Homotopy of tmf at the Prime 2

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Abstract

In this report, the author computed the homotopy ring of topological modular forms (tmf) at the prime 2 using the Adams spectral sequence under the assumption that tmf is an E_∞ ring spectrum and an H_∞ ring spectrum with $H^*(tmf;\mathbb{F}_2) = A \otimes_{A(2)} \mathbb{F}_2$.

Contents

1 Introduction

The Bott periodicity theorem is equivalent to the 8-periodicity of homotopy of the spectrum bo, the connective real K-theory. By theorem 5.7, $\pi_*(bo)$ can give some informations (imJ) of $\pi_*(S^0)$, while the latter is difficult to compute. It is interesting to note that $H^*(bo;\mathbb{F}_2) = A \otimes_{A(1)} \mathbb{F}_2$, where A is mod (2) Steenrod algebra and $A(1)$ is its sub-algebra generated by $\{Sq^1, Sq^2\}$. It is nature to consider if there is something with cohomology $A \otimes_{A(2)} \mathbb{F}_2$ and whether its homotopy can give us more information about $\pi_*(S^0)$. Acturally, tmf, a spectrum constructed by Hopkins, Mahowald, and Miller exactly has this property, and $\pi_*(tmf)$ detects many (v_2-)periodic families in $\pi_*(S^0)$ indeed.

In this article, I will compute the homotopy of tmf by the Adams spectral sequence. In section 2, the May spectral sequence for tmf will be constructed and the structure of its E_1-page will be obtained. In section 3, I will compute differentials in the May spectral sequence by the direct computation of cobar complex, algebraic Steenrod operations, and Massey products. Having obtained the structure of E_∞-page of May spectral sequence, we will face the extension problems, which I will solve in section 4 by shuffling Massey products in Adams E_2-page. In secction 5, I will compute differentials in the Adams spectral sequence using algebraic Steenrod operations and a "zig-zag" of differentials in the Mahowald square. In section 6, I will solve the extension problems in the Adams spectral sequence by information in the sphere spectrum and the ring of classical modular forms and by shuffling Toda brackets, which are possible to construct since tmf is a ring spectrum. Then the homotopy ring of tmf is obtained.

In this article, all spectra are implicitly localized at the prime 2, and we use $\mathbb{Z}_2$ to denote 2-adic integers.

Acknowledgments. I do not claim originality of any of the results of this paper. The computation was first done by Hopkins and Mahowald and the result has been published in various places.

I would like to thank Professor Houhong Fan, who leads me to get started in the field of algebraic topology and teaches me a lot. I am also very grateful to Zhouli Xu, who helps me to learn many knowledge about Adams spectral sequence and May spectral sequence, and suggests me to have a try to compute the homotopy of tmf.

2 The Construction of the May Spectral Sequence

To compute the homotopy of tmf, we will use the Adams spectral sequence which converges to the homotopy.

Theorem 2.1 (Adams spectral sequence)([Ada58]). *Let X be a connective CW spectrum of finite type. Then there is a spectral sequence $E_r^{s,t}$ with differentials $d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}$ such that*
$\cdot E_2^{s,t} = Ext_A^{s,t}(H^*(X), \mathbb{F}_2)$ where A is the mod(2) Steenrod algebra;
$\cdot E_\infty^{s,t}$ is the group associated with a certain filtration of $\pi_{t-s}(X) \otimes \mathbb{Z}_2$. □

There are three problems to be solved in the process of computing homotopy by Adams spectral sequence: the E_2-page, the differentials, and the extensions. For the computations of E_2-page, we will use the May spectral sequence. Before constructing the spectral sequence, we will first introduce some definitions about Hopf algebroids and some relative algebraic structures as shown in [Rav04].

Definition 2.2 (Hopf algerboid). *A* Hopf algebroid (A, Γ) *over a commutative ring K is a cogroupoid object in the category of commutative K-algebras. The structure maps are denoted as*

$$\begin{aligned} \textit{source} \quad & \eta_L : A \to \Gamma \\ \textit{target} \quad & \eta_R : A \to \Gamma \\ \textit{coproduct} \quad & \Delta : \Gamma \to \Gamma \otimes_A \Gamma \\ \textit{counit} \quad & \epsilon : \Gamma \to A \\ \textit{conjugation} \quad & c : \Gamma \to \Gamma \end{aligned}$$

A graded Hopf algebroid is said to be connected *if the left and right sub-A-modules generated by Γ_0 are both isomorphic to A.*

It can be easily seen that to give a Hopf algebroid (A, Γ) with $\eta_L = \eta_R$ is equivalent to give a commutative Hopf algebra Γ over A.

The following definition is an analogy of modules over an algebra.

Definition 2.3 (Comodule). *A left Γ-comodule M is a left A-module M together with a left A-linear map $\psi : M \to \Gamma \otimes_A M$ which is counitary and coassociative. A* right Γ-comodule *is similarly defined.*

To introduce more homological algebra in the category of Γ-comodules, we need the following theorem.

Theorem 2.4 ([HS71]). *Suppose Γ is flat as an A-module, then the category of left Γ-comodules is abelian.* □

In this article, we only consider the case that A is the field $\mathbb{F}_2$, when Γ is always flat over A, hence we may assume from now on that Γ is flat over A.

Definition 2.5 (Cotensor). *Let M be a right Γ-comodule, N be a left Γ-comodule. Their* cotensor product over Γ *is a K-module $M \square_\Gamma N$ defined by the exact sequence*

$$0 \to M \square_\Gamma N \to M \otimes_A N \xrightarrow{\psi_M \otimes N - M \otimes \psi_N} M \otimes_A \Gamma \otimes_A N$$

In fact, a left comodule M can be seen as a right comodule with the structure map

$$M \xrightarrow{\psi} \Gamma \otimes M \xrightarrow{T} M \otimes \Gamma \xrightarrow{M\otimes c} M \otimes \Gamma$$

where T interchanges two factors. Then it can be directly deduced by the definition that

Proposition 2.6. $M\Box_\Gamma N = N\Box_\Gamma M$ □

The relationship between Hom and cotensor is shown by the following theorem

Theorem 2.7 ([Rav04])**.** *Let M and N be left Γ-comodules, M projective over A, then $Hom_A(M, A)$ is a right Γ-comodule, and*

$$Hom_\Gamma(M, N) = Hom_A(M, A)\Box_\Gamma N$$

especially, $Hom_\Gamma(A, N) = A\Box_\Gamma N$. □

By analogy with Ext and Tor in the category of modules, Ext and $Cotor$ can be defined in the category of comodules as follows

Definition 2.8 (Ext and Cotor)**.** *For left Γ-comodules M, $Ext^i_\Gamma(M, -)$ is the i^{th} right derived functor of $Hom_\Gamma(M, -)$; for right Γ-comodules M, $Cotor^i_\Gamma(M, -)$ is the i^{th} right derived functor of $M\Box_\Gamma-$.*

The following lemma is a corollary of change-of-rings theorem, which will be of great help in the computation later.

Proposition 2.9 ([Rav04])**.** *Let K be a field and $f : (K, \Gamma) \to (K, \Sigma)$ is a surjective map of Hopf algebras. Then for any left Σ-comodule N,*

$$Ext_\Gamma(K, \Gamma\Box_\Sigma N) = Ext_\Sigma(K, N)$$

where the right Σ-comodule structure is induced by f. □

Let's turn to the case of tmf. We admit that $H^*(tmf) = A \otimes_{A(2)} \mathbb{F}_2$, where A is the $mod(2)$ Steenrod algebra and $A(n)$ is its sub-algebra generated by $\{Sq^1, Sq^2, \cdots, Sq^{2^n}\}$. The following theorem gives the Hopf algebra structure of A.

Theorem 2.10 ([Mil58])**.** *A is a graded associative, cocommutative, coassociative Hopf algebra with coproduct*

$$\Delta(Sq^k) = \sum_{i=0}^{k} Sq^i \otimes Sq^{k-i}$$

□

Notice that A is not commutative but is cocommutative, its dual A_* is commutative, hence can be seen as a Hopf algebroid over $\mathbb{F}_2$. Besides, as we will show, A_* has a clearer structure than A. Therefore, we are more interested in A_* and $A(n)_*$, the dual of $A(n)$.

The following discussion on the structure of A_* is shown in [Mil58].

Let $\xi_k \in A_*$ be the dual of $Sq^{I^k} \in A$, where I^k is the admissible sequence $(2^{k-1}, 2^{k-2}, \cdots, 1, 0, \cdots)$. Let $\mathcal{R}$ be the set of all infinite sequences of non-negative integers with finitely many non-zero entries. Let $\mathcal{J}$ be the subset of $\mathcal{R}$ consisting of all admissible sequences with lexicographical order from the right. Let ξ^R denote the element $\Pi_i\xi_i^{r_i}$ for each $R = (r_1, r_2, \cdots) \in \mathcal{R}$. Then there is a set isomorphism

$$\gamma : \mathcal{J} \to \mathcal{R}; \quad (a_1, a_2, \cdots, a_k, 0, \cdots) \mapsto (a_1 - 2a_2, a_2 - 2a_3, \cdots, a_k, 0, \cdots)$$

Theorem 2.11 ([Mil58]). *For $I, J \in \mathcal{J}$, if $I < J$, then $\langle \xi^{\gamma(J)}, Sq^I \rangle = 0$; if $I = J$, then $\langle \xi^{\gamma(J)}, Sq^I \rangle = 1$*

Proof. Let $J = (a_1, a_2, \cdots, a_k, 0, \cdots)$ and $I = (b_1, b_2, \cdots, b_k, 0, \cdots)$, where $a_k \geq 1$ and $a_k \geq b_k \geq 0$. Let $J' = J - I^k$, then $\xi^{\gamma(J)} = \xi^{\gamma(J')}\xi_k$. Therefore, we have

$$\begin{aligned} \langle \xi^{\gamma(J)}, Sq^I \rangle &= \langle \xi^{\gamma(J')}\xi_k, Sq^I \rangle \\ &= \langle \xi^{\gamma(J')} \otimes \xi_k, \Delta(Sq^I) \rangle \\ &= \sum_{I_1 + I_2 = I} \langle \xi^{\gamma(J')}, Sq^{I_1} \rangle \langle \xi_k, Sq^{I_2} \rangle \end{aligned}$$

Notice that $\langle \xi_k, Sq^{I_2} \rangle \neq 0$ if and only if Sq^{I_2} has a summand of Sq^{I^k}. However, consider Adem relations

$$Sq^a Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j \quad (0 < a < 2b)$$

It can be seen that if $a + b = 3j$ to make $Sq^{2j}Sq^j$ appear on the right hand side, $a - 2j$ must be negative because $a < 2b$. As a result, $Sq^{2j}Sq^j$ can not appear in the Adem relations, hence Sq^{I_2} has a summand of Sq^{I^k} exactly when $I_2 = I^k$.

If $b_k = 0$, then $I_2 \neq I^k$, and $\langle \xi^{\gamma(J)}, Sq^I \rangle = 0$. If $b_k \neq 0$, then $\langle \xi^{\gamma(J)}, Sq^I \rangle = \langle \xi^{\gamma(J - I^k)}, Sq^{I - I^k} \rangle$. By downward induction, we only need to consider the case that $I = (0, \cdots)$, when the proposition follows directly. □

Corollary 2.12 ([Mil58]). $A_* = \mathbb{F}_2[\xi_i : i = 1, 2, \cdots]$

Proof. By the previous theorem, $\{\xi^{\gamma(J)} : J \in \mathcal{J}\}$ forms a vector space basis of A_*, while $\{\xi^{\gamma(J)}\}$ runs over all the monomials generated by $\{\xi_i\}$, hence the result follows. □

The following theorem illustrates the coproduct structure of A_*.

Theorem 2.13 ([Mil58]).

$$\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$$

□

Notice that all the monomials of $\{\xi_i\}$ form a vector space basis of A_*, we can consider its dual basis in A, which is called *Milnor basis*. Denote the dual of $\xi_1^{r_1}\xi_2^{r_2}\cdots$ by $Sq(r_1, r_2, \cdots)$. There are some propositions about the Milnor bases.

Proposition 2.14. $Sq(i, 0, \cdots) = Sq^i$

Proof. Notice that $I_i = (i, 0, \cdots)$ is the minimal admissible sequence I such that Sq^I has degree i, by Theorem 2.11, $\langle x, Sq^{I_i} \rangle \neq 0$ if and only if $x = \xi^{\gamma^{-1}(I_i)} = \xi_1^i$, hence Sq^i is the Milnor basis dual to ξ_1^i. □

The product of two Milnor bases can be computed as follows.

Proposition 2.15 ([Mar83])**.**

$$Sq(r_1, r_2, \cdots)Sq(s_1, s_2, \cdots) = \sum_X \beta(X) Sq(t_1, t_2, \cdots) \tag{2.15.1}$$

where X is a matrix with non-negative integer entries

$$X = \left\| \begin{matrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & \cdots & \\ x_{20} & \vdots & & \\ \vdots & & & \end{matrix} \right\|$$

with $\sum_i x_{ij} = s_j$ and $\sum_j 2^j x_{ij} = r_i$;

$$\beta(X) = \prod_k (x_{k0}, x_{k-1,1}, \cdots, x_{0k}) \in \mathbb{Z}/2$$

where

$$(n_1, \cdots, n_k) = \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!} = \binom{n_1 + \cdots + n_k}{n_1} \binom{n_2 + \cdots + n_k}{n_2} \cdots \binom{n_k}{n_k}$$

and

$$t_k = \sum_{i+j=k} x_{ij}$$

□

If we denote $Sq(r_1, r_2, \cdots)$ with $r_i = 0$ for all i except that $r_t = 2^s$ by P_t^s, there is a direct corollary.

Corollary 2.16. *P_t^s is a summand of the right hand side of (2.15.1) if and only if the first factor of the left hand side is P_k^{s+t-k} and the second factor is P_{t-k}^s.*

Proof. In order to get the summand P_t^s, x_{ij} must be zero except the ones with $i + j = t$. To let $\beta(X) \neq 0$, there can only be one non-zero entry in X. Suppose that $x_{k,t-k} = 2^s$, then the only nonzero r_i is $r_k = 2^{s+t-k}$, and the only nonzero s_j is $s_{t-k} = 2^s$, which coincides with the corollary. □

Then we can obtain the structure of $A(n)_*$.

Theorem 2.17. $A(n)_* = \mathbb{F}_2[\xi_i : i = 1, 2, \cdots, n+1]/(\xi_i^{2^{n+2-i}})$

To prove this, we only need to prove that $A(n)$ has a vector space basis $\{Sq(r_1, r_2, \cdots) : r_i < 2^{n+2-i}\}$. The following theorem will help on this.

Theorem 2.18 ([AM74])**.** *The sub Hopf algebra of A generated by $\{P_t^s : s < h(t)\}$ as an algebra is spanned by $\{Sq(r_1, r_2, \cdots) : r_t < 2^{h(t)}\}$ as a vector space, where h is a function from $\{1, 2, \cdots\}$ to $\{0, 1, \cdots, \infty\}$.* □

We can let $h(t) = max\{n + 2 - t, 0\}$ in this theorem, then it suffices to show that $A(n)$ is generated by $\{P_t^s : s + t \leq n + 1\}$. By proposition 2.14, $A(n)$ is generated by $\{P_1^i : 0 \leq i \leq n\}$, hence the only remaining thing is that $\{P_t^s : s + t \leq n + 1\} \subset A(n)$.

proof of theorem 2.17. We will prove this by induction on n. For the case $n = 0$, $\{P_t^s : s + t \leq n + 1\} = \{P_1^0\}$ is a subset of $A(0)$ indeed. Suppose that $A(n-1)$ is spanned by $\{Sq(r_1, r_2, \cdots) : r_t < 2^{n+1-t}\}$, we will show that all the elements $P_t^{n+1-t} (t \geq 2)$ are decomposable in the terms of P_1^n and terms in $A(n-1)$.

By corollary 2.16, to obtain P_t^{n+1-t} as a summand, we need a product $P_k^{n+1-k} P_{t-k}^{n+1-t}$. The second factor is contained in $A(n-1)$ by induction hypothesis, while the only known case about the first factor is $P_1^n \in A(n)$. So consider $P_1^n P_{t-1}^{n+1-t}$, then we get the following equations of the matrix entries:

$$\begin{cases} x_{10} + 2^{t-1} x_{1,t-1} = 2^n \\ x_{0,t-1} + x_{1,t-1} = 2^{n+1-t} \end{cases} \Leftrightarrow \begin{cases} x_{10} = 2^n - 2^{t-1} x_{1,t-1} \\ x_{0,t-1} = 2^{n+1-t} - x_{1,t-1} \end{cases} \tag{2.18.1}$$

When $t \neq 2$, to make the corresponding summand not in $A(n-1)$, it is necessary that $x_{10} \geq 2^n$ or $x_{0,t-1} \geq 2^{n+2-t}$ or $x_{1,t-1} \geq 2^{n+1-t}$, hence $x_{1,t-1} = 2^{n+1-t}$ or $x_{1,t-1} = 0$. The case $x_{1,t-1} = 2^{n+1-t}$ corresponds to the summand P_t^{n+1-t}, while the case $x_{1,t-1} = 0$ corresponds to $Sq(2^n, 0, \cdots, 2^{n+1-t}, 0, \cdots)$. Therefore, it is natural to consider $P_{t-1}^{n+1-t} P_1^n$, which gives the equations:

$$\begin{cases} x_{t-1,0} + 2x_{t-1,1} = 2^{n+1-t} \\ x_{01} + x_{t-1,1} = 2^n \end{cases} \Leftrightarrow \begin{cases} x_{01} = 2^n - x_{t-1,1} \\ x_{t-1,0} = 2^{n+1-t} - 2x_{t-1,1} \end{cases} \tag{2.18.2}$$

To make the corresponding summand not in $A(n-1)$, it is necessary that $x_{t-1,1} = 0$, which is corresponding to $Sq(2^n, 0, \cdots, 2^{n+1-t}, 0, \cdots)$. Therefore, $P_t^{n+1-t} + P_1^n P_{t-1}^{n+1-t} + P_{t-1}^{n+1-t} P_1^n \in A(n-1)$.

When $t = 2$, we still claim that $P_2^{n-1} + P_1^n P_1^{n-1} + P_1^{n-1} P_1^n \in A(n-1)$. By (2.15.1),

$$\begin{aligned} P_1^n P_1^{n-1} &= \sum_{\substack{x_{10} = 2^n - 2x_{11} \\ x_{01} = 2^{n-1} - x_{11}}} (x_{10}, x_{01}) Sq(x_{01} + x_{10}, x_{11}, 0, \cdots) \\ &= \sum_{0 \leq x_{11} \leq 2^{n-1}/3} (2^n - 2x_{11}, 2^{n-1} - x_{11}) Sq(2^n + 2^{n-1} - 3x_{11}, x_{11}, 0, \cdots) \\ &\quad + P_2^{n-1} + \text{terms in } A(n-1) \end{aligned}$$

$$\begin{aligned} P_1^{n-1} P_1^n &= \sum_{\substack{x_{01} = 2^n - x_{11} \\ x_{10} = 2^{n-1} - 2x_{11}}} (x_{10}, x_{01}) Sq(x_{01} + x_{10}, x_{11}, 0, \cdots) \\ &= \sum_{0 \leq x_{11} \leq 2^{n-1}/3} (2^{n-1} - 2x_{11}, 2^n - x_{11}) Sq(2^n + 2^{n-1} - 3x_{11}, x_{11}, 0, \cdots) \\ &\quad + \text{terms in } A(n-1) \end{aligned}$$

Notice the fact that $(a + 2^k, b) = (a, b + 2^k) \in \mathbb{F}_2$ if $a, b < 2^k$, the proof is thus accomplished. $\square$

Back to the case of tmf. By the duality of modules and comodules, the E_2-page of Adams spectral sequence can be reformulated as

$$E_2^{s,t} = Ext_{A_*}^{s,t}(\mathbb{F}_2, H_*(tmf))$$

where $H_*(tmf)$ is the dual of $H^*(tmf) = A \otimes_{A(2)_*} \mathbb{F}_2$. By the duality of modules and comodules again, $H_*(tmf) = A_* \square_{A(2)_*} \mathbb{F}_2$. By lemma 2.9,

$$E_2^{s,t} = Ext^{s,t}_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2) = Cotor^{s,t}_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}(tmf) \otimes \mathbb{Z}_2$$

To compute the cotorsion, we may refer to the cobar resolution.

Definition 2.19 (Cobar resolution)**.** *For Hopf algebroid (A, Γ), a left Γ-comodule M, and a right Γ-comodule L which is projective over A, the* cobar resolution $C^*_\Gamma(L, M)$ *is defined by* $C^s_\Gamma(L, M) = L \otimes_A \bar{\Gamma}^s \otimes_A M$, *where* $\bar{\Gamma} = coker\ \eta_L$. *The coboundary map is given by*

$$\begin{aligned} d_s : C^s_\Gamma(L, M) &\to C^{s+1}_\Gamma(L, M); \\ l \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \longmapsto \quad & \psi_L(l) \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m \\ & + \sum_{i=1}^{s} (-1)^i l \otimes \gamma_1 \otimes \cdots \otimes \Delta(\gamma_i) \otimes \cdots \otimes \gamma_s \otimes m \\ & + (-1)^{s+1} l \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes \psi_M(m) \end{aligned}$$

The element $l \otimes \gamma_1 \otimes \cdots \otimes \gamma_s \otimes m$ is usually denoted by $l[\gamma_1|\cdots|\gamma_s]m$, where l or m can be omitted if $l = 1$ or $m = 1$ respectively.

Proposition 2.20 ([Rav04])**.** $H(C^*_\Gamma(L, M)) = Cotor_\Gamma(L, M)$ □

This proposition makes $Cotor^{s,t}_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$ computable, but in practice, it can be really cumbersome. Therefore, some other methods are necessary. A feasible one is the spectral sequence induced by filtration. The the case of decreasing filtration is similar to the increasing filtration, so we will only introduce the increasing one below.

Definition 2.21 (Filtration)**.** *An* increasing filtration *on a Hopf algebroid (A, Γ) is an increasing sequence of sub-K-modules*

$$K = F_0\Gamma \subset F_1\Gamma \subset \cdots$$

with $\Gamma = \bigcup F_s\Gamma$ such that

$\cdot F_s\Gamma \cdot F_t\Gamma \subset F_{s+t}\Gamma$,

$\cdot c(F_s\Gamma) \subset F_s\Gamma$, *and*

$\cdot \Delta(F_s\Gamma) \subset \bigoplus_{p+q=s} F_p\Gamma \otimes_A F_q\Gamma$.

A filtered Hopf algebroid *is one equipped with an filtration, where the filtration of A is induced by the one on Γ, i.e.,*

$$F_sA = \eta_L(A) \cap F_s\Gamma = \eta_R(A) \cap F_s\Gamma = \epsilon(F_s\Gamma)$$

An increasing filtration *on a Γ-comodule M is an increasing sequence of sub-K-modules*

$$K = F_0M \subset F_1M \subset \cdots$$

with $M = \bigcup F_sM$ such that

$\cdot F_sA \cdot F_tM \subset F_{s+t}\Gamma$, *and*

$\cdot \psi(F_sM) \subset \bigoplus_{p+q=s} F_p\Gamma \otimes_A F_qM$.

Definition 2.22 (Associated graded object)**.** *The* associated graded object *of A is E^0A with $E^0_sA = F_sA/F_{s-1}A$, where we let $F_{-1}A = 0$. $E^0\Gamma$ and E^0M can be similarly defined.*

Proposition 2.23 ([May66]). *$(E^0A, E^0\Gamma)$ is a graded Hopf algebroid and E^0M is a comodule over it.* □

Theorem 2.24 ([May66]). *Assume $E^0\Gamma$ is flat over E^0A, L, M are right and left comodules. Then there is a natural spectral sequence converging to $Cotor_\Gamma(L, M)$ with $E_1^{s,*} = Cotor^s_{E^0\Gamma}(E^0L, E^0M)$, where the second grading comes from the filtration, and the differentials $d_r : E_r^{s,t} \to E_r^{s+1,t-r}$.*

Proof. $Cotor^*_{E^0\Gamma}(E^0L, E^0M) = H(C_{E^0\Gamma}(E^0L, E^0M)) = H(E^0C_\Gamma(L, M))$, while $Cotor_\Gamma(L, M) = H(C_\Gamma(L, M))$. Therefore there is a spectral sequence induced by the filtration on $C_\Gamma(L, M)$, which is the one stated in the theorem, where elements in each page is given by

$$E_r^{s,u} = \{x \in F_uC^s : \delta x \in F_{u-r}C^{s+1}\}/(F_{u-1}C^s + \delta(F_{u+r-1}C^{s-1}))$$

where $C^* = C^*_\Gamma(L, M)$, and δ is the differentials in C^*. □

In [May74], May gives A_*, and hence $A(2)_*$, an increasing filtration called *May filtration.* As an algebra, $A_* = E(\xi_{i,j} : i \geq 1, j \geq 0)$, where $\xi_{i,j} = \xi_i^{2^j}$, and the coproduct is given by $\Delta(\xi_{i,j}) = \sum_k \xi_{i-k,k+j} \otimes \xi_{k,j}$, where $\xi_{0,j} = 1$. Then an increasing filtration can be given such that $\xi_{i,j}$ has a filtration degree $2i - 1$. As a result, $\Delta(\xi_{i,j}) = 1 \otimes \xi_{i,j} + \xi_{i,j} \otimes 1 +$ terms in $F_{2i-2}A_*$. Therefore, E^0A_* is an exterior algebra with all generators $\{\xi_{i,j}\}$ primitive. The following lemma then gives the structure of $Cotor_{E^0A_*}(\mathbb{F}_2, \mathbb{F}_2)$ and $Cotor_{E^0A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$.

Lemma 2.25 ([Rav04]). *Let Γ be a commutative, connected graded Hopf algebra of finite type over a field K which is an exterior algebra on primitive generators $x_1, x_2, \cdots$, each having odd degree if $char(K) \neq 2$. Then*

$$Cotor_\Gamma(K, K) = P(y_1, y_2, \cdots)$$

where $y_i \in Cotor^{1,|x_i|}$ is represented by $[x_i] \in C^1_\Gamma(K, K)$. □

We state the following theorem as an accomplishment of the construction of May spectral sequence.

Theorem 2.26 (May spectral sequence).
· ([May74]) *There is a spectral sequence $E_*^{s,t,u}$ converging to $Cotor^{s,t}_{A_*}(\mathbb{F}_2, \mathbb{F}_2)$ with $E_1^{*,*,*} = \mathbb{F}_2[h_{i,j} : i \geq 1, j \geq 0]$ and differentials $d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$, where $h_{i,j} \in E_1^{1,(2^i-1)2^j,2i-1}$ is represented by $[\xi_{i,j}]$.*
· *There is a spectral sequence $E_*^{s,t,u}$ converging to $Cotor^{s,t}_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$ with $E_1^{*,*,*} = \mathbb{F}_2[h_{i,j} : 1 \leq i + j \leq 3]$. And there is a map from the first one to the second one induced by the map $A_* \to A(2)_*$.* □

3 Differentials in the May Spectral Sequence

The differentials on the May spectral sequence are induced by the differentials on the cobar complex, which give the most direct way to compute the differentials.

