Filled Function Methods for Global Optimization Problems

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1. Preliminaries

Consider the following global optimization problem:

\[(GP) \min f(x) \quad x \in X,\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous and \( X \subset \mathbb{R}^n \).

- If \( X = \mathbb{R}^n \), then problem \((GP)\) is an unconstrained global optimization problem.

- If \( X \subset \mathbb{R}^n \), especially

\[ X := \{ x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \ldots, m \}, \]

then problem \((GP)\) is a constrained global optimization problem.
• For problem \((GP)\), there exist many mature local optimization methods to obtain a local minimizer \(x^*\) in literature.

• Our aim is to find a global minimizer.

• Among all the different types of global optimization algorithms available in the literature, one popular approach is called modified function method.

• Filled function method is a typical modified function method.

• The main idea of filled function method is as follows.

Let \(x^*\) be a local minimizer and not a global minimizer of problem \((GP)\). Then construct a filled function \(p_{x^*}(x)\) at \(x^*\) such that
1°. the local minimizer $x^*$ of problem $(GP)$ is a strictly local maximizer of function $p_{x^*}(x)$ on $X$;

2°. we can escape the current point $x^*$ to obtain a better point by solving the filled function problem $\min_{x \in X} p_{x^*}(x)$ via using some local methods.

See the Figure.

- How to construct a good filled function to satisfy the above conditions?

- This talk will introduce some new filled functions and some new filled function methods for global optimization problems and nonlinear equations.

- References:

[WLZY]: Z.Y. Wu, H.W.J. Lee, L.S. Zhang and X. M. Yang, A novel filled function method and quasi-


2. Filled function method for unconstrained global optimization problems

Consider the following unconstrained programming problem:

\[
(UGP) \quad \min f(x) \\
\text{s.t. } x \in \mathbb{R}^n,
\]

where \( f(x) \) is continuously differentiable on \( \mathbb{R}^n \).

- Reference [WLZY] propose a new filled function for problem \( (UGP) \). To introduce this new filled function, we need two auxiliary functions.

\[\nabla \text{ Let}\]
\[ f_r(t) = \begin{cases} 
  t + r & t \leq -r \\
  \frac{r - 2}{r^3} t^3 + \frac{r - 3}{r^2} t^2 + 1, & -r < t \leq 0 \\
  1, & t > 0 
\end{cases} \]  
(2.1)

\[ g_r(t) = \begin{cases} 
  0, & t \leq -r \\
  -\frac{2}{r^3} t^3 - \frac{3}{r^2} t^2 + 1, & -r < t \leq 0 \\
  1, & t > 0 
\end{cases} \]  
(2.2)

Functions \( f_r(t) \) and \( g_r(t) \) are continuously differentiable on \( R \), see Figure 2.1 and Figure 2.2.
Let

\[ H_{q,r,x^*}(x) = q \left( \exp \left( -\frac{||x - x^*||^2}{q} \right) g_r \left( f(x) - f(x^*) \right) \right) + f_r \left( f(x) - f(x^*) \right), \] (2.3)

where \( r > 0, q > 0 \) are parameters, \( x^* \) is the current local minimum and \( \exp(\cdot) \) is an exponential function.

- Function \( H_{q,r,x^*} \) has the following properties:
  - Suppose \( x^* \) is a local minimizer of problem \( (UGP) \), then \( x^* \) is a strictly local maximizer of \( H_{q,r,x^*}(x) \) on \( \mathbb{R}^n \) for any \( r > 0, q > 0 \).
  - Any local minimizer \( \bar{x} \) of \( H_{q,r,x^*}(x) \) on \( \mathbb{R}^n \) satisfies that \( f(\bar{x}) < f(x^*) \).
Suppose that $x^*$ is not a global minimizer of problem \((UGP)\). Let

$$L = \{ \bar{x} \mid \bar{x} \text{ is local minimizer satisfying } f(\bar{x}) < f(x^*) \}.$$  

Then for any $\bar{x} \in L$, $\bar{x}$ is a local minimizer of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ when the parameters $r$ satisfies some conditions.

From above properties, we know that: $x^*$ is a strictly local maximizer of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ and if $x^*$ is not a global minimizer of problem \((UGP)\), then any local minimizer $\bar{x}$ of $H_{q,r,x^*}(x)$ on $\mathbb{R}^n$ is a better point, i.e., $f(\bar{x}) < f(x^*)$. Thus, function $H_{q,r,x^*}(x)$ is a filled function of problem \((UGP)\) at $x^*$.

