Endoscopic Relative Orbital Integrals on $U_3$

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Automorphic representations

Global setting: $K$ is a number field, and $G$ is a reductive group over $K$.

- Consider the vector space

$$V = L^2([G]) = L^2(G(K) \backslash G(\mathbb{A}_K)).$$

- Right regular representation: an action of $G$ (thus a representation) $R : G(\mathbb{A}_K) \to \text{GL}(V)$ by translation

$$R(g) \cdot \varphi(x) = \varphi(xg).$$

- Ultimate goal: to study the decomposition and constituents of $(V, R)$. The isotypic components of $V$ are called *automorphic representations*. 
Distinction of automorphic representations

Let $H \subset G$ be a closed subgroup.

**Definition**

1. For any automorphic representation $\pi$ and $\varphi \in V_\pi$, define the period integral of $\varphi$ with respect to $H$ to be

$$P_H(\varphi) = \int_{[H]} \varphi(h) dh.$$ 

2. An automorphic representation $\pi$ is $H$-distinguished if $P_H(\varphi) \neq 0$ for some $\varphi \in V_\pi$. 
Examples of distinguished representations

1. $H$: a maximal unipotent. Then cuspidal representations $\iff$ not distinguished by $H$.

2. $L/K$ is quadratic. $G = \text{Res}_{L/K} \text{GL}_n$ and $H = \text{U}_n$. Then $H$-distinguished $\iff$ a base change lifting from $\text{GL}_n$ (Flicker, Mok and Zinoviev).

3. $G = \text{GL}_n$ and $H = \text{GO}_n$. Then $H$-distinguished $\iff$ a metaplectic lifting from $\text{GL}_n$ (conjecture of Jacquet, verified in $n = 3$ by Mao).

4. Distinction is also related to central $L$-values.
Motivation behind distinction problem

Distinction problem is fruitful from several perspectives:

1. The subject of the relative Langlands program introduced by Sakellaridis and Venkatesh.
2. To study algebraic cycles on Shimura varieties.
The Comparison of Relative Trace Formulae
A strategy for distinction

- Philosophy (Jacquet): to study distinction problems through the *comparison of relative trace formulae*.
- The proof of the unitary case of Gan-Gross-Prasad conjecture.
- An approach by Getz-Wambach that considers a triple of involutions.
  Distinction problems on classical groups $\longrightarrow$ Distinction problems on general linear groups.
The automorphic kernel

- For $f \in C_c^\infty(G(\mathbb{A}_K))$, we can define an operator $R(f)$ via
  
  $$R(f)\varphi(x) = \int_{G(\mathbb{A}_K)} f(y)\varphi(xy)dy.$$  

- The operator $R(f)$ has a kernel
  
  $$K_f(x,y) = \sum_{\gamma \in G(K)} f(x^{-1}\gamma y)$$

  referred to as the automorphic kernel associated to $f$.

- With some additional conditions, the automorphic kernel allows a spectral expansion
  
  $$K_f(x,y) = \sum_{\pi \in \hat{G}} m(\pi) \sum_{\phi \in \mathcal{B}_\pi} \pi(f)\phi(x)\overline{\phi(y)}.$$
Let $G_1, G_2$ be two algebraic subgroups of the reductive group $G$.

- We consider $\int_{[G_1 \times G_2]} K_f(x, y) dx dy$.
- If we uses the spectral expansion for $K_f(x, y)$, this integral is equal to

\[
\sum_{\pi \in \hat{G}} m(\pi) \sum_{\phi} \int_{[G_1]} \int_{[G_2]} \pi(f) \phi(x) \overline{\phi(y)} dx dy = \sum_{\pi \in \hat{G}} m(\pi) J_\pi(f).
\]
Relative trace: the geometric side

- On the other hand, the *geometric expansion* considers the same integral, but decomposed by equivalence classes.
- It becomes a sum over classes \( \gamma \in G_1(K)\setminus G(K)/G_2(K) \):

\[
\sum_{\gamma} a(\gamma) O_\gamma(f).
\]

Here

\[
O_\gamma(f) = \int_{I_\gamma(\mathbb{A}_K)\setminus G_1 \times G_2(\mathbb{A}_K)} f(x^{-1}\gamma y) \, dx \, dy
\]

is the *relative orbital integral*. 
The relative trace formula

- The *relative trace formula* thus asserts (roughly)
  \[ \sum_{\pi} m(\pi) J_\pi(f) = \sum_{\gamma} a(\gamma) O_\gamma(f), \]

- Here each \( J_\pi(f) \) are related to the period integrals of functions in \( V_\pi \) with respect to \( G_1 \) and \( G_2 \) because
  \[ J_\pi(f) = \sum_{\phi \in B_\pi} P_{G_1}(f \cdot \phi) \overline{P_{G_2}(\phi)}. \]

- Assume \( f = \prod_v f_v \), so that the relative orbital integrals \( O_\gamma(f) \) is a product of its local factors.

