封面

Twist Formula of epsilon factors of constructible étale sheaves.

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Thank you very much for the invitation.

I am very happy to back to Tokyo again!

I am enjoying this conference very much!!

I am sorry that the first half part of my talk was already given in Paris and also in Tokyo by Umezaki San. And I hope that the second half part of my talk would bring something New.

This is a joint work with ~~Umezaki and Zhao~~. Naoya Umezaki and Yigeng Zhao.

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§ 0 Notation

We fix some simple notation for this talk.

Let k be a finite field of characteristic $p>0$. (Sometimes, k is just assumed to be perfect)

Let $\ell \neq p$ be a prime.

Let Λ be a finite extension of $\mathbb{F}_\ell$ or $\mathbb{Q}_\ell$.

Let X/k be a separated and smooth scheme purely of dimension $d \geq 0$ over k

$$K_0(X/k,\Lambda) = \text{Grothendieck group of } D_c^b(X_{et},\Lambda)$$

$$= \frac{\text{Free abelian group generated by objects of } D_c^b(X_{et},\Lambda)}{\left\langle [\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}] \;\middle|\; \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \cdot \text{ distinguished triangle} \right\rangle}$$

<u>Characteristic class</u> $K_0(X/k,\Lambda) \xrightarrow{cc_{X/k}(-)} CH_0(X)$

$cc_{X/k}(\mathcal{F}) := 0_X^!(CC\mathcal{F})$ Gysin pullback of characteristic cycle of $\mathcal{F}$ by the zero-section $X \xrightarrow{0_X} T^*X$ of the cotangent bundle.

<u>Global epsilon factor</u> $K_0(X/k,\Lambda) \xrightarrow{\varepsilon(X,-)} \Lambda^\times$

The global epsilon factor of $\mathcal{F}$ is defined to be $\varepsilon(X,\mathcal{F}) = \det(-\mathrm{Frob}_k ; R\Gamma_c(X_{\bar{k}},\mathcal{F}))^{-1}$, where Frob_k is the geometric Frobenius, i.e., the inverse of the Frobenius substitution $x \mapsto x^{\# k}$ of $\bar{k}$ (viewed as an element of $\mathrm{Gal}(\bar{k}/k)$).

§1 Kato–T.Saito's conjecture (there are at least two Saito in this conference)

Now we assume that X is projective and smooth over the finite field k.

By Kato–T.Saito's unramified class field theory, there is a so-called reciprocity map

$$\begin{array}{ccc} CH_0(X) & \xrightarrow{\rho_X} & \pi_1^{ab}(X) \\ [s] & \longmapsto & [\mathrm{Frob}_s] \end{array} \quad \text{(sign choice)}$$

geo. Frobenius at s

ρ_X is injective with dense image!

The following (global) twist formula for $\varepsilon(X,\mathcal{F})$ was a modification of a conjecture of Kato and T.Saito in ~~2008~~ 2004.

Theorem A (Umezaki–Y–Zhao, 2017)

For any smooth sheaf $\mathcal{G}$ on X, we have

$$\varepsilon(X,\mathcal{F}\otimes\mathcal{G}) = \varepsilon(X,\mathcal{F})^{\mathrm{rank}\,\mathcal{G}}\cdot \det\mathcal{G}(\rho_X(-cc_{X/k}(\mathcal{F}))) \text{ in } \Lambda^\times,$$

where $CH_0(X) \xrightarrow{\rho_X} \pi_1^{ab}(X) \xrightarrow{\det\mathcal{G}} \Lambda^\times$

$$-cc_{X/k}(\mathcal{F}) \longmapsto \det\mathcal{G}(\rho_X(-cc_{X/k}(\mathcal{F}))).$$

Remark In Kato–T.Saito's paper 2008, they defined the Swan class $Sw^{ks}(\mathcal{F}) \in CH_0(\overline{U}\setminus U)\otimes\mathbb{Q}$ for a smooth sheaf $\mathcal{F}$ on $U \underset{\text{open}}{\subseteq} X/k$

$$\| $$

$$\varprojlim_{U\subseteq Y \text{ compactification}} CH_0(Y\setminus U)\otimes\mathbb{Q}$$

by using alteration and logarithmic blow-up.

In their paper, the global twist formula was written in terms of $Sw^{ks}(\mathcal{F})$.

Conjecture B (T. Saito, 2016)

k: any perfect field. $U \underset{\text{open dense}}{\overset{j}{\hookrightarrow}} X/k$ smooth. For any smooth sheaf $\mathcal{F}$ on U,

we have $Sw^{ks}(\mathcal{F}) = Sw^{cc}(\mathcal{F})$ in $CH_0(X\setminus U)$,

where $Sw^{cc}(\mathcal{F}) = 0^!_{X\setminus U}(\mathrm{rank}\,\mathcal{F}\cdot CC(j_!\Lambda) - CC(j_!\mathcal{F})) \in CH_d(X\setminus U)$, where $X\setminus U \xrightarrow{0_{X\setminus U}} T^*X\times_X(X\setminus U)$ and $\mathrm{rank}\,\mathcal{F}\cdot CC(j_!\Lambda) - CC(j_!\mathcal{F})$ is supported on $T^*X\times_X(X\setminus U)$.

