Strong Law of large number
Law of the iterated logarithm
for nonlinear probabilities

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Outline

♦ History of LLN and LIL for probabilities
♦ Why to study LLN and LIL for capacities
♦ Nonlinear probabilities and nonlinear expectations
♦ Main results
♦ Applications
0.1. History of LLN and LIL for probability

☆ Law of large number (LLN):

(1) Brahmagupta (598-668), Cardano (1501-1576)

(2) Jakob Bernoulli(1713), Poisson (1835)


☆ Law of iterated logarithm (LIL):

(1) Khintchine(1924) for Bernoulli model

Kolmogorov(1929), Hartman–Wintner(1941) (IID)

(2) Levy(1937) for Martingale

(3) Strassen(1964) for functional random variables.
0.2. **Strong LLN and LIL for probabilities**

**Assumption:** \( \{X_i\} \) IID, \( S_n/n := \sum_{i=1}^{n} X_i \), \( EX_1 = \mu \), Then

**Theorem 1:** Kolmogorov:

\[
P(\lim_{n \to \infty} S_n/n = \mu) = 1
\]

**Theorem 2:** Hartman–Wintner(1941): If \( EX_1 = 0, EX_1^2 = \sigma^2 \), Then

(a)

\[
P\left(\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = \sigma\right) = 1
\]

(b)

\[
P\left(\liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -\sigma\right) = 1
\]

(c) Suppose that \( C(\{x_n\}) \) is the cluster set of a sequence of \( \{x_n\} \) in \( R \), then

\[
P\left(C(\{\omega : S_n(\omega)/\sqrt{2n \log \log n}\}) = [-\sigma, \sigma]\right) = 1.
\]
0.3. Why to study LLN and LIL in Finance

**THEOREM 1 (Black-Scholes, 1973:)** In complete markets, there exists a unique probability measure $Q$, such that the pricing of option $\xi$ at strike date $T$ is given by $E_Q[\xi e^{-rT}]$. Where $r = 0$ is interest rate of bond.

Monte Carlo, $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i = E_Q[\xi]$.

- (Linear) expectation $\leftarrow$ **Black-Scholes** $\rightarrow$ Complete Markets
- $\inf_{Q \in \mathcal{P}} E_Q[\xi], \sup_{Q \in \mathcal{P}} E_Q[\xi] \iff$ Incomplete Markets, $Q$ is not unique, SET $\mathcal{P}$.
- **Super-pricing:** $\inf_{Q \in \mathcal{P}} E_Q[\xi], \sup_{Q \in \mathcal{P}} E_Q[\xi]$. Nonlinear expectation!
  $\lim_{n \to \infty} S_n / n = ?$
0.4. Bernoulli Trials with ambiguity

★ Bernoulli Trials:
Repeated independent trials are called Bernoulli trials if there are only two possible outcomes for each trial and their probabilities remain the same throughout the trials.

★ Let $X_i = 1$ if head occurs and $X_i = 0$ if tail occurs.
\[ P_\theta(X_i = 1) = \theta, \quad P_\theta(X_i = 0) = 1 - \theta, \quad S_n := \sum_{i=1}^{n} X_i \]

★ If $\theta = 1/2$ (Unbalance), LLN stats
\[ P_\theta(\lim_{n\to\infty} S_n/n = 1/2) = 1 \]

Or
\[ \lim_{n\to\infty} S_n/n = 1/2 \quad a.s \quad (P_\theta) \]
If a coin is balance. $P_\theta(X_i = 1) = \theta \in [1/3, 1/2]$.

Let $\mathcal{P} := \{P_\theta, \theta \in [1/3, 1/2]\}$.

$E_{P_\theta}[X_i] = \theta$ Unknown,

But $\max_{P \in \mathcal{P}} E_P[X_i] = 1/2$, $\min_{P \in \mathcal{P}} E_P[X_i] = 1/3$.