By theorem 2.26, the E_1-page of May SS for tmf is

$$E_1^{*,*,*} = \mathbb{F}_2[h_{10}, h_{11}, h_{12}, h_{20}, h_{21}, h_{30}]$$

Proposition 3.1. $d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}$

Proof. $h_{i,j}$ is represented by $[\xi_i^{2^j}] \in E^0_{2i-1}C_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$, where

$$\delta([\xi_i^{2^j}]) = \sum_{0<k<i} [\xi_{i-k}^{2^{k+j}}|\xi_k^{2^j}] \in F_{2i-2}C_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$$

Hence $d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j}h_{k,j}$. □

Table 3.1: May E_1-page generators and their differentials

generators	differentials
h_{10}	0
h_{11}	0
h_{12}	0
h_{20}	$h_{10}h_{11}$
h_{21}	$h_{11}h_{12}$
h_{30}	$h_{10}h_{21} + h_{12}h_{20}$

Now we have d_1-cycles h_{10}, h_{11}, h_{12}, h_{20}^2, h_{21}^2, h_{30}^2. We only need to consider the differentials of $h_{20}^i h_{21}^j h_{30}^k$, where $0 \le i, j, k \le 1$. In fact,

$$d_1(h_{20}h_{21}) = h_{10}h_{11}h_{21} + h_{11}h_{12}h_{20} = d_1(h_{11}h_{30})$$

hence $h_{11}h_{30} + h_{20}h_{21}$ is a d_1-cycle, which is denoted by $h_0(1)$, then $h_0(1)^2$ is decomposable in terms of other cycles $h_0(1)^2 = h_{11}^2h_{30}^2 + h_{20}^2h_{21}^2$. Besides,

$$d_1(h_{20}h_{30}) = h_{10}h_{11}h_{30} + h_{10}h_{20}h_{21} + h_{12}h_{20}^2 = h_{10}h_0(1) + h_{12}h_{20}^2$$

$$d_1(h_{21}h_{30}) = h_{11}h_{12}h_{30} + h_{10}h_{21}^2 + h_{12}h_{20}h_{21} = h_{12}h_0(1) + h_{10}h_{21}^2$$

$$d_1(h_{20}h_{21}h_{30}) = d_1(h_{30}h_0(1) + h_{11}h_{30}^2) = h_0(1)d_1(h_{30})$$

Therefore, we get the structure of E_2-page:

$$\text{generators: } \{h_{10},\ h_{11},\ h_{12},\ h_{20}^2,\ h_{21}^2,\ h_{30}^2,\ h_0(1)\}$$

$$\text{relations: } \{h_{10}h_{11},\ h_{11}h_{12},\ h_{10}h_0(1) + h_{12}h_{20}^2,$$
$$h_{12}h_0(1) + h_{10}h_{21}^2,\ h_0(1)^2 + h_{11}^2h_{30}^2 + h_{20}^2h_{21}^2\}$$

Then consider differentials on E_2-page. Since any generator $h_{i,j}$ has first grading and third grading odd, the first grading and the third grading of all elements have the same odevity. While

$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}$, the even pages do not admit nontrivial differentials. In particular, the elements in E_2-page are exactly the ones in E_3-page.

As for the differentials on E_3-page, we only need to consider the generators of E_1-page. Since $\delta([\xi_1^{2^j}]) = 0$ in $C_{A(2)_*}(\mathbb{F}_2, \mathbb{F}_2)$, there are no nontrivial differentials of $h_{1,j}$ admitted on any page, that is

Proposition 3.2. $d_r(h_{1,j}) = 0, \ \forall r$ □

Then consider $d_3(h_{20}^2)$, we have

Proposition 3.3. $d_3(h_{20}^2) = h_{11}^3 + h_{10}^2 h_{12}$

Proof.

$$0 \neq h_{20}^2 \in E_3^{2,6,6} = \{x \in F_6C^{2,6} : \delta x \in F_3C^{3,6}\}/(F_5C^{2,6} + \delta(F_8C^{1,6}))$$

where $C^{s,t} = C_{A(2)_*}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$. Notice that $F_6C^{2,6}$ is spanned by

Table 3.2: A basis of $F_6C^{2,6}$

elements	May filtration	differentials
$[\xi_1\|\xi_1^5]$	3	$[\xi_1\|\xi_1\|\xi_1^4] + [\xi_1\|\xi_1^4\|\xi_1]$
$[\xi_1\|\xi_1^2\xi_2]$	5	$[\xi_1\|\xi_1^2\|\xi_1^3] + [\xi_1\|\xi_1^2\|\xi_2] + [\xi_1\|\xi_2\|\xi_1^2] + [\xi_1\|\xi_1^4\|\xi_1]$
$[\xi_1^2\|\xi_1^4]$	2	0
$[\xi_1^2\|\xi_1\xi_2]$	5	$[\xi_1^2\|\xi_1\|\xi_2] + [\xi_1^2\|\xi_1^2\|\xi_1^2] + [\xi_1^2\|\xi_1^3\|\xi_1] + [\xi_1^2\|\xi_2\|\xi_1]$
$[\xi_1^3\|\xi_1^3]$	4	$[\xi_1\|\xi_1^2\|\xi_1^3] + [\xi_1^2\|\xi_1\|\xi_1^3] + [\xi_1^3\|\xi_1\|\xi_1^2] + [\xi_1^3\|\xi_1\|\xi_1^2]$
$[\xi_1^3\|\xi_2]$	5	$[\xi_1\|\xi_1^2\|\xi_2] + [\xi_1^2\|\xi_1\|\xi_2] + [\xi_1^3\|\xi_1^2\|\xi_1]$
$[\xi_2\|\xi_1^3]$	5	$[\xi_1^2\|\xi_1\|\xi_1^3] + [\xi_2\|\xi_1\|\xi_1^2] + [\xi_2\|\xi_1^2\|\xi_1]$
$[\xi_2\|\xi_2]$	6	$[\xi_1^2\|\xi_1\|\xi_2] + [\xi_2\|\xi_1^2\|\xi_1]$
$[\xi_1^4\|\xi_1^2]$	2	0
$[\xi_1\xi_2\|\xi_1^2]$	5	$[\xi_1\|\xi_2\|\xi_1^2] + [\xi_1^2\|\xi_1^2\|\xi_1^2] + [\xi_1^3\|\xi_1\|\xi_1^2] + [\xi_2\|\xi_1\|\xi_1^2]$
$[\xi_1^5\|\xi_1]$	3	$[\xi_1\|\xi_1^4\|\xi_1] + [\xi_1^4\|\xi_1\|\xi_1]$
$[\xi_1^2\xi_2\|\xi_1]$	5	$[\xi_1^2\|\xi_1^3\|\xi_1] + [\xi_1^2\|\xi_2\|\xi_1] + [\xi_2\|\xi_1^2\|\xi_1] + [\xi_1^4\|\xi_1\|\xi_1]$

For an element x representing h_{20}^2, x should have May filtration 6 with May filtration of dx being less than 3. From the table above, it can be figured out that h_{20}^2 can be represented by $[\xi_2|\xi_2] + [\xi_1^2|\xi_1\xi_2] + [\xi_1^2\xi_2|\xi_1]$, and hence $d_3(h_{20}^2)$ can be represented by $[\xi_1^2|\xi_1^2|\xi_1^2] + [\xi_1^4|\xi_1|\xi_1]$. Therefore, $d_3(h_{20}^2) = h_{11}^3 + h_{10}^2 h_{12}$. □

It can be seen that the direct computation is really complicated, some other indirect methods should be used, such as algebraic Steenrod operations and Massey products.

Like the ordinary Steenrod operations acting on the cohomology, there is an analogy that acts on the Cotor as follows.

Theorem 3.4 ([May70])**.** *Let Γ be a Hopf algebroid over $\mathbb{F}_2$ and M, N right and left Γ-comodule algebras. Denote $Cotor_\Gamma^{s,t}(M, N)$ by $H^{s,t}$, then there exists natural homomorphisms*

$$Sq^i : \ H^{s,t} \to H^{s+i,2t}$$

for $i \geq 0$, such that:

$\cdot Sq^i = 0$ *if $i > s$;*

$\cdot Sq^s(x) = x^2$ *for* $x \in H^{s,t}$*;*
$\cdot Sq^1$ *is the Bockstein homomorphism;*
$\cdot$*If* $x \in H^{s,t}$ *is represented by* $m[\gamma_1|\cdots|\gamma_s]n$*, then* $Sq^0(x)$ *is represented by* $m^2[\gamma_1^2|\cdots|\gamma_s^2]n^2$*;*
$\cdot Sq^i$ *satisfies Cartan formula and Adem relations, that is*

$$Sq^i(xy) = \sum_{j=0}^{i} Sq^j(x)Sq^{i-j}(y)$$

$$Sq^a Sq^b = \sum_{i\geq 0} \binom{b-i-1}{a-2i} Sq^{a+b-i}Sq^i \ (a < 2b)$$

□

It can be deduced immediately that

Corollary 3.5. *$Sq^0(h_{i,j}) = h_{i,j+1}$ where $h_{i,j}$ with $i+j > 3$ is viewed as* 0*.* □

Then several differentials can be computed by comparing the Adams SS and May SS.

Another proof of proposition 3.3. Notice that $h_{10}h_{11} = 0$ in Adams SS, we have

$$0 = Sq^1(h_{10}h_{11}) = h_{11}^3 + h_{10}^2 h_{12}$$

in Adams SS, hence it must be killed in May SS. It is an element in $E^{3,6,3}$, while $E_3^{4,6,*} = 0$ and $E_3^{2,6,*}$ is spanned by $h_{20}^2 \in E_3^{2,6,6}$, the only possibility is that $d_3(h_{20}^2) = h_{11}^3 + h_{10}^2 h_{12}$. □

Proposition 3.6. $d_3(h_{21}^2) = h_{12}^3$

Proof. $h_{12}^3 = Sq^1(h_{11}h_{12}) = 0$ in Adams SS, hence must be killed by some May differentials. Notice that h_{12}^3 cannot support a nontrivial differential by proposition 3.2, while $E_3^{2,12,*}$ is spanned by h_{21}^2, therefore, $d_3(h_{21}^2) = h_{12}^3$. □

Proposition 3.7. $d_3(h_{30}^2) = h_{11}h_{21}^2$

Proof. Whatever $d_3(h_0(1))$ is, $d_3(h_0(1)^2) = 0$, where $h_0(1)^2 = h_{11}^2 h_{30}^2 + h_{20}^2 h_{21}^2$. Therefore,

$$h_{11}^2 d_3(h_{30}^2) = d_3(h_{20}^2 h_{21}^2) = h_{12}(h_{10}h_{21} + h_{12}h_{20})^2 + h_{11}^3 h_{21}^2 = h_{11}^3 h_{21}^2$$

While $E_3^{3,14,*}$ is spanned by $h_{11}h_{21}^2$, the only possibility is that $d_3(h_{30}^2) = h_{11}h_{21}^2$. □

The following theorem will be useful in the computation of higher differentials.

Theorem 3.8 ([Nak72])**.** *For $x \in E_r^{s,t,u}$ ($r \geq 3$) in May SS and i such that $s-i$ is even, there exists an element in E_{2r+1}-page represented by $Sq^i(x)$ such that its differential is the element (possibly zero) to which $Sq^i(d_r(x))$ survives.* □

Proposition 3.9. $d_7(h_{30}^4) = h_{12}h_{21}^4$

Proof. Consider $h_{30}^2 \in E_3^{2,14,10}$, we have

$$\begin{aligned} d_7(h_{30}^4) =& d_7(Sq^2(h_{30}^2)) \\ =& Sq^2(d_3(h_{30}^2)) \\ =& Sq^2(h_{11}h_{21}^2) \\ =& h_{12}h_{21}^4 \end{aligned}$$

□

Then we will introduce the Massey product, which turns out to be useful in the computation. We will assume from now on that $2 = 0$ so that signs can be ignored.

Definition 3.10 (Massey products). *Let C be a differential graded algebra (DGA), $\alpha_1, \cdots, \alpha_n \in MH^*(C)$ such that the adjacent two matrices are compatible with entries of their products homogeneous. Their* Massey product *$\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined if there is a* defining system *$a_{i,j} \in MC$ for $0 \le i < j \le n$ such that:*

$\cdot \alpha_i$ can be represented by $a_{i-1,i}$;

$\cdot d(a_{i,j}) = \sum a_{i,k}a_{k,j}$ for $(i, j) \neq (0, n)$.

and $\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined to be $\{$the homology class represented by $\sum a_{0,k}a_{k,n}\}$. It is said to be strictly defined *if all lowerproducts have trivial indeterminacy (hence are $\{0\}$).*

About the indeterminacy, we have the following theorem.

Theorem 3.11 ([May69]). *Let $\langle \alpha_1, \cdots, \alpha_n \rangle$ be defined. For $1 \le k < n$ let the degree of $x_k \in MH^*(C)$ be one less than that of $\alpha_k \alpha_{k+1}$.*

$\cdot In\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \bigcup(\alpha_1 x_2 + x_1 \alpha_3)$

$\cdot$If $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is strictly defined, then

$In\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle = \bigcup(\langle \alpha_1, \alpha_2, x_3 \rangle + \langle \alpha_1, x_2, \alpha_4 \rangle + \langle x_1, \alpha_3, \alpha_4 \rangle)$ □

One reason that Massey product is useful is that there are relations between Massey products and ordinary products as stated in the following theorem, which will be used widely in dealing with extension problems.

Theorem 3.12 ([May69]).

$\cdot$If $\langle \alpha_1, \cdots, \alpha_n \rangle$ is defined then $\langle \alpha_1, \cdots, \alpha_n \rangle^\top = \langle \alpha_n^\top, \cdots, \alpha_1^\top \rangle$

$\cdot$If $\langle \alpha_2, \cdots, \alpha_n \rangle$ is defined then $\alpha_1 \langle \alpha_2, \cdots, \alpha_n \rangle \subset \langle \alpha_1\alpha_2, \alpha_3, \cdots, \alpha_n \rangle$

$\cdot$If $\langle \alpha_1\alpha_2, \alpha_3, \cdots, \alpha_n \rangle$ is defined then $\langle \alpha_1\alpha_2, \alpha_3, \cdots, \alpha_n \rangle \subset \langle \alpha_1, \alpha_2\alpha_3, \alpha_4, \cdots, \alpha_n \rangle$

$\cdot$If $\langle \alpha_1, \cdots, \alpha_{n-1} \rangle$ and $\langle \alpha_2, \cdots, \alpha_n \rangle$ are strictly defined then $\alpha_1 \langle \alpha_2, \cdots, \alpha_n \rangle = \langle \alpha_1, \cdots, \alpha_{n-1} \rangle \alpha_n$

$\cdot$If $\langle \alpha_1, \cdots, \alpha_k, \cdots, \alpha_n \rangle$ is defined, $\langle \alpha_1, \cdots, \alpha'_k, \cdots, \alpha_n \rangle$ is strictly defined then

$\langle \alpha_1, \cdots, \alpha_k, \cdots, \alpha_n \rangle \subset \langle \alpha_1, \cdots, \alpha'_k, \cdots, \alpha_n \rangle + \langle \alpha_1, \cdots, \alpha_k + \alpha'_k, \cdots, \alpha_n \rangle$

$\cdot$If $\langle \alpha_2, \cdots, \alpha_n \rangle$ is defined and l, $1 < l < n$, is given with $\langle \alpha_1, \cdots, \alpha_j \rangle = \{0\}$ for $1 < j < l$, then

$\alpha_1 \langle \alpha_2, \cdots, \alpha_n \rangle \subset \langle \langle \alpha_1, \cdots, \alpha_l \rangle, \alpha_{l+1}, \cdots, \alpha_n \rangle$ □

Another reason is that in the spectral sequence induced by a filtration on a DGA, the differentials of Massey products can be computed using the following theorem.

Theorem 3.13 ([May69]). *Let C be a DGA equipped with a regular increasing filtration with the inducing spectral sequence indexed such that $d_r : E_r^{p,q} \to E_r^{p+1,q-r}$. Let $\langle \alpha_1, \cdots, \alpha_n \rangle$ be defined in E_{r+1}, where each α_i is a matrix with entries being permanent cycles and α_i converges to $\beta_i \in$*

$MH^*(C)$. Let k be with $1 \le k \le n-2$ such that each $\langle \beta_i, \cdots, \beta_{i+k} \rangle$ is strictly defined in $H^*(C)$ and that if an entry of $a_{i,j}$ with $1 < j-i \le k$ in the defining system for $\langle \alpha_1, \cdots, \alpha_n \rangle$ has bidegree (p,q), then each element of $E_{r+m+1}^{p,q+m}$ with $m \ge 0$ is a permanent cycle. Let $s > r$ be such that for each (p,q) as above with $k < j-i < n$ and for each t with $r < t < s$, $E_t^{p+1,q-t} = 0$, and if $j-i > k+1$ then $E_{r+s-t}^{p+1,q-t} = 0$. Then for each $\alpha \in \langle \alpha_1, \cdots, \alpha_n \rangle$

$$d_t(\alpha) = 0 \quad \forall r < t < s$$

Besides, there are permanent cycles $\delta_i \in ME_{r+1}$ for $1 \le i \le n-k$ converging to elements in $\langle \beta_i, \cdots, \beta_{i+k} \rangle$ such that $\langle \gamma_1, \cdots, \gamma_{n-k} \rangle$ is defined in E_{r+1} and contains an element γ surviving to $d_s(\alpha)$, where

$$\gamma_1 = \begin{pmatrix} \delta_1 & \alpha_1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \alpha_{i+k} & 0 \\ \delta_i & \alpha_i \end{pmatrix} \ (1 < i < n-k), \quad \gamma_{n-k} = \begin{pmatrix} \alpha_n \\ \delta_{n-k} \end{pmatrix}$$

Assuming further that each δ_i is unique, that each $\langle \alpha_1, \cdots, \alpha_{i-1}, \delta_i, \alpha_{i+k+1}, \cdots, \alpha_n \rangle$ is strictly defined, and that all Massey products in sight, except possibly $\langle \beta_i, \cdots, \beta_{i+k} \rangle$, have zero indeterminacy, then we have

$$d_s(\langle \alpha_1, \cdots, \alpha_n \rangle) = \sum_{i=1}^{n-k} \langle \alpha_1, \cdots, \alpha_{i-1}, \delta_i, \alpha_{i+k+1}, \cdots, \alpha_n \rangle$$

□

There is also a converging theorem assuring that Massey products converge to the corresponding Massey products if some conditions are satisfied.

Theorem 3.14 (May's Convergence Theorem)([May69])**.** *With notation as above let $\langle \alpha_1, \cdots, \alpha_n \rangle$ be defined in E_{r+1}, where each α_i is a matrix with entries being permanent cycles and α_i converges to $\beta_i \in MH^*(C)$. If $\langle \beta_1, \cdots, \beta_n \rangle$ is strictly defined, and there are no crossing differentials, i.e., if an entry of $a_{i,j}$ with $1 < j-i < n$ in the defining system for $\langle \alpha_1, \cdots, \alpha_n \rangle$ has bidegree (p,q), then each element of $E_{r+m+1}^{p,q+m}$ with $m \ge 0$ is a permanent cycle. Then each element in $\langle \alpha_1, \cdots, \alpha_n \rangle$ is a permanent cycle converging to an element in $\langle \beta_1, \cdots, \beta_n \rangle$.* □

There is also an analogy of this theorem assuring that Massey products in Adams spectral sequence converge to the corresponding Toda brackets if similar conditions are satisfied, which is called Moss's Convergence Theorem ([Mos70]).

Then we can use the Massey products to compute some differentials in May SS.

Proposition 3.15. $d_3(h_0(1)) = h_{10}h_{12}^2$

Proof. Consider $\langle h_{11}, h_{10}, h_{11}, h_{12} \rangle$ in E_2-page of May SS, it has a defining system:

$$\begin{matrix} h_{11} & & h_{10} & & h_{11} & & h_{12} \\ & h_{20} & & h_{20} & & h_{21} & \\ & & 0 & & h_{30} & & \end{matrix}$$

Hence $h_0(1) \in \langle h_{11}, h_{10}, h_{11}, h_{12} \rangle$. By theorem 3.11, $\langle h_{11}, h_{10}, h_{11} \rangle$ and $\langle h_{10}, h_{11}, h_{12} \rangle$ have zero indeterminacy since $E_2^{1,3,4} = E_2^{1,6,4} = 0$, so $\langle h_{11}, h_{10}, h_{11}, h_{12} \rangle$ is strictly defined. By theorem 3.11 again, $\langle h_{11}, h_{10}, h_{11}, h_{12} \rangle$ has zero indeterminacy, thus $\langle h_{11}, h_{10}, h_{11}, h_{12} \rangle = \{h_0(1)\}$.

In theorem 3.13, let $r=1$. h_{10}, h_{11}, h_{12} are permanent cycles, hence converge to elements in Adams E_2-page, temporarily denoted by x, y, z respectively. Since $h_{10}h_{11}$ and $h_{11}h_{12}$ are killed by d_1 in May SS and there is no elements for extensions, xy and yz are 0 in Adams E_2-page. Therefore $\langle y,x,y\rangle$ and $\langle x,y,z\rangle$ are defined and hence strictly defined since every 2-fold product has trivial indeterminacy, so we can let $k=2$ in theorem 3.13. Nonexistence of crossing diffrentials is easy to be verified. For $h_{30}\in E_1^{1,7,5}$ in the defining system, $E_2^{2,7,*}=0$; for $0\in E_1^{1,5,5}$ in the defining system, $E_2^{2,5,3}=0$, but $E_3^{2,5,2}$ is spanned by $h_{10}h_{12}$, hence we can let $s=3$ in theorem 3.13.

Then consider $\langle y,x,y\rangle$ in Adams E_2-page. $E_2^{1,3}=0$, hence it has zero indeterminacy. There is a defining system

$$\begin{array}{ccccc} [\xi_1^2] & & [\xi_1] & & [\xi_1^2] \\ & [\xi_2] & & [\xi_2+\xi_1^3] & \end{array}$$

$\langle y,x,y\rangle=\{$the homology class of $[\xi_1^2|\xi_2+\xi_1^3]+[\xi_2|\xi_1^2]\}$, where $\delta([\xi_1^2\xi_2])+[\xi_1^4|\xi_1]=[\xi_1^2|\xi_2+\xi_1^3]+[\xi_2|\xi_1^2]$, hence $\langle y,x,y\rangle=\{xz\}$, to which only $h_{10}h_{12}$ converges. On the other hand, $E_2^{2,7,*}=0$ in May spectral sequence, hence the uniqueness of δ_i is satisfied, where $\delta_1=h_{10}h_{12}$ and $\delta_2=0$. Then by theorem 3.13, $d_3(h_0(1))=h_{10}h_{12}^2$. □

So far we have computed all the differentials in the E_3-page.

Table 3.3: May E_3-page generators and their differentials

generators	differentials
h_{10}	0
h_{11}	0
h_{12}	0
h_{20}^2	$h_{11}^3+h_{10}^2h_{12}$
h_{21}^2	h_{12}^3
h_{30}^2	$h_{11}h_{21}^2$
$h_0(1)$	$h_{10}h_{12}^2$

The differentials of monomials of the generators have the form of

$$h_{10}^{n_1}h_{11}^{n_2}h_{12}^{n_3}h_{20}^{4n_4}h_{21}^{4n_5}h_{30}^{4n_6}h_0(1)^{2n_7}d_3(h_{20}^{2i_1}h_{21}^{2i_2}h_{30}^{2i_3}h_0(1)^{i_4})$$

where $i_1,i_2,i_3,i_4\in\{0,1\}$, which can be listed as follows:

Table 3.4: The differentials of monomials of generators in E_3-page

monomials	differentials
h_{20}^2	$h_{11}^3+h_{10}^2h_{12}$
h_{21}^2	h_{12}^3
h_{30}^2	$h_{11}h_{21}^2$
$h_0(1)$	$h_{10}h_{12}^2$
$h_{20}^2h_{21}^2$	$h_{11}^2d_3(h_{30}^2)$
$h_{20}^2h_{30}^2$	$h_{11}h_0(1)^2+h_{10}^2h_{12}h_{30}^2$
$h_{20}^2h_0(1)$	$h_{11}^3h_0(1)$
(to be continued)	

(continued)

monomials	differentials
$h_{21}^2 h_{30}^2$	$h_{12}^3 h_{30}^2 + h_{11} h_{21}^4$
$h_{21}^2 h_0(1)$	0
$h_{30}^2 h_0(1)$	$h_{11} h_{21}^2 h_0(1) + h_{10} h_{12}^2 h_{30}^2$
$h_{20}^2 h_{21}^2 h_{30}^2$	$h_0(1)^2 d_3(h_{30}^2)$
$h_{20}^2 h_{21}^2 h_0(1)$	$h_0(1)^2 d_3(h_0(1))$
$h_{20}^2 h_{30}^2 h_0(1)$	$h_{11} h_0(1)^3$
$h_{21}^2 h_{30}^2 h_0(1)$	$h_{11} h_{21}^4 h_0(1)$
$h_{20}^2 h_{21}^2 h_{30}^2 h_0(1)$	$h_0(1)^2 d_3(h_0(1) h_{30}^2)$

Then we can get the structure of E_4-page:

$$\begin{aligned} \text{generators: } &\{h_{10},\ h_{11},\ h_{12},\ h_{11}h_0(1),\ h_{20}^4,\ h_{10}h_{30}^2,\\ &h_0(1)^2,\ h_{12}h_{30}^2,\ h_{21}^2h_0(1),\ h_{21}^4,\ h_{30}^4\}\\ \text{relations: } &\{h_{10}\cdot h_{11},\ h_{11}\cdot h_{12},\ h_{10}\cdot h_{11}h_0(1),\ h_{12}\cdot h_{11}h_0(1),\\ &h_{11}\cdot h_{10}h_{30}^2,\ h_{11}\cdot h_{12}h_{30}^2,\ h_{11}h_0(1)\cdot h_{10}h_{30}^2,\\ &h_{11}h_0(1)\cdot h_{12}h_{30}^2,\ h_{12}^2\cdot h_{20}^4 + h_{10}^2\cdot h_0(1)^2,\\ &h_{12}\cdot h_0(1)^2 + h_{10}\cdot h_{21}^2h_0(1),\ h_{12}\cdot h_{21}^2h_0(1) + h_{10}\cdot h_{21}^4,\\ &(h_0(1)^2)^2 + h_{20}^4\cdot h_{21}^4,\ h_{11}^3 + h_{10}^2\cdot h_{12},\ h_{12}^3,\ h_{10}\cdot h_{12}^2,\\ &h_{11}\cdot h_0(1)^2 + h_{10}^2\cdot h_{12}h_{30}^2,\ h_{11}^2\cdot h_{11}h_0(1),\\ &h_{12}^2\cdot h_{12}h_{30}^2 + h_{11}\cdot h_{21}^4,\ h_{11}\cdot h_{21}^2h_0(1) + h_{12}^2\cdot h_{10}h_{30}^2,\\ &h_{11}h_0(1)\cdot h_0(1)^2,\ h_{11}h_0(1)\cdot h_{21}^4\\ &\text{and relations which can be seen by the expressions}\} \end{aligned}$$

Notice that $h_{1,j} \in E^{1,2^j,1}$, $h_{i,j}^2 \in E^{2,(2^i-1)2^{j+1},4i-2}$, and $h_0(1) \in E^{2,9,6}$, each generator in $E_2^{s,t,u}$ has the property that $u-s$ is a multiple of 4. Hence for $r \geq 2$, d_r is nontrivial only when $r+1$ is a multiple of 4. So let us turn to E_7-page.