Both $Sw^{ks}(\mathcal{F})$ and $Sw^{cc}(\mathcal{F})$ satisfy the higher Grothendieck-Ogg-Shafarevich formula $\chi_c(U_{\bar{k}}, \mathcal{F}) = \mathrm{rank}\,\mathcal{F} \cdot \chi_c(U_{\bar{k}}, \Lambda) - \deg Sw(\mathcal{F})$ if X is projective. which is due to T. Saito for Sw^{cc} and due to Kato-T.Saito for Sw^{ks}.

<u>Theorem C (weak form of Conjecture B for surface)</u>

Assuming X is a projective smooth <u>surface</u> over a finite field k, and $U \overset{j}{\subseteq} X$ open dense. For any smooth sheaf $\mathcal{F}$ on U, we have

$$Sw^{ks}(\mathcal{F}) = Sw^{cc}(\mathcal{F}) \quad \text{in} \quad \underset{\text{(weak form)}}{\underline{CH_0(X)}}$$

The proof of theorem C is based on Brauer induction and a theorem of T. Saito and Yatagawa (Same ramification $\Rightarrow$ Same CC)

I will back to this again if I have time!

<u>~~Some~~ Previous results of twist formulas for epsilon factors</u>

① Local twist formula, due to Deligne and Henniart 1981.
Thm A can be viewed as a globalization of their formulae

② 1984, S. Saito proved an explicit formula for $\varepsilon(X, \mathcal{F})$ if $\mathcal{F}$ is smooth.
Our proof of thm A is based on his paper.

③ 1993, T. Saito —— explicit formula for tamely ramified sheaves.

④ 2009, I. Vidal (She) proved Thm A under the assumption that
- $\dim X = 2$ and $\mathrm{rank}\,\mathcal{F} = 1$
- some technical assumptions on the ramification of $\mathcal{F}$, e.g <u>cleanness</u>.

⑤ 2016, Tomoyuki Abe and Deepam Patel proved a K-spectrum version of twist formula (localization formula) for de Rham epsilon factor based on micro-local description of singular supports

$$K_S(X, \mathcal{D}_X) \longrightarrow K_S(T^*X), \quad S \subseteq T^*X$$

$\Uparrow$
K-spectrum of perfect coherent $\mathcal{D}_X$-modules with $SS \subseteq S$

$K_S(T^*X)$ = K-spectrum of coherent $\mathcal{O}_{T^*X}$-modules with support $\subseteq S$.

§2 Application to characteristic class

We first recall the definition of total characteristic class $cc_{X,\bullet}(\mathcal{F}) \in CH_\bullet(X)$.

$$\begin{tikzcd} K_0(X,\Lambda) \arrow[r, "\overline{CC}"] \arrow[rr, bend right, "cc_{X,\bullet}(-)"'] & CH_d(\mathbb{P}(T^*X \oplus \mathbb{A}^1_X)) \arrow[r, "\simeq"] & CH_\bullet(X) = \bigoplus_{i=0}^{d} CH_i(X) \end{tikzcd}$$

$$\overline{CC}(\mathcal{F}) = \mathbb{P}(\overline{CC\mathcal{F} \oplus \mathbb{A}^1_X}) \longmapsto cc_{X,\bullet}(\mathcal{F}).$$

We have $cc_{X,0}(\mathcal{F}) = cc_{X/k}(\mathcal{F})$

$cc_{X,d}(\mathcal{F}) = (-1)^d \operatorname{rank}\mathcal{F} \cdot [F]$ generic rank

$cc_{X,d-1}(\mathcal{F}) =$ Artin divisor class of $\mathcal{F}$.

If $k = \mathbb{C}$, by a theorem of V. Ginsburg, the following diagram is commutative for any projective morphism $f: X \to Y$ between smooth schemes over $k = \mathbb{C}$:

$$\begin{tikzcd} K_0(X,\Lambda) \arrow[r, "cc_{X,\bullet}"] \arrow[d, "f_*"'] \arrow[dr, phantom, "(*)"] & CH_\bullet(X) \arrow[d, "f_*"] \\ K_0(Y,\Lambda) \arrow[r, "cc_{Y,\bullet}"] & CH_\bullet(Y) \end{tikzcd}$$

But in the case where $\operatorname{char} k > 0$, $(*)$ is not commutative by a philosophy of Grothendieck, except for the degree zero part.

<u>Counter Example</u> $X = \mathbb{P}^d \xrightarrow{F} X = \mathbb{P}^d$ Frobenius map, radical and surjective. $F_*(\Lambda) \cong \Lambda$, $cc_{\mathbb{P}^d,\bullet}(\Lambda) = (-1)^d c(\Omega^1_{\mathbb{P}^d}) = \left((-1)^i \binom{d+1}{i+1}\right)_{\deg i}$
$\neq 0$

$CH_\bullet(\mathbb{P}^d) = \bigoplus_{i=0}^{d} \mathbb{Z} \xrightarrow{F_*} CH_\bullet(\mathbb{P}^d)$ is multiplication by p^i on the degree i-part.

