On the semi-continuity of characteristic cycles for étale sheaves

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§1 Motivation

Our motivation is to generalize Deligne–Laumon's semi-continuity for Swan conductors.

Consider

$$\begin{tikzcd} D \arrow[r, hook, "i"] \arrow[dr] & X \arrow[d, "f"] & U \arrow[l, hook', "j"] & \mathcal{F} \\ & S & & \end{tikzcd}$$

where S is an excellent Noetherian scheme over $\mathbb{Z}[\frac{1}{\ell}]$

f is separated and smooth of rel. dim 1

$f|_D$ is quasi-finite and flat.

— Let $\Lambda/\mathbb{F}_\ell$ be a finite extension

— Let $\mathcal{F}$ be a lisse sheaf of Λ-modules on U.

Consider the following function:

$$\varphi : S \longrightarrow \mathbb{Z}$$

$$t \longmapsto \sum_{x \in D_{\bar{t}}} \mathrm{dimtot}_x(j_!\mathcal{F}|_{X_{\bar{t}}}) \qquad \mathrm{dimtot} = \mathrm{Sw} + \mathrm{rank}.$$

Theorem A [Deligne–Laumon]

(1) φ is constructible and lower semi-continuous on S.

$\Downarrow$ $\exists\, S = \bigcup S_i$ (locally closed) such that $\varphi|_{S_i}$ locally constant

$\Downarrow$ $\forall t \in S$, $\exists\, V_t \ni t$ open neigh such that $\varphi|_{V_t} \geq \varphi(t)$.

(2) If $\varphi : S \longrightarrow \mathbb{Z}$ is locally constant, then f is (universally) locally acyclic with respect to $j_!\mathcal{F}$ ($\overset{\text{simple}}{\Longleftrightarrow}$ f is $j_!\mathcal{F}$-acyclic)

(3) If $S = \{s, \eta\}$ is a strict local trait with closed points s, generic point η.

If D_s consists of a single point x, then $\varphi(s) - \varphi(\eta) = -\dim_\Lambda \underbrace{R^1\Psi_x(j_!\mathcal{F})}_{\text{nearby cycle}}$.

For our purpose, we reformulate it in terms of "characteristic cycles" as follows:

In the following, we take $\underline{\mathcal{G} = j_!\mathcal{F}[1]}$.

(a1) $\exists\, A \underset{\substack{\text{closed}\\ \text{conical}}}{\subseteq} T^*(X/S)$ such that $A \longrightarrow S$ is an open mapping and of relative dimension 1 and that for any $t \in S$, we have

$$SS(\mathcal{G}|_{\underset{\substack{\uparrow\\ \text{alg. geo. point}}}{X_{\bar{t}}}}) = (A_{\bar{t}})_{red}.$$

Zero section.

Indeed, one can take $A = \left(T^*(X/S) \times_X D\right) \bigcup \left(T^*_X(X/S)\right)$

(a2) $\exists$ finite surjective morphism $S' \xrightarrow{\pi} S$, and a cycle B on $T^*(X'/S')$ supported on $A' = A \times_S S'$, here $X' = X \times_S S'$, such that

— On an open dense open ~~neigh~~ subscheme $W \subseteq S'$, we have

$$\forall\, t \in W,\ CC(\mathcal{G}|_{X_{\bar{t}}}) = B_{\bar{t}}$$

— In general, for all $t \in S'$, $B_{\bar{t}} - CC(\mathcal{G}|_{X_{\bar{t}}})$ is effective on $T^*X'_{\bar{t}}$.

(b) ~~iff~~ $f: X \longrightarrow S$ is ~~transversal~~ (universally) locally acyclic relatively to $\mathcal{G}$ ~~iff~~ iff for any $\bar{t} \longrightarrow S$, we have $CC(\mathcal{G}|_{X_{\bar{t}}}) = B_{\bar{t}}$.

(c) Index formula for vanishing cycles/nearby cycles.

