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The spine decomposition of branching processes and their applications

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Abstract: Due to their close relation to both biology and other branches of mathematics, branching processes, branching Markov processes and superprocesses, have been receiving more and more attention. Early studies of these processes mainly used analytic methods, which are not very transparent. More intuitive probabilistic arguments would be very helpful. In recent years, many people started to use probabilistic methods to study these processes. One of these probabilistic methods is the so-called spine method. In the supercritical cases, the spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a martingale change of measure, the transformed process can be decomposed in law as an immigration process along this spine. The spine method mainly uses the spine decomposition of the processes to reduce the study of a random number of sample paths of these processes to the study of one sample path. In this paper, we give a survey of the spine method and its applications to the three types of processes above.

Keywords: Spine decomposition, branching process, branching Markov process, superprocess, martingale change of measure

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1. Introductions

The spine method is a very intuitive probabilistic method, and it has been widely used to study the limit properties of various branching models. The spine is the trajectory of an immortal moving particle and the spine decomposition theorem says that, after a martingale change of measure, the transformed process can be decomposed in law as an immigration process along this spine. The behavior of the spine plays a very important role in studying the long term behaviors of branching systems. The spine decomposition can be used to give intuitive probabilistic proofs of deep results on branching models previous obtained via analytic methods. In this paper, we will survey the spine decompositions of branching processes, branching Markov

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processes and superprocesses, and their applications in the studying the limit behaviors of these models. We will mainly focus on the topics of exponential growth rate, extinction problems and probabilistic representations of solutions of Fisher–Kolmogorov–Petrovski–Piscounov (FKPP) equations related to branching Brownian motions and super Brownian motions.

The goal of this paper is to give a general introduction to the spine method. Since spine decompositions of branching processes, branching Markov processes and superprocesses can be seen as decompositions of the processes under martingale changed measures, some general facts on martingale change of a probability measure is presented in Section 2. Then in Sections 3–5, we introduce our three branching models: branching processes, branching Markov processes and superprocesses, some martingales related to these processes, and the spine decompositions of our three types of processes under martingale-changed measures. In the last section, we give some applications of the spine method in studying the existence, asymptotic and uniqueness of the traveling wave solutions to the FKPP equation related to branching Brownian motions and super Brownian motions.

2. Martingale change of measure

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a filtration $\{\mathcal{F}_t;\ t \in T\}$, where the index set T is either $\mathbb{N}$ or $\mathbb{R}_+$. Let $\{M_t;\ t \in T\}$ be a nonnegative martingale with respect to $\{\mathcal{F}_t;\ t \in T\}$ with $\mathbb{P}(M_0) = 1$. Here we use $\mathbb{P}(\cdot)$ to denote both the probability of an event and the expectation of a random variable with respect to the probability $\mathbb{P}$. On the σ field $\mathcal{F}_t$, $t \in T$, we define a probability measure Q_t by

$$\mathrm{d}Q_t = M_t \mathrm{d}\mathbb{P}|_{\mathcal{F}_t}.$$

We denote the σ field $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. Since $\{M_t;\ t \in T\}$ is a martingale with respect to $(\mathcal{F}_t)$, the measures $\{Q_t;\ t \in T\}$ are consistent with respect to $\{\mathcal{F}_t;\ t \in T\}$. By Kolmogorov's extension theorem, if Ω is a Polish space, then there exists a unique probability measure Q on $\mathcal{F}_\infty$ such that $Q|_{\mathcal{F}_t} = Q_t$. The notations $\mathbb{P}|_{\mathcal{F}_t}$ and $Q|_{\mathcal{F}_t}$ denote the restrictions of $\mathbb{P}$ and Q on $\mathcal{F}_t$. The measure Q is called a martingale change of $\mathbb{P}$. Since $\{M_t;\ t \in T\}$ is a nonnegative martingale, it has an almost sure limit under $\mathbb{P}$, denoted by M_∞. Then the following result holds (c.f. Durrett [14], Chapter 5).

Theorem 2.1. *For any $A \in \mathcal{F}_\infty$,*

$$Q(A) = \int_A M_\infty \mathrm{d}\mathbb{P} + Q\big(A \cap \{M_\infty = \infty\}\big).$$

If we define $Q_a(A) = \int_A M_\infty \mathrm{d}\mathbb{P}$, and $Q_s(A) = Q\big(A \cap \{M_\infty = \infty\}\big)$, then $Q = Q_a + Q_s$ is the Lebesgue–Radon–Nikodym decomposition of Q with respect to $\mathbb{P}$. In particular, *the following assertions hold:*

$$M_\infty = \infty, \quad Q\text{-a.s.} \Leftrightarrow M_\infty = 0, \quad \mathbb{P}\text{-a.s.} \tag{2.1}$$

$$M_\infty < \infty, \quad Q\text{-a.s.} \Leftrightarrow \int M_\infty \mathrm{d}\mathbb{P} = 1. \tag{2.2}$$

When one wants to prove the non-degeneracy of the limit of a nonnegative martingale, by the result above, one only needs to prove the finiteness of the limit under a martingale change of the original measure. The latter is always easier.

3. Branching processes

3.1. Galton–Watson processes and Galton–Watson trees

Galton–Watson process is the simplest and most important branching model. It can be described as follows. At time $t = 0$, there are Z_0 particles, each of which lives for one unit of time and splits independently of the others into a random number of offspring according to a given probability distribution $\mathrm{p} = \{p_k; k \in \mathbb{N}\}$. The total number Z_1 of particles thus produced is the sum of Z_0 random variables, each has distribution p. They constitute the first generation. They go on to produce the second generation of Z_2 particles, and so on. If at time $t = n$, there are Z_n particles, and at time $t = n + 1$ the k-th particle produces ξ_k^{n+1} children, then at time $t = n + 1$, the total number of particles is given by

$$Z_{n+1} = \sum_{k=1}^{Z_n} \xi_k^{n+1},$$

where $\{\xi_k^n; k = 1, 2, \ldots, n = 1, 2, \ldots\}$ are independent and identically distributed, and they are independent of Z_0. Each ξ_k^n is distributed as $\mathrm{p} = \{p_k; k \in \mathbb{N}\}$. The process $\mathrm{Z} = \{Z_n;\ n \in \mathbb{N}\}$ is called a Galton–Watson process (GW process for short). It is a Markov chain in $\mathbb{N} = \{0, 1, 2, \ldots\}$. The distribution law p is called the offspring distribution of Z. For properties of GW processes and of more complicated models such as multitype branching processes, continuous time branching process or continuous state branching processes, we refer our readers to Athreya and Ney [6], Harris [33].

The generating function of p is given by

$$f(s) = \sum_{k=0}^{\infty} p_k s^k, \quad |s| \le 1,$$

Then the mean of the offspring distribution is given by $m = f'(1) = \sum_{k=0}^{\infty} k p_k$. When $m > 1$ $(=,\ < 1)$, the GW process is called supercritical (critical, subcritical respectively). One of the basic problems in branching processes is the extinction probability, which is defined as $q := \lim_{n\to\infty} \mathbb{P}(Z_n = 0)$.

Theorem 3.1 (Steffensen (1930, 1932)). *If the GW process* Z *is critical or subcritical and* $p_1 < 1$*, then* $q = 1$*. When* Z *is supercritical,* q *is the unique solution to* $s = f(s)$ *in* $[0, 1)$*, thus with positive probability, the population system is alive forever. When* $m < \infty$*, the process* Z_n/m^n *is a martingale.*

Another way to describe Galton–Watson processes is using Galton–Watson trees. For this we refer our readers to Le Gall [41], Stanley [56] and the references therein. Here we define Galton–Watson trees by using the Ulam–Harris labels. Let $\mathbb{N}_0 = \{1, 2, \dots\}$. The Ulam–Harris label set is the set Γ defined by

$$\Gamma := \bigcup_{n=0}^{\infty} \mathbb{N}_0^n$$

(where $\mathbb{N}_0^0 = \{\emptyset\}$ is called the root). $\mathrm{u} = (u_1, \dots, u_n) \in \mathbb{N}_0^n$ is said to be a particle or a node, where $|\mathrm{u}| := n$ represents the length (or generation) of u. When $n \geq 1$, $\mathrm{u} - 1$ is the particle $(u_1, \dots, u_{n-1})$, which is the parent of the particle $\mathrm{u} = (u_1, \dots, u_n)$. For any $i \in \mathbb{N}_0$, and any particle $\mathrm{u} = (u_1, \dots, u_n) \in \mathbb{N}_0^n$, we denote the i-th child of u by $\mathrm{u}i = (u_1, \dots, u_n, i)$. If $\mathrm{u} = (u_1, \dots, u_n), \mathrm{v} = (v_1, \dots, v_m) \in \Gamma$, uv is the concatenation of u and v. It represents the particle $\mathrm{uv} = (u_1, \dots, u_n, v_1, \dots, v_m)$ ($\mathrm{u}\emptyset = \emptyset\mathrm{u} = \mathrm{u}$). If $m < n$, and $v_i = u_i, i = 1, \dots, m$, then we say v is an ancestor of u, and we denote this by $\mathrm{v} < \mathrm{u}$. Obviously $\mathrm{u} < \mathrm{uv}$ for all $\mathrm{v} \in \Gamma \backslash \{\emptyset\}$. Thus the set of all the ancestors of u is given by

$$\{\mathrm{v} \in \Gamma;\ \mathrm{v} < \mathrm{u}\} = \{\mathrm{v} \in \Gamma; \text{there exists } \mathrm{w} \in \Gamma \setminus \{\emptyset\} \text{ such that } \mathrm{vw} = \mathrm{u}\}.$$

We call a subset τ of Γ a Galton–Watson tree, if

(i) $\emptyset \in \tau$;

(ii) if $\mathrm{u}, \mathrm{v} \in \Gamma$, then $\mathrm{uv} \in \tau$ implies $\mathrm{u} \in \tau$;

(iii) for any node $\mathrm{u} \in \tau$, there exists a number $r_{\mathrm{u}} \in \mathbb{N}$ such that $\mathrm{u}j \in \tau$ if and only if $1 \leq j \leq r_{\mathrm{u}}$.

See Figure 1 for a Galton–Watson tree.

Any particle $\mathrm{v} \in \tau$ creates a subtree $\{\mathrm{v}\} \cup \{\mathrm{u} \in \tau; \mathrm{v} < \mathrm{u}\}$, which is the collection of v and particles that have v as an ancestor. Denote the set of all Galton–Watson trees by $\mathbb{T}$. For any $n \in \mathbb{N}$, we define the random variables Z_n on $\mathbb{T}$ by

$$Z_n(\tau) := \text{the number of individuals with length } n \text{ in } \tau,$$

The restriction of a tree τ to the first n generations is denoted by $[\tau]_n$. Define the σ field $\mathcal{F}_n := \sigma([\tau]_n)$, which consists of information about $[\tau]_n$, and the σ field $\mathcal{F}_\infty = \bigvee_{n\geq 0} \mathcal{F}_n$. It is obvious that Z_n is measurable with respect to $\mathcal{F}_n$. It is well known that there is a probability $\mathbb{P}$ on the measurable space $(\mathbb{T}, \mathcal{F}_\infty)$, under which

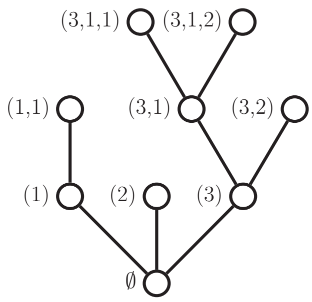

Figure 1: Galton–Watson branching tree

$\{Z_n; n \in \mathbb{N}\}$ is a Galton–Waston process, and the offspring distribution for any node u is $\mathbb{P}(r_{\mathrm{u}} = k) = p_k$. Theorem 3.1 says that, when $m > 1$, the branching tree is an infinite tree with positive probability. In this case, the number of nodes in generation n on the trees goes to infinity as the time n goes to infinity. The rate of increase is given by the following result due to Kesten and Stigum [36].