Question: what is the limit $S_n/n \to$?

(a) Capacity: If $V(A) := \max_{P \in \mathcal{P}} P(A)$, $v(A) := \min_{P \in \mathcal{P}} P(A)$

Can $S_n/n$ converge to $\max_{P \in \mathcal{P}} E_P[X_i]$ or $\min_{P \in \mathcal{P}} E_P[X_i]$ a.s. $V$ or $v$?

(b) The relation between the set of limit points of $S_n/n$ and the interval of $\min_{P \in \mathcal{P}} E_P[X_i]$ and $\max_{P \in \mathcal{P}} E_P[X_i]$.
0.5. Linear and Nonlinear Expectations

- **Kolmogorov**: Linear expectation: $P : \mathcal{F} \rightarrow [0, 1], P(A) = E[I_A]$

  \[ P(A + B) = P(A) + P(B), \quad A \cap B = \emptyset \iff E[\xi + \eta] = E[\xi] + E[\eta] \]

  Expectation is a linear functional of random variable.

- **Nonlinear probability (capacity)**: $V(\cdot) : \mathcal{F} \rightarrow [0, 1]$ but

  \[ V(A + B) \neq V(A) + V(B), \text{ even } A \cap B = \emptyset. \]

- **Nonlinear expectation**: $\mathbb{E}(\xi)$ is nonlinear functional in the sense of

  \[ \mathbb{E}[\xi + \eta] \neq \mathbb{E}[\xi] + \mathbb{E}[\eta]. \]

  Capacity $V(A) = \mathbb{E}[I_A]$ is nonlinear.
Modes of nonlinear expectations and capacity

(1) Choquet expectations (Choquet 1953, physics)

\[ C_V[X] := \int_0^\infty V(X \geq t)dt + \int_{-\infty}^0 [V(X \geq t) - 1]dt. \]

(2) \(g\)-expectation (Peng 1997)

(3) Sub-linear expectation (Peng 2007).

(a) Monotonicity: \( X \geq Y \) implies \( \mathbb{E}[X] \geq \mathbb{E}[Y] \).
(b) Constant preserving: \( \mathbb{E}[c] = c, \forall c \in \mathbb{R} \).
(c) Sub-additivity: \( \mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y] \).
(d) Positive homogeneity: \( \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0 \).

(1) Distorted probability measure: \( V(A) = g(P(A)), g : [0, 1] \rightarrow [0, 1] \).
(2) 2-alternating capacity: \( V(A \cup B) \leq V(A) + V(B) - V(A \cap B) \)
(3) \( V(A) = \max_{P \in \mathcal{P}} P(A), \mathcal{P} \) set of Probability.
1. Independence w.r.t probability or capacity

⋆ Linear: $A$ and $B$ independent $P(AB) = P(A)P(B)$

$$\iff E[\phi(I_A + I_B)] = E[E[\phi(x + I_B)]|x=I_A], \forall \phi(x)$$

⋆ Nonlinear: Epstein(2002), Marinacci(2005) $V(AB) = V(A)V(B)$

$$\iff \mathbb{E}[\phi(I_A + I_B)] = \mathbb{E}[\mathbb{E}[\phi(x + I_B)]|x=I_A]$$
2. Definition of IID under expectation

**Definition 1 (Peng 2007)**

**Independence:** A random variable \( X \in \mathcal{H} \) is said to be independent under \( \mathbb{E} \) to \( Y \), if for each \( \varphi \) such that \( \varphi(X, Y) \in \mathcal{H} \) and \( \varphi(X, y) \in \mathcal{H} \) for each \( y \in \mathbb{R} \)

\[
\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\varphi(Y)],
\]

where \( \varphi(y) := \mathbb{E}[\varphi(X, y)] \).

