WELL-POSEDNESS OF THE ERICKSEN-LESLIE SYSTEM

WEI WANG, PINGWEN ZHANG, AND ZHIFEI ZHANG

Abstract. In this paper, we prove the local well-posedness of the Ericksen-Leslie system, and the global well-posednss for small initial data under the physical constrain condition on the Leslie coefficients, which ensures that the energy of the system is dissipated. Instead of the Ginzburg-Landau approximation, we construct an approximate system with the dissipated energy based on a new formulation of the system.

1. Introduction

The hydrodynamic theory of liquid crystals was established by Ericksen [4, 5] and Leslie [9] in the 1960's. This theory treats the liquid crystal material as a continuum and completely ignores molecular details. Moreover, this theory considers perturbations to a presumed oriented sample. The configuration of the liquid crystals is described by a director field $\mathbf{n}(t,\mathbf{x})\in\mathbb{S}^2,\mathbf{x}\in\mathbb{R}^3$.

The general Ericksen-Leslie system takes the form

$$\begin{cases}\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+\dfrac{\gamma}{Re}\Delta\mathbf{v}+\dfrac{1-\gamma}{Re}\nabla\cdot\sigma,\\ \nabla\cdot\mathbf{v}=0,\\ \mathbf{n}\times\big(\mathbf{h}-\gamma_1\mathbf{N}-\gamma_2\mathbf{D}\cdot\mathbf{n}\big)=0,\end{cases}\tag{1.1}$$

where $\mathbf{v}$ is the velocity of the fluid, p is the pressure, Re is the Reynolds number and $\gamma\in(0,1)$. The stress σ is modeled by the phenomenological constitutive relation

$$\sigma=\sigma^L+\sigma^E,$$

where σ^L is the viscous (Leslie) stress

$$\sigma^L=\alpha_1(\mathbf{nn}:\mathbf{D})\mathbf{nn}+\alpha_2\mathbf{nN}+\alpha_3\mathbf{Nn}+\alpha_4\mathbf{D}+\alpha_5\mathbf{nn}\cdot\mathbf{D}+\alpha_6\mathbf{D}\cdot\mathbf{nn}\tag{1.2}$$

with $\mathbf{D}=\frac{1}{2}(\kappa^T+\kappa),\kappa=(\nabla\mathbf{v})^T$, and

$$\mathbf{N}=\mathbf{n}_t+\mathbf{v}\cdot\nabla\mathbf{n}+\mathbf{\Omega}\cdot\mathbf{n},\quad \mathbf{\Omega}=\frac{1}{2}(\kappa^T-\kappa).$$

The six constants $\alpha_1,\cdots,\alpha_6$ are called the Leslie coefficients. While, σ^E is the elastic (Ericksen) stress

$$\sigma^E=-\frac{\partial E_F}{\partial(\nabla\mathbf{n})}\cdot(\nabla\mathbf{n})^T,\tag{1.3}$$

where $E_F=E_F(\mathbf{n},\nabla\mathbf{n})$ is the Oseen-Frank energy with the form

$$E_F=\frac{k_1}{2}(\nabla\cdot\mathbf{n})^2+\frac{k_2}{2}|\mathbf{n}\times(\nabla\times\mathbf{n})|^2+\frac{k_3}{2}|\mathbf{n}\cdot(\nabla\times\mathbf{n})|^2.$$

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Here k_1, k_2, k_3 are the elastic constant. For the simplicity, we will consider the case $k_1 = k_2 = k_3 = 1$. In such case, $E_F = \frac{1}{2}|\nabla \mathbf{n}|^2$, and the molecular field $\mathbf{h}$ is given by

$$\mathbf{h} = -\frac{\delta E_F}{\delta \mathbf{n}} = \nabla \cdot \frac{\partial E_F}{\partial(\nabla \mathbf{n})} - \frac{\partial E_F}{\partial \mathbf{n}} = -\Delta \mathbf{n},$$
$$\big(\sigma^E\big)_{ij} = -\big(\nabla \mathbf{n} \odot \nabla \mathbf{n}\big)_{ij} = -\partial_i n_k \partial_j n_k.$$

Finally, the Leslie coefficients and γ_1, γ_2 satisfy the following relations

$$\alpha_2 + \alpha_3 = \alpha_6 - \alpha_5, \tag{1.4}$$

$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \tag{1.5}$$

where (1.4) is called Parodi's relation derived from the Onsager reciprocal relation [15]. These two relations ensure that the system has a basic energy law.

As the general Ericksen-Leslie system is very complicated, most of earlier works treated the simplified(or approximated) system of (1.1). Motivated by the work on the harmonic heat flow, Lin and Liu [12] add the penality term $\frac{1}{4\varepsilon^2}(|\mathbf{n}|^2 - 1)^2$ in W in order to remove some higher-order nonlinearities due to the constraint $|\mathbf{n}| = 1$. In such case, the system becomes

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \dfrac{\gamma}{Re} \Delta \mathbf{v} + \dfrac{1-\gamma}{Re} \nabla \cdot \sigma, \\ \mathbf{n}_t + \mathbf{v} \cdot \nabla \mathbf{n} + \mathbf{\Omega} \cdot \mathbf{n} - \mu_1 \Delta \mathbf{n} - \mu_2 \mathbf{D} \cdot \mathbf{n} - \dfrac{1}{\varepsilon^2}(|\mathbf{n}|^2 - 1)\mathbf{n} = 0. \end{cases} \tag{1.6}$$

This is so called the Ginzburg-Landau approximation. They proved the global existence of weak solution and the local existence and uniqueness of strong solution of the system (1.6) under certain strong constrains on the Leslie coefficients. We refer to [18] for a recent result about the role of Parodi's relation in the well-posedness and stability. However, whether the solution of (1.6) converges to that of (1.1) as ε tends to zero is still a challenging question. When neglecting the Leslie stress σ_L in (1.1), a simplest system preserving the basic energy law is the following

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} - \Delta \mathbf{v} + \nabla p = -\nabla \cdot \big(\nabla \mathbf{n} \odot \nabla \mathbf{n}\big), \\ \mathbf{n}_t + \mathbf{v} \cdot \nabla \mathbf{n} - \Delta \mathbf{n} = |\nabla \mathbf{n}|^2 \mathbf{n}. \end{cases} \tag{1.7}$$

For this system, the local existence and uniqueness of strong solution can be proved by using the standard energy method; see [16] for the well-posedness result with rough data. Huang and Wang [7] give the following BKM type blow-up criterion: Let T^* be the maximal existence time of the strong solution. If $T^* < \infty$, then it is necessary

$$\int_0^{T^*} \|\nabla \times \mathbf{v}(t)\|_{L^\infty} + \|\nabla \mathbf{n}(t)\|_{L^\infty}^2 \mathrm{d}t = +\infty.$$

In two dimensional case, the global existence of weak solution has been independently proved by Lin, Lin and Wang [13] and Hong [6], where they construct a class of weak solution with at most a finite number of singular times. The uniqueness of weak solution is proved by Lin-Wang [14] and Xu-Zhang [19]. The global existence of weak solution of (1.7) is a challenging open problem in three dimensional case. On the other hand, in the case when $|\nabla \mathbf{n}|^2 \mathbf{n}$ in (1.7) is replaced by $\frac{1}{\varepsilon^2}(|\mathbf{n}|^2 - 1)\mathbf{n}$, the global existence and partial regularity of weak solution were studied in [10, 11].

The purpose of this paper is to study the well-posedness of the general Ericksen-Leslie system. The first step is to understand the complicated energy-dissipation law of the system arising from the Leslie stress. Moreover, whether the energy defined in (2.1) is dissipated remains unknown in physics, since the Leslie coefficients are difficult to determine by using experimental results. We present a sufficient and necessary condition on the Leslie coefficients to ensure that the energy of the system is dissipated. The next step is to construct an approximate system with the dissipated energy under the physical condition on the Leslie coefficients. However, the Ginzburg-Landau approximation does not satisfy our requirement. We introduce a new equivalent formulation of the system (1.1). Based on this formulation, we can construct an approximate system such that the energy is still dissipated, although the key property $|\mathbf{n}| = 1$ is destroyed.

Our main results are stated as follows.

Theorem 1.1. *Let $s \geq 2$ be an integer. Assume that the Leslie coefficients satisfy (2.6), and the initial data $\nabla \mathbf{n}_0 \in H^{2s}(\mathbb{R}^3), \mathbf{v}_0 \in H^{2s}(\mathbb{R}^3)$. There exist $T > 0$ and a unique solution $(\mathbf{v}, \mathbf{n})$ of the Ericksen-Leslie system (1.1) such that*

$$\mathbf{v} \in C([0,T]; H^{2s}(\mathbb{R}^3)) \cap L^2(0,T; H^{2s+1}(\mathbb{R}^3)), \quad \nabla \mathbf{n} \in C([0,T]; H^{2s}(\mathbb{R}^3)).$$

Let T^* be the maximal existence time of the solution. If $T^* < +\infty$, then it is necessary

$$\int_0^{T^*} \|\nabla \times \mathbf{v}(t)\|_{L^\infty} + \|\nabla \mathbf{n}(t)\|_{L^\infty}^2 \mathrm{d}t = +\infty.$$

For small initial data, we prove the following global well-posedness.

Theorem 1.2. *With the same assumptions as in Theorem 1.1, there exists an $\varepsilon_0 > 0$ such that if*

$$\|\nabla \mathbf{n}_0\|_{H^{2s}} + \|\mathbf{v}_0\|_{H^{2s}} \leq \varepsilon_0,$$

then the solution obtained in Theorem 1.1 is global in time.

The other sections of this paper are organized as follows. In section 2, we derive the basic energy law of the system and give the physical constrain condition on the Leslie coefficients. In section 3, we introduce a new equivalent formulation. Section 4 is devoted to the proof of local well-posedness. In section 5, we prove the global well-posedness of the system for small initial data.

2. Basic energy-dissipation law

We first derive the basic energy law of the system (1.1).