Theorem 3.2 (Kesten and Stigum (1966))**.** *Suppose Z_n is a Galton–Watson tree with $Z_0 = 1$ with the mean of the offspring number $m \in (1, \infty)$. Let L be a random variable with the offspring probability law $\{p_k\}$. If W is the limit of the martingale Z_n/m^n, then $E[W] = 1$ if and only if $E[L \ln^+ L] < \infty$.*

Therefore, $L \ln^+ L$ moment is finite if and only if the process does not become extinct. In this case, Z_n increases with rate m^n.

In Kesten and Stigum [36], the proof of the result above is mainly based on the analysis of the generating functions of the process. It is purely analytic. In 1995, Lyons, Pemantle and Peres [47] gave a probabilistic proof of the result above. The method of [47] is now known as the spine method and will be introduced in the following subsection.

3.2. Spine decomposition of Galton–Watson processes

Let $\mathbb{P}$ be a probability on the trees space $(\mathbb{T}, \mathcal{F}_\infty)$ such that, under the probability $\mathbb{P}$, the process $\{Z_n(\tau), n \geq 0\}$ is a Galton–Waston process with offspring distribution p and initial value 1. Define the size-biased distribution: $\hat{\mathrm{p}} = \{\hat{p}_k; k \in \mathbb{N}\}$, where $\hat{p}_k := kp_k/m$, $k = 0, 1, 2, \ldots,$ Obviously $\hat{p}_0 = 0$. On the space $(\mathbb{T}, \mathcal{F}_\infty)$, we can construct another probability measure Q, under which every tree τ is an infinite tree almost surely. Such a tree can be constructed as follows. The tree τ starts with an initial particle $\xi_0 = \emptyset$. At time $t = 1$, the particle $\emptyset$ dies and produces $\hat{r}_\emptyset$ children according to the size-biased distribution $\hat{\mathrm{p}}$. Pick one of these children at random, say ξ_1. The other unchosen particles evolve independently according to $\mathbb{P}$, and ξ_1 gives

$\hat{r}_{\xi_1}$ children according to p̂. Again, pick one of the children of ξ_1 at random, call it ξ_2, and give the others ordinary Galton–Watson descendant trees. Continue in this way indefinitely. Note that size-biased Galton–Watson trees are always infinite (there is no extinction). The set $\xi = \{\emptyset, \xi_1, \dots\}$ is called the spine of τ. We call (τ, ξ) a size-biased tree with a spine. The right graph in Figure 2 is a tree with a spine.

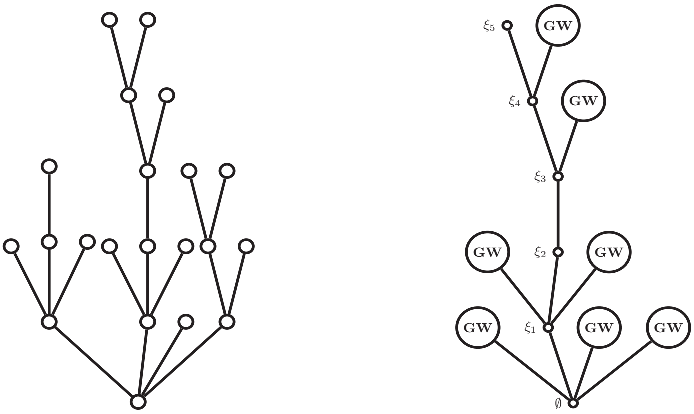

Figure 2: Galton–Watson tree and Galton–Watson tree with a spine

Let $\widetilde{Q}$ be the joint distribution of (τ, ξ). The measure Q is defined as the marginal distribution of Q on $(\mathbb{T}, \mathcal{F}_\infty)$. Under $\widetilde{Q}$, a tree τ is called a size-biased Galton–Watson tree. According to the construction of τ, the number of particles in generation n can be written as

$$Z_n = 1 + \sum_{i=1}^{n} Z_n^{(i)}, \tag{3.1}$$

where $\{Z_k^{(i)};\ k \geq i\}$, $i \geq 1$, under $\mathbb{P}$, is a sequence of GW processes with initial values $Z_i^{(i)} = \hat{r}_{\xi_{i-1}} - 1$. In this tree, $Z_n^{(i)}$ represents the number of particles in generation n, produced by the $\hat{r}_{\xi_{i-1}} - 1$ unchosen children of ξ_{i-1}, where for each $i = 0, 1, 2, \dots$, $\hat{r}_{\xi_i}$ is the random number of children given by ξ_i. The decomposition (3.1) is called the spine decomposition of Galton–Watson process.

Lyons, Pemantle and Peres [47] showed that Q is a martingale change of measure $\mathbb{P}$. Note that $W_n = Z_n/m^n$ is a martingale. The martingale change is given by

$$\mathrm{d}Q|_{\mathcal{F}_n} = W_n \mathrm{d}\mathbb{P}|_{\mathcal{F}_n}, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Now we explain why this is true. For any rooted tree τ and any $n \geq 0$, we denote by $[\tau]_n$ the set of rooted trees whose first n levels agree with those of τ. If u is a vertex at the n-th level of τ, then use $[\tau;\mathrm{u}]_n$ to denote the set of trees with distinguished paths such that the trees are in $[\tau]_n$ and the path starts from the root, does not backtrack and goes through u. Assume that τ is a tree of height at least $n+1$ and that the root of τ has k children with descendant trees $\tau^{(1)},\dots,\tau^{(k)}$. Any vertex u in level $n+1$ of τ is in one of these, say $\tau^{(i)}$. The measure $\widetilde{Q}$ clearly satisfies the recursion

$$\mathrm{d}\widetilde{Q}[\tau;\mathrm{u}]_{n+1} = \frac{kp_k}{m}\cdot\frac{1}{k}\mathrm{d}\widetilde{Q}[\tau^{(i)};\mathrm{u}]_n\prod_{j\neq i}\mathrm{d}\mathbb{P}[\tau^{(j)}]_n = \frac{p_k}{m}\mathrm{d}\widetilde{Q}[\tau^{(i)};\mathrm{u}]_n\prod_{j\neq i}\mathrm{d}\mathbb{P}[\tau^{(j)}]_n.$$

By induction, we obtain that

$$\mathrm{d}\widetilde{Q}[\tau;\mathrm{u}]_n = \frac{1}{m^n}\mathrm{d}\mathbb{P}[\tau]_n$$

for all n. Furthermore, since the particle ξ_n in the spine is uniformly chosen from the Z_n particles,

$$\mathrm{d}Q[\tau]_n = \mathrm{d}\widetilde{Q}[\tau]_n = \frac{Z_n}{m^n}\mathrm{d}\mathbb{P}[\tau]_n = W_n\mathbb{P}[\tau]_n.$$

In conclusion, under the probability Q, the Galton–Watson process $\{Z_n\}$ has a spine decomposition. We will use this decomposition to give a probabilistic proof of the Kesten–Stigum Theorem 3.2. For this purpose, we first introduce the following lemma, which is an immediate consequence of the Borel–Cantelli lemma, see Lemma 1.1 in [47].

Lemma 3.1. *If $X, X_1, X_2, \dots$ is a sequence of nonnegative independently identically distributed random variables, then*

$$\limsup_{n\to\infty}\frac{1}{n}X_n = \begin{cases} 0, & \text{if } E[X]<\infty; \\ \infty, & \text{if } E[X]=\infty. \end{cases} \tag{3.3}$$

Proof of Theorem 3.2. Using the spine decomposition (3.1) of Z_n, we rewrite the martingale W_n as

$$W_n = \frac{Z_n}{m^n} = \frac{1}{m^n} + \sum_{i=1}^{n}\frac{Z_n^{(i)}}{m^n}.$$

Let $\mathcal{G}$ be the σ-field generated by $=\sigma(Z_i^{(i)}: i\geq 1)$. Then

$$\widetilde{Q}\big[W_n|\mathcal{G}\big] = \frac{1}{m^n} + \sum_{i=1}^{n}\frac{Z_i^{(i)}}{m^i}.$$

According to Lemma 3.1,

$$\sum_{i=1}^{\infty}\frac{Z_i^{(i)}}{m^i}<\infty \quad \text{if and only if } \widetilde{Q}\left[\ln^+ Z_i^{(i)}\right]<\infty, \text{ or equivalently } \ Q[L\ln^+ L]<\infty.$$

By Fatou's lemma for conditional expectation, we get

$$\widetilde{Q}\big[W|\mathcal{G}\big] \leq \sum_{i=1}^{\infty}\frac{Z_i^{(i)}}{m^i}<\infty.$$

Therefore $W < \infty$, Q-a.s. which is equivalent to $\int W d\mathbb{P} = 1 = \mathbb{P}[W]$ by Theorem 2.1. $\square$

The above way of proving the Kesten–Stigum theorem is called spine method. It obviously makes the proof much easier than the purely analytic methods. This method goes like this. When we want to obtain a certain limit property of the branching process, we can try to find a nonnegative martingale $\{W_t, t \geq 0\}$ with mean 1. Then we define a martingale change of the original probability using the martingale $\{W_t, t \geq 0\}$. Under the new probability the branching process has a spine decomposition. This allows us to accomplish our task by analyzing properties of the spine instead of analyzing properties of a random number of paths.

The spine method can also be used to investigate other properties of Galton–Watson processes and other processes with branching property. When the Galton–Watson process is critical or subcritical, it will die out in a finite time. Geiger [24, 25], Roelly-Coppoletta and Rouault [53] proved that, conditional on non-extinction, the distribution of critical or subcritical GW processes are martingale changes of the original probabilities. The conditional process has spine decomposition. The authors applied the spine method to obtain the rate of the decrease of the population in their model. A continuous time and continuous state branching process is a scaling limit of Galton–Watson processes. Lambert [40] established a spine decomposition for continuous time and continuous state branching process. We also refer to Chu and Ren [10] for related researches. The spine method developed in Lyons, Pemantle and Peres [47] is a pioneering work, which have been extended to more general models in recent years. We will introduce some of our researches in this direction in Sections 4–6. Before we go to more general models, we first introduce the applications of the spine method in the study of multitype branching processes. When particles in the branching system have different types and different types of particles have different offspring distributions, the evolution of the system is described by a multitype branching process. The spine method also can be used to prove the Kesten–Stigum theorem for multitype branching processes which will be introduced in the next subsection.

3.3. Spine decomposition for multitype branching processes

We consider a branching particle system in which there are d $(2 \leq d < \infty)$ different types of particles. Let $S = \{1, 2, \dots, d\}$ be the set of types. To fully describe the multitype branching process, we need to introduce the concept of marked Galton–Watson trees. Suppose that each particle u on a tree τ has a mark $X_{\mathrm{u}} \in S$, called the type of u. We use (τ, X) to denote a marked Galton–Watson tree, and $\mathcal{T}$ the set of all marked trees. Define

$$Z_{j,n}(\tau) := \text{the number of type } j \text{ particles in generation } n \text{ of } \tau, \quad j = 1, \dots, d,$$

and $Z_n(\tau) = (Z_{1,n}(\tau), \dots, Z_{d,n}(\tau))$. As before we use $[\tau, X]_n$ to denote the set of marked trees whose first n generations agree with those of (τ, X). We define the σ-fields $\mathcal{F}_n = \sigma([\tau, X]_i; i \le n)$, $\mathcal{F}_\infty = \bigvee_{n \ge 0} \mathcal{F}_n$. Then there is a probability measure $\mathbb{P}$ on $(\mathcal{T}, \mathcal{F}_\infty)$ such that $\{Z_n, n \ge 0\}$ is a d-type Galton–Watson process.

Assume that $L^{i,j}$ is the number of type j particles given by a type i particle, $i, j = 1, 2, \dots, d$. For $\mathrm{k} = (k_1, \dots, k_d) \in \mathbb{N}^d$, let

$$p_{\mathrm{k}}^{(i)} = \mathbb{P}\left(L^{i,j} = k_j, j = 1, 2, \dots, d\right).$$

Suppose $m_{ij} = E\left[L^{i,j}\right], i, j \in S$, are finite and the mean matrix $M = (m_{ij})$ is indecomposable. Here we assume the principal eigenvalue ρ of M satisfies $\rho > 1$. In other words, here we consider an indecomposable supercritical d type branching process. If q_i is the extinction probability of a type i particle, then $q_i < 1$, $i = 1, 2, \dots, d$. Let d-dimensional vectors v and u be the left and right eigenvectors of the matrix M corresponding to ρ. By the Perron–Frobenius theorem, we can choose $v_i, u_i > 0$ for any $i = 1, 2, \dots, d$, and such that $\sum_{i=1}^d v_i = 1$.

