Large deviations and almost sure convergence for the extremes of branching Lévy processes

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Abstract

In this paper, we investigate the asymptotic behavior of supercritical branching Markov processes $\{\mathbb{X}_t, t \geq 0\}$ whose spatial motions are Lévy processes with regularly varying tails. Recently, Ren et al. [Appl. Probab. 61 (2024)] studied the weak convergence of the extremes of $\{\mathbb{X}_t, t \geq 0\}$. In this paper, we establish the large deviation of $\{\mathbb{X}_t, t \geq 0\}$ as well as some almost sure convergence results of the maximum of \mathbb{X}_t .

Keywords and Phrases: Branching Lévy processes, regularly varying tail, large deviation, almost surely convergence.

1 Introdution

1.1 Model and notation

A branching Lévy process on \mathbb{R} is a continuous-time particle system which can be described as follows. The system begins at time t=0 with a single particle located at x which moves according to a branching Lévy process $\{\xi_t, \mathbf{P}_x\}$ with Lévy exponent $\psi(\theta)$. After an independent exponential time with parameter β , the initial particle dies and gives birth to k new particles with probability p_k , $k \geq 0$. Each new particle moves according to the Lévy process ξ starting from the position of its parent's death, and branches independently with the same branching rate β and offspring distribution $\{p_k, k \geq 0\}$. All particles, once born, evolve independently of one another. The expectation with respect to \mathbf{P}_x will be denoted by \mathbf{E}_x . We write $\mathbf{P} := \mathbf{P}_0$ and $\mathbf{E} := \mathbf{E}_0$.

We label each particle using the classical Ulam-Harris labeling system. We denote by \mathbb{T} the set of all particles in the tree and use o represent the root of the tree. For any $u \in \mathbb{T}$,

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let N^u be the number of the offspring of u and τ_u denote the lifetime of u. Then $\{\tau_u, u \in \mathbb{T}\}$ are i.i.d. exponential random variables of parameter β .

We will use the notation u < v to mean that u is an ancestor of v, and $u \le v$ that either u < v or u = v. For any $u \in \mathbb{T}$, let $I_u^0 := \{v \in \mathbb{T} : v < u\}$ and $I_u := \{v \in \mathbb{T} : v \le u\}$. Let b_u and σ_u be the birth time and death time of u respectively. It is clear that

$$b_u = \sum_{v \in I_u^0} \tau_v, \qquad \sigma_u = b_u + \tau_u.$$

Let \mathcal{L}_t be the set of all particles alive at time t. Then $u \in \mathcal{L}_t$ means $b_u \leq t < \sigma_u$. For $u \in \mathbb{T}$ with $b_u \leq t$, define $\tau_{u,t} = \sigma_u \wedge t - b_u$. For any $u \in \mathcal{L}_t$, let ξ_t^u be the position of u at time t. The branching Lévy process $\{X_t : t \geq 0\}$ is the measure-valued process defined by

$$\mathbb{X}_t := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u}.$$

We use \mathbb{P}_x to denote the law of the branching Lévy process when the initial particle starts at position x. The expectation with respect to \mathbb{P}_x will be denoted by \mathbb{E}_x . We write $\mathbb{P} := \mathbb{P}_0$ and $\mathbb{E} := \mathbb{E}_0$. Let $\{\mathcal{F}_t\}$ be the natural filtration of \mathbb{X} and

$$\mathcal{F}_t^{\mathbb{T}} := \sigma(\{N^u : u \in \mathbb{T} \text{ with } \sigma_u \leq t\} \cup \{\tau_{u,t} : u \in \mathbb{T} \text{ with } b_u \leq t\}).$$

In this paper, we study supercritical branching Lévy processes, that is to say, we always assume that $\mu := \sum_k kp_k > 1$. Let \mathcal{S} be survival event. Then $\mathbb{P}_x(\mathcal{S}) > 0$ does not depend on the location x of the initial particle. The extinction probability $q := \mathbb{P}(\mathcal{S}^c)$ is the unique root in the interval [0,1) of the equation f(s) = s, where $f(s) := \sum_k p_k s^k$. For more details, see [5, Section III. 4]. For any $x \in \mathbb{R}$, we define $\mathbb{P}_x^*(\cdot) := \mathbb{P}_x(\cdot|\mathcal{S})$ and denote the corresponding expectation by \mathbb{E}_x^* . We write $\mathbb{P}^* := \mathbb{P}_0^*$ and $\mathbb{E}^* := \mathbb{E}_0^*$.