Question: Can one generalize the above theorem to the case where $\mathrm{rel.dim}(f) \geq 2$?

In order to answer this question, we need a relative version of Beilinson's singular support.

§2 Relative Singular Support

Let S be a connected Noetherian scheme.

$X \xrightarrow{\cancel{\pi}} S$ smooth, cotangent bundle $T^*(X/S)$.

We work in the category S-$\underline{\underline{\mathrm{Sm}}}$ of schemes, which are smooth over S.

For a conical closed subset $C \subseteq T^*(X/S)$, similar to Beilinson's talk,

replace Spec$k$ by S
replace T^*X by $T^*(X/S)$

we can define C-transversal test pair
$$\begin{tikzcd} X \arrow[dr] & Y \arrow[l, "f"'] \arrow[r, "g"] & Z \arrow[dl] \\ & S & \end{tikzcd}$$
in S-$\underline{\mathrm{Sm}}$.

Let $\mathcal{F} \in D^b_c(X, \Lambda)$.

Let $C(\mathcal{F}, X/S) = \left\{ \underset{\substack{\text{conical}\\ \text{closed}}}{C \subseteq T^*(X/S)} \;\middle|\; \begin{array}{l} \text{Any } C\text{-transversal test pair on } X \text{ in } S\text{-Sm} \\ \text{is } \mathcal{F}\text{-acyclic} \end{array} \right\}$

$\Updownarrow$

in the above, g is (universally) locally acyclic relatively to $f^*\mathcal{F}$.

One can show that

$T^*(X/S) \in C(\mathcal{F}, X/S) \overset{\text{if and only if}}{\Longleftrightarrow} X \xrightarrow{\pi} S$ is universally locally acyclic relatively to $\mathcal{F}$.

(For the direct "$\Rightarrow$", note that $X \xleftarrow{\mathrm{id}} X \xrightarrow{f} S$ is $T^*(X/S)$-transversal)

If $C(\mathcal{F}, X/S)$ has a smallest element, we denote it by $SS(\mathcal{F}, X/S)$, and we call it the relative singular support of $\mathcal{F}$ ~~(relative to S)~~.

Beilinson's method also implies the following:

Proposition (1) If $X \xrightarrow{f} S$ is projective smooth and $\mathcal{F}$-acyclic for $\mathcal{F} \in D_c^b(X, \Lambda)$, then the relative singular support $SS(\mathcal{F}, X/S)$ of $\mathcal{F}$ exists, and we
~~(2) In general, after replacing S by a Zariski open~~ have

$$\forall t \in S, \quad SS(\mathcal{F}|_{X_t}) \subseteq (SS(\mathcal{F}, X/S) \times_S t)_{red}.$$

(2) In general After replacing S by a Zariski open dense subscheme, the singular support $SS(\mathcal{F}, X/S)$ exists, and we have

$$SS(\mathcal{F}|_{X_t}) \subseteq (SS(\mathcal{F}, X/S) \times_S t)_{red} \quad \text{for any } t \in S.$$

And equality holds for ~~t~~ in an open dense subscheme of S.

Definition $X \longrightarrow S$ smooth, $\mathcal{F} \in D_c^b(X, \Lambda)$.

Assume that the relative singular support $SS(\mathcal{F}, X/S)$ exists.

A cycle $Z = \sum_{i \in I} m_i [Z_i]$ in $T^*(X/S)$ is called the relative characteristic cycle of $\mathcal{F}$ if ~~for any~~ — each Z_i is a subset of $SS(\mathcal{F}, X/S)$, and each $Z_i \to S$ is an open mapping and equidimensional.

— for any $t \in S$, we have

$$Z_t \left(\overset{\text{def}}{=} \sum_{i \in I} m_i [Z_{i\,t}] \right) = CC(\mathcal{F}|_{X_t}).$$

We denote by $CC(\mathcal{F}, X/S)$ the relative characteristic cycle of $\mathcal{F}$ on X.