Corollary D (degree 0-part of (*)) push-formula / ~~Macc~~

For any projective morphism $f: X \longrightarrow Y$ between smooth projective schemes over a finite field k, the diagram (*) commutes, i.e, for any $\mathcal{F} \in D_c^b(X, \Lambda)$, we have

$$f_* cc_{X/k}(\mathcal{F}) = cc_{Y/k}(Rf_*\mathcal{F}) \quad \text{in } CH_0(Y).$$

proof $\Lambda = \overline{\mathbb{Q}_\ell}^*$.

For any 1-dim (continuous) rep $\chi: \pi_1^{ab}(Y) \longrightarrow \overline{\mathbb{Q}_\ell}^\times$,

$$\begin{tikzcd} cc_{X/k}(\mathcal{F}) & CH_0(X) \arrow[r, "\rho_X"] \arrow[d, "f_*"] & \pi_1^{ab}(X) \arrow[d, "f_*"] \arrow[dr, "f^*\chi"] & \\ cc_{Y/k}(Rf_*\mathcal{F}) & CH_0(Y) \arrow[r, "\rho_Y"] & \pi_1^{ab}(Y) \arrow[r, "\chi"] & \overline{\mathbb{Q}_\ell}^\times \end{tikzcd}$$

$$\chi(\rho_Y(-\underline{cc_{Y/k}(Rf_*\mathcal{F})})) \xlongequal[\text{twist formula}]{\text{Thm A}} \frac{\varepsilon(Y, Rf_*\mathcal{F}\otimes\chi)}{\varepsilon(Y, Rf_*\mathcal{F})} \xlongequal{\text{proj formula}} \frac{\varepsilon(X, \mathcal{F}\otimes f^*\chi)}{\varepsilon(X, \mathcal{F})}$$

$$\xlongequal[\text{twist formula}]{\text{Thm A}} (f^*\chi)(\rho_X(-cc_{X/k}(\mathcal{F}))) \xlongequal[\text{field theory}]{\text{Functoriality of class}} \chi(\rho_Y(-\underline{f_* cc_{X/k}(\mathcal{F})}))$$

Since χ is arbitrary, ρ_Y is injective with dense image

$$\Rightarrow f_* cc_{X/k}\mathcal{F} = cc_{Y/k}(Rf_*\mathcal{F}).$$ ☒

Even though (*) is not commutative in general, we would expect the following holds:

— Fix an integer $r \geq 0$ and a smooth connected scheme S of dimension r over a perfect field k.

— Let $X \xrightarrow{f} S$ be a smooth morphism purely of relative dimension n.

Let $D_c^b(X/S,\Lambda) \subseteq D_c^b(X,\Lambda)$ be the thick sub-triangulated category consisting of objects $\mathcal{F}$ such that $X \xrightarrow{f} S$ is $SS\mathcal{F}$-transversal.

Let $K_0(X/S,\Lambda)$ be the Grothendieck group of $D_c^b(X/S,\Lambda)$.

For any proper morphism
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$
between smooth schemes over S,

there is a well-defined map
$$K_0(X/S,\Lambda) \xrightarrow{h_*} K_0(Y/S,\Lambda).$$

We expect that the following diagram is commutative:
$$\begin{tikzcd} K_0(X/S,\Lambda) \arrow[r, "cc_{X/k,r}"] \arrow[d, "h_*"'] \arrow[dr, phantom, "(**)"] & CH_r(X) \arrow[d, "h_*"] \\ K_0(Y/S,\Lambda) \arrow[r, "cc_{Y/k,r}"'] & CH_r(Y) \end{tikzcd}$$

If $r=0$, this is the previous case.

If $r=\dim X$,
$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$
étale covering.

Some evidences for the above question:

(1) We can prove a cohomological version of $(**)$.

(2) relative version of global-twist formula (Thm A).

(3) Relative singular support (Joint with Haoyu Hu)

§3 Cohomological Characteristic Class

Let S be a smooth connected scheme of dim r over a perfect field k.

Let $X \xrightarrow{f} S$ be a smooth morphism ~~over~~ purely of relative dimension $\underline{n}$.

$\mathcal{K}_{X/S} = Rf^! \Lambda$.

We can construct a map, which is compatible with proper push-forward:

$$\begin{array}{ccc} K_0(X/S, \Lambda) & \xrightarrow{cc_{X/S}} & H^0(X, \mathcal{K}_{X/S}) = H^{2n}(X, \Lambda(n)) \\ \downarrow & & \\ K_0(X, \Lambda) & & \end{array}$$

which is called the cohomological characteristic class.

