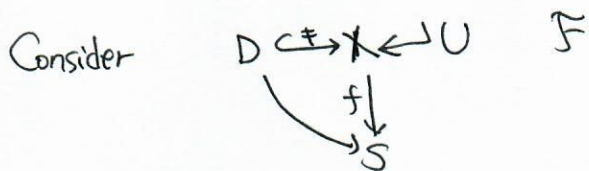


§1 Motivation

Our motivation is to generalize Deligne-Laumon's semi-continuity for Swan conductors.



where S is an excellent Noetherian scheme over $\mathbb{Z}[\frac{1}{\ell}]$

f is separated and smooth of rel. dim 1

$f|_D$ is quasi-finite and flat.

— Let $\Lambda/\ell\mathbb{Z}$ be a finite extension

— Let \mathcal{F} be a lisse sheaf of Λ -modules on U .

Consider the following function:

$$\begin{array}{ccc}
 \varphi: S & \longrightarrow & \mathbb{Z} \\
 t \longmapsto & \sum_{x \in D_{\bar{\mathbb{Z}}}} & \dim_{\text{tot}}(j_! \mathcal{F}|_{X_{\bar{\mathbb{Z}}}}) \quad \dim_{\text{tot}} = S_w + \text{rank}.
 \end{array}$$

Theorem A [Deligne-Laumon]

(1) φ is constructible and lower semi-continuous on S .
 \Downarrow $\forall t \in \mathbb{Z}, \exists V_t \supset t$ open neigh such that $\varphi|_{V_t} \geq \varphi(t)$.
 \Downarrow $\exists S = \cup S_i$ such that $\varphi|_{S_i}$ locally constant (locally closed)

(2) If $\varphi: S \rightarrow \mathbb{Z}$ is locally constant, then f is (universally) locally acyclic with respect to $j_! \mathcal{F}$ (simple \iff f is $j_! \mathcal{F}$ -acyclic)

(3) If $S = \{s, \eta\}$ is a strict local trait with closed point s , generic point η .

If D_s consists of a single point x , then $\varphi(s) - \varphi(\eta) = -\dim_{\mathbb{1}} R^1 \mathbb{P}_x(j_! \mathcal{F})$
 \uparrow nearby cycle.

For our purpose, we reformulate it in terms of "characteristic cycles" as follows:

In the following, we take $\mathcal{G} = j_! F[1]$.

(a1) $\exists A \subseteq_{\text{closed conical}} T^*(X/S)$ such that $A \rightarrow S$ is an open mapping and of relative dimension 1 and that for any $\epsilon \in S$, we have

$$SS(\mathcal{G}|_{X_\epsilon}) = (A_\epsilon)_{\text{red}}$$

\uparrow
alg. geo. point

Zero section.

Indeed, one can take $A = (T^*(X/S) \times_X D) \cup (T^*_X(X/S))$

(a2) \exists finite surjective morphism $S' \xrightarrow{\pi} S$, and a cycle B on $T^*(X'/S')$ supported on $A' = A \times_S S'$, here $X' = X \times_S S'$, such that

— On an open dense open ~~sub~~ subscheme $W \subseteq S'$, we have

$$\forall \epsilon \in W, CC(\mathcal{G}|_{X_\epsilon}) = B_\epsilon$$

— In general, for all $\epsilon \in S'$, $B_\epsilon - CC(\mathcal{G}|_{X_\epsilon})$ is effective on $T^*X'_\epsilon$.

(b) $f: X \rightarrow S$ is ~~transversal~~ (universally) locally acyclic relatively to \mathcal{G} iff for any $\epsilon \rightarrow S$, we have $CC(\mathcal{G}|_{X_\epsilon}) = B_\epsilon$.

(c) Index formula for vanishing cycles/nearby cycles.

Question Can one generalize the above theorem to the case where $\text{rel. dim}(f) \geq 2$?

In order to answer this question, we need a relative version of Beilinson's singular support.

