On time-reversibility of linear processes

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Summary

A necessary and sufficient condition for time-reversibility of stationary linear processes is given which, contrary to previous results, does not require existence of moments of order higher than two.

Some key words: Non-Gaussian linear process; Time-reversibility.

1. Introduction

Let \( x_t \) be a stationary linear process:

\[
x_t = w_t * u_t = \sum_{s \in Z} w_s u_{t-s}, \tag{1.1}
\]

where \( u_t \) is a sequence of independent and identically distributed random variables with \( E(u_t) = 0 \), \( E(u_t^2) = \sigma^2 \), \( w_t \) is a square-summable sequence of constants, and \( Z \) is the set of all integers. The relationship between \( w_t \) and the spectral density \( S_x(\omega) \) of \( x_t \) is

\[
S_x(\omega) = |W(\omega)|^2 E(u_t^2), \tag{1.2}
\]

where \( W(\omega) \) and \( S_x(\omega) \) are the Fourier transforms of \( w_t \) and \( E(x_{t+s} x_s) \) respectively. In this paper, we will assume that

\[
S_x(\omega) = |W(\omega)|^2 E(u_t^2) \neq 0 \tag{1.3}
\]

almost everywhere.

A stationary process \( x_t \) is time-reversible if, for any positive integer \( n \) and any \( t_1, \ldots, t_n \in Z \), \((x_{t_1}, \ldots, x_{t_n})\) and \((x_{-t_1}, \ldots, x_{-t_n})\) have the same joint probability distributions.

Weiss (1975) shows that, for causal autoregressive moving-average linear processes, time-reversibility is essentially restricted to Gaussian processes under certain conditions. Hallin, Lefèvre & Puri (1988) extend this result to general linear processes under some conditions including the existence of moments of all orders.

In this paper, a necessary and sufficient condition for time-reversibility of stationary linear processes is given which, contrary to previous results, does not require existence of moments of order higher than two.

In § 2, we prove a basic lemma. In § 3, we provide a basic theorem which gives a necessary and sufficient condition for distributional equivalence of two linear processes without requiring the existence of higher-order moments. The relation between distributional equivalence and time-reversibility is investigated. The time-reversibility theorem is given as a consequence of the basic theorem.
2. The basic lemma

Lemma 1. Let $u_t$ and $v_t$, for $t \in \mathbb{Z}$, be two independent and identically distributed processes with zero means and finite variances, and let

$$v_t = h_t * u_t = \sum_{s \in \mathbb{Z}} h_s u_{t-s}, \quad (2.1)$$

where $h_t$ is a constant sequence. If $v_t$ is non-Gaussian, then $h_t = a \delta_{t-t_0}$, where $a$ is a nonzero constant, $t_0$ is an integer, $\delta_0 = 1$ and $\delta_t = 0$ for $t \neq 0$.

Proof. Since $u_t$ and $v_t$ are independent and identically distributed, we have

$$E(u_t u_s) = E(u_0^2 \delta_{t-s}), \quad E(v_t v_s) = E(v_0^2 \delta_{t-s}). \quad (2.2)$$

From (2.1), we get

$$E(v_t^2 \delta_t) = E(v_t v_0) = E\left( \sum_{s} h_s u_{t-s} \sum_{l} h_l u_{t-l} \right) = \sum_{s} h_s h_l E(u_{t-s} u_{t-l})$$

$$= \sum_{s} h_s h_l E(u_0^2 \delta_{t-s-l}) = E(u_0^2) \sum_{s} h_s h_{-(t-s-l)} = E(u_0^2) h_t * h_{-t};$$

that is,

$$h_t * h_{-t} = b \delta_t, \quad (2.3)$$

where $b = E(v_0^2 / E(u_0^2)$.

It follows from (2.1) and (2.3) that

$$u_t = b^{-1} h_{-t} * v_t = b^{-1} \sum_{l} h_{-l} v_{t-l}. \quad (2.4)$$

Let $f(\lambda)$ and $g(\lambda)$ denote the characteristic functions of $v_t$ and $u_t$ respectively. By independence, from (2.1) and (2.4) we get

$$f(\lambda) = \prod_{s} g(h_s \lambda), \quad g(\lambda) = \prod_{l} f(b^{-1} h_{-l} \lambda) \quad (2.5)$$

and thereby

$$f(\lambda) = \prod_{s} \prod_{l} f(b^{-1} h_s h_{-l} \lambda). \quad (2.6)$$

Applying Theorem 5.6.1 of Kagan, Linnik & Rao (1973) yields that there exists $(t_0, t_1)$ such that

$$h_{t_0} h_{-t_1} \neq 0, \quad h_t h_{-l} = 0, \quad (s, l) \neq (t_0, t_1).$$

That means $h_t = 0$, for $t \neq t_0$. This shows that $h_t = a \delta_{t-t_0}$ and the proof is complete. \qed

3. The basic theorem and time-reversibility

Two processes $x_t$ and $y_t$ are said to be distributionally equivalent if, for any positive integer $n$ and any $t_1, t_2, \ldots, t_n \in \mathbb{Z}$, $(x_{t_1}, \ldots, x_{t_n})$ and $(y_{t_1}, \ldots, y_{t_n})$ have the same joint probability distribution.

