# Spine decomposition for branching Markov processes and its applications 

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#### Abstract

In the literature, the spine decomposition of branching Markov processes was constructed under the assumption that each individual has at least one child. In this paper, we give a detailed construction of the spine decomposition of general branching Markov processes allowing the possibility of no offspring when a particle dies. Then we give some applications of the spine decomposition.


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## 1 Introduction

The spine decomposition theorem is a very useful tool in studying the asymptotic behaviors of various branching models. This method was first introduced in [20] for GaltonWatson processes to give probabilistic proofs of the Kesten-Stigum $L \log L$ theorem in supercritical case and results on the rate of decay of the survival probability in the critical and subcritical cases. Since then, this method has been generalized to various other models with the branching property, see [1, 4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 23, 24, 27, for instance.

The spine decomposition theorems for Galton-Watson processes and superprocesses are pretty satisfactory. However, the spine decomposition theorem for branching Markov processes was only proved under the assumption that each individual has at least one

[^0]child, see [10, 18]. In [21, 26], the spine decomposition theorem for branching Markov processes without the assumption above about the offspring distribution was used, but no detailed construction of the spine decomposition was given. The purpose of this paper is to give a detailed construction of the spine decomposition for branching Markov processes without assuming that each individual has at least one child. We also do not assume that the branching Markov process is supercritical, so our spine decomposition works also in the critical and subcritical case. We also give some applications of the spine decomposition theorem.

We now introduce the setup of this paper. We always assume that $E$ is a locally compact separable metric space and $m$ is a $\sigma$-finite Borel measure on $E$ with full support. We will use $E_{\Delta}:=E \cup\{\Delta\}$ to denote the one-point compactification of $E$. We will use $\mathcal{B}(E)$ and $\mathcal{B}\left(E_{\Delta}\right)$ to denote the Borel $\sigma$-fields on $E$ and $E_{\Delta}$ respectively. $\mathcal{B}_{b}(E)$ (respectively, $\mathcal{B}^{+}(E)$ ) will denote the set of all bounded (respectively, non-negative) $\mathcal{B}(E)$ measurable functions on $E$. All functions $f$ on $E$ will be automatically extended to $E_{\Delta}$ by setting $f(\Delta)=0$. Let $\mathbf{M}_{p}(E)$ be the space of finite point measures on $E$, that is, measures of the form $\mu=\sum_{i=1}^{n} \delta_{x_{i}}$ where $n=0,1,2, \ldots$ and $x_{i} \in E, i=1, \ldots, n$. (When $n=0, \mu$ is the trivial zero measure.) For any function $f$ on $E$ and any measure $\mu \in \mathbf{M}_{p}(E)$, we use $\langle f, \mu\rangle$ or $\mu(f)$ to denote the integral of $f$ with respect to $\mu$.

We will always assume that $Y=\left\{Y_{t}, \Pi_{x}, \zeta\right\}$ is a Hunt process on $E$ with reference measure $m$, where $\zeta=\inf \left\{t>0: Y_{t}=\Delta\right\}$ is the lifetime of $Y$. Let $\left\{P_{t}, t \geq 0\right\}$ be the transition semigroup of $Y$ :

$$
P_{t} f(x)=\Pi_{x}\left[f\left(Y_{t}\right)\right] \quad \text { for } f \in \mathcal{B}^{+}(E)
$$

$\left\{P_{t}, t \geq 0\right\}$ can be extended to a strongly continuous semigroup on $L^{2}(E, m)$.
Consider a branching system determined by the following three parameters:
(a) a Hunt process $Y=\left\{Y_{t}, \Pi_{x}, \zeta\right\}$ with state space $E$;
(b) a nonnegative bounded Borel function $\beta$ on $E$;
(c) offspring distributions $\left\{\left(p_{n}(x)\right)_{n=0}^{\infty} ; x \in E\right\}$, such that, for each $n \geq 0$, the function $p_{n}(x)$ is Borel.

Put

$$
\begin{equation*}
\psi(x, z)=\sum_{n=0}^{\infty} p_{n}(x) z^{n}, \quad z \geq 0 \tag{1.1}
\end{equation*}
$$

$\psi(x, \cdot)$ is the generating function of the distribution $\left(p_{n}(x)\right)_{n=0}^{\infty}$.
The branching Hunt process is characterized by the following properties:
(i) Each particle has a random birth and a random death time.
(ii) Given that a particle is born at $x \in E$, the conditional distribution of its path after birth is determined by $\Pi_{x}$.
(iii) Given the path $Y$ of a particle and given that the particle is alive at time $t$, its probability of dying in the interval $[t, t+\mathrm{d} t)$ is $\beta\left(Y_{t}\right) \mathrm{d} t+o(\mathrm{~d} t)$.
(iv) When a particle dies at $x \in E$, it splits into $n$ particles at $x$ with probability $p_{n}(x)$.
(v) The point $\Delta$ is a cemetery. When a particle reaches $\Delta$, it stays at $\Delta$ forever and there is no branching at $\Delta$.

In this paper, to avoid triviality, we always assume that $m(\{x \in E, \beta(x)>0\})>0$. We assume that $A(x):=\psi^{\prime}(x, 1)=\sum_{n=0}^{\infty} n p_{n}(x)$ is bounded.

For any $c \in \mathcal{B}_{b}(E)$, we define

$$
e_{c}(t)=\exp \left(-\int_{0}^{t} c\left(Y_{s}\right) \mathrm{d} s\right)
$$

Let $X_{t}(B)$ be the number of particles which are alive and located in $B \in \mathcal{B}(E)$ at time $t$. A particle which dies at time $t$ is not counted in $X_{t}(B)$ even if the death location is in $B$. $\left\{X_{t}, t \geq 0\right\}$ is a Markov process in $\mathbf{M}_{p}(E)$. This process is called a $(Y, \beta, \psi)$-branching Hunt process. For any $\mu \in \mathbf{M}_{p}(E)$, let $\mathbf{P}_{\mu}$ be the law of $\left\{X_{t}, t \geq 0\right\}$ when $X_{0}=\mu$. Then we have

$$
\begin{equation*}
\mathbf{P}_{\mu} \exp \left\langle-f, X_{t}\right\rangle=\exp \left\langle\log u_{t}(\cdot), \mu\right\rangle \tag{1.2}
\end{equation*}
$$

where $u_{t}(x)$ satisfies the equation

$$
\begin{equation*}
u_{t}(x)=\Pi_{x}\left[e_{\beta}(t) \exp \left(-f\left(Y_{t}\right)\right)+\int_{0}^{t} e_{\beta}(s) \beta\left(Y_{s}\right) \psi\left(Y_{s}, u_{t-s}\left(Y_{s}\right)\right) \mathrm{d} s\right] \quad \text { for } t \geq 0 \tag{1.3}
\end{equation*}
$$

The formula (1.3) deals with a process started at time 0 with one particle located at $x$, and it has a clear heuristic meaning: the first term in the brackets corresponds to the case when the particle is still alive at time $t$; the second term corresponds to the case when it dies before $t$. The formula (1.3) implies that

$$
\begin{equation*}
u_{t}(x)=\Pi_{x} \int_{0}^{t}\left[\psi\left(Y_{s}, u_{t-s}\left(Y_{s}\right)\right)-u_{t-s}\left(Y_{s}\right)\right] \beta\left(Y_{s}\right) \mathrm{d} s+\Pi_{x} \exp \left(-f\left(Y_{t}\right)\right) \quad \text { for } t \geq 0 \tag{1.4}
\end{equation*}
$$

(see [8, Section 2.3]). For any $\mu \in \mathbf{M}_{p}(E), f \in \mathcal{B}_{b}^{+}(E)$ and $t \geq 0$, we have

$$
\begin{equation*}
\mathbf{P}_{\mu}\left[\left\langle f, X_{t}\right\rangle\right]=\Pi_{\mu}\left[e_{(1-A) \beta}(t) f\left(Y_{t}\right)\right] . \tag{1.5}
\end{equation*}
$$

Let $\left\{P_{t}^{(1-A) \beta}, t \geq 0\right\}$ be the Feynman-Kac semigroup defined by

$$
P_{t}^{(1-A) \beta} f(x):=\Pi_{x}\left[e_{(1-A) \beta}(t) f\left(Y_{t}\right)\right], \quad f \in \mathcal{B}(E) .
$$

Throughout this paper we assume that

Assumption 1.1 There exist a strictly positive Borel function $\phi$ and a constant $\lambda_{1} \in$ $(-\infty, \infty)$ such that

$$
\begin{equation*}
\phi(x)=e^{-\lambda_{1} t} P_{t}^{(1-A) \beta} \phi(x), \quad x \in E . \tag{1.6}
\end{equation*}
$$

Let $\mathcal{E}_{t}=\sigma\left(Y_{s} ; s \leq t\right)$. Note that

$$
\frac{\phi\left(Y_{t}\right)}{\phi(x)} e^{-\lambda_{1} t} e_{(1-A) \beta}(t), \quad t \geq 0
$$

is a martingale under $\Pi_{x}$, and so we can define a martingale change of measure by

$$
\begin{equation*}
\left.\frac{\mathrm{d} \Pi_{x}^{\phi}}{\mathrm{d} \Pi_{x}}\right|_{\mathcal{E}_{t}}=\frac{\phi\left(Y_{t}\right)}{\phi(x)} e^{-\lambda_{1} t} e_{(1-A) \beta}(t) \tag{1.7}
\end{equation*}
$$

For any nonzero measure $\mu \in \mathbf{M}_{p}(E)$, we define

$$
M_{t}(\phi):=e^{-\lambda_{1} t} \frac{\left\langle\phi, X_{t}\right\rangle}{\langle\phi, \mu\rangle} \quad t \geq 0
$$

Lemma 1.2 For any nonzero measure $\mu \in \mathbf{M}_{p}(E),\left\{M_{t}(\phi), t \geq 0\right\}$ is a non-negative martingale under $\mathbf{P}_{\mu}$, and therefore there exists a limit $M_{\infty}(\phi) \in[0, \infty), \mathbf{P}_{\mu}$-a.s.

Proof. By the Markov property of $X$, (1.5) and (1.6),

$$
\begin{aligned}
\mathbf{P}_{\mu}\left[M_{t+s}(\phi) \mid \mathcal{F}_{t}\right] & =\frac{1}{\langle\phi, \mu\rangle} e^{-\lambda_{1} t} \mathbf{P}_{X_{t}}\left[e^{-\lambda_{1} s}\left\langle\phi, X_{s}\right\rangle\right] \\
& =\frac{1}{\langle\phi, \mu\rangle} e^{-\lambda_{1} t}\left\langle e^{-\lambda_{1} s} \Pi .\left[e_{(1-A) \beta}(s) \phi\left(Y_{s}\right)\right], X_{t}\right\rangle \\
& =\frac{1}{\langle\phi, \mu\rangle} e^{-\lambda_{1} t}\left\langle\phi, X_{t}\right\rangle=M_{t}(\phi)
\end{aligned}
$$

This proves that $\left\{M_{t}(\phi), t \geq 0\right\}$ is a non-negative $\mathbf{P}_{\mu}$-martingale and so it has an almost sure limit $M_{\infty}(\phi) \in[0, \infty)$ as $t \rightarrow \infty$.

It follows from the branching property that when $\mu \in \mathbf{M}_{p}(E)$ is given by $\mu=$ $\sum_{i=1}^{n} \delta_{x_{i}}, n=1,2, \ldots,\left\{x_{i} ; i=1, \cdots, n\right\} \subset E$, we have

$$
M_{t}(\phi)=\sum_{i=1}^{n} e^{-\lambda_{1} t} \frac{\left\langle\phi^{t}, X_{t}^{i}\right\rangle}{\phi\left(x_{i}\right)} \cdot \frac{\phi\left(x_{i}\right)}{\langle\phi, \mu\rangle},
$$

where, for each $i=1, \ldots, n,\left\{X_{t}^{i}, t \geq 0\right\}$ is a branching Hunt process starting from $\delta_{x_{i}}$. If a certain assertion holds under $\mathbf{P}_{\delta_{x}}$ for all $x \in E$, then it also holds for general $\mu$. So in the remainder of this paper, we assume that the initial measure is of the form $\mu=\delta_{x}, x \in E$, and $\mathbf{P}_{\delta_{x}}$ will be denoted as $\mathbf{P}_{x}$.

## 2 Spine decomposition

Let $\mathbb{N}=\{1,2, \ldots\}$. We will use

$$
\Gamma:=\bigcup_{n=0}^{\infty} \mathbb{N}^{n}
$$

(where $\mathbb{N}^{0}=\{\emptyset\}$ ) to describe the genealogical structure of our branching Hunt process. The length (or generation) $|u|$ of each $u \in \mathbb{N}^{n}$ is defined to be $n$. When $n \geq 1$ and $u=\left(u_{1}, \ldots, u_{n}\right)$, we denote $\left(u_{1}, \ldots, u_{n-1}\right)$ by $u-1$ and call it the parent of $u$. For each $i \in \mathbb{N}$ and $u=\left(u_{1}, \ldots, u_{n}\right)$, we write $u i=\left(u_{1}, \ldots, u_{n}, i\right)$ for the $i$-th child of $u$. More generally, for $u=\left(u_{1}, \ldots, u_{n}\right), v=\left(v_{1}, \ldots, v_{m}\right) \in \Gamma$, we will use $u v$ to stand for the concatenation $\left(u_{1}, \ldots, u_{n}, v_{1}, \ldots, v_{m}\right)$ of $u$ and $v$. We will use the notation $v<u$ to mean that $v$ is an ancestor of $u$. The set of all ancestors of $u$ is given by $\{v \in \Gamma: v<u\}=$ $\{v \in \Gamma: \exists w \in \Gamma \backslash\{\emptyset\}$ such that $v w=u\}$. The notation $v \leq u$ has the obvious meaning that either $v<u$ or $v=u$.

A subset $\tau \subset \Gamma$ is called a Galton-Watson tree if a) $\emptyset \in \tau$; b) if $u, v \in \Gamma$, then $u v \in \tau$ implies $u \in \tau$; c) for all $u \in \tau$, there exists $r^{u} \in \mathbb{N} \cup\{0\}$ such that when $j \in \mathbb{N}, u j \in \tau$ if and only if $1 \leq j \leq r^{u}$. We will denote the collection of Galton-Watson trees by $\mathbb{T}$. Each $u \in \tau$ is called a node of $\tau$ or an individual in $\tau$ or just a particle.

To fully describe the branching Hunt process $X$, we need to introduce the concept of marked Galton-Watson trees. We suppose that each individual $u \in \tau$ has a mark $\left(Y^{u}, \sigma^{u}, r^{u}\right)$ where:
(i) $\sigma^{u}$ is the lifetime of $u$, which, along with the lifetimes of its ancestors, determines the fission time or the death time of the particle $u$ as $\zeta^{u}=\sum_{v \leq u} \sigma^{v}\left(\zeta^{\emptyset}=\sigma^{\emptyset}\right)$, and the birth time of $u$ as $b^{u}=\sum_{v<u} \sigma^{v}\left(b^{\emptyset}=0\right)$;
(ii) $Y^{u}:\left[b^{u}, \zeta^{u}\right] \rightarrow E_{\Delta}$ gives the location of $u$ and $Y_{b^{u}}^{u}=Y_{\zeta^{u-1}}^{u-1}$.
(iii) $r^{u}$ gives the number of the offspring of $u$ when it dies. $r^{u}$ depends on $Y_{\zeta_{u}}^{u}$ in general.

We will use $(\tau, Y, \sigma, r)$ (or simply $(\tau, M)$ ) to denote a marked Galton-Watson tree. We denote the set of all marked Galton-Watson trees by $\mathcal{T}=\{(\tau, M): \tau \in \mathbb{T}\}$.