§2 Relative Singular Support

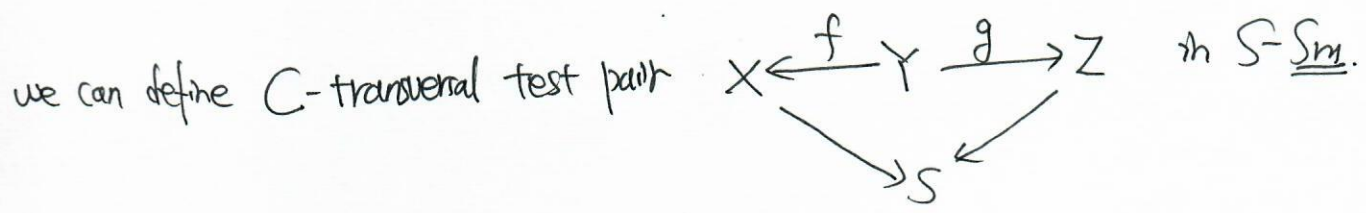
Let S be a connected Noetherian scheme.

$X \xrightarrow{\pi} S$ smooth ~~affine~~, cotangent bundle $T^*(X/S)$.

We work in the category $S\text{-Sm}$ of schemes, which are smooth over S .

For a conical closed subset $C \subseteq T^*(X/S)$, similar to Beilinson's talk,

replace $\text{Spec } k$ by S
replace $T^* X$ by $T^*(X/S)$



Let $\mathcal{F} \in D_c^b(X, \mathbb{N})$.

Let $C(\mathcal{F}, X/S) = \left\{ \begin{array}{l} C \subseteq T^*(X/S) \\ \text{conical} \\ \text{closed} \end{array} \right\}$ Any C -transversal test pair on X in $S\text{-Sm}$ is \mathcal{F} -acyclic
 \uparrow
in the above, g is (universally) locally acyclic relatively to $f^* \mathcal{F}$.

One can show that

$T^*(X/S) \in C(\mathcal{F}, X/S) \iff$ if and only if $X \xrightarrow{\pi} S$ is universally locally acyclic relatively to \mathcal{F} .

(For the direct " \Rightarrow ", note that $X \xleftarrow{\cong} X \xrightarrow{\pi} S$ is $T^*(X/S)$ -transversal)

If $C(\mathcal{F}, X/S)$ has a smallest element, we denote it by

$SS(\mathcal{F}, X/S)$, and we call it the relative singular support of \mathcal{F} (~~relative to S~~).

Beilinson's method also implies the following:

Proposition (1) If $X \xrightarrow{f} S$ is projective smooth and \mathcal{F} -acyclic for $\mathcal{F} \in D_c^b(X, \Lambda)$, then the relative singular support $SS(\mathcal{F}, X/S)$ of \mathcal{F} exists, and we

~~(2) In general, after replacing S by a Zariski open~~ have

$$\forall t \in S, \quad SS(\mathcal{F}|_{X_t}) \subseteq (SS(\mathcal{F}, X/S)|_{X_t})_{\text{red}}.$$

(2) ^{In general} After replacing S by a Zariski open dense subscheme, the singular support

$SS(\mathcal{F}, X/S)$ exists, and we have

$$SS(\mathcal{F}|_{X_t}) \subseteq (SS(\mathcal{F}, X/S)|_{X_t})_{\text{red}} \quad \text{for any } t \in S.$$

And equality holds for t in an open ^{dense} subscheme of S .

Definition $X \rightarrow S$ smooth, $\mathcal{F} \in D_c^b(X, \Lambda)$.

Assume that the relative singular support $SS(\mathcal{F}, X/S)$ exists.

A cycle $Z = \sum_{i \in I} m_i [Z_i]$ in $T^*(X/S)$ is called the relative characteristic cycle

of \mathcal{F} if ~~for any~~ — each Z_i is a subset of $SS(\mathcal{F}, X/S)$, and each $Z_i \rightarrow S$ is an open mapping and equidimensional.

— for any $t \in S$, we have

$$Z_t \stackrel{\text{def}}{=} \sum_{i \in I} m_i [Z_{i,t}] = CC(\mathcal{F}|_{X_t}).$$

We denote by $CC(\mathcal{F}, X/S)$ the ^{relative} characteristic cycle of \mathcal{F} on X . 15

Note that the relative char cycle does not always exist.

But under the following strong condition, $CC(\mathcal{F}, X/S)$ exists!