We expect the following diagram commutes ($S = \operatorname{Spec} k$, due to T. Saito 2016)

$$\begin{tikzcd} K_0(X/S, \Lambda) \arrow[d, "(-1)^r cc_{X,r}"'] \arrow[drr, "ccc_{X/S}"] & & \\ CH_r(X) = CH^n(X) \arrow[rr, "cl"', "\text{cycle class map}"] & & H^{2n}(X, \Lambda(n)) \end{tikzcd}$$

[The construction of $ccc_{X/S}$ is based on the method of [SGA5, Exposé III, Illusie].
See also Abbes-T.Saito 2003, or Martin Olsson 2016 for a motivic version.]
"$S = \operatorname{Spec} k$"

重要

Key-Lemma For any $\mathcal{F}, \mathcal{G} \in D^b_c(X/S, \Lambda)$, we have an isomorphism

$$\mathcal{F} \boxtimes^{L}_{S} D_{X/S}(\mathcal{G}) \xrightarrow{\cong} R\mathcal{H}om(pr_2^* \mathcal{G}, pr_1^! \mathcal{F})$$

where $D_{X/S}(\mathcal{G}) = R\mathcal{H}om(\mathcal{G}, \mathcal{K}_{X/S})$ and

$$\begin{tikzcd} X \times_S X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, phantom, "\square"] & X \arrow[d] & \mathcal{G} \\ X \arrow[r] & S & \end{tikzcd}$$
$\mathcal{F}$

Example

$U \overset{j}{\hookrightarrow} X = S$

$$\begin{array}{ccc} X & = & X \\ \| & & \| \\ X & = & X \end{array}$$

Λ ~~$j_!\mathcal{F}$~~ $\otimes R\mathcal{H}om(j_! \mathcal{F}, \Lambda)$

$\nparallel$?

$R\mathcal{H}om(j_! \mathcal{F}, \Lambda)$

$R\mathcal{H}om(j_! \Lambda, \Lambda) \cong j_* R\mathcal{H}om(\Lambda, j^* \Lambda)$
$\cong j_* \Lambda$

$j_! \Lambda \otimes j_* \Lambda \not\cong j_* \Lambda$
$\|$
$j_! \Lambda \cong j_!(\Lambda \otimes j^* j_* \Lambda)$

<u>proof</u> $S = \operatorname{Spec} k$ true by SGA5.
general case can be reduced to this case by applying a result of T. Saito to $X \times_S X \longrightarrow X \times_k X$ which is $\mathcal{F} \boxtimes^{L}_{k} D_{X/k}(\mathcal{G})$-transversal for any $\mathcal{F}$ and $\mathcal{G} \in D^b_c(X/S, \Lambda)$.

$$\begin{tikzcd} X \times_S X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \times_k X \arrow[d] \\ S \arrow[r] & S \times_k S \end{tikzcd}$$

<u>T. Saito's result</u> $Y \xrightarrow{h} W$ $\mathcal{F} \in D^b_c(W, \Lambda)$. If h is $SS\mathcal{F}$-transversal, then h is ~~$\mathcal{F}$-transversal~~ $\mathcal{F}$-transversal

Construction of $K_0(X/S,\Lambda) \xrightarrow{cc_{X/S}} H^0(X, \mathcal{K}_{X/S})$

Consider
$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\square"] & X \arrow[d, "\delta"] \\ X \arrow[r, "\delta"] & X\times_S X \end{tikzcd},$$

$$R\mathcal{H}om(\mathcal{F},\mathcal{F}) \cong \delta^! R\mathcal{H}om(pr_2^*\mathcal{G}, pr_1^!\mathcal{F}) \overset{\text{key-Lemma}}{\cong} \delta^!(\mathcal{F}\boxtimes_S D_{X/S}\mathcal{F})$$

$$\xrightarrow{1\to\delta_*\delta^*} \delta^!\delta_*\delta^*(\mathcal{F}\boxtimes_S D_{X/S}\mathcal{F}) = \delta^!\delta_*(\mathcal{F}\otimes D_{X/S}\mathcal{F}) \xrightarrow{\text{evaluation}} \delta^!\delta_*\mathcal{K}_{X/S} \xrightarrow[\text{base change.}]{\cong} \mathcal{K}_{X/S}$$

We get a map $\mathrm{Hom}(\mathcal{F},\mathcal{F}) \xrightarrow{Tr} H^0(X, \mathcal{K}_{X/S}) = H^{2n}(X,\Lambda(n))$.

