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Well-Posedness for the Dumbbell Model of Polymeric Fluids

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Abstract: The dumbbell model is a coupled hydrodynamic-kinetic model for polymeric fluids in which the configurations of the dumbbells are described by stochastic differential equations. We prove well-posedness of this model by deriving directly a priori estimates on the stochastic model. Our results can be used to analyze stochastic simulation methods such as the ones that are based on Brownian configuration fields.

1. Introduction

The dumbbell model is the simplest model of polymeric fluids that takes into account the microscopic behavior of the solute polymers [1]. It models the dilute polymers by dumbbells, each with two beads connected by a spring. The configuration of the spring then specifies the conformation of the polymer. Denote by $\mathbf{u}$ and p the velocity and pressure of the fluid, and $\mathbf{Q}$ the configuration of the spring, and hence the dumbbell, then $\mathbf{Q}$ obeys the following equation:

$$\frac{\partial \mathbf{Q}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\nabla \mathbf{u})^T \mathbf{Q} = -\mathbf{F}(\mathbf{Q}) + \dot{\mathbf{w}}(t). \tag{1}$$

Here $\dot{\mathbf{w}}(t)$ is Gaussian white noise in time.

This equation is the result of the balance between the friction force (caused by the viscous fluid) on the left hand side and the spring and Brownian force on the right hand side. For simplicity we will set all physical constants to be 1, and we will write the spring force in the form $\mathbf{F}(\mathbf{Q}) = \gamma(|Q|^2)\mathbf{Q}$.

In writing down (1) for an individual dumbbell we made the crucial assumption that the polymer-polymer interaction can be neglected. Thermal noise is then naturally

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expressed as white noise in time. In this description, the conformation of the dumbbell is described by a stochastic field $\mathbf{Q}(\mathbf{x}, t)$: for any fixed $\mathbf{x}$ in the flow domain, the ensemble $\{\mathbf{Q}(\mathbf{x}, t)\}$ represents possible dumbbell conformation at $\mathbf{x}$. $\mathbf{Q}$ is called Brownian configuration fields. An alternative description, which was used in earlier stochastic simulation methods such as CONNFFESSIT[10], attempts to keep track of each individual dumbbells, which are then subjected to independent thermal noises. The relation between these two descriptions are not fully understood.

The complete model for the full polymer-polymer system is then given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{2}$$

where τ is the polymer contribution to stress, which is expressed via the Kramers expression

$$\tau(\mathbf{x}, t) = \mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}). \tag{3}$$

In contrast to traditional models of complex fluids which express polymer stress τ using empirical constitutive relations, (1)-(3) expresses the polymer stress in terms of the microscopic conformations of the polymers using (3). To this end, a new dynamic equation (1) is added to the model which describes the evolution of the internal degrees of freedom of the polymers. Since the polymeric stress is computed directly from the configuration distribution of the polymers, there is no need to introduce ad hoc constitutive relations.

Equations (1)–(3) is a system of stochastic differential equations in which the dynamics of $(\mathbf{u}, p)$ is deterministic. One way of studying such systems is to use the equivalent Fokker-Planck equation for the $(\mathbf{x}, \mathbf{Q})$ distribution function, denoted by ψ, of the dumbbells

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla)\psi + \nabla_{\mathbf{Q}} \cdot \{(\nabla \mathbf{u})^T \mathbf{Q}\psi - \mathbf{F}(\mathbf{Q})\psi\} = \Delta_{\mathbf{Q}}\psi. \tag{4}$$

In (4), $\mathbf{Q}$ is an independent variable and we use the subscript $\mathbf{Q}$ to denote differentiation with respect to $\mathbf{Q}$. Without the subscript, the differentiation is understood to be in $\mathbf{x}$.

The novelty of Eqs. (2)–(4) is that the macroscopic fluid equation is coupled with the mesoscopic Fokker-Planck equation in kinetic theory. Mathematical study of such systems is still in its infancy. In [16, 17, 11], the local existence and uniqueness has been established.

In the special case when the spring force is linear, $\mathbf{F}(\mathbf{Q}) = H\mathbf{Q}$, we get from (1)-(3) a reduced system of equations for $\mathbf{u}$ and τ,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0, \tag{5}$$

$$\frac{\partial \tau}{\partial t} + (\mathbf{u} \cdot \nabla)\tau - (\nabla \mathbf{u})^T \tau - \tau \nabla \mathbf{u} + \tau - I = 0. \tag{6}$$

In this way, one eliminates $\mathbf{Q}$ as an independent variable. This is the well-known Oldroyd-B model. Its well-posedness has been studied by Saut et al. [5], Lions and Masmoudi [13]. However, their methods do not seem to extend to more general cases when closed systems of equations such as (4)–(5) are not available.

Despite the fact that a purely deterministic analysis based on the Fokker-Planck equation is possible, it is of great interest to treat directly the stochastic system (1)–(3) for several reasons:

1. Equation (1) gives a more direct and intuitive description of the conformation and dynamics of the polymers.
2. One may expect that analysis based on (1) is easier to generalize to more general models of polymers such as liquid crystal polymers, bead and spring models. Some evidence is already provided in [12].
3. There has been a great deal of interest in designing stochastic modelling techniques for polymeric fluids [15]. One of our interest is in the analysis of such stochastic methods. So far this is only done in [7] for one-dimensional shear flows using the specified structure of the shear flow system. The numerical analysis in the general case will depend crucially on proving that (1)-(3) is locally well-posed which is the main purpose of the present paper.

To avoid complications from the boundary of the physical domain, we assume that the physical domain is $D = [0, 1]^d$ with periodic boundary conditions. We will denote by $\mathbf{Q}_0(\mathbf{x}) = \mathbf{Q}(\mathbf{x}, 0)$ the initial condition for the configuration field. We take it to be deterministic. But certainly our results can be extended to the case when it is random.

Our main result is the following:

Theorem 1.1. *Assume that the spring force* $\mathbf{F}$ *and the initial value satisfy the following conditions (A) and (B):*

(A) The function γ *is* C^∞*-smooth from* $[0, +\infty)$ *to* $(0, +\infty)$*, and* $\gamma'(|\mathbf{Q}|^2) \geq 0$*, and the derivative of* $\mathbf{F}$ *satisfies that* $|\nabla^m_{\mathbf{Q}} \mathbf{F}(\mathbf{Q})| \leq C(1 + |\mathbf{Q}|^p)$ $(m = 0, 1, 2, 3, 4)$*, where* C *is a constant and* p *is a certain non-negative integer.*

(B)

$$\|\mathbf{u}_0\|_{H^4} \leq \ Const., \tag{7}$$

$$|\nabla^m \mathbf{Q}_0| \leq \ Const. \ (m = 0, 1, 2, 3, 4). \tag{8}$$

Then there exists a T^* *such that for* $t \leq T^*$*,* (1)-(3) *has a unique strong solution* $\mathbf{u}$ *with* $\mathbf{u} \in C^1([0, T^*$

Here $\mathbb{E}$ denotes expectation with respect to the statistics of the Gaussian white noise $\dot{\mathbf{w}}(t)$.

Equations (9), (10), and (11) are supplemented with initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{12}$$

$$\mathbf{Q}(\mathbf{x}, 0) = \mathbf{Q}_0(\mathbf{x}), \tag{13}$$

which are assumed to be smooth. $\{\mathbf{Q}_0\}$ can be random, however.