(Saito) $\left\{ \begin{array}{l} S/k \text{ smooth, } k = \bar{k}. \\ X \xrightarrow{f} S \text{ is smooth and } SS(\mathcal{F})\text{-transversal.} \\ \text{Each irr. component of } SS(\mathcal{F}) \text{ is open and equidimensional over } S. \end{array} \right.$

$\Rightarrow SS(\mathcal{F}, X/S)$ and $CC(\mathcal{F}, X/S)$ exists, and we have

$$SS(\mathcal{F}, X/S) = \Theta_*(SS(\mathcal{F}))$$

$$CC(\mathcal{F}, X/S) = \Theta_*(CC(\mathcal{F})) \cdot (-1)^{\dim S} \quad \boxed{\text{for } \Theta: T^*X \rightarrow T^*(X/S)}$$

In general, we have

Theorem B (Hu-Y)

Let S be a Noetherian scheme, $X \xrightarrow{f} S$ smooth of finite type, and $\mathcal{F} \in D_c^b(X, \mathbb{A})$. Then there exists a dominant and quasi-finite morphism $\pi: S' \rightarrow S$ such that the relative characteristic cycle $CC(\mathcal{F}, X/S')$ exists, where $X' = X \times_S S'$.

▷ The proof is based on Saito's construction of CC and his reformulation of Deligne-Laumon's semi-continuity.

▷ In the above theorem, in order to guarantee the existence of relative characteristic cycle, it is not enough to replace S by a Zariski open dense subscheme. In general, one has to take a quasi-finite and flat base change.

Example $k = \mathbb{F}_p^{\text{alg}}$ of char $p \geq 3$

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$$U = X \setminus D \xrightarrow{j} X = \text{Spec } k[x, y] \longleftarrow D = \text{Spec } \frac{k[x, y]}{(x^p - y)}$$

$$\downarrow$$

$$S = \text{Spec } k[y]$$

Fix a non-trivial additive character $\psi: \mathbb{F}_p \rightarrow \Lambda^\times$.

\mathcal{G} : smooth sheaf on U defined by $t^P - t = \frac{1}{x^p - y}$ and ψ .

$\mathcal{F} = j_! \mathcal{G}[1]$. (问题: $X \rightarrow S$ 是否 \mathcal{F} -ample)

We claim that the relative characteristic cycle $CC(\mathcal{F}, X/S)$ do not exist.

However, after taking ~~a~~ ^{the following} quasi-finite base change

$$S' = \text{Spec } k[y'] \longrightarrow S = \text{Spec } k[y]$$

$$= \text{Spec } \frac{k[y, y']}{(y'^p - y)}$$

then $CC(\mathcal{F}|_{X'}, X'/S')$ exists, and we have

$$CC(\mathcal{F}|_{X'}, X'/S') = [T_{X'}^*(X'/S')] + 2[T_{D'}^*(X'/S')].$$

Indeed, for each closed point $t \in S$, we have

$$(1) \quad CC(\mathcal{F}|_{X_t}) = [T_{X_t}^*(X_t)] + 2[T_{D_t}^*(X_t)]$$

$(r_x=1, s_w=1)$

Now we consider the generic point η of S , and let $K = k(y) = \text{Frac}(S)$, \bar{K}/K an algebraic closure, $\bar{\eta} = \text{Spec } \bar{K}$.

Let $\pi: X_{\bar{\eta}} \rightarrow X_\eta$ be the canonical map.

$$\text{Then } D_{\bar{\eta}} = \text{Spec } \frac{\bar{K}[x]}{(x - y^{1/p})^p} \quad \text{and} \quad D_{\bar{\eta}, \text{red}} = \text{Spec } \frac{\bar{K}[x]}{(x - y^{1/p})}$$

We have $\pi^* D_\eta = p \cdot D_{\eta, \text{red}}$, and

$$(2) \quad CC(\mathcal{F}|_{X_\eta}) = [T_{X_\eta}^* X_\eta] + 2[T_{D_{\eta, \text{red}}}^* X_\eta] = \pi^* ([T_{X_\eta}^* X_\eta] + \frac{2}{p} [T_{D_\eta}^* X_\eta])$$

(1) and (2) implies that the cycle $CC(\mathcal{F}, X/S)$ does not exist! ($\mathbb{Z}/p \notin \mathbb{Z}$).