Recently, many people studied the extreme of branching Lévy processes defined by

$$R_t := \max_{u \in \mathcal{L}_t} \xi_t^u, \quad t > 0.$$

Here we use the convention that $\max \emptyset = -\infty$. Among these, branching Brownian motions garnered the most attention. For branching Brownian motions, Bramson [12] (see also [13]) proved that, under some moment conditions on the offspring distribution, $\mathbb{P}(R_t - m(t) \le x) \to 1 - w(x)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$ and w(x) is a traveling wave solution. For the large deviation of R_t , Chauvin and Rouault [14, 15] studied the asymptotic behavior of $P(R_t > \sqrt{2}\delta t)$ for $\delta \ge 1$. Derrida and Shi [17, 18] studied the lower large deviation of R_t , i.e, the asymptotic behavior of $\frac{1}{t}\log P(R_t \le \sqrt{2}\delta t)$ for $\delta < 1$, and found that the rate function has a phase transition at $1 - \sqrt{2}$. Subsequently, Chen, He and Mallein [16] studied the asymptotic behavior of $P(R_t \le \sqrt{2}\delta t)$ for $\delta < 1$. Recently, [2, 3, 4] studied the extremal processes of branching Brownian motions.

In this paper we study the case when the spatial motion is a Lévy process with regularly varying Lévy exponent. We now state our assumptions.

Let Z_t be the total number of the particles alive at time t. It is well known that $\{Z_t : t \geq 0\}$ is a continuous time Galton-Watson process. The following two quantities will play

important roles in this paper:

$$\lambda := \beta(\mu - 1), \quad \vartheta := \int_0^\infty e^{-\lambda r} \mathbb{P}(Z_r > 0) dr. \tag{1.1}$$

By the Markov property and the branching property, the process $\{e^{-\lambda t}Z_t, t \geq 0\}$ is a non-negative martingale with respect to $\{\mathcal{F}_t^{\mathbb{T}}\}$. Thus it has an almost sure limit

$$\lim_{t \to \infty} e^{-\lambda t} Z_t =: W.$$

It is well known that W is non-degenerate if and only if the following $L \log L$ criterien holds:

Assumption 1.
$$\sum_{k=1}^{\infty} (k \log k p_k) < \infty$$
.

Moreover, under **Assumption** 1, $\mathbb{P}(W > 0) = \mathbb{P}(S) > 0$. For more details, see [5, Section III.7].

We will always assume that the spatial motion satisfies:

Assumption 2. There exist $\alpha \in (0,2)$, a complex constant c_* with $\text{Re}(c_*) > 0$ and a function $L(x) : \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that $\psi(\theta) \sim -c_*\theta^{\alpha}L(\theta^{-1})$ as $\theta \to 0_+$.

Strictly α -stable processes satisfy **Assumption** 2. By using the tables of complete Bernstein functions in [32], we can come up a lot of subordinate Brownian motions satisfying **Assumption** 2. Further discussions of **Assumption** 2 can be found in the Appendix.

In the Appendix, we will show that, under **Assumption** 2, the function $e^{-c_*|\theta|^{\alpha}}$, $\theta \in \mathbb{R}$, is the characteristic function of an α -stable random variable with Lévy measure v_{α} , where

$$v_{\alpha}(dx) = q_1 x^{-1-\alpha} \mathbf{1}_{(0,\infty)}(x) dx + q_2 |x|^{-1-\alpha} \mathbf{1}_{(-\infty,0)}(x) dx, \tag{1.2}$$

with q_1 and q_2 being nonnegative numbers, uniquely determined by the following equation: if $\alpha \neq 1$

$$c_* = \alpha \Gamma(1 - \alpha) \left(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2} \right),$$

and if $\alpha = 1$

$$q_1 = q_2 = \Re(c_*)/\pi.$$

It has been proved in [33, Remark 2.1] that for any s > 0,

$$\mathbf{P}(\xi_s \ge x) \sim \frac{q_1}{\alpha} s x^{-\alpha} L(x), \quad \mathbf{P}(\xi_s \le -x) \sim \frac{q_2}{\alpha} s x^{-\alpha} L(x), \quad x \to \infty, \tag{1.3}$$

that is, ξ_s has regularly varying tails. To ensure the right tail of ξ_s is regularly varying, we always assume that

Assumption 3. $q_1 > 0$.

In this paper, we always assume that **Assumptions** 1-3 hold, and that $\alpha \in (0, 2)$, c_* and L are as specified in **Assumption** 2.

The following variant of ϑ will also play a role later in this paper:

$$\vartheta^* := \frac{q_1}{\alpha} \int_0^\infty e^{-\lambda r} \mathbb{P}(Z_r > 0) dr = \frac{q_1}{\alpha} \vartheta.$$
 (1.4)