Theorem 3.3. *There is a one dimensional random variable W such that almost surely the following limit holds*

$$\lim_{n \to \infty} \frac{Z_n}{\rho^n} = W \mathrm{v}.$$

Moreover, $\mathbb{P}(W > 0) > 0$ if and only if for any $i, j = 1, 2, \dots, d$,

$$\mathbb{P}\left[\sum_{i,j=1}^d L^{i,j} \ln^+ L^{i,j}\right] < \infty.$$

For any tree τ, define the process

$$W_n(\tau) = \rho^{-n} \frac{\mathrm{Z}_n(\tau) \cdot \mathrm{u}}{\mathrm{Z}_0(\tau) \cdot \mathrm{u}}.$$

Then $\{W_n\}$ is a nonnegative martingale. Define the martingale change of measure:

$$\mathrm{d}Q\big|_{\mathcal{F}_n} = W_n \mathrm{d}\mathbb{P}\big|_{\mathcal{F}_n}.$$

Under Q, the d type branching process has the following spine decomposition: Assume the root of tree τ is of type l_0, $l_0 \in \{1, 2, \dots, d\}$. After a unit time, the particle dies, and gives birth to k children according to the size-biased distribution

$$\hat{p}_{\mathrm{k}}^{(l_0)} := \frac{p_{\mathrm{k}}^{(l_0)} \mathrm{k} \cdot \mathrm{u}}{\rho u_{l_0}},$$

where $\mathrm{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, and k_i is the number of type i children given by the root particle for each $i \in S$. It is obvious that $\hat{p}_0^{(i)} = 0$, $i = 1, 2, \ldots, d$. From the $|\mathrm{k}| = \sum_{i=1}^{d} k_i$ new born particles, one type i particle is chosen to be the first generation of the spine with probability $\frac{u_i}{\mathrm{k} \cdot \mathrm{u}}$, $i = 1, 2, \ldots, d$. The unchosen particles will evolve according to $\mathbb{P}$ to produce d type branching subtrees rooted on themselves. Assume the chosen particle is of type l_1, then the particle will die after one unit time and give birth to k' children with probability $\hat{p}_{\mathrm{k}'}^{(l_1)}$. The particle in the second generation in the spine is chosen from its children with probability $\frac{u_i}{\mathrm{k} \cdot \mathrm{u}}$ if it is of type i, $i = 1, 2, \ldots, d$. The family tree grows in this way indefinitely. For any nonnegative integer k, let l_k be the type of the spine at generation k. Thus the number of the particles in generation n is given by

$$\mathrm{Z}_n = \mathrm{e}_{l_n} + \sum_{i=1}^{n} \mathrm{Z}_n^{[i]},$$

where e_{l_n} represents a d dimensional unit vector whose r_n-th entry is 1, while the other entries are 0, $1 \leq l_n \leq d$. $\mathrm{Z}_n^{[i]}$ is the number of particles in the n-th generation on the tree generated by the particles who are the children, except ξ_i, of the spine particle ξ_{i-1}.

Proof of Theorem 3.3. The key step to prove this theorem is to justify that the almost limit of W_n is an $L^1(\mathbb{P})$ limit as well. So we mainly discuss this. Similar to the monotype case, let $\mathcal{G} = \sigma(l_i, \mathrm{Z}_i^{[i]}, i = 1, 2, \ldots)$, then

$$\widetilde{Q}(W_n | \mathcal{G}) = \frac{u_{l_n}}{\rho^n} + \sum_{i=1}^{n} \frac{\mathrm{Z}_i^{[i]} \cdot \mathrm{u}}{\rho^i}.$$

Here $\mathrm{Z}_i^{[i]}$, $i \in \mathbb{N}$, are independent random variables distributed according to the size-biased offspring distribution $\{\hat{p}_{\mathrm{k}}^{(l_{i-1})}, \mathrm{k} \in \mathbb{N}^d\}$. Since d is finite, we may use the argument for monotype branching process to handle each component, and then get the conclusion of the theorem. □

For details of the above we refer to Biggins and Kyprianou [7]. Georgii and Baake [26] used the spine decomposition of multitype branching processes to study the history of the type of ancestors. Olofsson [49] applied the spine method to other more general branching models which satisfy finite $L \log L$ moment conditions.

If the particles in the system are not only branching, but also moving in the space according to some Markovian rule during their life time, and the life time of each particle is a non-negative real-valued random variable, then we can use branching Markov process to describe the evolution of the spacial distribution of particles in the system. Continuous time branching Markov process is a very important probabilistic model, which will be discussed in the next section.

4. Branching Markov processes

4.1. Branching Markov process and its spine decomposition

In the beginning, people were interested in particular branching Markov processes such as branching random walks and branching diffusions. In 1968 and 1969, Ikada, Nagasawa and Watanaba [35] defined general branching Markov processes in their three paper series.

Let E be a locally compact metric space, and $E_\Delta := E \cup \{\Delta\}$ be its one point compactification. Let $\mathcal{B}_E$ and $\mathcal{B}_{E_\Delta}$ be the Borel σ-fields on E and E_Δ respectively. All functions f on E will be automatically extended to E_Δ by setting $f(\Delta) = 0$. As usual, $\mathbb{N}$ stands for the set of natural numbers. $\mathbb{N}_0$ consists of the positive natural numbers. A finite point measure μ on E means the $\mathbb{N}$-valued finite measure only supported on finite points on E. So it can be written as $\mu = \sum_{i=1}^n \delta_{x_i}$, where $x_i \in E, i = 1, \ldots, n$ and $n \in \mathbb{N}$. (When $n = 0$, μ is the trivial zero measure.) Denote the set of all finite point measures on E by $\mathcal{M}_p(E)$. Then equipped with the weak topology, $\mathcal{M}_p(E)$ is a locally compact metric space. A branching Markov process on $(E, \mathcal{B}_E)$ is a Markov process taking values in $\mathcal{M}_p(E)$.

Consider a branching system determined by the following three ingredients.

(i) A Markov process $Y = \{Y_t; t \geq 0\}$ on $(E, \mathcal{B}_E)$.

(ii) A non-negative bounded measurable function β on E.

(iii) An offspring distribution $\{(p_k(x))_{k\in\mathbb{N}}; x \in E\}$ such that for each fixed $k \in \mathbb{N}$, $p_k(x)$ is measurable with respect to $\mathcal{B}_E$, and the mean function $M(x) = \sum_{k=0}^\infty kp_k(x)$ is finite everywhere.

To fully describe the branching Markov process, we need to introduce the concept of marked Galton–Watson trees. We suppose that each individual $u \in \tau$ has a mark $(Y_{\mathrm{u}}, \sigma_{\mathrm{u}}, r_{\mathrm{u}})$ where

(i) σ_{u} is the lifetime of u, which determines the fission time or the death time of particle u as $\zeta_{\mathrm{u}} = \sum_{\nu\leq \mathrm{u}} \sigma_\nu$ ($\zeta_\emptyset = \sigma_\emptyset$), and the birth time of u as $b_{\mathrm{u}} = \sum_{\nu<\mathrm{u}} \sigma_\nu$ ($\sigma_\emptyset = 0$);

(ii) Y_{u}: $[b_{\mathrm{u}}, \zeta_{\mathrm{u}}) \to E_\Delta$ gives the location of u during its lifetime.

(iii) $r_{\mathrm{u}} : E_\Delta \to \mathbb{N}$ gives the number of the offsprings born by u when it dies.

We use (τ, Y, σ, r) (or simply (τ, N)) to denote a marked Galton–Watson tree. We denote the set of all marked Galton–Watson trees by $\mathcal{T} = \{(\tau, N); \tau \in \mathbb{T}\}$. For any time $t > 0$, define the σ-field on $\mathcal{T}$:

$$\mathcal{F}_t := \sigma\big(\{\mathrm{u}, r_{\mathrm{u}}, \sigma_{\mathrm{u}}, (Y_{\mathrm{u}}(s), s \in [b_{\mathrm{u}}, \zeta_{\mathrm{u}})) : \mathrm{u} \in \tau \in \mathbb{T} \text{ where } \zeta_{\mathrm{u}} \leq t\} \cup \\ \{\mathrm{u}, (Y_{\mathrm{u}}(s), s \in [b_{\mathrm{u}}, t]) : \mathrm{u} \in \tau \in \mathbb{T} \text{ and } t \in [b_{\mathrm{u}}, \zeta_{\mathrm{u}})\} \big).$$

Furthermore, we can define the σ-algebra $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t$. We use $L_t = \{\mathrm{u} \in \tau; b_\mathrm{u} \leq t < \zeta_\mathrm{u}\}$ to denote the set of particles that are alive at time t in the tree τ. There is a probability $\mathbb{P}^x$ on $\mathcal{F}_\infty$ such that the marked GW tree (τ, N) starts at location $Y_\emptyset(0) \equiv x$ at time $t = 0$, and grows according to the following rules.

(i) Given $Y_{\mathrm{u}-1}(\zeta_{\mathrm{u}-1}-)$ and b_u, $\{Y_\mathrm{u}(t); t \in [b_\mathrm{u}, \zeta_\mathrm{u})\}$ is a copy of the Markov process Y starting from $Y_{\mathrm{u}-1}(\zeta_{\mathrm{u}-1}-)$ at time b_u, i.e., a process with law $\Pi_{Y_{\mathrm{u}-1}(\zeta_{\mathrm{u}-1}-)}$ shifted by b_u. All the particles alive move independently in E.

(ii) Given the path Y_u of a particle u and given that the particle is alive at time t, its probability of dying in the interval $[t, t + \mathrm{d}t)$ is $\beta(Y_\mathrm{u}(t))\mathrm{d}t + \circ(\mathrm{d}t)$.

(iii) When the particle u dies at location $x \in E$, it produces r_u children according to $\mathbb{P}^x\,(r_\mathrm{u} = k) = p_k(Y_\mathrm{u}(\zeta_\mathrm{u}-))$ where $k \in \{0, 1, 2, \ldots\}$. The new born particles start at $Y_\mathrm{u}(\zeta_\mathrm{u})$ and follow the rules of their parents.

(iv) The point Δ is a cemetery. When a particle reaches Δ, it stays at Δ forever and there is no branching at Δ.

Y is called the underlying process, and the function β is called the branching rate. Under the probability $\mathbb{P}^x$, the process defined by

$$X_t = \sum_{\mathrm{u}\in L_t} \delta_{Y_\mathrm{u}(t)}$$

is a branching Markov process adapted to $\{\mathcal{F}_t; t \geq 0\}$ with initial value δ_x. We can also define branching Markov processes (X_t) with more general initial values of the form $\mu = \sum_{k=1}^n \delta_{x_k} \in \mathcal{M}_p(E)\backslash\{0\}$. Let X^i be a branching Markov process with initial value δ_{x_i}, $i = 1, 2, \ldots, n$, and assume they are independent. Then $X_t = \sum_{i=1}^n X_t^i$ is a branching Markov processes (X_t) with initial value μ. We use $\mathbb{P}^\mu$ to represent the probability of X. In particular, $\mathbb{P}^{\delta_x} = \mathbb{P}^x$.

If the underlying process Y is a random walk, then X is called a branching random walk. If the underlying process Y is a Hunt process, then X is called a branching Hunt process. In particular, if Y is a diffusion, X is called a branching diffusion, and when Y is a Brownian motion, X is called a branching Brownian motion.