Define

$$
\begin{aligned}
\mathcal{F}_{t}:= & \sigma\left\{\left[u, r^{u}, \sigma^{u},\left(Y_{s}^{u}, s \in\left[b^{u}, \zeta^{u}\right]\right): u \in \tau \in \mathbb{T} \text { with } \zeta^{u} \leq t\right]\right. \text { and } \\
& {\left.\left[u,\left(Y_{s}^{u}, s \in\left[b^{u}, t\right]\right): u \in \tau \in \mathbb{T} \text { with } t \in\left[b^{u}, \zeta^{u}\right)\right]\right\} }
\end{aligned}
$$

Set $\mathcal{F}=\bigcup_{t \geq 0} \mathcal{F}_{t}$. Let $\left\{\mathbf{P}_{x}: x \in E\right\}$ be probability measures on $(\mathcal{T}, \mathcal{F})$ such that $\left(\mathcal{T}, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \geq 0},\left(\mathbf{P}_{x}\right)_{x \in E}\right)$ is the canonical model for $X$, the branching Hunt process in $E$. For each $x \in E, \mathbf{P}_{x}$ stands for the law of the branching Hunt process starting from one particle located at $x$. For detailed constructions of the probability measures $\left\{\mathbf{P}_{x}: x \in E\right\}$,
we refer our readers to [2, 3, 22]. Under $\mathbf{P}_{x}$, the evolution of our branching Hunt process can be described as follows.
(i) The root moves according $\Pi_{x}$.
(ii) Given the path $Y^{u}$ of a particle $u$ and given that $u$ is alive at time $t$, its probability of dying in the interval $[t, t+\mathrm{d} t)$ is $\beta\left(Y_{t}^{u}\right) \mathrm{d} t+o(\mathrm{~d} t)$.
(iii) When a particle $u$ dies, it is replaced by $r^{u}$ number of offspring. The distribution of $r^{u}$ is given by $P\left(Y_{\zeta_{u}}^{u}\right)=\left(p_{k}\left(Y_{\zeta_{u}}^{u}\right)\right)_{k \in \mathbb{N}}$. The offspring of $u$ move independently according to $\Pi_{Y_{u}}$. More precisely, $\left(Y_{s}^{u}, s \in\left[b^{u}, \zeta^{u}\right]\right)$ is the restriction to $\left[b^{u}, \zeta^{u}\right]$ of a copy of the Hunt process starting from $Y_{\zeta^{u-1}}^{u-1}$ at time $b^{u}$.

For a marked tree $(\tau, Y, \sigma, r)$, we let $L_{t}=\left\{u \in \tau: b^{u} \leq t<\zeta^{u}\right\}$ be the set of particles alive at time $t$. Then

$$
X_{t}=\sum_{u \in L_{t}} \delta_{Y_{t}^{u}} .
$$

$\left\{M_{t}(\phi), t \geq 0\right\}$ is a $\mathbf{P}_{x}$-martingale for each $x \in E$, and so we can use $\left\{M_{t}(\phi), t \geq 0\right\}$ to define a martingale change of measure of $\mathbf{P}_{x}$. We are interested in an interpretation of the new measure, i.e., we want to know how the process $X$ evolves under the new measure. To this end, we need to define a new sample space $\widetilde{\mathcal{T}}$, which is the space of marked trees with distinguished spines. For a marked tree $(\tau, Y, \sigma, r)$, we let $D_{t}=\left\{u \in \tau: \zeta^{u} \leq t, r^{u}=0\right\}$ be the set of particles that died, before or at time $t$, with no offspring. Let $\dagger$ be a fictitious node not in $\tau$. A spine $\xi$ on a marked tree $(\tau, Y, \sigma, r)$ is a subset of $\tau \cup\{\dagger\}$ such that

- $\emptyset \in \xi$ and $\left|\xi \cap\left(L_{t} \cup\{\dagger\}\right)\right|=1$ for all $t \geq 0$.
- If $v \in \xi$ and $u<v$, then $u \in \xi$.
- If $v \in \xi$ and $r^{v}>0$, then there exists a unique $j=1, \ldots, r^{v}$ with $v j \in \xi$. If $v \in \xi$ and $r^{v}=0$, then $\xi \cap L_{t}$ is empty for all $t \geq \zeta^{v}$. In this case, we will write $v=\dagger-1$.

Note that the spine only contains information about the nodes along the spine, does not know the fission times or the number of offspring at these fission times. The fictitious particle (or node) $\dagger$ might move in space, but its movement will be of no concern to us. Thus we call $\zeta^{\dagger-1}$ the "lifetime" of the spine. $\dagger$ lives on forever.

We write

$$
\widetilde{\mathcal{T}}=\{(\tau, Y, \sigma, r, \xi):(\tau, Y, \sigma, r) \in \mathcal{T} \text { and } \xi \text { is a spine on }(\tau, Y, \sigma, r)\}
$$

for the space of marked trees with distinguished spines.
Given $(\tau, Y, \sigma, r, \xi) \in \widetilde{\mathcal{T}}$ and $t \geq 0$, we let $\xi_{t}:=v$ be the unique element $v \in \xi \cap$ $\left(L_{t} \cup\{\dagger\}\right)$. We will use $\widetilde{Y}=\left(\widetilde{Y}_{t}\right)_{t \geq 0}$ to denote the spatial path followed by the spine and
$n=\left(n_{t}: t \geq 0\right)$ to denote the counting process of fission times along the spine. More precisely, $\widetilde{Y}_{t}=Y_{t}^{u}$ and $n_{t}=|u|$, if $u \in L_{t} \cap \xi$. If $\xi_{t}=\dagger$, we set $\widetilde{Y}_{t}=Y_{t}^{\dagger}$ and write $u<\dagger$ if $u \in L_{s}$ and $u=\xi_{s}$ for some $s<t$.

If $v \in \xi \cap L_{t}$ and $r^{v}>0$, then at the fission time $\zeta^{v}$, exactly one of its offspring continues the spine. Let $O_{v}$ be the set of offspring of $v$ except the one belonging to the spine, then for any $j=1, \ldots, r^{v}$ such that $v j \in O_{v}$, we will use $(\tau, M)_{j}^{v}$ to denote the marked subtree rooted at $v j$.

Now we introduce two filtrations $\left\{\widetilde{\mathcal{F}}_{t}\right\}_{t \geq 0}$ and $\left\{\mathcal{G}_{t}\right\}_{t \geq 0}$ on $\widetilde{\mathcal{T}}$ by

$$
\widetilde{\mathcal{F}}_{t}:=\sigma\left(\mathcal{F}_{t}, \xi_{s}, s \leq t\right), \quad \mathcal{G}_{t}:=\sigma\left(\widetilde{Y}_{s}: s \in[0, t]\right), \quad t \geq 0 .
$$

$\widetilde{\mathcal{F}}_{t}$ knows everything about the marked tree up to time $t$ and the nodes on the spine up to time $t$ (and thus everything about the spine up to $t \wedge \zeta^{\dagger-1}$, including which nodes make up the spine, when they were born, when they died, and their family sizes). $\mathcal{G}_{t}$ contains all information about the path of the spine up to time $t$.

Set $\widetilde{\mathcal{F}}:=\bigcup_{t \geq 0} \widetilde{\mathcal{F}}_{t}, \mathcal{G}:=\sigma\left(\widetilde{Y}_{s}: s \geq 0\right), \widehat{\mathcal{G}}:=\sigma\left(\left(\widetilde{Y}_{s}, \xi_{s}: s \geq 0\right),\left(\zeta^{u}: u<\dagger\right)\right)$ and $\widetilde{\mathcal{G}}:=\sigma\left(\mathcal{G},\left(\xi_{s}: s \geq 0\right),\left(\zeta^{u}, u<\dagger\right),\left(r^{u}: u<\dagger\right)\right)$. The $\sigma$-field $\mathcal{G}$ knows everything about the path of the spine, the $\sigma$-field $\widehat{\mathcal{G}}$ knows everything about the path of the spine and the fission times along the spine, and the $\sigma$-field $\widetilde{\mathcal{G}}$ knows everything about the path of the spine, the fission times along the spine and the number of offspring born at these fission times.

As noted by Hardy and Harris [10], it is convenient to consider $\left\{\mathbf{P}_{x}, x \in E\right\}$ as measures on the enlarged space $(\widetilde{\mathcal{T}}, \mathcal{F})$, rather than on $(\mathcal{T}, \mathcal{F})$.

We need to extend the probability measures $\mathbf{P}_{x}$ to probability measures $\widetilde{\mathbf{P}}_{x}$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ so that the spine is a single genealogical line of descent chosen from the underlying tree. We will assume that at each fission time along the spine we make a uniform choice among the offspring, if there is at least one offspring, to decide which line of descent continues the spine $\xi$. If at some fission time of the spine, there is no offspring produced, we assume the spine continues with the fictitious particle $\dagger$. Then for $u \in \tau$ we have

$$
\operatorname{Prob}(u \in \xi)=\prod_{v<u} \frac{1}{r^{v}} .
$$

It is easy to see that

$$
\sum_{u \in L_{t}} \prod_{v<u} \frac{1}{r^{v}}+\sum_{u \in D_{t}} \prod_{v<u} \frac{1}{r^{v}}=1
$$

We first give the following representation, which is an extension of the one given in [19] for the case that $p_{0}=0$.
Theorem 2.1 Every $f \in \widetilde{\mathcal{F}}_{t}$ can be written as

$$
\begin{equation*}
f=\sum_{u \in L_{t}} f^{u}(\tau, M) \mathbf{1}_{\left\{\xi_{t}=u\right\}}+\sum_{u \in D_{t}} f^{u}(\tau, M) \mathbf{1}_{\{\dagger-1=u\}}, \tag{2.1}
\end{equation*}
$$

where $f^{u} \in \mathcal{F}_{t}$.
Proof. Suppose $f(\tau, M, \xi) \in \widetilde{\mathcal{F}}_{t}$. For every $t>0$, there is a unique $u \in L_{t} \cup\{\dagger\}$ such that $\xi_{t}=u$, and if $\xi_{t}=\dagger$, then there is one unique $u \in D_{t}$ such that $\dagger-1=u$. Thus we have $\sum_{u \in L_{t}} \boldsymbol{1}_{\left\{\xi_{t}=u\right\}}+\sum_{t \in D_{t}} \boldsymbol{1}_{\{\dagger-1=u\}}=1$, and hence

$$
\begin{aligned}
f & =\sum_{u \in L_{t}} f\left(\tau, M, \xi_{t}\right) \mathbf{1}_{\left\{\xi_{t}=u\right\}}+\sum_{u \in D_{t}} f(\tau, M, \dagger-1) \mathbf{1}_{\{\dagger-1=u\}} \\
& =\sum_{u \in L_{t}} f(\tau, M, u) \mathbf{1}_{\left\{\xi_{t}=u\right\}}+\sum_{u \in D_{t}} f(\tau, M, u) \mathbf{1}_{\{\dagger-1=u\}} .
\end{aligned}
$$

Since $f \in \widetilde{\mathcal{F}}_{t}$, for each fixed $u \in L_{t} \cup D_{t}$, we have $f^{u}:=f(\tau, M, u) \in \mathcal{F}_{t}$. Thus (2.1) is valid.

We define the measure $\widetilde{\mathbf{P}}_{x}$ on $\widetilde{\mathcal{F}}_{t}$ by

$$
\begin{align*}
& \left.\mathrm{d} \widetilde{\mathbf{P}}_{x}(\tau, M, \xi)\right|_{\widetilde{\mathcal{F}}_{t}} \\
& =\mathbf{1}_{\left\{\xi_{t} \in \tau\right\}} \mathrm{d} \Pi_{x}(\widetilde{Y}) \mathrm{d} L^{\beta(\widetilde{Y})}(\mathbf{n}) \prod_{v<\xi_{t}} p_{r^{v}}\left(\widetilde{Y}_{\zeta^{v}}\right) \prod_{v<\xi_{t}} \frac{1}{r^{v}} \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\tilde{Y}_{\zeta^{v}}}^{t-\zeta^{v}}\left((\tau, M)_{j}^{v}\right) \\
& +\mathbf{1}_{\left\{\xi_{t}=\dagger\right\}} \mathrm{d} \prod_{x}(\widetilde{Y}) \mathrm{d} L^{\beta(\widetilde{Y})}(\mathbf{n}) \prod_{v<\dagger-1} p_{r^{v}}\left(\widetilde{Y}_{\zeta^{v}}\right) \prod_{v<\dagger-1} \frac{1}{r^{v}} \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\widetilde{\zeta}_{\zeta^{v}}}^{t-\zeta^{v}}\left((\tau, M)_{j}^{v}\right), \tag{2.2}
\end{align*}
$$

where $\Pi_{x}(\widetilde{Y})$ is the law of the Hunt process $\widetilde{Y}$ starting from $x \in E, L^{\beta(\widetilde{Y})}(\mathbf{n})$ is the law of a Poisson random measure $\mathbf{n}=\left\{\left\{\sigma_{i}: i=1, \cdots, n_{t}\right\}: t \geq 0\right\}$ with intensity $\beta\left(\widetilde{Y}_{t}\right) \mathrm{d} t$ along the path of $\widetilde{Y}$ which gives the fission times along the spine, $p_{r^{v}}(y)=\sum_{k \geq 0} p_{k}(y) \mathbf{1}_{\left(r^{v}=k\right)}$ is the probability that the individual $v$, on the spine and located at $y \in E$, has $r^{v}$ offspring, and $\left.\mathbf{P}_{x}^{t-s}\left((\tau, M)_{j}^{v}\right)\right)$ stands for the law of a branching Hunt process on the marked tree $(\tau, M)_{j}^{v}$, with initial particle located at $x$ time shifted by $s$.

It follows from Theorem 2.1] that for any bounded $f \in \widetilde{\mathcal{F}}_{t}$,

$$
\begin{aligned}
\widetilde{\mathbf{P}}_{x}\left(f \mid \mathcal{F}_{t}\right) & =\widetilde{\mathbf{P}}_{x}\left(\sum_{u \in L_{t}} f^{u}(\tau, M) \mathbf{1}_{\left\{\xi_{t}=u\right\}}+\sum_{u \in D_{t}} f^{u}(\tau, M) \mathbf{1}_{\{\dagger-1=u\}} \mid \mathcal{F}_{t}\right) \\
& =\sum_{u \in L_{t}} f^{u}(\tau, M) \prod_{v<u} \frac{1}{r^{v}}+\sum_{u \in D_{t}} f^{u}(\tau, M) \prod_{v<u} \frac{1}{r^{v}}
\end{aligned}
$$

Thus we have for any $t \geq 0$ and bounded $f \in \widetilde{\mathcal{F}}_{t}$,

$$
\begin{equation*}
\widetilde{\mathbf{P}}_{x}(f)=\mathbf{P}_{x}\left(\sum_{u \in L_{t}} f^{u}(\tau, M) \prod_{v<u} \frac{1}{r^{v}}+\sum_{u \in D_{t}} f^{u}(\tau, M) \prod_{v<u} \frac{1}{r^{v}}\right) \tag{2.3}
\end{equation*}
$$

In particular,

$$
\widetilde{\mathbf{P}}_{x}(\widetilde{\mathcal{T}})=\mathbf{P}_{x}\left(\sum_{u \in L_{t}} \prod_{v<u} \frac{1}{r^{v}}+\sum_{u \in D_{t}} \prod_{v<u} \frac{1}{r^{v}}\right)=\mathbf{P}_{x}(1)=1,
$$

which implies $\widetilde{\mathbf{P}}_{x}$ is a probability measure. $\widetilde{\mathbf{P}}_{x}$ is an extension of $\mathbf{P}_{x}$ onto $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ and for any bounded $f \in \widetilde{\mathcal{F}}_{t}$ we have

$$
\begin{equation*}
\int_{\widetilde{\mathcal{T}}} f \mathrm{~d} \widetilde{\mathbf{P}}_{x}=\int_{\widetilde{\mathcal{T}}}\left(\sum_{u \in L_{t}} f^{u} \prod_{v<u} \frac{1}{r^{v}}+\sum_{u \in D_{t}} f^{u} \prod_{v<u} \frac{1}{r^{v}}\right) \mathrm{d} \mathbf{P}_{x} \tag{2.4}
\end{equation*}
$$

The decomposition (2.2) of $\widetilde{\mathbf{P}}_{x}$ suggests the following intuitive construction of the system under $\widetilde{\mathbf{P}}_{x}$ :
(i) the root of $\tau$ is at $x$ at time 0 , and the spine process $\widetilde{Y}_{t}$ moves according to $\Pi_{x}$;
(ii) given the trajectory $\widetilde{Y}$. of the spine, the fission times along the spine are distributed according to $L^{\beta(\widetilde{Y})}$, where $L^{\beta(\widetilde{Y})}$ is the law of a Poisson random measure with intensity $\beta\left(\widetilde{Y}_{t}\right) \mathrm{d} t$;
(iii) at the fission time of a node $v$ on the spine, the single spine particle is replaced by a random number $r^{v}$ of offspring with $r^{v}$ being distributed according to the law $P\left(\widetilde{Y}_{\zeta^{v}}\right)=\left(p_{k}\left(\widetilde{Y}_{\zeta^{v}}\right)\right)_{k \geq 1} ;$
(vi) if $r^{v}>0$, the spine is chosen uniformly from the $r^{v}$ offspring of $v$ at the fission time of $v$; if $r^{v}=0$, the spine continues as $\dagger$.
(v) if $r^{v} \geq 2$, the remaining $r^{v}-1$ particles $v j \in O_{v}$ give rise to independent subtrees $(\tau, M)_{j}^{v}$, which evolve as independent subtrees determined by the probability measure $\mathbf{P}_{\tilde{Y}_{\zeta^{v}}}$ shifted to the time of creation.