**Identical distribution:** Random variables \( X \) and \( Y \) are said to be identically distributed, if for each \( \varphi \) such that \( \varphi(X), \varphi(Y) \in \mathcal{H} \),

\[
\mathbb{E}[\varphi(X)] = \mathbb{E}[\varphi(Y)].
\]

**Mutual independence:** \( X \) and \( Y \) are mutually independent

\[
\mathbb{E}[\phi(X + Y)] = \mathbb{E}[\mathbb{E}[\phi(X + y)]|_{y=Y}]
\]
3. Definition: capacity and nonlinear expectation

(1) Probability space: \((\Omega, \mathcal{F}, P) \Rightarrow (\Omega, \mathcal{F}, \mathcal{P})\). Where \(\mathcal{P} := \{P_\theta : \theta \in \Theta\}\).

(2) Capacity: \(P \Rightarrow (v, V)\), where

\[
v(A) = \inf_{Q \in \mathcal{P}} Q(A), \quad V(A) = \sup_{Q \in \mathcal{P}} Q(A).
\]

(3) Property:

\[
V(A) + V(A^c) \geq 1, \quad v(A) + v(A^c) \leq 1
\]

but

\[
V(A) + v(A^c) = 1.
\]

(4) Nonlinear expectations: Lower-upper expectation \(\mathcal{E}[\xi]\) and \(\mathbb{E}[\xi]\)

\[
\mathcal{E}[\xi] = \inf_{Q \in \mathcal{P}} E_Q[\xi], \quad \mathbb{E}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi]
\]
4. **LLN for sub-linear expectations**

★ Weak LLN:

**Theorem 2 (Peng 2007, 2008)** \( \{X_i\}_{i=1}^{\infty} \) IID random variables, \( \overline{\mu} := \mathbb{E}[X_1], \mu := \mathcal{E}[X_1]. \) Then for any continuous and linear growth function \( \phi, \)

\[
\mathbb{E} \left[ \phi \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \right] \to \sup_{\mu \leq x \leq \overline{\mu}} \phi(x), \text{ as } n \to \infty.
\]

★ Theorem (Peng, 2006, 2007). **CLT for IID**
\[ V(AB) = V(A)V(B), \quad v(AB) = v(A)v(B) \]

\[ \star \text{Theorem (Epstein, 02, Marinacci, 99, 05). } \xi \text{ Bounded, } \Omega \text{ Polish, } C_v[X_i] = \underline{\mu}, C_V[X_i] = \bar{\mu}. \text{ } \{X_i\} \text{ IID, then} \]

\[ v \left( \underline{\mu} \leq \lim \inf_{n \to \infty} S_n/n \leq \lim \sup_{n \to \infty} S_n/n \leq \bar{\mu} \right) = 1. \]

Where \( V \) is totally 2-alternating \( V(A \cup B) \leq V(A) + V(B) - V(AB) \),

here \( C_v \) and \( C_V \) is Choquet are integrals.

Note \( C_v[X] \leq \mathcal{E}[X] \leq \mathbb{E}[X] \leq C_V[X], \forall X. \)
4.1. Limit theorem 1

Theorem: If \( \{X_i\} \) is IID, then \( \frac{S_n}{n} \) converges as \( n \to \infty \) a.s. \( v \) if and only if

\[
\mathcal{E}[X_1] = \mathbb{E}[X_1].
\]

In this case,

\[
\lim S_n/n = \mathcal{E}[X_1], \quad a.s. \quad v.
\]
5. Main results

**Theorem 3** \( \{X_i\}_{i=1}^n \) IID under nonlinear expectation \( \mathbb{E} \). Set \( \overline{\mu} := \mathbb{E}[X_i] \), \( \underline{\mu} := \mathbb{E}[X_i] \) and \( S_n := \sum_{i=1}^n X_i \). If \( \mathbb{E}[|X_i|^{1+\alpha}] < \infty \) for \( \alpha > 0 \). Then