Using the given filled function $H_{q,r,x^*}(x)$, we can design
a global optimization method for problem \((UGP)\).

**Algorithm 1: Filled function method for Problem \((UGP)\):**

**Step 0.** Let \(k_0\) be a positive number and let \(e_i, i = 1, \cdots, k_0\) almost uniformly distribute over the unit sphere \(B = \{x \in R^n | \|x\| = 1\}\). Let \(M\) be a very large number and \(\mu\) be a very small number. Choose an initial point \(x_0^1 \in R^n\). Set \(k := 1\).

**Step 1.** Let \(x_k^*\) be a local minimizer of problem \((UGP)\) starting from \(x_0^1\). Set \(i := 1\) and take a positive number \(\delta_0 > 0\), let \(\delta := \delta_0\).

**Step 2.** Let \(\bar{x}_k^* = x_k^* + \delta e_i\). If \(f(\bar{x}_k^*) < f(x_k^*)\), then set \(x_{k+1}^0 := \bar{x}_k^*, k := k + 1\) and go to **Step 1**; otherwise, go to **Step 3**.
Step 3. Let

\[ H_{q,r,x^*_k}(x) = q \left( \exp \left( -\frac{\|x - x^*_k\|^2}{q} \right) g_r \left( f(x) - f(x^*_k) \right) + f_r \left( f(x) - f(x^*_k) \right) \right), \]

where \( g_r(t) \) and \( f_r(t) \) are decided by (2.2) and (2.1), respectively. Solve the problem:

\[
\min_{x \in \mathbb{R}^n} H_{q,r,x^*_k}(x) \tag{2.4}
\]

by a local search method starting from the point \( x^*_k \). If we can find a local minimizer \( y^*_k \), then we have that \( f(y^*_k) < f(x^*_k) \), then let \( x^0_{k+1} := y^*_k \) and goto step 1; otherwise, goto Step 4. We have two cases: one is that problem (2.4) has no local
minimizer, then $x_k^*$ is already a global minimizer; another one is that (2.4) has local minimizer, but we can not find them, then we need to change the direction $e_i$, or change the parameters $q$ or $r$.

**Step 4.** If $i < k_0$, then let $i := i + 1$, go to Step 2; otherwise, go to Step 5.

**Step 5.** If $q < M$ and $r > \mu$, increase $q$ and decrease $r$; otherwise, go to Step 6.

**Step 6.** Stop and $x_k^*$ is a global minimizer of problem $(UGP)$. 
Figure 2.1: The behavior of $f_r(t)$ with $r = 0.5$, 0.4 and 0.3, respectively
Figure 2.2: The behavior of $g_r(t)$ with $r = 0.5$, 0.4 and 0.3, respectively.
3. **Filled function method for constrained global optimization problems**

Consider the following constrained global optimization problems:

\[
(CGP) \quad \text{min } f(x) \quad (3.1)
\]
\[
\text{s.t. } g_i(x) \leq 0, \ i = 1, \ldots, m
\]
\[
x \in X,
\]

where \( f : X \rightarrow \mathbb{R}, g_i : X \rightarrow \mathbb{R}, i = 1, 2, \ldots, m, \) are continuously differentiable on \( X, X \) is a box.

- Let

\[
S = \{ x \in X | g_i(x) \leq 0, \ i = 1, \ldots, m \},
\]
\[
S^\circ = \{ x \in \text{int}X | g_i(x) < 0, \ i = 1, \ldots, m \},
\]

where \( \text{int}A \) denotes the interior of set \( A \).
Assumption 1. Assume that $S^\circ \neq \emptyset$, $\text{cl} S^\circ = S$, where $\text{cl} A$ denotes the closure of set $A$.

By Assumption 1, we know that for any $x_0 \in S'$, there exists a sequence $\{x_n\} \subset S^\circ$, such that $\lim_{n \to \infty} x_n = x_0$.

• Reference [WHBY] proposes a filled function method to solve problem $(CGP)$. Here we also need the following two auxiliary functions.

$\triangle$ For $r > 0$, $c > 0$, let

$$f_{r,c}(t) = \begin{cases} 
  c, & t \geq 0 \\
  -\frac{2c}{r^3}t^3 - \frac{3c}{r^2}t^2 + c, & -r < t \leq 0 \\
  0, & t \leq -r
\end{cases}$$

$$(3.2)$$
\[ h_r(t) = \begin{cases} 
\frac{r - 4}{r^3}t^3 + \frac{2r - 6}{r^2}t^2 + t + 2 & t \geq 0 \\
0 & -r < t < 0 \\
-2r & t \leq -r 
\end{cases} \]  
(3.3)

△ Let

\[ p_{r,c,q,x^*}(x) = \frac{1}{\|x - x^*\|_2^2 + 1} f_{r,c} \left( h_r \left( f(x) - f(x^*) \right) \right) + \sum_{i=1}^{m} h_{r,q} \left( g_i(x) \right) - 2r \]  
(3.4)

where \( c > 0, \ r > 0 \) and \( q > 0 \) are parameters.