Put $\overline{\mathbb{R}}_0 = [-\infty, \infty] \setminus \{0\}$ with the topology generated by the set

$$\{(a,b), (-b,-a), (a,\infty], [-\infty,-a) : 0 < a < b \le \infty\}.$$

Let $C_c^+(\overline{\mathbb{R}}_0)$ be the family of all non-negative continuous functions g on $\overline{\mathbb{R}}_0$ with $g \equiv 0$ on $(-\delta,0) \cup (0,\delta)$ for some $\delta > 0$. Denote by $\mathcal{M}(\overline{\mathbb{R}}_0)$ the space of all Radon measures on $\overline{\mathbb{R}}_0$ endowed with the topology of vague convergence (denoted by $\stackrel{v}{\rightarrow}$), generated by the maps $\nu \to \int f d\nu$ for all $f \in C_c^+(\overline{\mathbb{R}}_0)$. For any $g \in \mathcal{B}_b^+(\overline{\mathbb{R}}_0)$, $\nu \in \mathcal{M}(\overline{\mathbb{R}}_0)$, we write $\nu(g) := \int_{\overline{\mathbb{R}}_0} g(x)\nu(\mathrm{d}x)$. A sequence of random elements ν_n in $\mathcal{M}(\overline{\mathbb{R}}_0)$ converges weakly to ν , denoted as $\nu_n \stackrel{d}{\to} \nu$, if and only if for all $g \in C_c^+(\overline{\mathbb{R}}_0)$, $\nu_n(g)$ converges weakly to $\nu(g)$. Let $\mathcal{B}_1(\mathbb{R})$ be the set of all the Borel functions $\varphi : \mathbb{R} \to [0,1]$ with $\varphi \equiv 1$ on $[-\delta,\delta]$ for some $\delta > 0$.

For any $x \neq 0$ and a measure ν , we denote by ν/x the measure defined by

$$(\nu/x)(g) = \int g(y/x)\nu(dy).$$

It is well known that there exists a continuous function $\tilde{L}: \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that $\lim_{x\to\infty}\frac{\tilde{L}(x)}{L(x)}=1$, $\tilde{L}(0+)\in(0,\infty)$ and $x^{-\alpha}\tilde{L}(x)$ is strictly decreasing on $(0,\infty)$. In this paper, we always assume that L satisfies this property. Note that $\lim_{x\to 0} x^{-\alpha}L(x)=\infty$ and $\lim_{x\to\infty} x^{-\alpha}L(x)=0$. Let $H(y):(0,\infty)\to(0,\infty)$ be the inverse function of $x^{-\alpha}L(x)$. Then

$$H(y)^{-\alpha}L(H(y)) = y, \quad y > 0.$$
 (1.5)

It is well known (see, [11, Theorem 1.5.12] for instance) that

$$H(y) = y^{-1/\alpha} \bar{L}(y^{-1}),$$

with \bar{L} being slowly varying at ∞ . From now on, \bar{L} always stands for the function above. In [33], we studied the weak convergence of extremes of \mathbb{X} . Let $h(t) := H(e^{-\lambda t})$, that is,

$$h(t)^{-\alpha}L(h(t)) = e^{-\lambda t}. (1.6)$$

Note that h is strictly increasing. Define

$$\mathcal{N}_t := \mathbb{X}_t/h(t) = \sum_{v \in \mathcal{L}_t} \delta_{h(t)^{-1}\xi_t^v}.$$

In [33, Theorem 1.1], we proved that \mathcal{N}_t converges weakly to a random measure \mathcal{N}_{∞} . More precisely, for any $g \in C_c^+(\overline{\mathbb{R}}_0)$,

$$\lim_{t \to \infty} \mathbb{E}(e^{-\mathcal{N}_t(g)}) = E(e^{-\mathcal{N}_\infty(g)}) = \mathbb{E}\left(\exp\left\{-C(e^{-g})W\right\}\right),\,$$

where

$$C(\varphi) := \int_0^\infty e^{-\lambda r} \int_{\mathbb{R}_0} \mathbb{E}(1 - \varphi(x)^{Z_r}) v_\alpha(dx) dr < \infty, \quad \forall \varphi \in \mathcal{B}_1(\mathbb{R}).$$
 (1.7)

Moreover, $\mathcal{N}_{\infty} = \sum_{j} T_{j} \delta_{e_{j}}$, where given W, $\sum_{j} \delta_{e_{j}}$ is a Poisson random measure with intensity $\vartheta W v_{\alpha}(dx)$, $\{T_{j}, j \geq 1\}$ are i.i.d. copies of a random variable T with

$$P(T=k) = \vartheta^{-1} \int_0^\infty e^{-\lambda r} \mathbb{P}(Z_r=k) dr, \quad k \ge 1,$$
(1.8)

where $v_{\alpha}(dx)$ is given by (1.2), ϑ is defined in (1.1), and $\sum_{j} \delta_{e_{j}}$ and $\{T_{j}, j \geq 1\}$ are independent. As a consequence, we proved in [33, Corollary 1.2] that under \mathbb{P}^{*} , $\frac{R_{t}}{h(t)}$ converges weakly. More precisely,

$$\lim_{t \to \infty} \mathbb{P}^* \left(\frac{R_t}{h(t)} \le x \right) = \begin{cases} \mathbb{E}^* \left(e^{-\vartheta^* W x^{-\alpha}} \right), & x > 0; \\ 0, & x \le 0. \end{cases}$$
 (1.9)