The cohomological char class of $\mathcal{F}$ (relative to $X\to S$) is defined to be
$$cc_{X/S}(\mathcal{F}) = Tr(1) \in H^{2n}(X,\Lambda(n)).$$

Following SGA5, we have the following formal property:

<u>Proposition E</u> For any proper map $\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$ between smooth schemes over S,

we have a commutative diagram

$$\begin{tikzcd} K_0(X/S,\Lambda) \arrow[r, "cc_{X/S}"] \arrow[d, "h_*"'] & H^0(X, \mathcal{K}_{X/S}) \arrow[d, "h_*"] \\ K_0(Y/S,\Lambda) \arrow[r, "cc_{Y/S}"] & H^0(Y, \mathcal{K}_{Y/S}) \end{tikzcd}
\quad \overset{定义式}{=\!=} \quad
\begin{tikzcd} Rh_*\mathcal{K}_{X/S} \arrow[r] \arrow[d, equal] & \mathcal{K}_{Y/S} \arrow[d, equal] \\ Rh_*Rf^!\Lambda \arrow[d, equal] & Rg^!\Lambda \arrow[dd, equal] \\ Rh_*Rh^!(Rg^!\Lambda) \arrow[d, equal] & \\ Rh_!Rh^!(Rg^!\Lambda) \arrow[r, "\text{adjunction.}"'] & Rg^!(\Lambda) \end{tikzcd}$$

§4 Relative twist formula

Let k be a finite field.

S/k smooth of dimension $r \geq 0$, $X \longrightarrow S$ smooth projective of relative dimension $n \geq 0$.

By a result of T. Saito, there exists a unique way to attach a pairing

$$CH^n(X) \times \pi_1^{ab}(S) \longrightarrow \pi_1^{ab}(X) \qquad (***)$$

satisfying the following conditions:

(1) when $S = \operatorname{Spec} k$ is a point, for a closed point $x \in |X|$, the pairing with the class $[x]$ is the map

inseparable degree

$$\mathrm{Gal}(k^{ab}/k) \xrightarrow{\text{transfer}_{k(x)/k} \,\cdot\, [k(x):k]_i} \mathrm{Gal}(k(x)^{ab}/k(x)) \xrightarrow[i:x\to X]{i_{x*}} \pi_1^{ab}(X).$$

$\mathrm{Gal}(\overline{k(x)}/k(x)) \subseteq \mathrm{Gal}(\bar{k}/k)$.

(2) For any point $s \in |S|$, the following diagram commutes:

$$\begin{tikzcd}
CH^n(X) \times \pi_1^{ab}(S) \arrow[r] \arrow[d, shift right=8] & \pi_1^{ab}(X) \\
CH^n(X_s) \times \pi_1^{ab}(s) \arrow[r] \arrow[u, shift right=8] & \pi_1^{ab}(X_s) \arrow[u]
\end{tikzcd}$$

Corollary F Let $\mathcal{F} \in D_c^b(X/S, \Lambda)$ such that $X \xrightarrow{f} S$ is properly $SS\mathcal{F}$-transversal, ~~i.e.~~ ("properly" means that $\forall s \in S$, the fiber $SS\mathcal{F} \times_S s$ is of dimension $\dim X_s$)

Then for any smooth sheaf $\mathcal{G}$ on X, we have

$$\det Rf_*(\mathcal{F} \otimes \mathcal{G}) \cong (\det Rf_*\mathcal{F})^{\otimes \operatorname{rank} \mathcal{G}} \otimes \det\mathcal{G}((-1)^r CC_{X,r}(\mathcal{F})) \quad \text{in } K_0(S, \Lambda).$$

where $\det\mathcal{G}((-1)^r CC_{X,r}\mathcal{F})$ is a smooth sheaf of rank 1 on S, defined via the pairing (***):

$$\pi_1^{ab}(S) \xrightarrow{(-1)^r CC_{X,r}\mathcal{F}} \cancel{CH}\ \pi_1^{ab}(X) \xrightarrow{\det\mathcal{G}} \cancel{E}.\Lambda^\times.$$

$\Lambda = \overline{\mathbb{Q}}_\ell$

proof Use Chebotarev density and pull-back of CC by properly $SS\mathcal{F}$-transversal map, then use Theorem A. $\square$

I don't know if there is a canonical isom for Corollary F.

§5 Proof of Kato–T.Saito's Conjecture (follow S.Saito)

Use T.Saito–Yatagawa's theorem on the existence of good Lefschetz pencils relative to $SS\mathcal{F}$, we can reduce the question to the key case:

$$\begin{tikzcd} X_L \arrow[r] \arrow[d, "\text{blow up}"'] & L \subseteq \check{\mathbb{P}}^N \\ X \subseteq \mathbb{P}^N & \end{tikzcd}$$

$X \xrightarrow{f} C$ $\quad$ C: smooth projective curve

$\omega \neq 0$ meromorphic 1-form on C.

$X \xrightarrow{f} C$ is $SS\mathcal{F}$-transversal outside a finite set of closed points $\{x_v\}_{v\in\Sigma}$, $\Sigma \subseteq |C|$ (each fiber X_v contains at most one isolated char point x_v)

— For $v\in\Sigma$, x_v is k-rational.

— ω has neither poles or zeros at Σ.

— For $v\in|C|$ with $\mathrm{ord}_v(\omega)\neq 0$, X_v is smooth and $X_v \xrightarrow{i_v} X$ is properly $SS\mathcal{F}$-transversal.

