ON SYSTEMS OF POLYNOMIALS WITH AT LEAST ONE POSITIVE REAL ZERO

JIE WANG

Abstract. In this paper, we prove several theorems on systems of polynomial equations with at least (exactly) one positive real zero, which can be viewed as some kind of multivariate Descartes' rule. Moreover, we give a class of polynomials attaining minimums, which is useful in polynomial optimization.

1. Introduction

An interesting problem in real algebraic geometry is bounding the numbers of real zeros and positive real zeros of polynomials or polynomial systems. In the univariate case, the well-known Descartes' rule accomplishes this task.

Descartes rule Given a univariate real polynomial $f(x)$ such that the terms of $f(x)$ are ordered by descending variable exponent, the number of positive real roots of f (counted with multiplicity) is bounded from above by the number of sign variations between consecutive nonzero coefficients. Additionally, the difference between these two numbers (the number of positive real roots and the number of sign variations) is even.

However, no complete multivariate generalization of Descartes' rule is known, except for a conjecture proposed by Itenberg and Roy in 1996 ([5]) and subsequently disproven by T.Y. Li in 1998 ([8]). In [6], a special case for polynomial systems with at most one positive real zero was considered through the theory of oriented matroids. Based on this method, a partially multivariate generalization of Descartes' rule for polynomial systems supported on circuits can be found in [2, 3].

In this paper, by virtue of the connection with nonnegative polynomials, we prove several theorems on systems of polynomial equations with at least (exactly) one positive real zero, which can be viewed as some kind of multivariate Descartes' rule. In polynomial optimization, the existence of minimizers of objective polynomials is often formulated as an assumption for some of the algorithmic approaches ([1, 11, 9]). We give a class of polynomials attaining minimums as a byproduct.

2. Preliminaries

2.1. **Nonnegative polynomials.** Let $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \ldots, x_n]$ be the ring of real n-variate polynomial, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Let $\mathbb{R}_+$ be the set of positive real numbers. For a finite set $\mathscr{A} \subset \mathbb{N}^n$, we denote by $\text{conv}(\mathscr{A})$ the convex hull of $\mathscr{A}$, and by $V(\mathscr{A})$ the vertices of the convex hull of $\mathscr{A}$. Also we denote by $V(P)$ the vertex set of

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a polytope P. For a polynomial $f \in \mathbb{R}[\mathbf{x}]$ of the form $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ with $c_{\boldsymbol{\alpha}} \in \mathbb{R}, \mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, the support of f is $\mathrm{supp}(f) := \{\boldsymbol{\alpha} \in \mathscr{A} \mid c_{\boldsymbol{\alpha}} \neq 0\}$ and the Newton polytope of f is defined as $\mathrm{New}(f) = \mathrm{conv}(\mathrm{supp}(f))$. For a polytope P, we use P° to denote the interior of P.

A polynomial $f \in \mathbb{R}[\mathbf{x}]$ which is nonnegative over $\mathbb{R}^n$ is called a *nonnegative polynomial*. A nonnegative polynomial must satisfy the following necessary conditions.

Proposition 2.1. *(*[10, Theorem 3.6]*) Let $\mathscr{A} \subset \mathbb{N}^n$ and $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with* $\mathrm{supp}(f) = \mathscr{A}$*. Then f is nonnegative only if the followings hold:*

(1) $V(\mathscr{A}) \subset (2\mathbb{N})^n$;
(2) If $\boldsymbol{\alpha} \in V(\mathscr{A})$, then the corresponding coefficient $c_{\boldsymbol{\alpha}}$ is positive.

2.2. **Coercive polynomials.** A polynomial $f \in \mathbb{R}[\mathbf{x}]$ is called a *coercive polynomial*, if $f(\mathbf{x}) \to +\infty$ holds whenever $\|\mathbf{x}\| \to +\infty$, where $\|\cdot\|$ denotes some norm on $\mathbb{R}^n$. Obviously the coercivity of f implies the existence of minimizers of f over $\mathbb{R}^n$. Necessary conditions ([1, Theorem 2.8]) and sufficient conditions ([1, Theorem 3.4]) for a polynomial to be coercive were given in [1].

Theorem 2.2. *(*[1, Theorem 2.8]*) Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}]$ with* $\mathrm{supp}(f) = \mathscr{A}$ *be a coercive polynomial and $c_{\mathbf{0}} > 0$. Then the following three conditions hold:*

(1) $V(\mathscr{A}) \subseteq (2\mathbb{N})^n$;
(2) If $\boldsymbol{\alpha} \in V(\mathscr{A})$, then the corresponding coefficient $c_{\boldsymbol{\alpha}}$ is positive;
(3) For every i, $1 \le i \le n$, there exists a vector $2k_i\boldsymbol{e}_i \in V(\mathscr{A})$ with $k_i \in \mathbb{N}^*$, where $\boldsymbol{e}_i$ is the standard basis vector.

