gibbs measure now concentrates as $N \to \infty$ on a neighborhood of $\mu_{V,\beta}$, for a proof see García-Zelada [2017]. Again the Coulomb nature of the interaction is not really needed. One can also refer to Rougerie [2014, 2016] for the mean-field and chaos aspects with a particular focus on their adaptation to the quantum setting.

## 4 Beyond the mean field limit: next order study

We have seen that studying systems with Coulomb (or more general) interactions at leading order leads to a good understanding of their limiting macroscopic behavior. One would like to go further and describe their microscopic behavior, at the scale of the typical inter-distance between the points, $N^{-1/\mathsf{d}}$. This in fact comes as a by-product of a next-to-leading order description of the energy $\mathcal{H}_N$, which also comes together with a next-to-leading order expansion of the free energy in the case (1-7).

Thinking of energy minimizers or of typical configurations under (1-7), since one already knows that $\sum_{i=1}^{N}\delta_{x_i} - N\mu_V$ is small, one knows that the so-called *discrepancy*

in balls $B_r(x)$ for instance, defined as

$$D(x,r) := \int_{B_r(x)} \sum_{i=1}^N \delta_{x_i} - N\, d\mu_V$$

is $o(r^{\mathsf{d}}N)$ as long as $r > 0$ is fixed. Is this still true at the mesoscopic scales for $r$ of the order $N^{-\alpha}$ with $\alpha < 1/\mathsf{d}$? Is it true down to the microscopic scale, i.e. for $r = RN^{-1/\mathsf{d}}$ with $R \gg 1$? Does it hold regardless of the temperature? This would correspond to a *rigidity result*. Note that point processes with discrepancies growing like the perimeter of the ball have been called *hyperuniform* and are of interest to physicists for a variety of applications, cf. Torquato [2016], see also Ghosh and Lebowitz [2017] for a review of the link between rigidity and hyperuniformity. An addition question is: how much of the microscopic behavior depends on $V$ or in another words is there a form of universality in this behavior? Such questions had only been answered in details in the one-dimensional case (1-5) as we will see below.

**4.1 Expanding the energy to next order.** The first step that we will describe is how to expand the energy $\mathcal{H}_N$ around the measure $N\mu_V$, following the approach initiated in Sandier and Serfaty [2015b] and continued in Sandier and Serfaty [2015a], Rougerie and Serfaty [2016], Petrache and Serfaty [2017], and Leblé and Serfaty [2017]. It relies on a splitting of the energy into a fixed leading order term and a next order term expressed in terms of the charge fluctuations, and on a rewriting of this next order term via the "electric potential" generated by the points. More precisely, exploiting the quadratic nature of the interaction, and letting $\triangle$ denote the diagonal in $\mathbb{R}^{\mathsf{d}} \times \mathbb{R}^{\mathsf{d}}$, let us expand

$$\begin{aligned}
\mathcal{H}_N(x_1,\dots,x_N) &= \frac12 \sum_{i\neq j} \mathsf{g}(x_i - x_j) + N\sum_{i=1}^N V(x_i)\\
&= \frac12 \iint_{\triangle^c} \mathsf{g}(x-y) d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x) d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d\Big(\sum_{i=1}^N \delta_{x_i}\Big)(x)\\
&= \frac{N^2}{2} \iint_{\triangle^c} \mathsf{g}(x-y) d\mu_V(x) d\mu_V(y) + N^2 \int_{\mathbb{R}^{\mathsf{d}}} V d\mu_V\\
&+ N\iint_{\triangle^c} \mathsf{g}(x-y) d\mu_V(x)\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)\\
&+ \frac12 \iint_{\triangle^c} \mathsf{g}(x-y) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y).
\end{aligned} \tag{4-1}$$

Recalling that $\mu_V$ is characterized by (3-5), we see that the middle term

$$\begin{aligned}
N\iint \mathsf{g}(x-y) d\mu_V(x) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y) + N\int_{\mathbb{R}^{\mathsf{d}}} V d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)&\\
= N\int_{\mathbb{R}^{\mathsf{d}}} (h^{\mu_V} + V) d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)&
\end{aligned} \tag{4-2}$$

can be considered as vanishing (at least it does if all the points $x_i$ fall in the support of $\mu_V$). We are then left with

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2 \mathcal{J}_V(\mu_V) + F_N^{\mu_V}(x_1,\dots,x_N) \tag{4-3}$$

with

(4-4)
$$F_N^{\mu_V}(x_1,\dots,x_N) = \frac{1}{2}\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y).$$

The relation (4-3) is a next-order expansion of $\mathcal{H}_N$ (recall (3-9)), valid for arbitrary configurations. The "next-order energy" $F_N^{\mu_V}$ can be seen as the Coulomb energy of the neutral system formed by the $N$ positive point charges at the $x_i$'s and the diffuse negative charge $-N\mu_V$ of same mass. To further understand $F_N^{\mu_V}$ let us introduce the potential generated by this system, i.e.

(4-5)
$$H_N(x) = \int_{\mathbb{R}^{\mathsf{d}}} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y)$$

(compare with (3-6)) which solves the linear elliptic PDE (in the sense of distributions)

(4-6)
$$-\Delta H_N = \mathsf{c}_{\mathsf{d}}\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)$$

and use for the first time crucially the Coulomb nature of the interaction to write

(4-7)
$$\iint_{\triangle^c} \mathsf{g}(x-y)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(x)d\Big(\sum_{i=1}^N \delta_{x_i} - N\mu_V\Big)(y)$$
$$\simeq -\frac{1}{\mathsf{c}_{\mathsf{d}}}\int_{\mathbb{R}^{\mathsf{d}}} H_N\Delta H_N = \frac{1}{\mathsf{c}_{\mathsf{d}}}\int_{\mathbb{R}^{\mathsf{d}}} |\nabla H_N|^2$$

after integrating by parts by Green's formula. This computation is in fact incorrect because it ignores the diagonal terms which must be removed from the integral, and yields a divergent integral $\int |\nabla H_N|^2$ (it diverges near each point $x_i$ of the configuration). However, this computation can be done properly by removing the infinite diagonal terms and "renormalizing" the infinite integral, replacing $\int |\nabla H_N|^2$ by

$$\int_{\mathbb{R}^{\mathsf{d}}} |\nabla H_{N,\eta}|^2 - N\mathsf{c}_{\mathsf{d}} g(\eta)$$

where we replace $H_N$ by $H_{N,\eta}$, its "truncation" at level $\eta$ (here $\eta = \alpha N^{-1/\mathsf{d}}$ with $\alpha$ a small fixed number) — more precisely $H_{N,\eta}$ is obtained by replacing the Dirac masses in (4-5) by uniform measures of total mass 1 supported on the sphere $\partial B(x_i,\eta)$ — and then removing the appropriate divergent part $\mathsf{c}_{\mathsf{d}}\mathsf{g}(\eta)$. The name *renormalized energy* originates in the work of Bethuel, Brezis, and Hélein [1994] in the context of two-dimensional Ginzburg–Landau vortices, where a similar (although different) renormalization procedure was introduced. Such a computation allows to replace the double integral, or sum of pairwise interactions of all the charges and "background", by a single integral, which is local in the potential $H_N$. This transformation is very useful, and uses crucially the fact that $\mathsf{g}$ is the kernel of a local operator (the Laplacian).

This electric energy $\int_{\mathbb{R}^{\mathsf{d}}} |\nabla H_{N,\eta}|^2$ is coercive and can thus serve to control the "fluctuations" $\sum_{i=1}^N \delta_{x_i} - N\mu_V$, in fact it is formally $\frac{1}{\mathsf{c}_{\mathsf{d}}} \|\nabla \Delta^{-1}(\sum_{i=1}^N \delta_{x_i} - N\mu_V)\|_{L^2}^2$. The relations (4-3)–(4-7) can be inserted into the Gibbs measure (1-7) to yield so-called "concentration results" in the case with temperature, see Serfaty [2014] (for prior such concentration results, see Maïda and Maurel-Segala [2014], Borot and Guionnet [2013a], and Chafai, Hardy, and Maida [2018]).

**4.2 Blow-up and limiting energy.** As we have seen, the configurations we are interested in are concentrated on (or near) the support of $\mu_V$ which is a set of macroscopic size and dimension $\mathsf{d}$, and the typical distance between neighboring points is $N^{-1/\mathsf{d}}$. The next step is then to blow-up the configurations by $N^{1/\mathsf{d}}$ and take the $N \to \infty$ limit in $F_N^{\mu_V}$. This leads us to a renormalized energy that we define just below. It allows to compute a total Coulomb interaction for an infinite system of discrete point charges in a constant neutralizing background of fixed density 1. Such a system is often called a *jellium* in physics, and is sometimes considered as a toy model for matter, with a uniform electron sea and ions whose positions remain to be optimized.

From now on, we assume that $\Sigma$, the support of $\mu_V$ is a set with a regular boundary and $\mu_V(x)$ is a regular density function in $\Sigma$. Centering at some point $x$ in $\Sigma$, we may blow-up the configuration by setting $x_i' = (N\mu_V(x))^{1/\mathsf{d}}(x_i - x)$ for each $i$. This way we expect to have a density of points equal to 1 after rescaling. Rescaling and taking $N \to \infty$ in (4-6), we are led to $H_N \to H$ with $H$ solving an equation of the form

$$-\Delta H = \mathsf{c}_{\mathsf{d}}(\mathcal{C} - 1) \tag{4-8}$$

where $\mathcal{C}$ is a locally finite sum of Dirac masses.

**Definition 4.1** (Sandier and Serfaty [2015b,a], Rougerie and Serfaty [2016], and Petrache and Serfaty [2017])**.** The (Coulomb) renormalized energy of $H$ is

$$\mathcal{W}(H) := \lim_{\eta \to 0} \mathcal{W}_\eta(H) \tag{4-9}$$

where we let

$$\mathcal{W}_\eta(H) := \limsup_{R \to \infty} \frac{1}{R^{\mathsf{d}}} \int_{[-\frac{R}{2}, \frac{R}{2}]^{\mathsf{d}}} |\nabla H_\eta|^2 - \mathsf{c}_{\mathsf{d}} \mathsf{g}(\eta) \tag{4-10}$$

and $H_\eta$ is a truncation of $H$ performed similarly as above.
We define the renormalized energy of a point configuration $\mathcal{C}$ as

$$\mathbb{W}(\mathcal{C}) := \inf\{\mathcal{W}(H) \mid -\Delta H = \mathsf{c}_{\mathsf{d}}(\mathcal{C} - 1)\} \tag{4-11}$$

with the convention $\inf(\varnothing) = +\infty$.