Proposition 2.1. *If $(\mathbf{v}, \mathbf{n})$ is a smooth solution of (1.1), then it holds that*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{Re}{2(1-\gamma)}|\mathbf{v}|^2 + E_F \mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^3} \Big(\frac{\gamma}{1-\gamma}|\nabla \mathbf{v}|^2 + (\alpha_1 + \frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2 + \alpha_4 \mathbf{D}:\mathbf{D} \\ + (\alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1})|\mathbf{D}\cdot\mathbf{n}|^2 + \frac{1}{\gamma_1}|\mathbf{n}\times\mathbf{h}|^2 \Big)\mathrm{d}\mathbf{x}.\end{aligned} \tag{2.1}$$

Proof. Using the first equation of (1.1) and $\nabla \cdot \mathbf{v} = 0$, we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{Re}{2(1-\gamma)}|\mathbf{v}|^2 + E_F \mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \frac{Re}{1-\gamma}\mathbf{v}\cdot\mathbf{v}_t \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3} \frac{\delta E_F}{\delta \mathbf{n}} \cdot \mathbf{n}_t \mathrm{d}\mathbf{x} \\ &= -\int_{\mathbb{R}^3} \frac{\gamma}{1-\gamma}|\nabla \mathbf{v}|^2 + (\sigma^L + \sigma^E) : \nabla \mathbf{v} \mathrm{d}\mathbf{x} + \int_{\mathbb{R}^3} \frac{\delta E_F}{\delta \mathbf{n}} \cdot (\dot{\mathbf{n}} - \mathbf{v}\cdot\nabla\mathbf{n})\mathrm{d}\mathbf{x},\end{aligned} \tag{2.2}$$

where $\dot{\mathbf{n}} = \mathbf{n}_t + \mathbf{v}\cdot\nabla\mathbf{n}$. Using $\nabla\cdot\mathbf{v} = 0$ again, we have

$$\begin{aligned}&\int_{\mathbb{R}^3} \sigma^E : \nabla \mathbf{v} + \frac{\delta E_F}{\delta \mathbf{n}} \cdot (\mathbf{v}\cdot\nabla\mathbf{n})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} (-\frac{\partial E_F}{\partial(\nabla \mathbf{n})} \cdot (\nabla \mathbf{n})^T) : \nabla \mathbf{v} - \Big(\nabla \cdot \frac{\partial E_F}{\partial(\nabla \mathbf{n})} - \frac{\partial E_F}{\partial \mathbf{n}}\Big) \cdot (\mathbf{v}\cdot\nabla\mathbf{n})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \frac{\partial E_F}{\partial(\nabla \mathbf{n})} : (\mathbf{v}\cdot\nabla^2\mathbf{n}) + \frac{\partial E_F}{\partial \mathbf{n}} \cdot (\mathbf{v}\cdot\nabla\mathbf{n})\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}^3} \mathbf{v}\cdot\nabla E_F(\mathbf{n}, \nabla\mathbf{n})\mathrm{d}\mathbf{x} = 0.\end{aligned} \tag{2.3}$$

Due to (1.2), (1.4) and (1.5), we find

$$\begin{aligned}
&\int_{\mathbb{R}^3}\sigma^L:\nabla\mathbf{v}\mathrm{d}\mathbf{x}\\
=&\int_{\mathbb{R}^3}\Big((\alpha_1(\mathbf{nn}\cdot\mathbf{D})\mathbf{nn}+\alpha_2\mathbf{nN}+\alpha_3\mathbf{Nn}+\alpha_4\mathbf{D}+\alpha_5\mathbf{nn}\cdot\mathbf{D}+\alpha_6\mathbf{D}\cdot\mathbf{nn}):(\mathbf{D}+\mathbf{\Omega})\Big)\mathrm{d}\mathbf{x}\\
=&\int_{\mathbb{R}^3}\Big(\alpha_1(\mathbf{nn}:\mathbf{D})^2+\alpha_4\mathbf{D}:\mathbf{D}+(\alpha_5+\alpha_6)|\mathbf{D}\cdot\mathbf{n}|^2+(\alpha_2+\alpha_3)\mathbf{n}\cdot(\mathbf{D}\cdot\mathbf{N})\\
&\qquad+(\alpha_2-\alpha_3)\mathbf{n}\cdot(\mathbf{\Omega}\cdot\mathbf{N})-(\alpha_5-\alpha_6)(\mathbf{D}\cdot\mathbf{n})\cdot(\mathbf{\Omega}\cdot\mathbf{n})\Big)\mathrm{d}\mathbf{x}\\
=&\int_{\mathbb{R}^3}\Big(\alpha_1(\mathbf{nn}:\mathbf{D})^2+\alpha_4\mathbf{D}:\mathbf{D}+(\alpha_5+\alpha_6)|\mathbf{D}\cdot\mathbf{n}|^2+\gamma_2\mathbf{n}\cdot(\mathbf{D}\cdot\mathbf{N})\\
&\qquad-\gamma_1\mathbf{n}\cdot(\mathbf{\Omega}\cdot\mathbf{N})+\gamma_2(\mathbf{D}\cdot\mathbf{n})\cdot(\mathbf{\Omega}\cdot\mathbf{n})\Big)\mathrm{d}\mathbf{x}\\
=&\int_{\mathbb{R}^3}\Big(\alpha_1(\mathbf{nn}:\mathbf{D})^2+\alpha_4\mathbf{D}:\mathbf{D}+(\alpha_5+\alpha_6)|\mathbf{D}\cdot\mathbf{n}|^2+\gamma_2\mathbf{N}\cdot(\mathbf{D}\cdot\mathbf{n})\\
&\qquad+\gamma_1\mathbf{N}\cdot(\mathbf{\Omega}\cdot\mathbf{n})+\gamma_2(\mathbf{D}\cdot\mathbf{n})\cdot(\mathbf{\Omega}\cdot\mathbf{n})\Big)\mathrm{d}\mathbf{x},
\end{aligned}$$

and

$$-\int_{\mathbb{R}^3}\frac{\delta E_F}{\delta\mathbf{n}}\cdot\dot{\mathbf{n}}\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^3}\mathbf{h}\cdot\dot{\mathbf{n}}\mathrm{d}\mathbf{x}=\int_{\mathbb{R}^3}\mathbf{h}\cdot(\mathbf{N}-\mathbf{\Omega}\cdot\mathbf{n})\mathrm{d}\mathbf{x}.$$

The third equation of (1.1) implies that

$$\int_{\mathbb{R}^3}(\mathbf{\Omega}\cdot\mathbf{n})\cdot\big(\gamma_1\mathbf{N}+\gamma_2(\mathbf{D}\cdot\mathbf{n})-\mathbf{h}\big)\mathrm{d}\mathbf{x}=0,$$

and direct calculations show that

$$\begin{aligned}
\int_{\mathbb{R}^3}\big(\gamma_2\mathbf{N}\cdot(\mathbf{D}\cdot\mathbf{n})+\mathbf{h}\cdot\mathbf{N}\big)\mathrm{d}\mathbf{x}&=\int_{\mathbb{R}^3}\big(\mathbf{n}\times\mathbf{N}\big)\cdot\big(\mathbf{n}\times\mathbf{h}+\gamma_2\mathbf{n}\times\mathbf{D}\cdot\mathbf{n}\big)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\frac{1}{\gamma_1}\big(\mathbf{n}\times\mathbf{h}-\gamma_2\mathbf{n}\times\mathbf{D}\cdot\mathbf{n}\big)\cdot\big(\mathbf{n}\times\mathbf{h}+\gamma_2\mathbf{n}\times\mathbf{D}\cdot\mathbf{n}\big)\mathrm{d}\mathbf{x}\\
&=\int_{\mathbb{R}^3}\frac{1}{\gamma_1}|\mathbf{n}\times\mathbf{h}|^2-\frac{\gamma_2^2}{\gamma_1}|\mathbf{D}\cdot\mathbf{n}|^2+\frac{\gamma_2^2}{\gamma_1}|\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}|^2\mathrm{d}\mathbf{x}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\sigma^L:\nabla\mathbf{v}-\frac{\delta E_F}{\delta\mathbf{n}}\cdot\dot{\mathbf{n}}\mathrm{d}\mathbf{x}\\
=&\int_{\mathbb{R}^3}\Big((\alpha_1+\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}:\mathbf{nn}|^2+\alpha_4\mathbf{D}:\mathbf{D}+(\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1})|\mathbf{D}\cdot\mathbf{n}|^2+\frac{1}{\gamma_1}|\mathbf{n}\times\mathbf{h}|^2\Big)\mathrm{d}\mathbf{x}.
\end{aligned}\tag{2.4}$$

Then the energy law (2.1) follows from (2.2)-(2.4). □

The following proposition presents a sufficient and necessary condition on the Leslie coefficients to ensure that the energy is dissipated; see also [12] for the related discussions on the choice of the Leslie coefficients. We denote

$$\beta_1=\alpha_1+\frac{\gamma_2^2}{\gamma_1},\quad \beta_2=\alpha_4,\quad \beta_3=\alpha_5+\alpha_6-\frac{\gamma_2^2}{\gamma_1}.$$

Proposition 2.2. *The following dissipation relation holds*

$$\beta_1(\mathbf{nn}:\mathbf{D})^2+\beta_2\mathbf{D}:\mathbf{D}+\beta_3|\mathbf{D}\cdot\mathbf{n}|^2\geq 0\tag{2.5}$$

for any symmetric trace free matrix $\mathbf{D}$ *and unit vector* $\mathbf{n}$*, if and only if*

$$\beta_2 \geq 0, \quad 2\beta_2 + \beta_3 \geq 0, \quad \frac{3}{2}\beta_2 + \beta_3 + \beta_1 \geq 0. \tag{2.6}$$

Proof. By the rotation invariance, we may assume $\mathbf{n} = (0,0,1)^T$ and $\mathbf{D} = (D_{ij})_{3\times 3}$ with $D_{11} + D_{22} + D_{33} = 0$. It is easy to get

$$\begin{aligned}
&\beta_1(\mathbf{nn}:\mathbf{D})^2 + \beta_2\mathbf{D}:\mathbf{D} + \beta_3|\mathbf{D}\cdot\mathbf{n}|^2\\
&= \beta_1 D_{33}^2 + \beta_2(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{32}^2 + 2D_{31}^2) + \beta_3(D_{31}^2 + D_{32}^2 + D_{33}^2)\\
&= 2\beta_2 D_{12}^2 + (2\beta_2+\beta_3)(D_{31}^2 + D_{32}^2) + \beta_2(D_{11}^2 + D_{22}^2) + (\beta_1+\beta_2+\beta_3)D_{33}^2\\
&= 2\beta_2 D_{12}^2 + (2\beta_2+\beta_3)(D_{31}^2 + D_{32}^2) + \beta_2(D_{11}^2 + D_{22}^2) + (\beta_1+\beta_2+\beta_3)(D_{11}+D_{22})^2.
\end{aligned}$$

The inequality holds

$$2\beta_2 D_{12}^2 + (2\beta_2+\beta_3)(D_{31}^2 + D_{32}^2) \geq 0$$

for all D_{12}, D_{31}, and D_{32}, if and only if $\beta_2 \geq 0$, and $2\beta_2 + \beta_3 \geq 0$.

As $D_{11}^2 + D_{22}^2 \geq \frac{1}{2}(D_{11}+D_{22})^2$, the inequality holds

$$\beta_2(D_{11}^2 + D_{22}^2) + (\beta_1+\beta_2+\beta_3)(D_{11}+D_{22})^2 \geq 0$$

for all D_{11} and D_{22}, if and only if $\frac{3}{2}\beta_2 + \beta_3 + \beta_1 \geq 0$. □

In [17], we show that if the Ericksen-Leslie system is derived from the Doi-Onsager equation, then the energy (2.1) is indeed dissipated. Let us make it precise. The nondimensional Doi-Onsager equation takes as follows

$$\begin{cases}
\dfrac{\partial f^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon\cdot\nabla f^\varepsilon = \dfrac{1}{\varepsilon}\mathcal{R}\cdot(\mathcal{R}f^\varepsilon + f^\varepsilon\mathcal{R}\mathcal{U}_\varepsilon f^\varepsilon) - \mathcal{R}\cdot(\mathbf{m}\times\kappa^\varepsilon\cdot\mathbf{m}f^\varepsilon),\\
\dfrac{\partial \mathbf{v}^\varepsilon}{\partial t} + \mathbf{v}^\varepsilon\cdot\nabla\mathbf{v}^\varepsilon = -\nabla p^\varepsilon + \dfrac{\gamma}{Re}\Delta\mathbf{v}^\varepsilon + \dfrac{1-\gamma}{2Re}\nabla\cdot\big(\mathbf{D}^\varepsilon:\langle\mathbf{mmmm}\rangle_{f^\varepsilon}\big)\\
\qquad\qquad + \dfrac{1-\gamma}{\varepsilon Re}(\nabla\cdot\tau_\varepsilon^e + \mathbf{F}_\varepsilon^e),
\end{cases} \tag{2.7}$$

where ε is the Deborah number, $\kappa^\varepsilon = (\nabla v^\varepsilon)^T, \mathbf{D}^\varepsilon = \frac{1}{2}\big(\kappa^\varepsilon + (\kappa^\varepsilon)^T\big)$, and

$$\begin{aligned}
&\tau_\varepsilon^e = -\langle\mathbf{mm}\times\mathcal{R}\mu_\varepsilon\rangle_{f^\varepsilon}, \quad \mathbf{F}_\varepsilon^e = -\langle\nabla\mu_\varepsilon\rangle_{f^\varepsilon}, \quad \mu_\varepsilon = \ln f^\varepsilon + \mathcal{U}_\varepsilon f,\\
&\mathcal{U}_\varepsilon f = \alpha\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|\mathbf{m}\times\mathbf{m}'|^2\frac{1}{\sqrt{\varepsilon}^3}g\big(\frac{\mathbf{x}-\mathbf{x}'}{\sqrt{\varepsilon}}\big)f(\mathbf{x}',\mathbf{m}',t)\mathrm{d}\mathbf{m}'\mathrm{d}\mathbf{x}'.
\end{aligned}$$