Consider all possible differentials on generators of E_7-page, the only possibly nontrivial one is $d_7(h_{30}^4) = h_{12}h_{21}^4$, which is proposition 3.9, while differentials on other generators must be trivial because of degree reasons.

Then we can get the structure of E_8-page:

$$\begin{aligned} \text{generators: } &\{h_{10},\ h_{11},\ h_{12},\ h_{11}h_0(1),\ h_{20}^4,\ h_{10}h_{30}^2,\ h_0(1)^2,\\ &h_{12}h_{30}^2,\ h_{21}^2h_0(1),\ h_{21}^4,\ h_{11}h_{30}^4,\ h_{11}h_{30}^4h_0(1),\ h_{30}^8\} \end{aligned}$$

the list of relations will be given in next section where we will discuss the extension problems.

All these generators can not support higher differentials because of degree reasons, so the E_8-page is the E_∞-page.

4 Extensions in the May Spectral Sequence

Although we get the structure of E_∞-page of May spectral sequence, we only obtain the structure of associated graded object instead of the object it converges to, in other words, $E_\infty^{s,t,u} = F_u H^{s,t}/F_{u-1}H^{s,t}$. Then the extension problem comes out that even if an element is 0 in $E_\infty^{s,t,u}$, there may be the case that it is nonzero in $H^{s,t}$ but with a lower filtration degree.

To distinguish the elements in May SS and Adams SS, $x_{i,j}$ will be used to represent the element in $E_2^{j,i+j}$ of Adams SS to which an generator of May E_∞-page converges.

Table 4.1: Names of elements in Adams SS and May SS

Adams SS	May SS
x_{01}	h_{10}
x_{11}	h_{11}
x_{31}	h_{12}
x_{83}	$h_{11}h_0(1)$
x_{84}	h_{20}^4
x_{123}	$h_{10}h_{30}^2$
x_{144}	$h_0(1)^2$
x_{153}	$h_{12}h_{30}^2$
x_{174}	$h_{21}^2h_0(1)$
x_{204}	h_{21}^4
x_{255}	$h_{11}h_{30}^4$
x_{327}	$h_{11}h_{30}^4h_0(1)$
x_{488}	h_{30}^8

Then consider generators of all relations in E_∞-page of May SS, where the extension problems may live in.

Table 4.2: Extension problems in May SS

relations	$(t-s,s,u)$	extensions	proof
$x_{01}x_{11}$	$(1,2,2)$	0	degree reasons
$x_{01}^2x_{31}+x_{11}^3$	$(3,3,3)$	0	degree reasons
$x_{11}x_{31}$	$(4,2,2)$	0	degree reasons
$x_{01}x_{31}^2$	$(6,3,3)$	0	degree reasons
$x_{01}x_{83}$	$(8,4,8)$	0	degree reasons
x_{31}^3	$(9,3,3)$	0	degree reasons
$x_{11}^2x_{83}$	$(10,5,9)$	0	degree reasons
$x_{31}x_{83}$	$(11,4,8)$	0	degree reasons
$x_{11}x_{123}$	$(13,4,12)$	0	degree reasons
$x_{01}^2x_{144}+x_{31}^2x_{84}$	$(14,6,14)$	0	degree reasons
$x_{01}x_{153}+x_{31}x_{123}$	$(15,4,12)$	0	degree reasons
$x_{11}x_{144}+x_{01}x_{31}x_{123}$	$(15,5,13)$	0	degree reasons
$x_{11}x_{153}$	$(16,4,12)$	0	degree reasons
x_{83}^2	$(16,6,14)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x_{01}x_{174}+x_{31}x_{144}$	$(17,5,13)$	0	degree reasons
$x_{11}x_{174}+x_{31}^2x_{123}$	$(18,5,13)$	0	degree reasons
$x_{01}x_{204}+x_{31}x_{174}$	$(20,5,13)$	0	degree reasons
$x_{83}x_{123}$	$(20,6,18)$	$x_{31}^2x_{144}$	proposition 4.1
$x_{11}x_{204}+x_{31}^2x_{153}$	$(21,5,13)$	0	degree reasons
$x_{83}x_{144}$	$(22,7,19)$	0	degree reasons
$x_{31}x_{204}$	$(23,5,13)$	0	degree reasons
$x_{83}x_{153}$	$(23,6,18)$	0	degree reasons
$x_{01}x_{255}$	$(25,6,22)$	0	degree reasons
$x_{83}x_{174}$	$(25,7,19)$	0	degree reasons
$x_{31}x_{84}x_{153}+x_{01}x_{123}x_{144}$	$(26,8,24)$	0	degree reasons
$x_{11}^2x_{255}+x_{31}x_{123}^2$	$(27,7,23)$	0	degree reasons
$x_{31}x_{255}$	$(28,6,22)$	0	degree reasons
$x_{83}x_{204}$	$(28,7,19)$	0	degree reasons
$x_{84}x_{204}+x_{144}^2$	$(28,8,24)$	0	degree reasons
$x_{123}x_{174}+x_{144}x_{153}$	$(29,7,23)$	0	degree reasons
$x_{31}x_{123}x_{153}$	$(30,7,23)$	0	degree reasons
$x_{123}x_{204}+x_{153}x_{174}$	$(32,7,23)$	0	degree reasons
$x_{01}x_{327}$	$(32,8,28)$	$x_{31}x_{144}x_{153}$	proposition 4.5
$x_{31}x_{153}^2$	$(33,7,23)$	0	degree reasons
$x_{11}x_{327}+x_{83}x_{255}$	$(33,8,28)$	0	degree reasons
$x_{144}x_{204}+x_{174}^2$	$(34,8,24)$	0	degree reasons
$x_{11}x_{83}x_{255}$	$(34,9,29)$	$x_{01}x_{144}x_{204}$	proposition 4.8
$x_{31}x_{153}x_{174}$	$(35,8,24)$	0	degree reasons
$x_{31}x_{327}$	$(35,8,28)$	0	degree reasons
$x_{123}x_{255}$	$(37,8,32)$	$x_{174}x_{204}$	proposition 4.11
$x_{84}x_{153}^2+x_{123}^2x_{144}$	$(38,10,34)$	0	degree reasons
$x_{144}x_{255}+x_{123}^2x_{153}$	$(39,9,33)$	0	degree reasons
$x_{31}x_{123}^3$	$(39,10,34)$	0	degree reasons
$x_{153}x_{255}$	$(40,8,32)$	x_{204}^2	proposition 4.14
$x_{83}x_{327}$	$(40,10,34)$	0	degree reasons
$x_{174}x_{255}+x_{123}x_{153}^2$	$(42,9,33)$	0	degree reasons
$x_{123}x_{327}$	$(44,10,38)$	0	proposition 4.15
$x_{204}x_{255}+x_{153}^3$	$(45,9,33)$	0	degree reasons
$x_{144}x_{327}$	$(46,11,39)$	0	proposition 4.17
$x_{153}x_{327}$	$(47,10,38)$	0	proposition 4.18
$x_{01}^4x_{488}+x_{123}^4$	$(48,12,44)$	$x_{84}x_{204}^2$	proposition 4.19
$x_{174}x_{327}$	$(49,11,39)$	0	proposition 4.20
$x_{11}^2x_{488}+x_{255}^2$	$(50,10,42)$	$x_{153}^2x_{204}$	proposition 4.21
$x_{204}x_{327}$	$(52,11,39)$	0	proposition 4.22
$x_{11}x_{83}x_{488}+x_{255}x_{327}$	$(57,12,48)$	0	proposition 4.23
x_{327}^2	$(64,14,54)$	0	proposition 4.24

The extension problems are mainly proved by shuffling Massey products and analyzing the product structure.

Proposition 4.1. $x_{83}x_{123} = x_{31}^2 x_{144}$

Lemma 4.2. $x_{83} = \langle x_{11}, x_{01}, x_{31}^2\rangle$

Proof. Since $E_2^{1,3} = E_2^{2,9} = 0$, $\langle x_{11}, x_{01}, x_{31}^2\rangle$ is strictly defined with zero indeterminacy, while $\langle h_{11}, h_{10}, h_{12}^2\rangle$ is defined in May E_4-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccc} h_{11} & & h_{10} & & h_{12}^2 \\ & 0 & & h_0(1) & \end{array}$$

So x_{83}, the element to which $h_{11}h_0(1)$ converges, must belong to the Massey product $\langle x_{11}, x_{01}, x_{31}^2\rangle$, which has exactly one element. □

Lemma 4.3. $x_{144} = \langle x_{123}, x_{11}, x_{01}\rangle$

Proof. As proved in proposition 3.15, $\langle x_{11}, x_{01}, x_{11}\rangle = x_{01}x_{31}$. Hence we have

$$x_{11}x_{144} = x_{01}x_{31}x_{123} = x_{123}\langle x_{11}, x_{01}, x_{11}\rangle = \langle x_{123}, x_{11}, x_{01}\rangle x_{11}$$

Then the lemma follows form the fact that the multiplication of x_{11} on $E_2^{4,18}$ is an injection. □

Similarly, there are

Lemma 4.4. $x_{174} = \langle x_{153}, x_{11}, x_{01}\rangle$

Proof. $x_{11}x_{174} = x_{01}x_{31}x_{153} = x_{153}\langle x_{11}, x_{01}, x_{11}\rangle = \langle x_{153}, x_{11}, x_{01}\rangle x_{11}$
Then the lemma follows form the fact that the multiplication of x_{11} on $E_2^{4,21}$ is an injection. □

Proof of proposition 4.1.

$$x_{83}x_{123} = x_{123}\langle x_{11}, x_{01}, x_{31}^2\rangle = \langle x_{123}, x_{11}, x_{01}\rangle x_{31}^2 = x_{31}^2 x_{144}$$

□

Proposition 4.5. $x_{01}x_{327} = x_{31}x_{144}x_{153}$

Lemma 4.6. $x_{327} = \langle x_{83}, x_{204}, x_{31}\rangle$

Proof. Since $E_2^{6,35} = E_2^{4,28} = 0$, $\langle x_{83}, x_{204}, x_{31}\rangle$ is strictly defined with zero indeterminacy, while $\langle h_{11}h_0(1), h_{21}^4, h_{12}\rangle$ is defined in May E_8-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccc} h_{11}h_0(1) & & h_{21}^4 & & h_{12} \\ & 0 & & h_{30}^4 & \end{array}$$

So x_{327}, the element to which $h_{11}h_{30}^4h_0(1)$ converges, must belong to the Massey product $\langle x_{83}, x_{204}, x_{31}\rangle$, which has exactly one element. □

Lemma 4.7. $x_{144}x_{153} = \langle x_{01}, x_{83}, x_{204}\rangle$

Proof. Since $E_2^{3,12} = E_2^{6,35} = 0$, $\langle x_{01}, x_{83}, x_{204}\rangle$ is strictly defined with zero indeterminacy, while $\langle h_{10}, h_{11}h_0(1), h_{21}^4\rangle$ is defined in May E_4-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccc} h_{10} & & h_{11}h_0(1) & & h_{21}^4 \\ & 0 & & h_{21}^2 h_{30}^2 h_0(1) & \end{array}$$

So $x_{144}x_{153}$, the element to which $h_{10}h_{21}^2h_{30}^2h_0(1) = h_{12}h_{30}^2h_0(1)^2$ converges, must belong to the Massey product $\langle x_{01}, x_{83}, x_{204}\rangle$, which has exactly one element. □

Proof of proposition 4.5.

$$x_{01}x_{327} = x_{01}\langle x_{83}, x_{204}, x_{31}\rangle = \langle x_{01}, x_{83}, x_{204}\rangle x_{31} = x_{31}x_{144}x_{153}$$

□

Proposition 4.8. $x_{11}x_{83}x_{255} = x_{01}x_{144}x_{204}$

Lemma 4.9. $x_{255} = \langle x_{11}, x_{31}, x_{204}\rangle$

Proof. Since $E_2^{2,7} = E_2^{4,28} = 0$, $\langle x_{11}, x_{31}, x_{204}\rangle$ is strictly defined with zero indeterminacy, while $\langle h_{11}, h_{12}, h_{21}^4\rangle$ is defined in May E_8-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccc} h_{11} & & h_{12} & & h_{21}^4 \\ & 0 & & h_{30}^4 & \end{array}$$

So x_{255}, the element to which $h_{11}h_{30}^4$ converges, must belong to the Massey product $\langle x_{11}, x_{31}, x_{204}\rangle$, which has exactly one element. □

Lemma 4.10. $x_{01}x_{144} = \langle x_{11}x_{83}, x_{11}, x_{31}\rangle$

Proof. Since $E_2^{4,15} = E_2^{1,7} = 0$, $\langle x_{11}x_{83}, x_{11}, x_{31}\rangle$ is strictly defined with zero indeterminacy, while $\langle h_{11}^2h_0(1), h_{11}, h_{12}\rangle$ is defined in May E_4-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccc} h_{11}^2h_0(1) & & h_{11} & & h_{12} \\ & h_{20}^2h_0(1) & & 0 & \end{array}$$

So $x_{01}x_{144}$, the element to which $h_{12}h_{20}^2h_0(1) = h_{10}h_0(1)^2$ converges, must belong to the Massey product $\langle x_{11}x_{83}, x_{11}, x_{31}\rangle$, which has exactly one element. □

Proof of proposition 4.8.

$$x_{11}x_{83}x_{255} = x_{11}x_{83}\langle x_{11}, x_{31}, x_{204}\rangle = \langle x_{11}x_{83}, x_{11}, x_{31}\rangle x_{204} = x_{01}x_{144}x_{204}$$

□

Proposition 4.11. $x_{123}x_{255} = x_{174}x_{204}$

Lemma 4.12. $x_{174} = \langle x_{123}, x_{11}, x_{31}\rangle$

Proof. Since $E_2^{3,17} = E_2^{2,7} = 0$, $\langle x_{123}, x_{11}, x_{01}x_{31}\rangle$ has zero indeterminacy. Thus

$$x_{01}x_{174} = x_{31}x_{144} = \langle x_{123}, x_{11}, x_{01}\rangle x_{31} = \langle x_{123}, x_{11}, x_{01}x_{31}\rangle = \langle x_{123}, x_{11}, x_{31}\rangle x_{01}$$

Then the lemma follows form the fact that the multiplication of x_{01} on $E_2^{4,21}$ is an injection. □

Similarly, there are

Lemma 4.13. $x_{204} = \langle x_{153}, x_{11}, x_{31}\rangle$

Proof. Since $E_2^{3,20} = E_2^{2,7} = 0$, $\langle x_{153}, x_{11}, x_{01}x_{31}\rangle$ has zero indeterminacy. Thus

$$x_{01}x_{204} = x_{31}x_{174} = \langle x_{153}, x_{11}, x_{01}\rangle x_{31} = \langle x_{153}, x_{11}, x_{01}x_{31}\rangle = \langle x_{153}, x_{11}, x_{31}\rangle x_{01}$$

Then the lemma follows form the fact that the multiplication of x_{01} on $E_2^{4,24}$ is an injection. □

Proof of proposition 4.11.

$$x_{123}x_{255} = x_{123}\langle x_{11}, x_{31}, x_{204}\rangle = \langle x_{123}, x_{11}, x_{31}\rangle x_{204} = x_{174}x_{204}$$

□

Proposition 4.14. $x_{153}x_{255} = x_{204}^2$

Proof.

$$x_{153}x_{255} = x_{153}\langle x_{11}, x_{31}, x_{204}\rangle = \langle x_{153}, x_{11}, x_{31}\rangle x_{204} = x_{204}^2$$

□

Proposition 4.15. $x_{123}x_{327} = 0$

Lemma 4.16. $x_{123} = \langle x_{01}, x_{11}, x_{31}, x_{31}^2\rangle$

Proof. Since $E_2^{2,7} = E_2^{3,14} = 0$, the lower threefold products of $\langle x_{01}, x_{11}, x_{31}, x_{31}^2\rangle$ are 0, hence it is strictly defined. Since $E_2^{1,7} = E_2^{2,14} = 0$, it has zero indeterminacy. $\langle h_{10}, h_{11}, h_{12}, h_{12}^2\rangle$ is defined in May E_4-page, and there are no crossing differentials. It has a defining system:

$$\begin{array}{ccccccc} h_{10} & & h_{11} & & h_{12} & & h_{12}^2 \\ & 0 & & 0 & & h_{21}^2 & \\ & & 0 & & h_{30}^2 & & \end{array}$$

So x_{123}, the element to which $h_{10}h_{30}^2$ converges, must belong to the Massey product $\langle x_{01}, x_{11}, x_{31}, x_{31}^2\rangle$, which has exactly one element. □

Proof of proposition 4.15.

$$x_{123}x_{327} = \langle x_{01}, x_{11}, x_{31}, x_{31}^2\rangle x_{327} \subset \langle x_{01}, x_{11}, x_{31}, x_{31}^2 x_{327}\rangle = \langle x_{01}, x_{11}, x_{31}, 0\rangle$$

Since $E_2^{2,7} = E_2^{10,53} = 0$, the lower threefold products of the right-hand-side are 0, hence it is strictly defined. Since $E_2^{1,7} = E_2^{9,53} = 0$, it has zero indeterminacy. However, there is a defining system $\{a_{i,j}\}$ with $a_{i,4} = 0$, hence the only element it contains is 0, thus $x_{123}x_{327} = 0$. □

Proposition 4.17. $x_{144}x_{327} = 0$

Proof.
$$x_{144}x_{327} = \langle x_{01}, x_{11}, x_{123}\rangle x_{327} \subset \langle x_{01}, x_{11}, x_{123}x_{327}\rangle = \langle x_{01}, x_{11}, 0\rangle = 0$$
The last equality holds since $E_2^{1,3} = E_2^{10,56} = 0$. □

Proposition 4.18. $x_{153}x_{327} = 0$

Proof.
$$x_{153}x_{327} = x_{153}\langle x_{83}, x_{204}, x_{31}\rangle = \langle x_{153}, x_{83}, x_{204}\rangle x_{31}$$
where $\langle x_{153}, x_{83}, x_{204}\rangle \in E_2^{9,53} = 0$. □

Proposition 4.19. $x_{01}^4 x_{488} + x_{123}^4 = x_{84}x_{204}^2$

Proof. Notice that $x_{123}^4 x_{255} = x_{123}^3 x_{174}x_{204} \neq 0$, while $x_{01}^4 x_{488}x_{255} = 0$, hence $x_{01}^4 x_{488} + x_{123}^4 \neq 0$. The only possibility is that $x_{01}^4 x_{488} + x_{123}^4 = x_{84}x_{204}^2$. □

Proposition 4.20. $x_{174}x_{327} = 0$

Proof.
$$x_{174}x_{327} = \langle x_{01}, x_{11}, x_{153}\rangle x_{327} \subset \langle x_{01}, x_{11}, x_{153}x_{327}\rangle = \langle x_{01}, x_{11}, 0\rangle = 0$$
The last equality holds since $E_2^{1,3} = E_2^{10,59} = 0$. □

Proposition 4.21. $x_{11}^2 x_{488} + x_{255}^2 = x_{153}^2 x_{204}$

Proof. Notice that $x_{11}^2 x_{488} x_{123}^2 = 0$, while $x_{255}^2 x_{123}^2 = x_{174}^2 x_{204}^2 \neq 0$, hence $x_{11}^2 x_{488} + x_{255}^2 \neq 0$. The only possibility is that $x_{11}^2 x_{488} + x_{255}^2 = x_{153}^2 x_{204}$. □

Proposition 4.22. $x_{204}x_{327} = 0$

Proof.
$$x_{204}x_{327} = \langle x_{31}, x_{11}, x_{153}\rangle x_{327} \subset \langle x_{31}, x_{11}, x_{153}x_{327}\rangle = \langle x_{31}, x_{11}, 0\rangle = 0$$
The last equality holds since $E_2^{1,6} = E_2^{10,59} = 0$. □

Proposition 4.23. $x_{11}x_{83}x_{488} + x_{255}x_{327} = 0$

Proof. If $x_{11}x_{83}x_{488} + x_{255}x_{327} \neq 0$, then $x_{11}x_{83}x_{488} + x_{255}x_{327} = x_{174}x_{204}^2$. However, $x_{11}x_{83}x_{488}x_{204} = x_{255}x_{327}x_{204} = 0$, while $x_{174}x_{204}^3 \neq 0$, which prevent this from happening. □

Proposition 4.24. $x_{327}^2 = 0$

Proof.
$$x_{327}^2 = x_{327}\langle x_{83}, x_{204}, x_{31}\rangle = \langle x_{327}x_{83}, x_{204}, x_{31}\rangle = \langle 0, x_{204}, x_{31}\rangle = 0$$
The last equality holds since $E_2^{13,74} = E_2^{4,28} = 0$. □

5 Differentials in the Adams Spectral Sequence

After accomplishment of the computation of extension problems in May spectral sequence, we now get the full structure of the Adams E_2-page as an algebra. Actually, it is a free $\mathbb{F}_2[x_{84}, x_{488}]$-module (see table A.1 for its basis) and a finitely-generated algebra. Now we will determine the differentials.

The diffrentials of other generators as an algebra are 0 because of degree reasons except $x_{11}, x_{123}, x_{153}, x_{255}, x_{488}$.

Proposition 5.1. *x_{11} is a permanent cycle.*

Proof. Suppose that x_{11} supports a nontrivial differential, say $d_r(x_{11}) = x_{01}^{r+1}$, then $0 = d_r(x_{01}x_{11}) = x_{01}^{r+2}$, which leads to a contradiction. □

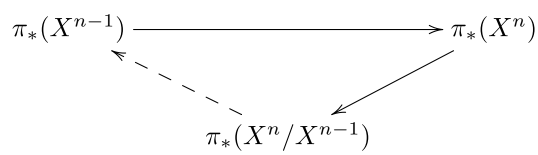
Proposition 5.2. $d_2(x_{255}) = 0$

Proof. Suppose that x_{255} supports a nontrivial d_2, then $d_2(x_{255}) = x_{01}x_{123}^2$. However, $x_{01}x_{255} = 0$ while $x_{01}^2 x_{123}^2 \neq 0$, which leads to a contradiction. □

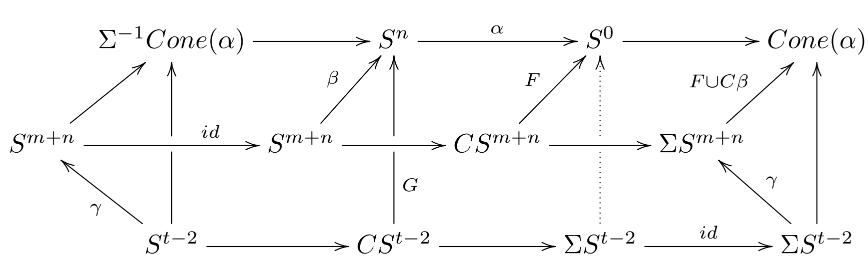
To achieve other differentials, we may refer to Atiyah-Hirzebruch spectral sequence (AHSS).

Theorem 5.3 (AHSS)([AH61])**.** *For a spectrum E and a CW spectrum X, there is a spectral sequence with $E^2_{p,q} = H_p(X; E_q(pt))$ converging strongly to $E_*(X)$.* □

Let E be the sphere spectrum, then $E^2_{p,q} = H_p(X; \pi^S_q)$ where π^S_* is the stable homotopy of spheres. It is actually induced by the exact couple



It can be seen that the construction of the differentials in AHSS exactly coincides with the construction of Toda brackets. For example, let X be a CW spectrum with three cells in the dimension 0, $n+1$, $n+m+2$ whose attaching maps can be detected by $\alpha \in \pi^S_n$ and $\beta \in \pi^S_m$ respectively. Let $x[n]$ denote the element in $\pi_{n+t}(S^n)$ for an n-cell and $x \in \pi^S_t$. Consider the differentials of $\gamma[m+n+2] \in \pi_t(X^{m+n+2}/X^{m+n+1}) = \pi_t(S^{m+n+2})$. Since the attaching map is detected by β, $d_{m+1}(\gamma[m+n+2]) = \beta\gamma[n+1]$. If $\beta\gamma = 0 \in \pi^S_*$, $d_{m+n+2}(\gamma[m+n+2])$ is constructed by

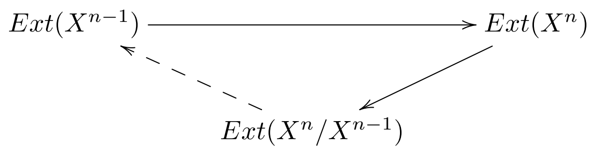


where F is the homotopy between $\alpha\beta$ and 0, and G is the homotopy between $\beta\gamma$ and 0. Hence $d_{m+n+2}(\gamma[m+n+2]) = \langle \alpha, \beta, \gamma \rangle[0]$.

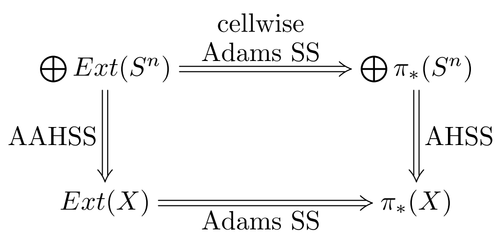
Actually, there is an analogy of AHSS called algebraic Atiyah-Hirzebruch spectral sequence (AAHSS) defined for CW spectrum X with short exact sequences

$$0 \to H^*(X^n/X^{n-1}) \to H^*(X^n) \to H^*(X^{n-1}) \to 0$$

The spectral sequence is then induced by the exact couple



where $Ext(X^n) = Ext_A(H^*(X^n), \mathbb{F}_2)$. The differentials in this spectral sequence, similarly, coincide with Massey products. A trick due to Mahowald is the Mahowald square as below:



Mahowald Square

Then we can finish the computations of d_2 in Adams SS.