Therefore, X_t , normalized by h(t), converges to a random measure. In particular, the largest position R_t is of order h(t) as $t \to \infty$. In this paper, we consider a function $\Lambda(t)$ which grows faster than the function h(t) in the sense that $\lim_{t\to\infty} \frac{\Lambda(t)}{h(t)} = \infty$, or slower than

h(t) in the sense that $\lim_{t\to\infty}\frac{\Lambda(t)}{h(t)}=0$. When $\Lambda(t)$ grows faster than h(t), we find the rate that $\mathbb{P}(R_t>\Lambda(t))$ converges to 0 an $t\to\infty$, and describe the limit of $\mathbb{X}_t/\Lambda(t)$, conditioned on $\{R_t>\Lambda(t)\}$, at $t\to\infty$. When $\Lambda(t)$ grows slower than h(t), we find the rate that $\mathbb{P}(R_t\leq\Lambda(t))$ converges to 0 and describe the limit of $\mathbb{X}_t/\Lambda(t)$ conditioned on $\{R_t\leq\Lambda(t)\}$. In this paper, we also study the almost sure asymptotic behavior of R_t .

1.2 Main results

In this subsection, we state our main results. Let $\mathcal{H}(\mathbb{R})$ denote the family of uniformly continuous functions $\varphi: \mathbb{R} \to [0,1]$ with $\varphi \equiv 1$ in some neighborhood of 0. Let $\mathcal{H}_0(\mathbb{R})$ denote the family of all the functions $\varphi \in \mathcal{H}(\mathbb{R})$ with $\varphi \equiv 0$ on (c, ∞) for some c > 0. Note that if $g \in C_c^+(\overline{\mathbb{R}}_0)$ then $e^{-g} \in \mathcal{H}(\mathbb{R})$.

Theorem 1.1. If $\Lambda:[0,\infty)\to(0,\infty)$ satisfies $\lim_{t\to\infty}\frac{\Lambda(t)}{h(t)}=\infty$, then for any $\varphi\in\mathcal{H}(\mathbb{R})$,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E} \left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u / \Lambda(t)) \right) \right) = C(\varphi),$$

where $C(\varphi)$ is defined in (1.7). In particular,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \mathbb{P}(R_t > \Lambda(t)) = \vartheta^*,$$

where ϑ^* is defined in (1.4).

In [35], Shiozawa studied the upper deviation of the maximal displacement of a branching symmetric stable process with spatially inhomogeneous branching structure, and proved some weak convergence results.

Corollary 1.2. If $\Lambda : [0, \infty) \to (0, \infty)$ satisfies $\lim_{t \to \infty} \frac{\Lambda(t)}{h(t)} = \infty$, then conditioned on $\{R_t > \Lambda(t)\}$,

- (1) $R_t/\Lambda(t)$ converges weakly to a random variable R^* with density $\alpha x^{-1-\alpha} \mathbf{1}_{(1,\infty)}(x)$.
- (2) $\mathbb{X}_t/\Lambda(t)$ converges weakly to $T\delta_{R^*}$, where the law of T is given in (1.8), and T and R^* are independent.

Recall that $f(s) = \sum_k p_k s^k$ and $q = \mathbb{P}(\mathcal{S}^c)$. Note that $f'(q) \in [0, 1)$. Put

$$\rho := \beta(1 - f'(q)). \tag{1.10}$$

For any $\theta \geq 0$, define

$$\phi(\theta) = \mathbb{E}(e^{-\theta W}). \tag{1.11}$$

Theorem 1.3. Let $\Lambda:[0,\infty)\to(0,\infty)$ be a non-decreasing function. Assume that

$$\sum_{n=1}^{\infty} n\Lambda(n)^{-\alpha} L(\Lambda(n)) < \infty \ and \ \lim_{t \to \infty} \frac{\Lambda(t)}{h(t)} = 0,$$

and if $p_k = 0$ for all $k \geq 3$, we further assume that $\Lambda(t) > e^{\gamma t}$ with some $\gamma > 0$ for t sufficiently large. Then for any $\varphi \in \mathcal{H}_0(\mathbb{R})$,

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{E}^* \left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u / \Lambda(t)) \right) = \frac{1}{1-q} A \left[\phi(C(\varphi)) \right],$$

where r(t) is defined by $h(r(t)) = \Lambda(t)$, $C(\varphi)$ is defined in (1.7), and A(s) is defined in (3.1) below. In particular,

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{P}^* \left(R_t \le \Lambda(t) \right) = \frac{1}{1-q} A \left[\phi(\vartheta^*) \right].$$

Corollary 1.4. Let $\Lambda:[0,\infty)\to(0,\infty)$ be a non-decreasing function. If

$$\sum_{n=1}^{\infty} n\Lambda(n)^{-\alpha} L(\Lambda(n)) < \infty \text{ and } \lim_{t \to \infty} \frac{\Lambda(t)}{h(t)} = 0,$$

then under \mathbb{P}^* , conditioned on $\{R_t \leq \Lambda(t)\}$, $\mathbb{X}_t/\Lambda(t)$ converges weakly to some random measure $\Xi = \sum_{k=1}^{K} \bar{\mathcal{N}}_{\infty}^{(k)}$, where