It is not a priori clear how to define a total Coulomb interaction of such a jellium system, because of the infinite size of the system and because of its lack of local charge neutrality. The definitions we presented avoid having to go through computing the sum of pairwise interactions between particles (it would not even be clear how to sum them), but instead replace it with (renormalized variants of) the extensive quantity $\int |\nabla H|^2$.

The energy $\mathbb{W}$ can be proven to be bounded below and to have a minimizer; moreover, its minimum can be achieved as the limit of energies of periodic configurations (with larger and larger period), for all these aspects see for instance Serfaty [2015].

**4.3 Crystallization questions for minimizers.** Determining the value of $\min \mathbb{W}$ is an open question, with the exception of the one-dimensional analogues for which the minimum is achieved at the lattice $\mathbb{Z}$ Sandier and Serfaty [2015a] and Leblé [2016].

The only question that we can completely answer so far is that of the minimization over the restricted class of lattice configurations in dimension $\mathsf{d} = 2$, i.e. configurations which are exactly a lattice $\mathbb{Z}\vec{u} + \mathbb{Z}\vec{v}$ with $det(\vec{u}, \vec{v}) = 1$.

**Theorem 4.** *The minimum of $\mathbb{W}$ over lattices of volume 1 in dimension 2 is achieved uniquely by the triangular lattice.*

Here the triangular lattice means $\mathbb{Z} + \mathbb{Z}e^{i\pi/3}$, properly scaled, i.e. what is called the Abrikosov lattice in the context of superconductivity. This result is essentially equivalent (see Osgood, Phillips, and Sarnak [1988] and Chiu [1997]) to a result on the minimization of the Epstein $\zeta$ function of the lattice

$$\zeta_s(\Lambda) := \sum_{p\in\Lambda\setminus\{0\}} \frac{1}{|p|^s}$$

proven in the 50's by Cassels, Rankin, Ennola, Diananda, cf. Montgomery [1988] and references therein. It corresponds to the minimization of the "height" of flat tori, in the sense of Arakelov geometry. In dimension $\mathsf{d} \geq 3$ the minimization of $\mathbb{W}$ restricted to the class of lattices is an open question, except in dimensions 4, 8 and 24 where a strict local minimizer is known Sarnak and Strömbergsson [2006] (it is the $E_8$ lattice in dimension 8 and the Leech lattice in dimension 24, which were already mentioned before).

One may ask whether the triangular lattice does achieve the global minimum of $\mathbb{W}$ in dimension 2. The fact that the Abrikosov lattice is observed in superconductors, combined with the fact that $\mathbb{W}$ can be derived as the limiting minimization problem of Ginzburg–Landau, see Sandier and Serfaty [2012], justify conjecturing this.

**Conjecture 4.2.** The triangular lattice is a global minimizer of $\mathbb{W}$ in dimension 2.

It was also recently proven in Bétermin and Sandier [2018] that this conjecture is equivalent to a conjecture of Brauchart, Hardin, and Saff [2012] on the next order term in the asymptotic expansion of the minimal logarithmic energy on the sphere (an important problem in approximation theory, also related to Smale's "7th problem for the 21st century"), which is obtained by formal analytic continuation, hence by very different arguments. In addition, the result of Coulangeon and Schürmann [2012] essentially yields the local minimality of the triangular lattice within all periodic (with possibly large period) configurations.

Note that the triangular lattice, the $E_8$ lattice in dimension 8 and Leech lattice in dimension 24, mentioned above, are also conjectured by Cohn and Kumar [2007] to have

universally minimizing properties i.e. to be the minimizer for a broad class of interactions. The proof of this conjecture in dimensions 8 and 24 was recently announced by the authors of Cohn, Kumar, Miller, Radchenko, and Viazovska [2017], and it should imply that these lattices also minimize $\mathbb{W}$.

One may expect that in general low dimensions, the minimum of $\mathbb{W}$ is achieved by some particular lattice. Folklore knowledge is that lattices are not minimizing in large enough dimensions, as indicated by the situation for the sphere packing problem mentioned above.

These questions belongs to the more general family of crystallization problems, see Blanc and Lewin [2015] for a review. A typical such question is, given an interaction kernel $\mathsf{g}$ in any dimension, to determine the point positions that minimize

$$\sum_{i\neq j} \mathsf{g}(x_i - x_j)$$

(with some kind of boundary condition), or rather

$$\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{i\neq j, x_i, x_j \in B_R} \mathsf{g}(x_i - x_j),$$

and to determine whether the minimizing configurations are lattices. Such questions are fundamental in order to understand the crystalline structure of matter. There are very few positive results in that direction in the literature, with the exception of Theil [2006] generalizing Radin [1981] for a class of very short range Lennard–Jones potentials, which is why the resolution of the sphere packing problem and the Cohn–Kumar conjecture are such breakthroughs.

**4.4 Convergence results for minimizers.** Given a (sequence of) configuration(s) $(x_1, \dots, x_N)$, we examine as mentioned before the blow-up point configurations

$$\{(\mu_V(x)N)^{1/d}(x_i - x)\}$$

and their infinite limits $\mathcal{C}$. We also need to let the blow-up center $x$ vary over $\Sigma$, the support of $\mu_V$. Averaging near the blow-up center $x$ yields a "point process" $P_N^x$: a point process is precisely defined as a probability distribution on the space of possibly infinite point configurations, denoted Config. Here the point process $P_N^x$ is essentially the Dirac mass at the blown-up configuration $\{(\mu_V(x)N)^{1/d}(x_i - x)\}$. This way, we form a "tagged point process" $P_N$ (where the tag is the memory of the blow-up center), probability on $\Sigma \times$ Config, whose "slices" are the $P_N^x$. Taking limits $N \to \infty$ (up to subsequences), we obtain limiting tagged point processes $P$, which are all stationary, i.e. translation-invariant. We may also define the renormalized Coulomb energy at the level of tagged point processes as

$$\overline{\mathbb{W}}(P) := \frac{1}{2\mathsf{C_d}} \int_\Sigma \int \mathbb{W}(\mathcal{C}) dP^x(\mathcal{C}) dx.$$

In view of (4-3) and the previous discussion, we may expect the following informally stated result (which we state only in the Coulomb cases, for extensions to (1-5) see Sandier and Serfaty [2015a] and to (1-6) see Petrache and Serfaty [2017]).

**Theorem 5** (Sandier and Serfaty [2015b] and Rougerie and Serfaty [2016]). *Consider configurations such that*

$$\mathcal{H}_N(x_1,\dots,x_N) - N^2\mathcal{J}_V(\mu_V) \le CN^{2-\frac{2}{\mathsf{d}}}.$$

*Then up to extraction $P_N$ converges to some $P$ and*

$$\mathcal{H}_N(x_1,\dots,x_N) \simeq N^2\mathcal{J}_V(\mu_V) + N^{2-\frac{2}{\mathsf{d}}}\overline{\mathbb{W}}(P) + o(N^{2-\frac{2}{\mathsf{d}}}) \tag{4-12}$$

[5] *and in particular*

$$\min \mathcal{H}_N = N^2\mathcal{J}_V(\mu_V) + N^{2-\frac{2}{\mathsf{d}}}\min\overline{\mathbb{W}} + o(N^{2-\frac{2}{\mathsf{d}}}). \tag{4-13}$$

Since $\overline{\mathbb{W}}$ is an average of $\mathbb{W}$, the result (4-13) can be read as: after suitable blow-up around a point $x$, for a.e. $x \in \Sigma$, the minimizing configurations converge to minimizers of $\mathbb{W}$. If one believes minimizers of $\mathbb{W}$ to ressemble lattices, then it means that minimizers of $\mathcal{H}_N$ should do so as well. In any case, $\mathbb{W}$ can distinguish between different lattices (in dimension 2, the triangular lattice has less energy than the square lattice) and we expect $\mathbb{W}$ to be a good quantitative measure of disorder of a configuration (see Borodin and Serfaty [2013]).

The analogous result was proven in Sandier and Serfaty [2012] for the vortices in minimizers of the Ginzburg–Landau energy (2-1): they also converge after blow-up to minimizers of $\mathbb{W}$, providing a first rigorous justification of the Abrikosov lattice observed in experiments, modulo Conjecture 4.2. The same result was also obtained in Goldman, Muratov, and Serfaty [2014] for a two-dimensional model of small charged droplets interacting logarithmically called the Ohta–Kawasaki model – a sort of variant of Gamov's liquid drop model, after the corresponding mean-field limit results was established in Goldman, Muratov, and Serfaty [2013].

One advantage of the above theorem is that it is valid for generic configurations and not just for minimizers. When using the minimality, better "rigidity results" (as alluded to above) of minimizers can be proven: points are separated by $\frac{C}{(N\|\mu_V\|_\infty)^{1/d}}$ for some fixed $C > 0$ and there is uniform distribution of points and energy, down to the microscopic scale, see Petrache and Serfaty [2017], Nodari and Serfaty [2015], and Petrache and Rota Nodari [2018].

Theorem 5 relies on two ingredients which serve to prove respectively a lower bound and an upper bound for the next-order energy. The first is a general method for proving lower bounds for energies which have two intrinsic scales (here the macroscopic scale 1 and the microscopic scale $N^{-1/\mathsf{d}}$) and which is handled via the introduction of the probability measures on point patterns $P_N$ described above. This method (see Sandier and Serfaty [2015b] and Serfaty [2015]), inspired by Varadhan, is reminiscent of Young measures and of Alberti and Müller [2001]. The second is a "screening procedure" which allows to exploit the local nature of the next-order energy expressed in terms of $H_N$, to paste together configurations given over large microscopic cubes and compute their next-order energy additively. To do so, we need to modify the configuration in a neighborhood of the boundary of the cube so as to make the cube neutral in charge and to

[5] In dimension $\mathsf{d} = 2$, there is an additional additive term $\frac{N}{4}\log N$ in both relations

make $\nabla H_N$ tangent to the boundary. This effectively screens the configuration in each cube in the sense that it makes the interaction between the different cubes vanish, so that the energy $\int |\nabla H_N|^2$ becomes proportional to the volume. One needs to show that this modification can be made while altering only a negligible fraction of the points and a negligible amount of the energy. This construction is reminiscent of Alberti, Choksi, and Otto [2009]. It is here crucial that the interaction is Coulomb so that the energy is expressed by a local function of $H_N$, which itself solves an elliptic PDE, making it possible to use the toolbox on estimates for such PDEs.