□ The term \( \sum_{i=1}^{m} h_{r,q} \left( g_i(x) \right) \) is used to penalize the unfeasible points. The term \( h_r \left( f(x) - f(x^*) \right) \) is used to penalize the points \( x \) which satisfy that
\[ f(x) \geq f(x^*). \] So here the function \( p_{r,c,q,x^*}(x) \) is not only a filled function, but also a penalty function for problem \((CGP)\).

\[ \triangle \] Function \( p_{r,c,q,x^*}(x) \) has the following properties.

* If \( x^* \) is a local minimizer of problem \((CGP)\), then for any \( c > 0, \ q > 0 \) and \( 0 < r \leq 1 \), \( x^* \) is a strictly local maximizer of \( p_{r,c,q,x^*}(x) \) on \( X \).

* For any \( x \in X \) with \( x \neq x^* \), if \( \nabla p_{r,c,q,x^*}(x) = 0 \), then \( f(x) < f(x^*) \) and \( x \in S \).

* When \( r \leq 1 \), any local minimizer \( \bar{x} \) of \( p_{r,c,q,x^*}(x) \) on \( X \) satisfies that

\[ f(\bar{x}) < f(x^*) \text{ and } \bar{x} \in S, \]

or

\[ \bar{x} \text{ is a vertex of } X. \]

* If \( x^* \) is not a global minimizer of problem \((CGP)\),
then there exist $r_0 > 0$, $q_0 > 0$ and $\bar{x} \in S^o$, such that $\bar{x}$ is a local minimizer of $p_{r,c,q,x^*}(x)$ on $X$ and $f(\bar{x}) < f(x^*)$ when $r \leq r_0$ and $q \geq q_0$.

- By the given filled function $p_{r,c,q,x^*}(x)$, the local optimization methods for constrained problem ($CGP$) and the local optimization methods for the unconstrained filled function problems (with only box constraint): $\min_{x \in X} p_{r,c,q,x^*}(x)$, we can obtain the following global optimization method for problem ($CGP$).

**Algorithm 2: Filled function method for Problem ($CGP$):**

**Step 0. a).** Choose a small positive numbers $\mu$, and a large positive number $M$. Choose a positive integer number $K$ and directions $e_1, \ldots, e_K$. Choose the initial values $q_1, c_1, \text{ and } r_1$ for the parameters $q, c,
and \( r \), respectively.

b). Choose an initial point \( x_1^0 \in X \) (here \( x_1^0 \) may not be a feasible point), then use penalty function methods to find the first local minimizer \( x_1^* \) of the original problem \((CGP)\). Let \( k := 1, j := 1 \) and \( \lambda := 1 \), and go to \textit{Step 1}.

\textbf{Step 1.} Let

\[
p_{r_k,c_k,q_k,x_k^*}(x) = \frac{1}{\|x - x_k^*\|^2 + 1} f_{r_k,c_k} \left( g_{r_k} \left( f(x) - f(x_k^*) \right) \right) + \sum_{i=1}^{m} g_{r_k,q_k} \left( g_i(x) \right) - 2r_k \right),
\]

where \( f_{r,c}(t) \) and \( g_r(t) \) are defined in \((3.2)\) and \((3.3)\) respectively. Go to \textit{Step 2}.
Step 2. If \( j \leq K \), choose a nonnegative \( \lambda \) with \( \lambda \leq 1 \) such that \( y^j_k := x^*_k + \lambda e_j \in X \), and go to Step 3; otherwise, go to Step 5.

Step 3. Search for a local minimizer of the following filled function problem starting from \( y^j_k \):

\[
\min_{x \in X} \quad pr_{k,c,k,q,k,x^*_k}(x). \tag{3.6}
\]

Let \( \bar{x}^*_k \) be an obtained local minimizer of problem (4.3). If \( \bar{x}^*_k \) satisfies \( f(\bar{x}^*_k) < f(x^*_k) \) and \( \bar{x}^*_k \in S \), then let \( x^0_{k+1} := \bar{x}^*_k, k := k + 1 \), and go to Step 4; otherwise, let \( j := j + 1 \) and go to Step 2.