Definition 2.2 Suppose that $(\Omega, \mathcal{H}, P)$ is a probability space, $\left\{\mathcal{H}_{t}, t \geq 0\right\}$ is a filtration on $(\Omega, \mathcal{H})$ and that $\mathcal{K}$ is a sub- $\sigma$-field of $\mathcal{H}$. A real-valued process $\left\{U_{t}, t \geq 0\right\}$ on $(\Omega, \mathcal{F}, P)$ is called a $P(\cdot \mid \mathcal{K})$-martingale with respect to $\left\{\mathcal{H}_{t}, t \geq 0\right\}$ if (i) it is adapted to $\left\{\mathcal{H}_{t} \vee \mathcal{K}, t \geq 0\right\}$; (ii) for any $t \geq 0, E\left(\left|U_{t}\right|\right)<\infty$ and (iii) for any $t>s$,

$$
E\left(U_{t} \mid \mathcal{H}_{s} \vee \mathcal{K}\right)=U_{s}, \quad \text { a.s. }
$$

We also say that $\left\{U_{t}, t \geq 0\right\}$ is a martingale with respect to $\left\{\mathcal{H}_{t}, t \geq 0\right\}$, given $\mathcal{K}$.
The following result is [18, Lemma 2.3].

Lemma 2.3 Suppose that $(\Omega, \mathcal{H}, P)$ is a probability space, $\left\{\mathcal{H}_{t}, t \geq 0\right\}$ is a filtration on $(\Omega, \mathcal{H})$ and that $\mathcal{K}_{1}, \mathcal{K}_{2}$ are two sub- $\sigma$-fields of $\mathcal{H}$ such that $\mathcal{K}_{1} \subset \mathcal{K}_{2}$. Assume that $\left\{U_{t}^{1}, t \geq 0\right\}$ is a $P\left(\cdot \mid \mathcal{K}_{1}\right)$-martingale with respect to $\left\{\mathcal{H}_{t}, t \geq 0\right\}$, $\left\{U_{t}^{2}, t \geq 0\right\}$ is a $P\left(\cdot \mid \mathcal{K}_{2}\right)$ martingale with respect to $\left\{\mathcal{H}_{t}, t \geq 0\right\}$. If $U_{t}^{1} \in \mathcal{K}_{2}, U_{t}^{2} \in \mathcal{H}_{t}$, and $E\left(\left|U_{t}^{1} U_{t}^{2}\right|\right)<\infty$ for any $t \geq 0$, then the product $\left\{U_{t}^{1} U_{t}^{2}, t \geq 0\right\}$ is a $P\left(\cdot \mid \mathcal{K}_{1}\right)$-martingale with respect to $\left\{\mathcal{H}_{t}, t \geq 0\right\}$.

Lemma 2.4 Suppose that, given the path of $\widetilde{Y}, \mathbf{n}=\left\{\left\{\zeta_{i}: i=1, \cdots, n_{t}\right\}: t \geq 0\right\}$ is a Poisson random measure with intensity $\beta\left(\widetilde{Y}_{t}\right) \mathrm{d} t$ along the path of $\widetilde{Y}$. Then

$$
\eta_{t}^{(1)}:=\prod_{i \leq n_{t}} A\left(\widetilde{Y}_{\zeta_{i}}\right) \cdot \exp \left(-\int_{0}^{t}((A-1) \beta)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right), \quad t \geq 0
$$

is an $L^{\beta(\widetilde{Y})}$-martingale with respect to the natural filtration $\left\{\mathcal{L}_{t}, t \geq 0\right\}$ of $\mathbf{n}$.
Proof. First note that

$$
\begin{equation*}
L^{\beta(\widetilde{Y})}\left[\prod_{i \leq n_{t}} A\left(\widetilde{Y}_{\zeta_{i}}\right)\right]=\exp \left(\int_{0}^{t}((A-1) \beta)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right) \tag{2.5}
\end{equation*}
$$

which implies that $L^{\beta(\widetilde{Y})}\left(\eta_{t}^{(1)}\right)=1$. It is easy to check that $\left\{\eta_{t}^{(1)}, t \geq 0\right\}$ is a martingale under $L^{\beta(\widetilde{Y})}$ by using the Markov property of $\mathbf{n}$. We omit the details.

It follows from the lemma above that we can define a measure $L^{(A \beta)(\widetilde{Y})}$ by

$$
\left.\frac{\mathrm{d} L^{(A \beta)(\widetilde{Y})}}{\mathrm{d} L^{\beta(\tilde{Y})}}\right|_{\mathcal{L}_{t}}=\prod_{i \leq n_{t}} A\left(\widetilde{Y}_{\zeta_{i}}\right) \cdot \exp \left(-\int_{0}^{t}((A-1) \beta)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right)
$$

Lemma 2.5 For any $x \in E$ and $t \geq 0$, we have

$$
\begin{equation*}
\widetilde{\mathbf{P}}_{x}\left[\left.\prod_{v<\xi_{t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \right\rvert\, \widehat{\mathcal{G}}\right]=1 \tag{2.6}
\end{equation*}
$$

Proof. It follows from (2.2) that, given $\widehat{\mathcal{G}}$, for each $v<\xi_{t}$,

$$
\widetilde{\mathbf{P}}_{x}\left(r^{v} \mid \widehat{\mathcal{G}}\right)=A\left(\widetilde{Y}_{\zeta^{v}}\right)
$$

Since, given $\widehat{\mathcal{G}},\left\{r^{v}, v<\xi_{n_{t}}\right\}$ are independent, we have

$$
\widetilde{\mathbf{P}}_{x}\left(\left.\prod_{v<\xi_{t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \right\rvert\, \widehat{\mathcal{G}}\right)=1
$$

Lemma 2.6 (1) The process

$$
\widetilde{\eta}_{t}^{(1)}:=\prod_{v<\xi_{t}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \cdot \exp \left(-\int_{0}^{t \wedge \varsigma^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right), \quad t \geq 0
$$

is a $\widetilde{\mathbf{P}}_{x}\left(\cdot \mid \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right)$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$.
(2) The process

$$
\widetilde{\eta}_{t}^{(2)}:=\prod_{v<\xi_{t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)}=\mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)}, \quad t \geq 0
$$

is a $\widetilde{\mathbf{P}}_{x}(\cdot \mid \widehat{\mathcal{G}})$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$, where the last equality holds because if $\xi_{t}=\dagger$, then $r^{v}=0$ for $v=\dagger-1$.

Proof. (1) First note that if $\xi_{t} \in L_{t}$ then $\zeta^{\dagger-1}>t$, and if $\xi_{t}=\dagger$ then $\zeta^{\dagger-1} \leq t$. For $s, t \geq 0$, by the Markov property, we have

$$
\begin{aligned}
& \widetilde{\mathbf{P}}_{x}\left[\widetilde{\eta}_{t+s}^{(1)} \mid \widetilde{\mathcal{F}}_{t} \vee \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right] \\
= & \widetilde{\mathbf{P}}_{x}\left[\prod_{v<\xi_{t+s}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \cdot \exp \left(-\int_{0}^{(t+s) \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r}\right) \mathrm{d} r\right) \mid \widetilde{\mathcal{F}}_{t} \vee \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right] \\
= & \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \cdot \exp \left(-\int_{0}^{t \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r}\right) \mathrm{d} r\right) \\
& \cdot \widetilde{\mathbf{P}}_{x}\left[\prod_{\xi_{t} \leq v<\xi_{t+s}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \cdot \exp \left(-\int_{0}^{s \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r+t}\right) \mathrm{d} r\right) \mid \widetilde{\mathcal{F}}_{t} \vee \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right] \\
& +\mathbf{1}_{\left\{\xi_{t}=\dagger\right\}} \prod_{v<\xi_{t}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \cdot \exp \left(-\int_{0}^{t \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r}\right) \mathrm{d} r\right) \\
= & \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \widetilde{\eta}_{t}^{(1)} \exp \left(-\int_{0}^{s \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r+t}\right) \mathrm{d} r\right) \widetilde{\mathbf{P}}^{x}\left[\prod_{\xi_{t} \leq v<\xi_{t+s}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \mid \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right] \\
& +\mathbf{1}_{\left\{\xi_{t}=\dagger\right\}} \widetilde{\eta}_{t}^{(1)} .
\end{aligned}
$$

For fixed $t>0$, given the path of $\tilde{Y}$, the collection of fission times $\left\{\left\{\zeta^{v}: \xi_{t} \leq v<\xi_{t+s}\right\}\right.$ : $s \geq 0\}$ is a Poisson random measure with intensity $\beta\left(\widetilde{Y}_{t+s}\right) \mathrm{d} s$, and has law $L^{\beta\left(\widetilde{Y}_{t+\cdot}\right)}$. It follows from (2.5) that

$$
\widetilde{\mathbf{P}}_{x}\left[\prod_{\xi_{n_{t}} \leq v<\xi_{t+s}} A\left(\widetilde{Y}_{\zeta^{v}}\right) \mid \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right]=\exp \left(\int_{0}^{s \wedge \zeta^{\dagger-1}}((A-1) \beta)\left(\widetilde{Y}_{r+t}\right) \mathrm{d} r\right)
$$

Thus

$$
\widetilde{\mathbf{P}}_{x}\left[\widetilde{\eta}_{t+s}^{(1)} \mid \widehat{\mathcal{F}}_{t} \vee \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right]=\widetilde{\eta}_{t}^{(1)} .
$$

(2) For $s, t \geq 0$, by the Markov property, we have

$$
\begin{aligned}
\widetilde{\mathbf{P}}_{x}\left[\widetilde{\eta}_{t+s}^{(2)} \mid \widetilde{\mathcal{F}}_{t} \vee \widehat{\mathcal{G}}\right] & =\widetilde{\mathbf{P}}_{x}\left[\left.\prod_{v<\xi_{t+s}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \right\rvert\, \widetilde{\mathcal{F}}_{t} \vee \widehat{\mathcal{G}}\right] \\
& =\mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \cdot \widetilde{\mathbf{P}}_{x}\left[\left.\prod_{\xi_{t} \leq v<\xi_{s+t}} \frac{r^{v}}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \right\rvert\, \widehat{\mathcal{G}}\right] \\
& =\widetilde{\eta}_{t}^{(2)},
\end{aligned}
$$

where in the last equality we used (2.6). Thus we have

$$
\widetilde{\mathbf{P}}_{x}\left[\widetilde{\eta}_{t+s}^{(2)} \mid \widetilde{\mathcal{F}}_{t} \vee \widehat{\mathcal{G}}\right]=\widetilde{\eta}_{t}^{(2)} .
$$

The effect of a change of measure using the martingale $\left\{\widetilde{\eta}_{t}^{(1)}, t \geq 0\right\}$ will change the fission rate along the spine from $\beta\left(\widetilde{Y}_{t}\right)$ to $(A \beta)\left(\widetilde{Y}_{t}\right)$. The effect of a change of measure using the martingale $\left\{\widetilde{\eta}_{t}^{(2)}, t \geq 0\right\}$ will change the offspring distribution from $P\left(\widetilde{Y}_{\zeta_{i}}\right)=$ $\left(p_{k}\left(\widetilde{Y}_{\zeta_{i}}\right)\right)_{k \geq 1}$ to the size-biased distribution $\dot{P}\left(\widetilde{Y}_{\zeta_{i}}\right)=\left(\dot{p}_{k}\left(Y_{\zeta_{i}}\right)\right)_{k \geq 1}$, where $\dot{p}_{k}(y)$ is defined by

$$
\dot{p}_{k}(y)=\frac{k p_{k}(y)}{A(y)}, \quad k \geq 1, y \in E .
$$

Define

$$
\widetilde{\eta}_{t}^{(3)}(\phi):=\frac{\phi\left(\widetilde{Y}_{t \wedge \zeta^{\dagger-1}}\right)}{\phi(x)} \exp \left(-\int_{0}^{t \wedge \zeta^{\dagger-1}}\left(\lambda_{1}-(A-1) \beta\right)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right), \quad t \geq 0
$$

$\left\{\widetilde{\eta}_{t}^{(3)}(\phi), t \geq 0\right\}$ is a $\widetilde{\mathbf{P}}_{x}$-martingale with respect to $\left\{\mathcal{G}_{t} \vee \sigma\left(\zeta^{\dagger-1}\right), t \geq 0\right\}$, and it is also a $\widetilde{\mathbf{P}}_{x}$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$, since $\widetilde{\eta}_{t}^{(3)}(\phi)$ can be expressed as

$$
\begin{align*}
\widetilde{\eta}_{t}^{(3)}(\phi)= & \sum_{u \in L_{t}} \phi(x)^{-1} \phi\left(\widetilde{Y}_{t}^{u}\right) \exp \left(-\int_{0}^{t}\left(\lambda_{1}-(A-1) \beta\right)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right) \mathbf{1}_{\left\{\xi_{t}=u\right\}} \\
& +\sum_{u \in D_{t}} \phi(x)^{-1} \phi\left(\widetilde{Y}_{\zeta^{u}}^{u}\right) \exp \left(-\int_{0}^{\zeta^{u}}\left(\lambda_{1}-(A-1) \beta\right)\left(\widetilde{Y}_{s}\right) \mathrm{d} s \mathbf{1}_{\{\dagger-1=u\}}\right) . \tag{2.7}
\end{align*}
$$

Define

$$
\eta_{t}(\phi):=\widetilde{\eta}_{t}^{(1)} \widetilde{\eta}_{t}^{(2)} \widetilde{\eta}_{t}^{(3)}(\phi), \quad t \geq 0
$$

It is easy to check, by the definition of $\widetilde{\eta}_{t}^{(1)}, \widetilde{\eta}_{t}^{(2)}$, and $\widetilde{\eta}_{t}^{(3)}(\phi)$, that

$$
\begin{equation*}
\widetilde{\eta}_{t}(\phi)=\mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} r^{v} \frac{\phi\left(\widetilde{Y}_{t}\right)}{\phi(x)} e^{-\lambda_{1} t} . \tag{2.8}
\end{equation*}
$$

Lemma $2.7\left\{\widetilde{\eta}_{t}(\phi), t \geq 0\right\}$ is a $\widetilde{\mathbf{P}}_{x}$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$.
Proof. $\quad\left\{\widetilde{\eta}_{t}^{(1)}, t \geq 0\right\}$ is a $\widetilde{\mathbf{P}}_{x}\left(\cdot \mid \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right)$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq\right.$ $0\}$, and $\left\{\widetilde{\eta}_{t}^{(2)}, t \geq 0\right\}$ is a $\widetilde{\mathbf{P}}_{x}(\cdot \mid \widehat{\mathcal{G}})$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$. Note that $\mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right) \subset \widehat{\mathcal{G}}$, and $\widetilde{\eta}_{t}^{(1)} \in \widehat{\mathcal{G}}, \widetilde{\eta}_{t}^{(2)} \in \widetilde{\mathcal{F}}_{t}$ for any $t \geq 0$. Using Lemma 2.3, $\left\{\widetilde{\eta}_{t}^{(1)} \widetilde{\eta}_{t}^{(2)}, t \geq\right.$ $0\}$ is a $\widetilde{\mathbf{P}}_{x}\left(\cdot \mid \mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)\right)$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$. Note that $\widetilde{\eta}_{t}^{(3)}(\phi) \in$ $\mathcal{G} \vee \sigma\left(\zeta^{\dagger-1}\right)$ and $\widetilde{\eta}_{t}^{(1)} \widetilde{\eta}_{t}^{(2)} \in \widetilde{\mathcal{F}}_{t}$ for any $t \geq 0$. Using Lemma 2.3 again, we see that $\left\{\widetilde{\eta}_{t}(\phi)=\widetilde{\eta}_{t}^{(1)} \widetilde{\eta}_{t}^{(2)} \widetilde{\eta}_{t}^{(3)}(\phi), t \geq 0\right\}$ is a $\widetilde{\mathbf{P}}_{x}$-martingale with respect to $\left\{\widetilde{\mathcal{F}}_{t}, t \geq 0\right\}$.