§3 Failure of the lower semi-continuity for CC

Let $k = \mathbb{F}_p^{\text{alg}}$ be a field of characteristic $p \geq 3$.

$$\begin{array}{ccc} X = \text{Spec } k[x, y, z] & \xleftarrow{(z)} & D = \text{Spec } k[x, y] \\ \downarrow f & & U = X \setminus D = \text{Spec } k[x, y, z^\pm] \\ S = \text{Spec } k[x] & & \end{array}$$

$\psi: \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ non-trivial

\mathcal{G} = smooth on U , associated to the Artin-Schreier ~~sub~~ covering defined by $t^p - t = \frac{xy}{z^p}$.

$$\mathcal{F} = j_! \mathcal{G}[z].$$

For each $s \in S(k)$, we have $X_s = \text{Spec } k[y, z]$, and $U_s = \text{Spec } k[y, z^\pm]$, and

$$\mathcal{G}|_{U_s} \text{ is defined by } t^p - t = \frac{sy}{z^p}$$

If $s=0$, $\mathcal{G}|_{U_s} = \Lambda_{U_s}$.

$$\text{Then we have } SS(\mathcal{F}|_{X_s}) = \begin{cases} [T_{X_s}^* X_s] \cup D_s \langle dz \rangle & \text{if } s=0 \\ [T_{X_s}^* X_s] \cup D_s \langle dy \rangle & \text{if } s \neq 0 \end{cases}$$

where $D_s \langle dz \rangle$ (resp $D_s \langle dy \rangle$) denotes the subline bundle of $T_{X_s}^* X_s$ spanned by the section dz (resp. dy).

Thus (a1) of Thm A cannot be generalized to \mathcal{F} .

$$\left(\begin{array}{l} \exists A \subseteq T^*(X/S), \left\{ \begin{array}{l} A \rightarrow S \text{ of relative dim } 2, \exists \forall t, A_t = \text{SS}(\mathcal{F}|_{X_t}) \\ \text{Example } A_{s=0} \text{ etc. } \dots \end{array} \right. \end{array} \right)$$

Example for (a2)

Now let $\mathcal{G}_1 = \Lambda_U$, $\mathcal{G}_2 = \text{smooth sheaf } U \text{ defined by } t^P - t = \frac{y}{z^P}$.

We put $\mathcal{F} = j_!(\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}) [2]$.

Then for any $t \in |S|$, we have

$$\text{SS}(\mathcal{F}|_{X_t}) = T_{X_t}^* X_t \cup D_t \langle dy \rangle \cup D_t \langle dz \rangle$$

Thus the closed subset $A = T_X^* X \cup D \langle dy \rangle \cup D \langle dz \rangle$
 $\subseteq T^*(X/S) \times_X D$

satisfies (a1) of Theorem A for \mathcal{F} .

$$\text{However, } \text{CC}(\mathcal{F}|_{X_t}) = \begin{cases} 3[T_{X_t}^* X_t] + 2[D_t \langle dz \rangle] + p[D_t \langle dy \rangle] & \text{if } s=0 \\ 3[T_{X_t}^* X_t] + [D_t \langle dz \rangle] + 2p[D_t \langle dy \rangle] & \text{if } s \neq 0 \end{cases}$$

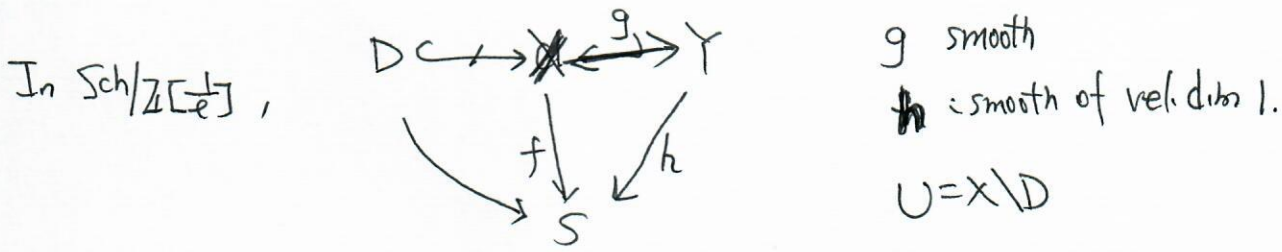
It implies that \mathcal{F} does not have a lower semi-continuous property at the origin of S .