- (i) $\{\bar{\mathcal{N}}_{\infty}^{(k)}, k \geq 1\}$ are i.i.d. with the same law as $P(\mathcal{N}_{\infty} \in \cdot | \mathcal{N}_{\infty}(\mathbb{R}) \neq 0, \mathcal{N}_{\infty}((1, \infty)) = 0);$
- (ii) K is a positive integer valued random variable with generating function

$$E\left(s^{\mathcal{K}}\right) = \frac{A((\phi(\vartheta^*) - q)s + q)}{A(\phi(\vartheta^*))};$$

(iii) $\{\bar{\mathcal{N}}_{\infty}^{(k)}, k \geq 1\}$ and \mathcal{K} are independent.

Remark 1.5. (1) It is interesting that, by Theorem 1.1 and Corollary 1.2, conditioned on $\{R_t > \Lambda(t)\}$, the limits of $R_t/\Lambda(t)$ and $\mathbb{X}_t/\Lambda(t)$ do not depend on the function $\Lambda(t)$: for any $\Lambda(t)$ satisfying $\lim_{t\to\infty}\frac{\Lambda(t)}{h(t)}=\infty$, the limits are the same. The limit of the point process $\mathbb{X}_t/\Lambda(t)$, conditioned on $\{R_t > \Lambda(t)\}$, is a point measure supported on one point R^* .

- (2) By Theorem 1.3 and and Corollary 1.4, the limit of $\mathbb{X}_t/\Lambda(t)$, conditioned on the event $\{R_t \leq \Lambda(t)\}$, does not depend on $\Lambda(t)$: for any $\Lambda(t)$ satisfying $\lim_{t\to\infty} \frac{\Lambda(t)}{h(t)} = 0$ and $\sum_{n=1}^{\infty} n\Lambda(n)^{-\alpha}L(\Lambda(n)) < \infty$, the limit Ξ does not depends on Λ . Comparing Ξ with \mathcal{N}_{∞} (the limit of $\mathbb{X}_t/h(t)$), we see that Ξ is a random sum of independent copies of $\bar{\mathcal{N}}_{\infty}^{(k)}$, with common law equal to that of \mathcal{N}_{∞} condition on $\{\mathcal{N}_{\infty}(\mathbb{R}) \neq 0, \mathcal{N}_{\infty}((1,\infty)) = 0\}$.
- (3) In the special case that $L \equiv 1$, we have $h(t) = e^{\frac{\lambda}{\alpha}t}$. Consider $\Lambda(t) = e^{c\frac{\lambda}{\alpha}t}$ for some constant c > 0. If c > 1, by Theorem 1.1,

$$\lim_{t \to \infty} e^{(c-1)\lambda t} \mathbb{P}(R_t > \Lambda(t)) = \vartheta^*.$$

If 0 < c < 1, then r(t) = ct. By Theorem 1.3,

$$\lim_{t \to \infty} e^{((1-c)\rho t} \mathbb{P}^* (R_t \le \Lambda(t)) = A \left(\phi \left(\vartheta^* \right) \right).$$

We now state some almost sure convergence results of R_t .

Theorem 1.6. It holds that

$$\liminf_{t \to \infty} \frac{R_t}{H(e^{-\lambda t} \log t)} = (\vartheta^* W)^{\frac{1}{\alpha}}, \qquad \mathbb{P}^*\text{-a.s.}$$

Theorem 1.7. Suppose that $G:[0,\infty)\to (0,\infty)$ is a non-decreasing function satisfying $\lim_{t\to\infty}\frac{G(t)}{h(t)}=\infty$.

(1) If $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, then

$$\limsup_{t \to \infty} \frac{R_t}{G(t)} = 0, \qquad \mathbb{P}^* \text{-a.s.}$$

(2) If $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$, then

$$\limsup_{t \to \infty} \frac{R_t}{G(t)} = \infty, \qquad \mathbb{P}^* \text{-a.s.}$$

This implies that, for a non-decreasing function G satisfying $\lim_{t\to\infty}\frac{G(t)}{h(t)}=\infty$, either $\limsup_{t\to\infty}\frac{R_t}{G(t)}=0$ \mathbb{P}^* -a.s., or $\limsup_{t\to\infty}\frac{R_t}{G(t)}=\infty$ \mathbb{P}^* -a.s.. Similar result holds for a subordinator. If ξ is a subordinator with infinite mean, then for any $G:[0,\infty)\to(0,\infty)$ being an

increasing function such that $\frac{G(t)}{t}$ increases, then $\limsup_{t\to\infty} \frac{\xi_t}{G(t)} = 0$ or ∞ almost surely, see [6, Theorem 13].