2.1. A priori estimates for **u**. Here we recall some standard estimates for Navier-Stokes type of equations.

Consider

$$\begin{cases} \mathbf{u}_t + (\mathbf{h} \cdot \nabla)\mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot \tau, \quad \nabla \cdot \mathbf{u} = 0 \\ \nabla \cdot \mathbf{h} = 0 \end{cases}, \tag{14}$$

where **u**, **h** and f are assumed to be smooth. Then we have

$$\|\mathbf{u}(\cdot, t)\|_{L^2}^2 + \int_0^t \|\nabla \mathbf{u}(\cdot, s)\|_{L^2}^2 ds \leq \|\mathbf{u}_0\|_{L^2}^2 + \int_0^t \|\tau(\cdot, s)\|_{L^2}^2 ds. \tag{15}$$

Lemma 2.1. *Let* $\mathbf{v} = \nabla \mathbf{u}$, $\mathbf{v}_0 = \nabla \mathbf{u}_0$, *then*

$$\|\mathbf{v}(\cdot, t)\|_{L^2}^2 \leq e^{\int_0^t \|\nabla \mathbf{h}(\cdot,s)\|_{L^\infty} ds} \left(\|\mathbf{v}_0\|_{L^2}^2 + \int_0^t \|\nabla \tau(\cdot, s)\|_{L^2}^2 ds \right), \tag{16}$$

$$\int_0^t \|\nabla \mathbf{v}(\cdot, s)\|_{L^2}^2 ds \leq \|\mathbf{v}_0\|_{L^2}^2 + \int_0^t ds \left(\|\nabla \mathbf{h}(\cdot, s)\|_{L^\infty} \|\mathbf{v}(\cdot, s)\|_{L^2}^2 + \|\nabla \tau(\cdot, s)\|_{L^2}^2 \right). \tag{17}$$

Lemma 2.2. *For* $\alpha = 0, 1, 2, 3, 4$,

$$\|\mathbf{u}(\cdot, t)\|_{H^\alpha}^2 \leq e^{\int_0^t \|\mathbf{h}(\cdot,s)\|_{H^4} ds} \left(\|\mathbf{u}_0\|_{H^\alpha}^2 + \int_0^t \|\tau(\cdot, s)\|_{H^\alpha}^2 ds \right), \tag{18}$$

$$\int_0^t \|\mathbf{u}(\cdot, s)\|_{H^{\alpha+1}}^2 ds \leq \|\mathbf{u}_0\|_{H^\alpha}^2 + \int_0^t (\|\mathbf{h}(\cdot, s)\|_{H^4} \|\mathbf{u}(\cdot, s)\|_{H^\alpha}^2 + \|\tau(\cdot, s)\|_{H^\alpha}^2) ds. \tag{19}$$

2.2. A priori estimates for **Q**. Consider

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} = \kappa \mathbf{Q} - \mathbf{F}(\mathbf{Q}) + \dot{\mathbf{w}}(t), \tag{20}$$

where $\kappa = (\nabla \mathbf{u})^T$, **F** is a smooth function. To be precise (20) should be written as

$$d\mathbf{Q} = (-\mathbf{u} \cdot \nabla \mathbf{Q} + \kappa \mathbf{Q} - \mathbf{F}(\mathbf{Q}))dt + d\mathbf{w}. \tag{21}$$

u is assumed to be a given smooth deterministic velocity field, s.t. $\nabla \cdot \mathbf{u} = 0$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space upon which the Wiener process $\mathbf{w}(\cdot)$ is defined. We will use ω to denote realizations of the Brownian path.

We will always assume that **F** satisfies the growth condition (A) in Theorem 1.1.

Denote by $\mathbf{X}(\boldsymbol{\alpha}, t)$ the Eulerian-Lagrangian flow map induced by the velocity field $\mathbf{u}$:

$$\begin{cases} \frac{d}{dt}\mathbf{X}(\boldsymbol{\alpha}, t) = \mathbf{u}(\mathbf{X}(\boldsymbol{\alpha}, t), t) \\ \mathbf{X}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha} \end{cases} . \tag{22}$$

Let $\mathbf{Q}$ be the solution of (20). Fix a $\boldsymbol{\alpha} \in R^d$, let $\mathbf{q}(\boldsymbol{\alpha}, t)$ be the solution of

$$d\mathbf{q} = (k(t)\mathbf{q} - \mathbf{F}(\mathbf{q}))dt + d\mathbf{w}, \quad \mathbf{q}(\boldsymbol{\alpha}, 0) = \mathbf{Q}_0(\boldsymbol{\alpha}), \tag{23}$$

where $k(t) = \kappa(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. Then

Lemma 2.3. *For almost all $\omega \in \Omega$,*

$$\mathbf{q}(\boldsymbol{\alpha}, t) = \mathbf{Q}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$$

for all $\boldsymbol{\alpha} \in R^d$.

Proof. Let $\bar{\mathbf{q}}_\alpha(t) = \mathbf{Q}(\mathbf{X}(\boldsymbol{\alpha}, t), t)$. It is easy to see that $\bar{\mathbf{q}}_\alpha(\cdot)$ is a solution of (23) under the $C^1(D)$-smooth condition of $\mathbf{u}$ which will be shown later. Hence Lemma 2.3 follows from the uniqueness results for (23). □

Lemma 2.4. *There exists a unique solution to the SDEs (23) with values $C([0, +\infty), \mathbb{R}^d)$.*

Proof. It is a standard procedure to prove the existence and uniqueness of the SDEs before the stopping time $\tau_N = \inf\{t|\ |\mathbf{Q}| > N\}$ for the smoothness of $\mathbf{F}$ and $\mathbf{u}$. The only needed thing is to show that the lifespan of the solution of (23) is $\mathbb{R}^+$ a.s., i.e. $\lim_{N\to+\infty} \tau_N = +\infty$. Because the stretching term $k(t)\mathbf{q}$ is just a linear growth term of $\mathbf{q}$, we only need to show that the equation

$$d\mathbf{q} = -\mathbf{F}(\mathbf{q})dt + d\mathbf{w} \tag{24}$$

will not blow up in finite time. We use Feller's test for explosion to deal with this problem [8, 9].

Consider the equation for $X_t = |\mathbf{q}|^2$ by applying Itô's formula

$$\begin{aligned} dX_t &= 2\mathbf{q} \cdot d\mathbf{q} + ndt \\ &= -2\mathbf{q} \cdot \mathbf{F}(\mathbf{q})dt + ndt + 2\mathbf{q} \cdot d\mathbf{w} \\ &= (n - 2X_t\gamma(X_t))dt + 2\sqrt{X_t}dB_t, \end{aligned} \tag{25}$$

where B_t is a Brownian motion by Paul Lévy characterization. $n = d$ is the spatial dimension. Define the scale function $p(x)$ satisfies

$$2(n - 2\gamma(x)x)p'(x) + 4xp''(x) = 0, \tag{26}$$

and, we obtain

$$p(x) = \int_c^x y^{-\frac{n}{2}} e^{h(y)} dy, \tag{27}$$

where $h'(x) = \gamma(x)$, c is a fixed positive number. Clearly $h(x)$ is a monotonely increasing function of x. The speed measure is defined as

$$m(dx) = \frac{1}{2} x^{\frac{n}{2}-1} e^{-h(x)} dx; \tag{28}$$

thus if $n = 2$, we have

$$
\begin{aligned}
v(x) &= \int_c^x \Big(p(x) - p(y)\Big) m(dy) \\
&= \int_c^x \Big(\frac{1}{2}\int_y^x \frac{y}{z}\cdot\frac{1}{y} e^{(h(z)-h(y))} dz\Big) dy \\
&\geq \frac{1}{2}\int_c^x \ln\frac{x}{y} dy = \frac{x}{2}\int_1^{\frac{c}{x}} \ln z dz.
\end{aligned} \tag{29}
$$

It is clear that $v(+\infty) = +\infty$.

If $n = 3$, we have

$$
\begin{aligned}
v(x) &= \int_c^x \Big(p(x) - p(y)\Big) m(dy) \\
&= \int_c^x \Big(\frac{1}{2}\int_y^x \Big(\frac{y}{z}\Big)^{\frac{3}{2}}\cdot\frac{1}{y} e^{(h(z)-h(y))} dz\Big) dy \\
&\geq \frac{1}{2}\int_c^x \sqrt{y}\big(-2z^{-\frac{1}{2}}|_y^x\big) dy = \int_c^x \left(1 - \sqrt{\frac{y}{x}}\right) dy \\
&= x\left(1 - \int_{\frac{c}{x}}^1 z^{\frac{1}{2}} dz\right) - c.
\end{aligned} \tag{30}
$$

It is clear that $v(+\infty) = +\infty$.