Proposition 5.4. $d_2(x_{123}) = x_{31}x_{84}$

Proof. Notice that tmf is a CW spectrum with $H^*(tmf) = A \otimes_{A(2)} \mathbb{F}_2$, by minimal cell structure theorem, the 13-skeleton of tmf is homotopy equivalent to a CW spectrum with three cells on dimension 0, 8, 12. Consider the action of Steenrod algebra on H^*, their attaching maps can be detected by Sq^8 and Sq^4 respectively. Thus the attaching maps are (odd times of) σ and ν. Consider tha differentials of 8[12]. $d_4(8[12]) = 8\nu[8] = 0$, then turn to d_{12}:

$$\begin{aligned}
d_{12}(8[12]) &= \langle 8, \nu, \sigma\rangle[0] && (ind = 8\pi_{11}^S + \sigma\pi_4^S = 0) \\
&\subset \langle 2, 4\nu, \sigma\rangle[0] && (ind = 2\pi_{11}^S + \sigma\pi_4^S = ind\{Ph_2\}) \\
&= (\langle 2, \eta^3, \sigma\rangle + \langle 2, \eta, \nu\rangle\nu^2)[0] && (ind = ind\{Ph_2\} + 0) \\
&= (\langle 2, \eta, \eta^2\sigma\rangle + \langle 2, \eta, \nu^3\rangle)[0] && (ind = ind\{Ph_2\} + ind\{Ph_2\}) \\
&= \langle 2, \eta, \eta\epsilon\rangle[0] && \text{[Tod62]}(ind = ind\{Ph_2\}) \\
&= \{Ph_2\}[0]
\end{aligned}$$

The last equality is deduced by using May's convergence theorem and Moss's convergence theorem on $\langle h_{10}, h_{11}, h_{11}^2 h_0(1)\rangle = h_{12}h_{20}^4$ in May E_4-page. Thus $x_{31}x_{84}$ can not survive, the only possibility is that $d_2(x_{123}) = x_{31}x_{84}$. ☐

Proposition 5.5. $d_2(x_{153}) = x_{01}x_{144}$

Proof.

$$x_{01}d_2(x_{153}) = d_2(x_{01}x_{153}) = d_2(x_{31}x_{123}) = x_{31}^2x_{84} = x_{01}^2x_{144}$$

Then the proposition follows from the fact that the multiplication of x_{01} on $E_2^{4,18}$ is an injection. □

Proposition 5.6. $d_2(x_{488}) = x_{153}^2x_{174}$

Proof. By information of Adams E_4-page, $d_4(x_{11}^2x_{488}) = d_4(x_{153}^2x_{204} + x_{255}^2) \neq 0$. However, if $d_2(x_{488}) = 0$, then x_{488} survives to E_5-page by degree reasons, then $d_4(x_{11}^2x_{488}) = 0$, which leads to a contradiction. Hence the only possibility is that $d_2(x_{488}) = x_{153}^2x_{174}$.

Note that the argument is not circular since we only need to use the information of E_4 in the range $t - s < 48$, the inference process of which is independent of $d_2(x_{488})$. □

So far we have computed all the differentials in Adams E_2-page.

Table 5.1: Adams E_2-page generators and their differentials

generators	differentials
x_{01}	0
x_{11}	0
x_{31}	0
x_{83}	0
x_{84}	0
x_{123}	$x_{31}x_{84}$
x_{144}	0
x_{153}	$x_{01}x_{144}$
x_{174}	0
x_{204}	0
x_{255}	0
x_{327}	0
x_{488}	$x_{153}^2x_{174}$

After listing all elements in Adams E_2-page and computing their differentials, we can list all the elements in Adams E_3-page and find its generators as an algebra:

$$\begin{aligned}\text{generators: } \{&x_{01},\ x_{11},\ x_{31},\ x_{83},\ x_{84},\ x_{01}^3x_{123},\ x_{144},\ x_{174},\\ &x_{204},\ x_{123}^2,\ x_{255},\ x_{123}x_{153},\ x_{153}^2,\ x_{153}x_{174},\\ &x_{327},\ x_{01}x_{123}^3,\ x_{01}x_{488},\ x_{11}x_{488},\ x_{31}x_{488},\\ &x_{83}x_{488},\ x_{01}^3x_{123}x_{488},\ x_{327}x_{488},\ x_{01}x_{123}^3x_{488},\ x_{488}^2\}\end{aligned}$$

On the premise that differentials of all generators with lower $t-s$ degree have been computed, differentials of other generators are 0 because of degree reasons except x_{174}, x_{123}^2, x_{153}^2, $x_{11}x_{488}$, x_{488}^2.

To compute $d_3(x_{174})$, we will first refer to the following theorem which gives a description of imJ.

Theorem 5.7 ([Ada66] and [Qui71]). *$J : \pi_k(SO) \to \pi_k^S$ is a monomorphism for $k \equiv 0$ or 1 mod 8 and $J(\pi_{4k-1}(SO))$ is a cyclic group whose 2-component is $\mathbb{Z}_2/(8k)$. If we denote by x_k the generator* *in dimension $4k-1$, then ηx_{2k} and $\eta^2 x_{2k}$ are generators of imJ in dimensions $8k$ and $8k+1$, respectively.* □

Proposition 5.8. $d_3(x_{174}) = x_{83}x_{84}$

Proof. By the previous theorem, π_{15}^S has a $\mathbb{Z}/32$ summand for imJ, whose generator $\rho_{15} \in \{h_0^3 h_4\}$ must support an η-extension. By the commutativity of π_*^S, notice that σ has an odd degree, $2\sigma^2$ must be zero, hence $h_0 h_3^2$ must be killed. The only possibility is that $d_2(h_4) = h_0 h_3^2$. Both $h_0 h_4$ and $h_0^2 h_4$ can not survive in Adams SS because of the order on imJ, thus they must support nontrivial d_3. Therefore d_0 detects an element $\kappa \in \pi_{14}^S$ with $2\kappa = 0$. By Toda brackets shuffling, $\eta^2\kappa = \langle 2, \eta, 2\rangle \kappa = 2\langle \eta, 2, \kappa\rangle \in \pi_{16}^S$, hence $\eta^2\kappa = 0$ and $\eta\rho_{15}$ is detected by Pc_0. The image of ρ_{15} under the Hurewicz map $\pi_*(S^0) \to \pi_*(tmf)$ is 0, hence $x_{83}x_{84}$ can not survive. The only possibility is that $d_3(x_{174}) = x_{83}x_{84}$. □

Proposition 5.9. $d_3(x_{123}^2) = x_{11}x_{84}x_{144}$

Proof. Consider $h_0^4[Sq^8 Sq^{16}]$ in AAHSS, $d_{24}(h_0^4[Sq^8 Sq^{16}]) = \langle h_0^4, h_3, h_4\rangle[0]$, where $\langle h_0^4, h_3, h_4\rangle = h_{20}^4 h_4$ in May E_8-page and converges to $Ph_4 = h_2 g$ by May's convergence theorem. This differential passes to $d_{24}(16[Sq^8 Sq^{16}]) = \nu\bar{\kappa}[0]$ in AHSS by cellwise Adams SS according to Mahowald square. Then $d_{24}(64[Sq^8 Sq^{16}]) = 4\nu\bar{\kappa}[0]$ in AHSS. However, there exists an extension that $4\nu\bar{\kappa}$ can be detected by $h_1 P d_0$ [MT67], hence $\{h_1 P d_0\}[0]$ is killed in AHSS and $x_{11}x_{84}x_{144}$ is killed in Adams SS. The only possibility is that $d_3(x_{123}^2) = x_{11}x_{84}x_{144}$. □

We will introduce another method used frequently to determine Adams differentials that uses Steenrod operations in Adams SS.

Theorem 5.10 ([BMMS86])**.** *Let Y be an H_∞ ring spectrum, $x \in E_r^{s,t}$ in the Adams SS for Y. Then*

$$d_* Sq^j x = Sq^{j+r-1} d_r x \dot{+} T_2$$

For A with filtration degree s, B_1 with filtration degree $s+r_1$ and B_2 with filtration degree $s+r_2$,

$$d_* A = B_1 \dot{+} B_2 \iff \begin{cases} d_{r_1} A = B_1, & \text{if } r_1 < r_2; \\ d_r A = B_1 + B_2, & \text{if } r_1 = r = r_2; \\ d_{r_2} A = B_2, & \text{if } r_1 > r_2. \end{cases}$$

$$T_2 = \begin{cases} 0, & v > k+1 \text{ or } 2r-2 < v < k; \\ \bar{a} x d_r x, & v = k+1; \\ \bar{a} Sq^{j+v} x, & v = k \text{ or } (v < k \text{ and } v \le 10). \end{cases}$$

where $k = s-j$, $v = v_2(t-j) = 8p + 2^q$ if the exponent of 2 in the prime factorization of $t-j+1$ is $4p+q$ with $0 \le q \le 3$; $a = a_2(t-j) \in \pi_{v-1}^S$ which is the map of degree 2 if $v = 1$; and $\bar{a}$ is the element in Adams E_∞-page for S^0 detecting a. □

Notice that $Sq^s x = x^2$ for $x \in E^{s,t}$, we can compute $d_*(x^2)$ using the information $d_*(x)$.

Another proof of proposition 5.9.

$$d_*(x_{123}^2) = d_*(Sq^3 x_{123}) = Sq^4(d_2 x_{123}) \dot{+} T_2 = Sq^4(x_{31}x_{84}) \dot{+} T_2$$

Then in the previous theorem, $k = 0$, $v = 1$, and a is the map of degree 2, hence $T_2 = x_{01}x_{31}x_{84}x_{123} = x_{11}x_{84}x_{144}$. On the other hand, by Cartan formula,

$$Sq^4(x_{31}x_{84}) = Sq^1x_{31}Sq^3x_{84} + Sq^0x_{31}Sq^4x_{84}$$

where $Sq^0x_{31} \in E^{1,8} = 0$ and $Sq^3x_{84} \in E^{7,24} = 0$. Therefore $Sq^4(x_{31}x_{84}) = 0$, and $x_{11}x_{84}x_{144} = d_*(x_{123}^2) = d_3(x_{123}^2)$. □

Proposition 5.11. $d_3(x_{153}^2) = x_{11}x_{84}x_{204}$

Proof.

$$x_{84}d_3(x_{153}^2) = d_3(x_{84}x_{153}^2) = d_3(x_{123}^2x_{144}) = x_{11}x_{84}^2x_{204}$$

Hence $d_3(x_{153}^2) \neq 0$, the only possibility is that $d_3(x_{153}^2) = x_{11}x_{84}x_{204}$. □

Proposition 5.12. $d_3(x_{11}x_{488}) = x_{84}x_{204}^2$

Proof. The candidates for $d_3(x_{11}x_{488})$ are $\{0, x_{01}^4x_{488}, x_{123}^4, x_{84}x_{204}^2\}$. Notice that $x_{01}d_3(x_{11}x_{488}) = d_3(x_{01}x_{11}x_{488}) = 0$, $x_{01}^4x_{488}$ and x_{123}^4 are excluded. If $d_3(x_{11}x_{488}) = 0$, $x_{11}x_{488}$ survives to E_5-page, and hence $d_4(x_{11}^2x_{488}) = 0$, which leads to a contradiction. The only possibility is that $d_3(x_{11}x_{488}) = x_{84}x_{204}^2$. □

Proposition 5.13. $d_3(x_{488}^2) = x_{153}x_{204}^4$

Proof. Again, we refer to Bruner's method.

$$d_*(x_{488}^2) = d_*(Sq^8x_{488}) = Sq^9(d_2x_{488})\dot{+}T_2 = Sq^9(x_{153}^2x_{174})\dot{+}T_2$$

$k = 0$, $v = 1$, hence $T_2 = x_{01}x_{153}^2x_{174}x_{488} = 0$. On the other hand, by Cartan formula,

$$Sq^9(x_{153}^2x_{174}) = (Sq^3x_{153})^2Sq^3x_{174} = x_{153}^4Sq^3x_{174}$$

By [Mil72], $Sq^3e_0 = m$ in Adams SS for S^0. By naturality of squaring operations, $Sq^3x_{174} = x_{153}x_{204}$. Hence $x_{153}^5x_{204} = x_{153}x_{204}^4 = d_*(x_{488}^2) = d_3(x_{488}^2)$. □

So far we have computed all the differentials in Adams E_3-page.

Table 5.2: Adams E_3-page generators and their differentials

generators	differentials
x_{01}	0
x_{11}	0
x_{31}	0
x_{83}	0
x_{84}	0
$x_{01}^3x_{123}$	0
x_{144}	0
x_{174}	$x_{83}x_{84}$
x_{204}	0
x_{123}^2	$x_{11}x_{84}x_{144}$

(to be continued)

(continued)	
generators	differentials
x_{255}	0
$x_{123}x_{153}$	0
x_{153}^2	$x_{11}x_{84}x_{204}$
$x_{153}x_{174}$	0
x_{327}	0
$x_{01}x_{123}^3$	0
$x_{01}x_{488}$	0
$x_{11}x_{488}$	$x_{84}x_{204}^2$
$x_{31}x_{488}$	0
$x_{83}x_{488}$	0
$x_{01}^3x_{123}x_{488}$	0
$x_{327}x_{488}$	0
$x_{01}x_{123}^3x_{488}$	0
x_{488}^2	$x_{153}x_{204}^4$

After computing differentials in Adams E_3-page (see table A.2), we can list all the elements in Adams E_4-page and find its generators as an algebra:

$$\begin{aligned}\text{generators: } \{&x_{01},\ x_{11},\ x_{31},\ x_{83},\ x_{84},\ x_{01}^3x_{123},\ x_{144},\ x_{204},\\ &x_{01}x_{123}^2,\ x_{255},\ x_{123}x_{153},\ x_{144}x_{174},\ x_{153}x_{174},\\ &x_{327},\ x_{01}x_{123}^3,\ x_{174}x_{204},\ x_{144}x_{153}^2,\ x_{01}x_{488},\\ &x_{153}x_{174}^2,\ x_{153}^2x_{204},\ x_{31}x_{488},\ x_{153}x_{204}^2,\\ &x_{83}x_{488},\ x_{01}^3x_{123}x_{488},\ x_{01}x_{123}^2x_{488},\ x_{327}x_{488},\\ &x_{01}x_{123}^3x_{488},\ x_{01}x_{488}^2,\ x_{11}x_{488}^2,\ x_{31}x_{488}^2,\ x_{83}x_{488}^2,\\ &x_{84}x_{488}^2,\ x_{01}^3x_{123}x_{488}^2,\ x_{144}x_{488}^2,\ x_{01}x_{123}^2x_{488}^2,\\ &x_{123}x_{153}x_{488}^2,\ x_{144}x_{174}x_{488}^2,\ x_{153}x_{174}x_{488}^2,\ x_{327}x_{488}^2,\\ &x_{01}x_{123}^3x_{488}^2,\ x_{174}x_{204}x_{488}^2,\ x_{144}x_{153}^2x_{488}^2,\ x_{01}x_{488}^3,\\ &x_{153}x_{174}^2x_{488}^2,\ x_{11}^2x_{488}^3,\ x_{31}x_{488}^3,\ x_{83}x_{488}^3,\\ &x_{01}^3x_{123}x_{488}^3,\ x_{01}x_{123}^2x_{488}^3,\ x_{327}x_{488}^3,\ x_{01}x_{123}^3x_{488}^3,\ x_{488}^4\}\end{aligned}$$

In Adams E_4-page, differentials of other generators are 0 because of degree reasons except the ones computed in the following propositions.

Proposition 5.14. $d_4(x_{174}x_{204}) = x_{84}^2x_{204}$

Proof. By [MT67], $d_4(e_0g) = Pd_0^2$ in Adams SS for S^0, hence by naturality, $d_4(x_{174}x_{204}) = x_{84}^2x_{204}$. □

Proposition 5.15. $d_4(x_{144}x_{174}) = x_{84}^2x_{144}$

Proof.

$$x_{204}d_4(x_{144}x_{174}) = d_4(x_{144}x_{174}x_{204}) = x_{84}^2x_{144}x_{204}$$

thus $d_4(x_{144}x_{174}) \neq 0$, the only possibility is that $d_4(x_{144}x_{174}) = x_{84}^2x_{144}$. □

Proposition 5.16. $d_4(x_{144}x_{153}^2) = x_{84}^2x_{123}x_{153}$

Proof.
$$x_{144}d_4(x_{144}x_{153}^2) = d_4(x_{144}^2x_{153}^2) = d_4(x_{123}x_{153}x_{144}x_{174}) = x_{84}^2x_{123}x_{144}x_{153}$$
thus $d_4(x_{144}x_{153}^2) \neq 0$, the only possibility is that $d_4(x_{144}x_{153}^2) = x_{84}^2x_{123}x_{153}$. □

Proposition 5.17. $d_4(x_{01}x_{488}) = x_{84}x_{123}^2x_{153}$

Proof.
$$x_{144}d_4(x_{01}x_{488}) = d_4(x_{01}x_{144}x_{488}) = d_4(x_{174}x_{204}x_{255}) = x_{84}^2x_{153}^3$$
where the second equality holds since $d_2(x_{153}x_{488}) = x_{01}x_{144}x_{488}+x_{174}x_{204}x_{255}$. Hence $d_4(x_{01}x_{488}) \neq 0$, the only possibility is that $d_4(x_{01}x_{488}) = x_{84}x_{123}^2x_{153}$. □

Proposition 5.18. $d_4(x_{153}x_{174}^2) = x_{84}^2(x_{153}x_{174} + x_{327})$

Proof. The candidates for $d_4(x_{153}x_{174}^2)$ are spanned by $x_{84}^2x_{153}x_{174}$, $x_{84}^2x_{327}$, and $x_{01}^7x_{488}$. Notice that $x_{01}d_4(x_{153}x_{174}^2) = d_4(x_{01}x_{153}x_{174}^2) = 0$, the remaining candidates are 0 and $x_{84}^2(x_{153}x_{174} + x_{327})$. However, it can not be zero since $x_{144}d_4(x_{153}x_{174}^2) = d_4(x_{144}x_{174}x_{153}x_{174}) = x_{84}^2x_{144}x_{153}x_{174}$. □

Proposition 5.19. $d_4(x_{153}^2x_{204}) = x_{84}x_{123}x_{144}x_{153}$

Proof.
$$x_{84}d_4(x_{153}^2x_{204}) = d_4(x_{84}x_{153}^2x_{204}) = d_4(x_{123}x_{153}x_{144}x_{174}) = x_{84}^2x_{123}x_{144}x_{153}$$
thus $d_4(x_{153}^2x_{204}) \neq 0$, the only possibility is that $d_4(x_{153}^2x_{204}) = x_{84}x_{123}x_{144}x_{153}$. □

Proposition 5.20. $d_4(x_{153}x_{204}^2) = x_{84}x_{144}x_{153}x_{174}$

Proof.
$$x_{84}d_4(x_{153}x_{204}^2) = d_4(x_{84}x_{153}x_{204}^2) = d_4(x_{144}x_{174}x_{153}x_{174}) = x_{84}^2x_{144}x_{153}x_{174}$$
thus $d_4(x_{153}x_{204}^2) \neq 0$, the only possibility is that $d_4(x_{153}x_{204}^2) = x_{84}x_{144}x_{153}x_{174}$. □

Proposition 5.21. $d_4(x_{144}x_{174}x_{488}^2) = x_{84}^2x_{144}x_{488}^2$

Proof.
$$x_{144}d_4(x_{144}x_{174}x_{488}^2) = d_4(x_{144}^2x_{174}x_{488}^2) = d_4(x_{144}x_{174}x_{144}x_{488}^2) = x_{84}^3x_{204}x_{488}^2$$
thus $d_4(x_{144}x_{174}x_{488}^2) \neq 0$, the only possibility is that $d_4(x_{144}x_{174}x_{488}^2) = x_{84}^2x_{144}x_{488}^2$. □

Proposition 5.22. $d_4(x_{174}x_{204}x_{488}^2) = x_{84}^2x_{204}x_{488}^2$

Proof.
$$x_{144}d_4(x_{174}x_{204}x_{488}^2) = d_4(x_{144}x_{174}x_{204}x_{488}^2) = d_4(x_{174}x_{204}x_{144}x_{488}^2) = x_{84}^2x_{174}^2x_{488}^2$$
thus $d_4(x_{174}x_{204}x_{488}^2) \neq 0$, the only possibility is that $d_4(x_{174}x_{204}x_{488}^2) = x_{84}^2x_{204}x_{488}^2$. □

Proposition 5.23. $d_4(x_{144}x_{153}^2x_{488}^2) = x_{84}^2x_{123}x_{153}x_{488}^2$

Proof.
$$x_{144}d_4(x_{153}^2x_{144}x_{488}^2) = d_4(x_{144}x_{153}^2x_{144}x_{488}^2) = x_{84}^2x_{123}x_{144}x_{153}x_{488}^2$$
thus $d_4(x_{144}x_{153}^2x_{488}^2) \neq 0$, the only possibility is that $d_4(x_{144}x_{153}^2x_{488}^2) = x_{84}^2x_{123}x_{153}x_{488}^2$. □

Proposition 5.24. $d_4(x_{01}x_{488}^3) = x_{84}x_{123}^2x_{153}x_{488}^2$

Proof.
$$x_{144}d_4(x_{01}x_{488}^3) = d_4(x_{144}x_{01}x_{488}^3) = d_4(x_{01}x_{488}x_{144}x_{488}^2) = x_{84}x_{153}^3x_{488}^2$$
thus $d_4(x_{01}x_{488}^3) \neq 0$, the only possibility is that $d_4(x_{01}x_{488}^3) = x_{84}x_{123}^2x_{153}x_{488}^2$. □

Proposition 5.25. $d_4(x_{153}x_{174}^2x_{488}^2) = x_{84}^2(x_{153}x_{174} + x_{327})x_{488}^2$

Proof. The candidates for $d_4(x_{153}x_{174}^2x_{488}^2)$ are spanned by $x_{84}^2x_{153}x_{174}x_{488}^2$, $x_{84}^2x_{327}x_{488}^2$, and $x_{01}^7x_{488}^3$. Since $x_{01}d_4(x_{153}x_{174}^2x_{488}^2) = d_4(x_{01}x_{153}x_{174}^2x_{488}^2) = 0$, the remaining candidates are 0 and $x_{84}^2(x_{153}x_{174}+x_{327})x_{488}^2$. However, it can not be zero since

$$x_{144}d_4(x_{153}x_{174}^2x_{488}^2) = d_4(x_{153}x_{174}^2x_{144}x_{488}^2) = x_{84}^2x_{144}x_{153}x_{174}x_{488}^2$$

□

Proposition 5.26. $d_4(x_{11}^2x_{488}^3) = x_{84}x_{123}x_{144}x_{153}x_{488}^2$

Proof.

$$\begin{aligned} x_{84}d_4(x_{11}^2x_{488}^3) &= d_4(x_{84}x_{11}^2x_{488}^3) = d_4(x_{84}x_{255}^2x_{488}^2) + d_4(x_{84}x_{153}^2x_{204}x_{488}^2) \\ &= d_4(x_{255}^2x_{84}x_{488}^2)d_4(x_{144}x_{174}x_{123}x_{153}x_{488}^2) \\ &= 0 + x_{84}^2x_{123}x_{144}x_{153}x_{488}^2 = x_{84}^2x_{123}x_{144}x_{153}x_{488}^2 \end{aligned}$$

thus $d_4(x_{11}^2x_{488}^3) \neq 0$, the only possibility is that $d_4(x_{11}^2x_{488}^3) = x_{84}x_{123}x_{144}x_{153}x_{488}^2$. □

So far we have computed all the differentials in Adams E_4-page.

Table 5.3: Adams E_4-page generators and their differentials

generators	differentials
x_{01}	0
x_{11}	0
x_{31}	0
x_{83}	0
x_{84}	0
$x_{01}^3x_{123}$	0
x_{144}	0
x_{204}	0
$x_{01}x_{123}^2$	0
x_{255}	0
$x_{123}x_{153}$	0

(to be continued)

(continued)

generators	differentials
$x_{144}x_{174}$	$x_{84}^2x_{144}$
$x_{153}x_{174}$	0
x_{327}	0
$x_{01}x_{123}^3$	0
$x_{174}x_{204}$	$x_{84}^2x_{204}$
$x_{144}x_{153}^2$	$x_{84}^2x_{123}x_{153}$
$x_{01}x_{488}$	$x_{84}x_{123}^2x_{153}$
$x_{153}x_{174}^2$	$x_{84}^2(x_{153}x_{174}+x_{327})$
$x_{153}^2x_{204}$	$x_{84}x_{123}x_{144}x_{153}$
$x_{31}x_{488}$	0
$x_{153}x_{204}^2$	$x_{84}x_{144}x_{153}x_{174}$
$x_{83}x_{488}$	0
$x_{01}^3x_{123}x_{488}$	0
$x_{01}x_{123}^2x_{488}$	0
$x_{327}x_{488}$	0
$x_{01}x_{123}^3x_{488}$	0
$x_{01}x_{488}^2$	0
$x_{11}x_{488}^2$	0
$x_{31}x_{488}^2$	0
$x_{83}x_{488}^2$	0
$x_{84}x_{488}^2$	0
$x_{01}^3x_{123}x_{488}^2$	0
$x_{144}x_{488}^2$	0
$x_{01}x_{123}^2x_{488}^2$	0
$x_{123}x_{153}x_{488}^2$	0
$x_{144}x_{174}x_{488}^2$	$x_{84}^2x_{144}x_{488}^2$
$x_{153}x_{174}x_{488}^2$	0
$x_{327}x_{488}^2$	0
$x_{01}x_{123}^3x_{488}^2$	0
$x_{174}x_{204}x_{488}^2$	$x_{84}^2x_{204}x_{488}^2$
$x_{144}x_{153}^2x_{488}^2$	$x_{84}^2x_{123}x_{153}x_{488}^2$
$x_{01}x_{488}^3$	$x_{84}x_{123}^2x_{153}x_{488}^2$
$x_{153}x_{174}^2x_{488}^2$	$x_{84}^2(x_{153}x_{174}+x_{327})x_{488}^2$
$x_{11}^2x_{488}^3$	$x_{84}x_{123}x_{144}x_{153}x_{488}^2$
$x_{31}x_{488}^3$	0
$x_{83}x_{488}^3$	0
$x_{01}^3x_{123}x_{488}^3$	0
$x_{01}x_{123}^2x_{488}^3$	0
$x_{327}x_{488}^3$	0
$x_{01}x_{123}^3x_{488}^3$	0
x_{488}^4	0

After computing differentials in Adams E_4-page (see table A.3), we can list all the elements in Adams E_5-page and find its generators as an algebra:

$$\begin{aligned}\text{generators: } \{ & x_{01},\ x_{11},\ x_{31},\ x_{83},\ x_{84},\ x_{01}^3 x_{123},\ x_{144},\ x_{204}, \\ & x_{01}x_{123}^2,\ x_{255},\ x_{123}x_{153},\ x_{153}x_{174},\ x_{327},\ x_{01}x_{123}^3, \\ & x_{01}^2 x_{488},\ x_{31}x_{488},\ x_{83}x_{488},\ x_{01}x_{84}x_{488} + x_{123}x_{144}x_{153}^2, \\ & x_{01}^3 x_{123}x_{488},\ x_{01}x_{123}^2 x_{488},\ x_{327}x_{488},\ x_{01}x_{123}^3 x_{488}, \\ & x_{01}x_{488}^2,\ x_{11}x_{488}^2,\ x_{31}x_{488}^2,\ x_{83}x_{488}^2,\ x_{84}x_{488}^2, \\ & x_{01}^3 x_{123}x_{488}^2,\ x_{144}x_{488}^2,\ x_{01}x_{123}^2 x_{488}^2,\ x_{123}x_{153}x_{488}^2, \\ & x_{153}x_{174}x_{488}^2,\ x_{327}x_{488}^2,\ x_{01}x_{123}^3 x_{488}^2,\ x_{01}^2 x_{488}^3, \\ & x_{31}x_{488}^3,\ x_{83}x_{488}^3,\ x_{01}x_{84}x_{488}^3 + x_{123}x_{144}x_{153}^2 x_{488}^2, \\ & x_{01}^3 x_{123}x_{488}^3,\ x_{01}x_{123}^2 x_{488}^3,\ x_{327}x_{488}^3,\ x_{01}x_{123}^3 x_{488}^3,\ x_{488}^4 \}\end{aligned}$$

Notice that for a permanent cycle x with $xy = 0$, there is $xd_r(y) = 0$. By letting $x = x_{01}^n$, x_{84}^n, or $x_{84}x_{488}^2$ (when analyzing $d_*(x_{11}x_{488}^2)$), and by degree reasons, all generators in E_5-page cannot support higher differentials, hence the E_5-page is the E_∞-page (see table A.4). E_∞-page is a free $\mathbb{F}_2[x_{488}^4]$-module, and there is no relations involving x_{488}^4. Therefore, there is an element $x[192, 32] \in \pi_{192}(tmf)$ to which x_{488}^4 converges, such that $\forall y \in \pi_*(tmf)$ nontrivial, $x[192, 32]y$ is also nontrivial.