When ε is small, the solution $(f^\varepsilon, \mathbf{v}^\varepsilon)$ of the system (2.7) has the expansion

$$\begin{aligned}
f^\varepsilon &= f_0(\mathbf{m}\cdot\mathbf{n}) + \varepsilon f_1 + \cdots,\\
\mathbf{v}^\varepsilon &= \mathbf{v}_0 + \varepsilon\mathbf{v}_1 + \cdots,
\end{aligned}$$

where $(\mathbf{v}_0, \mathbf{n})$ is determined by (1.1) with the Leslie coefficients given by

$$\begin{aligned}
&\alpha_1 = -\frac{S_4}{2}, \quad \alpha_2 = -\frac{1}{2}\big(1+\frac{1}{\lambda}\big)S_2, \quad \alpha_3 = -\frac{1}{2}\big(1-\frac{1}{\lambda}\big)S_2,\\
&\alpha_4 = \frac{4}{15} - \frac{5}{21}S_2 - \frac{1}{35}S_4, \quad \alpha_5 = \frac{1}{7}S_4 + \frac{6}{7}S_2, \quad \alpha_6 = \frac{1}{7}S_4 - \frac{1}{7}S_2.
\end{aligned} \tag{2.8}$$

Here $S_2 = \langle P_2(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\eta_1,\mathbf{n}}}$, $S_4 = \langle P_4(\mathbf{m}\cdot\mathbf{n})\rangle_{h_{\eta_1,\mathbf{n}}}$ with $P_k(x)$ the k-th Legendre polynomial and

$$h_{\eta_1,\mathbf{n}}(\mathbf{m}) = \frac{\mathrm{e}^{\eta_1(\mathbf{m}\cdot\mathbf{n})^2}}{\int_{\mathbb{S}^2}\mathrm{e}^{\eta_1(\mathbf{m}\cdot\mathbf{n})^2}\mathrm{d}\mathbf{m}}.$$

Here η_1 and λ are constants depending only on α. When the Leslie coefficients are given by (2.8), we show that the dissipation relation (2.5) holds; see [17, 8, 3] for the details.

3. A New formulation of the Ericksen-Leslie system

Set $\mu_1 = \frac{1}{\gamma_1}, \mu_2 = -\frac{\gamma_2}{\gamma_1}$. The third equation of (1.1) is equivalent to

$$\mathbf{n}_t + \mathbf{v}\cdot\nabla\mathbf{n} + \mathbf{\Omega}\cdot\mathbf{n} - (\mathbf{I}-\mathbf{nn})\cdot(\mu_1\mathbf{h}+\mu_2\mathbf{D}\cdot\mathbf{n}) = 0,$$

which can be written as

$$\mathbf{n}_t + \mathbf{v}\cdot\nabla\mathbf{n} + \mathbf{n}\times\big((\mathbf{\Omega}\cdot\mathbf{n} - \mu_1\mathbf{h} - \mu_2\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big) = 0. \tag{3.1}$$

Substituting them into (1.2), we get

$$\begin{aligned}\sigma^L =&\beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} - \frac{1}{2}(1+\mu_2)\mathbf{n}(\mathbf{I}-\mathbf{nn})\cdot\mathbf{h} + \frac{1}{2}(1-\mu_2)(\mathbf{I}-\mathbf{nn})\cdot\mathbf{hn}\\ &+\beta_2\mathbf{D} + \frac{\beta_3}{2}(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn})\\ =&\beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} - \frac{1}{2}(1+\mu_2)\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}(1-\mu_2)\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\\ &+\beta_2\mathbf{D} + \frac{\beta_3}{2}(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn}).\end{aligned}\tag{3.2}$$

With the new formulation (3.1) and (3.2), we can derive the same energy law (2.1) without using the constrain $|\mathbf{n}| = 1$. To see it, we need the following important cancelation relations.

Lemma 3.1. *It holds that*

$$\begin{aligned}&\Big(-\frac{1}{2}\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\Big):(\mathbf{D}+\mathbf{\Omega}) - \big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) = 0,\\ &-\Big(\frac{1}{2}\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\Big):(\mathbf{D}+\mathbf{\Omega}) + (\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big) = 0.\end{aligned}$$

Proof. Direct calculations show that

$$\begin{aligned}&\Big(-\frac{1}{2}\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\Big):(\mathbf{D}+\mathbf{\Omega}) - \big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big)\\ &= \big(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\big):\mathbf{\Omega} - \big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big)\\ &= \big(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\big)\cdot(\mathbf{\Omega}\cdot\mathbf{n}) - \big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) = 0,\end{aligned}$$

and

$$\begin{aligned}&-\Big(\frac{1}{2}\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\Big):(\mathbf{D}+\mathbf{\Omega}) + (\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\\ &= -\big(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\big):\mathbf{D} + \big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot(\mathbf{h}\times\mathbf{n})\\ &= -\big(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\big)\cdot(\mathbf{D}\cdot\mathbf{n}) + \big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot(\mathbf{h}\times\mathbf{n}) = 0.\end{aligned}$$

The proof is finished. □

Now we derive the energy law (2.1) by using (3.1) and (3.2), since the derivation will be helpful to understand the energy estimates in the next section. Thanks to (1.1) and (3.1), we have

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{Re}{1-\gamma}|\mathbf{v}|^2 + |\nabla\mathbf{n}|^2\mathrm{d}\mathbf{x} = -\int_{\mathbb{R}^3}\frac{Re}{1-\gamma}\mathbf{v}\cdot\mathbf{v}_t - \Delta\mathbf{n}\cdot\mathbf{n}_t\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^3}\frac{\gamma}{1-\gamma}|\nabla\mathbf{v}|^2 + (\sigma^L+\sigma^E):\nabla\mathbf{v} - (\mathbf{v}\cdot\nabla\mathbf{n})\cdot\mathbf{h}\\ &\qquad + \Big(\mathbf{n}\times\big((\mu_1\mathbf{h}+\mu_2\mathbf{D}\cdot\mathbf{n} - \mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\Big)\cdot\mathbf{h}\mathrm{d}\mathbf{x}\\ &= \int_{\mathbb{R}^3}\frac{\gamma}{1-\gamma}|\nabla\mathbf{v}|^2 + \mu_1|\mathbf{h}\times\mathbf{n}|^2 + \sigma^E:\nabla\mathbf{v} - (\mathbf{v}\cdot\nabla\mathbf{n})\cdot\mathbf{h}\\ &\qquad + \sigma^L:\nabla\mathbf{v} - \big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) + \mu_2(\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\mathrm{d}\mathbf{x}.\end{aligned}$$

For the Ericksen stress term, we have

$$\int_{\mathbb{R}^3} \sigma^E : \nabla \mathbf{v} - (\mathbf{v}\cdot\nabla\mathbf{n})\cdot\mathbf{h}\mathrm{d}\mathbf{x} = \int_{\mathbb{R}^3} -\partial_i n_k \partial_j n_k \partial_i v_j - v_j \partial_j n_k \partial_{ii} n_k \mathrm{d}\mathbf{x}$$
$$= \int_{\mathbb{R}^3} v_j \partial_j \partial_i n_k \partial_i n_k - \partial_i (v_j \partial_j n_k \partial_i n_k)\mathrm{d}\mathbf{x} = 0,$$

while for the Leslie stress term, we get by (3.2) and Lemma 3.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^3} \sigma^L : \nabla\mathbf{v} - \big((\boldsymbol{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) + \mu_2(\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^3} \Big(\beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} + \frac{1}{2}(-1-\mu_2)\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}(1-\mu_2)\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n} + \beta_2\mathbf{D}\\
&\qquad + \frac{\beta_3}{2}(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn})\Big) : (\mathbf{D}+\boldsymbol{\Omega}) - \big((\boldsymbol{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) + \mu_2(\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^3} \beta_1(\mathbf{nn}:\mathbf{D})^2 + \beta_2\mathbf{D}:\mathbf{D} + \beta_3|\mathbf{D}\cdot\mathbf{n}|^2 - \big((\boldsymbol{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\cdot\big(\mathbf{h}\times\mathbf{n}\big) + \mu_2(\mathbf{h}\times\mathbf{n})\cdot\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\\
&\qquad + \Big(\frac{1}{2}(-1-\mu_2)\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}(1-\mu_2)\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\Big) : (\mathbf{D}+\boldsymbol{\Omega})\mathrm{d}\mathbf{x}\\
&= \int_{\mathbb{R}^3} \beta_1(\mathbf{nn}:\mathbf{D})^2 + \beta_2\mathbf{D}:\mathbf{D} + \beta_3|\mathbf{D}\cdot\mathbf{n}|^2\mathrm{d}\mathbf{x}.
\end{aligned}$$

Then the energy law (2.1) follows from the above identities.