2.3. **Circuit polynomials.** A subset $\mathscr{A} \subseteq (2\mathbb{N})^n$ is called a *trellis* if $\mathscr{A}$ comprises the vertices of a simplex.

Definition 2.3. *Let $\mathscr{A}$ be a trellis and $f \in \mathbb{R}[\mathbf{x}]$. Then f is called a* circuit polynomial *if it is of the form*

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}}, \tag{2.1}$$

with $c_{\boldsymbol{\alpha}} > 0$ and $\boldsymbol{\beta} \in \mathrm{conv}(\mathscr{A})^\circ$. Assume

$$\boldsymbol{\beta} = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}}\boldsymbol{\alpha} \text{ with } \lambda_{\boldsymbol{\alpha}} > 0 \text{ and } \sum_{\boldsymbol{\alpha}\in\mathscr{A}} \lambda_{\boldsymbol{\alpha}} = 1. \tag{2.2}$$

For every circuit polynomial f, we define the corresponding circuit number *as $\Theta_f := \prod_{\boldsymbol{\alpha}\in\mathscr{A}}(c_{\boldsymbol{\alpha}}/\lambda_{\boldsymbol{\alpha}})^{\lambda_{\boldsymbol{\alpha}}}$.*

The nonnegativity of a circuit polynomial f is decided by its circuit number alone.

Theorem 2.4. *(*[4, Theorem 3.8]*) Let $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - d\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$ be a circuit polynomial and Θ_f its circuit number. Then f is nonnegative if and only if $\boldsymbol{\beta} \notin (2\mathbb{N})^n$ and $|d| \le \Theta_f$, or $\boldsymbol{\beta} \in (2\mathbb{N})^n$ and $d \le \Theta_f$.*

3. Polynomials attaining minimums

In this section, we prove a theorem on polynomials with global minimizers in $\mathbb{R}_+^n$.

Let $g(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}^{\pm}]$. For an invertible matrix $T \in \mathrm{GL}_n(\mathbb{Q})$, the polynomial obtained by applying T to the exponent vectors of g is denoted by $g^T = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{x}^{T\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}^{\pm}]$.

Let Δ be a polytope of dimension d. For a vertex $\boldsymbol{\alpha}$ of Δ, if $\boldsymbol{\alpha}$ is the intersection of precisely d edges, then we say Δ is *simple* at $\boldsymbol{\alpha}$. Obviously, a polygon is simple at every vertex.

Lemma 3.1. *Suppose* $f = c_{\mathbf{0}} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}^{\pm}]$ *such that* $\dim(\mathrm{New}(f)) = n$, $\mathscr{B} \subseteq \mathrm{New}(f)^{\circ}$, $\mathbf{0} \in V(\mathrm{New}(f))$ *and* $\mathrm{New}(f)$ *is simple at* $\mathbf{0}$. *Then there exists* $\mathscr{A}_0 = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subseteq V(\mathrm{New}(f))$ *and* $T \in \mathrm{GL}_n(\mathbb{Q})$ *such that* $f^T = c_{\mathbf{0}} + \sum_{i=1}^n c_{\boldsymbol{\alpha}_i} x_i^{2k_i} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}\backslash\mathscr{A}_0} c_{\boldsymbol{\alpha}} \mathbf{x}^{T\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{T\boldsymbol{\beta}}$, *where* $k_i \in \mathbb{N}^*$, $T\boldsymbol{\alpha} \in (2\mathbb{N})^n$ *for each* $\boldsymbol{\alpha} \in \mathscr{A}\backslash\mathscr{A}_0$ *and* $T\boldsymbol{\beta} \in \mathrm{New}(f^T)^{\circ} \cap \mathbb{N}^n$ *for each* $\boldsymbol{\beta} \in \mathscr{B}$.

Proof. Since $\mathrm{New}(f)$ is simple at $\mathbf{0}$, $\mathbf{0}$ is the intersection of precisely n edges. Let $\mathscr{A}_0 = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subseteq V(\mathrm{New}(f))$ be the other extreme points of these n edges. Let $T' \in \mathrm{GL}_n(\mathbb{Q})$ such that $T'(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n) = \mathrm{diag}(k'_1, \ldots, k'_n)$, where $k'_i \in \mathbb{N}^*$. Suppose $\mu \in \mathbb{N}^*$ is the least common multiple of the denominators of the coordinates of $T\boldsymbol{\alpha}$ and $T\boldsymbol{\beta}$, for $\boldsymbol{\alpha} \in \mathscr{A}\backslash\mathscr{A}_0$ and $\boldsymbol{\beta} \in \mathscr{B}$. Let $T = 2\mu T'$. Then $T\boldsymbol{\alpha} \in (2\mathbb{Z})^n$ for each $\boldsymbol{\alpha} \in \mathscr{A}\backslash\mathscr{A}_0$ and $T\boldsymbol{\beta} \in \mathbb{Z}^n$ for each $\boldsymbol{\beta} \in \mathscr{B}$. Moreover, since T keeps convex combinations, we have $T\boldsymbol{\alpha} \in (2\mathbb{N})^n$ and $T\boldsymbol{\beta} \in \mathrm{New}(f^T)^{\circ} \cap \mathbb{N}^n$. □