Thus we obtain the existence and uniqueness on $\mathbb{R}^+$. $\square$

Here and in the following we will often use the Frobenius-norm of the vectors or the tensors A which is defined as

$$|A|_F \triangleq \Big(\sum_{i,j,\dots,k} a_{i,j,\dots,k}^2\Big)^{\frac{1}{2}}. \tag{31}$$

It is not difficult to find that this norm satisfies the common triangle inequality and the product inequality

$$|A * B|_F \leq |A|_F |B|_F, \tag{32}$$

where $*$ may be $\cdot$, : or higher order contraction operators. We will still abbreviate $|\cdot|_F$ as $|\cdot|$ through the paper.

Lemma 2.5. *Define:*

$$\mathcal{Q}_m^{(0)}(t) = sup_{\boldsymbol{\alpha}} \mathbb{E}|\mathbf{q}(\boldsymbol{\alpha}, t)|^m = sup_{\mathbf{x}} \mathbb{E}|\mathbf{Q}|^m, \tag{33}$$

then we have the following recursive inequality

$$\mathcal{Q}_m^{(0)}(t) \leq \mathcal{Q}_m^{(0)}(0) + m\int_0^t \|\nabla\mathbf{u}\|_{L^\infty} \mathcal{Q}_m^{(0)}(s) ds + \frac{1}{2}(mn + m(m-2))\int_0^t \mathcal{Q}_{m-2}^{(0)}(s) dt. \tag{34}$$

If $m = 2$, then we have

$$\mathcal{Q}_2^{(0)}(t) \leq \mathcal{Q}_2^{(0)}(0) + nt + e^{\int_0^t 2\|\nabla\mathbf{u}\|_{L^\infty} ds}\int_0^t 2(\mathcal{Q}_2^{(0)}(0) + ns)\|\nabla\mathbf{u}\|_{L^\infty} ds, \tag{35}$$

where $n = d$ is the spatial dimension.

Proof. For $|\mathbf{q}|^m$, we use Itô's formula

$$d|\mathbf{q}|^m = m|\mathbf{q}|^{m-2}\mathbf{q}\cdot d\mathbf{q} + \frac{1}{2}(mn + m(m-2))|\mathbf{q}|^{m-2}dt. \tag{36}$$

Paying attention that the term $\mathbf{q}\cdot\mathbf{F}(\mathbf{q}) \geq 0$, we obtain

$$d|\mathbf{q}|^m \leq m|\nabla\mathbf{u}||\mathbf{q}|^m dt + m|\mathbf{q}|^{m-2}\mathbf{q}\cdot d\mathbf{w} + \frac{1}{2}(mn + m(m-2))|\mathbf{q}|^{m-2}dt. \tag{37}$$

Integrating on $[0, t]$, and taking the expectation on both sides, we get

$\mathcal{Q}($ $^{(\mathrm{mn}}$

Notice that

$$\int_D (\mathbf{u}\cdot\nabla)R * R|R|^{m-2}dx = 0; \tag{46}$$

we get

$$\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(1)} \le \|\nabla u\|_{L^\infty}\mathcal{Q}_m^{(1)} + \int_D |\nabla^2\mathbf{u}|\mathbb{E}(|\mathbf{Q}||R|^{m-1}). \tag{47}$$

Since

$$\begin{aligned}\int_D \mathbb{E}(|\nabla^2\mathbf{u}||\mathbf{Q}||R|^{m-1}) &\le \int_D (\mathbb{E}|R|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^2\mathbf{u}|^m|\mathbf{Q}|^m)^{\frac{1}{m}}dx\\ &\le \frac{m-1}{m}\mathcal{Q}_m^{(1)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m}\int_D |\nabla^2\mathbf{u}|^m dx,\end{aligned} \tag{48}$$

finally we get

$$\mathcal{Q}_m^{(1)}(t) \le e^{\int_0^t(\|\nabla\mathbf{u}\|_{L^\infty}+c)ds}\left(\mathcal{Q}_m^{(1)}(0) + \int_0^t \mathcal{Q}_m^{(0)}(s)\|\nabla^2\mathbf{u}(\cdot,s)\|_{L^m}^m ds\right). \tag{49}$$

□

Remark 2. Notice that if $m < +\infty$, we have $\|\nabla^2\mathbf{u}\|_{L^m} \le \|\mathbf{u}\|_{H^4} \in L^\infty(0,t)$. The estimate of $\mathcal{Q}_m^{(1)}(t)$ is valid.

Lemma 2.7. *Define*

$$\mathcal{Q}_m^{(2)}(t) = \int_D \mathbb{E}|\nabla^2\mathbf{Q}|^m dx, \tag{50}$$

then we have

$$\begin{aligned}\mathcal{Q}_m^{(2)}(t) \le e^{\int_0^t(\|\nabla\mathbf{u}\|_{L^\infty}+c)ds}\cdot\Big(\mathcal{Q}_m^{(2)}(0) + \int_0^t (\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m}\\ +\mathcal{Q}_m^{(0)}(s)\|\nabla^3\mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4p}^{(0)}(s) + \mathcal{Q}_{4m}^{(1)}(s))ds\Big).\end{aligned} \tag{51}$$

Proof. Define $S \triangleq \nabla R = \nabla^2\mathbf{Q}$, then

$$S_t + (\mathbf{u}\cdot\nabla)S + 2(\nabla\mathbf{u}\cdot\nabla)R + (\nabla^2\mathbf{u}\cdot\nabla)\mathbf{Q} = \nabla^2\kappa\cdot\mathbf{Q} + 2\nabla\kappa\cdot R + \kappa\cdot S - \nabla^2\mathbf{F}(\mathbf{Q}), \tag{52}$$

$$\nabla^2\mathbf{F}(\mathbf{Q}) = \nabla_x(\nabla_{\mathbf{Q}}\mathbf{F}\cdot\nabla_x\mathbf{Q}) = \nabla_{\mathbf{Q}}^2\mathbf{F}\nabla_x\mathbf{Q}\nabla_x\mathbf{Q} + \nabla_{\mathbf{Q}}\mathbf{F}\cdot\nabla_x^2\mathbf{Q}. \tag{53}$$

Taking the inner product on both sides with $S|S|^{m-2}$, and noting that

$$(\nabla_{\mathbf{Q}}\mathbf{F})_{il}S_{ljk}S_{ijk} = 2\gamma'(|\mathbf{Q}|^2)Q_iQ_l\partial_{lj}Q_k\partial_{ij}Q_k + \gamma(|\mathbf{Q}|^2)|S|^2 \ge 0, \tag{54}$$

where the summation convention is applied, thus we get

$$\begin{aligned}\frac{1}{m}|S|_t^m \le \|\nabla\mathbf{u}\|_{L^\infty}|S|^m - (\mathbf{u}\cdot\nabla)S * S|S|^{m-2} + |\nabla^2\mathbf{u}||\nabla\mathbf{Q}||S|^{m-1}\\ +|\nabla^3\mathbf{u}||\mathbf{Q}||S|^{m-1} + (1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^2|S|^{m-1}.\end{aligned} \tag{55}$$

Integrating on D and taking the expectation

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(2)} \leq \|\nabla \mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(2)} &- \int_D \mathbb{E}((\mathbf{u}\cdot\nabla)S * S|S|^{m-2})dx \\ &+ \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\nabla\mathbf{Q}||S|^{m-1})dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}||\mathbf{Q}||S|^{m-1})dx \\ &+ \int_D \mathbb{E}\Big((1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^2|S|^{m-1}\Big)dx, \end{aligned} \tag{56}$$

and using

$$\int_D (\mathbf{u}\cdot\nabla)S * S|S|^{m-2}dx = 0, \tag{57}$$

we get

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(2)} &\leq \|\nabla \mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(2)} + \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\nabla\mathbf{Q}||S|^{m-1})dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}||\mathbf{Q}||S|^{m-1})dx \\ &+ \int_D \mathbb{E}\Big((1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^2|S|^{m-1}\Big)dx \\ &= P1 + P2 + P3 + P4, \end{aligned} \tag{58}$$

and we have

$$\begin{aligned} P2 &\leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^2\mathbf{u}|^m|\nabla\mathbf{Q}|^m)^{\frac{1}{m}}dx \\ &\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{1}{2m}(\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(1)}(t)), \end{aligned} \tag{59}$$

$$\begin{aligned} P3 &\leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^3\mathbf{u}|^m|\mathbf{Q}|^m)^{\frac{1}{m}}dx \\ &\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m}\|\nabla^3\mathbf{u}\|_{L^m}^m, \end{aligned} \tag{60}$$

$$\begin{aligned} P4 &\leq \int_D (\mathbb{E}|S|^m)^{\frac{m-1}{m}}\Big(\mathbb{E}((1+|\mathbf{Q}|^{2p})|\nabla\mathbf{Q}|^{2m})\Big)^{\frac{1}{m}}dx \\ &\leq \frac{m-1}{m}\mathcal{Q}_m^{(2)} + \frac{1}{2m}(\mathcal{Q}_{4m}^{(1)}(t) + 2\mathcal{Q}_{2m}^{(1)}(t) + \mathcal{Q}_{4p}^{(0)}(t)). \end{aligned} \tag{61}$$