Step 4. Find a local minimizer \( x^*_k \) of the original constrained problem \((CGP)\) by local search methods starting from \( x^0_k \). Go to step 1.

Step 5. If \( q_k \leq M \), increase \( q_k \) and let \( j := 1 \), go to Step 1;
otherwise, go to Step 6.

Step 6. If $c_k \leq M$, increase $c_k$ and let $q_k := q_1$, $j := 1$, go to Step 1; otherwise, go to Step 7.

Step 7. If $r_k \geq \mu$, decrease $r_k$ and let let $c_k := c_1$, $q_k := q_1$, go to Step 1; otherwise, stop and $x_k^*$ is a global minimizer or an approximate global minimizer of problem (CGP).
4. Filled function method for Nonlinear Equations

Consider the following nonlinear equations:

\[(NE) \quad F(x) = 0 \quad x \in X,\]

where \(F : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is continuous and \(X \subset \mathbb{R}^n\).

- The typical methods for solving \((NE)\) are optimization-based methods in which \((NE)\) is reformulated as an optimization problem.

- The most popular optimization-based methods involve solving the following optimization problem \((OP)\) to
find solutions of equations \((NE)\).

\[(OP) \quad \min \varphi(x) := \frac{1}{2} \| F(x) \|_2^2 \]

s.t. \(x \in X\).

\(\heartsuit\) If \((NE)\) exists a solution in \(X\), then \(\bar{x} \in X\) is a solution of \((NE)\) if and only if \(\bar{x}\) is a global optimal solution of problem \((OP)\) with the zero optimal value.

\(\heartsuit\) Generally, the traditional optimization-based methods for solving nonlinear system \((NE)\) are frequently stuck at a stationary point or a local minimizer of the corresponding optimization problem, which is not necessarily a solution of the original system.

\(\heartsuit\) Recently, great efforts have been made to overcome the difficulty caused by non-global minimiz-
ers.

Reference: (C. Kanzow, “Global optimization techniques for mixed complementarity problems,” *Journal of Global Optimization*, vol. 16, pp. 1-21, 2000) incorporated two well-known global optimization algorithms, namely a tunneling method and a filled function method, into a standard nonsmooth Newton-type method to solve an unconstrained nonsmooth nonlinear system \((NE)\) which is a reformulation of the mixed complementarity problem.

Reference [WMBYE] proposes a new filled function method to solve nonlinear equations \((NE)\) with box constraint. Here I will introduce this method. First, we need several definitions.

A point \(x \in X\) is said to be a vertex of box \(X\)
if \( x = \lambda x_1 + (1 - \lambda)x_2 \) with \( x_1, x_2 \in X \) and \( \lambda \in (0, 1) \) implies that \( x = x_1 = x_2 \).

\[ \triangle \text{Definition of Filled Function for } (NE): \text{ A continuously differentiable function } P_{x^*}(x) \text{ is said to be a filled function of } (NE) \text{ at a point } x^* \text{ with } \varphi(x^*) > 0, \text{ if:} \]

1\(^{\circ} \) \( x^* \) is a strict local maximizer of \( P_{x^*}(x) \) on \( X \);

2\(^{\circ} \) Any local minimizer \( \bar{x} \) of \( P_{x^*}(x) \) on \( X \) satisfies

\[ \varphi(\bar{x}) < \frac{\varphi(x^*)}{2} \text{ or } \bar{x} \text{ is a vertex of } X; \]

3\(^{\circ} \) Any local minimizer \( \bar{x} \) of problem \((OP)\) with

\[ \varphi(\bar{x}) \leq \frac{\varphi(x^*)}{4} \]

is a local minimizer of \( P_{x^*}(x) \) on \( X \);

4\(^{\circ} \) Any \( \bar{x} \in X \) with \( \nabla P_{x^*}(\bar{x}) = 0 \) implies \( \varphi(\bar{x}) < \frac{\varphi(x^*)}{2} \).
Note that in the definition of filled function for $(NE)$, it is not necessary to require that $x^*$ is a local minimizer of optimization problem $(OP)$. It just needs that $\varphi(x^*) > 0$

Construct a Filled Function for $(NE)$: Using the two auxiliary functions $g_r(t)$ and $f_r(t)$ defined by (2.2) and (2.1), for a given $x^* \in X$ with $f(x^*) > 0$, let

$$
\Psi_{q,x^*}(x) = \frac{1}{\|x - x^*\|^2} + 1 \left( \frac{g_{\varphi(x^*)}}{4} \left( \varphi(x) - \frac{\varphi(x^*)}{2} \right) \right) + q \frac{f_{\varphi(x^*)}}{4} \left( \varphi(x) - \frac{\varphi(x^*)}{2} \right). \quad (4.1)
$$

Function $\Psi_{q,x^*}(x)$ has the following properties:

Let $\varphi(x^*) > 0$, $q > 0$. Then $x^*$ is a strict global
maximizer of $\Psi_{q,x^*}(x)$ on $X$.