Consider the bijective componentwise exponential function

$$\exp : \mathbb{R}^n \to \mathbb{R}^n_+, \quad \mathbf{x} = (x_1, \ldots, x_n) \mapsto e^{\mathbf{x}} = (e^{\mathbf{x}_1}, \ldots, e^{\mathbf{x}_n}). \tag{3.1}$$

The image of $g(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ under the map exp is $g(e^{\mathbf{x}}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} e^{\langle \boldsymbol{\alpha}, \mathbf{x}\rangle}$, where $\langle \boldsymbol{\alpha}, \mathbf{x}\rangle = \boldsymbol{\alpha}^{\intercal}\mathbf{x}$ is the inner product of $\boldsymbol{\alpha}$ and $\mathbf{x}$. Obviously, the range of $g(\mathbf{x})$ over $\mathbb{R}^n_+$ is same as the range of $g(e^{\mathbf{x}})$ over $\mathbb{R}^n$.

Lemma 3.2. *Let* $g(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{R}[\mathbf{x}^{\pm}]$ *and* $T \in \mathrm{GL}_n(\mathbb{Q})$. *Then the infimums of* $g(\mathbf{x})$ *and* $g^T(\mathbf{x})$ *over* $\mathbb{R}^n_+$ *are the same. The minimizers (or the zeros) of* $g(\mathbf{x})$ *and* $g^T(\mathbf{x})$ *over* $\mathbb{R}^n_+$ *are in a one-to-one correspondence.*

Proof. We only need to show that the same conclusions hold for $g(e^{\mathbf{x}})$ and $g^T(e^{\mathbf{x}})$ over $\mathbb{R}^n$, which follow from the equalities $g(e^{\mathbf{x}}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} e^{\langle \boldsymbol{\alpha}, \mathbf{x}\rangle} = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} e^{\langle T\boldsymbol{\alpha}, T^*\mathbf{x}\rangle} = g^T(e^{T^*\mathbf{x}})$ and $g^T(e^{\mathbf{x}}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} e^{\langle T\boldsymbol{\alpha}, \mathbf{x}\rangle} = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} e^{\langle \boldsymbol{\alpha}, T^{\intercal}\mathbf{x}\rangle} = g(e^{T^{\intercal}\mathbf{x}})$, where $T^* = (T^{-1})^{\intercal} = (T^{\intercal})^{-1}$. □

Lemma 3.3. *Suppose* $f = c_{\mathbf{0}} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}^{\pm}]$, $c_{\mathbf{0}}, c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ *such that* $\dim(\mathrm{New}(f)) = n$, $\mathscr{B} \subseteq \mathrm{New}(f)^{\circ}$, $\mathbf{0} \in V(\mathrm{New}(f))$ *and* $\mathrm{New}(f)$ *is simple at* $\mathbf{0}$. *Assume* $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}}$ *is not nonnegative over* $\mathbb{R}^n_+$. *Then* f *has a minimizer over* $\mathbb{R}^n_+$.

Proof. By Lemma 3.1, there exists $\mathscr{A}_0 = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subseteq V(\mathrm{New}(f))$ and $T \in \mathrm{GL}_n(\mathbb{Q})$ such that $f^T = c_{\mathbf{0}} + \sum_{i=1}^n c_{\boldsymbol{\alpha}_i} x_i^{2k_i} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}\backslash\mathscr{A}_0} c_{\boldsymbol{\alpha}} \mathbf{x}^{T\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} \mathbf{x}^{T\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$. By Theorem 3.4 in [1], f^T is a coercive polynomial, and hence has a global minimizer over $\mathbb{R}^n$. Note that $f^T(|\mathbf{x}|) = c_{\mathbf{0}} + \sum_{i=1}^n c_{\boldsymbol{\alpha}_i} |x_i|^{2k_i} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}\backslash\mathscr{A}_0} c_{\boldsymbol{\alpha}} |\mathbf{x}|^{T\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}} |\mathbf{x}|^{T\boldsymbol{\beta}} \leq f^T(\mathbf{x})$, where $|\mathbf{x}| = (|x_1|, \ldots, |x_n|)$. So we can assume the global minimizer of f^T is in $\mathbb{R}^n_{\geq 0}$. Since $f - c_{\mathbf{0}}$ is not nonnegative over $\mathbb{R}^n_+$, by Lemma 3.2, $f^T - c_{\mathbf{0}}$ is not nonnegative over $\mathbb{R}^n_+$. It follows that the minimum of f^T is lower than $c_{\mathbf{0}}$ and the global minimizer of f^T can not lie in the coordinate axes. Thus f^T has a minimizer over $\mathbb{R}^n_+$ and so does f by Lemma 3.2. □