6 Extensions in the Adams Spectral Sequence

The only problem remained in the computation of $\pi_*(tmf)$ is the extension problem in the Adams SS.

To distinguish the elements in Adams SS and the homotopy ring, we will use $x[i,j]$ or $y[i,j]$ to represent the element in $\pi_i(tmf)$ with Adams filtration degree j to which an generator of Adams E_∞-page converges. Explicitly,

Table 6.1: Names of elements in homotopy and Adams SS

homotopy	Adams SS
2	x_{01}
$y[1,1]$	x_{11}
$y[3,1]$	x_{31}
$y[8,3]$	x_{83}
$x[8,4]$	x_{84}
$x[12,6]$	$x_{01}^3x_{123}$
$y[14,4]$	x_{144}
$y[20,4]$	x_{204}
$x[24,7]$	$x_{01}x_{123}^2$
$y[25,5]$	x_{255}
$y[27,6]$	$x_{123}x_{153}$
$x[32,7]$	x_{327}
$y[32,7]$	$x_{327}+x_{153}x_{174}$
$x[36,10]$	$x_{01}x_{123}^3$
$x[48,10]$	$x_{01}^2x_{488}$
$y[51,9]$	$x_{31}x_{488}$
$x[56,11]$	$x_{83}x_{488}$
$x[56,13]$	$x_{01}x_{84}x_{488}+x_{123}x_{144}x_{153}^2$
$x[60,14]$	$x_{01}^3x_{123}x_{488}$
$x[72,15]$	$x_{01}x_{123}^2x_{488}$
$x[80,15]$	$x_{327}x_{488}$
$x[84,18]$	$x_{01}x_{123}^3x_{488}$
$x[96,17]$	$x_{01}x_{488}^2$
$y[97,17]$	$x_{11}x_{488}^2$
$y[99,17]$	$x_{31}x_{488}^2$
$y[104,19]$	$x_{83}x_{488}^2$
$x[104,20]$	$x_{84}x_{488}^2$
$x[108,22]$	$x_{01}^3x_{123}x_{488}^2$
$y[110,20]$	$x_{144}x_{488}^2$
$x[120,23]$	$x_{01}x_{123}^2x_{488}^2$
$y[123,22]$	$x_{123}x_{153}x_{488}^2$
$x[128,23]$	$x_{327}x_{488}^2$
$y[128,23]$	$(x_{327}+x_{153}x_{174})x_{488}^2$
$x[132,26]$	$x_{01}x_{123}^3x_{488}^2$
$x[144,26]$	$x_{01}^2x_{488}^3$

(to be continued)

(continued)

homotopy	Adams SS
$y[147,25]$	$x_{31}x_{488}^3$
$x[152,27]$	$x_{83}x_{488}^3$
$x[152,29]$	$x_{01}x_{84}x_{488}^3 + x_{123}x_{144}x_{153}^2x_{488}^2$
$x[156,30]$	$x_{01}^3x_{123}x_{488}^3$
$x[168,31]$	$x_{01}x_{123}^2x_{488}^3$
$x[176,31]$	$x_{327}x_{488}^3$
$x[180,34]$	$x_{01}x_{123}^3x_{488}^3$
$x[192,32]$	x_{488}^4

As shown in [DFHH14], there is a ring homomorphism $\phi : \pi_*(tmf) \to MF_*$, where $MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$ is the ring of classical modular forms. The cokernel of the map can be described explicitly as

$$coker(\phi) \otimes \mathbb{Z}_2 = \begin{cases} \mathbb{Z}/\frac{8}{gcd(8,k)}, & n = 24k; \\ (\mathbb{Z}/2)^{[\frac{n+12}{24}]}, & n \equiv 4 \ (mod\ 8); \\ 0, & otherwise. \end{cases}$$

which are generated by Δ^k and $\Delta^a c_4^b c_6$ respectively.

Consider the elements with nontrivial image under this homomorphism. In dimension 8, there must be a generator of the homotopy ring with image c_4. Notice that c_4^2 is in the image, while $\pi_{16}(tmf)$ is generated by $x[8,4]^2$, $x[8,4]$ must have nontrivial image. Meanwhile, $y[8,3]$ can be chosen to be the image of $\epsilon \in \pi_8(S^0)$ under the Hurwicz homomorphism, then $2\epsilon = 0$, and $2y[8,3]$ is trivial. Then $y[8,3]$ has the trivial image under ϕ.

Similarly, $x[12,6]$ and $x[24,7]$ have image $2c_6$ and 8Δ respectively. In dimension 32, there should be a generator of the homotopy ring with image $c_4\Delta$. Both $2x[32,7]$ and $2(x[32,7]+y[32,7])$ are not zero, then we can change $y[32,7]$ to be the representative of $x_{327} + x_{153}x_{174}$ such that $2y[32,7] = 0$, and hence $y[32,7]$ cannot have nontrivial image under ϕ. (Actually, we will show that the $y[32,7]$ chosen by this is in the image of Hurwicz homomorphism.) Therefore, the image of $x[32,7]$ is $c_4\Delta$, and $8x[32,7]$ has image $8c_4\Delta$, hence supports an extension. All elements with higher filtration degree has nontrivial image, and the only one with image $8c_4\Delta$ is $x[8,4]x[24,7]$. Then we get the extension $8x[32,7] = x[8,4]x[24,7]$.

It can be similarly checked that all $x[i,j]$'s have nontrivial image, and all $y[i,j]$'s have trivial image. All extensions only involving $x[i,j]$'s can be derived similarly since all candidates for extensions have nontrivial image.

It needs to be note that $x[56,13] = 2x[56,11]$ and $x[152,29] = 2x[152,27]$. Therefore, they are not generators of the homotopy ring.

Then we only need to deal with extension problems involving $y[i,j]$'s. All relations in Adams E_∞-page should be considered, which are listed as follows:

Table 6.2: Extension problems in Adams SS

relations	$(t-s,s)$	extensions	proof
$2y[1,1]$	$(1,2)$	0	degree reasons
$4y[3,1]+y[1,1]^3$	$(3,3)$	0	degree reasons
$y[1,1]y[3,1]$	$(4,2)$	0	degree reasons
$2y[3,1]^2$	$(6,3)$	0	degree reasons
$2y[8,3]$	$(8,4)$	0	proved above
$y[3,1]^3$	$(9,3)$	$y[1,1]y[8,3]$	proposition 6.1
$y[1,1]^2y[8,3]$	$(10,5)$	0	$x[8,4]$-stable
$y[3,1]y[8,3]$	$(11,4)$	0	degree reasons
$y[3,1]x[8,4]$	$(11,5)$	0	degree reasons
$y[1,1]x[12,6]$	$(13,7)$	0	degree reasons
$2y[14,4]$	$(14,5)$	0	degree reasons
$y[3,1]x[12,6]$	$(15,7)$	0	degree reasons
$y[8,3]^2$	$(16,6)$	0	$im(\phi)$
$y[1,1]^2y[14,4]$	$(16,6)$	0	$im(\phi)$
$y[8,3]x[8,4]$	$(16,7)$	0	$im(\phi)$
$4y[20,4]+y[3,1]^2y[14,4]$	$(20,6)$	0	changing representatives
$y[8,3]x[12,6]$	$(20,9)$	0	$im(\phi)$
$y[1,1]^2y[20,4]$	$(22,6)$	$x[8,4]y[14,4]$	$\pi_*(S^0)$
$y[8,3]y[14,4]$	$(22,7)$	$x[8,4]y[14,4]$	proposition 6.2
$y[3,1]y[20,4]$	$(23,5)$	0	degree reasons
$y[1,1]x[8,4]y[14,4]$	$(23,9)$	0	degree reasons
$2y[25,5]$	$(25,6)$	0	$y[1,1]$-product
$y[1,1]x[24,7]$	$(25,8)$	0	$x[8,4]$-product
$x[12,6]y[14,4]$	$(26,10)$	0	$x[8,4]$-stable
$2y[27,6]+y[1,1]^2y[25,5]$	$(27,7)$	0	degree reasons
$y[3,1]x[24,7]$	$(27,8)$	0	degree reasons
$y[3,1]y[25,5]$	$(28,6)$	$y[14,4]^2$	proposition 6.3
$y[8,3]y[20,4]$	$(28,7)$	$y[14,4]^2$	proposition 6.3
$y[1,1]y[27,6]$	$(28,7)$	$y[14,4]^2$	proposition 6.3
$x[8,4]y[20,4]+y[14,4]^2$	$(28,8)$	0	$im(\phi)$
$y[1,1]y[14,4]^2$	$(29,9)$	0	degree reasons
$y[3,1]y[27,6]$	$(30,7)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[8,4]^2y[14,4]$	$(30,12)$	0	degree reasons
$y[3,1]y[14,4]^2$	$(31,9)$	0	degree reasons
$2y[32,7]$	$(32,8)$	0	changing representatives
$x[12,6]y[20,4]$	$(32,10)$	0	$im(\phi)$
$y[8,3]x[24,7]$	$(32,10)$	0	$im(\phi)$
$y[1,1]x[32,7]+y[8,3]y[25,5]$	$(33,8)$	$x[8,4]y[25,5]$	changing representatives
$y[1,1]y[32,7]+y[8,3]y[25,5]$	$(33,8)$	0	$x[8,4]$-product
$y[1,1]y[8,3]y[25,5]$	$(34,9)$	0	$x[8,4]$-product
$y[3,1]x[32,7]$	$(35,8)$	0	$y[25,5]$-product
$y[3,1]y[32,7]$	$(35,8)$	$x[8,4]y[27,6]$	proposition 6.4
$y[1,1]y[14,4]y[20,4]$	$(35,9)$	$x[8,4]y[27,6]$	proposition 6.4
$y[8,3]y[27,6]$	$(35,9)$	$x[8,4]y[27,6]$	$y[25,5]$-product
$y[1,1]^2x[8,4]y[25,5]$	$(35,11)$	0	degree reasons
$x[8,4]y[14,4]^2$	$(36,12)$	0	$im(\phi)$
$x[12,6]y[25,5]$	$(37,11)$	0	degree reasons
$y[1,1]x[36,10]$	$(37,11)$	0	degree reasons
$y[14,4]x[24,7]$	$(38,11)$	0	degree reasons
$y[3,1]x[36,10]$	$(39,11)$	0	degree reasons
$x[12,6]y[27,6]$	$(39,12)$	0	degree reasons
$2y[20,4]^2$	$(40,9)$	$x[8,4]y[32,7]$	proposition 6.4
$y[1,1]y[14,4]y[25,5]$	$(40,10)$	$x[8,4]y[32,7]$	proposition 6.4
$y[8,3]x[32,7]$	$(40,10)$	0	$y[20,4]$-product
$y[8,3]y[32,7]$	$(40,10)$	$x[8,4]y[32,7]$	proposition 6.4
$y[1,1]y[20,4]^2$	$(41,9)$	$y[14,4]y[27,6]$	$y[25,5]$-product
$x[8,4]^2y[27,6]$	$(43,14)$	0	degree reasons
$y[20,4]x[24,7]$	$(44,11)$	0	$im(\phi)$
$x[12,6]y[32,7]$	$(44,13)$	0	$im(\phi)$
$y[8,3]x[36,10]$	$(44,13)$	0	$im(\phi)$
$y[1,1]y[20,4]y[25,5]$	$(46,10)$	$y[14,4]y[32,7]$	$y[14,4]$-product
$y[14,4]x[32,7]$	$(46,11)$	0	degree reasons
$y[20,4]y[27,6]$	$(47,10)$	0	degree reasons
$x[8,4]y[14,4]y[25,5]$	$(47,13)$	0	degree reasons
$y[14,4]^2y[20,4]$	$(48,12)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[8,4]^2y[32,7]$	$(48,15)$	0	$im(\phi)$
$y[1,1]x[48,10]$	$(49,11)$	0	$x[8,4]$-product
$x[24,7]y[25,5]$	$(49,12)$	0	$x[8,4]$-product
$x[8,4]y[14,4]y[27,6]$	$(49,14)$	0	$y[1,1]$-product
$y[14,4]x[36,10]$	$(50,14)$	0	$x[8,4]$-stable
$y[3,1]x[48,10]+y[1,1]y[25,5]^2$	$(51,11)$	0	degree reasons
$4y[51,9]+y[1,1]y[25,5]^2$	$(51,11)$	0	degree reasons
$x[24,7]y[27,6]$	$(51,13)$	0	degree reasons
$y[1,1]y[51,9]$	$(52,10)$	$y[25,5]y[27,6]$	proposition 6.5
$y[20,4]x[32,7]$	$(52,11)$	0	$im(\phi)$
$y[20,4]y[32,7]+y[25,5]y[27,6]$	$(52,11)$	0	$im(\phi)$
$y[1,1]^2y[25,5]^2$	$(52,12)$	0	$im(\phi)$
$2y[3,1]y[51,9]$	$(54,11)$	$y[14,4]y[20,4]^2$	proposition 6.5
$y[27,6]^2+y[14,4]y[20,4]^2$	$(54,12)$	0	degree reasons
$x[8,4]y[14,4]y[32,7]$	$(54,15)$	0	degree reasons
$y[8,3]x[48,10]$	$(56,13)$	0	$im(\phi)$
$x[24,7]y[32,7]$	$(56,14)$	0	$im(\phi)$
$y[20,4]x[36,10]$	$(56,14)$	0	$im(\phi)$
$y[14,4]^4$	$(56,16)$	0	$im(\phi)$
$y[3,1]^2y[51,9]$	$(57,11)$	$y[25,5]y[32,7]$	$y[3,1]$-product
$y[1,1]x[56,11]+y[25,5]x[32,7]$	$(57,12)$	0	$x[8,4]$-product
$y[8,3]y[25,5]^2$	$(58,13)$	0	$x[8,4]$-stable
$y[8,3]y[51,9]$	$(59,12)$	$y[14,4]y[20,4]y[25,5]$	$y[1,1]$-product
$y[3,1]x[56,11]$	$(59,12)$	0	$y[1,1]$-product
$y[27,6]x[32,7]$	$(59,13)$	0	degree reasons
$y[27,6]y[32,7]+y[14,4]y[20,4]y[25,5]$	$(59,13)$	0	degree reasons
$x[8,4]y[51,9]+y[14,4]y[20,4]y[25,5]$	$(59,13)$	0	degree reasons
$y[1,1]x[8,4]y[25,5]^2$	$(59,15)$	0	degree reasons
$y[14,4]^2y[32,7]+x[8,4]y[25,5]y[27,6]$	$(60,15)$	0	$im(\phi)$
$y[25,5]x[36,10]$	$(61,15)$	0	degree reasons
$y[14,4]x[48,10]$	$(62,14)$	0	degree reasons
$x[12,6]y[51,9]$	$(63,15)$	0	degree reasons
$y[27,6]x[36,10]$	$(63,16)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s, s, u)$	extensions	proof
$y[14,4]y[25,5]^2$	$(64,14)$	0	$im(\phi)$
$y[32,7]^2$	$(64,14)$	0	$im(\phi)$
$x[32,7]y[32,7]$	$(64,14)$	0	$im(\phi)$
$y[8,3]x[56,11]$	$(64,14)$	0	$im(\phi)$
$x[8,4]y[25,5]y[32,7]$	$(65,16)$	0	$x[8,4]$-stable
$y[14,4]^3y[25,5]$	$(67,17)$	0	degree reasons
$y[3,1]y[14,4]y[51,9]+y[20,4]x[48,10]$	$(68,14)$	0	$im(\phi)$
$y[32,7]x[36,10]$	$(68,17)$	0	$im(\phi)$
$y[14,4]x[56,11]$	$(70,15)$	0	degree reasons
$y[20,4]y[51,9]$	$(71,13)$	0	degree reasons
$y[14,4]y[25,5]y[32,7]$	$(71,16)$	0	degree reasons
$y[25,5]x[48,10]$	$(73,15)$	0	$x[8,4]$-product
$y[1,1]x[72,15]$	$(73,16)$	0	$x[8,4]$-product
$y[14,4]y[20,4]^3$	$(74,16)$	0	$x[8,4]$-stable
$y[27,6]x[48,10]$	$(75,16)$	0	degree reasons
$x[24,7]y[51,9]$	$(75,16)$	0	degree reasons
$y[3,1]x[72,15]$	$(75,16)$	0	degree reasons
$y[25,5]y[51,9]$	$(76,14)$	0	$im(\phi)$
$y[20,4]x[56,11]$	$(76,15)$	0	$im(\phi)$
$y[1,1]y[25,5]^3$	$(76,16)$	0	$im(\phi)$
$y[25,5]^2y[27,6]$	$(77,16)$	0	degree reasons
$y[27,6]y[51,9]$	$(78,15)$	0	degree reasons
$y[14,4]y[20,4]^2y[25,5]$	$(79,17)$	0	degree reasons
$y[32,7]x[48,10]$	$(80,17)$	0	$im(\phi)$
$y[8,3]x[72,15]$	$(80,17)$	0	$im(\phi)$
$y[1,1]x[80,15]+y[25,5]x[56,11]$	$(81,16)$	0	$x[8,4]$-stable
$y[25,5]^2y[32,7]$	$(82,17)$	0	$x[8,4]$-stable
$x[32,7]y[51,9]$	$(83,16)$	0	degree reasons
$y[32,7]y[51,9]$	$(83,16)$	0	degree reasons
$y[3,1]x[80,15]$	$(83,16)$	0	degree reasons
$y[27,6]x[56,11]$	$(83,17)$	0	degree reasons
$x[8,4]y[25,5]^3$	$(83,19)$	0	degree reasons
$y[1,1]x[84,18]$	$(85,19)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s, s, u)$	extensions	proof
$y[14,4]x[72,15]$	$(86,19)$	0	degree reasons
$x[36,10]y[51,9]$	$(87,19)$	0	degree reasons
$y[3,1]x[84,18]$	$(87,19)$	0	degree reasons
$y[20,4]^2x[48,10]$	$(88,18)$	0	$im(\phi)$
$y[32,7]x[56,11]$	$(88,18)$	0	$im(\phi)$
$y[8,3]x[80,15]$	$(88,18)$	0	$im(\phi)$
$y[20,4]x[72,15]$	$(92,19)$	0	$im(\phi)$
$y[8,3]x[84,18]$	$(92,21)$	0	$im(\phi)$
$y[14,4]x[80,15]$	$(94,19)$	0	degree reasons
$y[20,4]y[25,5]^3$	$(95,19)$	0	degree reasons
$y[1,1]x[96,17]$	$(97,18)$	0	$x[8,4]$-product
$2y[97,17]$	$(97,18)$	0	$y[1,1]$-product
$y[25,5]x[72,15]$	$(97,20)$	0	$x[8,4]$-product
$y[14,4]x[84,18]$	$(98,22)$	0	$x[8,4]$-stable
$2y[99,17]+y[3,1]x[96,17]$	$(99,18)$	0	changing representatives
$2y[3,1]x[96,17]+x[48,10]y[51,9]$	$(99,19)$	0	degree reasons
$y[1,1]^2y[97,17]+x[48,10]y[51,9]$	$(99,19)$	0	degree reasons
$y[27,6]x[72,15]$	$(99,21)$	0	degree reasons
$y[3,1]y[97,17]$	$(100,18)$	$y[20,4]^5$	proposition 6.6
$y[1,1]y[99,17]$	$(100,18)$	$y[20,4]^5$	proposition 6.6
$y[20,4]x[80,15]$	$(100,19)$	0	proposition 6.7
$y[25,5]^4+y[20,4]^5$	$(100,20)$	0	$im(\phi)$
$y[3,1]y[99,17]+y[51,9]^2$	$(102,18)$	0	degree reasons
$2y[51,9]^2$	$(102,19)$	0	degree reasons
$y[3,1]^2x[96,17]$	$(102,19)$	0	degree reasons
$y[8,3]x[96,17]$	$(104,20)$	0	$im(\phi)$
$2y[104,19]$	$(104,20)$	0	proposition 6.8
$y[32,7]x[72,15]$	$(104,22)$	0	$im(\phi)$
$y[20,4]x[84,18]$	$(104,22)$	0	$im(\phi)$
$y[3,1]y[51,9]^2$	$(105,19)$	$y[8,3]y[97,17]+y[20,4]^4y[25,5]$	proposition 6.9
$y[8,3]y[97,17]+y[25,5]x[80,15]$	$(105,20)$	$x[8,4]y[97,17]$	$x[8,4], y[20,4]$-product
$y[1,1]y[104,19]+y[25,5]x[80,15]$	$(105,20)$	$x[8,4]y[97,17]$	$x[8,4], y[20,4]$-product
$y[1,1]x[104,20]+x[8,4]y[97,17]$	$(105,21)$	0	changing representatives

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[25,5]^2x[56,11]$	$(106,21)$	$y[1,1]^2x[104,20]$	$x[8,4]^2$-product
$y[51,9]x[56,11]$	$(107,20)$	0	degree reasons
$y[8,3]y[99,17]$	$(107,20)$	0	degree reasons
$y[3,1]y[104,19]$	$(107,20)$	0	degree reasons
$y[27,6]x[80,15]$	$(107,21)$	0	degree reasons
$x[8,4]y[99,17]$	$(107,21)$	0	degree reasons
$y[3,1]x[104,20]$	$(107,21)$	0	degree reasons
$y[25,5]^3x[32,7]$	$(107,22)$	0	degree reasons
$y[25,5]x[84,18]$	$(109,23)$	0	degree reasons
$x[12,6]y[97,17]$	$(109,23)$	0	degree reasons
$y[1,1]x[108,22]$	$(109,23)$	0	degree reasons
$y[14,4]x[96,17]$	$(110,21)$	$y[20,4]^3y[25,5]^2$	proposition 6.10
$2y[110,20]$	$(110,21)$	$y[20,4]^3y[25,5]^2$	proposition 6.10
$y[1,1]y[110,20]+y[14,4]y[97,17]$	$(111,21)$	0	degree reasons
$x[12,6]y[99,17]$	$(111,23)$	0	degree reasons
$y[3,1]x[108,22]$	$(111,23)$	0	degree reasons
$y[27,6]x[84,18]$	$(111,24)$	0	degree reasons
$y[32,7]x[80,15]$	$(112,22)$	0	$im(\phi)$
$y[1,1]y[14,4]y[97,17]$	$(112,22)$	0	$im(\phi)$
$y[8,3]y[104,19]$	$(112,22)$	0	$im(\phi)$
$x[8,4]y[104,19]$	$(112,23)$	0	$im(\phi)$
$y[8,3]x[104,20]$	$(112,23)$	0	$im(\phi)$
$y[3,1]y[110,20]+y[14,4]y[99,17]$	$(113,21)$	0	$x[8,4]$-stable
$x[8,4]y[25,5]x[80,15]$	$(113,24)$	$y[1,1]x[32,7]x[80,15]$	$x[80,15]$-division
$2y[20,4]x[96,17]+y[14,4]y[51,9]^2$	$(116,22)$	0	$im(\phi)$
$y[32,7]x[84,18]$	$(116,25)$	0	$im(\phi)$
$x[12,6]y[104,19]$	$(116,25)$	0	$im(\phi)$
$y[8,3]x[108,22]$	$(116,25)$	0	$im(\phi)$
$y[1,1]y[20,4]y[97,17]$	$(118,22)$	$y[14,4]x[104,20]$	proposition 6.11
$y[14,4]y[104,19]$	$(118,23)$	$y[14,4]x[104,20]$	proposition 6.11
$y[8,3]y[110,20]$	$(118,23)$	$y[14,4]x[104,20]$	$y[20,4]$-product
$x[8,4]y[110,20]+y[14,4]x[104,20]$	$(118,24)$	0	degree reasons
$y[20,4]y[99,17]$	$(119,21)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[8,4]y[14,4]y[97,17]$	$(119,25)$	0	degree reasons
$y[20,4]^6$	$(120,24)$	0	$im(\phi)$
$y[25,5]x[96,17]$	$(121,22)$	0	$x[8,4]$-product
$x[24,7]y[97,17]$	$(121,24)$	0	$x[8,4]$-product
$y[1,1]x[120,23]$	$(121,24)$	0	$x[8,4]$-product
$y[14,4]x[108,22]$	$(122,26)$	0	$x[8,4]$-stable
$x[12,6]y[110,20]$	$(122,26)$	0	$x[8,4]$-stable
$y[1,1]y[25,5]y[97,17]+y[27,6]x[96,17]$	$(123,23)$	0	degree reasons
$2y[123,22]+y[27,6]x[96,17]$	$(123,23)$	0	degree reasons
$y[51,9]x[72,15]$	$(123,24)$	0	degree reasons
$x[24,7]y[99,17]$	$(123,24)$	0	degree reasons
$y[3,1]x[120,23]$	$(123,24)$	0	degree reasons
$y[25,5]y[99,17]$	$(124,22)$	$y[20,4]x[104,20]$	$y[1,1]$-product
$y[27,6]y[97,17]$	$(124,23)$	$y[20,4]x[104,20]$	$y[1,1]$-product
$y[20,4]y[104,19]$	$(124,23)$	$y[20,4]x[104,20]$	$y[14,4]$-product
$y[1,1]y[123,22]$	$(124,23)$	$y[20,4]x[104,20]$	proposition 6.12
$y[14,4]y[110,20]+y[20,4]x[104,20]$	$(124,24)$	0	$im(\phi)$
$y[14,4]^2y[97,17]+y[20,4]^5y[25,5]$	$(125,25)$	0	degree reasons
$y[27,6]y[99,17]$	$(126,23)$	0	degree reasons
$y[3,1]y[123,22]$	$(126,23)$	0	degree reasons
$x[8,4]y[14,4]x[104,20]$	$(126,28)$	0	degree reasons
$y[14,4]^2y[99,17]$	$(127,25)$	0	degree reasons
$y[32,7]x[96,17]$	$(128,24)$	0	$im(\phi)$
$2y[128,23]$	$(128,24)$	0	changing representatives
$x[24,7]y[104,19]$	$(128,26)$	0	$im(\phi)$
$y[20,4]x[108,22]$	$(128,26)$	0	$im(\phi)$
$y[8,3]x[120,23]$	$(128,26)$	0	$im(\phi)$
$y[32,7]y[97,17]+x[32,7]y[97,17]$	$(129,24)$	$y[25,5]x[104,20]$	$x[8,4]^2$-product
$y[25,5]y[104,19]+x[32,7]y[97,17]$	$(129,24)$	$y[25,5]x[104,20]$	$x[8,4]$-product
$y[1,1]x[128,23]+x[32,7]y[97,17]$	$(129,24)$	0	$x[8,4]$-product
$y[1,1]y[128,23]+x[32,7]y[97,17]$	$(129,24)$	$y[25,5]x[104,20]$	$x[8,4]^2$-product
$y[25,5]^2x[80,15]$	$(130,25)$	$y[1,1]y[25,5]x[104,20]+y[20,4]^4y[25,5]^2$	$y[25,5]$-division
$y[51,9]x[80,15]$	$(131,24)$	0	$y[25,5]$-product