Although the energy law can be derived without using the property $|\mathbf{n}| = 1$, this property is vital for the dissipation relation (2.5) under the condition (2.6). Hence, it is important to construct an approximate system preserving the energy law and $|\mathbf{n}| = 1$ in order to prove the local well-posedness of (1.1). It is usually difficult. For this, we introduce a modified stress tensor so that the energy is still dissipated for the modified system under the condition (2.6). The modified Leslie stress tensor takes the form

$$\begin{aligned}
\widetilde{\sigma}^L =& \beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} + \frac{1}{2}(-1-\mu_2)\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}(1-\mu_2)\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\\
&+ \beta_2|\mathbf{n}|^4\mathbf{D} + \frac{\beta_3}{2}|\mathbf{n}|^2(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn}).
\end{aligned}$$

It is obvious that $\widetilde{\sigma}^L = \sigma^L$ if $|\mathbf{n}| = 1$. An important fact is that for any traceless symmetric $\mathbf{D}$ and vector $\mathbf{n}$ (not necessary unit), it still holds

$$\big\langle \beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} + \beta_2|\mathbf{n}|^4\mathbf{D} + \frac{\beta_3}{2}|\mathbf{n}|^2(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn}), \mathbf{D}\big\rangle \geq 0 \tag{3.3}$$

under the condition (2.6). We denote

$$\begin{aligned}
\sigma_1(\mathbf{v},\mathbf{n}) &= \beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn} + \beta_2|\mathbf{n}|^4\mathbf{D} + \frac{\beta_3}{2}|\mathbf{n}|^2(\mathbf{nD}\cdot\mathbf{n} + \mathbf{D}\cdot\mathbf{nn}),\\
\sigma_2(\mathbf{n}) &= \frac{1}{2}(-1-\mu_2)\mathbf{nn}\times(\mathbf{h}\times\mathbf{n}) + \frac{1}{2}(1-\mu_2)\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}.
\end{aligned}$$

The reformulated new system takes

$$\begin{cases}
\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \nu\Delta\mathbf{v} + \nabla\cdot\big(\sigma_1(\mathbf{v},\mathbf{n}) + \sigma_2(\mathbf{n}) + \sigma^E\big),\\
\mathbf{n}_t + \mathbf{v}\cdot\nabla\mathbf{n} + \mathbf{n}\times\big((\boldsymbol{\Omega}\cdot\mathbf{n} - \mu_1\mathbf{h} - \mu_2\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big) = 0.
\end{cases} \tag{3.4}$$

Here we set $\nu=\frac{\gamma}{Re}$ and take $\frac{1-\gamma}{Re}=1$. Similar to Proposition 2.1, we can show that the system (3.4) obeys the following energy-dissipation law:

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathbf{v}|^2+|\nabla\mathbf{n}|^2\mathrm{d}\mathbf{x}=-\int_{\mathbb{R}^3}\Big(\nu|\nabla\mathbf{v}|^2+\beta_1|\mathbf{D}:\mathbf{nn}|^2+\beta_2|\mathbf{n}|^4\mathbf{D}:\mathbf{D}\\
+\beta_3|\mathbf{n}|^2|\mathbf{D}\cdot\mathbf{n}|^2+\mu_1|\mathbf{n}\times\mathbf{h}|^2\Big)\mathrm{d}\mathbf{x},
\end{aligned}
\tag{3.5}
$$

which is dissipated under the condition (2.6) by (3.3).

4. Local well-posedness and blow-up criterion

This section is devoted to proving the local well-posedness of the system (1.1). The following lemma will frequently used.

Lemma 4.1. *For any $\alpha,\beta\in\mathbb{N}^3$, it hods that*

$$
\begin{aligned}
&\|D^\alpha(fg)\|_{L^2}\le C\sum_{|\gamma|=|\alpha|}\big(\|f\|_{L^\infty}\|D^\gamma g\|_{L^2}+\|g\|_{L^\infty}\|D^\gamma f\|_{L^2}\big),\\
&\big\|[D^\alpha,f]D^\beta g\big\|_{L^2}\le C\Big(\sum_{|\gamma|=|\alpha|+|\beta|}\|D^\gamma f\|_{L^2}\|g\|_{L^\infty}+\sum_{|\gamma|=|\alpha|+|\beta|-1}\|\nabla f\|_{L^\infty}\|D^\gamma g\|_{L^2}\Big).
\end{aligned}
$$

This lemma can be easily proved by using Bony's decomposition; see [1] for example. The proof of Theorem 1.1 is split into several steps.

Step 1. Construction of the approximate solutions

The construction is based on the classical Friedrich's method. Define the smoothing operator

$$
\mathcal{J}_\varepsilon f=\mathcal{F}^{-1}(\mathbf{1}_{|\xi|\le\frac{1}{\varepsilon}}\mathcal{F}f),
\tag{4.6}
$$

where $\mathcal{F}$ is the usual Fourier transform. Let $\mathbb{P}$ be the operator which projects a vector field to its solenoidal part. We introduce the following approximate system of (3.4):

$$
\begin{cases}
\dfrac{\partial\mathbf{v}_\varepsilon}{\partial t}+\mathcal{J}_\varepsilon\mathbb{P}(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon)=\nu\Delta\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon+\nabla\cdot\mathcal{J}_\varepsilon\mathbb{P}\big(\sigma_1(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon,\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)+\sigma_2(\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)+\sigma^E(\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)\big),\\
\dfrac{\partial\mathbf{n}_\varepsilon}{\partial t}+\mathcal{J}_\varepsilon\Big(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon\cdot\nabla\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon+\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon\times\big[(\mathcal{J}_\varepsilon\mathbf{\Omega}_\varepsilon\cdot\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon-\mu_1\mathcal{J}_\varepsilon\mathbf{h}_\varepsilon-\mu_2\mathcal{J}_\varepsilon\mathbf{D}_\varepsilon\cdot\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)\times\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon\big]\Big)=0,\\
(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)|_{t=0}=(\mathcal{J}_\varepsilon\mathbf{v}_0,\mathcal{J}_\varepsilon\mathbf{n}_0).
\end{cases}
$$

The above system can be viewed as an ODE system on $L^2(\mathbb{R}^3)$. Then we know by the Cauchy-Lipschitz theorem that there exist a strictly maximal time T_ε and a unique solution $(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)$ which is continuous in time with value in $H^k(\mathbb{R}^3)$ for any $k\ge0$. As $\mathcal{J}_\varepsilon^2=\mathcal{J}_\varepsilon$, we know that $(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon,\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)$ is also a solution. Therefore, $(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)=(\mathcal{J}_\varepsilon\mathbf{v}_\varepsilon,\mathcal{J}_\varepsilon\mathbf{n}_\varepsilon)$. Thus, $(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)$ satisfies the following system

$$
\begin{cases}
\dfrac{\partial\mathbf{v}_\varepsilon}{\partial t}+\mathcal{J}_\varepsilon\mathbb{P}(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{v}_\varepsilon)=\nu\Delta\mathbf{v}_\varepsilon+\nabla\cdot\mathcal{J}_\varepsilon\mathbb{P}\big(\sigma_1(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)+\sigma_2(\mathbf{n}_\varepsilon)+\sigma^E(\mathbf{n}_\varepsilon)\big),\\
\dfrac{\partial\mathbf{n}_\varepsilon}{\partial t}+\mathcal{J}_\varepsilon\Big(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{n}_\varepsilon+\mathbf{n}_\varepsilon\times\big[(\mathbf{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon-\mu_1\mathbf{h}_\varepsilon-\mu_2\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big]\Big)=0,\\
(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon)|_{t=0}=(\mathcal{J}_\varepsilon\mathbf{v}_0,\mathcal{J}_\varepsilon\mathbf{n}_0).
\end{cases}
\tag{4.7}
$$

Step 2. Uniform energy estimates

We define

$$
E_s(\mathbf{v},\mathbf{n})\overset{\text{def}}{=}\|\mathbf{n}-\mathbf{n}_0\|_{L^2}^2+\|\nabla\mathbf{n}\|_{L^2}^2+\|\nabla\Delta^s\mathbf{n}\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2+\|\Delta^s\mathbf{v}\|_{L^2}^2.
$$

First of all, we get by the second equation of (4.7) that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{n}_\varepsilon-\mathbf{n}_0\|_{L^2}^2=2\big\langle\partial_t\mathbf{n}_\varepsilon,\mathbf{n}_\varepsilon-\mathbf{n}_0\big\rangle\\
&=\big\langle\mathbf{v}_\varepsilon\cdot\nabla\mathbf{n}_\varepsilon+\mathbf{n}_\varepsilon\times\big[(\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon-\mu_1\mathbf{h}_\varepsilon-\mu_2\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big],\mathcal{J}_\varepsilon(\mathbf{n}_\varepsilon-\mathbf{n}_0)\big\rangle\\
&=\big\langle\mathbf{v}_\varepsilon\cdot\nabla\mathbf{n}_0+\mathbf{n}_\varepsilon\times\big[(\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon-\mu_1\mathbf{h}_\varepsilon-\mu_2\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big],\mathcal{J}_\varepsilon(\mathbf{n}_\varepsilon-\mathbf{n}_0)\big\rangle\\
&\le C\big[\|\nabla\mathbf{n}_0\|_{L^\infty}\|\mathbf{v}_\varepsilon\|_{L^2}+\|\mathbf{n}_\varepsilon\|_{L^\infty}^2\big(\|\mathbf{n}_\varepsilon\|_{L^\infty}\|\nabla\mathbf{v}_\varepsilon\|_{L^2}+\|\Delta\mathbf{n}_\varepsilon\|_{L^2}\big)\big]\|\mathbf{n}_\varepsilon-\mathbf{n}_0\|_{L^2}\\
&\le C\big(\|\nabla\mathbf{n}_0\|_{L^\infty}+\|\mathbf{n}_\varepsilon\|_{L^\infty}^2+\|\mathbf{n}_\varepsilon\|_{L^\infty}^3\big)E_s(\mathbf{v}_\varepsilon,\mathbf{n}_\varepsilon).
\end{aligned}\tag{4.8}
$$

The following energy law still holds for the approximate system (4.7):

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathbf{v}_\varepsilon|^2+|\nabla\mathbf{n}_\varepsilon|^2\mathrm{d}\mathbf{x}=-\int_{\mathbb{R}^3}\Big(&\nu|\nabla\mathbf{v}_\varepsilon|^2+\beta_1|\mathbf{D}_\varepsilon:\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon|^2+\beta_2|\mathbf{n}_\varepsilon|^4\mathbf{D}_\varepsilon:\mathbf{D}_\varepsilon\\
&+\beta_3|\mathbf{n}_\varepsilon|^2|\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon|^2+\mu_1|\mathbf{n}_\varepsilon\times\mathbf{h}_\varepsilon|^2\Big)\mathrm{d}\mathbf{x}.
\end{aligned}\tag{4.9}
$$

Now we turn to the estimate of the higher order derivative for $\mathbf{n}_\varepsilon$.

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla\Delta^s\mathbf{n}_\varepsilon,\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle&=\big\langle\Delta^s(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{n}_\varepsilon),\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle+\big\langle\Delta^s\big[\mathbf{n}_\varepsilon\times\big((\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big)\big],\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\\
&\quad-\mu_2\big\langle\Delta^s\big[\mathbf{n}_\varepsilon\times\big((\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big)\big],\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle-\mu_1\big\langle\Delta^s\big[\mathbf{n}_\varepsilon\times\big(\Delta\mathbf{n}_\varepsilon\times\mathbf{n}_\varepsilon\big)\big],\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\\
&=I_1+I_2+I_3+I_4.
\end{aligned}\tag{4.10}
$$

As $\nabla\cdot\mathbf{v}_\varepsilon=0$, we get by Lemma 4.1 that

$$
\begin{aligned}
I_1&=-\big\langle\nabla\Delta^s(\mathbf{v}_\varepsilon\cdot\nabla\mathbf{n}_\varepsilon),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle+\big\langle\mathbf{v}_\varepsilon\cdot\nabla(\nabla\Delta^s\mathbf{n}_\varepsilon),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\\
&=-\big\langle[\nabla\Delta^s,\mathbf{v}_\varepsilon]\cdot\nabla\mathbf{n}_\varepsilon,\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\\
&\le\|[\nabla\Delta^s,\mathbf{v}_\varepsilon]\cdot\nabla\mathbf{n}_\varepsilon\|_{L^2}\|\nabla\Delta^s\mathbf{n}_\varepsilon\|_{L^2}\\
&\le C\big(\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty}+\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}\big)\|\nabla\Delta^s\mathbf{n}_\varepsilon\|_{L^2}\\
&\le C_\delta\big(\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty}+\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}^2.
\end{aligned}\tag{4.11}
$$