Theorem 3.4. *Suppose* $f=\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}\in\mathbb{R}[\mathbf{x}]$, $c_{\boldsymbol{\alpha}},d_{\boldsymbol{\beta}}>0$ *such that* $\dim(\mathrm{New}(f))=n$, $\mathscr{A}\subseteq(2\mathbb{N})^n$, $\mathscr{B}\subseteq\mathrm{New}(f)^{\circ}$. *Assume that* $\mathrm{conv}(\mathscr{A}\cup\{\mathbf{0}\})$ *is simple at* $\mathbf{0}$. *If* $\mathbf{0}$ *is not a global minimizer of* f, *then* f *has a global minimizer in* $\mathbb{R}_+^n$.

Proof. Since $f(|\mathbf{x}|)=\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}|\mathbf{x}|^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}|\mathbf{x}|^{\boldsymbol{\beta}}\le f(\mathbf{x})$, we only need to search the global minimizer of f in $\mathbb{R}_{\ge 0}^n$, or equivalently in $\mathbf{0}\cup\mathbb{R}_+^n$. If $\mathbf{0}\in\mathscr{A}$ and $f-c_{\mathbf{0}}$ is nonnegative, then $\mathbf{0}$ is a global minimizer of f. If $\mathbf{0}\in\mathscr{A}$ and $f-c_{\mathbf{0}}$ is not nonnegative, then by Lemma 3.3, f has a minimizer over $\mathbb{R}_+^n$, which is also a global minimizer. If $\mathbf{0}\notin\mathscr{A}$ and f is nonnegative, then $\mathbf{0}$ is a global minimizer of f. If $\mathbf{0}\notin\mathscr{A}$ and f is not nonnegative, consider the polynomial $f+c$, $c>0$. By Lemma 3.3, $f+c$ has a minimizer over $\mathbb{R}_+^n$. It follows f has a minimizer over $\mathbb{R}_+^n$, which is also a global minimizer. □

4. SYSTEMS OF POLYNOMIAL EQUATIONS WITH POSITIVE REAL ZEROS

A *positive real zero* of a polynomial or a system of polynomial equations is a zero with positive coordinates. Note that the positive real zeros of the polynomials $f(x_1,\dots,x_n)$ and $f(x_1^2,\dots,x_n^2)$ are in a one-to-one correspondence. Since we only consider positive real zeros in this paper, we can apply the map $x_i\mapsto x_i^2$ to every polynomial and assume that supports of polynomials appearing in this paper are in $(2\mathbb{N})^n$.

Theorem 4.1. *Let* F *be the following system of polynomial equations*

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0},\tag{4.1}$$

where $c_{\boldsymbol{\alpha}},d_{\boldsymbol{\beta}}>0$ *and* $\boldsymbol{\gamma}\in V(\mathscr{A}\cup\{\boldsymbol{\gamma}\})$, $\mathscr{B}\subseteq\mathrm{conv}(\mathscr{A}\cup\{\boldsymbol{\gamma}\})^{\circ}$. *Let* $\Delta=\mathrm{conv}(\mathscr{A}\cup\{\boldsymbol{\gamma}\})$. *Assume that* $\dim(\Delta)=n$, Δ *is simple at* $\boldsymbol{\gamma}$ *and* $\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ *is not nonnegative over* $\mathbb{R}_+^n$. *Then* F *has at least one positive real zero.*

Proof. Consider the polynomial $f=c_{\boldsymbol{\gamma}}\mathbf{x}^{\boldsymbol{\gamma}}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. Let $f'=f/\mathbf{x}^{\boldsymbol{\gamma}}=c_{\boldsymbol{\gamma}}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\gamma}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\gamma}}$. Then by Lemma 3.3, f' has a minimizer over $\mathbb{R}_+^n$. Assume the minimum of f' over $\mathbb{R}_+^n$ is ξ. Then $f'(\mathbf{x})-\xi$ is nonnegative over $\mathbb{R}_+^n$ and has a positive real zero. It follows that $f-\xi\mathbf{x}^{\boldsymbol{\gamma}}=(c_{\boldsymbol{\gamma}}-\xi)\mathbf{x}^{\boldsymbol{\gamma}}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is nonnegative over $\mathbb{R}_+^n$ and has a positive real zero, which implies that the system of $f-\xi\mathbf{x}^{\boldsymbol{\gamma}}=0$ and $\nabla(f-\xi\mathbf{x}^{\boldsymbol{\gamma}})=\mathbf{0}$ has a positive real zero. Multiplying $f-\xi\mathbf{x}^{\boldsymbol{\gamma}}=0$ by $\boldsymbol{\gamma}$ gives

$$(c_{\boldsymbol{\gamma}}-\xi)\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}.\tag{4.2}$$

