Laplace Operator and Heat Kernel for Shape Analysis

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Laplace Operator

- on $\mathbb{R}^k$, the standard Laplace operator:
  
  $\Delta_{\mathbb{R}^k} f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_k^2}$

  or $\Delta_{\mathbb{R}^k} f := \text{div}\nabla f$
Laplace Operator

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  - $\Delta_{\mathbb{R}^k} f := \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_k^2}$
  - $\Delta_{\mathbb{R}^k} f := \text{div} \nabla f$

- on a Riemannian manifold $(M, g)$, Laplace-Beltrami operator:
  - $\Delta_M f := \text{div} \nabla f$
  - $\Delta_M f := \frac{1}{\sqrt{\det g}} \sum_{i,j} \frac{\partial}{\partial x^j} \left( g^{ij} \sqrt{\det g} \frac{\partial f}{\partial x^i} \right)$
Laplace Operator

- on a Riemannian manifold, Laplace-Beltrami operator:
  
  - Exponential map: \( \exp : V \subset \mathbb{R}^k \rightarrow U \) by \( \exp(x) = \gamma(p, x, 1) \)
  
  - \( \tilde{f}(x) = f(\exp(x)) \)
  
  - \( \Delta_M f := \Delta_{\mathbb{R}^k} \tilde{f} = \frac{\partial^2 \tilde{f}}{\partial x_1^2} + \cdots + \frac{\partial^2 \tilde{f}}{\partial x_k^2} \)
  
  - Laplace-Beltrami operator is invariant under the map preserving geodesics
Eigenvalues and eigenfunctions

- $\Delta_M \phi = \lambda \phi$
  - For compact manifold, $\Delta_M$ is compact
  - $0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, $\infty$ is the only accumulating point
Eigenvalues and eigenfunctions

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  - For compact manifold, \( \Delta_M \) is compact
  - \( 0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots, \infty \) is the only accumulating point

- Spectrum: eigenvalues
  - \( \lambda_n \sim 4\pi^2 \left( \frac{n}{w_d \text{Vol}(M)} \right)^2 / d \) as \( n \uparrow \infty \)
  - heat trace: \( \sum_i e^{\lambda_i t} = \frac{1}{(4\pi t)^{d/2}} \sum_i c_i t. \)
    - \( c_0 = \text{vol}(M), c_1 = \frac{1}{3} \int s. \)
Spectrum

- isospectrality
  - “Can you hear the shape of a drum” [Kac 1966]
  - “Does the spectrum determines the shape upto isometry”
Spectrum

• isospectrality
  o “Can you hear the shape of a drum” [Kac 1966]
  o “Does the spectrum determines the shape upto isometry
  o negative [Gordon et al. 1992, Buser et al. 1992]
Spectrum

- Spectrum: shape DNA [Reuter et al. 2006]
Eigenfunctions

- $\Delta_M \phi = \lambda \phi$, $(M, g)$ is $C^\infty$-manifold
  - nodal set: $\phi^{-1}(0)$,
    nodal domain: the connected component of $M \setminus \phi^{-1}(0)$
  - Nodal domain theorem [Courant and Hilbert 1953, Cheng 1976]: \# of nodal domains of the i-th eigenfunction $\leq i + 1$
  - Properties of Nodal Set [Cheng 1976]: Except on a closed set of lower dimension (i.e., dim $< d - 1$) the nodal set off forms an $(d - 1)$-dim $C^\infty$-manifold.
Examples

- sphere \((x^2 + y^2 + z^2 = 1)\)

\[
\lambda_1 = 2(1.91) \quad \lambda_2 = 2(1.92) \quad \lambda_1 = 2(1.93)
\]
• star
Examples

• human
Intrinsic Symmetry Detection
Intrinsic Symmetry

- intrinsic symmetry: a self map preserving geodesic distances
Intrinsic Symmetry

- intrinsic symmetry: a self map preserving geodesic distances
  - invariant under non-rigid transformations
Intrinsic Symmetry

- intrinsic symmetry: a self map preserving geodesic distances
  - invariant under non-rigid transformations

- extrinsic symmetry: rotation and reflection
  - preserve Euclidean distances
  - invariant only under rigid transformations
Related Work

• extrinsic symmetry

[Podolak et al. 06] [Mitra et al. 06] [Shimari et al. 06] [Martinet et al. 07]

• intrinsic symmetry
  ○ difficulty: no simple characterization
Global Point Signature

• our strategy: reduce intrinsic to extrinsic
• our tool: eigenfunctions \( \phi_i \) and eigenvalues \( \lambda_i \) of \( \Delta_M \)
Global Point Signature

- our strategy: reduce intrinsic to extrinsic
- our tool: eigenfunctions $\phi_i$ and eigenvalues $\lambda_i$ of $\Delta_M$
- for each point $p$ on $M$, its GPS [Rustamov 07]

$$s(p) = \begin{pmatrix} \phi_1(p) \\ \sqrt{\lambda_1} \\ \phi_2(p) \\ \sqrt{\lambda_2} \\ \vdots \\ \phi_i(p) \\ \sqrt{\lambda_i} \\ \vdots \end{pmatrix}$$
Global Point Signature