Lemma 2.8 $M_{t}(\phi)$ is the projection of $\widetilde{\eta}_{t}(\phi)$ onto $\mathcal{F}_{t}$, i.e.,

$$
M_{t}(\phi)=\widetilde{\mathbf{P}}_{x}\left(\widetilde{\eta}_{t}(\phi) \mid \mathcal{F}_{t}\right)
$$

Proof. By (2.8),

$$
\tilde{\eta}_{t}(\phi)=\sum_{u \in L_{t}} \prod_{v<u} r^{v} e^{-\lambda_{1} t} \phi(x)^{-1} \phi\left(Y_{t}^{u}\right) \mathbf{1}_{\left\{\xi_{t}=u\right\}} .
$$

Thus

$$
\begin{aligned}
\widetilde{\mathbf{P}}_{x}\left(\widetilde{\eta}_{t}(\phi) \mid \mathcal{F}_{t}\right) & =\sum_{u \in L_{t}} e^{-\lambda_{1} t} \phi(x)^{-1} \phi\left(Y_{t}^{u}\right) \prod_{v<u} r^{v} \widetilde{\mathbf{P}}_{x}\left(\mathbf{1}_{\left\{\xi_{t}=u\right\}} \mid \mathcal{F}_{t}\right) \\
& =\sum_{u \in L_{t}} e^{-\lambda_{1} t} \phi(x)^{-1} \phi\left(Y_{t}^{u}\right)=M_{t}(\phi),
\end{aligned}
$$

where in the second equality we used the fact that

$$
\widetilde{\mathbf{P}}_{x}\left(\mathbf{1}_{L_{t}}(u) \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mid \mathcal{F}_{t}\right)=\mathbf{1}_{L_{t}}(u) \mathbf{1}_{\left\{\xi_{t}=u\right\}} \prod_{v<u} \frac{1}{r^{v}}
$$

Now we define a probability measure $\widetilde{\mathbf{Q}}_{x}$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}})$ by

$$
\begin{equation*}
\left.\frac{\mathrm{d} \widetilde{\mathbf{Q}}_{x}}{\mathrm{~d} \widetilde{\mathbf{P}}_{x}}\right|_{\widetilde{\mathcal{F}}_{t}}=\widetilde{\eta}_{t}(\phi), \quad t \geq 0 \tag{2.9}
\end{equation*}
$$

which, by (2.8), says that on $\widetilde{\mathcal{F}}_{t}$,

$$
\mathrm{d} \widetilde{\mathbf{Q}}_{x}=\widetilde{\eta}_{t}(\phi) \mathrm{d} \widetilde{\mathbf{P}}_{x}=\mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} r^{v} \frac{\phi\left(\widetilde{Y}_{t}\right)}{\phi(x)} e^{-\lambda_{1} t} \mathrm{~d} \widetilde{\mathbf{P}}_{x}
$$

Hence we have $\widetilde{\mathbf{Q}}_{x}\left(\xi_{t} \in L_{t}\right)=1$ for any $t \geq 0$, which implies that $\widetilde{\mathbf{Q}}_{x}\left(\xi_{t} \in L_{t}, \forall t \geq 0\right)=1$.

$$
\begin{aligned}
\mathrm{d} \widetilde{\mathbf{Q}}_{x}= & \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \frac{\phi\left(\widetilde{Y}_{t}\right)}{\phi(x)} \exp \left(-\int_{0}^{t}\left(\lambda_{1}-(A-1) \beta\right)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right) \mathrm{d} \Pi_{x}(\widetilde{Y}) \\
& \times \exp \left(-\int_{0}^{t}((A-1) \beta)\left(\widetilde{Y}_{s}\right) \mathrm{d} s\right) \mathrm{d} L^{\beta(\widetilde{Y})} \prod_{v<\xi_{t}} p_{r^{v}}\left(\widetilde{Y}_{\zeta^{v}}\right) \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\widetilde{Y}_{\zeta^{v}}^{t-\zeta^{v}}}^{t\left((\tau, M)_{j}^{v}\right)} \\
= & \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \mathrm{d} \Pi_{x}^{\phi}(\widetilde{Y}) \mathrm{d} L^{A \beta(\widetilde{Y})}(\mathbf{n}) \prod_{v<\xi_{t}} \frac{p_{r_{v}}\left(\widetilde{Y}_{\zeta^{v}}\right)}{A\left(\widetilde{Y}_{\zeta^{v}}\right)} \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\widetilde{Y}_{\zeta^{v}}^{t-\zeta^{v}}}\left((\tau, M)_{j}^{v}\right) \\
= & \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \mathrm{d} \Pi_{x}^{\phi}(\widetilde{Y}) \mathrm{d} L^{A \beta(\widetilde{Y})}(\mathbf{n}) \prod_{v<\xi_{t}}^{\dot{p}_{r^{v}}\left(\widetilde{Y}_{\zeta^{v}}\right)} \prod_{v<\xi_{t}} \frac{1}{r^{v}} \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\widetilde{Y}_{\zeta^{v}}^{t-\zeta^{v}}}\left((\tau, M)_{j}^{v}\right) \\
= & \mathrm{d} \Pi_{x}^{\phi}(\widetilde{Y}) \mathrm{d} L^{A \beta(\widetilde{Y})}(\mathbf{n}) \prod_{v<\xi_{t}} \dot{p}_{r^{v}}\left(\widetilde{Y}_{\zeta^{v}}\right) \prod_{v<\xi_{t}} \frac{1}{r^{v}} \prod_{j: v j \in O_{v}} \mathrm{~d} \mathbf{P}_{\widetilde{Y}_{\zeta^{v}}^{t-\zeta^{v}}}\left((\tau, M)_{j}^{v}\right) .
\end{aligned}
$$

Thus the change of measure from $\widetilde{\mathbf{P}}_{x}$ to $\widetilde{\mathbf{Q}}_{x}$ has three effects: the spine will be changed to a Hunt process with law $\Pi_{x}^{\phi}$, its fission times will be changed and the distribution of its family sizes will be sized-biased. More precisely, under $\widetilde{\mathbf{Q}}_{x}$ :
(i) the root of $\tau$ is at $x$ at time 0 , and the spine process $\widetilde{Y}_{t}$ moves according to $\Pi_{x}^{\phi}$;
(ii) given the trajectory $\tilde{Y}$. of the spine, the fission times along the spine are distributed according to $L^{(A \beta)(\widetilde{Y})}$, where $L^{(A \beta)(\widetilde{Y})}$ is the law of a Poisson random measure with intensity $(A \beta)\left(\widetilde{Y}_{t}\right) \mathrm{d} t$;
(iii) at the fission time of node $v$ on the spine, the single spine particle is replaced by a random number $r^{v}$ of offspring with $r^{v}$ being distributed according to the law $\dot{P}\left(\widetilde{Y}_{\zeta^{v}}\right):=\left(\dot{p}_{k}\left(\widetilde{Y}_{\zeta^{v}}\right)\right)_{k \geq 1} ;$
(vi) the spine is chosen uniformly from the $r^{v}$ offspring of $v$ at the fission time of $v$;
(v) the remaining $r^{v}-1$ particles $v j \in O_{v}$ give rise to independent subtrees $(\tau, M)_{j}^{v}$, which evolve as independent subtrees determined by the probability measure $\mathbf{P}_{\tilde{Y}_{C^{v}}}$ shifted to the time of creation.

We define a measure $\mathrm{Q}_{x}$ on $(\widetilde{\mathcal{T}}, \mathcal{F})$ by

$$
\mathrm{Q}_{x}:=\left.\widetilde{\mathbf{Q}}_{x}\right|_{\mathcal{F}}
$$

Theorem 2.9 (Spine decomposition) $\mathrm{Q}_{x}$ is a martingale change of measure by the martingale $\left\{M_{t}(\phi), t>0\right\}$ : for any $t>0$,

$$
\left.\frac{\mathrm{d} \mathbf{Q}_{x}}{\mathrm{~d} \mathbf{P}_{x}}\right|_{\mathcal{F}_{t}}=M_{t}(\phi)
$$

Proof. The result actually follow from a more general observation that if $\widetilde{\mu}_{1}$ and $\widetilde{\mu}_{2}$ are two measures defined on a measure space $(\Omega, \widetilde{\mathcal{S}})$ with Radon-Nikodym derivative

$$
\frac{\mathrm{d} \widetilde{\mu}_{2}}{\mathrm{~d} \widetilde{\mu}_{1}}=f
$$

and if $\mathcal{S}$ is a sub--algebra of $\widetilde{\mathcal{S}}$, then the two measures $\mu_{1}:=\left.\widetilde{\mu}_{1}\right|_{S}$ and $\mu_{2}:=\left.\widetilde{\mu}_{2}\right|_{S}$ on $(\Omega, \mathcal{S})$ are related by the conditional expectation operation:

$$
\frac{\mathrm{d} \mu_{2}}{\mathrm{~d} \mu_{1}}=\widetilde{\mu}_{1}(f \mid \mathcal{S})
$$

For each fixed $t>0$, applying this general result with $(\Omega, \widetilde{\mathcal{S}})=\left(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}}_{t}\right), \mathcal{S}=\mathcal{F}_{t}, \widetilde{\mu}_{2}=\widetilde{\mathbf{Q}}_{x}$, and $\widetilde{\mu}_{1}=\widetilde{\mathbf{P}}_{x}$, and using Lemma 2.8 yield the desired result.

We still use $X_{t}(B)$ to denote the number of particles located in $B \in \mathcal{B}(E)$ at time $t$ in the marked tree with distinguished spine. Note that

$$
X_{t}(B)=\mathbf{1}_{B}\left(\widetilde{Y}_{t}\right)+\sum_{u \in L_{t}, u \neq \xi_{t}} \mathbf{1}_{B}\left(Y_{t}^{u}\right)
$$

The individuals $\left\{u \in L_{t}, u \neq \xi_{t}\right\}$ can be partitioned into subtrees created from fissions along the spines, and regarded as immigrants. We may use the language of immigration to describe the system as follows: under $\mathbf{Q}_{x}$, (i) the spine process $\widetilde{Y}$. starts at $x$ at tome 0 , and moves according to $\Pi_{x}^{\phi}$ and thus has infinite lifetime; (ii) given the trajectory $\widetilde{Y}$. of the spine, the fission times along the spine are distributed according to $L^{(A \beta)(\widetilde{Y})}$; (iii) at the fission time of node $v$ on the spine, $r^{v}-1$ particles are immigrated to the system at $\widetilde{Y}_{\zeta^{v}}$, the position of the spine, with $r^{v}$ being distributed according to the law $\dot{P}\left(\widetilde{Y}_{\zeta^{v}}\right):=\left(\dot{p}_{k}\left(\widetilde{Y}_{\zeta^{v}}\right)\right)_{k \geq 1} ;($ vi $)$ the immigrated particles give rise to the independent subtrees, which evolve as independent subtrees determined by the probability measure $\mathbf{P}_{\tilde{Y}_{\zeta_{v}}}$ shifted to the time of creation. The above Theorem 2.9 says that $\mathbf{Q}_{x}$ is the measure change of $\mathbf{P}_{x}$ by the martingale $\left\{M_{t}(\phi), t \geq 0\right\}$.

Theorem 2.10 We have the following decomposition for the martingale $\left\{M_{t}(\phi), t \geq 0\right\}$ :

$$
\begin{equation*}
\widetilde{\mathbf{Q}}_{x}\left[\phi(x) M_{t}(\phi) \mid \widetilde{\mathcal{G}}\right]=\phi\left(\widetilde{Y}_{t}\right) e^{-\lambda_{1} t}+\sum_{u<\xi_{t}}\left(r^{u}-1\right) \phi\left(\widetilde{Y}_{\zeta^{u}}\right) e^{-\lambda_{1} \zeta^{u}} \tag{2.10}
\end{equation*}
$$

Proof. We first decompose the martingale $\left\{\phi(x) M_{t}(\phi), t \geq 0\right\}$ as

$$
\phi(x) M_{t}(\phi)=e^{-\lambda_{1} t} \phi\left(\tilde{Y}_{t}\right)+e^{-\lambda_{1} t} \sum_{u \in L_{t}, u \neq \xi_{t}} \phi\left(Y_{t}^{u}\right) .
$$

The individuals $\left\{u \in L_{t}, u \neq \xi_{t}\right\}$ can be partitioned into subtrees created from fissions along the spines. That is, each node $u<\xi_{t}$ in the spine $\xi$ has given birth at time $\zeta^{u}$ to $r^{u}$
offspring among which one has been chosen as a node of the spine while the other $r^{u}-1$ individuals go off independently to make the subtree $(\tau, M)_{j}^{u}$. Put

$$
X_{t}^{j}=\sum_{v \in L_{t}, v \in(\tau, M)_{j}^{u}} \delta_{Y_{t}^{v}}(\cdot), \quad t \geq \zeta^{u} .
$$

$\left\{X_{t}^{j}, t \geq \zeta^{u}\right\}$ is a $(Y, \beta, \psi)$-branching Hunt process with birth time $\zeta^{u}$ and starting point $\widetilde{Y}_{\zeta^{u}}$. Then

$$
\begin{equation*}
\phi(x) M_{t}(\phi)=e^{-\lambda_{1} t} \phi\left(\tilde{Y}_{t}\right)+\sum_{u<\xi_{t}} \sum_{j: u j \in O_{u}} M_{t}^{u, j}(\phi) \phi\left(\tilde{Y}_{\zeta^{u}}\right) e^{-\lambda_{1} \zeta^{u}}, \tag{2.11}
\end{equation*}
$$

where

$$
M_{t}^{u, j}(\phi):=e^{-\lambda_{1}\left(t-\zeta^{u}\right)} \frac{\left\langle\phi, X_{t-\zeta^{u}}^{j}\right\rangle}{\phi\left(\widetilde{Y}_{\zeta^{u}}\right)} .
$$

By definition (2.9), conditional on $\widetilde{\mathcal{G}}, u j \in O_{v}$ evolve as independent subtrees determined by the probability measure $\mathbf{P}_{\widetilde{Y}_{\zeta^{u}}}$ shifted to $\zeta^{u}$, the time of creation. Therefore, conditional on $\widetilde{\mathcal{G}},\left\{M_{t}^{u, j}(\phi), t \geq 0\right\}$ is a $\widetilde{\mathbf{Q}}_{x}$-martingale on the subtree $(\tau, M)_{j}^{u}$, and therefore

$$
\widetilde{\mathbf{Q}}_{x}\left(M_{t}^{u, j}(\phi) \mid \widetilde{\mathcal{G}}\right)=1
$$

Thus taking $\widetilde{\mathbf{Q}}_{x}$ conditional expectation of (2.11) gives

$$
\widetilde{\mathbf{Q}}_{x}\left[\phi(x) M_{t}(\phi) \mid \widetilde{\mathcal{G}}\right]=\phi\left(\widetilde{Y}_{t}\right) e^{-\lambda_{1} t}+\sum_{u<\xi_{t}}\left(r^{u}-1\right) \phi\left(\widetilde{Y}_{\zeta^{u}}\right) e^{-\lambda_{1} \zeta^{u}},
$$

which completes the proof.