$\triangle$ Let $\varphi(x^*) > 0$, $q > 0$. Any local minimizer $\bar{x}$ of $\Psi_{q,x^*}(x)$ on $X$ satisfies

$$\varphi(\bar{x}) < \frac{\varphi(x^*)}{2} \quad \text{or} \quad \bar{x} \text{ is a vertex of } X.$$

$\triangle$ Let $\varphi(x^*) > 0$, $q > 0$. Assume that system $(NE)$ has a solution. Then any local minimizer $\bar{x} \in X$ of problem $(OP)$ on $X$ with $\varphi(\bar{x}) < \frac{\varphi(x^*)}{4}$ is a local minimizer of $\Psi_{q,x^*}(x)$ on $X$.

$\triangle$ Let $\varphi(x^*) > 0$. Then any point $\bar{x} \in X \setminus \{x^*\}$ with $\nabla \Psi_{q,x^*}(\bar{x}) = 0$ implies that $\varphi(\bar{x}) < \frac{\varphi(x^*)}{2}$.

$\star$ Function $\Psi_{q,x^*}(x)$ is a filled function of $(NE)$. Using this filled function, we can obtain the following method to solve $(NE)$. 


Algorithm 3: Filled Function Method for Nonlinear equations ($NE$):

**Step 0.** Choose small positive numbers $\mu, \delta$ and a large positive number $M$ (such as, we take $\mu = 10^{-10}, \delta = \frac{1}{25}$ and $M = 10^{10}$). Choose a positive integer number $K$ and directions $e_1, \ldots, e_K$ (such as, we take $K = 2n$ and $e_i, \ i = 1, \ldots, K$, are the coordinate directions). Choose an initial value $q_0$ for the parameter $q$ (such as, we take $q_0 = 10$). Let $x_0 \in X$ be a given initial point and let $k := 0$. If $f(x_0) \leq \mu$, then let $x_k^* := x_0$ and go to Step 6. Otherwise, let $q := q_0$ and go to Step 1.

**Step 1.** Solve problem ($OP$) starting from $x_k$ using some local optimization methods. Let $x_k^*$ be a local minimizer. If $\varphi(x_k^*) \leq \mu$, go to Step 6; otherwise, set $i := 1$ and take a positive number $\delta_0 > 0$, let
\[ \delta := \delta_0. \text{goto Step 2.} \]

**Step 2.** Let \( \bar{x}^*_k = x^*_k + \delta e_i \). If \( f(\bar{x}^*_k) < f(x^*_k) \), then set \( x_{k+1} := \bar{x}^*_k, \ k := k + 1 \) and go to **Step 1**; otherwise, go to **Step 3**.

**Step 3.** Construct the following filled function

\[
\Psi_{q,x^*_k}(x) = \frac{1}{\|x - x^*_k\|^2 + 1} \left( g_{f(x^*_k)} \left( \varphi(x) - \frac{\varphi(x^*_k)}{2} \right) \right) + q f_{f(x^*_k)} \left( \varphi(x) - \frac{\varphi(x^*_k)}{2} \right), \tag{4.2}
\]

where \( g_r(t) \) and \( f_r(t) \) are defined by (2.2) and (2.1), respectively. Solve the problem

\[
\min_{x \in X} \Psi_{q,x^*_k}(x). \tag{4.3}
\]

by a local search method starting from the point \( \bar{x}^*_k \). Let \( y^*_k \) be a local minimizer. If \( f(y^*_k) < f(x^*_k) \),
then let $x_{k+1} := y_k^*, k := k + 1$, goto Step 1; otherwise, goto Step 4.

Step 4. If $i < K$, then let $i := i + 1$, go to Step 2; otherwise, go to Step 5.

Step 5. If $q < M$ and $r > \mu$, increase $q$ and decrease $r$; otherwise, go to Step 6.