Then we have an induction formula for $CC_{X/k}(\mathcal{F})$:

$$CC_{X/k}(\mathcal{F}) = -\sum_{v\in|\Sigma|} \mathrm{dimtot}\, R\Phi_{\bar{x}_v}(\mathcal{F},f)\cdot[x_v] - \sum_{v\in|C|\setminus|\Sigma|} \mathrm{ord}_v(\omega)\cdot CC_{X_v}(\mathcal{F}|_{X_v}).$$

Now we can prove theorem A by induction on $\dim(X)$ and by using Laumon's product formula: $\quad \mathcal{G}$: smooth on X.

$$\frac{\varepsilon(X,\mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rk}\,\mathcal{G}}} = \frac{\varepsilon(C, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon(C, Rf_*\mathcal{F})^{\mathrm{rk}\,\mathcal{G}}} \underset{\text{prod. formula}}{\overset{\text{Laumon's}}{=\!=}} \prod_{v\in|C|} \frac{\varepsilon_v(\omega, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\underset{\substack{\uparrow\\ \text{Local epsilon factor.}}}{\varepsilon_v(\omega, Rf_*)}^{\mathrm{rk}\,\mathcal{G}}}$$

By induction step and Local twist formula for local epsilon factor, we have

$$\frac{\varepsilon_v(\omega, Rf_*(\mathcal{F}\otimes\mathcal{G}))}{\varepsilon_v(\omega, Rf_*)^{\mathrm{rk}\,\mathcal{G}}} = \begin{cases} \det\mathcal{G}(\rho_X(\mathrm{ord}_v(\omega)\cdot CC_{X_v}(\mathcal{F}|_{X_v}))) & \text{if } v\notin\Sigma \text{ and } \mathrm{ord}_v(\omega)\neq 0 \\[2ex] \underset{\substack{\uparrow\\ \text{use } R\Gamma(X_{\bar{v}},\mathcal{F})\to R\Gamma(X_{\bar{\eta}_{\bar{v}}},\mathcal{F})\to R\Phi_{\bar{x}_v}(\mathcal{F},f)\to\cdot}}{\det\mathcal{G}}(\rho_X(\mathrm{dimtot}\, R\Phi_{\bar{x}_v}(\mathcal{F},f)\cdot[x_v])) & \text{if } v\in\Sigma \end{cases}$$

$$\Longrightarrow \frac{\varepsilon(X,\mathcal{F}\otimes\mathcal{G})}{\varepsilon(X,\mathcal{F})^{\mathrm{rk}\,\mathcal{G}}} = \det\mathcal{G}(\rho_X(-CC_{X/k}(\mathcal{F}))).$$

§ Appendix A: transfer map and relative reciprocity map

Transfer map $H < G$ closed subgroup of finite index between topological groups

$H^{ab} = H/H^c$, H^c = closure of the commutator subgroup of H.

The transfer map $t: G^{ab} \longrightarrow H^{ab}$ is defined as follows:

choose any section $s: H\backslash G \longrightarrow G$, then for any $g \in G$,

$$t(gG^c) = \prod_{x \in H\backslash G} h_{g,x} \mod H^c$$

where $h_{g,x} \in H$ is defined by $s(x) g = h_{g,x}\, s(xg)$.

relative reciprocity map $CH^n(X) \times \pi_1^{ab}(S) \longrightarrow \pi_1^{ab}(X)$:

duality $\rightsquigarrow$ Same to define $CH^n(X) \times H^1(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(S, \mathbb{Q}/\mathbb{Z})$.

We use the result: for any finite flat map $Y \xrightarrow{\pi} S$, we have the trace map

$$\mathrm{Tr}_{Y/S}: H^1(Y, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(S, \mathbb{Q}/\mathbb{Z})$$

The trace map commutes with arbitrary base change.

Case $S = \operatorname{Spec} k$

$$\begin{array}{ccc} Z_0(X) \times H^1(X, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & H^1(S, \mathbb{Q}/\mathbb{Z}) \\ ([x], \chi) & \longmapsto & \mathrm{Tr}_{k(x)/k}(\chi|_x) \end{array}$$

which factor through $CH_0(X) \times H^1(X, \mathbb{Q}/\mathbb{Z})$.

General case $K = \mathrm{Frac}(S)$. $H^1(S, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(K, \mathbb{Q}/\mathbb{Z})$ injective.

$$CH^n(X) \times H^1(X, \mathbb{Q}/\mathbb{Z}) \longrightarrow CH^n(X_K) \times H^1(X_K, \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1(K, \mathbb{Q}/\mathbb{Z})$$

Show image is contained in $H^1(S, \mathbb{Q}/\mathbb{Z})$.

Key-Case S = Spectrum of a discrete valuation ring.