Multiplying the i-th equation of $\nabla(f-\xi\mathbf{x}^{\boldsymbol{\gamma}})=\mathbf{0}$ by x_i gives

$$(c_{\boldsymbol{\gamma}}-\xi)\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}.\tag{4.3}$$

By (4.3)−(4.2), we obtain

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}}c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}}d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0},\tag{4.4}$$

which is F. Thus F has a positive real zero. □

Example 4.2. *The following system of polynomial equations satisfies the conditions of Theorem 4.1 with $\boldsymbol{\gamma} = (8, 8)$, and hence has a positive real zero.*

$$\begin{cases} -8y^8 - 4x^4y^4 - 8 + 18x^2y + 5x^3y^2 = 0 \\ -8x^8 - 4x^4y^4 - 8 + 21x^2y + 6x^3y^2 = 0 \end{cases}$$

Lemma 4.3. *Suppose $f_d = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + d\mathbf{x}^{\boldsymbol{\gamma}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \in \mathbb{R}[\mathbf{x}]$, $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ such that $\mathscr{B} \subseteq \operatorname{conv}(\mathscr{A})^\circ$ and $\boldsymbol{\gamma} \in \operatorname{conv}(\mathscr{A})$. Assume that $\operatorname{New}(f_d)$ is simple at $\boldsymbol{\alpha}_0 \in V(\mathscr{A})$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is not nonnegative over $\mathbb{R}^n_+$. Let $d^* = \inf\{d \mid f_d$ is nonnegative over $\mathbb{R}^n_+\}$. Then f_{d^*} has a positive real zero.*

Proof. Let $|\mathscr{B}| = l$. For each $\boldsymbol{\beta} \in \mathscr{B}$, since $\boldsymbol{\beta} \subseteq \operatorname{conv}(\mathscr{A})^\circ$, then there must exist a subset $A_{\boldsymbol{\beta}}$ of $\mathscr{A}$ such that $A_{\boldsymbol{\beta}} \cup \{\boldsymbol{\gamma}\}$ comprises the vertices of a simplex $\Delta_{\boldsymbol{\beta}}$ containing $\boldsymbol{\beta}$ as an interior point. For each $\boldsymbol{\alpha} \in \cup_{\boldsymbol{\beta}\in\mathscr{B}} A_{\boldsymbol{\beta}}$, count how many simplices contain $\boldsymbol{\alpha}$ and evenly distribute $c_{\boldsymbol{\alpha}}$. Then we can write

$$f_d = \sum_{\boldsymbol{\beta}\in\mathscr{B}} \Big(\sum_{\boldsymbol{\alpha}\in A_{\boldsymbol{\beta}}} c_{\boldsymbol{\beta}\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + \frac{d}{l}\mathbf{x}^{\boldsymbol{\gamma}} - d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} \Big) + \sum_{\boldsymbol{\alpha}\notin\cup_{\boldsymbol{\beta}\in\mathscr{B}} A_{\boldsymbol{\beta}}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} \tag{4.5}$$

as a sum of circuit polynomials. If d is large enough, then every circuit polynomial in (4.5) is nonnegative and hence f is nonnegative. So d^* exists.