After dropping some lower order terms, we obtain

$$\begin{aligned} \mathcal{Q}_m^{(2)}(t) \leq e^{\int_0^t(\|\nabla\mathbf{u}\|_{L^\infty}+c)ds}\cdot\Big(\mathcal{Q}_m^{(2)}(0) + \int_0^t(\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m} \\ + \mathcal{Q}_m^{(0)}(s)\|\nabla^3\mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4p}^{(0)}(s) + \mathcal{Q}_{4m}^{(1)}(s))ds\Big). \end{aligned} \tag{62}$$

□

Remark 3. Notice that if $m \leq 6$, we have $\|\nabla^2\mathbf{u}\|_{L^{2m}} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0,t)$, $\|\nabla^3\mathbf{u}\|_{L^m} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0,t)$. The estimate of $\mathcal{Q}_m^{(2)}(t)$ is valid.

Lemma 2.8. *Define*

$$\mathcal{Q}_m^{(3)}(t) = \int_D \mathbb{E}|\nabla^3 \mathbf{Q}|^m dx, \tag{63}$$

then we have

$$\begin{aligned}\mathcal{Q}_m^{(3)}(t) \le e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty}+c)ds} \cdot \Big(\mathcal{Q}_m^{(3)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} + \|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m} \\ + \mathcal{Q}_m^{(0)}(s)\|\nabla^4 \mathbf{u}\|_{L^m}^m + \mathcal{Q}_{4mp}^{(0)}(t) + \mathcal{Q}_{6m}^{(1)}(s) + \mathcal{Q}_{2m}^{(2)}(s))ds\Big).\end{aligned} \tag{64}$$

Proof. Define $T \triangleq \nabla S = \nabla^3 \mathbf{Q}$, then we have

$$\begin{aligned}&T_t + (\mathbf{u}\cdot\nabla)T + 3(\nabla \mathbf{u}\cdot\nabla)S + 3(\nabla^2\mathbf{u}\cdot\nabla)R + (\nabla^3\mathbf{u}\cdot\nabla)\mathbf{Q} \\ &= \nabla^3\kappa\cdot\mathbf{Q} + 3\nabla^2\kappa\cdot R + 3\nabla\kappa\cdot S + \kappa\cdot T - \nabla^3\mathbf{F}(\mathbf{Q}),\end{aligned} \tag{65}$$

$$\begin{aligned}\nabla_x^3\mathbf{F}(\mathbf{Q}) &= \nabla_x(\nabla_{\mathbf{Q}}^2\mathbf{F}\nabla_x\mathbf{Q}\nabla_x\mathbf{Q} + \nabla_{\mathbf{Q}}\mathbf{F}\cdot\nabla_x^2\mathbf{Q}) \\ &= \nabla_{\mathbf{Q}}^3\mathbf{F}\nabla_x\mathbf{Q}\nabla_x\mathbf{Q}\nabla_x\mathbf{Q} + \nabla_{\mathbf{Q}}^2\mathbf{F}\nabla_x^2\mathbf{Q}\nabla_x\mathbf{Q} + \nabla_{\mathbf{Q}}\mathbf{F}\nabla_x^3\mathbf{Q}.\end{aligned} \tag{66}$$

Taking the inner product on both sides of the equation for T with $T|T|^{m-2}$, we get

$$\begin{aligned}\frac{1}{m}|T|_t^m &\le \|\nabla\mathbf{u}\|_{L^\infty}|T|^m - (\mathbf{u}\cdot\nabla)T * T|T|^{m-2} - (\nabla_{\mathbf{Q}}\mathbf{F}\cdot T) * T|T|^{m-2} \\ &+|\nabla^2\mathbf{u}||\nabla^2\mathbf{Q}||T|^{m-1} + |\nabla^3\mathbf{u}||\nabla\mathbf{Q}||T|^{m-1} + |\nabla^4\mathbf{u}||\mathbf{Q}||T|^{m-1} \\ &+(|\nabla\mathbf{Q}|^3|T|^{m-1} + |\nabla^2\mathbf{Q}||\nabla\mathbf{Q}||T|^{m-1})(1+|\mathbf{Q}|^p).\end{aligned} \tag{67}$$

Integrating both sides and using the identity

$$\int_D (\mathbf{u}\cdot\nabla)T * T|T|^{m-2}dx = 0 \tag{68}$$

and the inequality $(\nabla_{\mathbf{Q}}\mathbf{F}\cdot T) * T \ge 0$, we get

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\mathcal{Q}_m^{(3)} &\le \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_m^{(3)} + \int_D \mathbb{E}(|\nabla^2\mathbf{u}||\nabla^2\mathbf{Q}||T|^{m-1})dx \\ &+ \int_D \mathbb{E}(|\nabla^3\mathbf{u}||\nabla\mathbf{Q}||T|^{m-1})dx + \int_D \mathbb{E}(|\nabla^4\mathbf{u}||\mathbf{Q}||T|^{m-1})dx \\ &+ \int_D \mathbb{E}((1+|\mathbf{Q}|^p)|\nabla\mathbf{Q}|^3|T|^{m-1})dx \\ &+ \int_D \mathbb{E}((1+|\mathbf{Q}|^p)|\nabla^2\mathbf{Q}||\nabla\mathbf{Q}||T|^{m-1})dx \\ &= P1 + P2 + P3 + P4 + P5 + P6.\end{aligned} \tag{69}$$

Note that

$$\begin{aligned}P2 &\le \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}}(\mathbb{E}|\nabla^2\mathbf{u}|^m|\nabla^2\mathbf{Q}|^m)^{\frac{1}{m}}dx \\ &\le \frac{m-1}{m}\mathcal{Q}_m^{(3)} + \frac{1}{2m}(\|\nabla^2\mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(2)}(t)),\end{aligned} \tag{70}$$

$$P3 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^3 \mathbf{u}|^m |\nabla \mathbf{Q}|^m)^{\frac{1}{m}} dx$$
$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{2m} (\|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m} + \mathcal{Q}_{2m}^{(1)}(t)), \tag{71}$$

$$P4 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^4 \mathbf{u}|^m |\mathbf{Q}|^m)^{\frac{1}{m}} dx$$
$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{\mathcal{Q}_m^{(0)}(t)}{m} \|\nabla^4 \mathbf{u}\|_{L^m}^m, \tag{72}$$

$$P5 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla \mathbf{Q}|^{3m} (1 + |\mathbf{Q}|^{mp}))^{\frac{1}{m}} dx$$
$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{2m} (\mathcal{Q}_{6m}^{(1)}(t) + 2\mathcal{Q}_{3m}^{(1)}(t) + \mathcal{Q}_{2mp}^{(0)}(t)), \tag{73}$$

$$P6 \leq \int_D (\mathbb{E}|T|^m)^{\frac{m-1}{m}} (\mathbb{E}|\nabla^2 \mathbf{Q}|^m |\nabla \mathbf{Q}|^m (1 + |\mathbf{Q}|^{mp}))^{\frac{1}{m}} dx$$
$$\leq \frac{m-1}{m} \mathcal{Q}_m^{(3)} + \frac{1}{4m} (2\mathcal{Q}_{2m}^{(2)}(t) + 2\mathcal{Q}_{2m}^{(1)}(t) + \mathcal{Q}_{4m}^{(1)}(t) + \mathcal{Q}_{4mp}^{(0)}(t)), \tag{74}$$

where some constants have been omitted. After dropping some lower order terms, we obtain

$$\mathcal{Q}_m^{(3)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \Big(\mathcal{Q}_m^{(3)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^{2m}}^{2m} + \|\nabla^3 \mathbf{u}\|_{L^{2m}}^{2m}$$
$$+ \mathcal{Q}_m^{(0)}(s) \|\nabla^4 \mathbf{u}\|_{L^m}^m + \mathcal{Q}_{6m}^{(1)}(s) + \mathcal{Q}_{4mp}^{(0)}(t) + \mathcal{Q}_{2m}^{(2)}(s)) ds \Big). \tag{75}$$