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[32,7]y[99,17]$	$(131,24)$	0	$y[25,5]$-product
$y[32,7]y[99,17]$	$(131,24)$	$y[27,6]x[104,20]$	$y[25,5]$-product
$y[3,1]x[128,23]$	$(131,24)$	0	$y[25,5]$-product
$y[3,1]y[128,23]$	$(131,24)$	$y[27,6]x[104,20]$	$y[25,5]$-product
$y[14,4]y[20,4]y[97,17]$	$(131,25)$	$y[27,6]x[104,20]$	$y[25,5]$-product
$y[27,6]y[104,19]$	$(131,25)$	$y[27,6]x[104,20]$	$y[25,5]$-product
$y[8,3]y[123,22]$	$(131,25)$	$y[27,6]x[104,20]$	$y[25,5]$-product
$x[8,4]y[123,22]+y[27,6]x[104,20]$	$(131,26)$	0	degree reasons
$y[14,4]^2x[104,20]$	$(132,28)$	0	$im(\phi)$
$x[36,10]y[97,17]$	$(133,27)$	0	degree reasons
$y[25,5]x[108,22]$	$(133,27)$	0	degree reasons
$y[1,1]x[132,26]$	$(133,27)$	0	degree reasons
$x[24,7]y[110,20]$	$(134,27)$	0	degree reasons
$y[14,4]x[120,23]$	$(134,27)$	0	degree reasons
$y[51,9]x[84,18]$	$(135,27)$	0	degree reasons
$x[36,10]y[99,17]$	$(135,27)$	0	degree reasons
$y[3,1]x[132,26]$	$(135,27)$	0	degree reasons
$y[27,6]x[108,22]$	$(135,28)$	0	degree reasons
$x[12,6]y[123,22]$	$(135,28)$	0	degree reasons
$y[20,4]^2x[96,17]$	$(136,25)$	$y[32,7]x[104,20]$	$y[14,4]$-product
$y[14,4]y[25,5]y[97,17]$	$(136,26)$	$y[32,7]x[104,20]$	$y[14,4]$-product
$x[32,7]y[104,19]$	$(136,26)$	0	$y[14,4]$-product
$y[32,7]y[104,19]$	$(136,26)$	$y[32,7]x[104,20]$	$y[14,4]$-product
$y[8,3]x[128,23]$	$(136,26)$	0	$y[14,4]$-product
$y[8,3]y[128,23]$	$(136,26)$	$y[32,7]x[104,20]$	$y[14,4]$-product
$x[8,4]y[128,23]+y[32,7]x[104,20]$	$(136,27)$	0	$im(\phi)$
$y[20,4]^2y[97,17]$	$(137,25)$	$y[27,6]y[110,20]$	$y[1,1]$-product
$y[14,4]y[123,22]+y[27,6]y[110,20]$	$(137,26)$	0	$x[8,4]$-stable
$x[8,4]x[32,7]y[97,17]$	$(137,28)$	$x[8,4]y[25,5]x[104,20]$	$x[8,4]$-stable
$x[8,4]y[27,6]x[104,20]$	$(139,30)$	0	degree reasons
$y[20,4]x[120,23]$	$(140,27)$	0	$im(\phi)$
$x[36,10]y[104,19]$	$(140,29)$	0	$im(\phi)$
$y[32,7]x[108,22]$	$(140,29)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[12,6]y[128,23]$	$(140,29)$	0	$im(\phi)$
$y[8,3]x[132,26]$	$(140,29)$	0	$im(\phi)$
$y[20,4]y[25,5]y[97,17]$	$(142,26)$	$y[32,7]y[110,20]$	$y[20,4]$-product
$x[32,7]y[110,20]$	$(142,27)$	0	degree reasons
$y[14,4]x[128,23]$	$(142,27)$	0	degree reasons
$y[14,4]y[128,23]+y[32,7]y[110,20]$	$(142,27)$	0	degree reasons
$y[20,4]y[123,22]$	$(143,26)$	0	degree reasons
$y[14,4]y[25,5]x[104,20]$	$(143,29)$	0	degree reasons
$y[20,4]^2x[104,20]$	$(144,28)$	0	$im(\phi)$
$x[8,4]y[32,7]x[104,20]$	$(144,31)$	0	$im(\phi)$
$x[48,10]y[97,17]$	$(145,27)$	0	$x[8,4]$-product
$y[1,1]x[144,26]$	$(145,27)$	0	$x[8,4]$-product
$y[25,5]x[120,23]$	$(145,28)$	0	$x[8,4]$-product
$y[14,4]y[27,6]x[104,20]$	$(145,30)$	0	$x[8,4]$-stable
$x[36,10]y[110,20]$	$(146,30)$	0	$x[8,4]$-stable
$y[14,4]x[132,26]$	$(146,30)$	0	$x[8,4]$-stable
$2y[147,25]+y[51,9]x[96,17]$	$(147,26)$	0	changing representatives
$y[25,5]^2y[97,17]+2y[51,9]x[96,17]$	$(147,27)$	0	degree reasons
$x[48,10]y[99,17]+2y[51,9]x[96,17]$	$(147,27)$	0	degree reasons
$y[3,1]x[144,26]+2y[51,9]x[96,17]$	$(147,27)$	0	degree reasons
$y[27,6]x[120,23]$	$(147,29)$	0	degree reasons
$x[24,7]y[123,22]$	$(147,29)$	0	degree reasons
$y[51,9]y[97,17]$	$(148,26)$	$y[25,5]y[123,22]$	$y[1,1]$-product
$y[1,1]y[147,25]$	$(148,26)$	$y[25,5]y[123,22]$	$y[1,1]^2$-product
$y[20,4]x[128,23]$	$(148,27)$	0	$im(\phi)$
$y[20,4]y[128,23]+y[25,5]y[123,22]$	$(148,27)$	0	$im(\phi)$
$y[3,1]y[147,25]+y[51,9]y[99,17]$	$(150,26)$	0	changing representatives
$y[3,1]y[51,9]x[96,17]$	$(150,27)$	$y[20,4]^2y[110,20]$	2-product
$x[48,10]y[51,9]^2$	$(150,28)$	$y[14,4]y[32,7]x[104,20]$	$y[51,9]$-division
$y[27,6]y[123,22]+y[20,4]^2y[110,20]$	$(150,28)$	0	2-product
$y[20,4]^5y[25,5]^2$	$(150,30)$	$y[14,4]y[32,7]x[104,20]$	$y[20,4]y[25,5]$-division
$x[48,10]y[104,19]$	$(152,29)$	0	$im(\phi)$
$y[8,3]x[144,26]$	$(152,29)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[32,7]x[120,23]$	$(152,30)$	0	$im(\phi)$
$x[24,7]y[128,23]$	$(152,30)$	0	$im(\phi)$
$y[20,4]x[132,26]$	$(152,30)$	0	$im(\phi)$
$y[51,9]^3$	$(153,27)$	$y[25,5]y[128,23]$	$y[3,1]$-product
$y[25,5]x[128,23]+x[56,11]y[97,17]$	$(153,28)$	0	$x[8,4]$-stable
$y[1,1]x[152,27]+x[56,11]y[97,17]$	$(153,28)$	0	$x[8,4]$-stable
$y[25,5]x[32,7]y[97,17]$	$(154,29)$	$y[25,5]^2x[104,20]$	$x[8,4]^2$-product
$x[56,11]y[99,17]$	$(155,28)$	0	$y[1,1]$-product
$y[51,9]y[104,19]$	$(155,28)$	$y[51,9]x[104,20]$	$y[1,1]$-product
$y[8,3]y[147,25]$	$(155,28)$	$y[51,9]x[104,20]$	$y[1,1]$-product
$y[3,1]x[152,27]$	$(155,28)$	0	$y[1,1]$-product
$y[20,4]y[25,5]y[110,20]+y[51,9]x[104,20]$	$(155,29)$	0	degree reasons
$x[32,7]y[123,22]$	$(155,29)$	0	degree reasons
$y[32,7]y[123,22]+y[51,9]x[104,20]$	$(155,29)$	0	degree reasons
$y[27,6]x[128,23]$	$(155,29)$	0	degree reasons
$y[27,6]y[128,23]+y[51,9]x[104,20]$	$(155,29)$	0	degree reasons
$x[8,4]y[147,25]+y[51,9]x[104,20]$	$(155,29)$	0	degree reasons
$2y[51,9]x[104,20]$	$(155,30)$	0	degree reasons
$x[60,14]y[97,17]$	$(157,31)$	0	degree reasons
$y[25,5]x[132,26]$	$(157,31)$	0	degree reasons
$y[1,1]x[156,30]$	$(157,31)$	0	degree reasons
$x[48,10]y[110,20]$	$(158,30)$	0	degree reasons
$y[14,4]x[144,26]$	$(158,30)$	0	degree reasons
$x[60,14]y[99,17]$	$(159,31)$	0	degree reasons
$y[51,9]x[108,22]$	$(159,31)$	0	degree reasons
$x[12,6]y[147,25]$	$(159,31)$	0	degree reasons
$y[3,1]x[156,30]$	$(159,31)$	0	degree reasons
$x[36,10]y[123,22]$	$(159,32)$	0	degree reasons
$y[27,6]x[132,26]$	$(159,32)$	0	degree reasons
$x[56,11]y[104,19]$	$(160,30)$	0	$im(\phi)$
$y[25,5]^2y[110,20]$	$(160,30)$	0	$im(\phi)$
$y[32,7]x[128,23]$	$(160,30)$	0	$im(\phi)$
$x[32,7]y[128,23]$	$(160,30)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[32,7]y[128,23]$	$(160,30)$	0	$im(\phi)$
$y[8,3]x[152,27]$	$(160,30)$	0	$im(\phi)$
$y[14,4]y[147,25]+y[51,9]y[110,20]$	$(161,29)$	0	$x[8,4]$-stable
$y[25,5]y[32,7]x[104,20]$	$(161,32)$	0	$x[8,4]$-stable
$y[20,4]x[144,26]+y[14,4]y[51,9]y[99,17]$	$(164,30)$	0	$im(\phi)$
$x[60,14]y[104,19]$	$(164,33)$	0	$im(\phi)$
$x[36,10]y[128,23]$	$(164,33)$	0	$im(\phi)$
$y[32,7]x[132,26]$	$(164,33)$	0	$im(\phi)$
$y[8,3]x[156,30]$	$(164,33)$	0	$im(\phi)$
$x[56,11]y[110,20]$	$(166,31)$	0	degree reasons
$y[14,4]x[152,27]$	$(166,31)$	0	degree reasons
$y[20,4]y[147,25]$	$(167,29)$	0	degree reasons
$y[25,5]x[144,26]$	$(169,31)$	0	$x[8,4]$-product
$x[72,15]y[97,17]$	$(169,32)$	0	$x[8,4]$-product
$y[25,5]y[32,7]y[110,20]$	$(169,32)$	0	$x[8,4]$-stable
$y[1,1]x[168,31]$	$(169,32)$	0	$x[8,4]$-product
$y[14,4]y[51,9]x[104,20]$	$(169,33)$	0	$x[8,4]$-stable
$y[20,4]^3y[110,20]$	$(170,32)$	0	$x[8,4]$-stable
$x[60,14]y[110,20]$	$(170,34)$	0	$x[8,4]$-stable
$y[14,4]x[156,30]$	$(170,34)$	0	$x[8,4]$-stable
$x[72,15]y[99,17]$	$(171,32)$	0	degree reasons
$y[51,9]x[120,23]$	$(171,32)$	0	degree reasons
$x[48,10]y[123,22]$	$(171,32)$	0	degree reasons
$y[27,6]x[144,26]$	$(171,32)$	0	degree reasons
$x[24,7]y[147,25]$	$(171,32)$	0	degree reasons
$y[3,1]x[168,31]$	$(171,32)$	0	degree reasons
$y[25,5]y[147,25]$	$(172,30)$	0	$im(\phi)$
$y[20,4]x[152,27]$	$(172,31)$	0	$im(\phi)$
$y[25,5]^2y[123,22]$	$(173,32)$	0	degree reasons
$y[51,9]y[123,22]$	$(174,31)$	0	degree reasons
$y[27,6]y[147,25]$	$(174,31)$	0	degree reasons
$y[20,4]y[25,5]^2x[104,20]$	$(174,34)$	0	degree reasons
$x[48,10]y[128,23]$	$(176,33)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[32,7]x[144,26]$	$(176,33)$	0	$im(\phi)$
$x[72,15]y[104,19]$	$(176,34)$	0	$im(\phi)$
$y[20,4]x[156,30]$	$(176,34)$	0	$im(\phi)$
$y[8,3]x[168,31]$	$(176,34)$	0	$im(\phi)$
$y[25,5]x[152,27]+x[80,15]y[97,17]$	$(177,32)$	0	$x[8,4]$-product
$y[1,1]x[176,31]+x[80,15]y[97,17]$	$(177,32)$	0	$x[8,4]$-product
$y[25,5]^2y[128,23]$	$(178,33)$	0	$x[8,4]$-stable
$x[80,15]y[99,17]$	$(179,32)$	0	degree reasons
$y[51,9]x[128,23]$	$(179,32)$	0	degree reasons
$y[51,9]y[128,23]$	$(179,32)$	0	degree reasons
$x[32,7]y[147,25]$	$(179,32)$	0	degree reasons
$y[32,7]y[147,25]$	$(179,32)$	0	degree reasons
$y[3,1]x[176,31]$	$(179,32)$	0	degree reasons
$x[56,11]y[123,22]$	$(179,33)$	0	degree reasons
$y[27,6]x[152,27]$	$(179,33)$	0	degree reasons
$y[25,5]^3x[104,20]$	$(179,35)$	0	degree reasons
$x[84,18]y[97,17]$	$(181,35)$	0	degree reasons
$y[25,5]x[156,30]$	$(181,35)$	0	degree reasons
$y[1,1]x[180,34]$	$(181,35)$	0	degree reasons
$x[72,15]y[110,20]$	$(182,35)$	0	degree reasons
$y[14,4]x[168,31]$	$(182,35)$	0	degree reasons
$x[84,18]y[99,17]$	$(183,35)$	0	degree reasons
$y[51,9]x[132,26]$	$(183,35)$	0	degree reasons
$x[36,10]y[147,25]$	$(183,35)$	0	degree reasons
$y[3,1]x[180,34]$	$(183,35)$	0	degree reasons
$x[60,14]y[123,22]$	$(183,36)$	0	degree reasons
$y[27,6]x[156,30]$	$(183,36)$	0	degree reasons
$x[80,15]y[104,19]$	$(184,34)$	0	$im(\phi)$
$x[56,11]y[128,23]$	$(184,34)$	0	$im(\phi)$
$y[32,7]x[152,27]$	$(184,34)$	0	$im(\phi)$
$y[8,3]x[176,31]$	$(184,34)$	0	$im(\phi)$
$y[20,4]x[168,31]$	$(188,35)$	0	$im(\phi)$
$x[84,18]y[104,19]$	$(188,37)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[60,14]y[128,23]$	$(188,37)$	0	$im(\phi)$
$y[32,7]x[156,30]$	$(188,37)$	0	$im(\phi)$
$y[8,3]x[180,34]$	$(188,37)$	0	$im(\phi)$
$x[80,15]y[110,20]$	$(190,35)$	0	degree reasons
$y[14,4]x[176,31]$	$(190,35)$	0	degree reasons
$x[96,17]y[97,17]$	$(193,34)$	0	$x[8,4]$-product
$y[25,5]x[168,31]$	$(193,36)$	0	$x[8,4]$-product
$y[97,17]^2+y[1,1]^2x[192,32]$	$(194,34)$	0	$x[8,4]^2$-product
$x[84,18]y[110,20]$	$(194,38)$	0	$x[8,4]$-stable
$y[14,4]x[180,34]$	$(194,38)$	0	$x[8,4]$-stable
$x[96,17]y[99,17]+2y[3,1]x[192,32]$	$(195,34)$	0	changing representatives
$y[51,9]x[144,26]+y[1,1]^3x[192,32]$	$(195,35)$	0	degree reasons
$x[48,10]y[147,25]+y[1,1]^3x[192,32]$	$(195,35)$	0	degree reasons
$x[72,15]y[123,22]$	$(195,37)$	0	degree reasons
$y[27,6]x[168,31]$	$(195,37)$	0	degree reasons
$y[97,17]y[99,17]$	$(196,34)$	0	$im(\phi)$
$y[20,4]x[176,31]$	$(196,35)$	0	$im(\phi)$
$y[99,17]^2+y[3,1]^2x[192,32]$	$(198,34)$	0	degree reasons
$y[51,9]y[147,25]+y[3,1]^2x[192,32]$	$(198,34)$	0	degree reasons
$y[51,9]^2x[96,17]$	$(198,35)$	0	degree reasons
$x[96,17]y[104,19]$	$(200,36)$	0	$im(\phi)$
$x[72,15]y[128,23]$	$(200,38)$	0	$im(\phi)$
$y[32,7]x[168,31]$	$(200,38)$	0	$im(\phi)$
$y[20,4]x[180,34]$	$(200,38)$	0	$im(\phi)$
$y[51,9]^2y[99,17]$	$(201,35)$	$y[1,1]y[8,3]x[192,32]$	$y[99,17]$-division
$y[97,17]y[104,19]+y[1,1]y[8,3]x[192,32]$	$(201,36)$	0	$x[8,4]$-stable
$y[25,5]x[176,31]+y[1,1]y[8,3]x[192,32]$	$(201,36)$	$y[1,1]x[8,4]x[192,32]$	$x[8,4]$-product
$y[97,17]x[104,20]+y[1,1]x[8,4]x[192,32]$	$(201,37)$	0	$x[8,4]$-product
$y[25,5]x[80,15]y[97,17]$	$(202,37)$	$y[1,1]^2x[8,4]x[192,32]$	$x[8,4]$-stable
$y[99,17]y[104,19]$	$(203,36)$	0	degree reasons
$x[56,11]y[147,25]$	$(203,36)$	0	degree reasons
$y[51,9]x[152,27]$	$(203,36)$	0	degree reasons
$y[99,17]x[104,20]$	$(203,37)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$x[80,15]y[123,22]$	$(203,37)$	0	degree reasons
$y[27,6]x[176,31]$	$(203,37)$	0	degree reasons
$y[97,17]x[108,22]$	$(205,39)$	0	degree reasons
$y[25,5]x[180,34]$	$(205,39)$	0	degree reasons
$x[96,17]y[110,20]$	$(206,37)$	0	degree reasons
$y[51,9]^2x[104,20]$	$(206,38)$	0	degree reasons
$y[97,17]y[110,20]+y[1,1]y[14,4]x[192,32]$	$(207,37)$	0	degree reasons
$y[99,17]x[108,22]$	$(207,39)$	0	degree reasons
$x[60,14]y[147,25]$	$(207,39)$	0	degree reasons
$y[51,9]x[156,30]$	$(207,39)$	0	degree reasons
$x[84,18]y[123,22]$	$(207,40)$	0	degree reasons
$y[27,6]x[180,34]$	$(207,40)$	0	degree reasons
$y[104,19]^2$	$(208,38)$	0	$im(\phi)$
$x[80,15]y[128,23]$	$(208,38)$	0	$im(\phi)$
$y[32,7]x[176,31]$	$(208,38)$	0	$im(\phi)$
$y[104,19]x[104,20]$	$(208,39)$	0	$im(\phi)$
$y[99,17]y[110,20]+y[3,1]y[14,4]x[192,32]$	$(209,37)$	0	$x[8,4]$-stable
$y[51,9]^2y[110,20]+y[3,1]^2y[14,4]x[192,32]$	$(212,38)$	0	$im(\phi)$
$y[104,19]x[108,22]$	$(212,41)$	0	$im(\phi)$
$x[84,18]y[128,23]$	$(212,41)$	0	$im(\phi)$
$y[32,7]x[180,34]$	$(212,41)$	0	$im(\phi)$
$y[104,19]y[110,20]$	$(214,39)$	$x[8,4]y[14,4]x[192,32]$	$y[20,4]$-product
$x[104,20]y[110,20]+x[8,4]y[14,4]x[192,32]$	$(214,40)$	0	degree reasons
$y[97,17]x[120,23]$	$(217,40)$	0	$x[8,4]$-product
$x[108,22]y[110,20]$	$(218,42)$	0	$x[8,4]$-stable
$x[96,17]y[123,22]+y[1,1]^2y[25,5]x[192,32]$	$(219,39)$	0	degree reasons
$y[99,17]x[120,23]$	$(219,40)$	0	degree reasons
$x[72,15]y[147,25]$	$(219,40)$	0	degree reasons
$y[51,9]x[168,31]$	$(219,40)$	0	degree reasons
$y[97,17]y[123,22]$	$(220,39)$	$y[14,4]^2x[192,32]$	$y[14,4]$-product
$y[110,20]^2+y[14,4]^2x[192,32]$	$(220,40)$	0	$im(\phi)$
$y[99,17]y[123,22]$	$(222,39)$	0	degree reasons
$x[96,17]y[128,23]$	$(224,40)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[104,19]x[120,23]$	$(224,42)$	0	$im(\phi)$
$y[97,17]x[128,23]+y[8,3]y[25,5]x[192,32]$	$(225,40)$	$x[8,4]y[25,5]x[192,32]$	$x[8,4]$-product
$y[97,17]y[128,23]+y[8,3]y[25,5]x[192,32]$	$(225,40)$	0	$x[8,4]$-stable
$y[99,17]x[128,23]$	$(227,40)$	0	$y[25,5]$-product
$y[99,17]y[128,23]$	$(227,40)$	$x[8,4]y[27,6]x[192,32]$	$y[25,5]$-product
$x[80,15]y[147,25]$	$(227,40)$	0	$y[25,5]$-product
$y[51,9]x[176,31]$	$(227,40)$	0	$y[25,5]$-product
$y[104,19]y[123,22]$	$(227,41)$	$x[8,4]y[27,6]x[192,32]$	$y[25,5]$-product
$x[104,20]y[123,22]+x[8,4]y[27,6]x[192,32]$	$(227,42)$	0	degree reasons
$y[97,17]x[132,26]$	$(229,43)$	0	degree reasons
$y[110,20]x[120,23]$	$(230,43)$	0	degree reasons
$y[99,17]x[132,26]$	$(231,43)$	0	degree reasons
$x[84,18]y[147,25]$	$(231,43)$	0	degree reasons
$y[51,9]x[180,34]$	$(231,43)$	0	degree reasons
$x[108,22]y[123,22]$	$(231,44)$	0	degree reasons
$y[104,19]x[128,23]$	$(232,42)$	0	$y[20,4]$-product
$y[104,19]y[128,23]$	$(232,42)$	$x[8,4]y[32,7]x[192,32]$	$y[20,4]$-product
$x[104,20]y[128,23]+x[8,4]y[32,7]x[192,32]$	$(232,43)$	0	$im(\phi)$
$y[110,20]y[123,22]+y[14,4]y[27,6]x[192,32]$	$(233,42)$	0	$x[8,4]$-stable
$x[108,22]y[128,23]$	$(236,45)$	0	$im(\phi)$
$y[104,19]x[132,26]$	$(236,45)$	0	$im(\phi)$
$y[110,20]x[128,23]$	$(238,43)$	0	degree reasons
$y[110,20]y[128,23]+y[14,4]y[32,7]x[192,32]$	$(238,43)$	0	degree reasons
$y[97,17]x[144,26]$	$(241,43)$	0	$x[8,4]$-product
$y[110,20]x[132,26]$	$(242,46)$	0	$x[8,4]$-stable
$x[96,17]y[147,25]+2y[51,9]x[192,32]$	$(243,42)$	0	changing representatives
$y[99,17]x[144,26]+y[1,1]y[25,5]^2x[192,32]$	$(243,43)$	0	degree reasons
$x[120,23]y[123,22]$	$(243,45)$	0	degree reasons
$y[97,17]y[147,25]$	$(244,42)$	$y[20,4]y[32,7]x[192,32]$	$y[1,1]$-product
$y[99,17]y[147,25]+y[3,1]y[51,9]x[192,32]$	$(246,42)$	0	changing representatives
$y[123,22]^2+y[14,4]y[20,4]^2x[192,32]$	$(246,44)$	0	degree reasons
$y[104,19]x[144,26]$	$(248,45)$	0	$im(\phi)$
$x[120,23]y[128,23]$	$(248,46)$	0	$im(\phi)$

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[97,17]x[152,27]+y[25,5]x[32,7]x[192,32]$	$(249,44)$	0	$x[8,4]$-product
$y[104,19]y[147,25]$	$(251,44)$	$y[14,4]y[20,4]y[25,5]x[192,32]$	$y[1,1]$-product
$y[99,17]x[152,27]$	$(251,44)$	0	$y[1,1]$-product
$y[123,22]x[128,23]$	$(251,45)$	0	degree reasons
$x[104,20]y[147,25]+y[14,4]y[20,4]y[25,5]x[192,32]$	$(251,45)$	0	degree reasons
$y[97,17]x[156,30]$	$(253,47)$	0	degree reasons
$y[110,20]x[144,26]$	$(254,46)$	0	degree reasons
$x[108,22]y[147,25]$	$(255,47)$	0	degree reasons
$y[99,17]x[156,30]$	$(255,47)$	0	degree reasons
$y[123,22]x[132,26]$	$(255,48)$	0	degree reasons
$y[128,23]^2$	$(256,46)$	0	$im(\phi)$
$x[128,23]y[128,23]$	$(256,46)$	0	$im(\phi)$
$y[104,19]x[152,27]$	$(256,46)$	0	$im(\phi)$
$y[110,20]y[147,25]+y[14,4]y[51,9]x[192,32]$	$(257,45)$	0	$x[8,4]$-stable
$y[128,23]x[132,26]$	$(260,49)$	0	$im(\phi)$
$y[104,19]x[156,30]$	$(260,49)$	0	$im(\phi)$
$y[110,20]x[152,27]$	$(262,47)$	0	$y[20,4]$-product
$y[97,17]x[168,31]$	$(265,48)$	0	$x[8,4]$-product
$y[110,20]x[156,30]$	$(266,50)$	0	$x[8,4]$-stable
$y[123,22]x[144,26]$	$(267,48)$	0	degree reasons
$x[120,23]y[147,25]$	$(267,48)$	0	degree reasons
$y[99,17]x[168,31]$	$(267,48)$	0	degree reasons
$y[128,23]x[144,26]$	$(272,49)$	0	$im(\phi)$
$y[104,19]x[168,31]$	$(272,50)$	0	$im(\phi)$
$y[97,17]x[176,31]+y[25,5]x[56,11]x[192,32]$	$(273,48)$	0	$x[8,4]$-product
$x[128,23]y[147,25]$	$(275,48)$	0	degree reasons
$y[99,17]x[176,31]$	$(275,48)$	0	degree reasons
$y[123,22]x[152,27]$	$(275,49)$	0	degree reasons
$y[97,17]x[180,34]$	$(277,51)$	0	degree reasons
$y[110,20]x[168,31]$	$(278,51)$	0	degree reasons
$x[132,26]y[147,25]$	$(279,51)$	0	degree reasons
$y[99,17]x[180,34]$	$(279,51)$	0	degree reasons
$y[123,22]x[156,30]$	$(279,52)$	0	degree reasons

(to be continued)

(continued)

relations	$(t-s,s,u)$	extensions	proof
$y[128,23]x[152,27]$	$(280,50)$	0	$im(\phi)$
$y[104,19]x[176,31]$	$(280,50)$	0	$im(\phi)$
$y[128,23]x[156,30]$	$(284,53)$	0	$im(\phi)$
$y[104,19]x[180,34]$	$(284,53)$	0	$im(\phi)$
$y[110,20]x[176,31]$	$(286,51)$	0	degree reasons
$y[110,20]x[180,34]$	$(290,54)$	0	$x[8,4]$-stable
$x[144,26]y[147,25]+x[48,10]y[51,9]x[192,32]$	$(291,51)$	0	degree reasons
$y[123,22]x[168,31]$	$(291,53)$	0	degree reasons
$y[147,25]^2+y[51,9]^2x[192,32]$	$(294,50)$	0	degree reasons
$y[128,23]x[168,31]$	$(296,54)$	0	$im(\phi)$
$y[147,25]x[152,27]$	$(299,52)$	0	degree reasons
$y[123,22]x[176,31]$	$(299,53)$	0	degree reasons
$y[147,25]x[156,30]$	$(303,55)$	0	degree reasons
$y[123,22]x[180,34]$	$(303,56)$	0	degree reasons
$y[128,23]x[176,31]$	$(304,54)$	0	$im(\phi)$
$y[128,23]x[180,34]$	$(308,57)$	0	$im(\phi)$
$y[147,25]x[168,31]$	$(315,56)$	0	degree reasons
$y[147,25]x[176,31]$	$(323,56)$	0	$y[25,5]$-product
$y[147,25]x[180,34]$	$(327,59)$	0	degree reasons

Proposition 6.1. $y[3,1]^3 = y[1,1]y[8,3]$

Proof. By [Tod62], $\epsilon + \eta\sigma = \langle \nu, \eta, \nu\rangle$ in sphere spectrum; by [Ada60], $\nu^2 = \langle \eta, \nu, \eta\rangle$, hence $y[8,3] = \langle y[3,1], y[1,1], y[3,1]\rangle$ and $y[3,1]^2 = \langle y[1,1], y[3,1], y[1,1]\rangle$.

$$y[1,1]y[8,3] = y[1,1]\langle y[3,1], y[1,1], y[3,1]\rangle = \langle y[1,1], y[3,1], y[1,1]\rangle y[3,1] = y[3,1]^3$$

□

Proposition 6.2. $y[8,3]y[14,4] = x[8,4]y[14,4]$

Proof. In sphere spectrum, $\kappa\nu^3 = 4\bar{\kappa}\nu$. Hence $\eta((\epsilon+\eta\sigma)\kappa+\eta^2\bar{\kappa}) = 0$. Then $(\epsilon+\eta\sigma)\kappa = \eta^2\bar{\kappa}$ in sphere spectrum by degree reasons, and therefore, $y[8,3]y[14,4] = y[1,1]^2 y[20,4] = x[8,4]y[14,4]$. □

Proposition 6.3. $y[3,1]y[25,5] = y[8,3]y[20,4] = y[1,1]y[27,6] = y[14,4]^2$

Proof. By the proposition above, $y[14,4]y[20,4](y[8,3]+x[8,4]) = 0$. However, $x[8,4]y[14,4]y[20,4] \neq 0$. Then $y[8,3]y[20,4]$ is nontrivial, and hence equals to $x[8,4]y[20,4] = y[14,4]^2$, since other candidates are $x[8,4]$-stable.