Now we give some intuitive idea for one of the main techniques of this paper. Let $Y_j := \xi_j - \xi_{j-1}, j \ge 1$. Then $\{Y_j\}$ are i.i.d. It is easy to see from (1.3) that

$$\mathbf{P}(\xi_n > x) \sim n\mathbf{P}(\xi_1 > x) \sim \mathbf{P}(\max_{j \le n} Y_j > x), \quad x \to \infty.$$

Thus the maximum $\max_{1 \leq j \leq n} Y_j$ plays a dominating role in the asymptotic behavior of ξ_n . For $t \geq 0$, $u \in \mathbb{T}$, we set $X_{u,t} := \xi^u_{\sigma_u \wedge t} - \xi^u_{b_u \wedge t}$. Then we have

$$\mathbb{X}_t = \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u} = \sum_{u \in \mathcal{L}_t} \delta_{\sum_{v \in I_u} X_{v,t}}.$$

We will see in Lemma 2.4 that the asymptotic behavior of X_t is governed by Y_t :

$$\mathbb{Y}_t := \sum_{u \in \mathcal{L}_t} \sum_{v \in I_u} \delta_{X_{v,t}}.$$

Thus, to prove Theorems 1.1, 1.6 and 1.7, we first establish the corresponding results for \mathbb{Y}_t . This technique has been employed in [33] for branching Lévy processes and in [9, 10, 20] for branching random walks with heavy tails. However this technique (Lemma 2.4)) does not work for the proof of lower deviation result in Theorem 1.3. We establish lower large deviation results of \mathbb{X}_t and \mathbb{Y}_t separately, and it turns out that the results are identical, which is somewhat surprising.

For branching random walks, several authors have studied the convergence of the extremes under an exponential moment assumption on the displacements of the offspring from the parent, see Aïdékon [1], Hu and Shi [29], and Madaule [31]. Recently, many researchers studied related topics for branching random walks with heavy-tailed displacements. Assume that the displacements of the offspring from the parents are i.i.d. with

$$P(X > x) \sim ae^{-L(x)x^r}, \quad x \to \infty,$$

where a > 0, L is slowly varying at ∞ and $r \in [0, 1)$. When $r \in (0, 1)$, the maximum M_n grows polynomially. For example, if L is a constant, then $M_n/n^{1/r}$ converges to a positive constant almost surely. See, [22, 23, 24, 26] for more related results. When r = 0, $\log t/L(t) \to 0$ (or $L(t) = o(\log t)$) as $t \to \infty$, the extremes have been investigated in [8]. When r = 0 and $L(x) = \alpha \log x - \log \tilde{L}(x)$ where \tilde{L} is slowly varying at ∞ , Durrett [20] proved that $a_n^{-1}M_n$ converges weakly, where $a_n = m^{n/\alpha}L_0(m^n)$ and L_0 is slowly varying at ∞ . Recently, the extremal processes of the branching random walks with regularly varying steps were studied by Bhattacharya et al. [9, 10]. It was proved in [9, 10] that the point random measures $\sum_{|v|=n} \delta_{a_n^{-1}S_v}$, where S_v is the position of v, converges weakly to a Cox cluster process, which are quite different from the case with exponential moments. Recently, Bhattacharya [7] studied the large deviations of extremes in branching random walk with regularly varying displacements, corresponding to our results for $\Lambda(t)$ growing fast than h(t).

2 Upper deviation of X_t and Y_t

It is well known (see [11, Theorem 1.5.6] for instance) that, for any $\epsilon > 0$, there exists $a_{\epsilon} > 0$ such that for any $x, y > a_{\epsilon}$,

$$\frac{L(y)}{L(x)} \le 2 \max\{(y/x)^{\epsilon}, (y/x)^{-\epsilon}\}, \qquad \frac{\bar{L}(y)}{\bar{L}(x)} \le 2 \max\{(y/x)^{\epsilon}, (y/x)^{-\epsilon}\}. \tag{2.1}$$

Let $C_b^0(\mathbb{R})$ be the space of all bounded continuous functions vanishing in a neighborhood of 0. Recall the definition of v_{α} in (1.2). In the following lemma, we present a generalization of [33, Lemma 2.1]. Since the proof follows a similar line of reasoning, we omit it.

Lemma 2.1. If **Assumption 2** holds, then for any $g \in C_b^0(\mathbb{R})$ and s > 0,

$$\lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbf{E} \left(g \left(\frac{\xi_s}{x} \right) \right) = s \int_{\mathbb{R}_0} g(x) v_{\alpha}(\mathrm{d}x).$$