Let $f'_d = f_d/\mathbf{x}^{\boldsymbol{\alpha}_0} = c_{\boldsymbol{\alpha}_0} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}\setminus\{\boldsymbol{\alpha}_0\}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}_0} + d\mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\alpha}_0} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\alpha}_0}$. By Lemma 3.1, there exists $\mathscr{A}_0 = \{\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n\} \subseteq V(\mathscr{A})\setminus\{\boldsymbol{\alpha}_0\}$ and $T \in \mathrm{GL}_n(\mathbb{Q})$ such that $f'^T_d = c_{\boldsymbol{\alpha}_0} + \sum_{i=1}^n c_{\boldsymbol{\alpha}_i} x_i^{2k_i} + \sum_{\boldsymbol{\alpha}\in\mathscr{A}\setminus(\mathscr{A}_0\cup\{\boldsymbol{\alpha}_0\})} c_{\boldsymbol{\alpha}}\, \mathbf{x}^{T(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)} + d\mathbf{x}^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{T(\boldsymbol{\beta}-\boldsymbol{\alpha}_0)} \in \mathbb{R}[\mathbf{x}]$. By Lemma 3.2, the nonnegativity of f'^T_d over $\mathbb{R}^n_+$ is the same as the nonnegativity of f'_d over $\mathbb{R}^n_+$, and hence is the same as the nonnegativity of f_d over $\mathbb{R}^n_+$. Let $d < d^*$ and by the definition of d^*, f'^T_d is not nonnegative over $\mathbb{R}^n_+$. That is to say, there exists $\mathbf{x}_d \in \mathbb{R}^n_+$ such that $f'^T_d(\mathbf{x}_d) < 0$. On the other hand, by Theorem 3.4 in [1], f'^T_d is a coercive polynomial. Hence there exists $N_d > 0$ such that for $\|\mathbf{x}\|_\infty > N_d$, $f'^T_d(\mathbf{x}) > 0$. It follows $\|\mathbf{x}_d\|_\infty \le N_d$. Let $d' > d$. Since $f'^T_{d'}(\mathbf{x}) - f'^T_d(\mathbf{x}) = (d'-d)\mathbf{x}^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)} > 0$ over $\mathbb{R}^n_+$, we have $f'^T_{d'}(\mathbf{x}) > f'^T_d(\mathbf{x})$ over $\mathbb{R}^n_+$. Thus for $\|\mathbf{x}\|_\infty > N_d$ and $\mathbf{x} \in \mathbb{R}^n_+$, $f'^T_{d'}(\mathbf{x}) > 0$. It follows $\|\mathbf{x}_{d'}\|_\infty \le N_d$. Let $d \to d^*$. Then we have $f'^T_d(\mathbf{x}_d) - f'^T_{d^*}(\mathbf{x}_d) = (d - d^*)\mathbf{x}_d^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)} \to 0$. Since $f'^T_{d^*}(\mathbf{x}_d) \ge 0$, we must have $f'^T_{d^*}(\mathbf{x}_d) \to 0$. Thus the infimum of f'_{d^*} over $\mathbb{R}^n_+$ is 0. It follows that $f'^T_{d^*} - c_{\boldsymbol{\alpha}_0}$ is not nonnegative over $\mathbb{R}^n_+$. So by Lemma 3.3, $f'^T_{d^*}$ has a minimizer over $\mathbb{R}^n_+$, which is a positive real zero. As a consequence, f_{d^*} has a positive real zero by Lemma 3.2. $\square$

Theorem 4.4. *Let F be the following system of polynomial equations*

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0}, \tag{4.6}$$

where $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ and $\mathscr{B} \subseteq \operatorname{conv}(\mathscr{A})^\circ$, $\boldsymbol{\gamma} \in \operatorname{conv}(\mathscr{A})$. Let $\Delta = \operatorname{conv}(\mathscr{A})$. Assume that $\dim(\Delta) = n$, Δ is simple at $\boldsymbol{\alpha}_0 \in V(\mathscr{A})$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is not nonnegative over $\mathbb{R}^n_+$. Then F has at least one positive real zero.

Proof. Consider the polynomial $f_d = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} + d\mathbf{x}^{\boldsymbol{\gamma}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. Define d^* as in Lemma 4.3. Then by Lemma 4.3, f_{d^*} has a positive real zero as a minimizer, which implies that the system of $f_{d^*} = 0$ and $\nabla(f_{d^*}) = \mathbf{0}$ has a positive real zero.

Multiplying $f_{d^*}=0$ by $\boldsymbol{\gamma}$ gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\alpha}}+d^*\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}. \tag{4.7}$$

Multiplying the i-th equation of $\nabla(f_{d^*})=\mathbf{0}$ by x_i gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\mathbf{x}^{\boldsymbol{\alpha}}+d^*\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}. \tag{4.8}$$

By (4.8)−(4.7), we obtain

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}, \tag{4.9}$$

which is F. Thus F has a positive real zero. □

Example 4.5. *The following system of polynomial equations satisfies the conditions of Theorem 4.4 with $\boldsymbol{\gamma}=(4,4)$, and hence has a positive real zero.*

$$\begin{cases}4x^8y^8+4x^8-4y^8-4+6x^2y+x^3y^2=0\\ 4x^8y^8-4x^8+4y^8-4+9x^2y+2x^3y^2=0\end{cases}$$