Note that $h_{11}h_{30}^4 = \langle h_{21}^4, h_{12}, h_{11}\rangle$ in May E_8-page, $y[25,5] \in \langle y[20,4], y[3,1], y[1,1]\rangle$, whose indeterminacy is annihilated by $y[3,1]$. Then

$$y[25,5]y[3,1] = \langle y[20,4], y[3,1], y[1,1]\rangle y[3,1] = y[20,4]\langle y[3,1], y[1,1], y[3,1]\rangle = y[20,4]y[8,3]$$

Note that $h_{10}h_{12}h_{30}^4 = \langle h_{21}^4, h_{12}, h_{10}h_{12}\rangle$ in May E_8-page, $y[27,6] = \langle y[20,4], y[3,1], 2y[3,1]\rangle$. On the other hand, in an E_∞-spectrum, $\langle y, x, y\rangle \cap \langle x, y, 2y\rangle \neq \emptyset$ for odd-dimensional classes y ([Tod62]). Therefore,

$$y[8,3] = \langle y[3,1], y[1,1], y[3,1]\rangle = \langle y[1,1], y[3,1], 2y[3,1]\rangle = \langle y[1,1], 2y[3,1], y[3,1]\rangle$$

and we have

$$y[27,6]y[1,1] = \langle y[20,4], y[3,1], 2y[3,1]\rangle y[1,1] = y[20,4]\langle y[3,1], 2y[3,1], y[1,1]\rangle = y[20,4]y[8,3]$$

□

Proposition 6.4. *$y[3,1]y[32,7] = y[1,1]y[14,4]y[20,4] = x[8,4]y[27,6]$;*
$2y[20,4]^2 = y[8,3]y[32,7] = x[8,4]y[32,7] = y[1,1]y[14,4]y[25,5]$

Proof. As shown in [BMT70], $\{q\}$ can be represented by $\langle \eta, \kappa^2, 2, \eta\rangle$ in the sphere spectrum. Consider $\gamma = \langle y[1,1], y[14,4]^2, 2, y[1,1]\rangle$, which is strictly defined and has zero indeterminacy. Note that

$$2\gamma = 2\langle y[1,1], y[14,4]^2, 2, y[1,1]\rangle = \langle 2, y[1,1], y[14,4]^2, 2\rangle y[1,1] \subset y[1,1]\pi_{31}(tmf) = 0$$

Then γ is either $y[32,7]$ or 0. Besides,

$$y[3,1]\gamma = \langle y[1,1], y[14,4]^2, 2, y[1,1]\rangle y[3,1] = y[1,1]\langle y[14,4]^2, 2, y[1,1], y[3,1]\rangle = y[1,1]y[14,4]y[20,4]$$

where the last equality holds since $y[20,4] = \langle y[14,4], 2, y[1,1], y[3,1]\rangle$ according to [MT63]. By [Tod62], if $\alpha \in \pi_k$ such that $(1-(-1)^k)\alpha = 0$, there is an element $\alpha^* \in \pi_{2k+1}$ such that $\alpha^*\beta \in \langle \alpha, \beta, \alpha\rangle$ for any β. Note that $\langle 2, y[1,1], 2\rangle = y[1,1]^2$,

$$\begin{aligned} y[8,3]\gamma &= \langle y[3,1], y[1,1], y[3,1]\rangle\gamma = \langle y[3,1], y[1,1], y[3,1]\gamma\rangle = \langle y[3,1], y[1,1], y[1,1]y[14,4]\rangle y[20,4] \\ &= \langle y[3,1], y[1,1], \langle 2, y[14,4], 2\rangle\rangle y[20,4] = \langle y[3,1], y[1,1], 2, y[14,4]\rangle 2y[20,4] = 2y[20,4]^2 \end{aligned}$$

If $2y[20,4]^2 = 0$, then $y[1,1]y[20,4]^2 = \langle 2, y[20,4]^2, 2\rangle$, and hence

$$x[8,4]y[14,4]y[20,4] = y[1,1]^2 y[20,4]^2 = y[1,1]\langle 2, y[20,4]^2, 2\rangle = 2\langle y[1,1], 2, y[20,4]^2\rangle$$

which is impossible by degree reasons. Then $2y[20,4]^2 \neq 0$. The only possibility is that $2y[20,4]^2 = x[8,4]y[32,7]$ since it has trivial image under ϕ. And $\gamma \neq 0$, hence $\gamma = y[32,7]$.

Finally, note that in Adams E_3-page, $x_{11}x_{144} = \langle x_{84}, x_{31}, x_{01}x_{31}\rangle$, and therefore $y[1,1]y[14,4] = \langle x[8,4], y[3,1], 2y[3,1]\rangle$. Then

$$y[1,1]y[14,4]y[25,5] = \langle x[8,4], y[3,1], 2y[3,1]\rangle y[25,5] = x[8,4]\langle y[3,1], 2y[3,1], y[25,5]\rangle$$

where

$$y[1,1]\langle y[3,1], 2y[3,1], y[25,5]\rangle = \langle y[1,1], y[3,1], 2y[3,1]\rangle y[25,5] = y[8,3]y[25,5]$$

Hence $\langle y[3,1], 2y[3,1], y[25,5]\rangle$ can be linearly generated by $y[32,7]$ and $2x[8,4]^4$. $y[1,1]y[14,4]y[25,5]$ must be $x[8,4]y[32,7]$ then, since it is not $x[8,4]$-stable. □

Proposition 6.5. *$y[1,1]y[51,9] = y[25,5]y[27,6]$; $2y[3,1]y[51,9] = y[14,4]y[20,4]^2$*

Proof. Consider $\langle y[20,4], y[27,6], y[3,1]\rangle \in \pi_{51}(tmf)$, which has indeterminacy generated by $4y[51,9]$. Note that

$$\langle y[20,4], y[27,6], y[3,1]\rangle y[1,1] = y[20,4]\langle y[27,6], y[3,1], y[1,1]\rangle$$

where

$$\langle y[27,6], y[3,1], y[1,1]\rangle y[3,1] = y[27,6]\langle y[3,1], y[1,1], y[3,1]\rangle = y[8,3]y[27,6] \neq 0$$

Hence there is a nontrivial $y[1,1]$-extension. Note that

$$\langle y[20,4], y[27,6], y[3,1]\rangle 2y[3,1] = y[20,4]\langle y[27,6], y[3,1], 2y[3,1]\rangle$$

where

$$\langle y[27,6], y[3,1], 2y[3,1]\rangle y[1,1] = y[27,6]\langle y[3,1], 2y[3,1], y[1,1]\rangle = y[8,3]y[27,6] \neq 0$$

Hence there is a nontrivial 2-extension. □

Proposition 6.6. *$y[3,1]y[97,17] = y[1,1]y[99,17] = y[20,4]^5$*

Proof. In Adams E_4-page, $\langle x_{255}^3, x_{204}, x_{31}\rangle = x_{31}x_{488}^2$, then $y[99,17] \in \langle y[25,5]^3, y[20,4], y[3,1]\rangle$, which has indeterminacy generated by $2y[99,17]$. Therefore,

$$y[1,1]y[99,17] = \langle y[25,5]^3, y[20,4], y[3,1]\rangle y[1,1] = y[25,5]^3\langle y[20,4], y[3,1], y[1,1]\rangle = y[25,5]^4$$

In Adams E_4-page, $\langle x_{255}, x_{204}x_{255}^2, x_{11}\rangle = x_{11}x_{488}^2$, then $y[97,17] \in \langle y[25,5], y[20,4]y[25,5]^2, y[1,1]\rangle$, which has indeterminacy $y[1,1]\pi_{96}(tmf)$. Therefore,

$$y[3,1]y[97,17] = \langle y[25,5], y[20,4]y[25,5]^2, y[1,1]\rangle y[3,1] = y[25,5]\langle y[20,4]y[25,5]^2, y[1,1], y[3,1]\rangle$$

Note that $\langle h_{21}^8, h_{11}, h_{12}\rangle = h_{12}^3 h_{30}^6$ in May E_8-page, $\langle x_{204}^2, x_{11}, x_{31}\rangle = x_{153}^3$ in Adams E_2-page, and hence $x_{153}^5 \in \langle x_{153}^2 x_{204}^2, x_{11}, x_{31}\rangle$, which has indeterminacy generated by $x_{31}x_{123}^2 x_{488}$. Therefore, $\langle y[20,4]y[25,5]^2, y[1,1], y[3,1]\rangle = y[25,5]^3$, and $y[3,1]y[97,17] = y[25,5]^4$. □

Proposition 6.7. $y[20,4]x[80,15] = 0$

Proof. In Adams E_3-page $x_{327}x_{488} = \langle x_{327}, x_{204}, x_{123}x_{153}\rangle$, then $x[80,15] \in \langle x[32,7], y[20,4], y[27,6]\rangle$. Therefore,

$$y[20,4]x[80,15] \in \langle x[32,7], y[20,4], y[27,6]\rangle y[20,4] = x[32,7]\langle y[20,4], y[27,6], y[20,4]\rangle$$

Then $y[20,4]x[80,15]$ is divided by $x[32,7]$, and hence cannot be $y[20,4]^5$. Thus it must be 0 by the reason of $im(\phi)$. □

Proposition 6.8. $2y[104,19] = 0$

Proof. Since $x_{83}x_{488}^2 = \langle x_{83}, x_{255}^3, x_{204}\rangle$ in Adams E_4-page, $y[104,19]$ can be chosen to be the element in $\langle y[8,3], y[25,5]^3, y[20,4]\rangle$. Then $2y[104,19] = \langle 2, y[8,3], y[25,5]^3\rangle y[20,4]$ is divided by $y[20,4]$, which must be zero because of $im(\phi)$. □

Proposition 6.9. $y[3,1]y[51,9]^2 = y[25,5]x[80,15] + y[20,4]^4 y[25,5] + x[8,4]y[97,17]$

Proof. Note that $y[3,1]y[51,9]^2$ is annihilated by $x[8,4]$ and $y[20,4]$, the only possible nontrivial candidate for it is $y[25,5]x[80,15] + y[20,4]^4 y[25,5] + x[8,4]y[97,17]$. Then

$$y[3,1]y[51,9]^2 = y[3,1]^2 y[99,17] \in \langle y[25,5]^3, y[20,4], y[3,1]^3\rangle = \langle y[25,5]^3, y[20,4], y[1,1]y[8,3]\rangle$$

Since $\langle x_{255}^3, x_{204}, x_{11}x_{83}\rangle = x_{11}x_{83}x_{488}^2$ in Adams E_4-page, $y[3,1]y[51,9]^2$ is nontrivial, then the proposition is derived. □

Proposition 6.10. $2y[110,20] = y[14,4]x[96,17] = y[20,4]^3 y[25,5]^2$

Proof. Suppose $2y[110,20] = 0$, then $y[1,1]y[110,20] = \langle 2, y[110,20], 2\rangle$, and hence

$$y[1,1]y[14,4]y[110,20] = \langle 2, y[110,20], 2\rangle y[14,4] = 2\langle y[110,20], 2, y[14,4]\rangle$$

which is impossible by degree reasons. Therefore $2y[110,20] = y[20,4]^3 y[25,5]^2$. Also note that

$$y[14,4]x[96,17] = \langle y[20,4]y[25,5]^2, y[25,5], 2\rangle y[14,4] = y[20,4]y[25,5]^2\langle y[25,5], 2, y[14,4]\rangle$$

where

$$2\langle y[25,5], 2, y[14,4]\rangle = \langle 2, y[25,5], 2\rangle y[14,4] = y[1,1]y[25,5]y[14,4] = 2y[20,4]^2$$

Hence $y[14,4]x[96,17] = y[20,4]^3 y[25,5]^2$. □

Proposition 6.11. $y[1,1]y[20,4]y[97,17] = y[14,4]y[104,19] = y[14,4]x[104,20]$

Proof.

$$\begin{aligned} y[1,1]y[20,4]y[97,17] &= \langle y[25,5], y[20,4]y[25,5]^2, y[1,1]^2 y[20,4]\rangle \\ &= \langle y[25,5], y[20,4]y[25,5]^2, x[8,4]y[14,4]\rangle = y[14,4]x[104,20] \end{aligned}$$

by the Massey products in Adams E_4-page. Note that $y[1,1]y[110,20] = \langle y[104,19], y[3,1], 2y[3,1]\rangle$ by the Massey product $h_{11}h_0(1)^2 = \langle h_{11}h_0(1), h_{12}, h_{10}h_{12}\rangle$ in May E_4-page, if $y[14,4]y[104,19] = 0$, then

$$y[1,1]y[14,4]y[110,20] = y[14,4]\langle y[104,19], y[3,1], 2y[3,1]\rangle = \langle y[14,4], y[104,19], y[3,1]\rangle 2y[3,1]$$

while $y[1,1]y[14,4]y[110,20]$ cannot be divided by 2 by degree reasons. □

Proposition 6.12. $y[1,1]y[123,22] = y[20,4]x[104,20]$

Proof. By the Massey product in Adams E_4-page, $y[123,22] \in \langle y[25,5]^3, y[20,4], y[27,6]\rangle$, which has indeterminacy generated by $2y[123,22]$. Therefore,

$$\begin{aligned} y[1,1]y[123,22] &= y[1,1]\langle y[25,5]^3, y[20,4], y[27,6]\rangle \\ &= \langle y[1,1], y[25,5]^3, y[20,4]\rangle y[27,6] = y[97,17]y[27,6] = y[20,4]x[104,20] \end{aligned}$$

□

Appendix A Tables and Charts

Table A.1: Basis of Adams E_2-page as an $\mathbb{F}_2[x_{84}, x_{488}]$-module

elements	$(t-s,s)$	elements	$(t-s,s)$	elements	$(t-s,s)$
x_{01}^n	$(0,n)$	$x_{01}x_{174}$	$(17,5)$	x_{327}	$(32,7)$
x_{11}	$(1,1)$	$x_{31}x_{153}$	$(18,4)$	$x_{01}x_{327}$	$(32,8)$
x_{11}^2	$(2,2)$	$x_{11}x_{174}$	$(18,5)$	$x_{01}^2x_{327}$	$(32,9)$
x_{31}	$(3,1)$	$x_{01}x_{204}$	$(20,5)$	$x_{11}x_{327}$	$(33,8)$
$x_{01}x_{31}$	$(3,2)$	$x_{01}^2x_{204}$	$(20,6)$	$x_{11}^2x_{327}$	$(34,9)$
x_{11}^3	$(3,3)$	$x_{11}x_{204}$	$(21,5)$	$x_{01}^nx_{123}^3$	$(36,9+n)$
x_{31}^2	$(6,2)$	$x_{01}^nx_{123}^2$	$(24,6+n)$	$x_{123}^2x_{153}$	$(39,9)$
x_{83}	$(8,3)$	x_{255}	$(25,5)$	$x_{123}x_{144}x_{153}$	$(41,10)$
$x_{11}x_{83}$	$(9,4)$	$x_{11}x_{255}$	$(26,6)$	$x_{123}x_{153}^2$	$(42,9)$
$x_{01}^nx_{123}$	$(12,3+n)$	$x_{123}x_{144}$	$(26,7)$	$x_{144}x_{153}^2$	$(44,10)$
x_{144}	$(14,4)$	$x_{123}x_{153}$	$(27,6)$	$x_{144}x_{153}x_{174}$	$(46,11)$
$x_{01}x_{144}$	$(14,5)$	$x_{31}x_{123}^2$	$(27,7)$	$x_{123}x_{144}x_{153}^2$	$(56,13)$
$x_{01}^2x_{144}$	$(14,6)$	$x_{144}x_{153}$	$(29,7)$	$x_{144}x_{153}^3$	$(59,13)$
$x_{01}x_{153}$	$(15,4)$	$x_{01}x_{144}x_{153}$	$(29,8)$	$x_{144}x_{153}^2x_{174}$	$(61,14)$
$x_{01}^2x_{144}$	$(15,5)$	$x_{144}x_{174}$	$(31,8)$	$x_{144}x_{153}^3x_{174}$	$(76,17)$
				$x_{153}^ix_{174}^jx_{204}^n$	$(0\le i,j\le 3)$

Table A.2: Basis of Adams E_3-page as an $\mathbb{F}_2[x_{488}^2]$-module

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
x_{01}^n	$(0,n)$	x_{31}	$(3,1)$
x_{11}	$(1,1)$	$x_{01}x_{31}$	$(3,2)$
x_{11}^2	$(2,2)$	x_{11}^3	$(3,3)$
x_{83}	$(8,3)$	x_{31}^2	$(6,2)$
$x_{11}x_{83}$	$(9,4)$	$x_{01}x_{174}$	$(17,5)$
$x_{01}^{n+3}x_{123}$	$(12,6+n)$	$x_{01}x_{204}$	$(20,5)$
x_{144}	$(14,4)$	$x_{01}^2x_{204}$	$(20,6)$
$x_{11}x_{144}$	$(15,5)$	$x_{31}x_{123}^2$	$(27,7)$
x_{174}	$(17,4)$	$x_{01}x_{31}x_{488}$	$(51,10)$
$x_{11}x_{174}$	$(18,5)$	$x_{11}^3x_{488}$	$(51,11)$
x_{204}	$(20,4)$	$x_{31}^2x_{488}$	$(54,10)$
$x_{11}x_{204}$	$(21,5)$	$x_{01}^2x_{204}x_{488}$	$(68,14)$
$x_{01}^nx_{123}^2$	$(24,6+n)$	$x_{153}^3x_{204}^{n+1}$	$(65+20n,13+4n)$
x_{255}	$(25,5)$	$x_{153}^2x_{204}^{n+2}$	$(70+20n,14+4n)$
$x_{11}x_{255}$	$(26,6)$	$x_{153}x_{204}^{n+3}$	$(75+20n,15+4n)$
$x_{123}x_{153}$	$(27,6)$	x_{204}^{n+4}	$(80+20n,16+4n)$
x_{153}^2	$(30,6)$		

(to be continued)

(continued)

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
$x_{144}x_{174}$	$(31,8)$		
x_{327}	$(32,7)$		
$x_{153}x_{174}$	$(32,7)$		
$x_{01}x_{327}$	$(32,8)$		
$x_{01}^2x_{327}$	$(32,9)$		
$x_{11}x_{327}$	$(33,8)$		
x_{174}^2	$(34,8)$		
$x_{01}^{n+1}x_{123}^3$	$(36,10+n)$		
$x_{174}x_{204}$	$(37,8)$		
$x_{123}^2x_{153}$	$(39,9)$		
x_{204}^2	$(40,8)$		
$x_{123}x_{144}x_{153}$	$(41,10)$		
$x_{123}x_{153}^2$	$(42,9)$		
$x_{144}x_{153}^2$	$(44,10)$		
x_{153}^3	$(45,9)$		
$x_{144}x_{153}x_{174}$	$(46,11)$		
$x_{01}^{n+1}x_{488}$	$(48,9+n)$		
$x_{11}x_{488}$	$(49,9)$		
$x_{153}x_{174}^2$	$(49,11)$		
$x_{11}^2x_{488}$	$(50,10)$		
$x_{153}^2x_{204}$	$(50,10)$		
$x_{31}x_{488}$	$(51,9)$		
x_{174}^3	$(51,12)$		
$x_{153}x_{174}x_{204}$	$(52,11)$		
$x_{174}^2x_{204}$	$(54,12)$		
$x_{153}x_{204}^2$	$(55,11)$		
$x_{83}x_{488}$	$(56,11)$		
$x_{123}x_{144}x_{153}^2$	$(56,13)$		
$x_{11}x_{83}x_{488}$	$(57,12)$		
$x_{174}x_{204}^2$	$(57,12)$		
x_{204}^3	$(60,12)$		
$x_{01}^{n+3}x_{123}x_{488}$	$(60,14+n)$		
$x_{01}x_{144}x_{488}$	$(62,13)$		
$x_{01}x_{31}x_{123}x_{488}$	$(63,13)$		
$x_{01}x_{174}x_{488}$	$(65,13)$		
$x_{31}^2x_{123}x_{488}$	$(66,13)$		
$x_{153}x_{174}^3$	$(66,15)$		
$x_{01}x_{204}x_{488}$	$(68,13)$		
$x_{11}x_{204}x_{488}$	$(69,13)$		
$x_{153}x_{174}^2x_{204}$	$(69,15)$		
$x_{01}^{n+1}x_{123}^2x_{488}$	$(72,15+n)$		
$x_{11}x_{255}x_{488}$	$(74,14)$		

(to be continued)

(continued)

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
$x_{31}x_{123}^2x_{488}$	$(75,15)$		
$x_{327}x_{488}$	$(80,15)$		
$x_{01}x_{327}x_{488}$	$(80,16)$		
$x_{01}^2x_{327}x_{488}$	$(80,17)$		
$x_{11}x_{327}x_{488}$	$(81,16)$		
$x_{11}^2x_{327}x_{488}$	$(82,17)$		
$x_{01}^{n+1}x_{123}^3x_{488}$	$(84,18+n)$		

Table A.3: Basis of Adams E_4-page as an $\mathbb{F}_2[x_{488}^4]$-module

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
x_{01}^n	$(0,n)$	x_{31}	$(3,1)$
x_{11}	$(1,1)$	$x_{01}x_{31}$	$(3,2)$
x_{11}^2	$(2,2)$	x_{11}^3	$(3,3)$
$x_{01}^{n+3}x_{123}$	$(12,6+n)$	x_{31}^2	$(6,2)$
x_{144}	$(14,4)$	x_{83}	$(8,3)$
x_{204}	$(20,4)$	$x_{11}x_{83}$	$(9,4)$
$x_{01}^{n+1}x_{123}^2$	$(24,7+n)$	$x_{11}x_{144}$	$(15,5)$
x_{255}	$(25,5)$	$x_{01}x_{174}$	$(17,5)$
$x_{11}x_{255}$	$(26,6)$	$x_{01}x_{204}$	$(20,5)$
$x_{123}x_{153}$	$(27,6)$	$x_{01}^2x_{204}$	$(20,6)$
$x_{144}x_{174}$	$(31,8)$	$x_{11}x_{204}$	$(21,5)$
x_{327}	$(32,7)$	$x_{31}x_{123}^2$	$(27,7)$
$x_{153}x_{174}$	$(32,7)$	$x_{11}x_{327}$	$(33,8)$
$x_{01}x_{327}$	$(32,8)$	x_{204}^2	$(40,8)$
$x_{01}^2x_{327}$	$(32,9)$	$x_{01}x_{31}x_{488}$	$(51,10)$
x_{174}^2	$(34,8)$	$x_{11}^3x_{488}$	$(51,11)$
$x_{01}^{n+1}x_{123}^3$	$(36,10+n)$	$x_{31}^2x_{488}$	$(54,10)$
$x_{174}x_{204}$	$(37,8)$	$x_{174}^2x_{204}$	$(54,12)$
$x_{123}^2x_{153}$	$(39,9)$	$x_{174}x_{204}^2+x_{11}x_{83}x_{488}$	$(57,12)$
$x_{123}x_{144}x_{153}$	$(41,10)$	x_{204}^3	$(60,12)$
$x_{144}x_{153}^2$	$(44,10)$	$x_{01}x_{174}x_{488}$	$(65,13)$
x_{153}^3	$(45,9)$	$x_{153}^3x_{204}$	$(65,13)$
$x_{144}x_{153}x_{174}$	$(46,11)$	$x_{01}^2x_{204}x_{488}$	$(68,14)$
$x_{01}^{n+1}x_{488}$	$(48,9+n)$	$x_{153}^2x_{204}^2$	$(70,14)$
$x_{153}x_{174}^2$	$(49,11)$	$x_{153}x_{204}^3$	$(75,15)$
$x_{11}^2x_{488}$	$(50,10)$	x_{204}^4	$(80,16)$
$x_{153}^2x_{204}$	$(50,10)$	$x_{153}^3x_{204}^2$	$(85,17)$
$x_{31}x_{488}$	$(51,9)$	$x_{153}^2x_{204}^3$	$(90,18)$
x_{174}^3	$(51,12)$	$x_{31}x_{488}^2$	$(99,17)$
$x_{153}x_{174}x_{204}$	$(52,11)$	$x_{01}x_{31}x_{488}^2$	$(99,18)$