Let $\{\mathbb{P}_t; t \geq 0\}$ be the transition semigroup of Y. Then

$$\mathbb{P}_t h(x) = \Pi_x[h(Y_t)], \quad h \in \mathcal{B}^+(E).$$

Suppose $\mathbf{A}$ is the infinitesimal generator of the semigroup $\{\mathbb{P}_t\}$. Put $\psi(x, \lambda) = \sum_{k=0}^\infty p_k(x)\lambda^k$, $\lambda \in [0, 1]$, which is the generating function for the number of offspring generated at point x. Then the Laplace functional of the transition probability of X is given as follows: for any nonnegative measurable function f on E,

$$u(t,x) := \mathbb{P}^x \exp\{-\langle f, X_t\rangle\} \tag{4.1}$$

is the minimal nonnegative solution of the following evolution equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathbf{A}u + \beta(x)(\psi(x,u) - u); \\ u(0,x) = e^{-f(x)}. \end{cases} \tag{4.2}$$

From the display above, we see that the process X is determined by the three ingredients (Y, β, ψ). We call the branching Markov process X described above a (Y, β, ψ)-branching Markov process.

If the underlying process is a Brownian motion on $\mathbb{R}$, or equivalently the infinitesimal generator

$$\mathbf{A} = \frac{1}{2}\frac{\partial^2}{\partial x^2}, \quad x \in \mathbb{R},$$

and if the offspring distribution p and the branching rate β are both independent of the space location, then the first equation in (4.2) is written as

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta\left(\psi(u) - u\right), \tag{4.3}$$

where $u : \mathbb{R}^+ \times \mathbb{R} \to [0,1]$. This equation is called a Fisher–Kolmogorov–Petrovskii–Piskounov equation, or FKPP equation for short.

In the remainder of this section, we assume that m is a positive Radon measure on E with full support, and $\{\mathbb{P}_t; t \geq 0\}$ is a strong semigroup in $L^2(E,m)$. Let $\{\widehat{\mathbb{P}}_t; t \geq 0\}$ be the dual semigroup of $\{\mathbb{P}_t; t \geq 0\}$ in $L^2(E,m)$. Then

$$\int_E h(x)\mathbb{P}_t g(x) m(\mathrm{d}x) = \int_E g(x)\widehat{\mathbb{P}}_t h(x) m(\mathrm{d}x) \quad \text{for any } h, g \in L^2(E,m).$$

Let $\mathbf{A}$ and $\widehat{\mathbf{A}}$ be the infinitesimal generators of $\{\mathbb{P}_t; t \geq 0\}$ and $\{\widehat{\mathbb{P}}_t; t \geq 0\}$ respectively in $L^2(E,m)$. Recall the definition of the mean function M of the offspring distribution that $M(x) := \sum_k k p_k(x)$, $x \in E$. We assume that M is a bounded measurable function on E. Define the Feynman–Kac semigroup $\{\mathbb{P}_t^{(M-1)\beta}; t \geq 0\}$:

$$\mathbb{P}_t^{(M-1)\beta} h(x) := \Pi_x \left[h(Y_t)\ \exp\left(\int_0^t ((M-1)\beta)(Y_r)\mathrm{d}r \right)\right], \quad h \in \mathcal{B}^+(E). \tag{4.4}$$

Then the infinitesimal generators of $\{\mathbb{P}_t^{(M-1)\beta}; t \geq 0\}$ and $\{\widehat{\mathbb{P}}_t^{(M-1)\beta}; t \geq 0\}$ in $L^2(E,m)$ are $\mathbf{A} + (M-1)\beta$ and $\widehat{\mathbf{A}} + (M-1)\beta$ respectively. For any nonnegative measurable function f, if $X_0 = \mu$, then

$$\mathbb{P}^\mu \langle f, X_t\rangle = \langle \mathbb{P}_t^{(M-1)\beta} f, \mu\rangle. \tag{4.5}$$

This formula is usually called the many-to-one formula for branching Markov process X (see, Harris and Roberts [32] for example).

Besides the assumptions above, we also assume the underlying processes satisfy the following assumptions.

Assumption 4.1. (1) There exists a family of continuous strictly positive functions $\{p(t,\cdot,\cdot); t>0\}$ on $E\times E$ such that any $(t,x)\in(0,\infty)\times E$,

$$\mathbb{P}_t f(x)=\int_E p(t,x,y)f(y)m(\mathrm{d}y),\quad \widehat{\mathbb{P}}_t f(x)=\int_E p(t,y,x)f(y)m(\mathrm{d}y).$$

(2) The semigroups $\{\mathbb{P}_t\}$ and $\{\widehat{\mathbb{P}}_t\}$ are ultracontractive, that is, for any $t>0$, there exists a constant $c_t>0$ such that $p(t,x,y)\le c_t$ for any $(x,y)\in E\times E$.

(3) The semigroup $\{\mathbb{P}_t^{(M-1)\beta}; t\ge 0\}$ and semigroup $\{\widehat{\mathbb{P}}_t^{(M-1)\beta}; t\ge 0\}$ are intrinsic ultracontractive, that is, there exists a constant $c_t>0$ such that

$$p^{(M-1)\beta}(t,x,y)\le c_t\phi(x)\widetilde{\phi}(y),\quad \forall(x,y)\in E\times E.$$

Here $p^{(M-1)\beta}(t,x,y)$ is the density of $\mathbb{P}_t^{(M-1)\beta}$.

It is easy to see that the above Assumption 4.1 (3) implies that there are positive constants c and ν,

$$\left|\frac{e^{-\lambda_1 t}p^{(M-1)\beta}(t,x,y)}{\phi(x)\tilde{\phi}(y)}-1\right|\le c\,e^{-\nu t},\quad x\in E,\, t>1. \tag{4.6}$$

Let λ_1 be the common principal eigenvalue for both $\mathbf{A}+(M-1)\beta$ and $\widehat{\mathbf{A}}+(M-1)\beta$, and let ϕ and $\widetilde{\phi}$ be eigenfunctions of $\mathbf{A}+(M-1)\beta$ and $\widehat{\mathbf{A}}+(M-1)\beta$ respectively corresponding to λ_1. Thus

$$\phi(x)=e^{-\lambda_1 t}\mathbb{P}_t^{(M-1)\beta}\phi(x),\quad \widetilde{\phi}(x)=e^{-\lambda_1 t}\widehat{\mathbb{P}}_t^{(M-1)\beta}\widetilde{\phi}(x),\quad x\in E. \tag{4.7}$$

We can choose ϕ and $\widetilde{\phi}$ to be strictly positive and continuous, and satisfy

$$\int_E \phi(x)\widetilde{\phi}(x)m(\mathrm{d}x)=1.$$

We assume that $\lambda_1>0$, which says we consider supercritical branching Markov process. Moreover, we assume the initial measure μ of the branching Markov process X satisfies $\langle\phi,\mu\rangle>0$. Define the process

$$M_t(\phi):=e^{-\lambda_1 t}\frac{\langle\phi,X_t\rangle}{\langle\phi,\mu\rangle},\quad t\ge 0. \tag{4.8}$$

Then $\{M_t(\phi); t\ge 0\}$ is a $\mathbb{P}^{\mu}$-nonnegative martingale with respect to $(\mathcal{F}_t)$. Then it has an almost sure limit, denoted by $M_\infty(\phi)$. We are concerned with the following classical question: under what condition is the limit $M_\infty(\phi)$ non-degenerate? If the limit $M_\infty(\phi)$ is not degenerate to 0, then the number of particles of the branching Markov process increases with exponential rate λ_1 when it is not extinct. At this point, it is necessary to investigate when $M_\infty(\phi)$ is non-degenerate. Here we still use the spine method to solve the problem.

Harris and Robert [32] gave a spine decomposition of X in the space of marked Galton–Watson trees which satisfies $p_0(x) = 0$ for any $x \in E$. Let $\tau \in \mathbb{T}$ be a marked tree. Choose one line of descents $\xi = \{\xi_0 = \emptyset,\ \xi_1, \xi_2, \ldots\}$, where $\xi_{n+1} \in \tau$ is one child of $\xi_n \in \tau$, $n = 0, 1, \ldots$. ξ is called a spine of tree τ. Denote the marked spine by (N, ξ). A particle u is said to be in the spine, if there is a number $i \geq 0$ such that $\mathrm{u} = \xi_i$. And this can be written as $\mathrm{u} \in \xi$. The set of marked Galton–Watson trees with different spines is given by

$$\widetilde{\mathcal{T}} = \{(\tau, Y, \sigma, r, \xi);\ \xi \subset \tau \in \mathbb{T}\}.$$

If $\widetilde{Y} = \{\widetilde{Y}_t; t \geq 0\}$ is the spatial path of the spine, then at any time t, the spine is located at $\widetilde{Y}_t$. If the spine branches n_t times before time t, then when $\mathrm{u} \in L_t \cap \xi$, $\widetilde{Y}_t = Y_{\mathrm{u}}(t)$ and $n_t = |\mathrm{u}|$. Therefore under $\mathbb{P}^x$, the process $\{\widetilde{Y}_t; t \geq 0\}$ is a Π_x Markov process. Let $\mathrm{node}_t((\tau, N, \xi))$ or $\mathrm{node}_t(\xi)$ represent the particle in the spine alive at time t. Then

$$\mathrm{node}_t(\xi) := \mathrm{node}_t((\tau, N, \xi)) := \mathrm{u}, \quad \text{if } \mathrm{u} \in \xi \cap L_t.$$

It is obvious that $\mathrm{node}_t(\xi) = \xi_{n_t}$. Let $\widetilde{\mathcal{F}}_t$ be the natural σ-field generated by the information of the marked Galton–Watson trees with a spine before time t, and define $\widetilde{\mathcal{F}}_\infty = \bigvee_{t\geq 0} \widetilde{\mathcal{F}}_t$. Define a probability measure $\widehat{\mathbb{P}}^x$ on the extended measurable space $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}}_\infty)$ as follows: if $\mathrm{v} \in \xi$, then at time ζ_{v}, v dies and produces r_{v} children. The spine is chosen uniformly from the r_{v} offspring of v at the fission time of v. The other unchosen children give rise to the independent subtrees, which evolve as independent subtrees determined by the probability $\mathbb{P}^{\widetilde{Y}_{\zeta_{\mathrm{v}}}}$. Let O_{v} be the set of children of v which are not chosen. If $\mathrm{v}j \in O_{\mathrm{v}}$, then denote by $(\tau, N)^{\mathrm{v}}_j$ the marked Galton–Watson tree rooted at $\mathrm{v}j$.

Then for any $t > 0$, the restriction of $\widehat{\mathbb{P}}^x$ on $\widetilde{\mathcal{F}}_t$ can be expressed as follows.

$$\begin{aligned}
&\mathrm{d}\widehat{\mathbb{P}}^x\,(\tau, N, \xi)\Big|_{\widetilde{\mathcal{F}}_t} \\
&\quad = \mathrm{d}\Pi_x(\widetilde{Y})\mathrm{d}L^{\beta(\widetilde{Y})}(\mathrm{n}) \prod_{\mathrm{v}<\xi_{n_t}} p_{r_{\mathrm{v}}}(\widetilde{Y}_{\zeta_{\mathrm{v}}}) \prod_{\mathrm{v}<\xi_{n_t}} \frac{1}{r_{\mathrm{v}}} \prod_{\{j:\mathrm{v}j\in O_{\mathrm{v}}\}} \mathrm{d}\mathbb{P}^{\widetilde{Y}_{\zeta_{\mathrm{v}}}}_{t-\zeta_{\mathrm{v}}}\left((\tau, N)^{\mathrm{v}}_j\right),
\end{aligned}$$

where $L^{\beta(\widetilde{Y})}(\mathrm{n})$ is the law of the Poisson point process $\mathrm{n} = \{n_t; t \geq 0\}$ with intensity $\beta(\widetilde{Y}_t)\mathrm{d}t$ along the path $\widetilde{Y}$, $\Pi_x(\widetilde{Y})$ is the law of $\widetilde{Y}$ starting from $x \in E$, and $p_{r_{\mathrm{v}}}(y) = \sum_{k\geq 1} p_k(y) I_{\{r_{\mathrm{v}}=k\}}$ is the probability that individual $\mathrm{v} \in \xi$, located at $y \in E$, has r_{v} children. From the definition above, we see that $\mathbb{P}^x$ is the marginal distribution of the probability $\widehat{\mathbb{P}}^x$ on $(\mathcal{T}, \mathcal{F}_\infty)$.