Step 6. Stop. $x_k^*$ is a solution or a $\mu$-approximate solution of $(NE)$. 
5. Numerical examples

EX1. Rastrigin (n = 2)

\[
\begin{align*}
\min \ f_R(x) &= x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2) \\
\text{s.t.} \quad -2 \leq x_i \leq 2, \ i = 1, 2.
\end{align*}
\]

\[
(5.1)
\]

◊ From Figure 5.3, we see that there are many local minima of this problem.

◊ Table 1 gives the numerical results obtained by Algorithm 1 [2] for problem (5.1).

◊ From Table 1, we see that the first local minimizer of problem (5.1) from the first initial point

\[
x_1^0 = (1, 1)^T
\]

is \(x_1^* = (1.0408, 1.0408)\). By the filled function, we find other several initial points:
Table 1: Results for Rastrigin by QFFM

<table>
<thead>
<tr>
<th>(k)</th>
<th>(x^*_k)</th>
<th>(f(x^*_k))</th>
<th>(\delta, e_i, q, r)</th>
<th>(\bar{x}_{q,r,x}^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 1)</td>
<td>(1.0408, 1.0408)</td>
<td>0.1798</td>
<td>(1/2^4, e_1, 10^9, 1) (1.1241, 1.0408)</td>
</tr>
<tr>
<td>2</td>
<td>(1.1241, 1.0408)</td>
<td>(0.3469, 1.0408)</td>
<td>-0.7890</td>
<td>(1/2^4, e_1, 10^9, 1) (0.7132, 1.0407)</td>
</tr>
<tr>
<td>3</td>
<td>(0.7132, 1.0407)</td>
<td>(-0.0000, 1.0408)</td>
<td>-0.9101</td>
<td>(1/2^4, e_1, 10^9, 1) (0.7515, 1.0405)</td>
</tr>
<tr>
<td>4</td>
<td>(0.7515, 1.0405)</td>
<td>(0.0000, 0.6938)</td>
<td>-1.5156</td>
<td>(1/2^3, e_1, 10^9, 1) (0.7512, 0.6936)</td>
</tr>
<tr>
<td>5</td>
<td>(0.7512, 0.6936)</td>
<td>(1.0 \times 10^{-6}(0.1434, -0.3560))</td>
<td>-2.0000</td>
<td>for any (e_i, i = 1, \ldots, 2n) (q \leq 10^{10}) and (r \geq 10^{-10}) (\bar{x}_{q,r,x}^*)</td>
</tr>
</tbody>
</table>

\[x^*_2 - x^*_5,\] then we obtain other several local minimizers \(x^*_2 - x^*_5\) of problem (5.1). The \(x^*_5\) is the approximate global minimizer of problem (5.1) obtained by Algorithm 1 in request of the precision \(\mu = 10^{-10}\) (since for any \(e_i, q, r\), we can not find better point, the point \(x^*_5\) is the global minimizer).

\(\diamond\) Reference [R1974]: Rastrigin, L., Systems of Extremal Control, Nauka, Moscow, 1974. also ob-
tained the same global minimizer.

**EX2. Two-dimensions Shubert III function** \((n = 2)\) (Shubert, 1972)

\[
\begin{align*}
\min f_S(x) &= \left( \sum_{i=1}^{5} i \cos[(i + 1)x_1 + i] \right) \\
&\quad \cdot \left( \sum_{i=1}^{5} i \cos[(i + 1)x_2 + i] \right) \\
&\quad + \left[ (x_1 + 1.42513)^2 + (x_2 + 0.80032)^2 \right] \\
\text{s.t. } -10 &\leq x_i \leq 10, \ i = 1, 2.
\end{align*}
\]

\((5.2)\)

◊ From Figure 5.4, we see that there are many local minima of this problem (there are about 760 minimums).
Table 2: Results for Shubert III function by QFFM

<table>
<thead>
<tr>
<th>k</th>
<th>$x_0^k$</th>
<th>$x_k^*$</th>
<th>$f(x_k^*)$</th>
<th>$\delta, e_i, q, r$</th>
<th>$x_{q,r,x_k^*}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 1)</td>
<td>(−0.8017, 2.7818)</td>
<td>−25.0600</td>
<td>$1/2^4, e_1, 10^{9}, 1$</td>
<td>(−1.4136, 2.2210)</td>
</tr>
<tr>
<td>2</td>
<td>(−1.4136, 2.2210)</td>
<td>(−1.4251, 2.2950)</td>
<td>−28.0619</td>
<td>$1/2^4, e_1, 10^{9}, 1$</td>
<td>(−1.4251, −0.8003)</td>
</tr>
<tr>
<td>3</td>
<td>(−1.4251, −0.8003)</td>
<td>(−1.4251, −0.8003)</td>
<td>−186.7309</td>
<td>for any $e_i, i = 1, \cdots, 2n$ \ $q \leq 10^{10}$ \ and $r \geq 10^{-10}$</td>
<td>$x_{q,r,x_k^*}$</td>
</tr>
</tbody>
</table>

Table 2 gives the numerical results obtained by Algorithm 1 [2] for the Two-dimensions Shubert III function.