The next order study has not at all been touched in the case of dynamics, but it has been tackled in the statistical mechanics setting of (1-7).

**4.5 Next-order with temperature.** Here the interesting temperature regime (to see nontrivial temperature effects) turns out to be $\beta_N = \beta N^{\frac{2}{d}-1}$.

In contrast to the macroscopic result, several observations (e.g. by numerical simulation, see Figure 3) suggest that the behavior of the system at the microscopic scale depends heavily on $\beta$, and one would like to describe this more precisely. In the particular case of (1-5) or (1-2) with $\beta = 2$, which both arise in Random Matrix Theory, many things can be computed explicitly, and expansions of $\log Z_{N,\beta}$ as $N \to \infty$, Central Limit Theorems for linear statistics, universality in $V$ (after suitable rescaling) of the microscopic behavior and local statistics of the points, are known Johansson [1998], Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], Bourgade, Erdős, and Yau [2014, 2012], Bekerman, Figalli, and Guionnet [2015], and Bekerman and Lodhia [2016]. Generalizing such results to higher dimensions and all $\beta$'s is a significant challenge.

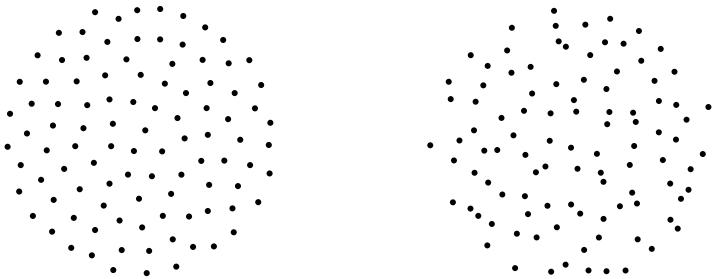

Figure 3: Case Log2 with $N = 100$ and $V(x) = |x|^2$, for $\beta = 400$ (left) and $\beta = 5$ (right).

**4.6 Large Deviations Principle.** A first approach consists in following the path taken for minimizers and using the next-order expansion of $\mathcal{H}_N$ given in (4-12). This

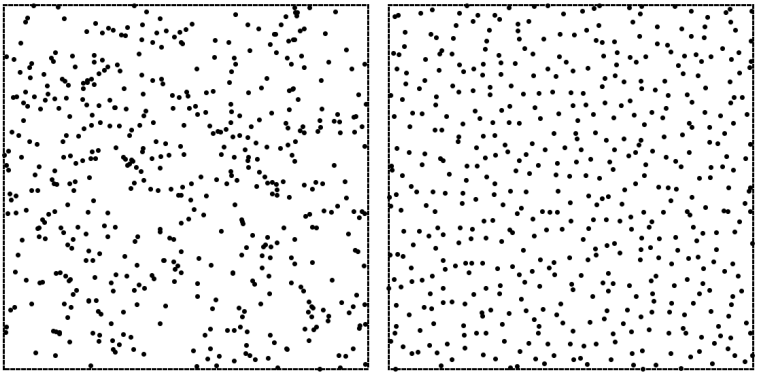

Figure 4: Simulation of the Poisson point process with intensity 1 (left), and the Ginibre point process with intensity 1 (right)

expansion can be formally inserted into (1-7), however this is not sufficient: to get a complete result, one needs to understand precisely how much volume in configuration space $(\mathbb{R}^{\mathsf{d}})^N$ is occupied near a given tagged point process $P$ — this will give rise to an entropy term — and how much error (in both volume and energy) the screening construction creates. At the end we obtain a Large Deviations Principle expressed at the level of the microscopic point processes $P$, instead of the macroscopic empirical measures $\mu$ in Theorem 3. This is sometimes called "type-III large deviations" or large deviations at the level of empirical fields. Such results can be found in Varadhan [1988], Föllmer [1988], the relative specific entropy that we will use is formalized in Föllmer and Orey [1988] (for the non-interacting discrete case), Georgii [1993] (for the interacting discrete case) and Georgii and Zessin [1993] (for the interacting continuous case).

To state the result precisely, we need to introduce the Poisson point process with intensity 1, denoted $\Pi$, as the point process characterized by the fact that for any bounded Borel set $B$ in $\mathbb{R}^{\mathsf{d}}$

$$\Pi\left(N(B)=n\right)=\frac{|B|^n}{n!}e^{-|B|}$$

where $N(B)$ denotes the number of points in $B$. The expectation of the number of points in $B$ can then be computed to be $|B|$, and one also observes that the number of points in two disjoint sets are independent, thus the points "don't interact", see Figure 4 for a picture. The "specific" relative entropy $\overline{\mathsf{ent}}$ with respect to $\Pi$ refers to the fact that it has to be computed taking an infinite volume limit, see Rassoul-Agha and Seppäläinen [2015] for a precise definition. One can just think that it measures how close the point process is to the Poisson one.

For any $\beta>0$, we then define a free energy functional $\overline{\mathcal{F}}_\beta$ as

$$\overline{\mathcal{F}}_\beta(P):=\frac{\beta}{2}\overline{\mathbb{W}}(P)+\overline{\mathsf{ent}}[P|\Pi]. \tag{4-14}$$

**Theorem 6** (Leblé and Serfaty [2017]). *Under suitable assumptions, for any $\beta > 0$ a Large Deviations Principle at speed $N$ with good rate function $\overline{\mathcal{F}}_\beta - \inf \overline{\mathcal{F}}_\beta$ holds in the sense that*

$$\mathbb{P}_{N,\beta}(P_N \simeq P) \simeq e^{-N(\overline{\mathcal{F}}_\beta(P) - \inf \overline{\mathcal{F}}_\beta)}$$

This way, the the Gibbs measure $\mathbb{P}_{N,\beta}$ concentrates on microscopic point processes which minimize $\overline{\mathcal{F}}_\beta$. This minimization problem corresponds to some balancing (depending on $\beta$) between $\overline{\mathbb{W}}$, which prefers order of the configurations (and expectedly crystallization in low dimensions), and the relative entropy term which measures the distance to the Poisson process, thus prefers microscopic disorder and decorrelation between the points. As $\beta \to 0$, or temperature gets very large, the entropy term dominates and one can prove Leblé [2016] that the minimizer of $\overline{\mathcal{F}}_\beta$ converges to the Poisson process. On the contrary, when $\beta \to \infty$, the $\mathbb{W}$ term dominates, and prefers regular and rigid configurations. (In the case (1-5) where the minimum of $\mathbb{W}$ is known to be achieved by the lattice, this can be made into a complete proof of crystallization as $\beta \to \infty$, cf. Leblé [ibid.]). When $\beta$ is intermediate then both terms are important and one does not expect crystallization in that sense nor complete decorrelation. For separation results analogous to those quoted about minimizers, one may see Ameur [2017] and references therein.

The existence of a minimizer to $\overline{\mathcal{F}}_\beta$ is known, it is certainly nonunique due to the rotational invariance of the problem, but it is not known whether it is unique modulo rotations, nor is the existence of a limiting point process $P$ (independent of the subsequence) in general. The latter is however known to exist in certain ensembles arising in random matrix theory: for (1-5) for any $\beta$, it is the so-called sine-$\beta$ process Kurzke and Spirn [2014] and Valkó and Virág [2009], and for (1-2) for $\beta = 2$ and $V$ quadratic, it is the Ginibre point process Ginibre [1965], shown in Figure 4. It was also shown to exist for the jellium for small $\beta$ in Imbrie [1982/83]. A consequence of Theorem 6 is to provide a variational interpretation to these point processes. One may hope to understand phase-transitions at the level of these processes, possibly via this variational interpretation, however this is completely open. While in dimension 1, the point process is expected to always be unique, in dimension 2, phase-transitions and symmetry breaking in positional or orientational order may happen. One would also like to understand the decay of the two-point correlation function and its possible change in rate, corresponding to a phase-transition. In the one-dimensional logarithmic case, the limits of the correlation functions are computed for rational $\beta$'s Forrester [1993] and indicate a phase-transition.

A second corollary obtained as a by-product of Theorem 6 is the existence of a next order expansion of the free energy $-\beta^{-1} \log Z_{N,\beta}$.

**Corollary 4.3** (Leblé and Serfaty [2017]).

$$-\beta^{-1} \log Z_{N,\beta} = N^{1+\frac{2}{\mathsf{d}}} \mathbb{1}_V(\mu_V) + N \min \overline{\mathcal{F}} + o(N) \tag{4-15}$$

*in the cases* (1-3)*; and in the cases* (1-2)*,* (1-5)*,*

$$-\beta^{-1} \log Z_{N,\beta} = N^2 \mathbb{1}_V(\mu_V) - \frac{N}{2\mathsf{d}} \log N + N \min \overline{\mathcal{F}}_\beta + o(N)$$

*or more explicitly*

$$-\beta^{-1}\log Z_{N,\beta} = \tag{4-16}$$
$$= N^2 \mathfrak{l}_V(\mu_V) - \frac{N}{2\mathsf{d}}\log N + NC_\beta + N\left(\frac{1}{\beta}-\frac{1}{2\mathsf{d}}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx + o(N),$$

*where $C_\beta$ depends only on $\beta$, but not on $V$.*

This formulae are to be compared with the results of Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], and Bekerman, Figalli, and Guionnet [2015] in the $\mathsf{Log}\,\mathsf{l}$ case, the semi-rigorous formulae in Zabrodin and Wiegmann [2006] in the dimension 2 Coulomb case, and are the best-known information on the free energy otherwise. We recall that understanding the free energy is fundamental for the description of the properties of the system. For instance, the explicit dependence in $V$ exhibited in (4-16) will be the key to proving the result of the next section.

Finally, note that a similar result to the above theorem and corollary can be obtained in the case of the two-dimensional two-component plasma alluded to in Section 2.4, see Leblé, Serfaty, and Zeitouni [2017].