Theorem 2.11 For any $u \in \Gamma$, it holds that

$$
\widetilde{\mathbf{Q}}_{x}\left(\xi_{t}=u \mid \mathcal{F}_{t}\right)=\mathbf{1}_{\left\{u \in L_{t}\right\}} \frac{\phi\left(Y_{t}^{u}\right)}{\left\langle\phi, X_{t}\right\rangle}
$$

Proof. It suffice to show that, for any $B \in \mathcal{F}_{t}$,

$$
\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathrm{d} \widetilde{\mathbf{Q}}_{x}=\int_{B} \mathbf{1}_{\left\{u \in L_{t}\right\}} \frac{\phi\left(Y_{t}^{u}\right)}{\left\langle\phi, X_{t}\right\rangle} \mathrm{d} \widetilde{\mathbf{Q}}_{x} .
$$

By definition (2.9),

$$
\begin{aligned}
\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathrm{d} \widetilde{\mathbf{Q}}_{x} & =\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathbf{1}_{\left\{\xi_{t} \in L_{t}\right\}} \prod_{v<\xi_{t}} r^{v} \frac{\phi\left(\widetilde{Y}_{t}\right)}{\phi(x)} e^{-\lambda_{1} t} \mathrm{~d} \widetilde{\mathbf{P}}_{x} \\
& =\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathbf{1}_{\left\{u \in L_{t}\right\}} \prod_{v<u} r^{v} \frac{\phi\left(Y_{t}^{u}\right)}{\phi(x)} e^{-\lambda_{1} t} \mathrm{~d} \widetilde{\mathbf{P}}_{x}
\end{aligned}
$$

By (2.3),

$$
\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathrm{d} \widetilde{\mathbf{Q}}_{x}=\int_{B} \mathbf{1}_{\left\{u \in L_{t}\right\}} \frac{\phi\left(Y_{t}^{u}\right)}{\phi(x)} e^{-\lambda_{1} t} \mathrm{~d} \mathbf{P}_{x}
$$

It follows from Theorem 2.9 that for any $A \in \mathcal{F}_{t}$,

$$
\mathbf{P}_{x}\left(A \cap\left(M_{t}(\phi)>0\right)\right)=\mathbf{P}_{x}\left(\frac{M_{t}(\phi)}{M_{t}(\phi)}, A \cap\left(M_{t}(\phi)>0\right)\right)=\mathbf{Q}_{x}\left(\frac{1}{M_{t}(\phi)}, A\right)
$$

Since $\left\{u \in L_{t}\right\} \subset\left(M_{t}(\phi)>0\right)$, we have

$$
\int_{B} \mathbf{1}_{\left\{\xi_{t}=u\right\}} \mathrm{d} \widetilde{\mathbf{Q}}_{x}=\int_{B} \mathbf{1}_{\left\{u \in L_{t}\right\}} \frac{\phi\left(Y_{t}^{u}\right)}{\left\langle\phi, X_{t}\right\rangle} \mathrm{d} \mathbf{Q}_{x}
$$

The proof is complete.
As consequences of the result above, we have the following
Corollary 2.12 If

$$
f=\sum_{u \in L_{t}} f^{u}(\tau, M) \mathbf{1}_{\left\{\xi_{t}=u\right\}}
$$

with $f^{u} \in \mathcal{F}_{t}$, then

$$
\widetilde{\mathbf{Q}}_{x}\left(f \mid \mathcal{F}_{t}\right)=\sum_{u \in L_{t}} f_{u} \frac{\phi\left(Y_{t}^{u}\right)}{\left\langle\phi, X_{t}\right\rangle} \quad \text { on } L_{t} \neq \emptyset
$$

Corollary 2.13 If $g$ is a Borel function on $E$ then

$$
\left\langle g \phi, X_{t}\right\rangle=\widetilde{\mathbf{Q}}_{x}\left(g\left(\widetilde{Y}_{t}\right) \mid \mathcal{F}_{t}\right)\left\langle\phi, X_{t}\right\rangle
$$

Proof. Writing $g\left(\widetilde{Y}_{t}\right)=\sum_{u \in L_{t}} g\left(Y_{t}^{u}\right) \mathbf{1}_{\left\{\xi_{t}=u\right\}}$ and applying Corollary 2.12, we immediately get the desired conclusion.

## 3 Applications

## 3.1 $L \log L$ criterion for supercritical branching Hunt processes

In this subsection, we will use the spine decomposition to prove the $L \log L$ theorem for branching Hunt processes without assuming that each individual has at least one child.

Let $\left\{\widehat{P}_{t}, t \geq 0\right\}$ be the dual semigroup of $\left\{P_{t}, t \geq 0\right\}$ on $L^{2}(E, m)$, that is

$$
\int_{E} f(x) P_{t} g(x) m(\mathrm{~d} x)=\int_{E} g(x) \widehat{P}_{t} f(x) m(\mathrm{~d} x), \quad f, g \in L^{2}(E, m) .
$$

We will use $\mathbf{A}$ and $\widehat{\mathbf{A}}$ to denote the generators of the semigroups $\left\{P_{t}\right\}$ and $\left\{\widehat{P}_{t}\right\}$ on $L^{2}(E, m)$ respectively. In this subsection, we will assume the following

Assumption 3.1 (i) There exists a family of continuous strictly positive functions $\{p(t, \cdot, \cdot) ; t>$ $0\}$ on $E \times E$ such that for any $(t, x) \in(0, \infty) \times E$ and $f \in \mathcal{B}^{+}(E)$, we have

$$
P_{t} f(x)=\int_{E} p(t, x, y) f(y) m(\mathrm{~d} y), \quad \widehat{P}_{t} f(x)=\int_{E} p(t, y, x) f(y) m(\mathrm{~d} y) .
$$

(ii) The semigroups $\left\{P_{t}\right\}$ and $\left\{\widehat{P}_{t}\right\}$ are ultracontractive, that is, for any $t>0$, there exists a constant $c_{t}>0$ such that

$$
p(t, x, y) \leq c_{t} \quad \text { for any }(x, y) \in E \times E
$$

Let $\left\{\widehat{P}_{t}^{(1-A) \beta}, t \geq 0\right\}$ be the dual semigroup of $\left\{P_{t}^{(1-A) \beta}, t \geq 0\right\}$ on $L^{2}(E, m)$. Under Assumption 3.1, we can easily show that the semigroups $\left\{P_{t}^{(1-A) \beta}\right\}$ and $\left\{\widehat{P}_{t}^{(1-A) \beta}\right\}$ are strongly continuous on $L^{2}(E, m)$. Moreover, there exists a family of continuous strictly positive functions $\left\{p^{(1-A) \beta}(t, \cdot, \cdot) ; t>0\right\}$ on $E \times E$ such that for any $(t, x) \in(0, \infty) \times E$ and $f \in \mathcal{B}^{+}(E)$, we have
$P_{t}^{(1-A) \beta} f(x)=\int_{E} p^{(1-A) \beta}(t, x, y) f(y) m(\mathrm{~d} y), \quad \widehat{P}_{t}^{(1-A) \beta} f(x)=\int_{E} p^{(1-A) \beta}(t, y, x) f(y) m(\mathrm{~d} y)$.
The generators of $\left\{P_{t}^{(1-A) \beta}\right\}$ and $\left\{\widehat{P}_{t}^{(1-A) \beta}\right\}$ can be formally written as $\mathbf{A}+(A-1) \beta$ and $\widehat{\mathbf{A}}+(A-1) \beta$ respectively.

Let $\sigma(\mathbf{A}+(A-1) \beta)$ and $\sigma(\widehat{\mathbf{A}}+(A-1) \beta)$ denote the spectra of the operators $\mathbf{A}+(A-1) \beta$ and $\widehat{\mathbf{A}}+(A-1) \beta$, respectively. It follows from Jentzch's Theorem (Theorem V.6.6 on page 333 of [25] ) and the strong continuity of $\left\{P_{t}^{(1-A) \beta}\right\}$ and $\left\{\widehat{P}_{t}^{(1-A) \beta}\right\}$ that the common value $\lambda_{1}:=\sup \operatorname{Re}(\sigma(\mathbf{A}+(A-1) \beta))=\sup \operatorname{Re}(\sigma(\widehat{\mathbf{A}}+(A-1) \beta))$ is an eigenvalue of multiplicity 1 for both $\mathbf{A}+(A-1) \beta$ and $\widehat{\mathbf{A}}+(A-1) \beta$, and that an eigenfunction $\phi$ of $\mathbf{A}+(A-1) \beta$ associated with $\lambda_{1}$ can be chosen to be strictly positive a.e. on $E$ and an eigenfunction $\widehat{\phi}$ of $\widehat{\mathbf{A}}+(A-1) \beta$ associated with $\lambda_{1}$ can be chosen to be strictly positive a.e. on $E$. By [12, Proposition 2.3] we know that $\phi$ and $\widehat{\phi}$ are strictly positive and continuous on $E$. We choose $\phi$ and $\widehat{\phi}$ so that $\int_{E} \phi^{2}(x) m(\mathrm{~d} x)=\int_{E} \phi(x) \widehat{\phi}(x) m(\mathrm{~d} x)=1$. Then

$$
\begin{equation*}
\phi(x)=e^{-\lambda_{1} t} P_{t}^{(1-A) \beta} \phi(x), \quad \widehat{\phi}(x)=e^{-\lambda_{1} t} \widehat{P}_{t}^{(1-A) \beta} \widehat{\phi}(x), \quad x \in E . \tag{3.1}
\end{equation*}
$$

Therefore Assumption 3.1 implies Assumption 1.1. We can define $\Pi_{x}^{\phi}, x \in E$, by a martingale change of measure, see (1.7). Then $\left\{Y, \Pi_{x}^{\phi}\right\}$ is a conservative Markov process, and $\phi \widehat{\phi}$ is the unique invariant probability density for the semigroup $P_{t}^{(1-A) \beta}$, that is, for any $f \in \mathcal{B}^{+}(E)$ and $t \geq 0$,

$$
\int_{E} \phi(x) \widehat{\phi}(x) P_{t}^{(1-A) \beta} f(x) m(\mathrm{~d} x)=\int_{E} f(x) \phi(x) \widehat{\phi}(x) m(\mathrm{~d} x)
$$

Let $p^{\phi}(t, x, y)$ be the transition density of $Y$ in $E$ under $\Pi_{x}^{\phi}$. Then

$$
p^{\phi}(t, x, y)=\frac{e^{-\lambda_{1} t}}{\phi(x)} p^{(1-A) \beta}(t, x, y) \phi(y)
$$

In this subsection, we also assume the following

Assumption 3.2 The semigroups $\left\{P_{t}^{(1-A) \beta}\right\}$ and $\left\{\widehat{P}_{t}^{(1-A) \beta}\right\}$ are intrinsic ultracontractive, that is, for any $t>0$ there exists a constant $c_{t}$ such that

$$
p^{(1-A) \beta}(t, x, y) \leq c_{t} \phi(x) \widehat{\phi}(y), \quad x, y \in E
$$

It follows from [12, Theorem 2.8] that

$$
\begin{equation*}
\left|\frac{e^{-\lambda_{1} t} p^{(1-A) \beta}(t, x, y)}{\phi(x) \widehat{\phi}(y)}-1\right| \leq c e^{-\nu t}, \quad x \in E \tag{3.2}
\end{equation*}
$$

for some positive constants $c$ and $\nu$, which is equivalent to

$$
\begin{equation*}
\sup _{x \in E}\left|\frac{p^{\phi}(t, x, y)}{\phi(y) \widehat{\phi}(y)}-1\right| \leq c e^{-\nu t} \tag{3.3}
\end{equation*}
$$

Thus for any $f \in \mathcal{B}_{b}^{+}(E)$ we have

$$
\sup _{x \in E}\left|\int_{E} p^{\phi}(t, x, y) f(y) m(\mathrm{~d} y)-\int_{E} \phi(y) \widehat{\phi}(y) f(y) m(\mathrm{~d} y)\right| \leq c e^{-\nu t} \int_{E} \phi(y) \widehat{\phi}(y) f(y) m(\mathrm{~d} y) .
$$

Consequently we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{\int_{E} p^{\phi}(t, x, y) f(y) m(\mathrm{~d} y)}{\int_{E} \phi(y) \widehat{\phi}(y) f(y) m(\mathrm{~d} y)}=1, \quad \text { uniformly for } f \in \mathcal{B}_{b}^{+}(E) \text { and } x \in E \tag{3.4}
\end{equation*}
$$

We also assume that
Assumption $3.3 \lambda_{1}>0$.
The above assumption says that the branching Hunt process is supercritical. There are many examples of Hunt processes satisfying Assumptions 3.1] and 3.2, see [18, Remark 1.4].

The purpose of this subsection is to extend the probabilistic proof of the Kesten-Stigum $L \log L$ theorem to branching Hunt processes without assuming that each individual has at least one child. Let

$$
\begin{equation*}
l(x)=\sum_{k=2}^{\infty} k \phi(x) \log ^{+}(k \phi(x)) p_{k}(x), \quad x \in E \tag{3.5}
\end{equation*}
$$

The main result of this subsection can be stated as follows.
Theorem 3.4 Suppose that $\left\{X_{t} ; t \geq 0\right\}$ is a $(Y, \beta, \psi)$-branching Hunt process and that Assumptions 3.1, 3.2 and 3.3 are satisfied. Then $M_{\infty}(\phi)$ is non-degenerate under $\mathbf{P}_{\mu}$ for any nonzero measure $\mu \in \mathbf{M}_{p}(E)$ if and only if

$$
\begin{equation*}
\int_{E} \widehat{\phi}(x) \beta(x) l(x) m(\mathrm{~d} x)<\infty \tag{3.6}
\end{equation*}
$$

where $l$ is defined in (3.5).

First, we give two lemmas. The first lemma is basically [7, Theorem 4.3.3].
Lemma 3.5 Suppose that $\mathbf{P}$ and $\mathbf{Q}$ are two probability measures on a measurable space $\left(\Omega, \mathcal{F}_{\infty}\right)$ with filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$, such that for some nonnegative martingale $\left\{Z_{t}, t \geq 0\right\}$,

$$
\left.\frac{\mathrm{d} \mathbf{Q}}{\mathrm{~d} \mathbf{P}}\right|_{\mathcal{F}_{t}}=Z_{t}
$$

The limit $Z_{\infty}:=\lim \sup _{t \rightarrow \infty} Z_{t}$ therefore exists and is finite almost surely under $\mathbf{P}$. Furthermore, for any $F \in \mathcal{F}_{\infty}$

$$
\mathbf{Q}(F)=\int_{F} Z_{\infty} \mathrm{d} \mathbf{P}+\mathbf{Q}\left(F \cap\left\{Z_{\infty}=\infty\right\}\right)
$$

and consequently,

$$
\begin{aligned}
& \text { (a) } \mathbf{P}\left(Z_{\infty}=0\right)=1 \Longleftrightarrow \mathbf{Q}\left(Z_{\infty}=\infty\right)=1 \\
& \text { (b) } \int Z_{\infty} \mathrm{d} \mathbf{P}=\int Z_{0} \mathrm{~d} \mathbf{P} \Longleftrightarrow \mathbf{Q}\left(Z_{\infty}<\infty\right)=1
\end{aligned}
$$

Now we are going to give a lemma which is the key to the proof of Theorem 3.4. To state this lemma, we need some more notation. Note that

$$
\widetilde{\mathbf{Q}}_{x}\left(\xi_{t} \neq \dagger, \forall t>0\right)=1
$$

and thus the lifetime the spine is $\infty$. We can select a line of descendants $\xi=\left\{\xi_{0}=\right.$ $\left.\emptyset, \xi_{1}, \xi_{2}, \cdots\right\}$, where $\xi_{n+1} \in \tau$ is an offspring of $\xi_{n} \in \tau, n=0,1, \cdots$, such that $\xi_{t}=$ $\xi_{n_{t}}, t \geq 0$. Under $\widetilde{\mathbf{Q}}_{x}$, given $\widetilde{\mathcal{G}}, N_{t}:=\left\{\left\{\left(\zeta^{\xi_{i}}, r^{\xi_{i}}\right): i=0,1,2, \cdots, n_{t}-1\right\}: t \geq 0\right\}$ is a Poisson point process with intensity measure $(A \beta)\left(\widetilde{Y}_{t}\right) \mathrm{d} t \mathrm{~d} \dot{P}\left(\widetilde{Y}_{t}\right)$, where for each $y \in E$, $\dot{P}(y)$ is the size-biased distribution of $P(y)$. To simplify notation, $\zeta^{\xi_{i}}$ and $r^{\xi_{i}}$ will be denoted as $\zeta_{i}$ and $r_{i}$, respectively.