From Table 2, we see that the first local minimizer of problem (5.2) from the first initial point $x_1^0 = (1, 1)^T$ is $x_1^* = (−0.8017, 2.7818)$. By the filled function, we find other two initial points: $x_2^0$ and $x_3^0$, then we obtain other two local minimizers $x_2^*$ and $x_3^*$ of problem (5.2). The $x_3^*$ is the
global minimizer of problem (5.2) obtained by Algorithm 1 in request of the precision $\mu = 10^{-10}$, which is the same as the global minimizer given by other references.

EX3. (Test Problem 14.1.1 in [1])

\[
\begin{align*}
4x_1^3 + 4x_1x_2 + 2x_2^2 - 42x_1 - 14 & = 0 \\
4x_2^3 + 2x_1^2 + 4x_1x_2 - 26x_2 - 22 & = 0 \\
-5 \leq x_1, x_2 & \leq 5 \\
\end{align*}
\] (5.3)

◊ There are 9 known solutions for this nonlinear equations as shown in [1] (Table 3):

Table 3: Known solutions for Example (5.3)

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>-3.7793</th>
<th>-3.0730</th>
<th>-2.8051</th>
<th>-0.2709</th>
<th>-0.1280</th>
<th>0.0867</th>
<th>3.0</th>
<th>3.3852</th>
<th>3.5844</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>-3.2832</td>
<td>-0.0814</td>
<td>3.1313</td>
<td>-0.9230</td>
<td>-1.9537</td>
<td>2.8843</td>
<td>2.0</td>
<td>0.0739</td>
<td>-1.8481</td>
</tr>
</tbody>
</table>
Table 4 records the numerical results of solving Example (5.3) by Algorithm 3 [4].

Table 4: Numerical results for Example 5.3

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>local minimizer $x^*_k$</th>
<th>$f(x^*_k)$</th>
<th>$F(x^*_k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1.0000, 3.0000)</td>
<td>(-2.8046, 3.1308)</td>
<td>3.0123 × 10^{-3}</td>
<td>(3.2415 × 10^{-2}, -4.4289 × 10^{-2})</td>
</tr>
<tr>
<td>1</td>
<td>(-2.8049, 3.1308)</td>
<td>(-2.8051, 3.1313)</td>
<td>3.3845 × 10^{-9}</td>
<td>(3.0661 × 10^{-5}, -3.0454 × 10^{-5})</td>
</tr>
<tr>
<td>0</td>
<td>(-5.0000, -3.0000)</td>
<td>(-0.2707, -0.9232)</td>
<td>3.8212 × 10^{-5}</td>
<td>(-5.9582 × 10^{-3}, 1.6470 × 10^{-3})</td>
</tr>
<tr>
<td>1</td>
<td>(-0.2708, -0.9232)</td>
<td>(-0.2708, -0.9230)</td>
<td>6.0267 × 10^{-13}</td>
<td>(-3.1757 × 10^{-7}, 7.0839 × 10^{-7})</td>
</tr>
</tbody>
</table>

It is clear that two solutions are obtained by our algorithm starting from two different initial points.
EX4. (Test Problem 14.1.2 in [1])

\[
\begin{align*}
    x_1 x_2 + x_1 - 3 x_5 &= 0 \\
    2x_1 x_2 + x_1 + 3 R_{10} x_2^2 + x_2 x_3^2 + R_7 x_2 x_3 \\
    + R_9 x_2 x_4 + R_8 x_2 - R x_5 &= 0 \\
    2x_2 x_3^2 + R_7 x_2 x_3 + 2 R_5 x_3^2 + R_6 x_3 - 8 x_5 &= 0 \\
    R_9 x_2 x_4 + 2 x_4^2 - 4 R x_5 &= 0 \\
    x_1 x_2 + x_1 + R_{10} x_2^2 + x_2 x_3^2 + R_7 x_2 x_3 + x_4^2 \\
    + R_9 x_2 x_4 + R_8 x_2 + R_5 x_3^2 + R_6 x_3 - 1 &= 0 \\
    0.0001 &\leq x_i \leq 100, \ i = 1, \ldots, 5,
\end{align*}
\]

where

\[
\begin{align*}
    R &= 10 \\
    R_5 &= 0.193 \\
    R_6 &= 4.10622 \times 10^{-4} \\
    R_7 &= 5.45177 \times 10^{-4} \\
    R_8 &= 4.4975 \times 10^{-7} \\
    R_9 &= 3.40735 \times 10^{-5} \\
    R_{10} &= 9.615 \times 10^{-7}.
\end{align*}
\]

\[\diamond\] The known solution of Example (5.4) as shown in
[1] is

$$(0.003431, 31.325636, 0.068352, 0.859530, 0.036963)^T.$$  

◊ Table 5 records the numerical results of solving Example (5.4) by Algorithm 3 [4].