(I) \[
\nu \left( \omega \in \Omega : \underline{\mu} \leq \lim \inf_{n \to \infty} S_n(\omega)/n \leq \lim \sup_{n \to \infty} S_n/n(\omega) \leq \overline{\mu} \right) = 1.
\]

(II) \[
V \left( \omega \in \Omega : \lim \sup_{n \to \infty} S_n(\omega)/n = \overline{\mu} \right) = 1
\]
\[
V \left( \omega \in \Omega : \lim \inf_{n \to \infty} S_n(\omega)/n = \underline{\mu} \right) = 1.
\]

(III) Suppose that \( C(\{S_n(\omega)/n\}) \) is the cluster set of a sequence of \( \{S_n(\omega)/n\} \), then \[
V \left( \omega \in \Omega : C(\{S_n(\omega)/n\}) = [\underline{\mu}, \overline{\mu}] \right) = 1
\]
6. Law of iterated logarithm for sub-linear expectations

**Theorem 4** \{X_n\} bounded IID. \(\mathbb{E}[X_1] = \mathbb{E}[X_1] = 0, \sigma^2 := \mathbb{E}[X_1^2], \bar{\sigma}^2 := \mathbb{E}[X_1^2].\) Let \(S_n := \sum_{i=1}^{n} X_i, a_n := \sqrt{2n \log \log n},\) then

(I) \[ v \left( \sigma \leq \limsup_{n} \frac{S_n}{a_n} \leq \bar{\sigma} \right) = 1; \]

(II) \[ v \left( -\bar{\sigma} \leq \liminf_{n} \frac{S_n}{a_n} \leq -\sigma \right) = 1. \]

(III) Suppose that \(C(\{x_n\})\) is the cluster set of a sequence of \(\{x_n\}\) in \(R,\) then

\[ v \left( C(\{S_n/\sqrt{2n\log\log n}\}) \supseteq (-\sigma, \sigma) \right) = 1. \]
7. Key of proof

**Theorem 5** Suppose \( \xi \) is distributed to \( G \) normal \( N(0; [\sigma^2, \overline{\sigma}^2]) \), where \( 0 < \sigma \leq \overline{\sigma} < \infty \). Let \( \phi \) be a bounded continuous function. Furthermore, if \( \phi \) is a positively even function, then, for any \( b \in \mathbb{R} \),

\[
e^{-\frac{b^2}{2\sigma^2}} \mathcal{E}[\phi(\xi)] \leq \mathcal{E}[\phi(\xi - b)].\]
8. Application

Total 100 balls in box, Black + Red + Yellow = 100,
Black = Red, Yellow ∈ [30, 40], then \( P_Y \in [3/10, 4/10] \).
Take a ball from this box,
\( X_i = 1 \), if ball is black, \( X_i = 0 \), if ball is Yellow, \( X_i = -1 \) for red.
\( S_n = \sum_{i=1}^{n} X_i \), is the excess frequency of black than Red.
Then
(a) \( \mathbb{E}[X_i] = \mathbb{E}[X_i] = 0 \)
(b)
\[
\sqrt{6/10} \leq \limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \lg \lg n}} \leq \sqrt{7/10}.
\]
9. Nonlinear expectation in Finance

In incomplete markets, there exists a set $\mathcal{P}$ of probability measures, such that the super-sub-hedging price of option $\xi$ at strike date $T$ are given by

$$\bar{\mu} := \inf_{Q \in \mathcal{P}} E_Q[\xi], \quad \underline{\mu} := \sup_{Q \in \mathcal{P}} E_Q[\xi].$$

then

(1) $$\underline{\mu} \leq \liminf_{n \to \infty} S_n(\omega)/n \leq \limsup_{n \to \infty} S_n(\omega)/n(\omega) \leq \bar{\mu}$$

(2) $$\limsup_{n \to \infty} S_n(\omega)/n = \bar{\mu}, \quad V,$$

$$\liminf_{n \to \infty} S_n(\omega)/n = \underline{\mu}, \quad V$$
Thank you!