Homotopy types of homeomorphism groups
and spaces of emebddings in 2-manifolds

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Abstract. In this talk we discuss on homotopy types of homeomorphism groups of non-compact 2-manifolds. Suppose M is a noncompact connected 2-manifold and X is a compact subpolyhedron of M. Let $\mathcal{H}_X(M)$ denote the group of homeomorphisms h of M onto itself with $h|_X = id$ equipped with the compact-open topology. For any subgroup $\mathcal{G}$ of $\mathcal{H}_X(M)$, the symbols $\mathcal{G}_0$ and $\mathcal{G}_1$ denote the connected component and the path component of id_M in $\mathcal{G}$. Let $\mathcal{G}^c = \{h \in \mathcal{G} \mid h \text{ has compact support}\}$ and let $(\mathcal{G}^c)_1^*$ denote the subgroup of $\mathcal{G}^c$ consisting of $h \in \mathcal{G}^c$ which admit a path h_t to id_M in $\mathcal{G}^c$ with a common compact support.

A subset A of a space X is said to be homotopy dense (HD) if there exists a homotopy $\varphi_t : X \to X$ $(0 \le t \le 1)$ such that $\varphi_0 = id_X$ and $\varphi_t(X) \subset A$ $(0 < t \le 1)$. In this case the inclusion $i : A \subset X$ is a homotopy equivalence with the homotopy inverse $\varphi_1 : X \to A$.

Theorem 0.1.

(1) $\mathcal{H}_X(M)_0$ *is a topological* ℓ_2*-manifold.* — *[3]*

(2) $$\mathcal{H}_X(M)_0 \simeq \begin{cases} \mathbb{S}^1 & \textit{if } (M,X) \cong (\mathbb{R}^2,\emptyset),\ (\mathbb{R}^2, 1pt),\ (\mathbb{S}^1\times\mathbb{R}^1,\emptyset),\ (\mathbb{S}^1\times[0,1),\emptyset) \textit{ or } (\mathbb{P}^2\setminus 1pt,\emptyset), \\ * & \textit{in all other cases.} \end{cases}$$

— *[3]*

(3) $\mathcal{H}_X^c(M)_1^* \subset \mathcal{H}_X(M)_0$ *: HD* — *[4]*

Suppose M is a noncompact connected PL 2-manifold. Let $\mathcal{H}_X^{\rm PL}(M) = \{h \in \mathcal{H}_X(M) \mid h : \text{PL-homeo}\}$.

Theorem 0.2. $\mathcal{H}_X^{\rm PL,c}(M)_1^* \subset \mathcal{H}_X(M)_0$ *: HD* — *[4]*

Suppose μ is a good Radon measure on M with $\mu(\operatorname{Fr} X \cup \partial M) = 0$. Let $\mathcal{H}_X(M,\mu) = \{h \in \mathcal{H}_X(M) \mid h \text{ preserves } \mu\}$, $\mathcal{H}_X(M,\mu\text{-reg}) = \{h \in \mathcal{H}_X(M) \mid h \text{ preserves } \mu\text{-null sets}\}$.

Theorem 0.3. — *[6]*

(1) There exists a PL-structure of M *such that* $\mathcal{H}_X^{\rm PL}(M) \subset \mathcal{H}_X(M,\mu\text{-reg})$.
$\mathcal{H}_X^{\rm PL,c}(M)_1^* \subset \mathcal{H}_X(M,\mu\text{-reg})_1^* \subset \mathcal{H}_X(M)_0$ *: HD*

(2) (after R. Berlanga) $\mathcal{H}_X(M;\mu)_0$ *is a strong deformation retract of* $\mathcal{H}_X(M)_0$.

We have also studied the groups of Lipschitz homeomorphisms and quasiconformal homeomorphisms of (noncompact) 2-manifolds ([4], [1]). In [5] we classified homotopy types of the components of spaces of embeddings of compact polyhedra into 2-manifolds.

References

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