For $z \in |X_K|$ and $\chi \in H^1(X, \mathbb{Q}/\mathbb{Z})$,

$$\begin{array}{c} Z = \overline{\{z\}} \subseteq X \\ \Big\downarrow {\scriptstyle \text{finite flat}} \\ S \end{array} \qquad \langle z, \chi|_{X_K} \rangle = \mathrm{Tr}_{Z/S}(\chi|_Z) \in H^1(S, \mathbb{Q}/\mathbb{Z}).$$

§ Appendix B: Swan class and uniqueness theorem

$U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X_{/k}$ proper smooth over a perfect field k.

$\mathcal{F}$: smooth étale sheaf on U.

In 2004, Kato and T. Saito defined the Swan classes $Sw^{ks}(\mathcal{F}) \in CH_0(X \setminus U) \otimes \mathbb{Q}$.

Assume resolution of singularities in the following, then $Sw^{ks}(\mathcal{F}) \in CH_0(X \setminus U)$

$Sw^{ks}(-)$ satisfies the following properties:

(1) If $\mathcal{F}$ and $\mathcal{G}$ have same wild ramification, then $Sw^{ks}(\mathcal{F}) = Sw^{ks}(\mathcal{G})$.

smooth on U

$\Downarrow$

universally same Euler number $\Rightarrow$ Same rank and Same Artin conductor by cut-by-curve.

(2) (push-forward) For any cartesian diagram
$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "f"'] & Y \arrow[d, "\bar{f}"] \\ U \arrow[r, hook] & X \end{tikzcd}$$
, where

X and Y are proper smooth,
f: finite étale
$\mathcal{G}$: any smooth sheaf on V

$\Rightarrow$ then we have
$$Sw^{ks}(f_*\mathcal{G}) = \bar{f}_* Sw^{ks}(\mathcal{G}) + \operatorname{rank}\mathcal{G} \cdot Sw^{ks}(f_*\Lambda)$$
in $CH_0(X)$ (更强，在 $CH_0(X \setminus U)$，但今天我们只讨论 weak form)

(3) (Initial condition)

If moreover $D = X \setminus U$ and $B = Y \setminus V$ are SNC divisors, then
$$Sw^{ks}(f_*\Lambda) = d^{\log} := (-1)^{\dim X - 1} \bar{f}_* \left\{ c_d(\Omega^1_{Y/k}(\log B)) - \bar{f}^* \Omega^1_{X/k}(\log D) \cap [Y] \right\}$$
in $CH_0(X)$

logarithmic differential zero class

(Hurwitz formula).

Remark For $Sw^{cc}(\mathcal{F}) = 0_X^!(\operatorname{rank}\mathcal{F} \cdot CC(j_!\Lambda) - CC(j_!\mathcal{F})) \in CH_0(X)$.

$Sw^{cc}(\mathcal{F})$ satisfies (3) $\sim$ tame sheaf $\boxed{\text{CC of}}$

$Sw^{cc}(\mathcal{F})$ satisfies (1) by T. Saito - Yatagawa, which is a generalization of Illusie and I. Vidal.

~~For~~ (2) is still open for $Sw^{cc}(\mathcal{F})$.

(True if k is finite and if X and Y are projective and smooth)

Theorem (Uniqueness) Assume resolution of singularities.

Any $Sw^{\cdot} : K_0(\text{local system on } U) \longrightarrow CH_0(X)$ satisfying (1)(2)(3) equals to Sw^{ks}.

proof Brauer induction $\Rightarrow$ WMA $\mathcal{F}$ is of rank 1, and is trivialized by a finite étale covering $V \longrightarrow U$ of Galois group $\mathbb{Z}/p^n\mathbb{Z}$.

By induction on n $\Rightarrow$ WMA $n=1$.

$\mathcal{F} \longleftrightarrow \chi$ character of $G = \mathbb{F}_p$.

We may assume $\overset{\mathcal{F}\neq\Lambda.}{\chi}$ is non-trivial

Fact (T. Saito-Yatagawa) For any non-trivial character χ and χ' of G (hence of same order) $\Rightarrow$ χ and χ' have same wild ramification.

~~Choose~~

Now $f_*\Lambda \cong \Lambda \oplus \bigoplus_{\chi\neq 1} \chi$

$$\Rightarrow (p-1)Sw^{\cdot}(\mathcal{F}) = \sum_{\chi\neq 1} Sw^{\cdot}(\chi) = Sw^{\cdot}(f_*\Lambda) \overset{(3)}{=\!=} Sw^{ks}(f_*\Lambda)$$
$$= (p-1)\, Sw^{ks}(\mathcal{F})$$

$$\Rightarrow Sw^{\cdot}(\mathcal{F}) = Sw^{ks}(\mathcal{F}).$$

choose $H \subseteq G$ unique subgroup of order p.

χ: character of G associated to $\mathcal{F}$.

$\chi_H = \chi|_H$. $\Rightarrow$ χ is of order p^n, $\chi|_H = \chi_H$ is of order p.