□

Remark 4. Notice that if $m \leq 3$, we have $\|\nabla^2 \mathbf{u}\|_{L^{2m}}, \|\nabla^3 \mathbf{u}\|_{L^{2m}} \leq \|\mathbf{u}\|_{H^4} \in L^\infty(0, t)$. If $m = 3$, we have

$$\|\nabla^4 \mathbf{u}\|_{L^3}^3 \leq \|\nabla^4 \mathbf{u}\|_{H^{\frac{1}{2}}}^3 \leq (\|\nabla^4 \mathbf{u}\|_{H^0}^{\frac{1}{2}})^3 (\|\nabla^4 \mathbf{u}\|_{H^1}^{\frac{1}{2}})^3 \leq \|\nabla^4 \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\mathbf{u}\|_{H^5}^{\frac{3}{2}}, \tag{76}$$

and a simple Hölder inequality shows that $\|\nabla^4 \mathbf{u}\|_{L^3}^3$ belongs to $L^1(0, t)$. The estimate of $\mathcal{Q}_m^{(3)}(t)$ is valid.

Lemma 2.9. *Define*

$$\mathcal{Q}_m^{(4)}(t) = \int_D \mathbb{E}|\nabla^4 \mathbf{Q}|^m dx, \tag{77}$$

then we have

$$\mathcal{Q}_2^{(4)}(t) \leq e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \Big(\mathcal{Q}_2^{(4)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^6}^6 + \|\nabla^3 \mathbf{u}\|_{L^4}^4 + \|\nabla^4 \mathbf{u}\|_{L^3}^3$$
$$+ \mathcal{Q}_2^{(0)}(s) \|\nabla^5 \mathbf{u}\|_{L^2}^2 + \mathcal{Q}_{12p}^{(0)}(s) + \mathcal{Q}_{16}^{(1)}(s) + \mathcal{Q}_6^{(2)}(s) + \mathcal{Q}_3^{(3)}(s)) ds \Big). \tag{78}$$

Proof. Define $U \triangleq \nabla T = \nabla^4 \mathbf{Q}$, then we have

$$\begin{aligned} &U_t + (\mathbf{u}\cdot\nabla)U + 4(\nabla\mathbf{u}\cdot\nabla)T + 6(\nabla^2\mathbf{u}\cdot\nabla)S + 4(\nabla^3\mathbf{u}\cdot\nabla)R + (\nabla^4\mathbf{u}\cdot\nabla)\mathbf{Q} \\ &\quad = \nabla^4\kappa\cdot\mathbf{Q} + 4\nabla^3\kappa\cdot R + 6\nabla^2\kappa\cdot S + 4\nabla\kappa\cdot T + \kappa\cdot U - \nabla^4\mathbf{F}(\mathbf{Q}). \end{aligned} \tag{79}$$

Straightforward calculation gives $\nabla_x^4\mathbf{F}(\mathbf{Q}) = \nabla_{\mathbf{Q}}\mathbf{F}\cdot U + Rem$, where

$$|Rem| \le C(|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}| + |\nabla^2\mathbf{Q}|^2 + |\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2 + |\nabla\mathbf{Q}|^4)(1+|\mathbf{Q}|^p). \tag{80}$$

Hence

$$\begin{aligned} \frac{1}{2}|U|_t^2 \le\ & \|\nabla\mathbf{u}\|_{L^\infty}|U|^2 - (\mathbf{u}\cdot\nabla)U * U - (\nabla_{\mathbf{Q}}\mathbf{F}\cdot U) * U + |\nabla^2\mathbf{u}||\nabla^3\mathbf{Q}||U| \\ & + |\nabla^3\mathbf{u}||\nabla^2\mathbf{Q}||U| + |\nabla^4\mathbf{u}||\nabla\mathbf{Q}||U| + |\nabla^5\mathbf{u}||\mathbf{Q}||U| + (|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}| \\ & + |\nabla^2\mathbf{Q}|^2 + |\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2 + |\nabla\mathbf{Q}|^4)(1+|\mathbf{Q}|^p|)|U|. \end{aligned} \tag{81}$$

Integrating both sides and using

$$\int (\mathbf{u}\cdot\nabla)U * U dx = 0 \tag{82}$$

and the inequality $(\nabla_{\mathbf{Q}}\mathbf{F}\cdot U) * U \ge 0$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\mathcal{Q}_2^{(4)}(t) \le\ & \|\nabla\mathbf{u}\|_{L^\infty}\mathcal{Q}_2^{(4)}(t) + \int_D \mathbb{E}|\nabla^2\mathbf{u}||\nabla^3\mathbf{Q}||U|dx + \int_D \mathbb{E}|\nabla^3\mathbf{u}||\nabla^2\mathbf{Q}||U|dx \\ & + \int_D \mathbb{E}|\nabla^4\mathbf{u}||\nabla\mathbf{Q}||U|dx + \int_D \mathbb{E}|\nabla^5\mathbf{u}||\mathbf{Q}||U|dx \\ & + \int_D \mathbb{E}|\nabla^3\mathbf{Q}||\nabla\mathbf{Q}|(1+|\mathbf{Q}|^p)|U|dx + \int_D \mathbb{E}|\nabla^2\mathbf{Q}|^2(1+|\mathbf{Q}|^p)|U|dx \\ & + \int_D \mathbb{E}|\nabla^2\mathbf{Q}||\nabla\mathbf{Q}|^2(1+|\mathbf{Q}|^p)|U|dx + \int_D \mathbb{E}|\nabla\mathbf{Q}|^4(1+|\mathbf{Q}|^p)|U|dx \\ \triangleq\ & P1 + P2 + P3 + P4 + P5 + P6 + P7 + P8 + P9, \end{aligned} \tag{83}$$

$$\begin{aligned} P2 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2\mathbf{u}|^2|\nabla^3\mathbf{Q}|^2)dx \\ &\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^2\mathbf{u}\|_{L^6}^6 + \mathcal{Q}_3^{(3)}(t), \end{aligned} \tag{84}$$

$$\begin{aligned} P3 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^3\mathbf{u}|^2|\nabla^2\mathbf{Q}|^2)dx \\ &\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^3\mathbf{u}\|_{L^4}^4 + \mathcal{Q}_4^{(2)}(t), \end{aligned} \tag{85}$$

$$\begin{aligned} P4 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^4\mathbf{u}|^2|\nabla\mathbf{Q}|^2)dx \\ &\le \mathcal{Q}_2^{(4)}(t) + \|\nabla^4\mathbf{u}\|_{L^3}^3 + \mathcal{Q}_6^{(1)}(t), \end{aligned} \tag{86}$$

$$
\begin{aligned}
P5 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^5 \mathbf{u}|^2 |\mathbf{Q}|^2) dx \\
&\le \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_2^{(0)}(t)\|\nabla^5 \mathbf{u}\|_{L^2}^2,
\end{aligned} \tag{87}
$$

$$
\begin{aligned}
P6 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^3 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^2 (1 + |\mathbf{Q}|^{2p})) dx \\
&\le \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{12p}^{(0)}(t) + \mathcal{Q}_{12}^{(1)}(t) + \mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_3^{(3)}(t),
\end{aligned} \tag{88}
$$

$$
\begin{aligned}
P7 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2 \mathbf{Q}|^4 (1 + |\mathbf{Q}|^{2p})) dx \\
&\le \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{6p}^{(0)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_6^{(2)}(t),
\end{aligned} \tag{89}
$$