Theorem 1. Let $x_t$ and $y_t$ be two non-Gaussian linear processes:

$$x_t = w_t * u_t, \quad y_t = w'_t * u'_t \quad (t \in \mathbb{Z}), \quad (3.1)$$

where $u_t$ and $u'_t$ are independent and identically distributed with zero means and finite variances, and $w_t$ and $w'_t$ are square-summable sequences satisfying (1.3). In order for $x_t$ and $y_t$ to be distributionally equivalent, it is necessary and sufficient that

(i) $w'_t = a^{-1} w_{t+t_0}, \quad (3.2)$

(ii) $u'_t$ and $au_{t-t_0}$ have the same distributions,

where $a$ is a nonzero constant and $t_0$ is an integer.
Proof: Necessity. Let $H_x$ and $H_y$ denote two Hilbert subspaces generated by $x_t$ and $y_t$, respectively. As a result of the equivalence of $x_t$ and $y_t$ under the correspondent relation $x_t \leftrightarrow y_t$, $H_x$ and $H_y$ are isometrically isomorphic. Then there exist $v_t \in H_y$ corresponding to $u_t \in H_x$. As a result of the equivalence and isometrical isomorphism, $v_t$ is independent and identically distributed with the same distribution as $u_t$, and $y_t = w_t \ast v_t$.

Then $y_t = w_t \ast v_t = w'_t \ast u'_t$. By formula (3.5) of Cheng (1990), we have

$$u'_t = c_t \ast v_t = \sum_s c_s v_{t-s}.$$

By Lemma 1, it follows that

$$u'_t = av_{t-t_0}, \quad w'_t = \frac{1}{a} w_{t+t_0}.$$

Hence, necessity holds. Sufficiency is obvious and the theorem is proved. \hfill \Box

The following corollary is immediate.

**Corollary 1.** Let $x_t$ and $y_t$ be two stationary linear processes:

$$x_t = w_t \ast u_t, \quad y_t = w'_t \ast u'_t \quad (t \in Z),$$

where $u_t$ and $u'_t$ are two sequences of independent and identically distributed random variables with zero means and finite variances, and $w_t$ and $w'_t$ are square-summable sequences satisfying (1.3). Then $x_t$ and $y_t$ are distributionally equivalent if and only if either

(a) $x_t$ and $y_t$ are Gaussian, and their spectral density functions $S_x(\omega)$ and $S_y(\omega)$ are identical, or

(b) $w'_t = a^{-1}w_{t+t_0}$, and $u'_t$ and $u_{t-t_0}$ have the same distribution, where $a$ is a nonzero constant and $t_0$ is an integer.

We now turn to discuss time-reversibility, which is in fact a special case of distributional equivalence. For a linear process $x_t = w_t \ast u_t$, we have $x_{-t} = w_{-t} \ast u_{-t}$. For an independent and identically distributed series $u_t$, $u_{-t}$ is independent and identically distributed too. If we set $y_t = x_{-t} = w_{-t} \ast u_{-t}$, by Theorem 1 the following theorem is immediate.

**Theorem 2.** Let $x_t$ be a non-Gaussian linear process, $x_t = w_t \ast u_t$, where the $u_t$ are independent and identically distributed, and $w_t$ is a square-summable sequence satisfying (1.3). Then $x_t$ is time-reversible if and only if

(i) $w_t = aw_{t_0-t}.$ \hfill (3.3)

(ii) $u_t$ and $au_t$ have the same distributions,

where $a$ is a nonzero constant and $t_0$ is an integer.

Note that $a = 1$ or $-1$, because $E(u_t^2) = E(a^2u_t^2)$. It follows from (3.3) that $a$ is unique.

Theorem 2 can be rewritten in the following form.

**Corollary 2.** Let $x_t = w_t \ast u_t$, where $t \in Z$, $u_t$ is independent and identically distributed with zero means and finite variances, and $w_t$ is square-summable and satisfies (1.3). Then $x_t$ is time-reversible if and only if either (a) $x_t$ is Gaussian, or (b) conditions (i) and (ii) of Theorem 2 hold.

Now we discuss the general solution of (3.3) when $w_t$ is real. Let $W(\omega)$ be the Fourier transform of $w_t$:

$$W(\omega) = A(\lambda) \exp \{ i\theta(\omega) \} = \sum_t w_t e^{-i\omega t}, \quad (3.4)$$

where $A(\lambda)$ and $\theta(\omega)$ are the amplitude spectrum and the phase spectrum of $w_t$, respectively, which satisfy

$$A(-\omega) = A(\omega), \quad \theta(-\omega) = -\theta(\omega). \quad (3.5)$$
From (3.3) it follows that
\[ \exp\{-i\theta(\omega)\} = a \exp\{it_0 \omega + i\theta(\omega)\}. \] (3.6)

Note that \( a = \exp\{i\pi(a - 1)/2\} \). Then the general solution of (3.6) is
\[ \theta(\omega) = \frac{1}{2} \{2\pi(1 - a) + t_0 \omega\} + \pi k(\omega) \quad (k(\omega) \in \mathbb{Z}). \] (3.7)

Hence, the general solution of (3.3) is \( W(\omega) = A(\omega) \exp\{i\theta(\omega)\} \), where \( A(\omega) \) is square integrable in \([-\pi, \pi]\) and positive almost everywhere, and \( \theta(\omega) \) satisfies (3.5) and (3.7).

For causal sequences \( w_t \), that is \( w_t = 0 \) for \( t < u \), from (3.3) we have
\[ w_t = \begin{cases} 
 0 & (t < 0 \text{ or } t > t_0), \\
 aw_{t_0 - t} & (0 \leq t \leq t_0),
\end{cases} \]
where \( t_0 \) is a unique nonnegative integer. Thus, \( x_t \) is a special moving average model.

References


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