$\mathcal{F}_{\chi_H}$: sheaf on U'

$V \xrightarrow{g} U' \xrightarrow{h} U$ étale coverings such that the Galois group of $V \xrightarrow{g} U'$ is H

By def $\mathcal{F}_{\chi_H} = h^*\mathcal{F}$.

$$h_*\mathcal{F}_{\chi_H} = h_*h^*\mathcal{F} \cong \mathcal{F} \otimes h_*\Lambda \simeq \mathcal{F} \otimes \bigoplus_{\psi} \mathcal{F}_\psi \cong \bigoplus_{\psi} (\mathcal{F} \otimes \mathcal{F}_\psi)$$

irr. rep. of G/H (cyclic)

We have $Sw^{ks}_X(\mathcal{F}) = Sw^{ks}_X(\mathcal{F} \otimes \mathcal{F}_\psi)$ thence condition (2).

of same order

Same wild ramification (T. Saito - Yatagawa)

X/k of finite type, $\Lambda = \mathbb{F}_\ell$, $\ell \neq p = \mathrm{char}\, k$.

Case 1: X is normal and separated

$\mathcal{F}$ and $\mathcal{F}'$ are locally constant constructible sheaves of Λ-modules.

We say $\mathcal{F}$ and $\mathcal{F}'$ have the <u>same wild ramification</u> if there exist proper

normal $\overline{X} \underset{\text{dense}}{\overset{\text{open}}{\supseteq}} X$ such that for all geometric $\bar{x} \longrightarrow \overline{X}$, we have:

— Let G be a finite quotient group of the inertia group $I_{\bar{x}} = \pi_1(\overline{X}_{(\bar{x})} \times_{\overline{X}} X, \bar{t})$

w.r.t a base point $\bar{t}$ such that the pull-back $\mathcal{F}|_{\overline{X}_{(\bar{x})} \times_{\overline{X}} X}$ and

$\mathcal{F}'|_{\overline{X}_{(\bar{x})} \times_{\overline{X}} X}$ correspond to G-modules M and M' respectively.

<u>Then</u> for every element $\sigma \in G$ of p-power order, we have an equality of the dimension of the σ-fixed parts:

$$\dim M^\sigma = \dim M'^\sigma$$

Case 2: In general

Let $\mathcal{F}$ and $\mathcal{F}'$ be constructible complexes of Λ-modules on X.

We say $\mathcal{F}$ and $\mathcal{F}'$ have <u>same wild ramification</u> if

$\exists$ finite partition $X = \coprod_{i \in I} X_i$ by locally closed normal and separated subschemes such that for every q and for every i, the restrictions $\mathcal{H}^q(\mathcal{F})|_{X_i}$ and $\mathcal{H}^q(\mathcal{F}')|_{X_i}$ are locally constant constructible and have the same wild ramification in the sense of 1.

Definition of Swan class Sw^{ks}

X variety over a perfect field k.

$U \underset{\text{open}}{\overset{j}{\hookrightarrow}} X$. U smooth. $\mathcal{F}$: smooth sheaf on U.

($\mathcal{F}$ mod ℓ を方は在) $\Downarrow$

Assume $\mathcal{F}$ is trivialized by a finite Galois covering $V \to U$ of Galois group G.

M = representation of G corresponding to $\mathcal{F}$.

Further assume there is a commutative diagram

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & Y \arrow[d, "f:\text{proper}"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

Y: smooth and $Y \setminus V$ is a SNC divisor.

In general, we consider $\mathcal{F}$ mod ℓ and use Brauer trace and "alteration"

$$\begin{tikzcd} Y \arrow[r, "\Delta_Y"', "\text{log-diagonal}"] & (Y\times Y)' \arrow[d] \\ & Y \times Y \end{tikzcd}$$

$(Y\times Y)'$ blow-up at $D_1 \times D_1, \dots, D_m \times D_m$ where $D = Y \setminus V = \bigcup_{i=1}^{m} D_i$

For any $\sigma \in G = \mathrm{Gal}(V/U)$ such that $\sigma \neq 1$, let Γ_σ = graph of σ and

$\overline{\Gamma_\sigma}$ be the closure of $\Gamma_\sigma \subseteq V \times_U V$ in $(Y\times Y)'$.

tame ramification $\Rightarrow$ no intersection
wild ramification $\Rightarrow$ non-empty intersection.

<u>Define</u> $S_{V/U}(\sigma) = -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(Y \setminus V)$,

$$S_{V/U}(1) := -\sum_{\sigma \neq 1} S_{V/U}(\sigma).$$

$$Sw(\mathcal{F}) := \frac{1}{|G|} \sum_{\sigma \in G} f_* S_{V/U}(\sigma) \cdot \mathrm{Tr}(\sigma : M) \in CH_0(X \setminus U) \otimes \mathbb{Q}.$$

$\Uparrow$ For $\mathcal{F}$ mod ℓ, use Brauer trace.

Theorem (Kato–T. Saito, Higher GOS)

If X is proper, $\chi_c(U_{\bar k}, \mathcal{F}) = \chi_c(U_{\bar k}) \operatorname{rank} \mathcal{F} - \deg Sw(\mathcal{F})$.

(use Lefschetz trace formula for open variety proved using a log product)

~~and~~