- Strategy: using the (local) comparison on the geometric side to study the spectral data.
The Getz-Wambach comparison

- Setting: let $L/K$ be a quadratic extension of number fields and $H = \text{Res}_{L/K} \text{GL}_n$.
- Given a pair of commuting involutions (automorphism of order 2) on $H$: $\theta$ and $\sigma$. Let $\tau = \sigma \circ \theta$.
- Twisted relative trace formula of $H \leftrightarrow$ Relative trace formula of $G = H^\tau$. 
A general principle

- With the previous setting, they suggested that (roughly speaking) one should have

**Ansatz (Getz-Wambach)**

An automorphic representation $\pi$ of $H(\mathbb{A}_K)$ is distinguished by both $H^\sigma$ and $H^\theta \iff \pi$ is a lifting of an $G^\sigma$-distinguished automorphic representation on $G(\mathbb{A}_K)$.

They proposed a relative trace formula method of proof.

- Relates directly to the relative Langlands program because symmetric subgroups are always spherical.
Studied examples

- The biquadratic case: a theorem of Getz-Wambach says that (with some additional condition) the ansatz holds for the case of \( U_n \subset \text{Res}_{L/K} U_n \), where \( L \) is a quadratic extension over \( K \).

- The unitary Friedberg-Jacquet case: for \( U_n \times U_n \subset U_{2n} \).
  
  (\( \Rightarrow \)) The work of S. Leslie and the work of J. Xiao and W. Zhang.
  
  (\( \Leftarrow \)) The work of Pollack-Wan-Zydor.
Case of interest and its conjecture

- Let $L/K$ be a quadratic extension. $H = \text{Res}_{L/K} \text{GL}_n$, $\sigma$ be a quasi-split orthogonal involution, and $\theta$ be the nontrivial Galois conjugate.
- Thus $G = \text{U}_n$ and $G^\sigma = \text{O}_n$ are quasi-split reductive groups.

Conjecture (L.)

A cuspidal automorphic representation $\pi$ of $\text{U}_n(\mathbb{A}_K)$ is distinguished by $\text{O}_n \iff$ its base change lifting to $\text{Res}_{L/K} \text{GL}_n(\mathbb{A}_K)$ is a metaplectic lifting from $\text{Res}_{L/K} \tilde{\text{GL}}_n(\mathbb{A}_K)$. 
The associated symmetric variety

- We consider the integrals on $H^\sigma \backslash H // H^\theta$ and the integrals on $G^\sigma \backslash G // G^\sigma$. Furthermore, we will be focusing on the latter for the rest of the talk.

- The symmetrization map

$$G \rightarrow G$$
$$g \mapsto gg^{-\sigma}$$

has kernel $G^\sigma$.

- Instead of considering $G // G^\sigma$, we consider the schematic image of this map, denoted as $S$.

- $S$ is a spherical variety under the adjoint action of $G$.

- Studying setting: fix $n = 3$ and consider the adjoint action of $G_1 := \text{SO}_3$ on the symmetric space $S$. 
Pre-stabilization and Endoscopy
The pre-stabilization of the relative trace formulae

The adelic ring: $\mathbb{A}_K = \prod_{v}^{\text{res}} K_v$. Fix a nonarchimedean local field $K_v = F$ and denote its valuation ring by $\mathcal{O}_F$.

- On the geometric side of the relative trace formula: relative orbital integrals with local factors

$$O_{\gamma}(f) = \int_{G_{1\gamma}(F)\backslash G_1(F)} f(g^{-1} \cdot \gamma) dg.$$

- Over the algebraic closure, there is a *norm map* to match orbits.

- Difficulty: the $F$-points of $G_{1\gamma}\backslash G_1$ can be different to $G_{1\gamma}(F)\backslash G_1(F)$.

- The solution to this issue is called the *pre-stabilization*. 