**4.7 A Central Limit Theorem for fluctuations.** Another approach to understanding the rigidity of configurations and how it depends on the temperature is to examine the behavior of the linear statistics of the fluctuations, i.e. consider, for a regular test function $f$, the quantity

$$\sum_{i=1}^N f(x_i) - N\int f d\mu_V.$$

**Theorem 7** (Leblé and Serfaty [2018]). *In the case (1-2), assume $V \in C^4$ and the previous assumptions on $\mu_V$ and $\partial\Sigma$, and let $f \in C_c^4(\mathbb{R}^2)$ or $C_c^3(\Sigma)$. If $\Sigma$ has $m \geq 2$ connected components $\Sigma_i$, add $m-1$ conditions $\int_{\partial\Sigma_i}\Delta f^\Sigma = 0$ where $f^\Sigma$ is the harmonic extension of $f$ outside $\Sigma$. Then*

$$\sum_{i=1}^N f(x_i) - N\int_\Sigma f\,d\mu_V$$

*converges in law as $N\to\infty$ to a Gaussian distribution with*

$$\textit{mean} = \frac{1}{2\pi}\left(\frac{1}{\beta}-\frac{1}{4}\right)\int_{\mathbb{R}^2}\Delta f\,(\mathbf{1}_\Sigma + \log\Delta V)^\Sigma \qquad \textit{variance} = \frac{1}{2\pi\beta}\int_{\mathbb{R}^2}|\nabla f^\Sigma|^2.$$

*The result can moreover be localized with $f$ supported on any mesoscale $N^{-\alpha}$, $\alpha < \frac{1}{2}$, and it is true as well for energy minimizers, taking formally $\beta = \infty$.*

This result can be interpreted in terms of the convergence of $H_N$ (of (4-5)) to a suitable so-called "Gaussian Free Field", a sort of two-dimensional analogue of Brownian motion. This theorem shows that if $f$ is smooth enough, the fluctuations of linear statistics are typically of order 1, i.e. much smaller than the sum of $N$ iid random variables

which is typically or order $\sqrt{N}$. This a manifestation of rigidity, which even holds down to the mesoscales. Note that the regularity of $f$ is necessary, the result is false if $f$ is discontinuous, however the precise threshold of regularity is not known.

In dimension 1, this theorem was first proven in Johansson [1998] for polynomial $V$ and $f$ analytic. It was later generalized in Shcherbina [2013], Borot and Guionnet [2013a], Borot and Guionnet [2013b], Bekerman and Lodhia [2016], Webb [2016], and Bekerman, Leblé, and Serfaty [2017]. In dimension 2, this result was proven for the determinantal case $\beta = 2$, first in Rider and Virág [2007] (for $V$ quadratic), Berman [n.d.] assuming just $f \in C^1$, and then Ameur, Hedenmalm, and Makarov [2011] under analyticity assumptions. It was then proven for all $\beta$ simultaneously as Leblé and Serfaty [2018] in Bauerschmidt, Bourgade, Nikula, and Yau [n.d.], with $f$ assumed to be supported in $\Sigma$. The approach for proving such results has generally been based on Dyson–Schwinger (or "loop") equations.

If the extra conditions do not hold, then the CLT is not expected to hold. Rather, the limit should be a Gaussian convolved with a discrete Gaussian variable, as shown in the $\mathsf{Log1}$ case in Borot and Guionnet [2013b].

To prove Theorem 7, following the approach pioneered by Johansson [1998], we compute the Laplace transform of these linear statistics and see that it reduces to understanding the ratio of two partition functions, the original one and that of a Coulomb gas with potential $V$ replaced by $V_t = V + tf$ with $t$ small. Thanks to Serfaty and Serra [n.d.] the variation of the equilibrium measure associated to this replacement is well understood. We are then able to leverage on the expansion of the partition function of (4-16) to compute the desired ratio, using also a change of variables which is a transport map between the equilibrium measure $\mu_V$ and the perturbed equilibrium measure. Note that the use of changes of variables in this context is not new, cf. Johansson [1998], Borot and Guionnet [2013a], Shcherbina [2013], and Bekerman, Figalli, and Guionnet [2015]. In our approach, it essentially replaces the use of the loop or Dyson–Schwinger equations.

**4.8 More general interactions.** It remains to understand how much of the behavior we described are really specific to Coulomb interactions. Already Theorems 5 and 6 were shown in Petrache and Serfaty [2017] and Leblé and Serfaty [2017] to hold for the more general Riesz interactions with $\mathsf{d} - 2 \leq s < \mathsf{d}$. This is thanks to the fact that the Riesz kernel is the kernel for a fractional Laplacian, which is not a local operator but can be interpreted as a local operator after adding one spatial dimension, according to the procedure of Caffarelli and Silvestre [2007]. The results of Theorem 6 are also valid in the *hypersingular* Riesz interactions $\mathsf{s} > \mathsf{d}$ (see Hardin, Leblé, Saff, and Serfaty [2018]), where the kernel is very singular but also decays very fast. The Gaussian behavior of the fluctuations seen in Theorem 7 is for now proved only in the logarithmic cases, but it remains to show whether it holds for more general Coulomb cases and even possibly more general interactions as well.

**Acknowledgments.** I am very grateful to Mitia Duerinckx, Thomas Leblé, Mathieu Lewin and Nicolas Rougerie for their helpful comments and suggestions on the first version of this text. I also thank Catherine Goldstein for the historical references and Alon Nishry for the pictures of Figure 4.

## References

Alexei A. Abrikosov (1957). "On the magnetic properties of superconductors of the second group". *Sov. Phys. JETP* 5, pp. 1174–1182 (cit. on p. 938).

A. Alastuey and B. Jancovici (1981). "On the classical two-dimensional one-component Coulomb plasma". *J. Physique* 42.1, pp. 1–12. MR: 604143 (cit. on p. 941).

Giovanni Alberti, Rustum Choksi, and Felix Otto (2009). "Uniform energy distribution for an isoperimetric problem with long-range interactions". *J. Amer. Math. Soc.* 22.2, pp. 569–605. MR: 2476783 (cit. on p. 958).

Giovanni Alberti and Stefan Müller (2001). "A new approach to variational problems with multiple scales". *Comm. Pure Appl. Math.* 54.7, pp. 761–825. MR: 1823420 (cit. on p. 957).

Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré (2005). *Gradient flows in metric spaces and in the space of probability measures*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, pp. viii+333. MR: 2129498 (cit. on p. 948).

Luigi Ambrosio and Sylvia Serfaty (2008). "A gradient flow approach to an evolution problem arising in superconductivity". *Comm. Pure Appl. Math.* 61.11, pp. 1495–1539. MR: 2444374 (cit. on p. 948).

Yacin Ameur (2017). "Repulsion in low temperature $\beta$-ensembles". arXiv: 1701.04796 (cit. on p. 960).

Yacin Ameur, Håkan Hedenmalm, and Nikolai Makarov (2011). "Fluctuations of eigenvalues of random normal matrices". *Duke Math. J.* 159.1, pp. 31–81. MR: 2817648 (cit. on p. 962).

Greg W. Anderson, Alice Guionnet, and Ofer Zeitouni (2010). *An introduction to random matrices*. Vol. 118. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, pp. xiv+492. MR: 2760897 (cit. on p. 943).

Julien Barré, Freddy Bouchet, Thierry Dauxois, and Stefano Ruffo (2005). "Large deviation techniques applied to systems with long-range interactions". *J. Stat. Phys.* 119.3-4, pp. 677–713. MR: 2151219 (cit. on p. 942).

Roland Bauerschmidt, Paul Bourgade, Miika Nikula, and Horng-Tzer Yau (n.d.). "The two-dimensional Coulomb plasma: quasi-free approximation and central limit theorem". arXiv: 1609.08582v2 (cit. on p. 962).

F. Bekerman, A. Figalli, and A. Guionnet (2015). "Transport maps for $\beta$-matrix models and universality". *Comm. Math. Phys.* 338.2, pp. 589–619. MR: 3351052 (cit. on pp. 958, 961, 962).

Florent Bekerman, Thomas Leblé, and Sylvia Serfaty (2017). "CLT for fluctuations of $\beta$-ensembles with general potential". arXiv: 1706.09663 (cit. on p. 962).

Florent Bekerman and Asad Lodhia (2016). "Mesoscopic central limit theorem for general $\beta$-ensembles". arXiv: 1605.05206 (cit. on pp. 958, 962).

G. Ben Arous and A. Guionnet (1997). "Large deviations for Wigner's law and Voiculescu's non-commutative entropy". *Probab. Theory Related Fields* 108.4, pp. 517–542. MR: 1465640 (cit. on p. 951).

Gérard Ben Arous and Ofer Zeitouni (1998). "Large deviations from the circular law". *ESAIM Probab. Statist.* 2, pp. 123–134. MR: 1660943 (cit. on p. 951).

R. J. Berman (n.d.). "Determinantal point processes and fermions on complex manifolds: Bulk universality". To appear in *Conference Proceedings on Algebraic and Analytic Microlocal Analysis (Northwestern, 2015)*, M. Hitrik, D. Tamarkin, B. Tsygan, and S. Zelditch, (eds.) Springer. arXiv: 0811.3341 (cit. on p. 962).

Robert J. Berman (2014). "Determinantal point processes and fermions on complex manifolds: large deviations and bosonization". *Comm. Math. Phys.* 327.1, pp. 1–47. MR: 3177931 (cit. on p. 951).

– (2017). "Large deviations for Gibbs measures with singular Hamiltonians and emergence of Kähler-Einstein metrics". *Comm. Math. Phys.* 354.3, pp. 1133–1172. MR: 3668617 (cit. on p. 944).

Robert J. Berman, Sébastien Boucksom, and David Witt Nyström (2011). "Fekete points and convergence towards equilibrium measures on complex manifolds". *Acta Math.* 207.1, pp. 1–27. MR: 2863909 (cit. on p. 944).

Robert J. Berman and Magnus Onnheim (2016). "Propagation of chaos for a class of first order models with singular mean field interactions". arXiv: 1610.04327 (cit. on p. 949).

– (2018). "Propagation of chaos, Wasserstein gradient flows and toric Kähler-Einstein metrics". *Anal. PDE* 11 (6), pp. 1343–1380. arXiv: 1501.07820 (cit. on p. 950).

Laurent Bétermin and Etienne Sandier (2018). "Renormalized Energy and Asymptotic Expansion of Optimal Logarithmic Energy on the Sphere". *Constr. Approx.* 47.1, pp. 39–74. MR: 3742809 (cit. on p. 955).

F. Bethuel, G. Orlandi, and D. Smets (2005). "Collisions and phase-vortex interactions in dissipative Ginzburg–Landau dynamics". *Duke Math. J.* 130.3, pp. 523–614. MR: 2184569 (cit. on p. 939).