Here and in what follows, δ denotes a positive constant to be determined later. We rewrite I_2 as

$$
\begin{aligned}
I_2=&\big\langle\mathbf{n}_\varepsilon\times\big((\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big),\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\\
&-\big\langle\nabla\Delta^s\big[\mathbf{n}_\varepsilon\times\big((\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big)\big],\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle+\big\langle\mathbf{n}_\varepsilon\times\big((\nabla\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\\
&+\big\langle(\nabla\mathbf{n}_\varepsilon)\times\big((\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle+\big\langle\mathbf{n}_\varepsilon\times\big((\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot(\nabla\mathbf{n}_\varepsilon))\times\mathbf{n}_\varepsilon\big),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\\
&+\big\langle\mathbf{n}_\varepsilon\times\big((\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times(\nabla\mathbf{n}_\varepsilon)\big),\nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle,
\end{aligned}
$$

from which and Lemma 4.1, it follows that

$$
\begin{aligned}
I_2\le&\big\langle\mathbf{n}_\varepsilon\times\big((\Delta^s\boldsymbol{\Omega}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big),\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\\
&+C_\delta\big(\|\mathbf{n}_\varepsilon\|_{L^\infty}^2\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty}+\|\mathbf{n}_\varepsilon\|_{L^\infty}^4\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}^2.
\end{aligned}\tag{4.12}
$$

Similarly, we have

$$
\begin{aligned}
I_3\le&-\mu_2\big\langle\mathbf{n}_\varepsilon\times\big((\Delta^s\mathbf{D}_\varepsilon\cdot\mathbf{n}_\varepsilon)\times\mathbf{n}_\varepsilon\big),\Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\\
&+C_\delta\big(\|\mathbf{n}_\varepsilon\|_{L^\infty}^2\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty}+\|\mathbf{n}_\varepsilon\|_{L^\infty}^4\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2+\delta\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}^2.
\end{aligned}\tag{4.13}
$$

For I_4, we have

$$
\begin{aligned}
I_4 =& -\mu_1\big\langle \mathbf{n}_\varepsilon \times \big(\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle \\
& - \big[\mu_1\big\langle \Delta^s\big[\mathbf{n}_\varepsilon \times \big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big)\big], \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle - \mu_1\big\langle \mathbf{n}_\varepsilon \times \Delta^s\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\big] \\
& - \big[\mu_1\big\langle \mathbf{n}_\varepsilon \times \Delta^s\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle - \mu_1\big\langle \mathbf{n}_\varepsilon \times \big(\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle\big] \\
=& \mu_1\big\langle \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon, \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big\rangle + I_{41} + I_{42}.
\end{aligned}
$$

We get by Lemma 4.1 that

$$
\begin{aligned}
I_{42} &\le C\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}\|\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^2} \\
&\le C_\delta\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta\|\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^2}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
I_{41} =& \mu_1\big[\big\langle \Delta^s\big[\nabla\mathbf{n}_\varepsilon \times \big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big)\big], \nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle - \big\langle (\nabla\mathbf{n}_\varepsilon) \times \Delta^s\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\big] \\
& \mu_1\big[\big\langle \Delta^s\big[\mathbf{n}_\varepsilon \times \nabla\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big)\big], \nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle - \big\langle \mathbf{n}_\varepsilon \times \nabla\Delta^s\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big), \nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle\big] \\
\le& C\big(\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}\|\Delta^s\big(\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big)\|_{L^2} + \|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}\|\Delta\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^\infty}\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}} \\
\le& C\big\{\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}(\|\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^2} + \|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}) \\
& \qquad + \|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}\|\Delta\mathbf{n}_\varepsilon\|_{L^\infty}\|\mathbf{n}_\varepsilon\|_{L^\infty}\big\}\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}},
\end{aligned}
$$

which imply that

$$
\begin{aligned}
I_4 \le& -\mu_1\big\langle \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon, \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big\rangle \\
& + C_\delta\big(\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2 + \|\Delta\mathbf{n}_\varepsilon\|_{L^\infty}\|\mathbf{n}_\varepsilon\|_{L^\infty}\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta\|\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^2}^2. \qquad (4.14)
\end{aligned}
$$

Substituting (4.11)-(4.14) into (4.10), we infer that

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle \nabla\Delta^s\mathbf{n}_\varepsilon, \nabla\Delta^s\mathbf{n}_\varepsilon\big\rangle + \mu_1\big\langle \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon, \Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\big\rangle \\
& \le \big\langle \mathbf{n}_\varepsilon \times \big((\Delta^s\mathbf{\Omega}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle - \mu_2\big\langle \mathbf{n}_\varepsilon \times \big((\Delta^s\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon) \times \mathbf{n}_\varepsilon\big), \Delta^{s+1}\mathbf{n}_\varepsilon\big\rangle \\
& \quad + C_\delta\big(\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty} + \|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2 + \|\Delta\mathbf{n}_\varepsilon\|_{L^\infty}\|\mathbf{n}_\varepsilon\|_{L^\infty}\big)\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2 \\
& \quad + \delta\big(\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}^2 + \|\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon\|_{L^2}^2\big). \qquad (4.15)
\end{aligned}
$$

Next we consider the estimate of the higher order derivative for $\mathbf{v}_\varepsilon$.

$$
\begin{aligned}
& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Delta^s\mathbf{v}_\varepsilon, \Delta^s\mathbf{v}_\varepsilon\rangle + \nu\langle \nabla\Delta^s\mathbf{v}_\varepsilon, \nabla\Delta^s\mathbf{v}_\varepsilon\rangle \\
&= -\big\langle \Delta^s(\mathbf{v}_\varepsilon \cdot \nabla\mathbf{v}_\varepsilon), \Delta^s\mathbf{v}_\varepsilon\big\rangle + \big\langle \Delta^s(\nabla\mathbf{n}_\varepsilon \odot \nabla\mathbf{n}_\varepsilon), \Delta^s\nabla\mathbf{v}_\varepsilon\big\rangle \\
& \quad - \Big\langle \Delta^s\Big(\beta_1(\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon : \mathbf{D}_\varepsilon)\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon + \beta_2|\mathbf{n}_\varepsilon|^4\mathbf{D}_\varepsilon + \frac{\beta_3}{2}|\mathbf{n}_\varepsilon|^2(\mathbf{n}_\varepsilon\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon + \mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon\mathbf{n}_\varepsilon) \\
& \qquad - \frac{1}{2}(1+\mu_2)\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon \times (\mathbf{h}_\varepsilon \times \mathbf{n}_\varepsilon) + \frac{1}{2}(1-\mu_2)\mathbf{n}_\varepsilon \times (\mathbf{h}_\varepsilon \times \mathbf{n}_\varepsilon)\mathbf{n}_\varepsilon\Big), \Delta^s\nabla\mathbf{v}_\varepsilon\Big\rangle \\
&= -\langle \Delta^s(\mathbf{v}_\varepsilon \cdot \nabla\mathbf{v}_\varepsilon), \Delta^s\mathbf{v}_\varepsilon\rangle + \langle \Delta^s(\nabla\mathbf{n}_\varepsilon \odot \nabla\mathbf{n}_\varepsilon), \Delta^s\nabla\mathbf{v}_\varepsilon\rangle \\
& \quad - \Big\langle \Delta^s\big(\beta_1(\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon : \mathbf{D}_\varepsilon)\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon + \beta_2|\mathbf{n}_\varepsilon|^4\mathbf{D}_\varepsilon + \frac{\beta_3}{2}|\mathbf{n}_\varepsilon|^2(\mathbf{n}_\varepsilon\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon + \mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon\mathbf{n}_\varepsilon)\big), \Delta^s\mathbf{D}_\varepsilon\Big\rangle \\
& \qquad + \mu_2\big\langle \Delta^s\big(\mathbf{n}_\varepsilon \times (\mathbf{h}_\varepsilon \times \mathbf{n}_\varepsilon)\mathbf{n}_\varepsilon\big), \Delta^s\mathbf{D}_\varepsilon\big\rangle - \big\langle \Delta^s(\mathbf{n}_\varepsilon \times (\mathbf{h}_\varepsilon \times \mathbf{n}_\varepsilon)\mathbf{n}_\varepsilon), \Delta^s\mathbf{\Omega}_\varepsilon\big\rangle \\
&= II_1 + II_2 + II_3 + II_4 + II_5.
\end{aligned}
$$

It follows from Lemma 4.1 that

$$
\begin{aligned}
II_1 &= \big\langle [\Delta^s, \mathbf{v}_\varepsilon] \cdot \nabla\mathbf{v}_\varepsilon, \Delta^s\mathbf{v}_\varepsilon\big\rangle \le C\|\nabla\mathbf{v}_\varepsilon\|_{L^\infty}\|\mathbf{v}_\varepsilon\|_{H^{2s}}^2, \\
II_2 &\le C_\delta\|\nabla\mathbf{n}_\varepsilon\|_{L^\infty}^2\|\nabla\mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta\|\nabla\mathbf{v}_\varepsilon\|_{H^{2s}}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
II_4 \le &\mu_2\big\langle \mathbf{n}_\varepsilon \times (\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon), \Delta^s\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon\big\rangle \\
&+ C_\delta \|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 \|\mathbf{n}_\varepsilon\|_{L^\infty}^4 \|\nabla \mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta \|\nabla \mathbf{v}_\varepsilon\|_{H^{2s}}^2, \\
II_5 \le &-\big\langle \mathbf{n}_\varepsilon \times (\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon), \Delta^s\mathbf{\Omega}_\varepsilon \cdot \mathbf{n}_\varepsilon\big\rangle \\
&+ C_\delta \|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 \|\mathbf{n}_\varepsilon\|_{L^\infty}^4 \|\nabla \mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta \|\nabla \mathbf{v}_\varepsilon\|_{H^{2s}}^2,
\end{aligned}
$$

and by (3.3),

$$
\begin{aligned}
II_3 \le &-\Big\langle \big(\beta_1(\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon : \Delta^s\mathbf{D}_\varepsilon)\mathbf{n}_\varepsilon\mathbf{n}_\varepsilon + \beta_2|\mathbf{n}_\varepsilon|^4\Delta^s\mathbf{D}_\varepsilon + \frac{\beta_3}{2}|\mathbf{n}_\varepsilon|^2(\mathbf{n}_\varepsilon\Delta^s\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon + \Delta^s\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon\mathbf{n}_\varepsilon)\big), \Delta^s\mathbf{D}_\varepsilon\Big\rangle \\
&+ C_\delta \|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 \|\mathbf{n}_\varepsilon\|_{L^\infty}^6 \|\mathbf{v}_\varepsilon\|_{H^{2s}}^2 + C_\delta \|\mathbf{v}_\varepsilon\|_{L^\infty}^2 \|\mathbf{n}_\varepsilon\|_{L^\infty}^6 \|\nabla \mathbf{n}_\varepsilon\|_{H^{2s}}^2 + \delta \|\nabla \mathbf{v}_\varepsilon\|_{H^{2s}}^2 \\
\le &C_\delta \|\mathbf{n}_\varepsilon\|_{L^\infty}^6 \big(\|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 + \|\mathbf{v}_\varepsilon\|_{L^\infty}^2\big)\big(\|\mathbf{v}_\varepsilon\|_{H^{2s}}^2 + \|\nabla \mathbf{n}_\varepsilon\|_{H^{2s}}^2\big) + \delta \|\nabla \mathbf{v}_\varepsilon\|_{H^{2s}}^2.
\end{aligned}
$$