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**Endoscopic Relative Orbital Integrals on $U_3$**

**Chung-Ru Lee**

**Distinction of Automorphic Representations**

**The Comparison of Relative Trace Formulae**

**Pre-stabilization and Endoscopy**

**Endoscopic Relative Orbital Integrals**
Stable orbits versus rational orbits

- Stable orbits: $F$-points of $G_1\gamma \backslash G_1 \leftrightarrow G_1(\overline{F})$-classes in $S(F)$.
- Rational orbits: $G_1\gamma(F) \backslash G_1(F) \leftrightarrow G_1(F)$-classes.
- The set of rational orbits inside a stable orbit is parametrized by a group
  $\mathcal{D}(F, G_1\gamma, G_1) := \ker[H^1(F, G_1\gamma) \to H^1(F, G_1)]$.
- Harmonic analysis: to stabilize, one should consider the local $\kappa$-orbital integrals.
Relative $\kappa$-orbital integrals

- For each $\kappa \in \mathcal{D}(F, G_1^\gamma, G_1)^D$, define the $\kappa$-orbital integral
  \[ SO_\gamma^\kappa(f) = \sum_{\gamma' \sim_{st} \gamma} \kappa(\gamma') O_\gamma(f). \]

- $SO_\gamma(f) := SO_1^\gamma(f)$ is called the stable relative orbital integral.

- We say that $SO_\gamma^\kappa(f)$ is an endoscopic relative orbital integral if $\kappa$ is nontrivial.

- Expectation: proceed inductively by relating the endoscopic relative orbital integrals to stable relative orbital integrals on other (simpler) spaces.
Endoscopic Relative Orbital Integrals
Unitary relative endoscopy

Local setting: let \( E = F[\xi] \) be an unramified quadratic extension of local fields. Denote the uniformizer by \( \varpi \).

- Recall: \( G = U_3 \) and \( G^\sigma = O_3 \) are quasi-split. \( S \) is the space of (twisted) symmetric unitary matrices. \( G_1 := SO_3 \).
- The generic stabilizer \( G_{1\gamma} \) is disconnected and finite.
- For any regular semisimple point \( \gamma \in S(F') \), we want to compute \( SO_{\gamma}^\kappa(f) \) for \( f = 1_S(\mathcal{O}_F) \) for \( \kappa \) nontrivial.
- In particular, we consider only those \( \gamma \) with a nontrivial \( \mathcal{D}(F, G_{1\gamma}, G_1) \).
Classification of stable orbits

- It turns out that $\mathcal{D}(\gamma, G_1\gamma, G_1)$ can be computed by the isomorphism classes of $G_\gamma$, which occurs in J. Rogawski’s work.

- Let

\[
T_\nu(R) = \left\{ \begin{pmatrix} x & z & y \\ \nu y & z & x \end{pmatrix} \in \text{U}_3(R) \mid x, y, z \in E \otimes F \right\}.
\]

**Lemma (L.)**

Any regular semisimple stable orbit with a nontrivial $\mathcal{D}(F, G_1\gamma, G_1)$ contains an element in (some unique) $T_\nu(F)$ with $\nu \in \{1, \xi^2, \varpi, \xi^2 \varpi\}$.
Types of the endoscopic stable orbits

- The stable orbits of interest are represented by elements in $T_\nu(F)$ for some $\nu \in \{1, \xi^2, \varpi, \xi^2 \varpi\}$.

- Types of tori: we say that $T_\nu$ is of

\[
\begin{cases} 
\text{Type I} & \text{if } \nu = \xi^2, \\
\text{Type II} & \text{if } \nu = 1, \\
\text{Type III} & \text{if } \nu = \varpi \text{ or } \xi^2 \varpi.
\end{cases}
\]

- Later, we will compute the formula on type I tori as an explicit example.
Related cohomological data

- Recall: \( \mathcal{D}(F, G_{1\gamma}, G_1) = \ker[H^1(F, G_{1\gamma}) \to H^1(F, G_1)] \) parametrizes rational orbits inside the stable orbits of \( \gamma \). In general, those are *Galois cohomology pointed sets*.

**Lemma (L.)**

*In our case, there exist a natural group structure on \( \mathcal{D}(F, G_{1\gamma}, G_1) \) so that*

\[
\mathcal{D}(F, G_{1\gamma}, G_1) \cong \begin{cases} 
F^\times \backslash N(E^\times) & \text{for Type I tori,} \\
F^\times \backslash (F^\times)^2 & \text{for Type II and III tori.}
\end{cases}
\]

- In particular, \( |\mathcal{D}(F, G_{1\gamma}, G_1)| = 2 \) for \( \gamma \) in a Type I torus.
Iwasawa decomposition

- Recall that we are considering

\[ O_\gamma(f) = \int_{G_1 \gamma(F) \backslash G_1(F)} 1_S(\mathcal{O}_F)(g^{-1} \cdot \gamma) dg. \]