– (2007). "Dynamics of multiple degree Ginzburg–Landau vortices". *Comm. Math. Phys.* 272.1, pp. 229–261. MR: 2291808 (cit. on p. 939).

Fabrice Bethuel, Haïm Brezis, and Frédéric Hélein (1994). *Ginzburg–Landau vortices*. Vol. 13. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, pp. xxviii+159. MR: 1269538 (cit. on pp. 939, 953).

Fabrice Bethuel, Robert L. Jerrard, and Didier Smets (2008). "On the NLS dynamics for infinite energy vortex configurations on the plane". *Rev. Mat. Iberoam.* 24.2, pp. 671–702. MR: 2459209 (cit. on p. 939).

Fabrice Bethuel and Tristan Rivière (1995). "Vortices for a variational problem related to superconductivity". *Ann. Inst. H. Poincaré Anal. Non Linéaire* 12.3, pp. 243–303. MR: 1340265 (cit. on p. 939).

W. Bietenholz and U. Gerber (2016). "Berezinskii–Kosterlitz–Thouless Transition and the Haldane Conjecture: Highlights of the Physics Nobel Prize 2016". *Revista Cubana de Física* 33.2, pp. 156–168. arXiv: 1612.06132 (cit. on p. 943).

Xavier Blanc and Mathieu Lewin (2015). "The crystallization conjecture: a review". *EMS Surv. Math. Sci.* 2.2, pp. 225–306. MR: 3429164 (cit. on p. 956).

C. Blondel and B. Wolff (2015). "Coulomb et la difficile gestion du "mélange du Calcul et de la Physique". In: Sous la direction de C. Gilain et A. Guilbaud, CNRS éditions, pp. 1750–1850 (cit. on p. 936).

Niklas Boers and Peter Pickl (2016). "On mean field limits for dynamical systems". *J. Stat. Phys.* 164.1, pp. 1–16. MR: 3509045 (cit. on p. 949).

Alexei Borodin (2011). "Determinantal point processes". In: *The Oxford handbook of random matrix theory*. Oxford Univ. Press, Oxford, pp. 231–249. MR: 2932631 (cit. on p. 944).

Alexei Borodin and Vadim Gorin (2016). "Lectures on integrable probability". In: *Probability and statistical physics in St. Petersburg*. Vol. 91. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, pp. 155–214. MR: 3526828 (cit. on p. 944).

Alexei Borodin and Sylvia Serfaty (2013). "Renormalized energy concentration in random matrices". *Comm. Math. Phys.* 320.1, pp. 199–244. MR: 3046995 (cit. on p. 957).

G. Borot and A. Guionnet (2013a). "Asymptotic expansion of $\beta$ matrix models in the one-cut regime". *Comm. Math. Phys.* 317.2, pp. 447–483. MR: 3010191 (cit. on pp. 954, 958, 961, 962).

Gaëtan Borot and Alice Guionnet (Mar. 2013b). "Asymptotic expansion of beta matrix models in the multi-cut regime". arXiv: 1303.1045 (cit. on pp. 958, 961, 962).

Paul Bourgade, László Erdős, and Horng-Tzer Yau (2012). "Bulk universality of general $\beta$-ensembles with non-convex potential". *J. Math. Phys.* 53.9, pp. 095221, 19. MR: 2905803 (cit. on p. 958).

– (2014). "Universality of general $\beta$-ensembles". *Duke Math. J.* 163.6, pp. 1127–1190. arXiv: 0907.5605. MR: 3192527 (cit. on p. 958).

J. S. Brauchart, D. P. Hardin, and Edward B. Saff (2012). "The next-order term for optimal Riesz and logarithmic energy asymptotics on the sphere". In: *Recent advances in orthogonal polynomials, special functions, and their applications*. Vol. 578. Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 31–61. MR: 2964138 (cit. on pp. 941, 955).

S. G. Brush, H. L. Sahlin, and E. Teller (1966). "Monte Carlo Study of a One-Component Plasma. I". *The Journal of Chemical Physics* 45.6, pp. 2102–2118 (cit. on p. 942).

Luis Caffarelli and Luis Silvestre (2007). "An extension problem related to the fractional Laplacian". *Comm. Partial Differential Equations* 32.7-9, pp. 1245–1260. MR: 2354493 (cit. on p. 962).

Luis Caffarelli, Fernando Soria, and Juan Luis Vázquez (2013). "Regularity of solutions of the fractional porous medium flow". *J. Eur. Math. Soc. (JEMS)* 15.5, pp. 1701–1746. MR: 3082241 (cit. on p. 948).

Luis Caffarelli and Juan Luis Vazquez (2011). "Nonlinear porous medium flow with fractional potential pressure". *Arch. Ration. Mech. Anal.* 202.2, pp. 537–565. MR: 2847534 (cit. on p. 948).

E. Caglioti, P.-L. Lions, C. Marchioro, and M. Pulvirenti (1992). "A special class of stationary flows for two-dimensional Euler equations: a statistical mechanics description". *Comm. Math. Phys.* 143.3, pp. 501–525. MR: 1145596 (cit. on p. 951).

J. M. Caillol, D. Levesque, J. J. Weis, and J. P. Hansen (1982). "A Monte Carlo study of the classical two-dimensional one-component plasma". *Journal of Statistical Physics* 28.2, pp. 325–349 (cit. on p. 942).

Alessandro Campa, Thierry Dauxois, and Stefano Ruffo (2009). "Statistical mechanics and dynamics of solvable models with long-range interactions". *Phys. Rep.* 480.3-6, pp. 57–159. MR: 2569625 (cit. on p. 942).

José Antonio Carrillo, Young-Pil Choi, and Maxime Hauray (2014). "The derivation of swarming models: mean-field limit and Wasserstein distances". In: *Collective dynamics from bacteria to crowds*. Vol. 553. CISM Courses and Lect. Springer, Vienna, pp. 1–46. MR: 3331178 (cit. on p. 949).

Emmanuel Cépa and Dominique Lépingle (1997). "Diffusing particles with electrostatic repulsion". *Probab. Theory Related Fields* 107.4, pp. 429–449. MR: 1440140 (cit. on p. 950).

Djalil Chafai, Adrien Hardy, and Myléne Maida (2018). "Concentration for Coulomb gases and Coulomb transport inequalities". *J. Funct. Anal.* 275, pp. 1447–1483. arXiv: 1610.00980 (cit. on p. 954).

Djalil Chafaï, Nathael Gozlan, and Pierre-André Zitt (2014). "First-order global asymptotics for confined particles with singular pair repulsion". *Ann. Appl. Probab.* 24.6, pp. 2371–2413. MR: 3262506 (cit. on p. 951).

S. J. Chapman, J. Rubinstein, and M. Schatzman (1996). "A mean-field model of superconducting vortices". *European J. Appl. Math.* 7.2, pp. 97–111. MR: 1388106 (cit. on p. 948).

Patrick Chiu (1997). "Height of flat tori". *Proc. Amer. Math. Soc.* 125.3, pp. 723–730. MR: 1396970 (cit. on p. 955).

Ph. Choquard and J. Clerouin (1983). "Cooperative phenomena below melting of the one-component two-dimensional plasma". *Physical review letters* 50.26, p. 2086 (cit. on p. 942).

Gustave Choquet (1958). "Diamètre transfini et comparaison de diverses capacités". *Séminaire Brelot–Choquet–Deny. Théorie du potentiel* 3.4, pp. 1–7 (cit. on p. 947).

Henry Cohn (2017). "A conceptual breakthrough in sphere packing". *Notices Amer. Math. Soc.* 64.2, pp. 102–115. MR: 3587715 (cit. on p. 941).

Henry Cohn and Abhinav Kumar (2007). "Universally optimal distribution of points on spheres". *J. Amer. Math. Soc.* 20.1, pp. 99–148. MR: 2257398 (cit. on p. 955).

Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, and Maryna S. Viazovska (2017). "The sphere packing problem in dimension 24". *Ann. of Math. (2)* 185.3, pp. 1017–1033. MR: 3664817 (cit. on pp. 941, 956).

J. E. Colliander and R. L. Jerrard (1998). "Vortex dynamics for the Ginzburg–Landau–Schrödinger equation". *Internat. Math. Res. Notices* 7, pp. 333–358. MR: 1623410 (cit. on p. 939).

– (1999). "Ginzburg–Landau vortices: weak stability and Schrödinger equation dynamics". *J. Anal. Math.* 77, pp. 129–205. MR: 1753485 (cit. on p. 939).

J. H. Conway and N. J. A. Sloane (1999). *Sphere packings, lattices and groups*. Third. Vol. 290. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. With additional contributions by E. Bannai, R. E.

Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Springer-Verlag, New York, pp. lxxiv+703. MR: 1662447 (cit. on p. 941).

Renaud Coulangeon and Achill Schürmann (2012). "Energy minimization, periodic sets and spherical designs". *Int. Math. Res. Not. IMRN* 4, pp. 829–848. MR: 2889159 (cit. on p. 955).

C. A. Coulomb (1785). "Premier Mémoire sur l'Electricité et le Magnétisme". In: pp. 569–577 (cit. on p. 936).

Thierry Dauxois, Stefano Ruffo, Ennio Arimondo, and Martin Wilkens (2002). "Dynamics and thermodynamics of systems with long-range interactions: an introduction". In: *Dynamics and thermodynamics of systems with long-range interactions (Les Houches, 2002)*. Vol. 602. Lecture Notes in Phys. Springer, Berlin, pp. 1–19. MR: 2008176.

David S. Dean, Pierre Le Doussal, Satya N. Majumdar, and Grégory Schehr (2016). "Noninteracting fermions at finite temperature in a d-dimensional trap: Universal correlations". *Physical Review A* 94.6, p. 063622 (cit. on p. 942).

P. A. Deift (1999). *Orthogonal polynomials and random matrices: a Riemann–Hilbert approach*. Vol. 3. Courant Lecture Notes in Mathematics. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, pp. viii+273. MR: 1677884 (cit. on p. 943).

Jean-Marc Delort (1991). "Existence de nappes de tourbillon en dimension deux". *J. Amer. Math. Soc.* 4.3, pp. 553–586. MR: 1102579 (cit. on p. 947).

Amir Dembo and Ofer Zeitouni (2010). *Large deviations techniques and applications*. Vol. 38. Stochastic Modelling and Applied Probability. Corrected reprint of the second (1998) edition. Springer-Verlag, Berlin, pp. xvi+396. MR: 2571413 (cit. on p. 950).