Note that the relative char cycle does not always exists.

But under the following strong condition, $CC(\mathcal{F}, X/S)$ exists!

(Saito)
- S/k smooth, $k=\bar{k}$.
- $X \xrightarrow{f} S$ is smooth and $SS(\mathcal{F})$-transversal.
- Each irr. component of $SS(\mathcal{F})$ is open and equi-dimensional over S.

$\Rightarrow$ $SS(\mathcal{F}, X/S)$ and $CC(\mathcal{F}, X/S)$ exists, and we have

$$SS(\mathcal{F}, X/S) = \theta_*(SS(\mathcal{F}))$$
$$CC(\mathcal{F}, X/S) = \theta_*(CC(\mathcal{F}))\cdot(-1)^{\dim S} \quad \boxed{\text{for } \theta : T^*X \longrightarrow T^*(X/S)}$$

In general, we have

<u>Theorem B (Hu–Y)</u>

Let S be a Noetherian scheme, $X \xrightarrow{f} S$ smooth of finite type, and $\mathcal{F} \in D_c^b(X, \Lambda)$. Then there exists a <u>dominant</u> and <u>quasi-finite</u> morphism $\pi : S' \longrightarrow S$ such that the relative characteristic cycle $CC(\mathcal{F}, X'/S')$ exists, where $X' = X \times_S S'$.

▷ The proof is based on Saito's construction of CC and his reformulation of Deligne–Laumon's semi-continuity.

▷ In the above theorem, in order to guarantee the existence of relative characteristic cycle, it is not enough to replace S by a Zariski open dense subscheme. In general, one has to take a quasi-finite and flat base change.

Example $k = \bar{k}^{alg}$ field of char $p \geq 3$

$$\begin{tikzcd} U = X \setminus D \arrow[r, hook, "j"] & X = \mathrm{Spec}\, k[x,y] \arrow[d] & D = \mathrm{Spec}\, \frac{k[x,y]}{(x^p - y)} \arrow[l, hook'] \\ & S = \mathrm{Spec}\, k[y] & \end{tikzcd}$$

Fix a non-trivial additive character $\psi : \mathbb{F}_p \to \Lambda^\times$.

$\mathcal{G}$: smooth sheaf on U defined by $t^p - t = \frac{1}{x^p - y}$ and ψ.

$\mathcal{F} = j_! \mathcal{G}[1]$. (问题: $X \to S$ 是否 $\mathcal{F}$-acyclic)

We claim that the relative characteristic cycle $CC(\mathcal{F}, X/S)$ do not exist.

However, after taking ~~a~~ the following quasi-finite base change

$$\begin{aligned} S' = \mathrm{Spec}\, k[y'] &\longrightarrow S = \mathrm{Spec}\, k[y] \\ = \mathrm{Spec}\, \frac{k[y,y']}{(y'^p - y)} & \end{aligned}$$

then $CC(\mathcal{F}|_{X'}, X'/S')$ exists, and we have

$$CC(\mathcal{F}|_{X'}, X'/S') = [T^*_{X'}(X'/S)] + 2[T^*_{D'_{red}}(X'/S')].$$

Indeed, for each closed point $t \in S$, we have

$$rk = 1, \; Sw = 1$$

$$(1) \qquad CC(\mathcal{F}|_{X_t}) = [T^*_{X_t} X_t] + 2[T^*_{D_t} X_t]$$

Now we consider the generic point η of S, and let $K = k(y) = \mathrm{Frac}(S)$, $\bar{K}/K$ an algebraic closure, $\bar{\eta} = \mathrm{Spec}\, \bar{K}$.

Let $\pi : X_{\bar{\eta}} \to X_\eta$ be the canonical map.