Lemma 4.6. *Suppose $f_d=\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}-d\mathbf{x}^{\boldsymbol{\gamma}}\in\mathbb{R}[\mathbf{x}]$, $c_{\boldsymbol{\alpha}},d_{\boldsymbol{\beta}}>0$ such that $\mathscr{B}\cup\{\boldsymbol{\gamma}\}\subseteq\operatorname{conv}(\mathscr{A})^\circ$. Assume that $\operatorname{New}(f_d)$ is simple at $\boldsymbol{\alpha}_0\in V(\mathscr{A})$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is nonnegative over $\mathbb{R}_+^n$. Let $d^*=\sup\{d\mid f_d$ is nonnegative over $\mathbb{R}_+^n\}$. Then f_{d^*} has a positive real zero.*

Proof. Obviously, d^* exists. Let $f_d'=f_d/\mathbf{x}^{\boldsymbol{\alpha}_0}=c_{\boldsymbol{\alpha}_0}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}\backslash\{\boldsymbol{\alpha}_0\}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\alpha}_0}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}-\boldsymbol{\alpha}_0}-d\mathbf{x}^{\boldsymbol{\gamma}-\boldsymbol{\alpha}_0}$. By Lemma 3.1, there exists $\mathscr{A}_0=\{\boldsymbol{\alpha}_1,\ldots,\boldsymbol{\alpha}_n\}\subseteq V(\mathscr{A})\backslash\{\boldsymbol{\alpha}_0\}$ and $T\in\mathrm{GL}_n(\mathbb{Q})$ s.t. $f_d'^T=c_{\boldsymbol{\alpha}_0}+\sum_{i=1}^n c_{\boldsymbol{\alpha}_i}x_i^{2k_i}+\sum_{\boldsymbol{\alpha}\in\mathscr{A}\backslash(\mathscr{A}_0\cup\{\boldsymbol{\alpha}_0\})} c_{\boldsymbol{\alpha}}\mathbf{x}^{T(\boldsymbol{\alpha}-\boldsymbol{\alpha}_0)}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{T(\boldsymbol{\beta}-\boldsymbol{\alpha}_0)}-d\mathbf{x}^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)}\in\mathbb{R}[\mathbf{x}]$. By Lemma 3.2, the nonnegativity of $f_d'^T$ over $\mathbb{R}_+^n$ is the same as the nonnegativity of f_d' over $\mathbb{R}_+^n$, and hence is the same as the nonnegativity of f_d over $\mathbb{R}_+^n$. Let $d>d^*$ and by the definition of d^*, $f_d'^T$ is not nonnegative over $\mathbb{R}_+^n$. That is to say, there exists $\mathbf{x}_d\in\mathbb{R}_+^n$ such that $f_d'^T(\mathbf{x}_d)<0$. On the other hand, by Theorem 3.4 in [1], $f_d'^T$ is a coercive polynomial. Hence there exists $N_d>0$ such that for $\|\mathbf{x}\|_\infty>N_d$, $f_d'^T(\mathbf{x})>0$. It follows $\|\mathbf{x}_d\|_\infty\le N_d$. Let $d'<d$. Since $f_{d'}'^T(\mathbf{x})-f_d'^T(\mathbf{x})=(d-d')\mathbf{x}^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)}>0$ over $\mathbb{R}_+^n$, we have $f_{d'}'^T(\mathbf{x})>f_d'^T(\mathbf{x})$ over $\mathbb{R}_+^n$. Thus for $\|\mathbf{x}\|_\infty>N_d$ and $\mathbf{x}\in\mathbb{R}_+^n$, $f_{d'}'^T(\mathbf{x})>0$. It follows $\|\mathbf{x}_{d'}\|_\infty\le N_d$. Let $d\to d^*$. Then we have $f_d'^T(\mathbf{x}_d)-f_{d^*}'^T(\mathbf{x}_d)=(d^*-d)\mathbf{x}_d^{T(\boldsymbol{\gamma}-\boldsymbol{\alpha}_0)}\to 0$. Since $f_{d^*}'^T(\mathbf{x}_d)\ge 0$, we must have $f_{d^*}'^T(\mathbf{x}_d)\to 0$. Thus the infimum of f_{d^*}' over $\mathbb{R}_+^n$ is 0. It follows that $f_{d^*}'^T-c_{\boldsymbol{\alpha}_0}$ is not nonnegative over $\mathbb{R}_+^n$. So by Lemma 3.3, $f_{d^*}'^T$ has a minimizer over $\mathbb{R}_+^n$, which is a positive real zero. As a consequence, f_{d^*} has a positive real zero by Lemma 3.2. □