In [33, Lemma 2.2], we have proved that, under **Assumption 2**, there exist $c_0 > 0$ and $x_0 > 0$ such that for any s > 0 and $x > x_0$,

$$\mathbf{P}(|\xi_s| > x) \le c_0 s x^{-\alpha} L(x). \tag{2.2}$$

Using (2.1) with $\epsilon = 1$, we see that, for any $c \in (0,1)$ and y > 0 sufficiently small so that $cH(y) > a_1 + x_0$, it holds that

$$\mathbf{P}(|\xi_s| > cH(y)) \le c_0 s c^{-\alpha} H(y)^{-\alpha} L(cH(y))$$

$$\le 2c_0 s c^{-\alpha - 1} H(y)^{-\alpha} L(H(y)) = 2c_0 c^{-\alpha - 1} s y, \tag{2.3}$$

where in the last equality we used (1.5). It was shown in [36] that, for any t > 0,

$$\lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbf{P} \left(\sup_{0 \le s \le t} \xi_s > x \right) = \lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbf{P} (\xi_t > x) = \frac{q_1}{\alpha}$$
 (2.4)

and

$$\lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbf{P} \left(\inf_{0 \le s \le t} \xi_s < -x \right) = \lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbf{P} (\xi_t < -x) = \frac{q_2}{\alpha}.$$
 (2.5)

We now recall a special many-to-one formula. For more general many-to-one formulas, see [28, Theorem 8.5]. For any $u \in \mathbb{T}$, let n^u be the number of particles in $I_u \setminus \{o\}$.

Lemma 2.2 (Many-to-one formula). Let $\{n_t\}$ be a Poisson process with parameter β on some probability space (Ω, \mathcal{G}, P) . Then for any $g \in \mathcal{B}_b^+(\mathbb{R})$,

$$\mathbb{E}\left(\sum_{v\in\mathcal{L}_t}g(n^v)\right)=e^{\lambda t}E\left(g(n_t)\right).$$

2.1 A key lemma on "one big jump"

In the remainder of the paper, we use $g_1(t) \stackrel{t}{\leq} g_2(t)$ to denote that $g_1(t) \leq g_2(t)$ for sufficiently large t. For any nonnegative function g and measure $\nu = \sum_{k=1}^n \delta_{x_k}$, define

$$\mathcal{I}(g,\nu) := \prod_{k=1}^{n} g(x_k).$$

Here we use the convention that $\prod_{k=1}^{0} g(x_k) = 1$. It is clear that $\mathcal{I}(g, \nu) = e^{\nu(\log g)}$. Recall that, for $t \geq 0$ and $u \in \mathbb{T}$,

$$X_{u,t} = \xi_{\sigma_u \wedge t}^u - \xi_{b_u \wedge t}^u.$$

Let $\mathcal{D}_t := \{u \in \mathbb{T} : b_u \leq t, Z_t^u > 0\}$, where Z_t^u is the number of offspring of particle u alive at time t. Define

$$M_t := \max_{v \in \mathcal{L}_t} \max_{u \in I_v} X_{u,t} = \max_{u \in \mathcal{D}_t} X_{u,t}$$
(2.6)

Recall that

$$\mathbb{Y}_t = \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v} \delta_{X_{u,t}} = \sum_{u \in \mathcal{D}_t} Z_t^u \delta_{X_{u,t}}.$$

Let 0 < s < t. The particles in \mathcal{D}_t can be divided into two groups: those born before time t - s and those born after t - s. We define

$$M_{s,t} := \max_{u \in \mathbb{T}: b_u < t-s} |X_{u,t}|$$

and

$$\mathbb{Y}_{s,t} := \sum_{u \in \mathcal{D}_t: t-s < b_u \le t} Z_t^u \delta_{X_{u,t}} = \sum_{u \in \mathbb{T}: t-s < b_u \le t} Z_t^u \delta_{X_{u,t}}.$$

Using the tree structure, we can categorize all particles born after t-s according to the branches formed by particles that were alive at time t-s. More precisely,

$$\mathbb{Y}_{s,t} = \sum_{v \in \mathcal{L}_{t-s}} \sum_{u: v < u, b_u \le t} Z_t^u \delta_{X_{u,t}} =: \sum_{v \in \mathcal{L}_{t-s}} \mathbb{Y}_{s,t}^v. \tag{2.7}$$

By the branching property and the Markov property, conditioned on \mathcal{F}_{t-s} , $\{\mathbb{Y}_{s,t}^v, v \in \mathcal{L}_{t-s}\}$ are i.i.d. with a common law equal to that of

$$\mathbb{Y}_s' := \sum_{u \in \mathcal{D}_s'} Z_s^u \delta_{X_{u,s}},\tag{2.8}$$

where $\mathcal{D}'_s := \mathcal{D}_s \setminus \{o\}$. We will also use the following notation later:

$$M_t' := \max_{u \in \mathcal{D}_t'} X_{u,t}. \tag{2.9}$$

Lemma 2.3. If a(t) is a positive function with $\lim_{t\to\infty} a(t) = \infty$, then

$$\lim_{s \to \infty} \limsup_{t \to \infty} e^{-\lambda t} a(t)^{\alpha} L(a(t))^{-1} \mathbb{P}(M_{s,t} > a(t)) = 0.$$

Proof: When a(t) = h(t), it has been proven in [33, (2.14)]) that

$$\mathbb{P}(M_{s,t} > a(t)) \stackrel{t}{\leq} c_0 e^{\lambda t} a(t)^{-\alpha} L(a(t)) \left(\lambda^{-1} e^{-\lambda s} + \frac{e^{-\beta s} - e^{-\lambda s}}{\lambda - \beta}\right). \tag{2.10}$$

For the general case, (2.10) also holds. The proof follows almost the same argument, so we omit the details. Then the desired result follows immediately.