$$
\begin{aligned}
P8 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla^2 \mathbf{Q}|^2 |\nabla \mathbf{Q}|^2 (1 + |\mathbf{Q}|^{2p})) dx \\
&\le \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_{8p}^{(0)}(t),
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
P9 &\le \int_D \mathbb{E}|U|^2 dx + \int_D \mathbb{E}(|\nabla \mathbf{Q}|^8 (1 + |\mathbf{Q}|^{2p})) dx \\
&\le \mathcal{Q}_2^{(4)}(t) + \mathcal{Q}_{16}^{(1)}(t) + \mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_{4p}^{(0)}(t),
\end{aligned} \tag{91}
$$

where some constants are omitted. After dropping some lower order terms, we obtain

$$
\begin{aligned}
\mathcal{Q}_2^{(4)}(t) \le e^{\int_0^t (\|\nabla \mathbf{u}\|_{L^\infty} + c) ds} \cdot \Big(\mathcal{Q}_2^{(4)}(0) + \int_0^t (\|\nabla^2 \mathbf{u}\|_{L^6}^6 + \|\nabla^3 \mathbf{u}\|_{L^4}^4 + \|\nabla^4 \mathbf{u}\|_{L^3}^3 \\
+ \mathcal{Q}_2^{(0)}(s)\|\nabla^5 \mathbf{u}\|_{L^2}^2 + \mathcal{Q}_{12p}^{(0)}(s) + \mathcal{Q}_{16}^{(1)}(s) + \mathcal{Q}_6^{(2)}(s) + \mathcal{Q}_3^{(3)}(s)) ds \Big).
\end{aligned} \tag{92}
$$

□

Lemma 2.10. *Assume that* **F** *satisfies Condition (A) as in Theorem 1.1, then*

$$|\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}| \le C(1 + |\mathbf{Q}|^{p_1}), \tag{93}$$

$$|\nabla_x (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \le C|\nabla \mathbf{Q}|(1 + |\mathbf{Q}|^{p_2}), \tag{94}$$

$$|\nabla_x^2 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \le C(|\nabla \mathbf{Q}|^2 + |\nabla^2 \mathbf{Q}|)(1 + |\mathbf{Q}|^{p_3}), \tag{95}$$

$$|\nabla_x^3 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \le C(|\nabla \mathbf{Q}|^3 + |\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}| + |\nabla^3 \mathbf{Q}|)(1 + |\mathbf{Q}|^{p_4}), \tag{96}$$

$$
\begin{aligned}
|\nabla_x^4 (\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})| \le C(&|\nabla^4 \mathbf{Q}| + |\nabla^3 \mathbf{Q}||\nabla \mathbf{Q}| + |\nabla^2 \mathbf{Q}|^2 \\
&+ |\nabla^2 \mathbf{Q}||\nabla \mathbf{Q}|^2 + |\nabla \mathbf{Q}|^4)(1 + |\mathbf{Q}|^{p_5}),
\end{aligned} \tag{97}
$$

where p_1, p_2, p_3, p_4 *and* p_5 *are suitable integers which are greater than* p.

Proof. This follows from a direct calculation. □

Lemma 2.11. *The stress τ has the following estimates:*

$$\|\tau\|_{L^2}^2 \leq C(1 + \mathcal{Q}_{2p_1}^{(0)}(t)) \cdot meas(D), \tag{98}$$

$$\|\nabla\tau\|_{L^2}^2 \leq C(1 + \mathcal{Q}_{2p_2}^{(0)}(t))\mathcal{Q}_2^{(1)}(t), \tag{99}$$

$$\|\nabla^2\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_3}^{(0)}(t))(\mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_2^{(2)}(t)), \tag{100}$$

$$\|\nabla^3\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_4}^{(0)}(t))(\mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_2^{(3)}(t)), \tag{101}$$

$$\|\nabla^4\tau\|_{L^2}^2 \leq (1 + \mathcal{Q}_{2p_5}^{(0)}(t))(\mathcal{Q}_8^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_3^{(3)}(t) + \mathcal{Q}_2^{(4)}(t)), \tag{102}$$

where some lower order terms have been omitted.

Proof. The estimates of different order derivatives of τ will be done in different steps:

Step 1. Estimate of $\|\tau\|_{L^2}$:

$$\begin{aligned}
\|\tau\|_{L^2}^2 &= \int |\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \leq \int (\mathbb{E}|\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q}|)^2 dx \\
&\leq C \int \left(\mathbb{E}(1 + |\mathbf{Q}|^{p_1})\right)^2 dx \leq C \int (1 + \mathbb{E}|\mathbf{Q}|^{2p_1}) dx \\
&= C \int (1 + \mathbb{E}|q|^{2p_1}) dx \leq C(1 + \mathcal{Q}_{2p_1}^{(0)}(t)) \cdot meas(D).
\end{aligned} \tag{103}$$

Step 2. Estimate of $\|\nabla\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla\tau\|_{L^2}^2 &= \int |\nabla\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(1 + |\mathbf{Q}|^{2p_2})\mathbb{E}|\nabla\mathbf{Q}|^2 dx \\
&\leq C(1 + \mathcal{Q}_{2p_2}^{(0)}(t))\mathcal{Q}_2^{(1)}(t).
\end{aligned} \tag{104}$$

Step 3. Estimate of $\|\nabla^2\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla^2\tau\|_{L^2}^2 &= \int |\nabla_x^2\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(|\nabla\mathbf{Q}|^4 + |\nabla^2\mathbf{Q}|^2)\mathbb{E}(1 + |\mathbf{Q}|^{2p_3}) dx \\
&\leq C(1 + \mathcal{Q}_{2p_3}^{(0)}(t))(\mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_2^{(2)}(t)).
\end{aligned} \tag{105}$$

Step 4. Estimate of $\|\nabla^3\tau\|_{L^2}$:

$$\begin{aligned}
\|\nabla^3\tau\|_{L^2}^2 &= \int |\nabla_x^3\mathbb{E}(\mathbf{F}(\mathbf{Q}) \otimes \mathbf{Q})|^2 dx \\
&\leq C \int \mathbb{E}(|\nabla\mathbf{Q}|^6 + |\nabla^2\mathbf{Q}|^2|\nabla\mathbf{Q}|^2 + |\nabla^3\mathbf{Q}|^2)\mathbb{E}(1 + |\mathbf{Q}|^{2p_4}) dx \\
&\leq C(1 + \mathcal{Q}_{2p_4}^{(0)}(t))(\mathcal{Q}_6^{(1)}(t) + \mathcal{Q}_4^{(1)}(t) + \mathcal{Q}_4^{(2)}(t) + \mathcal{Q}_2^{(3)}(t)).
\end{aligned} \tag{106}$$

Step 5. Estimate of $\|\nabla^4\tau\|_{L^2}$:

$$\begin{aligned}\|\nabla^4\tau\|_{L^2}^2 &= \int |\nabla_x^4\mathbb{E}(\mathbf{F}(\mathbf{Q})\otimes\mathbf{Q})|^2 dx\\ &\le C\int \mathbb{E}(|\nabla^4\mathbf{Q}|^2+|\nabla^3\mathbf{Q}|^2|\nabla\mathbf{Q}|^2+|\nabla^2\mathbf{Q}|^4+|\nabla^2\mathbf{Q}|^2|\nabla\mathbf{Q}|^4\\ &\quad +|\nabla\mathbf{Q}|^8)\mathbb{E}(1+|\mathbf{Q}|^{2p_5})dx\\ &\le C(1+\mathcal{Q}_{2p_5}^{(0)}(t))(\mathcal{Q}_8^{(1)}(t)+\mathcal{Q}_6^{(1)}(t)+\mathcal{Q}_4^{(2)}(t)+\mathcal{Q}_3^{(3)}(t)+\mathcal{Q}_2^{(4)}(t)).\end{aligned} \tag{107}$$

This completes the a priori estimates. □

3. The Local Well-Posedness

To prove local well-posedness, namely Theorem 1.1, we set up a standard iteration scheme. Let $\mathbf{u}^0(\mathbf{x},t) = \mathbf{u}_0(\mathbf{x})$, $\mathbf{Q}^0(\mathbf{x},t,\omega) = \mathbf{Q}_0(\mathbf{x},\omega)$. From $\{\mathbf{u}^0,\mathbf{Q}^0\}$, we obtain a sequence $\{\mathbf{u}^n,\mathbf{Q}^n\}$ by solving the following system

$$\begin{cases}\partial_t\mathbf{u}^{n+1}+(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}+\nabla p^{n+1}=\Delta\mathbf{u}^{n+1}+\nabla\cdot\tau^n,\\ \nabla\cdot\mathbf{u}^{n+1}=0,\\ \tau^n=\mathbb{E}(\mathbf{F}(\mathbf{Q}^n)\otimes\mathbf{Q}^n),\\ \partial_t\mathbf{Q}^{n+1}+(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{Q}^{n+1}=\kappa^{n+1}\mathbf{Q}^{n+1}-\mathbf{F}(\mathbf{Q}^{n+1})+\dot{\mathbf{w}}(t),\end{cases} \tag{108}$$

where $\kappa^{n+1}=\nabla\mathbf{u}^{n+1}$.