Cl. Deutsch and M. Lavaud (1974). "Equilibrium properties of a two-dimensional Coulomb gas". *Physical Review A* 9.6, p. 2598 (cit. on p. 943).

Mitia Duerinckx (2016). "Mean-field limits for some Riesz interaction gradient flows". *SIAM J. Math. Anal.* 48.3, pp. 2269–2300. MR: 3514713 (cit. on p. 949).

Ioana Dumitriu and Alan Edelman (2002). "Matrix models for beta ensembles". *J. Math. Phys.* 43.11, pp. 5830–5847. MR: 1936554 (cit. on p. 943).

Freeman J. Dyson (1962). "Statistical theory of the energy levels of complex systems. III". *J. Mathematical Phys.* 3, pp. 166–175. MR: 0143558 (cit. on p. 943).

Weinan E (1994a). "Dynamics of vortices in Ginzburg–Landau theories with applications to superconductivity". *Phys. D* 77.4, pp. 383–404. MR: 1297726 (cit. on p. 939).

– (1994b). "Dynamics of vortices in Ginzburg–Landau theories with applications to superconductivity". *Phys. D* 77.4, pp. 383–404. MR: 1297726 (cit. on p. 948).

László Erdős and Horng-Tzer Yau (2017). *A dynamical approach to random matrix theory*. Vol. 28. Courant Lecture Notes in Mathematics. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, pp. ix+226. MR: 3699468 (cit. on p. 944).

Gregory L. Eyink and Katepalli R. Sreenivasan (2006). "Onsager and the theory of hydrodynamic turbulence". *Rev. Modern Phys.* 78.1, pp. 87–135. MR: 2214822 (cit. on p. 940).

L. Fejes (1940). "Über einen geometrischen Satz". *Math. Z.* 46, pp. 83–85. MR: 0001587 (cit. on p. 941).

Hans Föllmer (1988). "Random fields and diffusion processes". In: *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*. Vol. 1362. Lecture Notes in Math. Springer, Berlin, pp. 101–203. MR: 983373 (cit. on p. 959).

Hans Föllmer and Steven Orey (1988). "Large deviations for the empirical field of a Gibbs measure". *Ann. Probab.* 16.3, pp. 961–977. MR: 942749 (cit. on p. 959).

P. J. Forrester (1993). "Exact integral formulas and asymptotics for the correlations in the $1/r^2$ quantum many-body system". *Phys. Lett. A* 179.2, pp. 127–130. MR: 1230810 (cit. on p. 960).

– (2010). *Log-gases and random matrices*. Vol. 34. London Mathematical Society Monographs Series. Princeton University Press, Princeton, NJ, pp. xiv+791. MR: 2641363 (cit. on pp. 937, 942, 943).

P. J. Forrester, N. E. Frankel, T. M. Garoni, and N. S. Witte (2003). "Finite one-dimensional impenetrable Bose systems: Occupation numbers". *Physical Review A* 67.4, p. 043607 (cit. on p. 942).

P. J. Forrester, B. Jancovici, and D. S. McAnally (2001). "Analytic properties of the structure function for the one-dimensional one-component log-gas". In: *Proceedings of the Baxter Revolution in Mathematical Physics (Canberra, 2000)*. Vol. 102. 3-4, pp. 737–780. MR: 1832069 (cit. on p. 942).

Nicolas Fournier, Maxime Hauray, and Stéphane Mischler (2014). "Propagation of chaos for the 2D viscous vortex model". *J. Eur. Math. Soc. (JEMS)* 16.7, pp. 1423–1466. MR: 3254330 (cit. on p. 950).

Jürg Fröhlich (1976). "Classical and quantum statistical mechanics in one and two dimensions: two-component Yukawa- and Coulomb systems". *Comm. Math. Phys.* 47.3, pp. 233–268. MR: 0434278 (cit. on p. 943).

Jürg Fröhlich and Thomas Spencer (1981). "The Kosterlitz–Thouless transition in two-dimensional abelian spin systems and the Coulomb gas". *Comm. Math. Phys.* 81.4, pp. 527–602. MR: 634447 (cit. on p. 943).

O. Frostman (1935). "Potential d'équilibre et capacité des avec quelques applications à la thórie des functions". *Meddelanden Mat. Sem. Univ. Lund* 3, p. 115 (cit. on p. 945).

David García-Zelada (Mar. 2017). "A large deviation principle for empirical measures on Polish spaces: Application to singular Gibbs measures on manifolds". arXiv: 1703.02680 (cit. on p. 951).

Hans-Otto Georgii (1993). "Large deviations and maximum entropy principle for interacting random fields on $\mathbf{Z}^d$". *Ann. Probab.* 21.4, pp. 1845–1875. MR: 1245292 (cit. on p. 959).

Hans-Otto Georgii and Hans Zessin (1993). "Large deviations and the maximum entropy principle for marked point random fields". *Probab. Theory Related Fields* 96.2, pp. 177–204. MR: 1227031 (cit. on p. 959).

Subhro Ghosh and Joel L. Lebowitz (2017). "Number rigidity in superhomogeneous random point fields". *J. Stat. Phys.* 166.3-4, pp. 1016–1027. MR: 3607597 (cit. on p. 952).

Jean Ginibre (1965). "Statistical ensembles of complex, quaternion, and real matrices". *J. Mathematical Phys.* 6, pp. 440–449. MR: 0173726 (cit. on pp. 943, 960).

M. D. Girardeau, E. M. Wright, and J. M. Triscari (2001). "Ground-state properties of a one-dimensional system of hard-core bosons in a harmonic trap". *Physical Review A* 63.3, p. 033601 (cit. on p. 942).

V. L. Girko (1983). "The circle law". *Teor. Veroyatnost. i Mat. Statist.* 28. English translation in Girko [1984], pp. 15–21. MR: 727271 (cit. on p. 969).

– (1984). "Circle law". *Theory Probab. Appl.* 29. Original in Russian in Girko [1983], pp. 694–706 (cit. on p. 969).

S. M. Girvin (1999). "The quantum Hall effect: novel excitations and broken symmetries". In: *Aspects topologiques de la physique en basse dimension/Topological aspects of low dimensional systems (Les Houches, 1998)*. EDP Sci., Les Ulis, pp. 53–175. MR: 1850417 (cit. on p. 941).

– (2004). "Introduction to the fractional quantum Hall effect". *Séminaire Poincaré* 2, pp. 54–74 (cit. on p. 941).

Dorian Goldman, Cyrill B. Muratov, and Sylvia Serfaty (2013). "The Γ-limit of the two-dimensional Ohta–Kawasaki energy. I. Droplet density". *Arch. Ration. Mech. Anal.* 210.2, pp. 581–613. MR: 3101793 (cit. on p. 957).

– (2014). "The Γ-limit of the two-dimensional Ohta–Kawasaki energy. Droplet arrangement via the renormalized energy". *Arch. Ration. Mech. Anal.* 212.2, pp. 445–501. MR: 3176350 (cit. on p. 957).

François Golse (2016). "On the dynamics of large particle systems in the mean field limit". In: *Macroscopic and large scale phenomena: coarse graining, mean field limits and ergodicity*. Vol. 3. Lect. Notes Appl. Math. Mech. Springer, [Cham], pp. 1–144. MR: 3468297 (cit. on pp. 945, 949).

Jonathan Goodman, Thomas Y. Hou, and John Lowengrub (1990). "Convergence of the point vortex method for the 2-D Euler equations". *Comm. Pure Appl. Math.* 43.3, pp. 415–430. MR: 1040146 (cit. on p. 948).

J. Gunson and L. S. Panta (1977). "Two-dimensional neutral Coulomb gas". *Comm. Math. Phys.* 52.3, pp. 295–304. MR: 0452403 (cit. on p. 943).

Thomas C. Hales (2005). "A proof of the Kepler conjecture". *Ann. of Math. (2)* 162.3, pp. 1065–1185. MR: 2179728 (cit. on p. 941).

Douglas P. Hardin, Thomas Leblé, Edward B. Saff, and Sylvia Serfaty (2018). "Large deviation principles for hypersingular Riesz gases". *Constr. Approx.* 48 (1), pp. 61–100 (cit. on p. 962).

Maxime Hauray (2009). "Wasserstein distances for vortices approximation of Euler-type equations". *Math. Models Methods Appl. Sci.* 19.8, pp. 1357–1384. MR: 2555474 (cit. on p. 949).

Maxime Hauray and Pierre-Emmanuel Jabin (2015). "Particle approximation of Vlasov equations with singular forces: propagation of chaos". *Ann. Sci. Éc. Norm. Supér. (4)* 48.4, pp. 891–940. MR: 3377068 (cit. on p. 949).

Maxime Hauray and Stéphane Mischler (2014). "On Kac's chaos and related problems". *J. Funct. Anal.* 266.10, pp. 6055–6157. MR: 3188710 (cit. on p. 945).

J. Ben Hough, Manjunath Krishnapur, Yuval Peres, and Bálint Virág (2009). *Zeros of Gaussian analytic functions and determinantal point processes*. Vol. 51. University Lecture Series. American Mathematical Society, Providence, RI, pp. x+154. MR: 2552864 (cit. on p. 944).

Jiaoyang Huang and Benjamin Landon (Dec. 19, 2016). "Local law and mesoscopic fluctuations of Dyson Brownian motion for general $\beta$ and potential". arXiv: 1612.06306.

Kerson Huang (1987). *Statistical mechanics*. Second. John Wiley & Sons, Inc., New York, pp. xiv+493. MR: 1042093 (cit. on p. 942).

John Z. Imbrie (1982/83). "Debye screening for jellium and other Coulomb systems". *Comm. Math. Phys.* 87.4, pp. 515–565. MR: 691043 (cit. on p. 960).

Pierre-Emmanuel Jabin (2014). "A review of the mean field limits for Vlasov equations". *Kinet. Relat. Models* 7.4, pp. 661–711. MR: 3317577 (cit. on pp. 940, 949, 950).

Pierre-Emmanuel Jabin and Zhenfu Wang (2016). "Mean field limit and propagation of chaos for Vlasov systems with bounded forces". *J. Funct. Anal.* 271.12, pp. 3588–3627. MR: 3558251 (cit. on pp. 949, 950).

– (June 2017). "Quantitative estimate of propagation of chaos for stochastic systems with $W^{-1,\infty}$ kernels". arXiv: 1706.09564 (cit. on pp. 949, 950).