Then $D_{\bar{\eta}} = \mathrm{Spec}\, \frac{\bar{K}[x]}{(x - y^{1/p})^p}$ and $D_{\bar{\eta}, red} = \mathrm{Spec}\, \frac{\bar{K}[x]}{(x - y^{1/p})}$

We have $\pi^* D_\eta = p \cdot D_{\bar\eta, \mathrm{red}}$, and

(2) $$CC(\mathcal{F}|_{X_{\bar\eta}}) = [T^*_{X_{\bar\eta}} X_{\bar\eta}] + 2[T^*_{D_{\bar\eta,\mathrm{red}}} X_{\bar\eta}] = \pi^*\left([T^*_{X_\eta} X_\eta] + \tfrac{2}{p}[T^*_{D_\eta} X_\eta]\right)$$

(1) and (2) implies that the cycle $CC(\mathcal{F}, X/S)$ does not exist! ($2/p \notin \mathbb{Z}$).

§3 Failure of the lower semi-continuity for CC

Let $k = \bar{k}^{\mathrm{alg}}$ be a field of characteristic $p \geq 3$.

$$\begin{array}{ccc} X = \mathrm{Spec}\, k[x,y,z] & \xleftarrow{(z)} & D = \mathrm{Spec}\, k[x,y] \\ \big\downarrow f & & U = X \setminus D = \mathrm{Spec}\, k[x,y,z^{\pm1}]. \\ S = \mathrm{Spec}\, k[x] & & \end{array}$$

$\psi : \mathbb{F}_p \to \Lambda^\times$ non-trivial

$\mathcal{G}$: smooth on U, associated to the Artin-Schreier covering defined by $t^p - t = \frac{xy}{z^p}$.

$\mathcal{F} = j_! \mathcal{G}[2]$.

For each $s \in S(k)$, we have $X_s = \mathrm{Spec}\, k[y,z]$, and $U_s = \mathrm{Spec}\, k[y, z^{\pm1}]$, and

$\mathcal{G}|_{U_s}$ is defined by $t^p - t = \frac{sy}{z^p}$

If $s=0$, $\mathcal{G}|_{U_s} = \Lambda_{U_s}$.

then we have $$SS(\mathcal{F}|_{X_s}) = \begin{cases} [T^*_{X_s} X_s] \cup D_s\langle dz\rangle & \text{if } s=0 \\ [T^*_{X_s} X_s] \cup D_s\langle dy\rangle & \text{if } s\neq 0 \end{cases}$$

where $D_s\langle dz\rangle$ (resp $D_s\langle dy\rangle$) denotes the subline bundle of $T^*X_s \times_{X_s} D_s$ spanned by the section dz (resp. dy).

Thus (a1) of Thm A cannot be generalized to $\mathcal{F}$.

(要求 $\exists A \subseteq T^*(X/S)$, $A \to S$ of relative dim 2，且 $\forall \bar{t}$, $A_{\bar{t}} = SS(\mathcal{F}|X_{\bar{t}})$.
上述例子中 $A_{s=0}$ 处 ……)

(五) Example for (a2)

Now let $\mathcal{G}_1 = \Lambda_U$, $\mathcal{G}_2 =$ smooth sheaf U defined by $t^p - t = \frac{y}{z^p}$.

We put $\mathcal{F} = j_!(\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G})[2]$.

Then for any $t \in |S|$, we have

$$SS(\mathcal{F}|_{X_t}) = T^*_{X_t} X_t \cup D_t\langle dy\rangle \cup D_t\langle dz\rangle$$

Thus the closed subset $A = \underset{\substack{\cap \\ T^*(X/S)|X \times_X D}}{T^*_X X \cup D\langle dy\rangle \cup D\langle dz\rangle}$

satisfies (a1) of Theorem A for $\mathcal{F}$.

$$\text{However, } CC(\mathcal{F}|_{X_t}) = \begin{cases} 3[T^*_{X_t}X_t] + 2[D_t\cdot\langle dz\rangle] + p[D_t\langle dy\rangle] & \text{if } s=0 \\ 3[T^*_{X_t}X_t] + [D_t\langle dz\rangle] + 2p[D_t\langle dy\rangle] & \text{if } s\neq 0 \end{cases}$$

It implies that $\mathcal{F}$ does not have a lower semi-continuous property at the origin of S.