We will prove that for short times, the sequence $\{\mathbf{u}^n\}$ is uniformly bounded in a high enough Sobolev norm, and contractive in the L^2 norm.

Define the norm:

$$|||\mathbf{u}^n||| = \left(\max_{s\le T^*}\|\mathbf{u}^n(s)\|_{H^4}^2+\int_0^{T^*}\|\mathbf{u}^n(s)\|_{H^5}^2 ds\right)^{\frac{1}{2}}, \tag{109}$$

where T^* is a constant to be determined later.

Step 1. Uniform boundedness of $\mathbf{u}^n$. It follows from Lemma 2.2 that

$$\begin{aligned}|||\mathbf{u}^{n+1}|||^2 &\le e^{\int_0^{T^*}\|\mathbf{u}^n\|_{H^4}ds}(\|\mathbf{u}_0\|_{H^4}^2+\int_0^{T^*}\|\tau^n\|_{H^4}^2 ds)+\|\mathbf{u}_0\|_{H^4}^2\\ &\quad+\int_0^{T^*}\{\|\mathbf{u}^n\|_{H^4}[e^{\int_0^{T^*}\|\mathbf{u}^n\|_{H^4}ds}(\|\mathbf{u}_0\|_{H^4}^2+\int_0^{T^*}\|\tau^n\|_{H^4}^2 ds)]+\|\tau^n\|_{H^4}^2\}ds.\end{aligned} \tag{110}$$

Define $C_0=\|\mathbf{u}_0\|_{H^4}^2$, assume $|||\mathbf{u}^n|||\le K$, then

$$\begin{aligned}|||\mathbf{u}^{n+1}|||^2 &\le e^{KT^*}\left(C_0+\int_0^{T^*}\|\tau^n\|_{H^4}^2 ds\right)+C_0\\ &\quad+\int_0^{T^*}\left(K[e^{KT^*}(C_0+\int_0^t\|\tau^n\|_{H^4}^2 ds)]+\|\tau^n\|_{H^4}^2\right)ds.\end{aligned} \tag{111}$$

We only need to consider $\int_0^t\|\tau^n\|_{H^4}^2 ds$.

From Lemma 2.11, the highest order terms in $\int_0^t \|\tau^n\|_{H^4}^2 ds$ are $\mathcal{Q}_6^{(2)}(t)$, $\mathcal{Q}_3^{(3)}(t)$, $\mathcal{Q}_2^{(4)}(t)$, which corresponds to the highest order spatial derivative terms. Their validity is easily obtained by using

$$\|\nabla^4 \mathbf{u}\|_{L^3}^3 \leq \|\nabla^4 \mathbf{u}\|_{H^{\frac{1}{2}}}^3 \leq (\|\nabla^4 \mathbf{u}\|_{H^0}^{\frac{1}{2}})^3 (\|\nabla^4 \mathbf{u}\|_{H^1}^{\frac{1}{2}})^3 \leq \|\nabla^4 \mathbf{u}\|_{L^2}^{\frac{3}{2}} \|\mathbf{u}\|_{H^5}^{\frac{3}{2}}. \tag{112}$$

We will simplify the term $\int_0^t \|\tau^n\|_{H^4}^2 ds \leq f(t, K)$, where $f(t, K)$ denotes the right hand side of the last inequality in Lemma 2.10. $f(t, K)$ is a monotonely increasing continuous function of t, and $f(0, K) = 0$, thus

$$|||\mathbf{u}^{n+1}|||^2 \leq e^{KT^*}(C_0 + f(T^*, K)) + C_0 + KT^* e^{KT^*}(C_0 + f(T^*, K)) + Kf(T^*, K). \tag{113}$$

Let $K = 2C_0 + 1$, and choose T^* sufficiently small such that

$$e^{KT^*}(C_0 + f(T^*, K)) + C_0 + KT^* e^{KT^*}(C_0 + f(T^*, K)) + Kf(T^*, K) \leq K^2, \tag{114}$$

then we have

$$|||\mathbf{u}^n||| \leq K. \tag{115}$$

Step 2. Contraction in the low norm:

Define $\mathbf{v}^{n+1} = \mathbf{u}^{n+1} - \mathbf{u}^n$, $\pi^{n+1} = p^{n+1} - p^n$, $\mathbf{R}^{n+1} = \mathbf{Q}^{n+1} - \mathbf{Q}^n$, then

$$\mathbf{v}_t^{n+1} + (\mathbf{u}^n \cdot \nabla)\mathbf{v}^{n+1} + (\mathbf{v}^n \cdot \nabla)\mathbf{u}^n + \nabla \pi^{n+1} = \Delta \mathbf{v}^{n+1} + \nabla \cdot (\tau^n - \tau^{n-1}), \tag{116}$$

$$\nabla \cdot \mathbf{v}^{n+1} = 0,$$

$$\begin{aligned} \mathbf{R}_t^{n+1} - (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{R}^{n+1} - (\mathbf{v}^{n+1} \cdot \nabla)\mathbf{Q}^n &= \kappa^{n+1}\mathbf{R}^{n+1} + \nabla(\mathbf{u}^{n+1} - \mathbf{u}^n)\mathbf{Q}^n \\ &\quad -(\mathbf{F}(\mathbf{Q}^{n+1}) - \mathbf{F}(\mathbf{Q}^n)). \end{aligned} \tag{117}$$

From the definition of τ we have

$$\begin{aligned} \|\tau^n - \tau^{n-1}\|_{L^2}^2 &\leq \int_D \mathbb{E}|\mathbf{F}(\mathbf{Q}^n)|^2 \mathbb{E}|\mathbf{Q}^n - \mathbf{Q}^{n-1}|^2 dx \\ &\quad + \int_D \mathbb{E}|\mathbf{Q}^{n-1} \otimes \nabla_{\mathbf{Q}} \mathbf{F}(\mathbf{Q}^\theta)|^2 \mathbb{E}|\mathbf{Q}^n - \mathbf{Q}^{n-1}|^2 dx \\ &\leq C(1 + \mathcal{Q}_{2p+2}^{(0)}(t)) \int_D \mathbb{E}|\mathbf{R}^n|^2 dx, \end{aligned} \tag{118}$$

where $\mathbf{Q}^\theta = \theta \mathbf{Q}^n + (1-\theta)\mathbf{Q}^{n-1}$, $\theta \in [0, 1]$. From (116)

$$\begin{aligned} \frac{d}{dt}\|\mathbf{v}^{n+1}\|_{L^2}^2 &\leq \|\mathbf{v}^{n+1}\|_{L^2}^2 - \|\nabla \mathbf{v}^{n+1}\|_{L^2}^2 + \|\tau^n - \tau^{n-1}\|_{L^2}^2 \\ &\quad + \|\nabla \mathbf{u}^n\|_{L^\infty}(\|\mathbf{v}^n\|_{L^2}^2 + \|\mathbf{v}^{n+1}\|_{L^2}^2) \\ &\leq \|\mathbf{v}^{n+1}\|_{L^2}^2 - \|\nabla \mathbf{v}^{n+1}\|_{L^2}^2 + C(1 + \mathcal{Q}_{2p+2}^{(0)}(t))\mathbb{E}\|\mathbf{R}^n\|_{L^2}^2 \\ &\quad + K(\|\mathbf{v}^n\|_{L^2}^2 + \|\mathbf{v}^{n+1}\|_{L^2}^2). \end{aligned} \tag{119}$$