Lemma 3.6 (1) If $\int_{E} \widehat{\phi}(y) \beta(y) l(y) m(\mathrm{~d} y)<\infty$, then

$$
\sum_{i=0}^{\infty} e^{-\lambda_{1} \zeta_{i}} r_{i} \phi\left(\widetilde{Y}_{\zeta_{i}}\right)<\infty, \quad \widetilde{\mathbf{Q}}_{x} \text {-a.s. }
$$

(2) If $\int_{E} \widehat{\phi}(y) \beta(y) l(y) m(\mathrm{~d} y)=\infty$, then

$$
\limsup _{i \rightarrow \infty} e^{-\lambda_{1} \zeta_{i}} r_{i} \phi\left(\widetilde{Y}_{\zeta_{i}}\right)=\infty, \quad \widetilde{\mathbf{Q}}_{x} \text {-a.s. }
$$

The proof of the above result goes along the same line as the proof of [18, Lemma 3.2]. We omit the details here.

Proof of Theorem 3.4. The proof heavily depends on the decomposition (2.10).

When $\int_{E} \widehat{\phi}(x) \beta(x) l(x) m(\mathrm{~d} x)<\infty$, the first conclusion of Lemma 3.6 says that

$$
\sup _{t>0} \widetilde{\mathbf{Q}}_{x}\left[\phi(x) M_{t}(\phi) \mid \widetilde{\mathcal{G}}\right] \leq \sum_{u \in \xi} r^{u} \phi\left(\widetilde{Y}_{\zeta^{u}}\right) e^{-\lambda_{1} \zeta^{u}}+\|\phi\|_{\infty}<\infty .
$$

Fatou's lemma for conditional probability implies that $\lim _{\inf }^{t \rightarrow \infty}$ $M_{t}(\phi)<\infty, \widetilde{\mathbf{Q}}_{x}$-a.s. The Radon-Nikodym derivative tells us that $\left\{M_{t}(\phi)^{-1}, t \geq 0\right\}$ is a nonnegative supermartingale under $\mathbf{Q}_{x}$ and therefore has a finite limit $\mathbf{Q}_{x}$-a.s. So $\lim _{t \rightarrow \infty} M_{t}(\phi)=M_{\infty}<\infty$, $\mathrm{Q}_{x}$-a.s. Lemma 3.5 implies that in this case,

$$
\mathbf{P}_{x}\left[M_{\infty}(\phi)\right]=\lim _{t \rightarrow \infty} \mathbf{P}_{x}\left[M_{t}(\phi)\right]=1
$$

When $\int_{E} \widehat{\phi}(x) \beta(x) l(x) m(\mathrm{~d} x)=\infty$, using the second conclusion in Lemma 3.6, we can get under $\mathbf{Q}_{x}$,

$$
\limsup _{t \rightarrow \infty} \phi(x) M_{t}(\phi) \geq \limsup _{t \rightarrow \infty} \phi\left(\widetilde{Y}_{\zeta_{n_{t}}}\right)\left(r_{n_{t}}-1\right) e^{-\lambda_{1} \zeta_{n_{t}}}=\infty
$$

This yields that $M_{\infty}(\phi)=\infty, \mathbf{Q}_{x^{-}}$a.s. Using Lemma 3.5 again, we get $M_{\infty}(\phi)=0, \mathbf{P}_{x^{-}}$ a.s. The proof is finished.

Theorem 3.7 Suppose that $\left\{X_{t} ; t \geq 0\right\}$ is a $(Y, \beta, \psi)$-branching Hunt process and that Assumptions 3.1, 3.2 and 3.3 are satisfied. Suppose (3.6) holds, then there exists $\Omega_{0} \subset \Omega$ with full probability (that is, $\mathbf{P}_{x}\left(\Omega_{0}\right)=1$ for every $x \in E$ ) such that, for every $\omega \in \Omega_{0}$ and for every bounded Borel function $f$ on $E$ with compact support whose set of discontinuous points has zero m-measure, we have

$$
\lim _{t \rightarrow \infty} e^{-\lambda_{1} t}\left\langle f, X_{t}\right\rangle=M_{\infty}(\phi) \int_{E} \widehat{\phi}(x) f(x) m(\mathrm{~d} x)
$$

With our spine decomposition theorem and Theorem 3.4, the proof of [26] goes through. We omit the details.

### 3.2 Kolmogorov type theorem for critical branching Hunt process

In this subsection, we use our spine decomposition to give a proof of a Kolmogorov type theorem for critical branching Hunt processes, see Theorem 3.10 below. The key to prove this result is Lemma 3.11 below, which says that studying the limit of $\frac{t \mathbf{P}_{x}\left(\left\langle\phi, X_{t}\right\rangle>0\right)}{\phi(x)}$ as $t \rightarrow \infty$ is equivalent to studying the limit of $\int_{E} t \mathbf{P}_{x}\left(\left\langle\phi, X_{t}\right\rangle>0\right) \widehat{\phi}(x) m(\mathrm{~d} x)$ as $t \rightarrow \infty$. The proof of Lemma 3.11 uses our spine decomposition.

Throughout this subsection, we assume that Assumptions 3.1 and 3.2 hold. Let $\lambda_{1}, \phi$ and $\widehat{\phi}$ be defined as in Subsection 3.1, Put

$$
\begin{equation*}
V(x):=\psi^{\prime \prime}(x, 1)=\sum_{k=2}^{\infty} k(k-1) p_{k}(x), \quad x \in E \tag{3.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma^{2}:=\int_{E} \beta(y) V(y) \phi^{2}(y) \widehat{\phi}(y) m(\mathrm{~d} y) . \tag{3.8}
\end{equation*}
$$

Let $\Psi$ be the operator on $\mathcal{B}_{E}^{+}$defined by

$$
(\Psi f)(x):=\psi(x, f(x)), \quad f \in \mathcal{B}^{+}(E), x \in E .
$$

Recall that $f$ is automatically extended to $E_{\Delta}$ by setting $f(\Delta)=0$. For $f \in \mathcal{B}^{+}(E)$, put

$$
V_{t}\left(e^{-f}\right)(x):=\mathbf{P}_{x}\left(\exp \left\langle-f, X_{t}\right\rangle\right), \quad t \geq 0, x \in E
$$

Then (1.4) can be written as

$$
V_{t}\left(e^{-f}\right)(x)=P_{t}\left(e^{-f} \mathbf{1}_{E}\right)(x)+\Pi_{x}(t \geq \zeta)+\int_{0}^{t} P_{r}\left[\left(\Psi\left(V_{t-r}\left(e^{-f}\right)\right)-V_{t-r}\left(e^{-f}\right)\right) \beta\right](x) \mathrm{d} s
$$

where we used the fact that $\beta(\Delta)=0$. Note that

$$
1=\Pi_{x}(t<\zeta)+\Pi_{x}(t \geq \zeta)=P_{t} \mathbf{1}_{E}(x)+\Pi_{x}(t \geq \zeta)
$$

Thus we have

$$
1-V_{t}\left(e^{-f}\right)(x)=P_{t}\left(\left(1-e^{-f}\right) \mathbf{1}_{E}\right)(x)+\int_{0}^{t} P_{r}\left[\left(-\Psi\left(V_{t-r}\left(e^{-f}\right)\right)+V_{t-r}\left(e^{-f}\right)\right) \beta\right](x) \mathrm{d} s
$$

which can be written as

$$
\begin{aligned}
& 1-V_{t}\left(e^{-f}\right)(x)=P_{t}\left(\left(1-e^{-f}\right) \mathbf{1}_{E}\right)(x) \\
& +\int_{0}^{t} P_{r}\left[\left(A V_{t-r}\left(e^{-f}\right)+1-A-\Psi\left(V_{t-r}\left(e^{-f}\right)\right)+(A-1)\left(1-V_{t-r}\left(e^{-f}\right)\right)\right) \beta\right](x) \mathrm{d} s,
\end{aligned}
$$

which in turn is equivalent to

$$
\begin{align*}
1-V_{t}\left(e^{-f}\right)(x)= & P_{t}^{(1-A) \beta}\left(\left(1-e^{-f}\right) \mathbf{1}_{E}\right)(x) \\
& +\int_{0}^{t} P_{r}^{(1-A) \beta}\left[\left(A V_{t-r}\left(e^{-f}\right)+1-A-\Psi\left(V_{t-r}\left(e^{-f}\right)\right) \beta\right] \mathrm{d} s .\right. \tag{3.9}
\end{align*}
$$

We first consider the asymptotic behavior of

$$
v_{t}(x):=\mathbf{P}_{x}\left(X_{t}(E)=0\right), \quad t>0, x \in E
$$

By monotone convergence, we have

$$
v_{t}(x)=\lim _{\theta \rightarrow \infty} V_{t}\left(e^{-\theta 1_{E}}\right)(x), \quad t>0, x \in E
$$

By the Markov property of $X$, we have

$$
\begin{align*}
V_{t} v_{s}(x) & =\mathbf{P}_{x}\left[e^{\left\langle X_{t}, \log \lim _{\theta \rightarrow \infty} V_{s}\left(e^{-\theta 1_{E}}\right)\right\rangle}\right]=\lim _{\theta \rightarrow \infty} \mathbf{P}_{x}\left[e^{\left\langle X_{t}, \log V_{s}\left(e^{-\theta 1_{E}}\right)\right\rangle}\right]  \tag{3.10}\\
& =\lim _{\theta \rightarrow \infty} V_{t} V_{s}\left(e^{-\theta \mathbf{1}_{E}}\right)(x)=v_{t+s}(x), \quad s, t>0, x \in E
\end{align*}
$$

Using (3.9), (3.10) and $f=-\log v_{s}$, we get

$$
\begin{align*}
1-v_{t+s}(x)= & P_{t}^{(1-A) \beta}\left(\left(1-v_{s}\right) \mathbf{1}_{E}\right)(x) \\
& +\int_{0}^{t} P_{r}^{(1-A) \beta}\left[\left(A v_{t-r+s}+1-A-\Psi\left(v_{t-r+s}\right)\right) \beta\right](x) \mathrm{d} s \tag{3.11}
\end{align*}
$$

Define

$$
v_{\infty}(x)=: \lim _{t \rightarrow \infty} v_{t}(x)=\mathbf{P}_{x}\left(\exists t>0 \text { such that } X_{t}(E)=0\right)
$$

Recall the quantities $V$ and $\sigma^{2}$ defined in (3.7) and (3.8). Throughout this subsection we assume that

Assumption 3.8 (i) The branching Hunt process $X$ is critical, i.e., $\lambda_{1}=0$;
(ii) $\sigma^{2}>0$;
(iii) the function $\phi V: x \rightarrow \phi(x) V(x)$ is bounded on $E$.

Lemma 3.9 Suppose that Assumptions 3.1, 3.2 and 3.8 (i-ii) hold. Then for any $x \in E$,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup _{x \in E} \frac{\mathbf{P}_{x}\left(X_{t}(E)>0\right)}{\phi(x)}=0 \tag{3.12}
\end{equation*}
$$

Proof. For any $f, g \in \mathcal{B}^{+}(E)$, we use $\langle f, g\rangle_{m}$ to denote $\int_{E} f(x) g(x) m(\mathrm{~d} x)$. Under Assumption 3.2, $\langle 1, \widehat{\phi}\rangle_{m}<\infty$. In fact, according to (3.2), for $t>0$ large enough, there is a $c_{t}^{\prime}>0$ such that

$$
\widehat{\phi}(y) \leq q(t, x, y)\left(c_{t}^{\prime}\right)^{-1} \phi^{-1}(x)
$$

and clearly, as a function of $y$, the right hand above is integrable with respect to $m$. Integrating (3.9) with respect to $\widehat{\phi}(x) m(\mathrm{~d} x)$, we get that

$$
\begin{equation*}
\left\langle 1-v_{t+s}, \widehat{\phi}\right\rangle_{m}=\left\langle 1-v_{s}, \widehat{\phi}\right\rangle_{m}+\int_{0}^{t}\left\langle\left(A v_{t-r+s}+1-A-\Psi\left(v_{t-r+s}\right)\right) \beta, \widehat{\phi}\right\rangle_{m} \mathrm{~d} s \tag{3.13}
\end{equation*}
$$

Letting $s \rightarrow \infty$, we get

$$
\left\langle 1-v_{\infty}, \widehat{\phi}\right\rangle_{m}=\left\langle 1-v_{\infty}, \widehat{\phi}\right\rangle_{m}+t\left\langle\left(A v_{\infty}+1-A-\Psi\left(v_{\infty}\right)\right) \beta, \widehat{\phi}\right\rangle_{m}
$$

Thus we have

$$
\left\langle A v_{\infty}+1-A-\Psi\left(v_{\infty}\right), \beta \widehat{\phi}\right\rangle_{m}=0
$$

It is easy to check that for any $x \in E, A z+1-A-\psi(x, z) \leq 0, \forall z \in[0,1]$. Since $\widehat{\phi}(x)>0$ on $E$, we must have

$$
\begin{equation*}
A v_{\infty}+1-A-\Psi\left(v_{\infty}\right)=0, \quad m \text {-a.e. on }\{x \in E, \beta(x)>0\} . \tag{3.14}
\end{equation*}
$$

Letting $s \rightarrow \infty$ in (3.11), we get

$$
1-v_{\infty}(x)=P_{t}^{(1-A) \beta}\left(\left(1-v_{\infty}\right) \mathbf{1}_{E}\right)(x)+\int_{0}^{t} P_{s}^{(1-A) \beta}\left[\left(A v_{\infty}+1-A-\psi\left(v_{\infty}\right)\right) \beta\right](x) \mathrm{d} s
$$

and thus $1-v_{\infty}(x)=P_{t}^{(1-A) \beta}\left(\left(1-v_{\infty}\right) \mathbf{1}_{E}\right)(x)$, which says that $1-v_{\infty}$ is an eigenfunction of $\mathbf{A}+(A-1) \beta$ corresponding to the eigenvalue $\lambda_{1}=0$. Since the eigenvalue $\lambda_{1}=0$ is simple, $1-v_{\infty}=c \phi$ on $E$ for some constant $c$. Note that, for each fixed $x \in E$, the function $\psi_{0}(x, z):=\psi(x, z)-A(x) z+A(x)-1$ is strictly decreasing for $z \in(0,1)$ with $\psi_{0}(x, 1)=0$ and $\psi_{0}(x, 0)=\sum_{k=2}^{\infty}(k-1) p_{k}(x) \geq 0$. Assumption 3.8 (ii) implies that $m\left(\left\{x \in E ; \beta(x)>0, \psi_{0}(x, 0)>0\right\}\right)>0$. Since $v_{\infty}$ satisfies (3.14), we must have $c=0$, or equivalently $v_{\infty} \equiv 1$. Thus

$$
\lim _{t \rightarrow \infty} \mathbf{P}_{x}\left(X_{t}(E)>0\right)=1-v_{\infty}(x)=0, \quad x \in E
$$