- our strategy: reduce intrinsic to extrinsic
- our tool: eigenfunctions $\phi_i$ and eigenvalues $\lambda_i$ of $\Delta_M$
- for each point $p$ on $M$, its GPS \cite{Rustamov07}

\[
\mathbf{s}(p) = \begin{pmatrix}
\frac{\phi_1(p)}{\sqrt{\lambda_1}} \\
\frac{\phi_2(p)}{\sqrt{\lambda_2}} \\
\vdots \\
\frac{\phi_i(p)}{\sqrt{\lambda_i}} \\
\vdots \\
\end{pmatrix}
\]
\[
\mathbf{s}(T(p)) = \begin{pmatrix}
\frac{\phi_1(T(p))}{\sqrt{\lambda_1}} \\
\frac{\phi_2(T(p))}{\sqrt{\lambda_2}} \\
\vdots \\
\frac{\phi_i(T(p))}{\sqrt{\lambda_i}} \\
\vdots \\
\end{pmatrix}
\]

○ relation between $\mathbf{s}(p)$ and $\mathbf{s}(T(p))$?
Theorem: For a compact manifold $M$, $T$ is an intrinsic symmetry if and only if there is a transformation $R$ such that $R(s(p)) = s(T(p))$ for each point $p \in M$ and $R$ restricting to any eigenspace is a rigid transformation.
Theorem: For a compact manifold $M$, $T$ is an intrinsic symmetry if and only if there is a transformation $R$ such that $R(s(p)) = s(T(p))$ for each point $p \in M$ and $R$ restricting to any eigenspace is a rigid transformation.

- “only if” part
1. $\phi = \phi \circ T$: **positive** eigenfunction
2. $\phi = -\phi \circ T$: **negative** eigenfunction
3. $\lambda$ is a repeated eigenvalue
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2. $\phi = -\phi \circ T$: negative eigenfunction
3. $\lambda$ is a repeated eigenvalue

Eigenfunctions and Intrinsic Symmetry

reflection
Eigenfunctions and Intrinsic Symmetry

1. $\phi = \phi \circ T$: positive eigenfunction
2. $\phi = -\phi \circ T$: negative eigenfunction
3. $\lambda$ is a repeated eigenvalue
Transforming Theorem

**Theorem:** For a compact manifold $M$, $T$ is an intrinsic symmetry if and only if there is a transformation $R$ such that $R(s(p)) = s(T(p))$ for each point $p \in M$ and $R$ restricting to any eigenspace is a rigid transformation.
Transforming Theorem

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- one to one correspondence between $T$ and $R$
  - $T$ is an identity $\iff R$ is an identity
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- one to one correspondence between $T$ and $R$
  - $T$ is an identity $\iff R$ is an identity

- detection of intrinsic symmetry reduced to that of extrinsic rigid transformation
  - detection of extrinsic symmetry

[Rus07, PSG06, MGP06, MSHS06]
Results

scans of a real person
(SCAPE dataset)
Limitation of Global Point Signature

- For any $x$, its GPS [Rus07]:
  \[ \text{GPS}_x = \left( \frac{\phi_1(x)}{\sqrt{\lambda_1}}, \frac{\phi_2(x)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_i(x)}{\sqrt{\lambda_i}}, \ldots \right) \]
  
  - global
  - not unique
    - orthonormal transformation within eigenspace
    - eigenfunction switching [GVL96]

courtesy of Jain and Zhang
Heat Kernel Signature
Heat Kernel Signature

- define HKS for any point $x$ as a function on $\mathbb{R}^+$:
  - $HKS_x : \mathbb{R}^+ \rightarrow \mathbb{R}^+$
Heat Kernel Signature

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  - $\text{HKS}_x : \mathbb{R}^+ \rightarrow \mathbb{R}^+$
- isometric invariant
Heat Kernel Signature

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  - $\text{HKS}_x : \mathbb{R}^+ \rightarrow \mathbb{R}^+$
- isometric invariant
- multi-scale

![Graph showing HKS for different points](image)
Heat Kernel Signature

- define HKS for any point $x$ as a function on $\mathbb{R}^+$:
  - $\text{HKS}_x : \mathbb{R}^+ \to \mathbb{R}^+$
- isometric invariant
- multi-scale
- informative
  - $\{\text{HKS}_x\}_{x \in M}$ characterizes almost all shapes up to isometry.
Heat Diffusion Process

- how heat diffuses over time

\[ H^t(f) = ? \]
Heat Diffusion Process

- how heat diffuses over time

\[ H^t(f) = ? \]

- heat kernel \( k_t(x, y) : \mathbb{R}^+ \times M \times M \rightarrow \mathbb{R}^+ \)
  - heat transferred from \( y \) to \( x \) in time \( t \)
  - \( H^t f(x) = \int_M k_t(x, y) f(y) dy \)
Heat Diffusion Process

- how heat diffuses over time

\[ f \quad \Rightarrow \quad H^t(f) =? \]