Summing up, we conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle \Delta^s\mathbf{v}_\varepsilon, \Delta^s\mathbf{v}_\varepsilon\rangle + \nu\langle \nabla\Delta^s\mathbf{v}_\varepsilon, \nabla\Delta^s\mathbf{v}_\varepsilon\rangle \\
&\le \mu_2\big\langle \mathbf{n}_\varepsilon \times (\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon), \Delta^s\mathbf{D}_\varepsilon \cdot \mathbf{n}_\varepsilon\big\rangle - \big\langle \mathbf{n}_\varepsilon \times (\Delta^{s+1}\mathbf{n}_\varepsilon \times \mathbf{n}_\varepsilon), \Delta^s\mathbf{\Omega}_\varepsilon \cdot \mathbf{n}_\varepsilon\big\rangle \\
&\quad + C_\delta\big(1 + \|\mathbf{n}_\varepsilon\|_{L^\infty}^6\big)\big(\|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 + \|\nabla \mathbf{v}_\varepsilon\|_{L^\infty} + \|\mathbf{v}_\varepsilon\|_{L^\infty}^2\big)E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon) + \delta\|\nabla \mathbf{v}_\varepsilon\|_{H^{2s}}^2.
\end{aligned} \tag{4.16}
$$

Summing up (4.8), (4.9), (4.15) and (4.16), then taking δ small enough, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon) + \frac{\nu}{2}\langle \nabla\mathbf{v}_\varepsilon, \nabla\mathbf{v}_\varepsilon\rangle + \frac{\nu}{2}\langle \nabla\Delta^s\mathbf{v}_\varepsilon, \nabla\Delta^s\mathbf{v}_\varepsilon\rangle \\
&\le C\big(1 + \|\nabla \mathbf{n}_0\|_{L^\infty} + \|\mathbf{n}_\varepsilon\|_{L^\infty}^6\big)\big(1 + \|\nabla \mathbf{n}_\varepsilon\|_{L^\infty}^2 + \|\nabla \mathbf{v}_\varepsilon\|_{L^\infty} + \|\mathbf{v}_\varepsilon\|_{L^\infty}^2\big)E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon).
\end{aligned} \tag{4.17}
$$

Step 3. Existence of the solution

As $s \ge 2$, we deduced from Sobolev embedding and (4.17) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon) + \nu\langle \nabla\mathbf{v}_\varepsilon, \nabla\mathbf{v}_\varepsilon\rangle + \nu\langle \nabla\Delta^s\mathbf{v}_\varepsilon, \nabla\Delta^s\mathbf{v}_\varepsilon\rangle \le \mathcal{F}(E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon)),
$$

where $\mathcal{F}$ is an increasing function with $\mathcal{F}(0) = 0$. This implies that there exists $T > 0$ depending only on $E_s(\mathbf{v}_0, \mathbf{n}_0)$ such that for any $t \in [0, \min(T, T_\varepsilon)]$,

$$
E_s(\mathbf{v}_\varepsilon, \mathbf{n}_\varepsilon) + \nu\langle \nabla\mathbf{v}_\varepsilon, \nabla\mathbf{v}_\varepsilon\rangle + \nu\langle \nabla\Delta^s\mathbf{v}_\varepsilon, \nabla\Delta^s\mathbf{v}_\varepsilon\rangle \le 2E_s(\mathbf{v}_0, \mathbf{n}_0),
$$

which in turn ensures that $T_\varepsilon \ge T$ by a continuous argument. Thus, we obtain an uniform estimate for the approximate solution on $[0, T]$. Then the existence of the solution can be deduced by a standard compactness argument.

Step 4. Uniqueness of the solution

Let $(\mathbf{v}_1, \mathbf{n}_1)$ and $(\mathbf{v}_1, \mathbf{n}_1)$ be two solutions of the system (1.1) with the same initial data. We denote

$$
\delta_{\mathbf{v}} = \mathbf{v}_1 - \mathbf{v}_2, \quad \delta_{\mathbf{n}} = \mathbf{n}_1 - \mathbf{n}_2, \quad \delta_{\mathbf{h}} = \mathbf{h}_1 - \mathbf{h}_2, \quad \delta_{\mathbf{D}} = \mathbf{D}_1 - \mathbf{D}_2, \quad \delta_{\mathbf{\Omega}} = \mathbf{\Omega}_1 - \mathbf{\Omega}_2.
$$

Then $(\delta_{\mathbf{v}}, \delta_{\mathbf{n}})$ satisfies

$$
\begin{aligned}
\frac{\partial \delta_{\mathbf{v}}}{\partial t} + \mathbf{v}_1 \cdot \nabla\delta_{\mathbf{v}} + \delta_{\mathbf{v}} \cdot \nabla\mathbf{v}_2 &= -\nabla p + \nu\Delta\delta_{\mathbf{v}} + \nabla \cdot \big(\sigma_1(\mathbf{v}_1, \mathbf{n}_1) - \sigma_1(\mathbf{v}_2, \mathbf{n}_2) \\
&\quad + \sigma_2(\mathbf{n}_1) - \sigma(\mathbf{n}_2) + \sigma^E(\mathbf{n}_1) - \sigma^E(\mathbf{n}_2)\big), \\
\frac{\partial \delta_{\mathbf{n}}}{\partial t} + \mathbf{v}_1 \cdot \nabla\delta_{\mathbf{n}} + \delta_{\mathbf{v}} \cdot \nabla\mathbf{n}_2 &= -\mathbf{n}_1 \times \big((\mathbf{\Omega}_1 \cdot \mathbf{n}_1 - \mu_1\mathbf{h}_1 - \mu_2\mathbf{D}_1 \cdot \mathbf{n}_1) \times \mathbf{n}_1\big) \\
&\quad + \mathbf{n}_2 \times \big((\mathbf{\Omega}_2 \cdot \mathbf{n}_2 - \mu_1\mathbf{h}_2 - \mu_2\mathbf{D}_2 \cdot \mathbf{n}_2) \times \mathbf{n}_2\big).
\end{aligned}
$$

We make L^2 energy estimate for $\delta_{\mathbf{v}}$ to get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta_{\mathbf{v}}\|_{L^2}^2+\nu\|\nabla\delta_{\mathbf{v}}\|_{L^2}^2=-\big\langle\delta_{\mathbf{v}}\cdot\nabla\mathbf{v}_2,\delta_{\mathbf{v}}\big\rangle+\big\langle\nabla\cdot(\sigma_1(\mathbf{v}_1,\mathbf{n}_1)-\sigma_1(\mathbf{v}_2,\mathbf{n}_2)),\delta_{\mathbf{v}}\big\rangle\\
&\quad+\big\langle\nabla\cdot\big(\sigma_2(\mathbf{n}_1)-\sigma_2(\mathbf{n}_2)\big),\delta_{\mathbf{v}}\big\rangle+\big\langle\nabla\cdot\big(\sigma^E(\mathbf{n}_1)-\sigma^E(\mathbf{n}_2)\big),\delta_{\mathbf{v}}\big\rangle\\
&=R_1+R_2+R_3+R_4,
\end{aligned}$$

and make H^1 energy estimate for $\delta_{\mathbf{n}}$ to get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\delta_{\mathbf{n}}\|_{L^2}^2=&\big\langle\mathbf{v}_1\cdot\nabla\delta_{\mathbf{n}}+\delta_{\mathbf{v}}\cdot\nabla\mathbf{n}_2,\Delta\delta_{\mathbf{n}}\big\rangle\\
&+\big\langle\mathbf{n}_1\times\big((\mathbf{\Omega}_1\cdot\mathbf{n}_1-\mu_1\mathbf{h}_1-\mu_2\mathbf{D}_1\cdot\mathbf{n}_1)\times\mathbf{n}_1\big)\\
&\qquad-\mathbf{n}_2\times\big((\mathbf{\Omega}_2\cdot\mathbf{n}_2-\mu_1\mathbf{h}_2-\mu_2\mathbf{D}_2\cdot\mathbf{n}_2)\times\mathbf{n}_2\big),\Delta\delta_{\mathbf{n}}\big\rangle\\
=&S_1+S_2.
\end{aligned}$$

Now we estimate $R_1,\cdots,R_4$. It is easy to see that

$$\begin{aligned}
&R_1\le\|\nabla\mathbf{v}_2\|_{L^\infty}\|\delta_{\mathbf{v}}\|_{L^2}^2,\\
&R_4\le C\big(\|\nabla\mathbf{n}_1\|_{L^\infty}+\|\nabla\mathbf{n}_2\|_{L^\infty}\big)\|\nabla\delta_{\mathbf{n}}\|_{L^2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}.
\end{aligned}$$

By (3.3), we have

$$\begin{aligned}
R_2&=-\big\langle\sigma_1(\delta_{\mathbf{v}},\mathbf{n}_1),\nabla\delta_{\mathbf{v}}\big\rangle-\big\langle\sigma_1(\mathbf{v}_2,\mathbf{n}_1)-\sigma_1(\mathbf{v}_2,\mathbf{n}_2),\nabla\delta_{\mathbf{v}}\big\rangle\\
&\le C\|\nabla\mathbf{v}_2\|_{L^3}\|\nabla\delta_{\mathbf{n}}\|_{L^2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}.
\end{aligned}$$

For R_3, we have

$$\begin{aligned}
R_3=&\mu_2\big\langle\mathbf{n}_1\times(\mathbf{h}_1\times\mathbf{n}_1)\mathbf{n}_1-\mathbf{n}_2\times(\mathbf{h}_2\times\mathbf{n}_2)\mathbf{n}_2,\delta_{\mathbf{D}}\big\rangle\\
&+\big\langle\mathbf{n}_1\times(\mathbf{h}_1\times\mathbf{n}_1)\mathbf{n}_1-\mathbf{n}_2\times(\mathbf{h}_2\times\mathbf{n}_2)\mathbf{n}_2,\delta_{\mathbf{\Omega}}\big\rangle\\
=&\,\mu_2\big\langle\mathbf{n}_1\times(\delta_{\mathbf{h}}\times\mathbf{n}_1)\mathbf{n}_1,\delta_{\mathbf{D}}\big\rangle+\big\langle\mathbf{n}_1\times(\delta_{\mathbf{h}}\times\mathbf{n}_1)\mathbf{n}_1,\delta_{\mathbf{\Omega}}\big\rangle\\
&+\mu_2\big\langle\mathbf{n}_1\times(\mathbf{h}_2\times\mathbf{n}_1)\mathbf{n}_1-\mathbf{n}_2\times(\mathbf{h}_2\times\mathbf{n}_2)\mathbf{n}_2,\delta_{\mathbf{D}}\big\rangle\\
&+\big\langle\mathbf{n}_1\times(\mathbf{h}_2\times\mathbf{n}_1)\mathbf{n}_1-\mathbf{n}_2\times(\mathbf{h}_2\times\mathbf{n}_2)\mathbf{n}_2,\delta_{\mathbf{\Omega}}\big\rangle\\
\le&\,\mu_2\big\langle\mathbf{n}_1\times(\delta_{\mathbf{h}}\times\mathbf{n}_1)\mathbf{n}_1,\delta_{\mathbf{D}}\big\rangle+\big\langle\mathbf{n}_1\times(\delta_{\mathbf{h}}\times\mathbf{n}_1)\mathbf{n}_1,\delta_{\mathbf{\Omega}}\big\rangle\\
&+C\|\Delta\mathbf{n}_2\|_{L^3}\|\nabla\delta_{\mathbf{n}}\|_{L^2}\|\nabla\delta_{\mathbf{v}}\|_{L^2}.
\end{aligned}$$