4.2. $L\log L$ conditions for supercritical branching Hunt processes

Suppose $X = \{X_t; t \geq 0\}$ is a (Y, β, ψ)-branching Markov process on the space $(E, \mathcal{B}_E, m)$. Then the process

$$\left\{ M_t(\phi) := \frac{\langle \phi, X_t \rangle}{\langle \phi, \mu \rangle} e^{-\lambda_1 t}; t \geq 0 \right\}$$

is a nonnegative martingale. It has almost sure limit $M_\infty(\phi)$ as t goes to infinity. We use the notation $\ln^+ t = \ln t \vee 0,\ t > 0$. Asmussen and Hering [5] gave an $L\log L$ criteria for $M_\infty(\phi)$ to be non-degenerate for some branching diffusions in terms of the triplet (Y, β, ψ). In their paper, the space $E = D \subset \mathbb{R}^d$ is a union of some bounded C^3 domains. When the underlying process is a regular diffusion process on D and $\lambda_1 > 0$, then the limit $M_\infty(\phi)$ of $M_t(\phi)$ is non-degenerate if and only if

$$\int_D m(\mathrm{d}y)\widetilde{\phi}(y)\beta(y) \sum_n p_n(y)(n\phi(y)) \ln^+(n\phi(y)) < \infty. \tag{4.9}$$

In Asmussen and Hering [5], the proof was given in an analytical way. We will generalize the result above to general branching Markov processes. We give a more probabilistic proof of the result, using the spine decomposition of measure. Liu, Ren and Song [45] proved the following result.

Theorem 4.2. *Let $\{X_t; t \geq 0\}$ be a (Y, β, ψ)-branching Hunt process, satisfying Assumption 4.1 and $\lambda_1 > 0$. Define*

$$l(x) = \sum_{k=1}^{\infty} k\phi(x) \ln^+(k\phi(x))\, p_k(x), \quad x \in E. \tag{4.10}$$

Then for any measure $\mu \in \mathcal{M}_p(E) \backslash \{0\}$, $M_\infty(\phi)$ is an $L^1(\mathbb{P}^\mu)$ limit if and only if

$$\int_E \beta(x) l(x) \widetilde{\phi}(x) m(\mathrm{d}x) < \infty. \tag{4.11}$$

Because of the branching property, we only need to discuss the special cases that the initial value $\mu = \delta_x$ for some $x \in E$. The above result was proved using a spine decomposition for branching Markov processes under a martingale changed probability. The martingale change Q^x of $\mathbb{P}^x$ in the spine method is given by

$$\mathrm{d}Q^x|_{\mathcal{F}_t} = \frac{M_t(\phi)}{\phi(x)} \mathrm{d}\mathbb{P}^x|_{\mathcal{F}_t}.$$

As before, under probability Q^x, the branching Hunt process X has a spine decomposition, see Hardy and Harris [27]. First of all, we introduce some processes and some notations. Define the process

$$\widetilde{\eta}_t^{(3)}(\phi) := \frac{\phi(\widetilde{Y}_t)}{\phi(x)} \exp\left(-\int_0^t (\lambda_1 - (M-1)\beta)(\widetilde{Y}_s) \mathrm{d}s \right),$$

and σ-field filtration $\mathcal{G}_t := \sigma\big(\widetilde{Y}_s : 0 \leq s \leq t\big)$. Then $\widetilde{\eta}_t^{(3)}(\phi)$ is a nonnegative Π_x martingale with respect to $\{\mathcal{G}_t; t \geq 0\}$. Define the probability Π_x^ϕ on the σ-field $\mathcal{G} = \bigvee_{t\geq 0} \mathcal{G}_t$, which is the martingale change of Π_x given by

$$\left.\frac{\mathrm{d}\Pi_x^\phi}{\mathrm{d}\Pi_x}\right|_{\mathcal{G}_t} = \widetilde{\eta}_t^{(3)}(\phi).$$

Then under measure Π_x^ϕ, the process $\widetilde{Y}$ is also a conservative Markov process, ergodic, and the function $\phi(x)\widetilde{\phi}(x)$ is the unique invariant probability density of the transition semigroup $\mathbb{P}_t^{(M-1)\beta}$. Let $p^{(M-1)\beta}(t,x,y)$ be the density of the measure $\mathbb{P}_t^{(M-1)\beta}(x,\cdot)$ for any $t > 0$. Then for any $h \in \mathcal{B}^+(E)$,

$$\mathbb{P}_t^{(M-1)\beta} h(x) = \int_E p^{(M-1)\beta}(t,x,y) h(y) m(\mathrm{d}y).$$

As a consequence, (Y, Π_x^ϕ) has transition densities given by

$$p^\phi(t,x,y) = \frac{e^{-\lambda_1 t}}{\phi(x)}\, p^{(M-1)\beta}(t,x,y)\ \phi(y).$$

Note that the IU property implies that for the transition densities $\{p^\phi(t,x,y); t > 0, x, y \in E\}$ converge uniformly to invariant probability density $\phi(y)\widetilde{\phi}(y)$.

Lemma 4.1. *Suppose that* $\mathrm{n} = \{\{\zeta_i, i = 1, \dots, n_t\}; t \geq 0\}$ *is a Poisson process with intensity* $\beta(\widetilde{Y}_t)\mathrm{d}t$ *along the path* $\widetilde{Y}$. *Then*

$$\widetilde{\eta}_t^{(1)} := \prod_{i \leq n_t} M(\widetilde{Y}_{\zeta_i}) \cdot \exp\left(-\int_0^t ((M-1)\beta)(\widetilde{Y}_s)\mathrm{d}s\right)$$

is an $L^{\beta(\widetilde{Y})}$*-martingale with respect to the natural filtration* $\mathcal{L}_t = \sigma(n_s : s \leq t)$.

Put $\mathcal{L} = \bigvee_{t \geq 0} \mathcal{L}_t$. Given the path $\widetilde{Y}$, define $L^{(M\beta)(\widetilde{Y})}$ on $\mathcal{L}$ using the martingale change of measure

$$\left.\frac{\mathrm{d}L^{(M\beta)(\widetilde{Y})}}{\mathrm{d}L^{\beta(\widetilde{Y})}}\right|_{\mathcal{L}_t} = \prod_{i \leq n_t} M(\widetilde{Y}_{\zeta_i}) \cdot \exp\left(-\int_0^t ((M-1)\beta)(\widetilde{Y}_s)\mathrm{d}s\right).$$

Then under measure $L^{(M\beta)(\widetilde{Y})}$, the process n turns out to be a Poisson point process with intensity function $M\beta$.

Define

$$\widetilde{\eta}_t^{(2)} := \prod_{i \leq n_t} \frac{r_{\zeta_i}}{M(\widetilde{Y}_{\zeta_i})}.$$

Then

$$\widetilde{\eta}_t := \widetilde{\eta}_t^{(1)} \times \widetilde{\eta}_t^{(2)} \times \widetilde{\eta}_t^{(3)}$$

is a $\widehat{\mathbb{P}}^x$-martingale with respect to $\widetilde{\mathcal{F}}_t$. Now we define a probability measure $\widetilde{Q}^x$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}}_\infty)$ by

$$\mathrm{d}\widetilde{Q}^x|_{\mathcal{F}_t} = \tilde{\eta}_t \mathrm{d}\widehat{\mathbb{P}}^x|_{\mathcal{F}_t}.$$

Define the size-biased off–spring distribution $\hat{p}_k$ as follows,

$$\hat{p}_k(y) = \frac{kp_k(y)}{M(y)}, \quad k \geq 0, \ y \in E.$$

The change of measure from $\widehat{\mathbb{P}}^x$ to $\widetilde{Q}^x$ has three effects: the spine will be changed to a Hunt process with law Π_x^ϕ, its fission times will be increased and the distribution of its family sizes will be sized-biased. More precisely, under $\widetilde{Q}^x$,

(i) the spine process $\widetilde{Y}_t$ moves according to the measure Π_x^ϕ;

(ii) the fission times along the spine occur at an accelerated intensity $(M\beta)(\widetilde{Y}_t)\mathrm{d}t$;

(iii) at the fission time of node v on the spine, the single spine particle is replaced by a random number r_{v} of offspring with size-biased offspring distribution $\widehat{\mathrm{p}}(\widetilde{Y}_{\zeta_{\mathrm{v}}}) := \left(\hat{p}_k(\widetilde{Y}_{\zeta_{\mathrm{v}}})\right)_{k\geq 1}$;

(iv) the spine is chosen uniformly from the r_{v} particles at the fission point of v;

(v) each of the remaining $r_{\mathrm{v}}-1$ particles $\mathrm{v}j \in O_{\mathrm{v}}$ gives rise to independent subtrees $(\tau, N)_j^{\mathrm{v}}$ which evolve as independent subtrees determined by the probability measure $\mathbb{P}^{\widetilde{Y}_{\zeta_{\mathrm{v}}}}$ shifted to the time of creation.

Then the probability measure $\widetilde{Q}^x$ can be expressed as

$$\begin{aligned} &\mathrm{d}\widetilde{Q}^x\left(\tau, N, \xi\right)\big|_{\widetilde{\mathcal{F}}_t} \\ &\quad = \mathrm{d}\Pi_x^\phi(\widetilde{Y})\mathrm{d}L^{M\beta(\widetilde{Y})}(\mathrm{n}) \prod_{\mathrm{v}<\xi_{n_t}} \hat{p}_{r_{\mathrm{v}}}(\widetilde{Y}_{\zeta_{\mathrm{v}}}) \prod_{\mathrm{v}<\xi_{n_t}} \frac{1}{r_{\mathrm{v}}} \prod_{\{j:\mathrm{v}j\in O_{\mathrm{v}}\}} \mathrm{d}\mathbb{P}_{t-\zeta_{\mathrm{v}}}^{\widetilde{Y}_{\zeta_{\mathrm{v}}}}\left((\tau, N)_j^{\mathrm{v}}\right). \end{aligned} \tag{4.12}$$

The probability Q^x is the restriction of probability $\widetilde{Q}^x$ on $(\mathcal{T}, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$.

Theorem 4.3 (Spine decomposition)**.** *Define the σ-field*

$$\widetilde{\mathcal{G}} = \sigma\left(\widetilde{Y}, \mathrm{node}_t(\xi), \zeta_{\mathrm{u}}, r_{\mathrm{u}}, \mathrm{u} < \mathrm{node}_t(\xi), 0 \leq t < \infty\right).$$

Then under $\widetilde{Q}^x$, for any $t > 0$, the conditional expectation of $M_t(\phi)$ has the following spine decomposition:

$$\widetilde{Q}^x\left[\phi(x)M_t(\phi)\middle|\widetilde{\mathcal{G}}\right] = \phi(\widetilde{Y}_t)e^{-\lambda_1 t} + \sum_{\mathrm{u}<\xi_{n_t}} (r_{\mathrm{u}}-1)\phi(\widetilde{Y}_{\zeta_{\mathrm{u}}})e^{-\lambda_1\zeta_{\mathrm{u}}}. \tag{4.13}$$

To simplify notation, ζ_{ξ_i} and r_{ξ_i} will be denoted as ζ_i and r_i, respectively. Recall that $l(x) = \sum_{i=2}^\infty (i\phi(x))\ln^+(i\phi(x))\,p_i(x)$. The following lemma is the main step to prove Theorem 4.2.

Lemma 4.2. (1) *If* $\int_E \widetilde{\phi}(y)\beta(y)l(y)m(\mathrm{d}y) < \infty$, *then*

$$\sum_{i=0}^{\infty} e^{-\lambda_1 \zeta_i} r_i \phi(\widetilde{Y}_{\zeta_i}) < \infty, \quad \widetilde{Q}^x\text{-a.s.}$$

(2) *If* $\int_E \widetilde{\phi}(y)\beta(y)l(y)m(\mathrm{d}y) = \infty$, *then*

$$\limsup_{i\to\infty} e^{-\lambda_1 \zeta_i} r_i \phi(\widetilde{Y}_{\zeta_i}) = \infty, \quad \widetilde{Q}^x\text{-a.s.}$$

Based on Lemma 4.2 and using arguments similar to that for branching process, we can prove Theorem 4.2, see [45]. For detailed discussions about the applications of the spine decompositions in the study of the growth rate of the supercritical branching Markov process, we refer to Engländer and Kyprianou [19], Engländer et al. [18, 20], Liu, Ren and Song [45], etc.