Hence we cannot find a finite surjective map $S' \xrightarrow{f} S$ and a cycle

B supported on $A \times_S S'$ that generalize part (a2) of Thm A for \mathcal{F} .

$$\forall t \in S', B_{\mathcal{F}} = \text{CC}(\mathcal{F}|_{X'_t}) \text{ effective on } T^*X'_t,$$

Proof of Thm B is based on the following reformulation of Deligne-Laumon's semi-continuity. 19



$\mathcal{F} \in D_c^b(X, \Lambda)$ such that f is universally locally acyclic relatively to \mathcal{F}
 $g|_U$ is --- to \mathcal{H}_U . (X)

Define a function $D \xrightarrow{\varphi = \varphi_{\mathcal{F}, g}} \mathbb{Z}$
 $x \mapsto \dim_{\text{tot}} R\Phi_x(\mathcal{F}|_{X_S}, g|_{X_S})$, where $s = f(x)$.

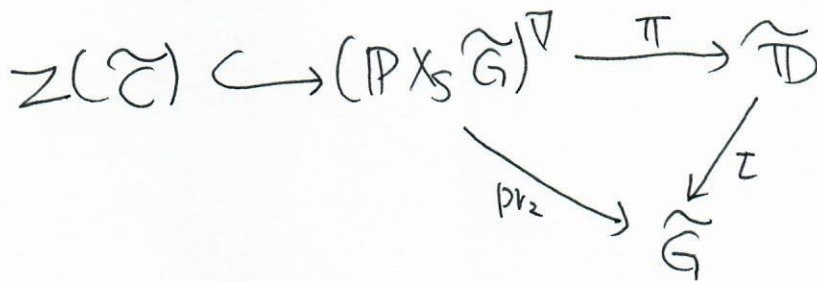
Prop

(1) Then the function $\varphi_{\mathcal{F}, g}: D \rightarrow \mathbb{Z}$ is constructible.

(2) If $f|_D: D \rightarrow S$ is finite étale, then

$$\begin{aligned}
 f_x(\varphi_{\mathcal{F}, g}) : S &\rightarrow \mathbb{Z} && \text{is locally constant on } S. \\
 s &\mapsto \sum_{x \in D_s} \varphi_{\mathcal{F}, g}(x)
 \end{aligned}$$

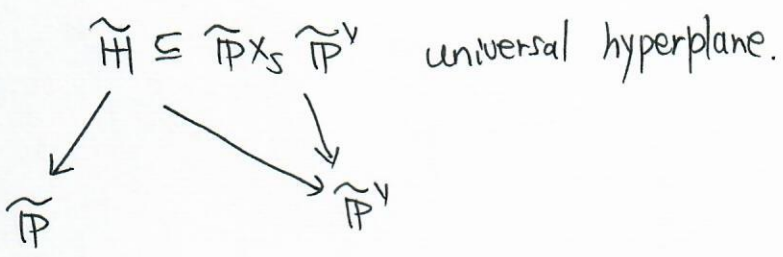
We will apply this to the following diagram: (Construction of CC also uses the same diagram for $S = \text{Spec } k$)



with S affine and integral, and that $X = \mathbb{P} = \mathbb{P}_S^n \rightarrow S$.

— Take a Veronese embedding of degree $N \gg 0$.