Let us turn to estimate S_1 and S_2. It is easy to see that

$$S_1\le\|\nabla\mathbf{v}_1\|_{L^\infty}\|\nabla\delta_{\mathbf{n}}\|_{L^2}^2+C\big(\|\nabla\mathbf{n}_2\|_{L^\infty}+\|\Delta\mathbf{n}_2\|_{L^3}\big)\|\nabla\delta_{\mathbf{v}}\|_{L^2}\|\nabla\delta_{\mathbf{n}}\|_{L^2},$$

and for S_2, we have

$$\begin{aligned}
S_2=&\big\langle\mathbf{n}_1\times\big((\delta_{\mathbf{\Omega}}\cdot\mathbf{n}_1-\mu_1\delta_{\mathbf{h}}-\mu_2\delta_{\mathbf{D}}\cdot\mathbf{n}_1)\times\mathbf{n}_1\big),\Delta\delta_{\mathbf{n}}\big\rangle\\
&+\big\langle\mathbf{n}_1\times\big((\mathbf{\Omega}_2\cdot\mathbf{n}_1-\mu_1\mathbf{h}_2-\mu_2\mathbf{D}_2\cdot\mathbf{n}_1)\times\mathbf{n}_1\big)\\
&\qquad-\mathbf{n}_2\times\big((\mathbf{\Omega}_2\cdot\mathbf{n}_2-\mu_1\mathbf{h}_2-\mu_2\mathbf{D}_2\cdot\mathbf{n}_2)\times\mathbf{n}_2\big),\Delta\delta_{\mathbf{n}}\big\rangle\\
\le&\big\langle\mathbf{n}_1\times\big((\delta_{\mathbf{\Omega}}\cdot\mathbf{n}_1-\mu_1\delta_{\mathbf{h}}-\mu_2\delta_{\mathbf{D}}\cdot\mathbf{n}_1)\times\mathbf{n}_1\big),\Delta\delta_{\mathbf{n}}\big\rangle\\
&+C\big(\|\nabla\mathbf{v}_2\|_{L^\infty}+\|\Delta\mathbf{v}_2\|_{L^3}+\|\Delta\mathbf{n}_2\|_{L^2}+\|\nabla\Delta\mathbf{n}_2\|_{L^3}\big)\|\nabla\delta_{\mathbf{n}}\|_{L^2}^2.
\end{aligned}$$

Summing up all the above estimates, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\delta_{\mathbf{v}}\|_{L^2}^2+\|\nabla\delta_{\mathbf{n}}\|_{L^2}^2\big)\le C\big(\|\delta_{\mathbf{v}}\|_{L^2}^2+\|\nabla\delta_{\mathbf{n}}\|_{L^2}^2\big),$$

which implies that $\delta_{\mathbf{v}}(t)=0$ and $\delta_{\mathbf{n}}(t)=0$ on $[0,T]$.

Step 5. Blow-up criterion

First of all, the solution of (1.1) satisfies $|\mathbf{n}|=1$ if $|\mathbf{n}_0|=1$. Thus, it holds that

$$\mathbf{n}\times(\Delta\mathbf{n}\times\mathbf{n})=\Delta\mathbf{n}+|\nabla\mathbf{n}|^2\mathbf{n}. \tag{4.18}$$

Hence, I_4 in (4.10) can be written as

$$\begin{aligned} I_4=&-\mu_1\big\langle\Delta^{s+1}\mathbf{n}+\Delta^s(|\nabla\mathbf{n}|^2\mathbf{n}),\Delta^{s+1}\mathbf{n}\big\rangle\\ =&-\mu_1\big\langle\Delta^{s+1}\mathbf{n},\Delta^{s+1}\mathbf{n}\big\rangle+\mu_1\big\langle\Delta^s\big(\nabla(|\nabla\mathbf{n}|^2)\mathbf{n}\big)+\Delta^s(|\nabla\mathbf{n}|^2\nabla\mathbf{n}),\nabla\Delta^s\mathbf{n}\big\rangle, \end{aligned}$$

which along with Lemma 4.1 gives

$$\begin{aligned} I_4\leq&-\mu_1\big\langle\Delta^{s+1}\mathbf{n},\Delta^{s+1}\mathbf{n}\big\rangle+C\|\nabla\mathbf{n}\|_{L^\infty}\|\nabla\mathbf{n}\|_{H^{2s}}\|\Delta^{s+1}\mathbf{n}\|_{L^2}\\ &+C\|\nabla\mathbf{n}\|_{L^\infty}^2\|\nabla\mathbf{n}\|_{H^{2s}}^2. \end{aligned}$$

On the other hand, we can bound II_3 as

$$II_3\leq C\big(\|\nabla\mathbf{n}\|_{L^\infty}^2+\|\nabla\mathbf{v}\|_{L^\infty}\big)\big(\|\nabla\mathbf{n}\|_{H^{2s}}^2+\|\mathbf{v}\|_{H^{2s}}^2\big)+\delta\|\nabla\Delta^s\mathbf{v}\|_{L^2}^2$$

by using the commutator estimate like

$$\|\nabla[\Delta^s,f]\nabla g\|_{L^2}\leq C\big(\|\Delta^s\nabla f\|_{L^2}\|\nabla g\|_{L^\infty}+\|\nabla f\|_{L^\infty}\|\Delta^s\nabla g\|_{L^2}\big).$$

From the proof in Step 2, we can deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s(\mathbf{v},\mathbf{n})\leq C\big(1+\|\nabla\mathbf{n}\|_{L^\infty}^2+\|\nabla\mathbf{v}\|_{L^\infty}\big)E_s(\mathbf{v},\mathbf{n}).$$

Recall the following Logarithmic Sobolev inequality from[2]:

$$\|\nabla\mathbf{v}\|_{L^\infty}\leq C\big(1+\|\nabla\mathbf{v}\|_{L^2}+\|\nabla\times\mathbf{v}\|_{L^\infty}\big)\log(2+\|\mathbf{v}\|_{H^k})$$

for any $k\geq 3$. Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_s(\mathbf{v},\mathbf{n})\leq C\big(1+\|\nabla\mathbf{v}\|_{L^2}+\|\nabla\mathbf{n}\|_{L^\infty}^2+\|\nabla\times\mathbf{v}\|_{L^\infty}\big)\log\big(2+E_s(\mathbf{v},\mathbf{n})\big)E_s(\mathbf{v},\mathbf{n}).$$

Applying Gronwall's inequality twice, we infer that

$$E_s(\mathbf{v},\mathbf{n})\leq E_s(\mathbf{v}_0,\mathbf{n}_0)\exp\exp\Big(C\int_0^t\big(1+\|\nabla\mathbf{v}\|_{L^2}+\|\nabla\mathbf{n}\|_{L^\infty}^2+\|\nabla\times\mathbf{v}\|_{L^\infty}\big)\mathrm{d}\tau\Big)$$

for any $t\in[0,T^*)$. Especially, if $T^*<+\infty$ and

$$\int_0^{T^*}\big(\|\nabla\mathbf{n}\|_{L^\infty}^2+\|\nabla\times\mathbf{v}\|_{L^\infty}\big)\mathrm{d}t<+\infty,$$

then $E_s(\mathbf{v},\mathbf{n})(t)\leq C$ for any $t\in[0,T^*)$. Thus, the solution can be extended after $t=T^*$, which contradicts the definition of T^*. The blow-up criterion follows.

5. Global well-posedness for small initial data

This section is devoted to the proof of Theorem 1.2. Assume that $(\mathbf{v},\mathbf{n})$ is the solution of the system (1.1) on $[0,T]$ obtained in Theorem 1.1. We define

$$\begin{aligned} E_s(\mathbf{v},\mathbf{n})&\stackrel{\mathrm{def}}{=}\|\nabla\mathbf{n}\|_{L^2}^2+\|\nabla\Delta^s\mathbf{n}\|_{L^2}^2+\|\mathbf{v}\|_{L^2}^2+\|\Delta^s\mathbf{v}\|_{L^2}^2,\\ D_s(\mathbf{v},\mathbf{n})&\stackrel{\mathrm{def}}{=}\mu_1\|\Delta\mathbf{n}\|_{L^2}^2+\mu_1\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2+\nu\|\nabla\mathbf{v}\|_{L^2}^2+\nu\|\Delta^s\nabla\mathbf{v}\|_{L^2}^2. \end{aligned}$$

By the interpolation, there exist $c_0>0$ and $C_0>0$ such that

$$\begin{aligned} &c_0\big(\|\nabla\mathbf{n}\|_{H^{2s}}^2+\|\mathbf{v}\|_{H^{2s}}^2\big)\leq E_s(\mathbf{v},\mathbf{n})\leq C_0\big(\|\nabla\mathbf{n}\|_{H^{2s}}^2+\|\mathbf{v}\|_{H^{2s}}^2\big),\\ &c_0\big(\|\Delta\mathbf{n}\|_{H^{2s}}^2+\|\nabla\mathbf{v}\|_{H^{2s}}^2\big)\leq D_s(\mathbf{v},\mathbf{n})\leq C_0\big(\|\Delta\mathbf{n}\|_{H^{2s}}^2+\|\nabla\mathbf{v}\|_{H^{2s}}^2\big). \end{aligned}$$

The basic energy-dissipation law tells us that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathbf{v}|^2+|\nabla\mathbf{n}|^2\mathrm{d}\mathbf{x}+\int_{\mathbb{R}^3}\big(\nu|\nabla\mathbf{v}|^2+\mu_1|\mathbf{n}\times\mathbf{h}|^2\big)\mathrm{d}\mathbf{x}\leq 0,$$

which along with (4.18) implies that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\mathbf{v}|^2+|\nabla\mathbf{n}|^2\mathrm{d}\mathbf{x}+\int_{\mathbb{R}^3}\big(\nu|\nabla\mathbf{v}|^2+\mu_1|\Delta\mathbf{n}|^2\big)\mathrm{d}\mathbf{x}\\ &\leq\mu_1\int_{\mathbb{R}^3}|\nabla\mathbf{n}|^4\mathrm{d}\mathbf{x}\leq C\|\nabla\mathbf{n}\|_{L^2}\|\Delta\mathbf{n}\|_{L^2}^3\leq CE_s(\mathbf{v},\mathbf{n})D_s(\mathbf{v},\mathbf{n}).\end{aligned}\tag{5.19}$$

Similar to (4.10), we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla\Delta^s\mathbf{n},\nabla\Delta^s\mathbf{n}\big\rangle&=\big\langle\Delta^s(\mathbf{v}\cdot\nabla\mathbf{n}),\Delta^{s+1}\mathbf{n}\big\rangle+\big\langle\Delta^s\big[\mathbf{n}\times\big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\big],\Delta^{s+1}\mathbf{n}\big\rangle\\ &\quad-\mu_2\big\langle\Delta^s\big[\mathbf{n}\times\big((\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big)\big],\Delta^{s+1}\mathbf{n}\big\rangle-\mu_1\big\langle\Delta^s\big[\mathbf{n}\times\big(\Delta\mathbf{n}\times\mathbf{n}\big)\big],\Delta^{s+1}\mathbf{n}\big\rangle\\ &=I_1+I_2+I_3+I_4.\end{aligned}\tag{5.20}$$