Hence we cannot find a finite surjective map $S' \xrightarrow{f} S$ and a cycle B supported on $A \times_S S'$ that generalize part (a2) of Thm A for $\mathcal{F}$.

$$\forall t \in S', \ B_{\bar{t}} - CC(\mathcal{F}|_{X_{\bar{t}}}) \text{ effective on } T^*X'_{\bar{t}},$$

Proof of Thm B is based on the following reformulation of Deligne-Laumon's semi-continuity.

In $Sch/\mathbb{Z}[\frac{1}{\ell}]$,

$$\begin{tikzcd} D \arrow[r, hook] \arrow[dr, bend right] & X \arrow[r, "g"] \arrow[d, "f"] & Y \arrow[dl, "h"] \\ & S & \end{tikzcd}$$

g smooth

h smooth of rel. dim 1.

$U = X \setminus D$

$\mathcal{F} \in D^b_c(X, \Lambda)$ such that f is universally locally acyclic relatively to $\mathcal{F}$

$g|_U$ is ——————————— to $\mathcal{F}|_U$. $(\ast)$

Define a function $D \xrightarrow{\varphi = \varphi_{\mathcal{F},g}} \mathbb{Z}$

$x \longmapsto \dim\mathrm{tot}\, R\Phi_x(\mathcal{F}|_{X_{\bar{s}}}, g|_{X_{\bar{s}}})$, where $s = f(x)$.

Prop

(1) Then the function $\varphi_{\mathcal{F},g} : D \longrightarrow \mathbb{Z}$ is constructible.

(2) If $f|_D : D \to S$ is finite étale, then

$f_*(\varphi_{\mathcal{F},g}) : S \longrightarrow \mathbb{Z}$ is locally constant on S.

$s \longmapsto \sum_{x \in D_{\bar{s}}} \varphi_{\mathcal{F},g}(x)$

We will apply this to the following diagram: (Construction of CC also uses the same diagram for $S = \mathrm{Spec}\, k$)

$$\begin{tikzcd} Z(\tilde{C}) \arrow[r, hook] & (\mathbb{P} \times_S \tilde{G})^{\nabla} \arrow[r, "\pi"] \arrow[dr, "pr_2"'] & \tilde{\mathbb{D}} \arrow[d, "\tau"] \\ & & \tilde{G} \end{tikzcd}$$

with S affine and integral, and that $X = \mathbb{P} = \mathbb{P}^n_S \longrightarrow S$.

— Take a Veronese embedding of degree $N \gg 0$.

$X = \mathbb{P} \hookrightarrow \tilde{\mathbb{P}}$

$\widetilde{\mathbb{H}} \subseteq \widetilde{\mathbb{P}} \times_S \widetilde{\mathbb{P}}^\vee$ universal hyperplane.

$$\begin{tikzcd} & \widetilde{\mathbb{H}} \arrow[dl] \arrow[dr] & \\ \widetilde{\mathbb{P}} & & \widetilde{\mathbb{P}}^\vee \end{tikzcd}$$

$\widetilde{G}$ = Grassmannian of projective lines in $\widetilde{\mathbb{P}}^\vee$

$\widetilde{\mathbb{D}} \subseteq \widetilde{\mathbb{P}}^\vee \times_S \widetilde{G}$ universal line such that

$\forall s \in S$, $\widetilde{\mathbb{D}}_s = \{ (x,y) \in \widetilde{\mathbb{P}}_s^\vee \times_s \widetilde{G}_s \mid x \in L_y \}$

L_y: proj line associated to $y \in \widetilde{G}_s$.