$$X = \mathbb{P} \hookrightarrow \widetilde{\mathbb{P}}$$



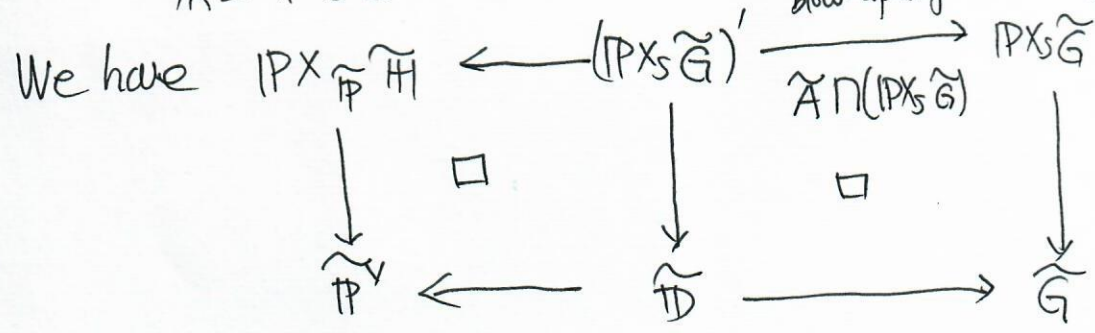
\tilde{G} = Grassmannian of projective lines in $\tilde{\mathbb{P}}^1$

$\tilde{D} \subseteq \tilde{\mathbb{P}}^1 \times \tilde{G}$ universal line such that

$$\forall s \in S, \tilde{D}_s = \{(x, y) \in \tilde{\mathbb{P}}^1 \times \tilde{G} \mid x \in L_y\}$$

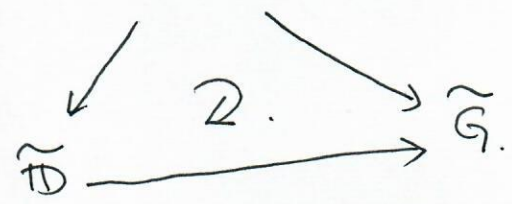
proj line associated to $y \in \tilde{G}_s$.

$\tilde{A} \subseteq \tilde{\mathbb{P}}^1 \times \tilde{G}$ universal axis



$$\forall s \in S, \tilde{A}_s = \{(x, y) \in \tilde{\mathbb{P}}^1 \times \tilde{G} \mid x \in L_z \text{ for any } z \in L_y\}$$

We put $(\mathbb{P} \times_S \tilde{G})^\circ = (\mathbb{P} \times_S \tilde{G}) \setminus (\tilde{A} \cap \mathbb{P} \times_S \tilde{G})$



Let $(\mathbb{P} \times_S \tilde{G})^\nabla$ be the largest open subset of $(\mathbb{P} \times_S \tilde{G})^\circ$ such that the inverse image $Z(\tilde{C}) = \mathbb{P}(\tilde{C}) \times_{\mathbb{P} \times_{\tilde{\mathbb{P}}^1} \tilde{H}}$ $(\mathbb{P} \times_S \tilde{G})$ is quasi-finite over \tilde{G} .

$\text{tr}_1 |_{\mathbb{P} \times_S \tilde{G}} \rightarrow \tilde{D}$ satisfies the the assumption (*) for $\mathcal{F} |_{(\mathbb{P} \times_S \tilde{G})^\nabla}$.

Now $\varphi_{S,\pi}: Z(\tilde{C}) \rightarrow Z$ is constructible. |||

$\exists Z'(\tilde{C}) \subseteq Z(\tilde{C})$ open dense such that $\varphi_{S,\pi}$ is locally constant on $Z'(\tilde{C})$.

We may assume $\forall \alpha \in I, Z'(\tilde{C}_\alpha)$ irreducible, $\forall s \in S$, the fiber

$Z'(\tilde{C}_\alpha)_s$ is non-empty and geo. irreducible.

$\forall \alpha \in I, \xi_\alpha =$ generic point of $P(\tilde{C}_\alpha)$

$\eta_\alpha =$ generic point of \tilde{D}_α .

$\varphi_\alpha =$ value of $\varphi_{S,\pi}$ on $Z'(\tilde{C}_\alpha)$

$$B = - \sum_{\alpha \in I} \frac{\varphi_\alpha}{[\xi_\alpha : \eta_\alpha]} [C_\alpha].$$

$$C = \cup C_\alpha.$$

$$\tilde{C} = \cup \tilde{C}_\alpha.$$

\uparrow
inverse image $(d\tilde{\tau})^{-1}(C)$

in $\mathbb{P}^n \times_{\mathbb{F}} T^*(\mathbb{F}/S)$

$$\tilde{\tau}: \mathbb{P}^n \hookrightarrow \tilde{\mathbb{P}}.$$