◊ Here, we find another different approximate solution for Example (5.4).

Table 5: Numerical results for Example (5.4)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>local minimizer $x_k^*$</th>
<th>$f(x_k^*)$</th>
<th>$F(x_k^*)$</th>
</tr>
</thead>
</table>
| 0   | 26.0000  
    10.0000 × 10^{-5}  
    6.0000  
    10.0000  
    10.0000 | 10.0000 × 10^{-5}  
    10.0000 × 10^{-5}  
    2.5706  
    4.7886  
    0.4109 | 1423.4870 | $\begin{pmatrix} -1.2325 \\ -4.1078 \\ -0.7338 \\ 29.4271 \\ 23.2078 \end{pmatrix}$ |
| 1   | 10.0000 × 10^{-5}  
    10.0000 × 10^{-5}  
    2.5706  
    4.7886  
    0.5119 | $1.1153 \times 10^{-2}$  
    9.3500  
    0.1243  
    0.8579  
    3.6804 × 10^{-2} | $3.3798 \times 10^{-5}$ | $\begin{pmatrix} 5.0180 \times 10^{-3} \\ -2.6706 \times 10^{-3} \\ 1.2161 \times 10^{-3} \\ 9.2324 \times 10^{-6} \\ -8.4303 \times 10^{-5} \end{pmatrix}$ |
EX4. (Test Problem 14.1.3 in [1])

\[
\begin{align*}
10^4 x_1 x_2 - 1 &= 0 \\
\exp(-x_1) + \exp(-x_2) - 1.001 &= 0 \\
5.49 \times 10^{-6} &\leq x_1 \leq 4.553 \\
2.196 \times 10^{-3} &\leq x_2 \leq 18.21
\end{align*}
\]  \tag{5.5}

◊ The known solution of Example (5.5) as shown in [1] is \((1.450 \times 10^{-5}, 6.8933353)\).

◊ Table 6 records the numerical results of solving Example (5.5) by Algorithm 3 [4].

◊ Another two different solutions for Example (5.5) are obtained by Algorithm 3.
Table 6: Numerical results for Example (5.5)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$x_k^*$</th>
<th>$f(x_k^*)$</th>
<th>$F(x_k^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(4.5000, 18.0000)</td>
<td>(5.7220 × 10^{-6}, 16.8750)</td>
<td>1.1847 × 10^{-3}</td>
<td>(−3.4405 × 10^{-2}, −1.0057 × 10^{-3})</td>
</tr>
<tr>
<td></td>
<td>(1.0000, 1.0000)</td>
<td>(2.1964 × 10^{-3}, 2.1960 × 10^{-3})</td>
<td>1.8951</td>
<td>(−0.9518, 0.9946)</td>
</tr>
<tr>
<td>1</td>
<td>(3.2049 × 10^{-5}, 3.1447)</td>
<td>(3.0075 × 10^{-5}, 3.3296)</td>
<td>1.2113 × 10^{-3}</td>
<td>(1.3975 × 10^{-3}, 3.4776 × 10^{-2})</td>
</tr>
<tr>
<td>2</td>
<td>(2.2287 × 10^{-5}, 4.4847)</td>
<td>(2.2132 × 10^{-5}, 4.5175)</td>
<td>9.7922 × 10^{-5}</td>
<td>(−1.7902 × 10^{-4}, 9.8939 × 10^{-3})</td>
</tr>
<tr>
<td>3</td>
<td>(1.5358 × 10^{-5}, 6.5332)</td>
<td>(1.4184 × 10^{-5}, 7.0500)</td>
<td>2.1563 × 10^{-8}</td>
<td>(−2.8178 × 10^{-7}, −1.4684 × 10^{-4})</td>
</tr>
</tbody>
</table>
Figure 5.3: The behavior of **Rastrigin**
Figure 5.4: The behavior of two-dimension Shubert III function
References

Thank You!