By (3.11) and (3.2), we have

$$
1-v_{t+s}(x) \leq P_{t}^{(1-A) \beta}\left(\left(1-v_{s}\right) \mathbf{1}_{E}\right)(x) \leq\left(1+c e^{-\nu t}\right) \phi(x) \int_{E} \widehat{\phi}(y)\left(1-v_{s}\right)(y) m(\mathrm{~d} y)
$$

which implies

$$
\frac{1-v_{t+s}(x)}{\phi(x)} \leq\left(1+c e^{-\nu t}\right) \int_{E} \widehat{\phi}(y)\left(1-v_{s}\right)(y) m(\mathrm{~d} y) .
$$

Using the monotonicity of $v_{t}$ in $t$, we get (3.12).
The following Kolmogorov type theorem is the main result of this subsection.
Theorem 3.10 Suppose that Assumptions 3.1, 3.2 and 3.8 hold. Then

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{t \mathbf{P}_{x}\left(\left\langle\phi, X_{t}\right\rangle>0\right)}{\phi(x)}=\frac{2}{\sigma^{2}} \tag{3.15}
\end{equation*}
$$

uniformly for $x \in E$.
We prove the above result by proving two lemmas first. Define

$$
\begin{equation*}
b(t):=\int_{E}\left(1-v_{t}(x)\right) \widehat{\phi}(x) m(\mathrm{~d} x)=\int_{E} \mathbf{P}_{x}\left(X_{t}(E)>0\right) \widehat{\phi}(x) m(\mathrm{~d} x) . \tag{3.16}
\end{equation*}
$$

Lemma 3.11 Under Assumptions 3.1, 3.2 and 3.8, we have

$$
\sup _{x \in E}\left|\frac{1-v_{t}(x)}{b(t) \phi(x)}-1\right| \underset{t \rightarrow \infty}{\longrightarrow} 0
$$

where $b(t)$ is defined in (3.16).
Proof. First note that

$$
\frac{1-v_{t}(x)}{\phi(x)}=\frac{\mathbf{P}_{x}\left(X_{t}(E)>0\right)}{\phi(x)}=\mathbf{Q}_{x}\left(\frac{\mathbf{1}_{\left(X_{t}(E)>0\right)}}{\left\langle\phi, X_{t}\right\rangle}\right)=\mathbf{Q}_{x}\left(X_{t}(\phi)^{-1}\right) .
$$

For $0<t_{0}<t<\infty$, define

$$
I_{t}^{\left(0, t_{0}\right]}=\sum_{u \in L_{t_{0}}, u \neq \xi_{t_{0}}} \delta_{Y_{t}^{u}} \quad \text { and } \quad I_{t}^{\left(t_{0}, t\right]}=\sum_{u \in L_{t} \backslash L_{t_{0}}} \delta_{Y_{t}^{u}} .
$$

Then we have

$$
\begin{equation*}
X_{t}=I_{t}^{\left(0, t_{0}\right]}+I_{t}^{\left(t_{0}, t\right]} \tag{3.17}
\end{equation*}
$$

Define

$$
\mathbf{Q}_{\phi \widehat{\phi} m}(\cdot):=\int_{E} \mathbf{Q}_{x}(\cdot) \phi(x) \widehat{\phi}(x) m(\mathrm{~d} x)
$$

Under $\mathbf{Q}_{\phi \widehat{\phi} m}, X_{0}=\delta_{Z}$ with $Z$ being an $E$-valued random variable with distribution $\phi \widehat{\phi} m$. It is easy to see, from the construction of $\mathbf{Q}_{x}$ and the Markov property of the immigration that for any $0<t_{0}<t<\infty$,

$$
\mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1} \mid \mathcal{G}_{t_{0}}\right]=\mathbf{Q}_{\widetilde{Y}_{t_{0}}}\left[\left(X_{t-t_{0}}(\phi)\right)^{-1}\right]=\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)\left(\widetilde{Y}_{t_{0}}\right) .
$$

Therefore, we have

$$
\mathbf{Q}_{\phi \widehat{\phi} m}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]=\mathbf{Q}_{\phi \widehat{\phi} m}\left[\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)\left(\widetilde{Y}_{t_{0}}\right)\right]=\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}
$$

and

$$
\begin{align*}
\mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right] & =\mathbf{Q}_{x}\left[\left(\phi^{-1}\left(1-v_{\left.t-t_{0}\right)}\right)\left(\widetilde{Y}_{t_{0}}\right)\right]\right. \\
& =\int_{E} p^{\phi}\left(t_{0}, x, y\right)\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)(y) m(\mathrm{~d} y) . \tag{3.18}
\end{align*}
$$

By the decomposition (3.17), we have

$$
\begin{align*}
\phi^{-1}\left(1-v_{t}(x)\right)= & \mathbf{Q}_{x}\left[\left(X_{t}(\phi)\right)^{-1}\right] \\
= & \mathbf{Q}_{\phi \widehat{\phi} m}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]+\left(\mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]-\mathbf{Q}_{\phi \widehat{\phi} m}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]\right) \\
& +\left(\mathbf{Q}_{x}\left[\left(X_{t}(\phi)\right)^{-1}-\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]\right)  \tag{3.19}\\
= & :\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}+\epsilon_{x}^{1}\left(t_{0}, t\right)+\epsilon_{x}^{2}\left(t_{0}, t\right) .
\end{align*}
$$

Suppose that $t_{0}>1$, and let $c, \nu>0$ be the constants in (3.3). Using (3.18), we have

$$
\begin{align*}
\left|\epsilon_{x}^{1}\left(t_{0}, t\right)\right| & =\left|\mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]-\mathbf{Q}_{\phi \widehat{\phi} m}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]\right| \\
& =\left|\int_{E} p^{\phi}\left(t_{0}, x, y\right)\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)(y) m(\mathrm{~d} y)-\left\langle 1-v_{t-t_{0}}, \widetilde{\phi}\right\rangle_{m}\right| \\
& \leq \int_{y \in E}\left|p^{\phi}\left(t_{0}, x, y\right)-(\phi \widehat{\phi})(y)\right|\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)(y) m(\mathrm{~d} y)  \tag{3.20}\\
& \leq c e^{-\nu t_{0}}\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m} .
\end{align*}
$$

We also have

$$
\begin{align*}
\left|\epsilon_{x}^{2}\left(t_{0}, t\right)\right| & =\left|\mathbf{Q}_{x}\left[\left(X_{t}(\phi)\right)^{-1}-\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]\right| \\
& =\mathbf{Q}_{x}\left[I_{t}^{\left(0, t_{0}\right]}(\phi) \cdot\left(X_{t}(\phi)\right)^{-1} \cdot\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right] \\
& \leq \mathbf{Q}_{x}\left[\mathbf{1}_{I_{t}^{\left(0, t_{0}\right]}(\phi)>0} \cdot\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1}\right]  \tag{3.21}\\
& =\mathbf{Q}_{x}\left(\mathbf{Q}_{x}\left[\mathbf{1}_{I_{t}^{\left(0, t_{0}\right]}(\phi)>0} \mid \mathcal{G}_{t_{0}}\right] \cdot \mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1} \mid \mathcal{G}_{t_{0}}\right]\right) .
\end{align*}
$$

Recall that $\zeta_{i}$ and $r_{i}$ are the shorthand notation for $\zeta_{\xi_{i}}$ and $r_{\xi_{i}}$ respectively. Note that

$$
\begin{aligned}
\mathbf{Q}_{x}\left[\mathbf{1}_{I_{t}^{\left(0, t_{0}\right]}(\phi)=0} \mid \mathcal{G}_{t_{0}}\right] & =\mathbf{Q}_{x}\left[\prod_{\zeta_{i} \leq t_{0}}\left(\mathbf{P}_{\tilde{Y}\left(\zeta_{i}\right)}\left(X_{t-\zeta_{i}}(E)=0\right)\right)^{r_{i}-1} \mid \mathcal{G}_{t_{0}}\right] \\
& \geq \mathbf{Q}_{x}\left[\prod_{\zeta_{i} \leq t_{0}}\left(\mathbf{P}_{\tilde{Y}\left(\zeta_{i}\right)}\left(X_{t-t_{0}}(E)=0\right)\right)^{r_{i}-1} \mid \mathcal{G}_{t_{0}}\right]
\end{aligned}
$$

and that

$$
\begin{align*}
\mathbf{Q}_{x}\left[\mathbf{1}_{I_{t}^{\left(0, t_{0}\right]}(\phi)>0} \mid \mathcal{G}_{t_{0}}\right] & \leq \mathbf{Q}_{x}\left[\prod_{\zeta_{i} \leq t_{0}}\left(r_{i}-1\right) \mathbf{P}_{\tilde{Y}\left(\zeta_{i}\right)}\left(X_{t-t_{0}}(E)>0\right) \mid \mathcal{G}_{t_{0}}\right] \\
& \leq \mathbf{Q}_{x}\left[\sum_{\zeta_{i} \leq t_{0}}\left(r_{i}-1\right)\left(1-v_{t-t_{0}}\right)\left(\widetilde{Y}\left(\zeta_{i}\right)\right)\right]  \tag{3.22}\\
& =\int_{0}^{t_{0}} \beta\left(\widetilde{Y}_{s}\right)(k-1) k p_{k}\left(\widetilde{Y}_{s}\right)\left(1-v_{t-t_{0}}\right)\left(\widetilde{Y}_{s}\right) \mathrm{d} s \\
& \leq t_{0}\|\beta V \phi\|_{\infty}\left\|\phi^{-1}\left(1-v_{t-t_{0}}\right)\right\|_{\infty} .
\end{align*}
$$

Thus by (3.21) and (3.22), we have

$$
\begin{align*}
\left|\epsilon_{x}^{2}\left(t_{0}, t\right)\right| & \left.\leq t_{0}\|\beta V \phi\|_{\infty}\left\|\phi^{-1}\left(1-v_{t-t_{0}}\right)\right\|_{\infty} \mathbf{Q}_{x}\left(\mathbf{Q}_{x}\left[\left(I_{t}^{\left(t_{0}, t\right]}(\phi)\right)^{-1} \mid \mathcal{G}_{t_{0}}\right]\right]\right) \\
& =t_{0}\|\beta V \phi\|_{\infty}\left\|\phi^{-1}\left(1-v_{t-t_{0}}\right)\right\|_{\infty} \int_{E} p^{\phi}\left(t_{0}, x, y\right)\left(\phi^{-1}\left(1-v_{t-t_{0}}\right)\right)(y) m(\mathrm{~d} y)  \tag{3.23}\\
& \leq t_{0}\|\beta V \phi\|_{\infty}\left\|\phi^{-1}\left(1-v_{t-t_{0}}\right)\right\|_{\infty}\left(1+c e^{-\nu t}\right)\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}
\end{align*}
$$

Combining (3.19), (3.20) and (3.23), we have that

$$
\begin{align*}
\left|\frac{\phi^{-1}\left(1-v_{t}(x)\right)}{\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}}-1\right| & \leq \frac{\left|\epsilon_{x}^{1}\left(t_{0}, t\right)\right|}{\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}}+\frac{\left|\epsilon_{x}^{2}\left(t_{0}, t\right)\right|}{\left\langle 1-v_{t-t_{0}}, \widehat{\phi}\right\rangle_{m}}  \tag{3.24}\\
& \leq c e^{-\gamma t_{0}}+t_{0}\|\beta V \phi\|_{\infty}\left\|\phi^{-1}\left(1-v_{t-t_{0}}\right)\right\|_{\infty}\left(1+c e^{-\nu t_{0}}\right)
\end{align*}
$$

Since we know from Lemma 3.9 that $\left\|\phi^{-1}\left(1-v_{t}\right)\right\|_{\infty} \rightarrow 0$ as $t \rightarrow \infty$, there exists a map $t \mapsto t_{0}(t)$ such that,

$$
t_{0}(t) \underset{t \rightarrow \infty}{\longrightarrow} \infty ; \quad t_{0}(t)\left\|\phi^{-1}\left(1-v_{t-t_{0}(t)}\right)\right\|_{\infty} \underset{t \rightarrow \infty}{\longrightarrow} 0
$$

Plugging this choice of $t_{0}(t)$ back into (3.24), we have that

$$
\begin{equation*}
\sup _{x \in E}\left|\frac{\phi^{-1}\left(1-v_{t}(x)\right)}{\left\langle 1-v_{t-t_{0}(t)}, \widehat{\phi}\right\rangle_{m}}-1\right| \underset{t \rightarrow \infty}{ } 0 . \tag{3.25}
\end{equation*}
$$

Now notice that

$$
\begin{align*}
\left|\frac{\left\langle 1-v_{t}, \widehat{\phi}\right\rangle_{m}}{\left\langle 1-v_{t-t_{0}(t)}, \widehat{\phi}\right\rangle_{m}}-1\right| & \leq \int\left|\frac{\phi^{-1}\left(1-v_{t}(x)\right)}{\left\langle 1-v_{t-t_{0}(t)}, \widehat{\phi}\right\rangle}-1\right| \phi \widehat{\phi}(x) m(\mathrm{~d} x)  \tag{3.26}\\
& \leq \sup _{x \in E}\left|\frac{\phi^{-1}\left(1-v_{t}(x)\right)}{\left\langle 1-v_{t-t_{0}(t)}, \widehat{\phi}\right\rangle_{m}}-1\right| \underset{t \rightarrow \infty}{\longrightarrow} 0 .
\end{align*}
$$

Finally, by (3.25), (3.26) and property of uniform convergence, we have

$$
\sup _{x \in E}\left|\frac{\phi^{-1}\left(1-v_{t}(x)\right)}{\left\langle 1-v_{t}, \widehat{\phi}\right\rangle_{m}}-1\right| \underset{t \rightarrow \infty}{ } 0
$$

as desired.