5. Superdiffusions

5.1. Definition of superdiffusion

Super Brownian motions were first introduced in Dawson [12] and Watanabe [58]. After that a series of papers described super Brownian motions in different ways, see Aldous [3, 4], Dawson [11], Duquesne and Le Gall [13], Dynkin [15, 16], Le Gall [41], Li [43] and Perkins [51]. In this paper, we only describe superdiffusions by the Laplace functional of their transition semigroups. Let $\mathcal{M}_F(\mathbb{R}^d)$ be the set of all finite measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. A superdiffusion $X = \{X_t; t \geq 0\}$ is a time homogeneous Markov processes taking values in $\mathcal{M}_F(\mathbb{R}^d)$. Similar to branching Markov process, the distribution of a superdiffusion is also determined by three items: branching rate, underlying process, branching mechanism. Now we set up the superdiffusions.

Let $a_{ij}, i, j = 1, \ldots, d$, be bounded functions in $C^1(\mathbb{R}^d)$ such that all their first partial derivatives are continuous. We assume that the matrix (a_{ij}) is symmetric and satisfies

$$0 < a|\upsilon|^2 \leq \sum_{i,j} a_{ij}\upsilon_i\upsilon_j \quad \text{for all } x \in \mathbb{R}^d \text{ and } \upsilon \in \mathbb{R}^d$$

for some positive constant a. Let $b_i, i = 1, \ldots, d$, be bounded Borel functions on $\mathbb{R}^d$.

We will use $(Y, \Pi_x, x \in \mathbb{R}^d)$ to denote a diffusion process on $\mathbb{R}^d$ corresponding to the operator

$$\mathbf{L} = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla.$$

Assume D is a bounded domain in $\mathbb{R}^d$. We will use $(Y^D, \Pi_x, x \in D)$ to denote the process obtained by killing Y upon exiting from D, that is,

$$Y_t^D = \begin{cases} Y_t, & \text{if } t < \tau_D, \\ \partial, & \text{if } t \geq \tau_D, \end{cases}$$

where $\tau_D = \inf\{t \geq 0; Y_t \notin D\}$ is the first exit time of D and ∂ is a cemetery point. Any function f on D is automatically extended to $D \cup \{\partial\}$ by setting $f(\partial) = 0$. The diffusion Y plays the role of underlying process.

Suppose α and β are bounded measurable functions on $\mathbb{R}^d$, β is a nonnegative measurable function on $\mathbb{R}^d$, and n is a measurable kernel from $\mathbb{R}^d$ to $(0, \infty)$ satisfying

$$\sup_{x \in \mathbb{R}^d} \int_0^\infty (r \wedge r^2) n(x, \mathrm{d}r) < \infty.$$

Define the function

$$\psi(x, \lambda) = \alpha(x)\lambda + \beta(x)\lambda^2 + \int_0^\infty \left(e^{-\lambda r} - 1 + \lambda r\right) n(x, \mathrm{d}r). \tag{5.1}$$

This function is called the branching mechanism of the superdiffusion.

If the branching rate is $\mathrm{d}t$ and the initial value is $\mu \in \mathcal{M}_F(\mathbb{R}^d)$, then the Laplace functional of the transition probability of the corresponding superdiffusion X is determined by a semilinear partial differential equation. For any $f \in C_b(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\big(\exp\{-\langle f, X_t\rangle\}\big) = \exp\langle -u_t(f), \mu\rangle, \tag{5.2}$$

The function $u_t(f)$ is the unique solution to the following semilinear partial differential equation:

$$\begin{cases} \dfrac{\partial u_t(f)}{\partial t} = \mathbf{L}u_t(f) - \psi(u_t(f)), \\ u_0(f) = f. \end{cases} \tag{5.3}$$

Dynkin [15] proved that when α, β and n in the definition of ψ satisfy the assumptions above, and $L = \frac{1}{2}\triangle$, the process X satisfying (5.2) exists, and is called a super Brownian motion. Dynkin [15] also considered the cases that the branching mechanisms depend on time. Here we only consider the time homogeneous cases. For the underlying (Y^D, Π_x), the superdiffusion also exists and is called a (Y^D, ψ)-superprocesses. The function u_t is the unique bounded solution of the following evolution equation:

$$u_t(f)(x) + \Pi_x \int_0^{t \wedge \tau_D} \psi(Y_s, u_{t-s}(f)(Y_s))\mathrm{d}s = \Pi_x(f(Y_t), t < \tau_D), \quad x \in D, \tag{5.4}$$

where τ_D is the first exit time of Y from D.

By differentiating the Laplace functional, we get the expectation formula for any function $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\big[\langle f, X_t\rangle\big] = \langle \Pi_\cdot\big(f(Y_t^D)e^{-\int_0^t \alpha(Y_s^D)\mathrm{d}s}\big), \mu\rangle. \tag{5.5}$$

This formula can be regarded as the many-to-one formula for (Y^D, ψ)-superprocesses. Define the following Feynman–Kac semigroup $\mathbb{P}_t^{-\alpha, D}$:

$$\mathbb{P}_t^{-\alpha, D} f(x) = \Pi_x\big(f(Y_t^D)e^{-\int_0^t \alpha(Y_s^D)\mathrm{d}s}\big), \quad x \in D.$$

Its dual semigroup is denoted as $\widehat{\mathbb{P}}_t^{-\alpha,D.}$

In the rest of this section, we assume that D is a bounded domain in $\mathbb{R}^d$. Let $\mathbf{A}^{\alpha,D}$ and $\widehat{\mathbf{A}}^{\alpha,D}$ be the infinitesimal generators of $\{\mathbb{P}_t^{-\alpha,D}; t \geq 0\}$ and $\{\widehat{\mathbb{P}}_t^{-\alpha,D}; t \geq 0\}$ respectively in $L^2(D)$. Let λ_1 be the common principle eigenvalue for both $\mathbf{A}^{\alpha,D}$ and $\widehat{\mathbf{A}}^{\alpha,D}$, and let ϕ and $\widetilde{\phi}$ be eigenfunctions of $\mathbf{A}^{\alpha,D}$ and $\widehat{\mathbf{A}}^{\alpha,D}$ respectively corresponding to λ_1.

Assume that $\lambda_1 > 0$ and the following intrinsic ultracontractive property holds:

Assumption 5.1. The semigroups $\{\mathbb{P}_t^{-\alpha,D}\}$ and $\{\widehat{\mathbb{P}}_t^{-\alpha,D}\}$ are intrinsic ultracontractive (IU in short), that is, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p^{-\alpha,D}(t,x,y) \leq c_t \phi(x)\widetilde{\phi}(y), \quad \forall (x,y) \in E \times E,$$

where $p^{-\alpha,D}(t,x,y)$ is the density of the semigroup $\mathbb{P}_t^{-\alpha,D}$.

Consequently, we have a nonnegative martingale defined by

$$M_t(\phi) := e^{-\lambda_1 t} \frac{\langle \phi, X_t \rangle}{\langle \phi, \mu \rangle}, \quad t \geq 0. \tag{5.6}$$

We also suppose that for each $x \in D$, there exists a σ-finite measure $\mathbb{N}_x$ and the Dynkin–Kuznetsov $\mathbb{N}$-measure for ψ-superdiffusion $(X_t, \mathbb{P}_{\delta_x})$ (c.f. Dynkin and Kuznetsov [17]), such that, for any $f \in C_b^+(\mathbb{R}^d)$ and $t \geq 0$,

$$\mathbb{N}_x\big(1 - e^{-\langle f, X_t \rangle}\big) = -\ln \mathbb{P}_{\delta_x}\big(e^{-\langle f, X_t \rangle}\big). \tag{5.7}$$

In fact, the equation (5.7) is the Lévy–Khinchine formula for X, and $\mathbb{N}_x$ plays the role of the Lévy measure.

5.2. $L \log L$ criteria for supercritical superdiffusions

Liu, Ren and Song [44, 46] discussed the rate of increase of supercritical (that is, $\lambda_1 > 0$) superdiffusions. The spine decompositions for superdiffusions under a martingale changed probability is established, which is quite similar to that of branching Markov process. The authors used this decomposition to get an $L \log L$ criteria for the non-degeneracy of the limit of martingale $\{M_t(\phi)\}$. Here we will not introduce it in detail. We will give the spine decomposition of superdiffusions for another type of martingale, see Section 6. For the rate of increase of supercritical superprocesses, we only state the results, for proofs see [44, 46]. Define the function l by

$$l(y) := \int_1^\infty r\phi(y) \ln^+(r\phi(y)) n(y, \mathrm{d}r), \quad y \in D. \tag{5.8}$$

Theorem 5.2. *Suppose $X = \{X_t; t \geq 0\}$ is a $(Y^D, \psi(\lambda))$-superdiffusion with initial value $\mu \in \mathcal{M}_F(D)$. Assume that $\lambda_1 > 0$ and that the Feynman–Kac semigroup $\mathbb{P}_t^{-\alpha,D}$ satisfies Assumption 5.1. Then the almost limit $M_\infty(\phi)$ of $M_t(\phi)$ is also the $L^1(D)$ limit if and only if $\int_D \widetilde{\phi}(y) l(y) \mathrm{d}y < \infty$.*

Furthermore we have the following almost sure limit result.

Theorem 5.3. *Assume the superdiffusion $\{X_t\}$ satisfies the assumptions in Theorem 5.2. Then there is a subset $\Omega_0 \subset \Omega$ with probability* 1 *($\mathbb{P}_\mu(\Omega_0) = 1$ for any $\mu \in \mathcal{M}_F(D)$) such that, for any $\omega \in \Omega_0$ and for every nontrivial nonnegative bounded Borel measurable function f on D such that $f \le c\phi$ for some constant $c > 0$ and that the set of discontinuous points of f has zero Lebesgue measure, we have*

$$\lim_{t\to\infty} e^{-\lambda_1 t}\langle f, X_t\rangle(\omega) = M_\infty(\phi)(\omega)\int_D \widetilde{\phi}(y)f(y)\mathrm{d}y. \tag{5.9}$$

Therefore when $\int_D \widetilde{\phi}(y)l(y)\mathrm{d}y < \infty$ and the process is not extinct, it will grow exponentially like $e^{\lambda_1 t}$.

Some discussions about the spine decompositions of superprocesses conditioned on non-extinction and related problems can be found in Etheridge and Williams [21], Evans [22], Evans and Perkins [23], Krone [37], Lee [42], Overbeck [50] and Serlet [54], etc.

6. Applications of spine methods in traveling wave solutions of FKPP equations

6.1. Solutions of the FKPP equations related to branching Brownian motions

The key of the spine method is to reconstruct the branching process under the new probability measure (martingale changed measure). As a result the study on properties of many paths can be translated to the study of one path. This idea was first applied to study the traveling wave solution in Chauvin and Rouault [8]. In [8], the process $\{X_t\}$ is a branching Brownian motion on $\mathbb{R}$, the branching rate is a constant $\beta > 0$ and the mean of the offspring number is $m > 1$. For any $\lambda \in \mathbb{R}$, define

$$W_t(\lambda) := \sum_{\mathrm{u}\in L_t} e^{-\beta(m-1)t}e^{-\lambda Y_{\mathrm{u}}(t)-1/2\lambda^2 t} = \sum_{\mathrm{u}\in L_t} e^{-\lambda(Y_{\mathrm{u}}(t)+c_\lambda t)},$$

where $c_\lambda = \lambda/2+\beta(m-1)/\lambda$. Then $\{W_t(\lambda)\}$ is a nonnegative martingale. Thus it has an almost sure limit $W_\infty(\lambda) = \lim_{t\to\infty} W_t(\lambda)$. Now, using the martingale $(W_t(\lambda), t \ge 0)$ we define a martingale change of measure. For any $t > 0$, $\mathcal{F}_t = \sigma(X_s; s \le t)$, define

$$\left.\frac{\mathrm{d}Q^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathcal{F}_t} = \frac{W_t(\lambda)}{W_0(\lambda)}.$$

Chauvin and Rouault [8] discussed the spine decomposition of the binary branching Brownian motion under the new probability measure Q^x, and used this decomposition to get the large deviation of the rightmost particle of the branching Brownian motion and a Yaglom-type theorem. Here we only introduce the spine decomposition and its application in proving the existence, asymptotic and uniqueness of the traveling wave solutions to the FKPP equation related to branching Brownian motion.