- heat kernel \( k_t(x, y) : \mathbb{R}^+ \times M \times M \to \mathbb{R}^+ \)
  - heat transferred from \( y \) to \( x \) in time \( t \)
  - \( H^t f(x) = \int_M k_t(x, y) f(y) dy \)
  - \( k_t(x, y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y) \)
Heat Kernel

- characterize shape up to isometry
  - $T : M \to N$ is isometric iff $k_t(x, y) = k_t(T(x), T(y))$.
  - heat kernel recovers geodesic distances.
    - $d^2_M(x, y) = -4 \lim_{t \to 0} t \log k_t(x, y)$
  - heat diffusion process governed by heat equation.
    - $\Delta_M u(t, x) = -\frac{\partial u(t, x)}{\partial t}$

- generate a Brownian motion on a manifold.
Heat Kernel

- multi-scale
  - for any $x$, a family of functions $\{k_t(x, \cdot)\}_t$

$\delta$ funct for any $x$, a family of functions $\{k_t(x, \cdot)\}_t$

$t = 0$: $\delta$ funct

$t \to \infty$: const funct

- local features in small $t$'s; global summaries in big $t$'s
Heat Kernel

- multi-scale
  - for any $x$, a family of functions $\{k_t(x, \cdot)\}_t$

$t = 0$: $\delta$ funct

$t \to \infty$: const funct

- local features in small $t$’s; global summaries in big $t$’s

- however, $\{k_t(x, \cdot)\}_t$’s complexity is extremely high
  - difficult to compare $\{k_t(x, \cdot)\}_t$ with $\{k_t(x', \cdot)\}_t$
Heat Kernel Signature

- HKS is the restriction of $\{k_t(x, \cdot)\}_t$ to the temporal domain
  - $\text{HKS}_x : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by $\text{HKS}_x(t) = k_t(x, x)$
Heat Kernel Signature

- HKS is the restriction of \( \{k_t(x, \cdot)\}_t \) to the temporal domain
  - \( \text{HKS}_x : \mathbb{R}^+ \to \mathbb{R}^+ \) by \( \text{HKS}_x(t) = k_t(x, x) \)
  - concise and commensurable
  - multi-scale
  - isometric invariant
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  - isometric invariant
  - informative?
Heat Kernel Signature

• \( \{ \text{HKS}_x \}_{x \in M} \) is informative

**Informative Theorem.** If the eigenvalues of \( M \) and \( N \) are not repeated, a homeomorphism \( T : M \rightarrow N \) is isometric iff

\[
k_t^M (x, x) = k_t^N (T(x), T(x))
\]

for any \( x \in M \) and any \( t > 0 \).
Heat Kernel Signature

- \{ \text{HKS}_x \}_{x \in M} \text{ is informative}

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\[ k_t^M(x,x) = k_t^N(T(x),T(x)) \]
for any \( x \in M \) and any \( t > 0 \).

- almost all shapes have no repeated eigenvalues [BU82]
Heat Kernel Signature

- \( \{ \text{HKS}_x \}_{x \in M} \) is informative

**Informative Theorem.** If the eigenvalues of \( M \) and \( N \) are not repeated, a homeomorphism \( T: M \to N \) is isometric iff

\[
k^M_t(x, x) = k^N_t(T(x), T(x))
\]

for any \( x \in M \) and any \( t > 0 \).

- almost all shapes have no repeated eigenvalues [BU82]
- the theorem fails if there are repeated eigenvalues
Relation to Curvature

- the polynomial expansion of HKS at small $t$:

$$HKS_x(t) = k_t(x, x) = (4\pi t)^{-d/2} (1 + \frac{1}{6} s(x)t + O(t^2))$$

plot of $k_t(x, x)$ for a fixed $t$
Relation to Diffusion Distance

- diffusion distance [Laf04]
  \[ d_t^2(x, y) = k_t(x, x) + k_t(y, y) - 2k_t(x, y) \]
  - eccentricity in terms of diffusion distance
    \[ ecc_t(x) = \frac{1}{A_M} \int_M d_t^2(x, y) dy = k_t(x, x) + H_M(t) - \frac{2}{A_M}, \]
    - \( ecc_t(x) \) and \( k_t(x, x) \) have the same level sets, in particular, extrema points
    - shape segmentation [dGGV08]
Multi-Scale Matching

scaled HKS:

\[
\frac{k_t(x,x)}{\int_M k_t(x,x) \, dx}
\]
Multi-Scale Matching
Multi-Scale Matching

(a) maxima of $k_t(x, x)$ for a fixed $t$. 

(b) $t = [0.1, 4]$ 

c) $t = [0.1, 80]$
Thank you for your attention

Questions?
Computation

- Laplace-Beltrami Operator:
  - based on its eigenfunctions and eigenvalues
    \[ k_t(x, y) = \sum_i e^{-\lambda_i t} \phi_i(x) \phi_i(y) \]
    \[ \Rightarrow HKS_x(t) = k_t(x, x) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \]

- discrete case:
  - build the discrete Laplace operator \( L = A^{-1}W \) [BSW08]
  - solve \( W \phi = \lambda A \phi \)
  - compute \( HKS_x(t) = \sum_i e^{-\lambda_i t} \phi_i^2(x) \)
first level bulletin
  second level bulletin
    third level bulletin