B. Jancovici (1995). "Classical Coulomb systems: screening and correlations revisited". *J. Statist. Phys.* 80.1-2, pp. 445–459. MR: 1340566 (cit. on p. 941).

B. Jancovici, Joel L. Lebowitz, and G. Manificat (1993). "Large charge fluctuations in classical Coulomb systems". *J. Statist. Phys.* 72.3-4, pp. 773–787. MR: 1239571 (cit. on p. 941).

Robert L. Jerrard and Daniel Spirn (2015). "Hydrodynamic limit of the Gross–Pitaevskii equation". *Comm. Partial Differential Equations* 40.2, pp. 135–190. MR: 3277924 (cit. on p. 949).

Robert Leon Jerrard and Halil Mete Soner (1998). "Dynamics of Ginzburg–Landau vortices". *Arch. Rational Mech. Anal.* 142.2, pp. 99–125. MR: 1629646 (cit. on p. 939).

Kurt Johansson (1998). "On fluctuations of eigenvalues of random Hermitian matrices". *Duke Math. J.* 91.1, pp. 151–204. MR: 1487983 (cit. on pp. 958, 962).

Michael K.-H. Kiessling (1993). "Statistical mechanics of classical particles with logarithmic interactions". *Comm. Pure Appl. Math.* 46.1, pp. 27–56. MR: 1193342 (cit. on pp. 941, 951).

– (2014). "The microscopic foundations of Vlasov theory for jellium-like Newtonian $N$-body systems". *J. Stat. Phys.* 155.6, pp. 1299–1328. MR: 3207739 (cit. on p. 949).

Michael K.-H. Kiessling and Herbert Spohn (1999). "A note on the eigenvalue density of random matrices". *Comm. Math. Phys.* 199.3, pp. 683–695. MR: 1669669 (cit. on p. 941).

Rowan Killip and Irina Nenciu (2004). "Matrix models for circular ensembles". *Int. Math. Res. Not.* 50, pp. 2665–2701. MR: 2127367 (cit. on p. 943).

Rowan Killip and Mihai Stoiciu (2009). "Eigenvalue statistics for CMV matrices: from Poisson to clock via random matrix ensembles". *Duke Math. J.* 146.3, pp. 361–399. MR: 2484278.

Semyon Klevtsov (2016). "Geometry and large $N$ limits in Laughlin states". In: *Travaux mathématiques. Vol. XXIV*. Vol. 24. Trav. Math. Fac. Sci. Technol. Commun. Univ. Luxemb., Luxembourg, pp. 63–127. MR: 3643934 (cit. on p. 944).

H. Kunz (1974). "The one-dimensional classical electron gas". *Ann. Physics* 85, pp. 303–335. MR: 0426742 (cit. on p. 936).

Matthias Kurzke, Christof Melcher, Roger Moser, and Daniel Spirn (2009). "Dynamics for Ginzburg–Landau vortices under a mixed flow". *Indiana Univ. Math. J.* 58.6, pp. 2597–2621. MR: 2603761.

Matthias Kurzke and Daniel Spirn (2014). "Vortex liquids and the Ginzburg–Landau equation". *Forum Math. Sigma* 2, e11, 63. MR: 3264248 (cit. on pp. 949, 960).

L. D. Landau and V. L. Ginzburg (1965). *On the theory of superconductivity,[in:] Collected papers of LD Landau, D. ter Haar [Ed.], 546–568* (cit. on p. 938).

Robert B. Laughlin (1983). "Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations". *Physical Review Letters* 50.18, p. 1395 (cit. on p. 941).

– (1999). "Nobel lecture: fractional quantization". *Rev. Modern Phys.* 71.4, pp. 863–874. MR: 1712324 (cit. on p. 941).

Dustin Lazarovici (2016). "The Vlasov–Poisson dynamics as the mean field limit of extended charges". *Comm. Math. Phys.* 347.1, pp. 271–289. MR: 3543184 (cit. on p. 949).

Dustin Lazarovici and Peter Pickl (2017). "A mean field limit for the Vlasov–Poisson system". *Arch. Ration. Mech. Anal.* 225.3, pp. 1201–1231. MR: 3667287 (cit. on p. 949).

T. Leblé and Sylvia Serfaty (2018). "Fluctuations of Two-Dimensional Coulomb Gases". *Geom. Funct. Anal. (GAFA)* 28 (2), pp. 443–508. arXiv: 1609.08088 (cit. on pp. 961, 962).

Thomas Leblé (2016). "Logarithmic, Coulomb and Riesz energy of point processes". *J. Stat. Phys.* 162.4, pp. 887–923. MR: 3456982 (cit. on pp. 955, 960).

Thomas Leblé and Sylvia Serfaty (2017). "Large deviation principle for empirical fields of Log and Riesz gases". *Invent. Math.* 210.3, pp. 645–757. MR: 3735628 (cit. on pp. 952, 960, 962).

Thomas Leblé, Sylvia Serfaty, and Ofer Zeitouni (2017). "Large deviations for the two-dimensional two-component plasma". *Comm. Math. Phys.* 350.1, pp. 301–360. MR: 3606477 (cit. on p. 961).

A. Lenard (1961). "Exact statistical mechanics of a one-dimensional system with Coulomb forces". *J. Mathematical Phys.* 2, pp. 682–693. MR: 0129874 (cit. on p. 936).

Andrew Lenard (1963). "Exact statistical mechanics of a one-dimensional system with Coulomb forces. III. Statistics of the electric field". *J. Mathematical Phys.* 4, pp. 533–543. MR: 0148411 (cit. on p. 936).

Elliott H. Lieb and Joel L. Lebowitz (1972). "The constitution of matter: Existence of thermodynamics for systems composed of electrons and nuclei". *Advances in Math.* 9, pp. 316–398. MR: 0339751 (cit. on p. 941).

Elliott H. Lieb and Michael Loss (1997). *Analysis*. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. xviii+278. MR: 1415616.

Elliott H. Lieb and Heide Narnhofer (1975). "The thermodynamic limit for jellium". *J. Statist. Phys.* 12, pp. 291–310. MR: 0401029 (cit. on p. 941).

Elliott H. Lieb, Nicolas Rougerie, and Jakob Yngvason (June 2018). "Local incompressibility estimates for the Laughlin phase". *Communications in Mathematical Physics*. arXiv: 1701.09064 (cit. on p. 941).

F.-H. Lin and J. X. Xin (1999a). "On the dynamical law of the Ginzburg–Landau vortices on the plane". *Comm. Pure Appl. Math.* 52.10, pp. 1189–1212. MR: 1699965 (cit. on p. 939).

– (1999b). "On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation". *Comm. Math. Phys.* 200.2, pp. 249–274. MR: 1674000 (cit. on p. 939).

Fang Hua Lin (1996). "Some dynamical properties of Ginzburg–Landau vortices". *Comm. Pure Appl. Math.* 49.4, pp. 323–359. MR: 1376654 (cit. on p. 939).

Fanghua Lin and Ping Zhang (2000). "On the hydrodynamic limit of Ginzburg–Landau vortices". *Discrete Contin. Dynam. Systems* 6.1, pp. 121–142. MR: 1739596 (cit. on p. 948).

Mylène Maïda and Édouard Maurel-Segala (2014). "Free transport-entropy inequalities for non-convex potentials and application to concentration for random matrices". *Probab. Theory Related Fields* 159.1-2, pp. 329–356. MR: 3201924 (cit. on p. 954).

Carlo Marchioro and Mario Pulvirenti (1994). *Mathematical theory of incompressible nonviscous fluids*. Vol. 96. Applied Mathematical Sciences. Springer-Verlag, New York, pp. xii+283. MR: 1245492 (cit. on p. 940).

Martial Mazars (2011). "Long ranged interactions in computer simulations and for quasi-2d systems". *Physics Reports* 500.2-3, pp. 43–116 (cit. on p. 942).

H. P. McKean Jr. (1967). "Propagation of chaos for a class of non-linear parabolic equations". In: *Stochastic Differential Equations (Lecture Series in Differential Equations, Session 7, Catholic Univ., 1967)*. Air Force Office Sci. Res., Arlington, Va., pp. 41–57. MR: 0233437 (cit. on p. 950).

Madan Lal Mehta (2004). *Random matrices*. Third. Vol. 142. Pure and Applied Mathematics (Amsterdam). Elsevier/Academic Press, Amsterdam, pp. xviii+688. MR: 2129906 (cit. on pp. 937, 943, 946).

Joachim Messer and Herbert Spohn (1982). "Statistical mechanics of the isothermal Lane–Emden equation". *J. Statist. Phys.* 29.3, pp. 561–578. MR: 704588 (cit. on p. 951).

Hugh L. Montgomery (1988). "Minimal theta functions". *Glasgow Math. J.* 30.1, pp. 75–85. MR: 925561 (cit. on p. 955).

Simona Rota Nodari and Sylvia Serfaty (2015). "Renormalized energy equidistribution and local charge balance in 2D Coulomb systems". *Int. Math. Res. Not. IMRN* 11, pp. 3035–3093. MR: 3373044 (cit. on p. 957).

B. Osgood, R. Phillips, and P. Sarnak (1988). "Extremals of determinants of Laplacians". *J. Funct. Anal.* 80.1, pp. 148–211. MR: 960228 (cit. on p. 955).

Felix Otto (2001). "The geometry of dissipative evolution equations: the porous medium equation". *Comm. Partial Differential Equations* 26.1-2, pp. 101–174. MR: 1842429 (cit. on p. 948).

O. Penrose and E. R. Smith (1972). "Thermodynamic limit for classical systems with Coulomb interactions in a constant external field". *Comm. Math. Phys.* 26, pp. 53–77. MR: 0303866 (cit. on p. 941).

L. Peres and J. Rubinstein (1993). "Vortex dynamics in U(1) Ginzburg–Landau models". *Phys. D* 64.1-3, pp. 299–309. MR: 1214555 (cit. on p. 939).

Mircea Petrache and Simona Rota Nodari (2018). "Equidistribution of Jellium Energy for Coulomb and Riesz Interactions". *Constr. Approx.* 47.1, pp. 163–210. MR: 3742814 (cit. on p. 957).

Mircea Petrache and Sylvia Serfaty (2017). "Next order asymptotics and renormalized energy for Riesz interactions". *J. Inst. Math. Jussieu* 16.3, pp. 501–569. MR: 3646281 (cit. on pp. 952, 954, 956, 957, 962).