The traveling wave solutions of semi-linear partial differential equations have been studied by many authors analytically, see Uchiyama [57] for a survey and the references therein. For probabilistic discussions of the traveling wave solutions to FKP-P equations, see Harris [28], Harris, Harris and Kyprianou [29], Harris and Harris [30, 31], Kyprianou [38], etc.

Under the new probability Q^x, the spine decomposition of $\{X_t\}$ is as follows.

(1) The spine starts from the initial position of the root, moves like a Brownian motion with drift $-\lambda$.

(2) Independent of its movement, the spine dies and gives birth to children after an exponential time at rate $m\beta$.

(3) The number of the children of the particle in the spine is distributed as the size-biased distribution $\{\hat{p}_k\}$.

(4) The particle in the spine of the next generation is chosen from the children of this particle randomly.

(5) The unchosen particles will evolve as a branching Brownian motion independently according to probability $\mathbb{P}$, starting at the location where they were born.

Using the spine decomposition above, we can prove the following result which answers the question when $W_\infty(\lambda)$ is non-degenerate. In the following we write $\mathbb{P}^0 = \mathbb{P}$ for simplicity.

Theorem 6.1. *Put* $\underline{\lambda} = \sqrt{2\beta(m-1)}$*. The limit* $W_\infty(\lambda)$ *satisfies the following properties:*

(i) *If* $|\lambda| \geq \underline{\lambda}$*, then* $W_\infty(\lambda) = 0$*,* $\mathbb{P}$-a.s.

(ii) *If* $|\lambda| < \underline{\lambda}$*, then* $W_\infty(\lambda) = 0$ $\mathbb{P}$-a.s. *or* $W_\infty(\lambda)$ *is an* $L^1(\mathbb{P})$ *limit according as* $\sum_{k=2}^\infty k \ln kp_k = \infty$ *or* $\sum_{k=2}^\infty k \ln kp_k < \infty$*.*

When $W_\infty(\lambda)$ is not degenerate, a functional of this limit will be a traveling wave solution of the corresponding FKPP equations, see below. The FKPP equation related to branching Brownian motion is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \beta\left(f(u) - u\right), \tag{6.1}$$

where $f(u) = \sum_{k=0}^{\infty} p_k u^k$, $u : \mathbb{R} \times \mathbb{R}^+ \to [0,1]$. By a traveling wave solution we mean a twice continuously differentiable increasing function $\Phi_c : \mathbb{R} \to [0,1]$ such that $\Phi_c(-\infty) = 0 = 1 - \Phi_c(\infty)$ with $u(x,t) = \Phi_c(x - ct)$ being a solution to the equation (6.1). The constant $c \in \mathbb{R}$ is called the wave speed. Now we state the result on the traveling wave solutions to the FKPP equation (6.1).

(1) Subcritical cases: When $|c| < \underline{c} := \sqrt{2\beta(m-1)} = \underline{\lambda}$, there is no traveling wave solution.

(2) Supercritical cases: When $|c| > \underline{c}$ and $\sum_{k=2}^{\infty} k \ln k p_k < \infty$, there is a unique traveling wave solution at speed c given by

$$\Phi_{c_\lambda}(x) = \mathbb{P}\big(\exp\left\{-e^{-\lambda x} W_\infty(\lambda)\right\} \big),$$

where $|\lambda| \in [0, \underline{\lambda})$ such that $c = c_\lambda$. Further this unique traveling wave solution has the asymptotic behavior:

$$1 - \Phi_{c_\lambda}(x) \sim \text{const} \times e^{-\lambda x},$$

as x goes to infinity.

For the critical case (i.e., $|c| = \underline{c}$), to discuss the traveling wave solution, some derivative martingales are needed. Recall that we have a collection of martingales $W_t(\lambda)$ with parameter λ. Taking a derivative with respect to λ, we get the following derivative martingales:

$$\partial W_t(\lambda) = -\frac{\partial}{\partial \lambda} W_t(\lambda) = \sum_{\mathrm{u} \in L_t} \left(Y_{\mathrm{u}}(t) + \lambda t\right) e^{-\lambda(Y_{\mathrm{u}}(t) + c_\lambda t)}.$$

Define the space-time barrier:

$$\Gamma^{(-x,\lambda)} := \left\{(y,t) \in \mathbb{R} \times \mathbb{R}^+ : \, y + \lambda t = -x\right\}.$$

For any time $t > 0$, define a subset of L_t, say $\widetilde{L}_t$, consisting of all particles alive at time t having ancestry (including themselves) whose special paths have not met $\Gamma^{(-x,\lambda)}$ by time t. Define

$$V_t^x(\lambda) = \sum_{\mathrm{u} \in \widetilde{L}_t} \frac{x + Y_{\mathrm{u}}(t) + \lambda t}{x} e^{-\lambda(Y_{\mathrm{u}}(t) + c_\lambda t)}.$$

Then $\{V_t^x(\lambda)\}$ is a nonnegative martingale with mean 1, and has an almost sure limit, denoted by $V_\infty^x(\lambda)$. Using this martingale, we define a martingale change of measure:

$$\left.\frac{\mathrm{d}Q^x}{\mathrm{d}\mathbb{P}^x}\right|_{\mathcal{F}_t} = V_t^x(\lambda).$$

Under the new measure Q^x, the spine decomposition of the branching Brownian motion is stated in the following.

(1) The diffusion along the spine is such that $\{x + Y_t + \lambda t; t \geq 0\}$ is a Bessel-3 process on $(0, \infty)$ starting at x.

(2) The points of fission along the spine form a Poisson process with accelerated rate $m\beta$.

(3) The distribution of offspring numbers at each point of fission on the spine has size-biased distribution:

$$\hat{p}_k = \frac{k}{m} p_k, \quad k \geq 0.$$

(4) The spine is chosen randomly so that at each fission point, the next individual to represent the spine is chosen with uniform probability from the offspring of the current representative.

(5) Individuals which do not carry the spine evolve as $\mathbb{P}$-branching Brownian motions.

Based on this spine decomposition, Yang and Ren [59] proved the following result on the almost sure limit $V_\infty^x(\lambda)$.

Theorem 6.2. *For $x > 0$, the almost sure limit $V_\infty^x(\lambda)$ of $\{V_t^x(\lambda)\}$ satisfies the following properties.*

(i) *If $\lambda > \underline{\lambda}$, then $V_\infty^x(\lambda) = 0$, $\mathbb{P}$-a.s.*

(ii) *If $\lambda = \underline{\lambda}$, then $V_\infty^x(\lambda) = 0$, $\mathbb{P}$-a.s. or is an $L^1(\mathbb{P})$ limit according as $\sum_{k=2}^{\infty} k(\ln k)^2 p_k = \infty$ or $\sum_{k=2}^{\infty} k(\ln k)^2 p_k < \infty$.*

(iii) *If $\lambda \in [0, \underline{\lambda})$, then $V_\infty^x(\lambda) = 0$, $\mathbb{P}$-a.s. or is an $L^1(\mathbb{P})$ limit according as $\sum_{k=2}^{\infty} (k \ln k) p_k = \infty$ or $\sum_{k=2}^{\infty} (k \ln k) p_k < \infty$.*

Kyprianou [38] proved that when $|\lambda| \geq \underline{\lambda}$, $\lim_{t\to\infty} \partial W_t(\lambda) = \lim_{t\to\infty} x V_t^x(\lambda)$ almost surely. Thus the derivative martingale has the same limit properties as $V_t^x(\lambda)$ as $t \to \infty$. As a consequence, the following theorem holds.

Theorem 6.3. *When $|\lambda| \geq \underline{\lambda}$, there is a random variable $\partial W(\lambda)$, such that $\partial W(\lambda) = \lim_{t\to\infty} \partial W_t(\lambda)$, $\mathbb{P}$-a.s.*

(i) *When $|\lambda| > \underline{\lambda}$, $\partial W(\lambda)$ is equal to 0 almost surely.*

(ii) *When $|\lambda| = \underline{\lambda}$, $\mathbb{P}(\partial W(\lambda) = 0) = q$ if and only if $\sum_{k=2}^{\infty} k(\ln k)^2 p_k < \infty$, where q is the extinction probability.*

Therefore, when $|c| = \underline{c}$, the traveling wave solution to the equation (6.1) has the probabilistic expression.

(3) When $|c| = \underline{c}$ and $\sum_{k=2}^{\infty} k(\ln k)^2 p_k < \infty$, the FKPP equation (6.1) has a unique traveling wave solution with speed $\underline{c}$. It can be expressed as

$$\Phi_{\underline{c}}(x) = \mathbb{P}\left(\exp\left\{-e^{-\underline{\lambda} x}\partial W(\lambda)\right\}\right).$$

Further the solution has the following asymptotic behavior:

$$1 - \Phi_{\underline{c}}(x) \sim \text{const} \times xe^{-\underline{\lambda} x},$$

as x goes to infinity.

Finally, we mention that when the underlying process is not a Brownian motion, the derivative martingale might not converge, see Chen [9] for related results for branching random walks.

6.2. Solutions of the FKPP equations related to super Brownian motions

In this subsection we introduce recent progress in the applications of the spine method in discussing the traveling wave solutions to the FKPP equations related to super Brownian motions. The contents mainly are from Kyprianou et al. [39] and the references therein.

Suppose $X = (X_t, \mathbb{P}_\mu)$, $\mu \in \mathcal{M}_F(\mathbb{R})$, is a super Brownian motion starting from μ. The branching mechanism ψ is given by for any $\lambda \geq 0$,

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) n(\mathrm{d}r), \tag{6.2}$$

where α, β are constants, and $\alpha < 0$, $\beta \geq 0$, n is a σ-finite measure on $(0,\infty)$ satisfying $\int_0^\infty (r \wedge r^2) n(\mathrm{d}r) < \infty$. Moreover we assume $\psi(\infty) = \infty$. Such a process X is called a ψ-super Brownian motion. Since the function ψ is strictly convex, $\psi(0) = 0$, and $\alpha < 0$, so ψ has a unique root λ^* in $(0, \infty)$. Therefore, with a positive probability the limit holds $\lim_{t\to\infty} X_t(1) = 0$. Set the event $\mathcal{E} := \{\lim_{t\to\infty} X_t(1) = 0\}$, then

$$\mathbb{P}_\mu(\mathcal{E}) = \exp\{-\lambda^* \mu(1)\}. \tag{6.3}$$

The FKPP equation related to X can be written as

$$\frac{\partial}{\partial t} u_t(x) = \frac{1}{2}\frac{\partial^2}{\partial x^2} u_t(x) - \psi(u_t(x)). \tag{6.4}$$

We are interested in non-increasing solutions to (6.4) of the form $\Phi_c(x - ct)$ where $\Phi_c \geq 0$ and $c \geq 0$ is the wave speed. That is to say we look for nonnegative $\Phi_c \in C^2(\mathbb{R})$ such that

$$\frac{1}{2}\Phi_c'' + c\Phi_c' - \psi(\Phi_c) = 0. \tag{6.5}$$

Moreover we assume that $\Phi_c(-\infty) = \lambda^*$, $\Phi_c(+\infty) = 0$.

Henceforth we shall say that any solution to (6.4) satisfying the aforementioned conditions of non-negativity, monotonicity and connecting the points λ^* at $-\infty$ and 0 at ∞ a traveling wave with wave speed c.