- Iwasawa decomposition: \( G(F) = N(F)A(F)G(\mathcal{O}_F) \). Since \( f \) is \( G(\mathcal{O}_F) \)-invariant we have (with the suitable choice of measure)

\[ O_\gamma(f) = \int_{F \times F} f \left( \left( \begin{array}{cc} t & -u^2/2 \\ 1 & -t^{-1}u \end{array} \right)^{-1} \cdot \gamma \right) \frac{dud \times t}{|t|}. \]
Relative orbital integral

Rational orbits in the stable orbit of $\gamma$ are given by
$$\gamma_\mu = \begin{pmatrix} x & \mu y \\ \mu^{-1} \nu y & x \end{pmatrix}.$$

The entries of
$$\left( \begin{array}{cc}
  t & -t^{-1} u^2/2 \\
  1 & -t^{-1} u \\
  t^{-1} & 
\end{array} \right)^{-1} \cdot \gamma_\mu$$
are (excluding the repeated ones):
1. $x - \frac{1}{2} u^2 \mu^{-1} \nu y$,
2. $t^{-1} (ux - uz - \frac{1}{2} u^3 \mu^{-1} \nu y)$,
3. $t^{-2} (\mu y - u^2 x + u^2 z + \frac{1}{4} u^4 \mu^{-1} \nu y)$,
4. $t u \mu^{-1} \nu y$,
5. $u^2 \mu^{-1} \nu y + z$,
6. $t^2 \mu^{-1} \nu y$.

Problem: computing the measure of those $u$ and $t$ that makes all these entries integral.
A few technical remarks

Notice that by Iwasawa decomposition, the orbital integral can be reduced to a double integral taken over $F \times F^\times$.

We will write $v(t) = m$ and $v(u) = k$, then separate the orbital integral accordingly:

\[ m k (A) \]

\[ k \]

\[ (B) \]

\[ (C) \]

\[ m \]
Some invariants

- Goal: express the relative orbital integrals in terms of invariants.
- Let $\lambda_i$ denote the eigenvalues of $\gamma$ (we fix an ordering so that $\lambda_2 = z$).
- Invariants associated to the stable orbit of $\gamma$:

\[
M_{ij} := v(\lambda_i - \lambda_j),
\]
\[
N_{ij} := v(\lambda_i + \lambda_j).
\]
The nature of computation varies on different parts of the orbital integral.

- On (A), it involves solving a quadratic equation
  \[ u^4 \equiv 4\mu^2\xi^{-2} \pmod{\varpi^{m+k-M_{13}+v(\mu)}}. \]

- On (B), it also depends on solving a quadratic equation, along with a combinatorial datum
  \[ 2k + 2M_{12} \geq 2m > 2k + M_{12}. \]

- On (C) all the conditions are combinatorial:
  \[ 2M_{12} - M_{13} + v(\mu) \geq 2m > -M_{13} + v(\mu) \]
  \[ 4m > 4k \geq 2m - M_{13} + v(\mu). \]
The formula for relative orbital integrals

- The expressions for $O_{\gamma \mu}(f)$ depends on $M_{i,j}$.
- I have computed the formula for $O_{\gamma \mu}(f)$ for any $\gamma \mu$.
- For instance, when $M_{13} > M_{12} > 0$, $O_{\gamma \mu}(1_S(O_F))$ is equal to

$$\frac{1}{2} \left( (M_{13} - M_{12} + \delta(M_{12}, 1)) q^{[M_{12}/2]} ight.$$  

$$+ 2 \left( 1 + \delta(M_{12} - M_{13}, \mu) \right) \frac{q^{[M_{12}/2]} - 1}{q-1}$$  

$$+ \delta(M_{13}, \mu) - 1 \right).$$

Here $\delta$ is a function that detects parity.
The formula

For every $\gamma$ in type I, II and III tori I have computed the formula for endoscopic relative orbital integrals. In particular,

**Theorem (L.)**

Let $\gamma$ be in a type I torus. The endoscopic orbital integral $SO_\gamma^\kappa(1_{S(O_F)})$ is computed as in the following table:

<table>
<thead>
<tr>
<th>$M_{13}$</th>
<th>$M_{13} &gt; M_{12} = 0$</th>
<th>$M_{13} &gt; M_{12} &gt; 0$</th>
<th>$M_{12} = M_{13} = M_{23} &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{13} = 0$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}(-1)^M_{13}$</td>
<td>$\frac{1}{2}(1 + \left(\frac{z^2_y - \xi^2}{F}\right))q^{[M_{12}/2]} - 1$ + $\frac{1}{2}(-1)^M_{13}$</td>
</tr>
<tr>
<td>$M_{13} &gt; M_{12} = 0$</td>
<td></td>
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<tr>
<td>$M_{12} = M_{13} = M_{23} &gt; 0$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$k \neq 1$
Thank you for your attention.