Lemma 3.12 Under Assumptions 3.1, 3.2 and 3.8, we have

$$
\frac{1}{t b(t)} \underset{t \rightarrow \infty}{\longrightarrow} \frac{1}{2}\langle\beta V \phi, \phi \widehat{\phi}\rangle_{m}
$$

where $b(t)=\left\langle 1-v_{t}, \widehat{\phi}\right\rangle_{m}$.
Proof. For $z \in[0,1]$, define

$$
\left.\psi_{0}(x, z):=\psi(x, z)-1-A(x)(z-1)\right)
$$

and

$$
R(x, z):=\psi_{0}(x, z)-\frac{1}{2} V(x)(z-1)^{2} .
$$

Note that $\psi^{\prime}(x, \theta) \leq 0$ and $\psi^{\prime \prime}(x, \theta) \geq 0$ for $\theta \in(z, 1]$. By the mean value theorem, $\psi_{0}(x, z)=\frac{1}{2} \psi^{\prime \prime}(x, \theta)(z-1)^{2} \leq \frac{1}{2} V(x)(z-1)^{2}$ with $\theta \in(z, 1]$. Thus

$$
R(x, z)=e(x, z)(z-1)^{2}, \quad \forall z \in[0,1],
$$

where $e(x, z)$ satisfies $|e(x, z)| \leq V(x), \forall z \in[0,1]$ and that

$$
\begin{equation*}
e(x, z) \underset{z \rightarrow 1}{\longrightarrow} 0, \quad x \in E \tag{3.27}
\end{equation*}
$$

Let $\Psi_{0}$ be the operator on $\mathcal{B}^{+}(E)$ defined by

$$
\left(\Psi_{0} f\right)(x):=\psi_{0}(x, f(x)), \quad f \in \mathcal{B}^{+}(E), x \in E
$$

Writing $l_{t}(x):=\left(1-v_{t}(x)\right)-b(t) \phi(x)$, Lemma 3.11 says that

$$
\begin{equation*}
\sup _{x \in E}\left|\frac{l_{t}(x)}{b(t) \phi(x)}\right| \underset{t \rightarrow \infty}{ } 0 \tag{3.28}
\end{equation*}
$$

Using (3.13), we see that $t \mapsto b(t)$ is differentiable on the set

$$
\mathbf{C}=\left\{t>s_{0}: \text { the function } t \mapsto\left\langle\Psi_{0}\left(v_{t}\right), \beta \widehat{\phi}\right\rangle_{m} \text { is continuous at } t\right\}
$$

and that

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t} b(t) & =-\left\langle\Psi_{0}\left(v_{t}\right), \widehat{\phi}\right\rangle_{m}=-\left\langle\frac{1}{2} V \cdot\left(1-v_{t}\right)^{2}+R\left(\cdot, v_{t}(\cdot)\right), \beta \hat{\phi}\right\rangle_{m} \\
& =-\left\langle\frac{1}{2} V \cdot\left(b(t) \phi+l_{t}\right)^{2}+R\left(\cdot, v_{t}(\cdot)\right), \beta \widehat{\phi}\right\rangle_{m}  \tag{3.29}\\
& =-b(t)^{2}\left[\frac{1}{2}\langle\beta V \phi, \phi \widehat{\phi}\rangle_{m}+g(t)\right], \quad t \in \mathbf{C}
\end{align*}
$$

where

$$
g(t)=\left\langle\frac{l_{t}}{b(t) \phi}, \beta V \phi^{2} \hat{\phi}\right\rangle_{m}+\frac{1}{2}\left\langle\left(\frac{l_{t}}{b(t) \phi}\right)^{2}, \beta V \phi^{2} \widehat{\phi}\right\rangle_{m}+\left\langle\frac{R\left(\cdot, v_{t}(\cdot)\right)}{b(t)^{2} \phi^{2}}, \phi^{2} \widehat{\phi}\right\rangle_{m}
$$

It follows from (3.29) that

$$
\frac{\mathrm{d}}{\mathrm{~d} t}\left(\frac{1}{b(t)}\right)=-\frac{\mathrm{d} b(t)}{b(t)^{2} \mathrm{~d} t}=\frac{1}{2}\langle\beta V \phi, \phi \widehat{\phi}\rangle_{m}+g(t), \quad t \in \mathbf{C} .
$$

Since the function $t \mapsto\left\langle\Psi_{0}\left(v_{t}\right), \beta \widehat{\phi}\right\rangle_{m}$ is non-increasing, $\left(s_{0}, \infty\right) \backslash \mathbf{C}$ has at most countably many points. Using (3.27) and (3.28), and repeating the argument in the proof of [23, Lemma 5.4], we obtain that $g(t) \rightarrow 0$ as $t \rightarrow \infty$, and thus

$$
\frac{1}{b(t) t} \underset{t \rightarrow \infty}{ } \frac{1}{2}\langle\beta V \phi, \phi \widehat{\phi}\rangle_{m}
$$

as desired.
Combining Lemmas 3.11 and 3.12, we immediately get Theorem 3.10,

### 3.3 Branching Brownian motion and traveling wave solution

We consider a branching Brownian motion on $\mathbb{R}$, i.e., the spatial motion $Y=\left\{Y_{t}, \Pi_{x}\right\}$ is a Brownian motion on $\mathbb{R}$. Suppose the branching rate $\beta>0$ is a constant, the offspring distribution $\left\{\left(p_{n}\right)_{n=0}^{\infty}\right\}$ does not depend on the spatial position and $A:=\sum_{n=0}^{\infty} n p_{n}<\infty$.

It is known that, for any $\lambda \in \mathbb{R}, \Pi_{x} e^{\lambda Y_{t}}=e^{\frac{1}{2} \lambda^{2} t}$. Put $\phi(x)=e^{-\lambda x}$. Then

$$
\phi(x)=e^{-\left[\frac{1}{2} \lambda^{2}+(A-1) \beta\right] t} P_{t}^{(1-A) \beta} \phi(x), \quad x \in \mathbb{R} .
$$

Therefore,

$$
\begin{equation*}
W_{t}(\lambda):=e^{-\left[\frac{1}{2} \lambda^{2}+(A-1) \beta\right] t}\left\langle\phi, X_{t}\right\rangle=e^{-\left[\frac{1}{2} \lambda^{2}+(A-1) \beta\right] t} \sum_{u \in L_{t}} e^{-\lambda Y_{u}(t)}, \quad t \geq 0 \tag{3.30}
\end{equation*}
$$

is a non-negative $\mathbf{P}_{x}$-martingale with respect to $\left\{\mathcal{F}_{t}, t \geq 0\right\}$. Thus, for any $x \in \mathbb{R}$, the limit $W_{\infty}(\lambda):=\lim _{t \rightarrow \infty} W_{t}(\lambda)$ exists $\mathbf{P}_{x}$-almost surely.

Under the assumption $p_{0}=0$, Kyprianou [14] used spine decomposition techniques to give necessary and sufficient conditions for the $L^{1}$-convergence of the martingales $\left\{W_{t}(\lambda), t \geq 0\right\}:$

Theorem 3.13 Suppose $p_{0}=0$. Let $\underline{\lambda}:=\sqrt{2 \beta(A-1)}$.
(1) if $|\lambda| \geq \underline{\lambda}, W_{\infty}(\lambda)=0 \mathbf{P}_{x}$-almost surely;
(2) if $|\lambda|<\underline{\lambda}$ and $\sum_{n=1}^{\infty} p_{n} n \log n=\infty$, then $W_{\infty}(\lambda)=0 \mathbf{P}_{x}$-almost surely;
(3) if $|\lambda|<\underline{\lambda}$ and $\sum_{n=1}^{\infty} p_{n} n \log n<\infty$, then $W_{t}(\lambda) \rightarrow W_{\infty}(\lambda) \mathbf{P}_{x}$-almost surely and in $L^{1}\left(\mathbf{P}_{x}\right)$.

Using spine techniques, Hardy and Harris [10] proved that in many cases where the martingale has a non-trivial limit, the convergence can be strengthen as $L^{p}\left(\mathbf{P}_{x}\right)$ convergence with some $p \in(1,2]$.

Theorem 3.14 Suppose $p_{0}=0$. For any $x \in \mathbb{R}$, and for each $p \in(1,2]$ we have
(1) As $t \rightarrow \infty, W_{t}(\lambda) \rightarrow W_{\infty}(\lambda) \mathbf{P}_{x}$-almost surely and in $L^{p}\left(\mathbf{P}_{x}\right)$ if $p \lambda^{2}<2(A-1) \beta$ and $\sum_{n=1}^{\infty} p_{n} n^{p}<\infty$;
(2) $\lim _{t \rightarrow \infty} \mathbf{P}_{x}\left(W_{t}(\lambda)\right)=\infty$ if $p \lambda^{2}>2(A-1) \beta$ or $\sum_{n=1}^{\infty} p_{n} n^{p}=\infty$.

Now using our spine decomposition in Section 2, the above Theorems 3.13 and 3.14 also hold for the case that $p_{0}>0$.

It is known that

$$
\partial W_{t}(\lambda):=e^{-\left[\frac{1}{2} \lambda^{2}+(A-1) \beta\right] t} \sum_{u \in L_{t}}\left(Y_{u}(t)+\lambda t\right) \mathrm{e}^{-\lambda Y_{u}(t)} \mathrm{e}^{-\lambda Y_{u}(t)}, \quad t \geq 0
$$

is a $\mathbf{P}_{x}$-martingale with respect to $\left\{\mathcal{F}_{t}, t \geq 0\right\}$, which is also referred to as the derivative martingale.

The martingale $\left\{\partial W_{t}(\lambda), t \geq 0\right\}$ is not non-negative. To establish the convergence of $\partial W_{t}(\lambda)$ as $t \rightarrow \infty$, we usually consider the following related non-negative martingale.

Let $\widetilde{L}_{t}$ denote the set of particles in $L_{t}$ that, along with their ancestors, have not met by time $t$ the space-time barrier $y+\sqrt{(A-1) \beta} t=-x$. Define

$$
\begin{equation*}
V_{t}^{x}(\lambda)=e^{-\left[\frac{1}{2} \lambda^{2}+(A-1) \beta\right] t} \sum_{u \in \widetilde{L}_{t}} \frac{x+Y_{u}(t)+\lambda t}{x} \mathrm{e}^{-\lambda Y_{u}(t)}, \quad t \geq 0 . . \tag{3.31}
\end{equation*}
$$

Under the condition that $p_{0}=0$, Kyprianou [14] proved that $\left\{V_{t}^{x}(\lambda), t \geq 0\right\}$ is a mean $1 \mathbf{P}_{x}$-martingale, and that when $\lambda \geq \sqrt{2(A-1) \beta}, \partial W(\lambda):=\lim _{t \rightarrow+\infty} \partial W_{t}(\lambda) \mathbf{P}_{x^{-}}$a.s. exists and is equal to $\lim _{t \rightarrow+\infty} x V_{t}^{x}(\lambda)$.

The importance of the limit $\partial W(\underline{\lambda})$ lies in that when $\partial W(\underline{\lambda})$ is non-degenerate, its rescaled Laplace transform provides a traveling wave solution to the KPP equation

$$
\frac{\partial u}{\partial t}=\frac{1}{2} \frac{\partial^{2} u}{\partial x^{2}}+\beta(f(u)-u)
$$

where $f(u):=\sum_{n=0}^{\infty} p_{n} u^{n}$ is the generating function of the distribution $\left\{p_{n}, n \geq 0\right\}$. By a traveling wave solution we mean a solution of the form $u(t, x)=w(x-c t)$, where $w$ is a monotone function connecting 0 at $-\infty$ to 1 at $+\infty$ and $c$ is called the speed of the wave. The following result of Yang and Ren [27] gives a necessary and sufficient condition for $\partial W(\underline{\lambda})$ being non-degenerate.

Theorem 3.15 Suppose $p_{0}=0$ and $\lambda=\underline{\lambda}$. For any $x \in \mathbb{R}$, we have
(1) $\partial W(\underline{\lambda})>0 \mathbf{P}_{x}$-almost surely if $\sum_{n=1}^{\infty} p_{n} n(\log n)^{2}<\infty$;
(2) $\partial W(\underline{\lambda})=0 \mathbf{P}_{x}$-almost surely if $\sum_{n=1}^{\infty} p_{n} n(\log n)^{2}=\infty$.

Corollary 3.16 When $c=\underline{\lambda}=\sqrt{2 \beta \beta(A-1)}$ and $\sum_{n=1}^{\infty} p_{n} n(\log n)^{2}<\infty$, there is a unique traveling wave with speed c given by $\Phi_{c}(x)=E^{x} \exp \left(-e^{\lambda x} \partial W(\lambda)\right)$.

Now using our spine decomposition in Section2, the above Theorem3.15and Corollary 3.16 also hold for the case that $p_{0}>0$.

### 3.4 Typed branching Brownian motion and traveling wave solution

Hardy and Harris [10] considered a typed branching diffusion in which the Hunt process, that is, the spatial motion, is described by $\left(Y_{t}, \eta_{t}\right)_{t \geq 0}$, where the type $\eta_{t}$ evolves as a Markov chain on $I:=\{1, \cdots, n\}$ with $Q$-matrix $\theta Q$, where $\theta>0$ is a constant, and the spatial location, $S_{t}$, moves as a driftless Brownian motion on $\mathbb{R}$ with diffusion coefficient $a(i)>0$ whenever $\eta_{t}$ is in state $i$. Any particle currently of type $i$ will undergo fission at rate $\beta(i)$ to be replaced by a random number of offspring with law $\left\{p_{n}(i), n \geq 0\right\}$. At birth,
offspring inherit the parent's spatial and type positions and then move off independently, repeating stochastically the parent's behaviour, and so on. Let $A(i):=\sum_{n=0}^{\infty} n p_{n}(i)<\infty$ be the mean of the distribution of offspring given by a type $i$ particle.

As usual, let the configuration of the whole branching diffusion at time $t$ be given by the $\mathbb{R} \times I$-valued point process $X_{t}=\sum_{u \in L_{t}} \delta_{(y, i)}$, where $L_{t}$ is the set of particles alive at time $t$. Let the probabilities for this process be given by $\left\{\mathbf{P}_{(y, i)},(y, i) \in \mathbb{R} \times I\right\}$ defined on the natural filtration, $\left(\mathcal{F}_{t}, t \geq 0\right\}$, where $\mathbf{P}_{(y, i)}$ is the law of the typed branching Brownian motion starting with one initial particle of type $i$ at spatial position $y$.

For this finite-type branching diffusion, a fundamental positive martingale is defined for this model:

$$
W_{\lambda}(t):=\sum_{u \in L_{t}} v_{\lambda}\left(\eta_{u}(t)\right) e^{\lambda Y_{u}(t)-E_{\lambda} t}, \quad t \geq 0,
$$

where $v_{\lambda}$ and $E_{\lambda}$ satisfy

$$
\left(\frac{1}{2} \lambda^{2} \Sigma+\theta Q+(A-1) R\right) v_{\lambda}=E_{\lambda} v_{\lambda}
$$

where $\Sigma:=\operatorname{diag}(a(i): i \in I), A:=\operatorname{diag}(A(i), i \in I)$ and $R:=\operatorname{diag}(\beta(i): i \in I)$. That is, $v_{\lambda}$ is the (Perron-Frobenius) eigenvector of the matrix $\frac{1}{2} \lambda^{2} A+\theta Q+R$, with eigenvalue $E_{\lambda}$. This martingale should be compared with the corresponding martingale (3.30) for branching Brownian motion.

Since $\left\{W_{t}(\lambda), t \geq 0\right\}$ is a strictly-positive martingale it is immediate that $W_{\infty}(\lambda):=$ $\lim _{t \rightarrow \infty} W_{t}(\lambda)$ exists and is finite almost-surely under $\mathbf{P}_{(y, i)}$. Under the condition that $p_{0}(i)=0, i \in I$, Hardy and Harris [10, Theorem 10.4] give a necessary and sufficient conditions for $L^{1}$-convergence of the martingale $\left\{W_{t}(\lambda), t \geq 0\right\}$. Using our general spine decomposition, the condition that $p_{0}(i)=0, i \in I$ can be dropped now. Once again, in many cases where the martingale has a non-trivial limit, the convergence will be much stronger than merely in $L^{1}\left(\mathbf{P}_{(y, i}\right)$, that is $L^{p}\left(\mathbf{P}_{(y, i}\right)(p \in(1,2])$-convergence, using the spine decomposition, see [10, Theorem 10.5].

In Harris and Williams [11], a continuous-typed branching diffusion, where the Hunt process, that is, the movement of the particles, is described by $\left(Y_{t}, V_{t}\right)_{t \geq 0}$, where the spatial motion $\left(Y_{t}\right)_{t \geq 0}$ is a driftless Brownian motion with instantaneous variance $a y^{2}$ with $a \geq 0$ being a fixed constant, and the type moves on the real line as an Orstein-Uhlenbeck process. A particle of type $v$ dies at rate $r v^{2}+\rho$ with $r, \rho \geq 0$ being fixed constant, and then produce two particles at the same space-type location as the parent. This model is similar in flavour to the finite-type model. There is also a strictly-positive martingale $\left\{W_{t}(\lambda), t \geq 0\right\}$. Hardy and Harris [10. Theorem 11.1], using the spine technique, give a necessary condition and sufficient condition for $L^{p}$-convergence with $p \in(1,2]$. We remark here that, their results also hold for general offspring distribution, that is when a particle of type $v \in \mathbb{R}$ dies, it gives birth a random number of particles according to
law $\left\{p_{n}(v), n \geq 0\right\}$ at the same space-type location as the parent. Under some moment condition on $\left\{p_{n}(v), n \geq 0\right\}, v \in \mathbb{R}$, allowing $p_{0}(v)=0, v \in \mathbb{R}$, results similar to [10, Theorem 11.1] remain true. We will not go to the details here.

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