(to be continued)

(continued)

x_{84}-stable elements	$(t-s,s)$	x_{84}-unstable elements	$(t-s,s)$
$x_{153}x_{204}^2$	$(55,11)$	$x_{11}^3x_{488}^2$	$(99,19)$
$x_{83}x_{488}$	$(56,11)$	x_{204}^5	$(100,20)$
$x_{123}x_{144}x_{153}^2$	$(56,13)$	$x_{31}^2x_{488}^2$	$(102,18)$
$x_{11}x_{83}x_{488}$	$(57,12)$	$x_{83}x_{488}^2$	$(104,19)$
$x_{01}^{n+3}x_{123}x_{488}$	$(60,14+n)$	$x_{11}x_{83}x_{488}^2$	$(105,20)$
$x_{01}x_{144}x_{488}$	$(62,13)$	$x_{153}^3x_{204}^3$	$(105,21)$
$x_{153}x_{174}^3$	$(66,15)$	$x_{153}^2x_{204}^4$	$(110,22)$
$x_{01}x_{204}x_{488}$	$(68,13)$	$x_{11}x_{144}x_{488}^2$	$(111,21)$
$x_{153}x_{174}^2x_{204}$	$(69,15)$	$x_{01}x_{174}x_{488}^2$	$(113,21)$
$x_{01}^{n+1}x_{123}^2x_{488}$	$(72,15)$	$x_{01}x_{204}x_{488}^2$	$(116,21)$
$x_{31}x_{123}^2x_{488}$	$(75,15)$	$x_{01}^2x_{204}x_{488}^2$	$(116,22)$
$x_{327}x_{488}$	$(80,15)$	$x_{11}x_{204}x_{488}^2$	$(117,21)$
$x_{01}x_{327}x_{488}$	$(80,16)$	$x_{31}x_{123}^2x_{488}^2$	$(123,23)$
$x_{01}^2x_{327}x_{488}$	$(80,17)$	$x_{11}x_{84}x_{204}x_{488}^2$	$(125,25)$
$x_{11}x_{327}x_{488}$	$(81,16)$	$x_{11}x_{327}x_{488}^2$	$(129,24)$
$x_{11}^2x_{327}x_{488}$	$(82,17)$	$x_{153}^2x_{204}^5$	$(130,26)$
$x_{01}^{n+1}x_{123}^3x_{488}$	$(84,18+n)$	$x_{01}x_{31}x_{488}^3$	$(147,26)$
$x_{01}^{n+1}x_{488}^2$	$(96,17+n)$	$x_{11}^3x_{488}^3$	$(147,27)$
$x_{11}x_{488}^2$	$(97,17)$	$x_{31}^2x_{488}^3$	$(150,26)$
$x_{11}^2x_{488}^2$	$(98,18)$	$x_{174}^2x_{204}x_{488}^2$	$(150,28)$
$x_{84}x_{488}^2$	$(104,20)$	$x_{174}x_{204}^2x_{488}^2+x_{11}x_{83}x_{488}^3$	$(153,28)$
$x_{01}^{n+3}x_{123}x_{488}^2$	$(108,22+n)$	$x_{01}x_{174}x_{488}^3$	$(161,29)$
$x_{144}x_{488}^2$	$(110,20)$	$x_{01}^2x_{204}x_{488}^3$	$(164,30)$
$x_{01}^{n+1}x_{123}^2x_{488}^2$	$(120,23+n)$		
$x_{11}x_{255}x_{488}^2$	$(122,22)$		
$x_{123}x_{153}x_{488}^2$	$(123,22)$		
$x_{84}x_{204}x_{488}^2$	$(124,24)$		
$x_{144}x_{174}x_{488}^2$	$(127,24)$		
$x_{327}x_{488}^2$	$(128,23)$		
$x_{153}x_{174}x_{488}^2$	$(128,23)$		
$x_{01}x_{327}x_{488}^2$	$(128,24)$		
$x_{01}^2x_{327}x_{488}^2$	$(128,25)$		
$x_{174}^2x_{488}^2$	$(130,24)$		
$x_{01}^{n+1}x_{123}^3x_{488}^2$	$(132,26+n)$		
$x_{174}x_{204}x_{488}^2$	$(133,24)$		
$x_{123}^2x_{153}x_{488}^2$	$(135,25)$		
$x_{123}x_{144}x_{153}x_{488}^2$	$(137,26)$		
$x_{144}x_{153}^2x_{488}^2$	$(140,26)$		
$x_{144}x_{153}x_{174}x_{488}^2$	$(142,27)$		
$x_{01}^{n+1}x_{488}^3$	$(144,25+n)$		
$x_{153}x_{174}^2x_{488}^2$	$(145,27)$		
$x_{11}^2x_{488}^3$	$(146,26)$		

(to be continued)

(continued)

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
$x_{31}x_{488}^3$	$(147,25)$		
$x_{174}^3x_{488}^2$	$(147,28)$		
$x_{153}x_{174}x_{204}x_{488}^2$	$(148,27)$		
$x_{84}x_{153}^3x_{488}^2$	$(149,29)$		
$x_{83}x_{488}^3$	$(152,27)$		
$x_{123}x_{144}x_{153}^2x_{488}^2$	$(152,29)$		
$x_{11}x_{83}x_{488}^3$	$(153,28)$		
$x_{84}x_{153}^2x_{204}x_{488}^2$	$(154,30)$		
$x_{01}^{n+3}x_{123}x_{488}^3$	$(156,30+n)$		
$x_{01}x_{144}x_{488}^3$	$(158,29)$		
$x_{84}x_{153}x_{204}^2x_{488}^2$	$(159,31)$		
$x_{153}x_{174}^3x_{488}^2$	$(162,31)$		
$x_{01}x_{204}x_{488}^3$	$(164,29)$		
$x_{153}x_{174}^2x_{204}x_{488}^2$	$(165,31)$		
$x_{01}^{n+1}x_{123}^2x_{488}^3$	$(168,31+n)$		
$x_{31}x_{123}^2x_{488}^3$	$(171,31)$		
$x_{327}x_{488}^3$	$(176,31)$		
$x_{01}x_{327}x_{488}^3$	$(176,32)$		
$x_{01}^2x_{327}x_{488}^3$	$(176,33)$		
$x_{11}x_{327}x_{488}^3$	$(177,32)$		
$x_{11}^2x_{327}x_{488}^3$	$(178,33)$		
$x_{01}^{n+1}x_{123}^3x_{488}^3$	$(180,34+n)$		

Table A.4: Basis of Adams E_∞-page as an $\mathbb{F}_2[x_{488}^4]$-module

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
x_{01}^n	$(0,n)$	x_{31}	$(3,1)$
x_{11}	$(1,1)$	$x_{01}x_{31}$	$(3,2)$
x_{11}^2	$(2,2)$	x_{11}^3	$(3,3)$
$x_{01}^{n+3}x_{123}$	$(12,6+n)$	x_{31}^2	$(6,2)$
$x_{01}^{n+1}x_{123}^2$	$(24,7+n)$	x_{83}	$(8,3)$
x_{255}	$(25,5)$	$x_{11}x_{83}$	$(9,4)$
$x_{11}x_{255}$	$(26,6)$	x_{144}	$(14,4)$
x_{327}	$(32,7)$	$x_{11}x_{144}$	$(15,5)$
$x_{01}x_{327}$	$(32,8)$	$x_{01}x_{174}$	$(17,5)$
$x_{01}^2x_{327}$	$(32,9)$	x_{204}	$(20,4)$
$x_{01}^{n+1}x_{123}^3$	$(36,10+n)$	$x_{01}x_{204}$	$(20,5)$
$x_{01}^{n+2}x_{488}$	$(48,10+n)$	$x_{01}^2x_{204}$	$(20,6)$
x_{255}^2	$(50,10)$	$x_{11}x_{204}$	$(21,5)$
$x_{83}x_{488}$	$(56,11)$	$x_{84}x_{144}$	$(22,8)$
$x_{01}x_{84}x_{488}+x_{123}x_{144}x_{153}^2$	$(56,13)$	$x_{123}x_{153}$	$(27,6)$

(to be continued)

(continued)

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
$x_{11}x_{83}x_{488}$	$(57,12)$	$x_{31}x_{123}^2$	$(27,7)$
$x_{01}^{n+3}x_{123}x_{488}$	$(60,14+n)$	$x_{84}x_{204}$	$(28,8)$
$x_{01}^{n+1}x_{123}^2x_{488}$	$(72,15+n)$	$x_{327}+x_{153}x_{174}$	$(32,7)$
$x_{327}x_{488}$	$(80,15)$	$x_{11}x_{327}$	$(33,8)$
$x_{01}x_{327}x_{488}$	$(80,16)$	x_{174}^2	$(34,8)$
$x_{01}^2x_{327}x_{488}$	$(80,17)$	$x_{84}x_{123}x_{153}$	$(35,10)$
$x_{11}x_{327}x_{488}$	$(81,16)$	$x_{123}^2x_{153}$	$(39,9)$
$x_{11}^2x_{327}x_{488}$	$(82,17)$	x_{204}^2	$(40,8)$
$x_{01}^{n+1}x_{123}^3x_{488}$	$(84,18+n)$	$x_{84}(x_{327}+x_{153}x_{174})$	$(40,11)$
$x_{01}^{n+1}x_{488}^2$	$(96,17+n)$	$x_{123}x_{144}x_{153}$	$(41,10)$
$x_{11}x_{488}^2$	$(97,17)$	$x_{84}x_{174}^2$	$(42,12)$
$x_{11}^2x_{488}^2$	$(98,18)$	x_{153}^3	$(45,9)$
$x_{84}x_{488}^2$	$(104,20)$	$x_{144}x_{153}x_{174}$	$(46,11)$
$x_{01}^{n+3}x_{123}x_{488}^2$	$(108,22+n)$	$x_{31}x_{488}$	$(51,9)$
$x_{01}^{n+1}x_{123}^2x_{488}^2$	$(120,23+n)$	$x_{01}x_{31}x_{488}$	$(51,10)$
$x_{11}x_{255}x_{488}^2$	$(122,22)$	$x_{11}^3x_{488}$	$(51,11)$
$x_{327}x_{488}^2$	$(128,23)$	$x_{153}x_{174}x_{204}$	$(52,11)$
$x_{01}x_{327}x_{488}^2$	$(128,24)$	$x_{84}x_{153}^3$	$(53,13)$
$x_{01}^2x_{327}x_{488}^2$	$(128,25)$	$x_{31}^2x_{488}$	$(54,10)$
$x_{84}x_{255}x_{488}^2$	$(129,25)$	$x_{174}^2x_{204}$	$(54,12)$
$x_{01}^{n+1}x_{123}^3x_{488}^2$	$(132,26+n)$	$x_{174}x_{204}^2+x_{11}x_{83}x_{488}$	$(57,12)$
$x_{01}^{n+2}x_{488}^3$	$(144,26+n)$	$x_{31}x_{84}x_{488}$	$(59,13)$
$x_{83}x_{488}^3$	$(152,27)$	x_{204}^3	$(60,12)$
$x_{01}x_{84}x_{488}^3+x_{123}x_{144}x_{153}^2x_{488}^2$	$(152,29)$	$x_{84}x_{153}x_{174}x_{204}$	$(60,15)$
$x_{11}x_{83}x_{488}^3$	$(153,28)$	$x_{01}x_{174}x_{488}$	$(65,13)$
$x_{84}x_{255}^2x_{488}^2$	$(154,30)$	$x_{153}^3x_{204}$	$(65,13)$
$x_{01}^{n+3}x_{123}x_{488}^3$	$(156,30+n)$	$x_{153}x_{174}^3$	$(66,15)$
$x_{01}^{n+1}x_{123}^2x_{488}^3$	$(168,31+n)$	$x_{01}^2x_{204}x_{488}$	$(68,14)$
$x_{327}x_{488}^3$	$(176,31)$	$x_{153}^2x_{204}^2$	$(70,14)$
$x_{01}x_{327}x_{488}^3$	$(176,32)$	$x_{153}x_{204}^3+x_{31}x_{123}^2x_{488}$	$(75,15)$
$x_{01}^2x_{327}x_{488}^3$	$(176,33)$	x_{204}^4	$(80,16)$
$x_{11}x_{327}x_{488}^3$	$(177,32)$	$x_{153}^3x_{204}^2$	$(85,17)$
$x_{11}^2x_{327}x_{488}^3$	$(178,33)$	$x_{153}^2x_{204}^3$	$(90,18)$
$x_{01}^{n+1}x_{123}^3x_{488}^3$	$(180,34+n)$	$x_{31}x_{488}^2$	$(99,17)$
		$x_{01}x_{31}x_{488}^2$	$(99,18)$
		$x_{11}^3x_{488}^2$	$(99,19)$
		x_{204}^5	$(100,20)$
		$x_{31}^2x_{488}^2$	$(102,18)$
		$x_{83}x_{488}^2$	$(104,19)$
		$x_{11}x_{83}x_{488}^2$	$(105,20)$
		$x_{153}^3x_{204}^3$	$(105,21)$
		$x_{144}x_{488}^2$	$(110,20)$

(to be continued)

(continued)

x_{84}-stable		x_{84}-unstable	
elements	$(t-s,s)$	elements	$(t-s,s)$
		$x_{153}^2x_{204}^4$	$(110,22)$
		$x_{11}x_{144}x_{488}^2$	$(111,21)$
		$x_{01}x_{174}x_{488}^2$	$(113,21)$
		$x_{01}x_{204}x_{488}^2$	$(116,21)$
		$x_{01}^2x_{204}x_{488}^2$	$(116,22)$
		$x_{11}x_{204}x_{488}^2$	$(117,21)$
		$x_{84}x_{144}x_{488}^2$	$(118,24)$
		$x_{123}x_{153}x_{488}^2$	$(123,22)$
		$x_{31}x_{123}^2x_{488}^2$	$(123,23)$
		$x_{84}x_{204}x_{488}^2$	$(124,24)$
		$x_{11}x_{84}x_{204}x_{488}^2$	$(125,25)$
		$(x_{327}+x_{153}x_{174})x_{488}^2$	$(128,23)$
		$x_{11}x_{327}x_{488}^2$	$(129,24)$
		$x_{174}^2x_{488}^2$	$(130,24)$
		$x_{153}^2x_{204}^5$	$(130,26)$
		$x_{84}x_{123}x_{153}x_{488}^2$	$(131,26)$
		$x_{123}^2x_{153}x_{488}^2$	$(135,25)$
		$x_{84}(x_{327}+x_{153}x_{174})x_{488}^2$	$(136,27)$
		$x_{123}x_{144}x_{153}x_{488}^2$	$(137,26)$
		$x_{84}x_{174}^2x_{488}^2$	$(138,28)$
		$x_{144}x_{153}x_{174}x_{488}^2$	$(142,27)$
		$x_{31}x_{488}^3$	$(147,25)$
		$x_{01}x_{31}x_{488}^3$	$(147,26)$
		$x_{11}^3x_{488}^3$	$(147,27)$
		$x_{153}x_{174}x_{204}x_{488}^2$	$(148,27)$
		$x_{84}x_{153}^3x_{488}^2$	$(149,29)$
		$x_{31}^2x_{488}^3$	$(150,26)$
		$x_{174}^2x_{204}x_{488}^2$	$(150,28)$
		$x_{84}x_{144}x_{153}x_{174}x_{488}^2$	$(150,31)$
		$x_{174}x_{204}^2x_{488}^2+x_{11}x_{83}x_{488}^3$	$(153,28)$
		$x_{31}x_{84}x_{488}^3$	$(155,29)$
		$x_{84}x_{153}x_{174}x_{204}x_{488}^2$	$(156,31)$
		$x_{01}x_{174}x_{488}^3$	$(161,29)$
		$x_{153}x_{174}^3x_{488}^2$	$(162,31)$
		$x_{01}^2x_{204}x_{488}^3$	$(164,30)$

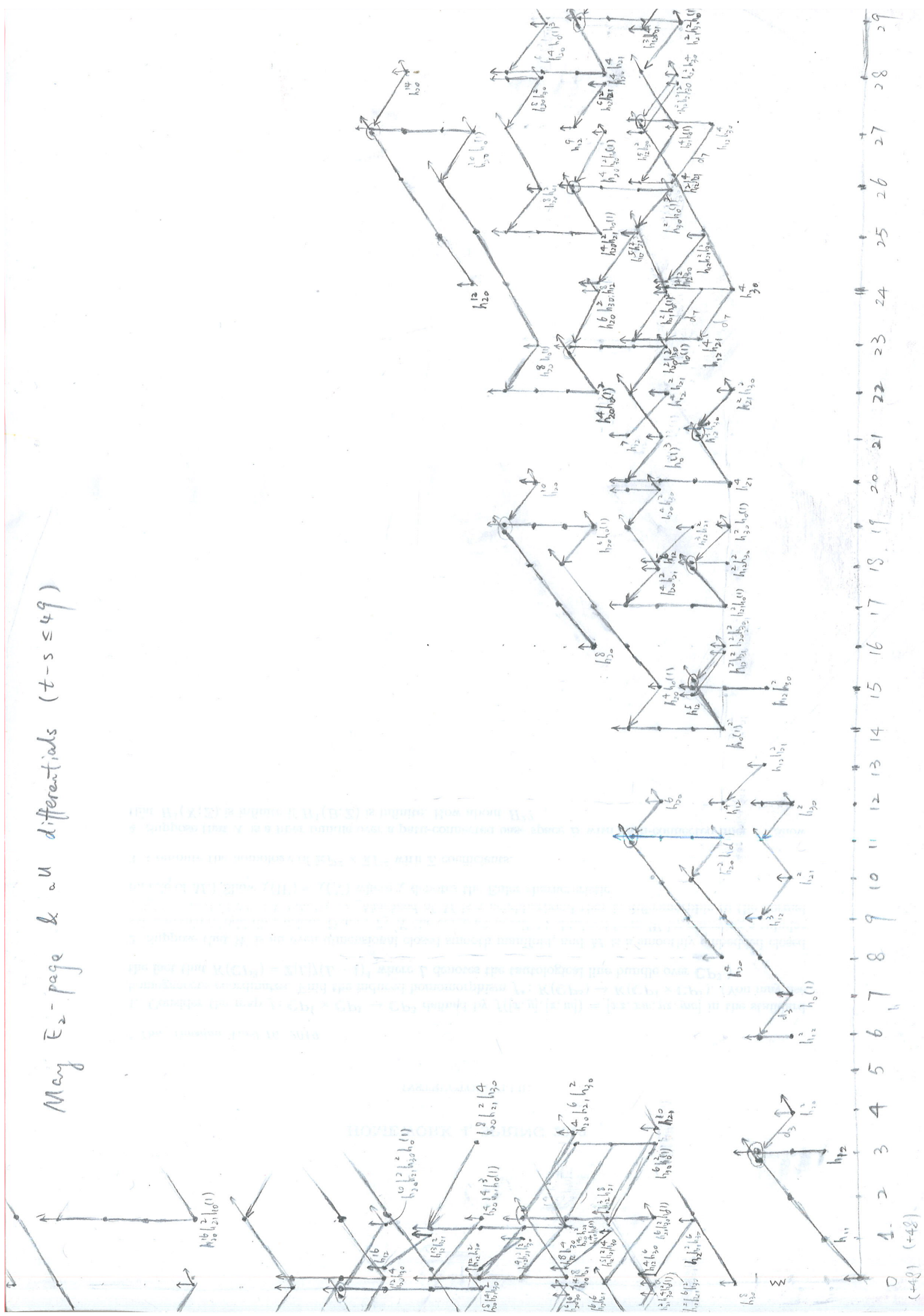


Figure 1: E_2-page of May SS and all differentials in the range of $t - s \leq 49$ (1)

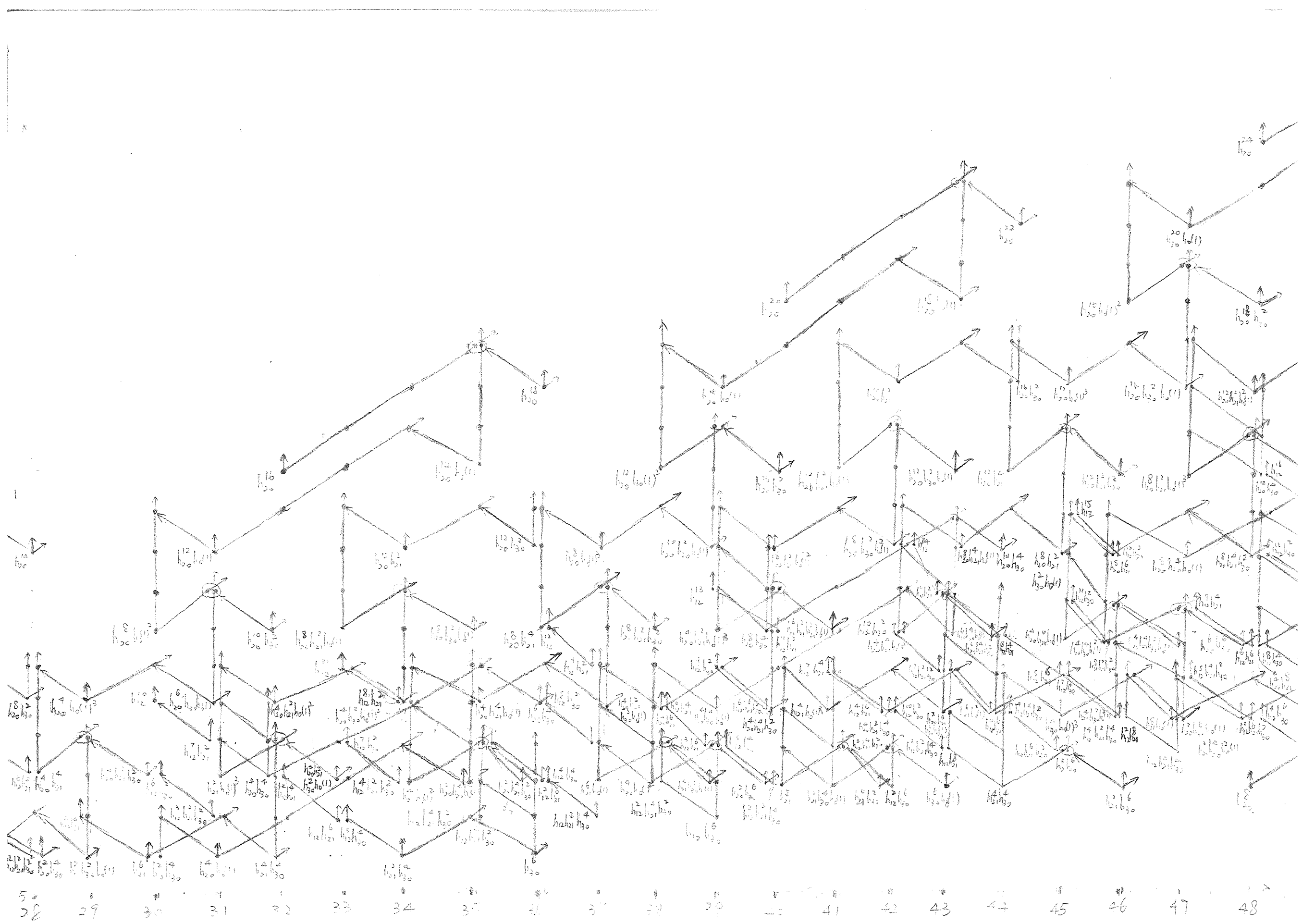


Figure 2: E_2-page of May SS and all differentials in the range of $t-s \leq 49$ (2)

References

[Ada58] John Frank Adams. On the structure and applications of the Steenrod algebra. *Commentarii Mathematici Helvetici*, 32:180–214, 1958.

[Ada60] John Frank Adams. On the non-existence of Hopf invariant one. *The Annals of Mathematics*, 72(1):20–104, 1960.

[Ada66] John Frank Adams. On the groups J(x), IV. *Topology*, 5:21–71, 1966.

[AH61] Michael Francis Atiyah and Friedrich Hirzebruch. Vector bundles and homogeneous spaces. *Proceedings of the Symposium in Pure Mathematics*, 3:7–38, 1961.

[AM74] John Frank Adams and Harvey Rogert Margolis. Sub-Hopf algebras of the Steenrod algebras. *Mathematical Proceedings of the Cambridge Philosophical Society*, 76:45–52, 1974.

[BMMS86] Robert Ray Bruner, Jon Peter May, James Edward McClure, and Mark Steinberger. *H_∞ ring spectrum and their applications*, volume 1176 of *Lecture Notes in Mathematics*. Springer, 1986.

[BMT70] Michael George Barratt, Mark Edward Mahowald, and Martin Charles Tangora. Some differentials in the Adams spectral sequence II. *Topology*, 9:309–316, 1970.

[DFHH14] Christopher Lee Douglas, John Francis, Andr Gil Henriques, and Michael Anthony Hill. *Topological modular forms*, volume 201 of *Mathematical Surveys and Monograph*. American Mathematical Society, 2014.

[HS71] Peter John Hilton and Urs Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1971.

[Mar83] Harvey Rogert Margolis. *Spectra and the Steenrod algebra: modules over the Steenrod algebra and the stable homotopy category*, volume 29 of *North-Holland mathematical library*. North-Holland, 1983.

[May66] Jon Peter May. The cohomology of the restricted Lie algebras and of Hopf algebras. *Journal of Algebra*, 3:123–146, 1966.

[May69] Jon Peter May. Matric Massey products. *Journal of Algebra*, 12:533–568, 1969.

[May70] Jon Peter May. A general algebraic approach to Steenrod operations. *Lecture Notes in Mathematics*, 168:153–231, 1970.

[May74] Jon Peter May. The Steenrod algebra and its associated graded algebra. University of Chicago preprint, 1974.

[Mil58] John Willard Milnor. The Steenrod algebra and its dual. *The Annals of Mathematics*, 67(1):150–171, January 1958.

[Mil72] Richard James Milgram. Group representations and the Adams spectral sequence. *Pacific Journal of Mathematics*, 41(1):157–182, 1972.

[Mos70] R. Michael F. Moss. Secondary compositions and the Adams spectral sequence. *Mathematische Zeitschrift*, 115:283–310, 1970.

[MT63] Mamoru Mimura and Hirosi Toda. The $(n + 20)$-th homotopy groups of n-spheres. *Journal of Mathematics of Kyoto University*, 3:37–58, 1963.

[MT67] Mark Edward Mahowald and Martin Charles Tangora. Some differentials in the Adams spectral sequence. *Topology*, 6:349–369, 1967.

[Nak72] Osamu Nakamura. On the squaring operations in the May spectral sequence. *Memoirs of the Faculty of Science, Kyushu University, Series A*, 26(2):293–308, 1972.

[Qui71] Daniel Gray Quillen. The Adams conjecture. *Topology*, 10:1–10, 1971.

[Rav04] Douglas Conner Ravenel. *Complex cobordism and stable homotopy groups of spheres.* AMS Chealsea Publishing, Providence, Rhode Island, 2004.

[Tod62] Hirosi Toda. *Composition methods in homotopy groups of spheres*, volume 49 of *Annals of Mathematical Studies.* Princeton University Press, 1962.