Dénes Petz and Fumio Hiai (1998). "Logarithmic energy as an entropy functional". In: *Advances in differential equations and mathematical physics (Atlanta, GA, 1997)*. Vol. 217. Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 205–221. MR: 1606719 (cit. on p. 951).

Charles Radin (1981). "The ground state for soft disks". *J. Statist. Phys.* 26.2, pp. 365–373. MR: 643714 (cit. on p. 956).

Firas Rassoul-Agha and Timo Seppäläinen (2015). *A course on large deviations with an introduction to Gibbs measures*. Vol. 162. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, pp. xiv+318. MR: 3309619 (cit. on p. 959).

Brian Rider and Bálint Virág (2007). "The noise in the circular law and the Gaussian free field". *Int. Math. Res. Not. IMRN* 2, Art. ID rnm006, 33. MR: 2361453 (cit. on p. 962).

Nicolas Rougerie (2014). "De Finetti theorems, mean-field limits and Bose-Einstein condensation". In: LMU lecture notes. arXiv: 1506.05263 (cit. on p. 951).

– (2016). *Théorèmes de De Finetti, limites de champ moyen et condensation de Bose-Einstein*. Paris: Les cours Peccot, Spartacus IHD (cit. on p. 951).

Nicolas Rougerie and Sylvia Serfaty (2016). "Higher-dimensional Coulomb gases and renormalized energy functionals". *Comm. Pure Appl. Math.* 69.3, pp. 519–605. MR: 3455593 (cit. on pp. 952, 954, 957).

Nicolas Rougerie, Sylvia Serfaty, and J. Yngvason (2014). "Quantum Hall phases and plasma analogy in rotating trapped Bose gases". *J. Stat. Phys.* 154.1-2, pp. 2–50. MR: 3162531 (cit. on p. 941).

Nicolas Rougerie and Jakob Yngvason (2015). "Incompressibility estimates for the Laughlin phase". *Comm. Math. Phys.* 336.3, pp. 1109–1140. MR: 3324139 (cit. on p. 941).

Edward B. Saff and A. B. J. Kuijlaars (1997). "Distributing many points on a sphere". *Math. Intelligencer* 19.1, pp. 5–11. MR: 1439152 (cit. on p. 941).

Edward B. Saff and Vilmos Totik (1997). *Logarithmic potentials with external fields*. Vol. 316. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Appendix B by Thomas Bloom. Springer-Verlag, Berlin, pp. xvi+505. MR: 1485778 (cit. on pp. 940, 947).

Etienne Sandier and Sylvia Serfaty (2007). *Vortices in the magnetic Ginzburg–Landau model*. Vol. 70. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, pp. xii+322. MR: 2279839 (cit. on pp. 939, 947).

Etienne Sandier and Sylvia Serfaty (2012). "From the Ginzburg–Landau model to vortex lattice problems". *Comm. Math. Phys.* 313.3, pp. 635–743. MR: 2945619 (cit. on pp. 939, 955, 957).
– (2015a). "1D log gases and the renormalized energy: crystallization at vanishing temperature". *Probab. Theory Related Fields* 162.3-4, pp. 795–846. MR: 3383343 (cit. on pp. 952, 954–956).
– (2015b). "2D Coulomb gases and the renormalized energy". *Ann. Probab.* 43.4, pp. 2026–2083. MR: 3353821 (cit. on pp. 952, 954, 957).
R. R. Sari and D. Merlini (1976). "On the $\nu$-dimensional one-component classical plasma: the thermodynamic limit problem revisited". *J. Statist. Phys.* 14.2, pp. 91–100. MR: 0449401 (cit. on p. 941).
Peter Sarnak and Andreas Strömbergsson (2006). "Minima of Epstein's zeta function and heights of flat tori". *Invent. Math.* 165.1, pp. 115–151. MR: 2221138 (cit. on p. 955).
David G. Schaeffer (1975). "A stability theorem for the obstacle problem". *Advances in Math.* 17.1, pp. 34–47. MR: 0380093.
Steven Schochet (1996). "The point-vortex method for periodic weak solutions of the 2-D Euler equations". *Comm. Pure Appl. Math.* 49.9, pp. 911–965. MR: 1399201 (cit. on p. 948).
Sylvia Serfaty (2007). "Vortex collisions and energy-dissipation rates in the Ginzburg–Landau heat flow. I. Study of the perturbed Ginzburg–Landau equation". *J. Eur. Math. Soc. (JEMS)* 9.2, pp. 177–217. MR: 2293954 (cit. on p. 939).
– (2011). "Gamma-convergence of gradient flows on Hilbert and metric spaces and applications". *Discrete Contin. Dyn. Syst.* 31.4, pp. 1427–1451. MR: 2836361 (cit. on p. 948).
– (2014). "Ginzburg–Landau vortices, Coulomb gases, and renormalized energies". *J. Stat. Phys.* 154.3, pp. 660–680. MR: 3163544 (cit. on p. 954).
– (2015). *Coulomb gases and Ginzburg–Landau vortices*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, pp. viii+157. MR: 3309890 (cit. on pp. 939, 946, 947, 951, 955, 957).
– (2017). "Mean field limits of the Gross–Pitaevskii and parabolic Ginzburg–Landau equations". *J. Amer. Math. Soc.* 30.3, pp. 713–768. MR: 3630086 (cit. on p. 949).
Sylvia Serfaty and Mitia Duerinckx (Mar. 2018). "Mean Field Limit for Coulomb Flows". arXiv: 1803.08345 (cit. on pp. 948, 949).
Sylvia Serfaty and Joaquim Serra (n.d.). "Quantitative stability of the free boundary in the obstacle problem". To appear in *Anal. and PDE*. arXiv: 1708.01490 (cit. on p. 962).
Sylvia Serfaty and Juan Luis Vázquez (2014). "A mean field equation as limit of nonlinear diffusions with fractional Laplacian operators". *Calc. Var. Partial Differential Equations* 49.3-4, pp. 1091–1120. MR: 3168624 (cit. on p. 948).
M. Shcherbina (2013). "Fluctuations of linear eigenvalue statistics of $\beta$ matrix models in the multi-cut regime". *J. Stat. Phys.* 151.6, pp. 1004–1034. MR: 3063494 (cit. on pp. 958, 961, 962).

Barry Simon (2008). "The Christoffel–Darboux kernel". In: *Perspectives in partial differential equations, harmonic analysis and applications*. Vol. 79. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, pp. 295–335. MR: 2500498 (cit. on p. 940).

Neil J. A. Sloane (1998). "The sphere packing problem". In: *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*. Extra Vol. III, pp. 387–396. MR: 1648172 (cit. on p. 941).

Didier Smets, Fabrice Bethuel, and Giandomenico Orlandi (2007). "Quantization and motion law for Ginzburg–Landau vortices". *Arch. Ration. Mech. Anal.* 183.2, pp. 315–370. MR: 2278409 (cit. on p. 939).

Thomas Spencer (1997). "Scaling, the free field and statistical mechanics". In: *The Legacy of Norbert Wiener: A Centennial Symposium (Cambridge, MA, 1994)*. Vol. 60. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, pp. 373–389. MR: 1460292 (cit. on p. 943).

Herbert Spohn (2004). *Dynamics of charged particles and their radiation field*. Cambridge University Press, Cambridge, pp. xvi+360. MR: 2097788 (cit. on p. 947).

S. M. Stishov (1998). "Does the phase transition exist in the one-component plasma model?" *JJour. Exp. Theor. Phys. Lett* 67.1, pp. 90–94 (cit. on p. 942).

Horst L. Stormer (1999). "Nobel lecture: the fractional quantum Hall effect". *Rev. Modern Phys.* 71.4, pp. 875–889. MR: 1712320 (cit. on p. 941).

Alain-Sol Sznitman (1991). "Topics in propagation of chaos". In: *École d'Été de Probabilités de Saint-Flour XIX—1989*. Vol. 1464. Lecture Notes in Math. Springer, Berlin, pp. 165–251. MR: 1108185 (cit. on p. 950).

Terence Tao and Van Vu (2010). "Random matrices: universality of ESDs and the circular law". *Ann. Probab.* 38.5. With an appendix by Manjunath Krishnapur, pp. 2023–2065. MR: 2722794.

– (2011). "Random matrices: universality of local eigenvalue statistics". *Acta Math.* 206.1, pp. 127–204. MR: 2784665 (cit. on p. 944).

Florian Theil (2006). "A proof of crystallization in two dimensions". *Comm. Math. Phys.* 262.1, pp. 209–236. MR: 2200888 (cit. on p. 956).

Salvatore Torquato (2016). "Hyperuniformity and its generalizations". *Phys. Rev. E* 94.2, p. 022122 (cit. on pp. 942, 952).

Benedek Valkó and Bálint Virág (2009). "Continuum limits of random matrices and the Brownian carousel". *Invent. Math.* 177.3, pp. 463–508. MR: 2534097 (cit. on p. 960).

Srinivasa R. S. Varadhan (1988). "Large deviations and applications". In: *École d'Été de Probabilités de Saint-Flour XV–XVII, 1985–87*. Vol. 1362. Lecture Notes in Math. Springer, Berlin, pp. 1–49. MR: 983371 (cit. on p. 959).

Maryna S. Viazovska (2017). "The sphere packing problem in dimension 8". *Ann. of Math. (2)* 185.3, pp. 991–1015. MR: 3664816 (cit. on p. 941).

Christian Webb (2016). "Linear statistics of the circular $\beta$-ensemble, Stein's method, and circular Dyson Brownian motion". *Electron. J. Probab.* 21, Paper No. 25, 16. MR: 3485367 (cit. on p. 962).

Eugene P. Wigner (1955). "Characteristic vectors of bordered matrices with infinite dimensions". *Ann. of Math. (2)* 62, pp. 548–564. MR: 0077805 (cit. on pp. 943, 946).

A. Zabrodin and P. Wiegmann (2006). "Large-$N$ expansion for the 2D Dyson gas". *J. Phys. A* 39.28, pp. 8933–8963. MR: 2240466 (cit. on p. 961).

Xuhuan Zhou and Weiliang Xiao (2017). "Well-posedness of a porous medium flow with fractional pressure in Sobolev spaces". *Electron. J. Differential Equations*, Paper No. 238, 7. MR: 3711191 (cit. on p. 948).

Received 2017-12-05.

Sylvia Serfaty
serfaty@cims.nyu.edu
Courant Institute of Mathematical Sciences
New York University
251 Mercer St.
New York, NY 10012