From (117)

$$\frac{d}{dt}\|\mathbf{R}^{n+1}\|_{L^2}^2 \leq \|\nabla \mathbf{u}^{n+1}\|_{L^\infty}\|\mathbf{R}^{n+1}\|_{L^2}^2 + 2\int_D (1+|\mathbf{Q}^{n+1}|^p + |\mathbf{Q}^n|^p)|\mathbf{R}^{n+1}|dx$$
$$+\int_D (\mathbf{v}^{n+1}\cdot\nabla\mathbf{Q}^n\cdot\mathbf{R}^{n+1})dx + \int_D (\nabla(\mathbf{u}^{n+1}-\mathbf{u}^n)\cdot\mathbf{Q}^n\cdot\mathbf{R}^{n+1})dx$$
$$\triangleq P1+P2+P3+P4, \tag{120}$$

$$\begin{aligned} P3 &\leq \int_D |\mathbf{v}^{n+1}|^2|\nabla\mathbf{Q}^n|^2dx + \|\mathbf{R}^{n+1}\|_{L^2}^2 \\ &\leq \|\nabla\mathbf{Q}^n\|_{L^\infty}^2\int_D |\mathbf{v}^{n+1}|^2dx + \|\mathbf{R}^{n+1}\|_{L^2}^2 \\ &\leq \|\mathbf{Q}^n\|_{H^3}^2\int_D |\mathbf{v}^{n+1}|^2dx + \|\mathbf{R}^{n+1}\|_{L^2}^2, \end{aligned} \tag{121}$$

$$P4 \leq C_1\int_D |\nabla\mathbf{v}^{n+1}|^2|\mathbf{Q}^n|^2dx + C_2\|\mathbf{R}^{n+1}\|_{L^2}^2, \tag{122}$$

where C_1, C_2 is chosen such that $C_1\mathcal{Q}_2^{(0)}(T^*) \leq 1$. Thus

$$\begin{aligned} \frac{d}{dt}\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 &\leq C(\|\nabla\mathbf{u}^{n+1}\|_{L^\infty} + \sqrt{\mathcal{Q}_{2p}^{(0)}(t)}+1)\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 \\ &\quad +\mathbb{E}\|\mathbf{Q}^n\|_{H^3}^2\int_D |\mathbf{v}^{n+1}|^2dx \\ &\quad +C_1\int_D |\nabla\mathbf{v}^{n+1}|^2\mathbb{E}|\mathbf{Q}^n|^2dx \\ &\leq C(\|\nabla\mathbf{u}^{n+1}\|_{L^\infty} + \sqrt{\mathcal{Q}_{2p}^{(0)}(T^*)}+1)\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 + C\|\mathbf{v}^{n+1}\|_{L^2}^2 \\ &\quad +C_1\mathcal{Q}_2^{(0)}(T^*)\int_D |\nabla\mathbf{v}^{n+1}|^2dx \\ &\leq C\mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2 + C\|\mathbf{v}^{n+1}\|_{L^2}^2 + \int_D |\nabla\mathbf{v}^{n+1}|^2dx. \end{aligned} \tag{123}$$

Hence we have, for some constant C_*,

$$\begin{aligned} \frac{d}{dt}(\|\mathbf{v}^{n+1}\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2) &\leq C_*(\|\mathbf{v}^{n+1}\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}\|_{L^2}^2) \\ &\quad +(\|\mathbf{v}^n\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n\|_{L^2}^2). \end{aligned} \tag{124}$$

Gronwall's inequality shows

$$\|\mathbf{v}^{n+1}(t)\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^{n+1}(t)\|_{L^2}^2 \leq e^{C_*t}\int_0^t (\|\mathbf{v}^n\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n\|_{L^2}^2)ds. \tag{125}$$

Let $\|\mathbf{v}^n\|_C = \max_{s\leq t}(\|\mathbf{v}^n(s)\|_{L^2}^2 + \mathbb{E}\|\mathbf{R}^n(s)\|_{L^2}^2)^{\frac{1}{2}}$. If

$$T^*e^{C_*T^*} = \beta < 1, \tag{126}$$

then we have

$$\|\mathbf{v}^{n+1}\|_C^2 \leq \beta \|\mathbf{v}^n\|_C^2. \quad (127)$$

This establishes the contraction property.

The rest of the proof follows from standard arguments, see for example [14].

4. Conclusion

In this paper, we give a proof of well-posedness of the stochastic model (1)-(3) for a dumbbell-solvent system. We believe that our method, which is based on analyzing directly the stochastic model (1) is of interest by itself. The results of this paper have been used in [4] for the numerical analysis of stochastic simulation methods based on Brownian configuration fields (BCF). Through our analysis, we also demonstrate how to handle directly the stochastic system (1)-(3).

The main technical restriction in our analysis is the fact that noise depends only on time. This is fine for dilute polymer solutions since polymer-polymer interaction can be neglected. But for semi-dilute or for concentrated solutions, this kind of technique can only handle systems under the mean field approximation (see [2]). The mean field approximation is typically made for liquid crystal polymer systems. Therefore a natural next step is to extend the results of the present paper for that case [12].

In the special case when $\mathbf{F}$ is linear in $\mathbf{Q}$, our results recover the local well-posedness results of Saut, but not the results of Lions and Masmoudi for the global existence of weak solutions for the Oldroyd model [13].

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References

1. Bird, R.B., Curtiss, C.F., Armstrong, R.C., Hassager, O.: *Dynamics of Polymeric Liquids, Vol. 2: Kinetic Theory*. New York: John Wiley, 1987
2. Doi, M., Edwards, S.F.: The theory of polymer dynamics. Oxford: Oxford University Press, 1986
3. E, W., Li, T., Zhang, P.: Convergence of a stochastic method for the modeling of polymeric fluids. Acta Mathematicae Applicatae Sinica, English Series, **18**(4), 529–536 (2002)
4. E, W., Li, T., Zhang, P.: Convergence analysis of BCF methods. In preparation
5. Guillope, C., Saut, J.C.: Existence results for the flow of viscoelastic fluids with a differnetial co-stitutive law. Nonlinear Anal. **15**, 849–869 (1990)
6. Hulsen, M.A., van Heel, A.P.G., van den Brule, B.H.A.A.: Simulation of viscoelastic flows using Brownian configuration fields. J. Non-Newtonian Fluid Mech. **70**, 79–101 (1997)
7. Jourdain, B., Lelièvre, T., Le Bris.: C.L Numerical analysis of micro-macro simulations of polymeric fluid flows: A simple case. Math. Models and Methods in Appl.Sci. **12**(9), 1205–1243 (2002)
8. Jourdain, B., Lelièvre, T., Le Bris, C.: Existence of solution for a micro-macro model of polymeric fluid: The FENE model. J. Funct. Anal. **209**, 162–193 (2004)
9. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus (2nd edition), New York: Springer-Verlag, 1991
10. Laso, M., Öttinger, H.C.: Calculation of viscoelastic flow using molecular models: The CONNF-FESSIT approach. J. Non-Newtonian Fluid Mech. **47**, 1–20 (1993)
11. Li, T., Zhang, H., Zhang, P.: Local existence for the dumbbell model of polymeric fluids. Comm. PDE **29**, 903–923 (2004)
12. Li, T., Zhang, P., Zhou, X.: Well-posedness and numerical analysis of stochastic models for liquid crystal polymers. Submitted to Comm. Math. Sci.

13. Lions, P.L., Masmoudi, N.: Global solutions for some Oldroyd models of non-Newtonian flows. Chin. Ann. of Math. **21B**, 131–146 (2000)
14. Majda, A.J.: Compressible fluid flow and systems of conservation laws in several space variables. New York: Springer-Verlag, 1984
15. Öttinger, H.C.: Stochastic processes in polymeric liquids. Berlin-Heidelberg-New York: Springer-Verlag, 1996
16. Renardy, M.: Local existence of solutions of the dirichlet initial-boundary value problem for incompressible hypoelastic materials. SIAM J. Math. Anal. **21**, 1369–1385 (1990)
17. Renardy, M.: An existence theorem for model equations resulting from kinetic theories of polymer solutions. SIAM J. Math. Anal. **22**, 313–327 (1991)

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