$\widetilde{\mathbb{A}} \subseteq \widetilde{\mathbb{P}} \times_S \widetilde{G}$ universal axis,

$\forall s \in S$, $\widetilde{\mathbb{A}}_s = \{ (x,y) \in \widetilde{\mathbb{P}}_s \times_s \widetilde{G}_s$ s.t. $x \in H_z$ for any $z \in L_y \}$

We have
$$\begin{tikzcd} \mathbb{P} \times_{\widetilde{\mathbb{P}}} \widetilde{\mathbb{H}} \arrow[d] \arrow[dr, phantom, "\square"] & (\mathbb{P} \times_S \widetilde{G})' \arrow[l] \arrow[r, "\text{blow-up along}", "\widetilde{\mathbb{A}} \cap (\mathbb{P} \times_S \widetilde{G})"'] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P} \times_S \widetilde{G} \arrow[d] \\ \widetilde{\mathbb{P}}^\vee & \widetilde{\mathbb{D}} \arrow[l] \arrow[r] & \widetilde{G} \end{tikzcd}$$

We put $(\mathbb{P} \times_S \widetilde{G})^\circ = (\mathbb{P} \times_S \widetilde{G}) \setminus (\widetilde{\mathbb{A}} \cap \mathbb{P} \times_S \widetilde{G})$

$$\begin{tikzcd} & (\mathbb{P} \times_S \widetilde{G})^\circ \arrow[dl] \arrow[dr] & \\ \widetilde{\mathbb{D}} \arrow[rr] \arrow[ur, phantom, "2." near end] & & \widetilde{G}. \end{tikzcd}$$

Let $(\mathbb{P} \times_S \widetilde{G})^\nabla$ be the largest open subset of $(\mathbb{P} \times_S \widetilde{G})^\circ$ such that the inverse image $Z(\widetilde{C}) = \mathbb{P}(\widetilde{C}) \times_{\mathbb{P} \times_{\widetilde{\mathbb{P}}} \widetilde{\mathbb{H}}} (\mathbb{P} \times_S \widetilde{G})$ is quasi-finite over $\widetilde{G}$.

$$\begin{tikzcd} Z(\widetilde{C}) \arrow[r, hook] & (\mathbb{P} \times_S \widetilde{G})^\nabla \arrow[r, "\pi"] \arrow[dr] & \widetilde{\mathbb{D}} \arrow[d] \\ & & \widetilde{G} \end{tikzcd}$$
satisfies the the assumption $(*)$ for $\mathcal{F}|_{(\mathbb{P} \times_S \widetilde{G})^\nabla}$.

Now $\varphi_{\mathcal{G},\pi} : Z(\widetilde{C}) \longrightarrow \mathbb{Z}$ is constructible.

$\exists\, Z'(\widetilde{C}) \subseteq Z(\widetilde{C})$ open dense such that $\varphi_{\mathcal{G},\pi}$ is locally constant on $Z'(\widetilde{C})$.

We may assume $\forall \alpha \in I$, $Z'(\widetilde{C}_\alpha)$ irreducible, $\forall s \in S$, the fiber

$Z'(\widetilde{C}_\alpha)_s$ is non-empty and geo. irreducible.

$\forall \alpha \in I$, ξ_α = generic point of $P(\widetilde{C}_\alpha)$

η_α = generic point of $\widetilde{D}_\alpha$.

$C = \bigcup C_\alpha$.

φ_α = value of $\varphi_{\mathcal{G},\pi}$ on $Z'(\widetilde{C}_\alpha)$

$\widetilde{C} = \bigcup \widetilde{C}_\alpha$.

$$B = -\sum_{\alpha \in I} \frac{\varphi_\alpha}{[\xi_\alpha : \eta_\alpha]} [C_\alpha].$$

$\widetilde{C}$ inverse image $(d\tilde{i})^{-1}(C)$ in $\mathbb{P} \times_{\widetilde{\mathbb{P}}} T^*(\widetilde{\mathbb{P}}/S)$

$\tilde{i} : \mathbb{P} \hookrightarrow \widetilde{\mathbb{P}}$.