For each $x \in \mathbb{R}$, we write $\mathbb{P}_x$ for $\mathbb{P}_{\delta_x}$ for simplicity. For convenience we write $\underline{\lambda} = \sqrt{-2\Psi'(0+)}$, and for each $\lambda \in \mathbb{R}$, define $c_\lambda = -\Psi'(0+)/\lambda + \lambda/2$. For $\lambda \in \mathbb{R}$, define

$$W_t(\lambda) := e^{-\lambda c_\lambda t}\langle e^{-\lambda\cdot}, X_t\rangle, \quad t \geq 0. \tag{6.6}$$

Then $\{W_t(\lambda); t \geq 0\}$ is a $\mathbb{P}_x$-martingale with respect to $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Since this martingale is nonnegative, it has an almost sure limit denoted by $W_\infty(\lambda)$. For any $\lambda, x \in \mathbb{R}$, define the new probability $\mathbb{P}_x^{-\lambda}$:

$$\left.\frac{\mathrm{d}\mathbb{P}_x^{-\lambda}}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{W_t(\lambda)}{W_0(\lambda)}, \quad t \geq 0.$$

Under the measure $\mathbb{P}_x^{-\lambda}$ the superprocess X can be decomposed into two parts. The first one is a copy of the original superprocess and the second one can be related to an independent process of immigration. As we shall demonstrate next, the process of immigration is governed by a spine or immortal particle along which two independent Poisson point processes of immigration occur. We need first to introduce some more notation.

To be more precisely, the spine decomposition of $(X_t, \mathbb{P}_x^{-\lambda})$ is like this.

(1) (Spine) We take a copy of the process $Y = \{Y_t : t \geq 0\}$ under $\Pi_x^{-\lambda}$ and henceforth refer to it as the *spine*, where $\Pi_x^{-\lambda}$ is the law of a Brownian motion with drift $-\lambda \in \mathbb{R}$ started from $x \in \mathbb{R}$.

(2) (Continuum immigration) Suppose that $\mathbf{n}$ is a Poisson point process such that, for $t \geq 0$, given the spine Y, $\mathbf{n}$ issues a superprocess $X^{\mathbf{n},t}$ at space-time position (ξ_t, t) with rate $2\beta\mathrm{d}t \times \mathrm{d}\mathbb{N}_{Y_t}$.

(3) (Jump immigration) Suppose that $\mathbf{m}$ is a Poisson point process such that, independently of $\mathbf{n}$, given the spine Y, $\mathbf{m}$ issues a superprocess $X^{\mathbf{m},t}$ at space-time point (Y_t, t) with initial mass r at rate $\mathrm{d}t \times rn(\mathrm{d}r) \times \mathrm{d}\mathbb{P}_{r\delta_{Y_t}}$.

We now define for $t \geq 0$,

$$\Lambda_t = X'_t + X_t^{(\mathbf{n})} + X_t^{(\mathbf{m})}, \tag{6.7}$$

where $\{X'_t : t \geq 0\}$ is an independent copy of $(X, \mathbb{P}_\mu)$,

$$X_t^{(\mathbf{n})} = \sum_{s \leq t : \mathbf{n}} X_{t-s}^{\mathbf{n},s},\ t \geq 0 \quad \text{and} \quad X_t^{(\mathbf{m})} = \sum_{s \leq t : \mathbf{m}} X_{t-s}^{\mathbf{m},s},\ t \geq 0.$$

The initial values of $X^{(\mathbf{n})}$ and $X^{(\mathbf{m})}$ are both δ_0, and $X'_0 = \delta_x$.

The distribution of $\{X_t, t \geq 0\}$ under $\mathbb{P}_x^{-\lambda}$ is still a Markov process. The Laplace transforms of the transition probabilities are given below.

Theorem 6.4. *For any* $\lambda \in \mathbb{R}$, $g \in C_b^+(\mathbb{R})$,

$$\mathbb{P}_x^{-\lambda}\big[e^{-\langle g, X_t\rangle}\big] = \mathbb{P}_x\big[e^{-\langle g, X_t\rangle}\big]\Pi_x^{-\lambda}\left[\exp\left\{-\int_0^t \phi(u_g(s, Y_{t-s}))\mathrm{d}s\right\}\right], \tag{6.8}$$

where $\phi(\lambda) = \psi'(\lambda) - \psi'(0+) = 2\beta\lambda + \int_0^\infty (1 - e^{-\lambda r}) rn(\mathrm{d}r)$, $\lambda \geq 0$, u_g *is the solution to the Log-Laplace equation* (6.5) *with initial value* g.

Now we describe the second family of martingales we are interested in by taking the negative derivative in λ of $W(\lambda)$. For any $\lambda \in \mathbb{R}$, define

$$\partial W_t(\lambda) := -\frac{\partial}{\partial\lambda} W_t(\lambda) = \langle (\lambda t + \cdot) e^{-\lambda(c_\lambda t + \cdot)}, X_t\rangle, \quad t \geq 0. \tag{6.9}$$

It can be proved that $\partial W_t(\lambda)$ is a martingale, which is a signed martingale which does not necessarily converge almost surely.

Similar to the previous arguments used for branching Brownian motion, the spine method can be used to prove the following results for these two types of martingales in a probabilistic way.

Theorem 6.5. *Assume that*

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^u \psi(\lambda)\mathrm{d}\lambda}} \mathrm{d}u < \infty,$$

then the following results hold.

(i) *The almost sure limit* $W_\infty(\lambda)$ *is an* $L^1(\mathbb{P}_x)$ *limit if and only if* $|\lambda| < \underline{\lambda}$, *and*

$$\int_1^\infty r(\ln r) n(\mathrm{d}r) < \infty.$$

When $W_\infty(\lambda)$ *is an* $L^1(\mathbb{P}_x)$ *limit the event* $\{W_\infty(\lambda) > 0\} = \mathcal{E}^c$, $\mathbb{P}_x$-a.s. *Otherwise, when it is not an* $L^1(\mathbb{P}_x)$ *limit, its limit is identically zero.*

(ii) *When* $|\lambda| \geq \underline{\lambda}$, $\partial W(\lambda)$ *has a nonnegative almost sure limit* $\partial W_\infty(\lambda)$, *which is identically zero when* $|\lambda| > \underline{\lambda}$ *and when* $|\lambda| = \underline{\lambda}$ *its limit is almost surely strictly positive on* $\mathcal{E}^c$ *if and only if* $\int_1^\infty r(\ln r)^2 n(\mathrm{d}r) < \infty$.

The methods to prove the above results are quite similar to that used for branching Brownian motions. To prove (ii) above we need to construct a nonnegative martingale $\{V_t^{-y}(\lambda), t \geq 0\}$, and then to give the spine decomposition of the super Brownian motion under a new probability obtained by martingale change of measure using the martingale $\{V_t^{-y}(\lambda), t \geq 0\}$. We will define the martingale and give the spine decomposition without proof. For details we refer to Kyprianou [39].

Suppose X^c is a super diffusion process whose underlying process is a Brownian motion with drift c, and whose branching mechanism is ψ. Consider the space-time domain $D_{-y}^t = \{(s,z) \in \mathbb{R}_+ \times \mathbb{R}; s < t, -z < y\}$ for $t, y > 0$. And consider the first exit measure $X_{D_{-y}^t}^c$ of X^c from D_{-y}^t. Set $b_\lambda = c_\lambda - \lambda = -\alpha/\lambda - \lambda/2$, $\lambda > 0$, and $\underline{\lambda} = \sqrt{-2\alpha}$. The nonnegative martingale we are trying to construct is $V_t^{-y}(\lambda)$ which is defined by

$$V_t^{-y}(\lambda) = e^{-\lambda b_\lambda t} y^{-1} \langle (y + \cdot) e^{-\lambda \cdot}, X_{D_{-y}^t}^\lambda \rangle, \quad t \geq 0, \tag{6.10}$$

Denote the almost sure limit of $V_t^{-y}(\lambda)$ by $V_\infty^{-y}(\lambda)$. Write the distribution $\mathbb{P}_{\delta_0}$ of X as $\mathbb{P}$ for short. Define

$$\left.\frac{\mathrm{d}\mathbb{P}^{-y}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = V_t^{-y}(\underline{\lambda}), \quad t \geq 0. \tag{6.11}$$

Then $(X_t, \mathbb{P}^{-y})$ has the following spine decomposition.

(1) (Spine) The spine process is $Y = (Y_t, t \geq 0)$ such that $y + Y_t + \underline{\lambda} t$ is a Bessel-3 process starting from position y.

(2) (Continuum immigration) Suppose that $\mathbf{n}$ is a Poisson point process such that, for $t \geq 0$, given the spine Y, $\mathbf{n}$ issues a superprocess $X^{\mathbf{n},t}$ at space-time position (Y_t, t) with rate $2\beta \mathrm{d}t \times \mathrm{d}\mathbb{N}_{Y_t}$.

(3) (Jump immigration) Suppose that $\mathbf{m}$ is a Poisson point process such that, independently of $\mathbf{n}$, given the spine Y, $\mathbf{m}$ issues a superprocess $X^{\mathbf{m},t}$ at space-time point (Y_t, t) with initial mass r at rate $\mathrm{d}t \times rn(\mathrm{d}r) \times \mathrm{d}\mathbb{P}_{r\delta_{Y_t}}$.

Using the spine decomposition we get the following properties of the martingale limit $\{V_\infty^{-y}(\lambda)\}$.

Theorem 6.6. *For any $y > 0$,*

(i) *If $\lambda > \underline{\lambda}$, then $V_\infty^{-y}(\lambda) = 0$, $\mathbb{P}$-a.s.*

(ii) *If $\lambda = \underline{\lambda}$, then $V_\infty^{-y}(\lambda)$ is an $L^1(\mathbb{P})$ limit if and only if $\int_1^\infty r(\ln r)^2 n(\mathrm{d}r) < \infty$, otherwise $V_\infty^{-y}(\lambda) = 0$, $\mathbb{P}$-a.s.*

(iii) *If $\lambda \in (0, \underline{\lambda})$, then $V_\infty^{-y}(\lambda)$ is an $L^1(\mathbb{P})$ limit if and only if $\int_1^\infty r(\ln r) n(\mathrm{d}r) < \infty$, otherwise $V_\infty^{-y}(\lambda) = 0$, $\mathbb{P}$-a.s.*

When $|\lambda| \geq \underline{\lambda}$, the difference between $V_t^{-y}(\lambda)$ and $\partial W_t(\lambda)$ is ignorable as t is sufficiently large. Thus Theorem 6.5 (ii) follows. The following theorem gives probabilistic representations of the traveling waves using the two types of martingale limits introduced above.

Theorem 6.7. (i) *Suppose that $\int_{[1,\infty)} r(\ln r)n(\mathrm{d}r) < \infty$ and $\lambda \in (0, \underline{\lambda})$. Then, up to an additive constant in its argument, the traveling wave Φ_{c_λ} to (6.5) is given by*

$$\Phi_{c_\lambda}(x) = -\ln \mathbb{P}\big[e^{-e^{-\lambda x} W_\infty(\lambda)}\big],$$

and there is a constant $k_\lambda \in (0,\infty)$ such that

$$\lim_{x\to\infty} \frac{\Phi_{c_\lambda}(x)}{e^{-\lambda x}} = k_\lambda.$$

(ii) *Suppose that $\int_{[1,\infty)} r(\ln r)^2 n(\mathrm{d}r) < \infty$ and $\lambda = \underline{\lambda}$. Then, the critical traveling wave $\Phi_{\underline{\lambda}}$ to (6.5) is given by*

$$\Phi_{\underline{\lambda}}(x) = -\ln \mathbb{P}\big[e^{-e^{-\underline{\lambda} x} \partial W_\infty(\underline{\lambda})}\big].$$

Moreover, there is a constant $k_\lambda \in (0,\infty)$ such that

$$\lim_{x\to\infty} \frac{\Phi_{c_\lambda}(x)}{x e^{-\lambda x}} = k_\lambda. \tag{6.12}$$

Besides the applications of the spine method in the analysis of the traveling wave for monotype process, Ren and Yang [52] discussed the topics for multitype processes. The spine method is also widely used to find the extreme point of branching random walk, see Aïdékon [1], Aïdékon and Shi [2], Hu and Shi [34], Maillard [48], Shi [55] and the references therein.

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