Theorem 4.7. *Let F be the following system of polynomial equations*

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}}=\mathbf{0}, \tag{4.10}$$

where $c_{\boldsymbol{\alpha}},d_{\boldsymbol{\beta}}>0$ and $\mathscr{B}\cup\{\boldsymbol{\gamma}\}\subseteq\operatorname{conv}(\mathscr{A})^\circ$. Let $\Delta=\operatorname{conv}(\mathscr{A})$. Assume that $\dim(\Delta)=n$, $\operatorname{conv}(\Delta)$ is simple at $\boldsymbol{\alpha}_0\in V(\mathscr{A})$ and $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}-\sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is nonnegative over $\mathbb{R}_+^n$ and has no zeros in $\mathbb{R}_+^n$. Then F has at least one positive real *zero. Moreover, assume that all $\boldsymbol{\beta}$'s and $\boldsymbol{\gamma}$ lie in the same side of every hyperplane determined by points among $\mathscr{A}$. Then F has exactly one positive real zero.*

Proof. Consider the polynomial $f_d = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}} - d\mathbf{x}^{\boldsymbol{\gamma}}$. Define d^* as in Lemma 4.6. Then $d^* > 0$ and by Lemma 4.6, f_{d^*} has a positive real zero as a minimizer, which implies that the system of $f_{d^*} = 0$ and $\nabla(f_{d^*}) = \mathbf{0}$ has a positive real zero. Multiplying $f_{d^*} = 0$ by $\boldsymbol{\gamma}$ gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\beta}} - d^*\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{0}. \tag{4.11}$$

Multiplying the i-th equation of $\nabla(f_{d^*}) = \mathbf{0}$ by x_i gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} - d^*\boldsymbol{\gamma}\mathbf{x}^{\boldsymbol{\gamma}} = \mathbf{0}. \tag{4.12}$$

By (4.12)−(4.11), we obtain

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}(\boldsymbol{\beta}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0}, \tag{4.13}$$

which is F. Thus F has a positive real zero.

If all $\boldsymbol{\beta}$'s and $\boldsymbol{\gamma}$ lie in the same side of every hyperplane determined by points among $\mathscr{A}$, then by Corollary 4.4 in [12], f_{d^*} is a sum of nonnegative circuit polynomials and has exactly one positive real zero. Suppose $\mathbf{x}_*$ is a positive real zero of F. Then $\mathbf{x}_*$ is also a positive real zero of f_{d^*}. Thus $\mathbf{x}_*$ is unique. □

Example 4.8. *The following system of polynomial equations satisfies the conditions of Theorem 4.7 with $\boldsymbol{\gamma} = (2,1)$, and hence has exactly one positive real zero.*

$$\begin{cases} 6x^8y^8 + 6x^8 - 2y^8 + 2x^4y^4 - 2 - x^3y^2 = 0 \\ 6x^8y^8 - x^8 + 7y^8 + 3x^4y^4 - 1 - x^3y^2 = 0 \end{cases}$$

Finally, the remaining case that $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is nonnegative and has a zero in $\mathbb{R}_+^n$ is easy.

Theorem 4.9. *Let F be the following system of polynomial equations*

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0}, \tag{4.14}$$

where $c_{\boldsymbol{\alpha}}, d_{\boldsymbol{\beta}} > 0$ and $\mathscr{B} \subseteq \operatorname{conv}(\mathscr{A})^\circ$. Assume $\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$ is nonnegative over $\mathbb{R}_+^n$ and has a zero in $\mathbb{R}_+^n$. Then F has at least one positive real zero.

Proof. Consider the polynomial $f = \sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\mathbf{x}^{\boldsymbol{\beta}}$. Since f is nonnegative and has a zero in $\mathbb{R}_+^n$, $\nabla(f) = \mathbf{0}$ has a positive real zero. Multiplying the i-th equation of $\nabla(f) = \mathbf{0}$ by x_i gives

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\mathbf{x}^{\boldsymbol{\alpha}} - \sum_{\boldsymbol{\beta}\in\mathscr{B}} d_{\boldsymbol{\beta}}\boldsymbol{\beta}\mathbf{x}^{\boldsymbol{\beta}} = \mathbf{0}, \tag{4.15}$$

which is F. Thus F has a positive real zero. □

Remark 4.10. *Birch's theorem ([6]) states that the following system of polynomial equations*

$$\sum_{\boldsymbol{\alpha}\in\mathscr{A}} c_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}-\boldsymbol{\gamma})\mathbf{x}^{\boldsymbol{\alpha}} = \mathbf{0}, \tag{4.16}$$

where $c_{\boldsymbol{\alpha}} > 0$ *for* $\boldsymbol{\alpha} \in \mathscr{A}$, $\boldsymbol{\gamma} \in \mathrm{conv}(\mathscr{A})^{\circ}$, $\dim(\mathrm{conv}(\mathscr{A})) = n$, *has exactly one positive real zero. Our theorems hence can be viewed as a generalization of Birch's theorem.*

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JIE WANG, SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY
E-mail address: wangjie212@pku.edu.cn