We get by Lemma 4.1 and Sobolev embedding that

$$\begin{aligned}I_1\leq&\|\Delta^s(\mathbf{v}\cdot\nabla\mathbf{v})\|_{L^2}\|\Delta^{s+1}\mathbf{n}\|_{L^2}\\ \leq&C\|\mathbf{v}\|_{L^\infty}\|\Delta^s\nabla\mathbf{v}\|_{L^2}\|\Delta^{s+1}\mathbf{n}\|_{L^2}\leq CE_s(\mathbf{v},\mathbf{n})^{\frac{1}{2}}D_s(\mathbf{v},\mathbf{n});\end{aligned}\tag{5.21}$$

and

$$\begin{aligned}I_2=&\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle\\ &+\big\langle\Delta^s\big[\mathbf{n}\times\big((\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big)\big],\Delta^{s+1}\mathbf{n}\big\rangle-\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle\\ \leq&\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle\\ &+C\big(\|\nabla\mathbf{n}\|_{L^\infty}\|\nabla\mathbf{v}\|_{H^{2s-1}}+\|\Delta^s\mathbf{n}\|_{L^2}\|\nabla\mathbf{v}\|_{L^\infty}\big)\|\Delta^{s+1}\mathbf{n}\|_{L^2}\\ \leq&\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle+CE_s(\mathbf{v},\mathbf{n})^{\frac{1}{2}}D_s(\mathbf{v},\mathbf{n});\end{aligned}\tag{5.22}$$

$$I_3\leq-\mu_2\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle+CE_s(\mathbf{v},\mathbf{n})^{\frac{1}{2}}D_s(\mathbf{v},\mathbf{n});\tag{5.23}$$

Similar to Step 4 in Section 4, we have

$$\begin{aligned}I_4\leq&-\mu_1\big\langle\Delta^{s+1}\mathbf{n},\Delta^{s+1}\mathbf{n}\big\rangle+C\big(\|\nabla\mathbf{n}\|_{L^\infty}+\|\nabla\mathbf{n}\|_{L^\infty}^2\big)\|\Delta^s\mathbf{n}\|_{H^1}\|\Delta^{s+1}\mathbf{n}\|_{L^2}\\ \leq&-\mu_1\big\langle\Delta^{s+1}\mathbf{n},\Delta^{s+1}\mathbf{n}\big\rangle+C\big(E_s(\mathbf{v},\mathbf{n})^{\frac{1}{2}}+E_s(\mathbf{v},\mathbf{n})\big)D_s(\mathbf{v},\mathbf{n}).\end{aligned}\tag{5.24}$$

Summing up (5.20)-(5.24), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla\Delta^s\mathbf{n},\nabla\Delta^s\mathbf{n}\big\rangle+\mu_1\big\langle\Delta^{s+1}\mathbf{n},\Delta^{s+1}\mathbf{n}\big\rangle\\ &\leq\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{\Omega}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle-\mu_2\big\langle\mathbf{n}\times\big((\Delta^s\mathbf{D}\cdot\mathbf{n})\times\mathbf{n}\big),\Delta^{s+1}\mathbf{n}\big\rangle\\ &\quad+C\big(E_s(\mathbf{v},\mathbf{n})^{\frac{1}{2}}+E_s(\mathbf{v},\mathbf{n})\big)D_s(\mathbf{v},\mathbf{n}).\end{aligned}\tag{5.25}$$

Now we consider the estimate for the velocity. By Step 2 in Section 4, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle\Delta^s\mathbf{v},\Delta^s\mathbf{v}\rangle+\nu\langle\nabla\Delta^s\mathbf{v},\nabla\Delta^s\mathbf{v}\rangle\\ &=-\langle\Delta^s(\mathbf{v}\cdot\nabla\mathbf{v}),\Delta^s\mathbf{v}\rangle+\langle\Delta^s(\nabla\mathbf{n}\odot\nabla\mathbf{n}),\Delta^s\nabla\mathbf{v}\rangle\\ &\quad-\Big\langle\Delta^s\big(\beta_1(\mathbf{nn}:\mathbf{D})\mathbf{nn}+\beta_2\mathbf{D}+\frac{\beta_3}{2}(\mathbf{nD}\cdot\mathbf{n}+\mathbf{D}\cdot\mathbf{nn})\big),\Delta^s\mathbf{D}\Big\rangle\\ &\quad+\mu_2\big\langle\Delta^s\big(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}\big),\Delta^s\mathbf{D}\big\rangle-\big\langle\Delta^s(\mathbf{n}\times(\mathbf{h}\times\mathbf{n})\mathbf{n}),\Delta^s\mathbf{\Omega}\big\rangle\\ &=II_1+II_2+II_3+II_4+II_5.\end{aligned}\tag{5.26}$$

We get by Lemma 4.1 and Sobolev embedding that

$$II_1 \le C\|\mathbf{v}\|_{L^\infty}\|\Delta^s\mathbf{v}\|_{L^2}\|\Delta^s\nabla\mathbf{v}\|_{L^2} \le CE_s(\mathbf{v},\mathbf{n})^{\frac12}D_s(\mathbf{v},\mathbf{n}); \tag{5.27}$$

$$II_2 \le C\|\nabla\mathbf{n}\|_{L^\infty}\|\Delta^s\nabla\mathbf{n}\|_{L^2}\|\Delta^s\nabla\mathbf{v}\|_{L^2} \le CE_s(\mathbf{v},\mathbf{n})^{\frac12}D_s(\mathbf{v},\mathbf{n}); \tag{5.28}$$

and by Proposition 2.2,

$$\begin{aligned} II_3 \le &- \Big\langle \big(\beta_1(\mathbf{nn}:\Delta^s\mathbf{D})\mathbf{nn} + \beta_2\Delta^s\mathbf{D} + \frac{\beta_3}{2}(\mathbf{n}\Delta^s\mathbf{D}\cdot\mathbf{n} + \Delta^s\mathbf{D}\cdot\mathbf{nn})\big), \Delta^s\mathbf{D}\Big\rangle \\ &+ C\big(\|\nabla\mathbf{v}\|_{L^\infty}\|\Delta^s\mathbf{n}\|_{L^2} + \|\nabla\mathbf{n}\|_{L^\infty}\|\nabla\mathbf{v}\|_{H^{2s-1}}\big)\|\Delta^s\nabla\mathbf{v}\|_{L^2} \\ \le & CE_s(\mathbf{v},\mathbf{n})^{\frac12}D_s(\mathbf{v},\mathbf{n}); \end{aligned} \tag{5.29}$$

Similarly, we have

$$\begin{aligned} II_4 + II_5 \le &\mu_2\big\langle \mathbf{n}\times(\Delta^{s+1}\mathbf{n}\times\mathbf{n}), \Delta^s\mathbf{D}\cdot\mathbf{n}\big\rangle - \big\langle \mathbf{n}\times(\Delta^{s+1}\times\mathbf{n}), \Delta^s\mathbf{\Omega}\cdot\mathbf{n}\big\rangle \\ &+ CE_s(\mathbf{v},\mathbf{n})^{\frac12}D_s(\mathbf{v},\mathbf{n}). \end{aligned} \tag{5.30}$$

Summing up (5.26)-(5.30), we obtain

$$\begin{aligned} &\frac12\frac{\mathrm{d}}{\mathrm{d}t}\langle\Delta^s\mathbf{v},\Delta^s\mathbf{v}\rangle + \nu\langle\nabla\Delta^s\mathbf{v},\nabla\Delta^s\mathbf{v}\rangle \\ &\le \mu_2\big\langle \mathbf{n}\times(\Delta^{s+1}\mathbf{n}\times\mathbf{n}), \Delta^s\mathbf{D}\cdot\mathbf{n}\big\rangle - \big\langle \mathbf{n}\times(\Delta^{s+1}\times\mathbf{n}), \Delta^s\mathbf{\Omega}\cdot\mathbf{n}\big\rangle \\ &\quad + CE_s(\mathbf{v},\mathbf{n})^{\frac12}D_s(\mathbf{v},\mathbf{n}). \end{aligned} \tag{5.31}$$

It follows from (5.19), (5.25) and (5.31) that

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}E_s(\mathbf{v},\mathbf{n}) + D_s(\mathbf{v},\mathbf{n}) \le C\big(E_s(\mathbf{v},\mathbf{n})^{\frac12} + E_s(\mathbf{v},\mathbf{n})\big)D_s(\mathbf{v},\mathbf{n}).$$

This implies that there exists an $\varepsilon_0>0$ such that if $E_s(\mathbf{v}_0,\mathbf{n}_0)\le\varepsilon_0$, then

$$E_s(\mathbf{v},\mathbf{n})(t) \le E_s(\mathbf{v}_0,\mathbf{n}_0) \quad \text{for any } t\in[0,T].$$

Thus, the solution is global in time by blow-up criterion in Theorem 1.1.

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References

[1] H. Bahouri, J.-Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Fundamental Principles of Mathematical Sciences, 343, Springer, Heidelberg, 2011.

[2] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equation*, Comm. Math. Phys., 94(1984), 61-66.

[3] W. E and P. Zhang, *A Molecular Kinetic Theory of Inhomogeneous Liquid Crystal Flow and the Small Deborah Number Limit*, Methods and Appications of Analysis, **13**(2006), 181-198.

[4] J. Ericksen, *Conservation laws for liquid crystals*, Trans. Soc. Rheol. , 5(1961), 22-34.

[5] J. Ericksen, *Liquid crystals with variable degree of orientation*, Arch. Rat. Mech. Anal., 113 (1991), 97-120.

[6] M. Hong, *Global existence of solutions of the simplified Ericksen-Leslie system in dimension two*, Calc. Var. Partial Differential Equations, 40(2011), 15-36.

[7] T. Huang and C. Wang, *Blow up criterion for nematic liquid crystal flows*, Communications in Partial Differential Equations, 37(2012), 875-884.

[8] N. Kuzuu and M. Doi, *Constitutive equation for nematic liquid crystals under weak velocity gradient derived from a molecular kinetic equation,* Journal of the Physical Society of Japan, 52(1983), 3486-3494.

[9] F. M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Rat. Mech. Anal., 28 (1968), 265-283.

[10] F.-H. Lin and C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure Appl. Math., 48(1995), 501-537.

[11] F.-H. Lin and C. Liu, *Partial regularities of the nonlinear dissipative systems modeling the fow of liquid crystals*, Disc. Conti. Dyna. Sys., 2 (1996), 1-23.
[12] F.-H. Lin and C. Liu, *Existence of solutions for the Ericksen-Leslie system*, Arch. Ration. Mech. Anal., 154(2000), 135-156.
[13] F.-H. Lin, J. Lin and C. Wang, *Liquid crystal flows in two dimensions*, Arch. Ration. Mech. Anal., 197 (2010), 297-336.
[14] F.-H. Lin and C. Wang, *On the uniqueness of heat flow of harmonic maps and hydrodynamic flow of nematic liquid crystals*, Chin. Ann. Math. Ser. B, 31(2010), 921-938.
[15] O. Parodi, *Stress tensor for a nematic liquid crystal,* Journal de Physique, 31 (1970), 581-584.
[16] C. Wang, *Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data*, Arch. Ration. Mech. Anal., 200(2011), 1-19.
[17] W. Wang, P. Zhang and Z. Zhang, *The small Deborah number limit of the Doi-Onsager equation to the Ericksen-Leslie equation*, arXiv:1206.5480.
[18] H. Wu, X. Xu and C. Liu, *On the general Ericksen Leslie system: Parodi's relation, well-posedness and stability*, arXiv:1105.2180.
[19] X. Xu and Z. Zhang, *Global regularity and uniqueness of weak solution for the 2-D liquid crystal flows*, J. Differential Equations, 252 (2012), 1169-1181.

School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: wangw07@pku.edu.cn

School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: pzhang@pku.edu.cn

School of Mathematical Sciences, Peking University, Beijing 100871, China
E-mail address: zfzhang@math.pku.edu.cn