The following key lemma says that X_t and Y_t have similar asymptotic behaviors. In the proof of Lemma 2.4, we show that with high probability, for all $v \in \mathcal{L}_t$, there exists at most one particle $u \in I_v$ that experiences a "large jump".

Lemma 2.4. If a(t) is a positive function such that $a(t) \stackrel{t}{>} e^{\epsilon t}$ for some $\epsilon \in (0,1)$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t \to \infty} e^{-\lambda t} a(t)^{\alpha} L(a(t))^{-1} \mathbb{E} \left(|\mathcal{I}(\varphi, \mathbb{X}_t / a(t)) - \mathcal{I}(\varphi, \mathbb{Y}_t / a(t))| \right) = 0.$$

Proof: We divide the proof into three steps.

Step 1 For any t > 0 and $\theta \in (0, 1]$, define

$$A_t(\theta) := \bigcap_{v \in \mathcal{L}_t} \left\{ \sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \le 1 \right\}.$$

We claim that for any $p \in (0, 2)$,

$$\lim_{t \to \infty} e^{-\lambda t} a(t)^{p\alpha} \mathbb{P}(A_t(\theta)^c) = 0. \tag{2.11}$$

Note that

$$\mathbb{P}(A_t(\theta)^c | \mathcal{F}_t^{\mathbb{T}}) \le \sum_{v \in \mathcal{L}_t} \mathbb{P}\left(\left\{\sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \ge 2\right\} | \mathcal{F}_t^{\mathbb{T}}\right). \tag{2.12}$$

By (2.2) we have that

$$\mathbb{P}\left(|X_{u,t}| > \theta a(t)/\log a(t)|\mathcal{F}_t^{\mathbb{T}}\right) = \mathbf{P}(|\xi_s| > \theta a(t)/\log a(t))|_{s=\tau_{u,t}}$$

$$\stackrel{t}{\leq} \left(c_0 \theta^{-\alpha} t \cdot a(t)^{-\alpha} [\log a(t)]^{\alpha} L(\theta a(t)/\log a(t))\right) \wedge 1 =: p_t. \tag{2.13}$$

Recall that, for any $v \in \mathbb{T}$, n^v is the number of particles in $I_v \setminus \{o\}$. Thus $|I_v| = n^v + 1$. Since, conditioned on $\mathcal{F}_t^{\mathbb{T}}$, $\{X_{u,t}, u \in I_v\}$ are independent, we have

$$\mathbb{P}\left(\sum_{u \in I_{v}} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \ge 2|\mathcal{F}_{t}^{\mathbb{T}}\right) \le \sum_{m=2}^{n^{v}+1} \binom{n^{v}+1}{m} p_{t}^{m} \\
\le p_{t}^{2} \sum_{m=0}^{n^{v}-1} n^{v} (n^{v}+1) \binom{n^{v}-1}{m} p_{t}^{m} \\
= p_{t}^{2} n^{v} (n^{v}+1) (1+p_{t})^{n^{v}-1}.$$

Thus by (2.12) and the many-to-one formula, we have

$$\mathbb{P}(A_t(\theta)^c) \le e^{\lambda t} p_t^2 E(n_t(n_t+1)(1+p_t)^{n_t-1}) = e^{\lambda t} p_t^2 (2+(1+p_t)\beta t)\beta t e^{\beta t p_t}$$

$$\sim (\beta t)^2 e^{\lambda t} p_t^2, \quad t \to \infty.$$
(2.14)

Here we used the fact $tp_t \to 0$ as $t \to \infty$. Now (2.11) follows from (2.13) and (2.14) immediately.

Step 2 In the remainder of this proof, we fix a constant $c > 2\alpha + e^2\beta/\epsilon$. Define $B_t := \bigcap_{v \in \mathcal{L}_t} \{n^v \le c \log a(t)\}$. Using the many-to-one formula, we get

$$\mathbb{P}(B_t^c) = \mathbb{P}\left(\bigcup_{v \in \mathcal{L}_t} \{n^v > c \log a(t)\}\right) \leq \mathbb{E}\left(\sum_{v \in \mathcal{L}_t} \mathbf{1}_{\{n^v > c \log a(t)\}}\right) \\
= e^{\lambda t} P(n_t > c \log a(t)) \leq e^{\lambda t} \inf_{r > 0} e^{-rc \log a(t)} E(e^{rn_t}) \\
= e^{\lambda t} e^{-(\log \log a(t) + \log c - \log \beta t - 1)c \log a(t) - \beta t}.$$
(2.15)

Since $a(t) \stackrel{t}{>} e^{\epsilon t}$, we have

$$\log \log a(t) + \log c > \log(c\epsilon t) \ge \log(\beta t) + 2.$$

Thus by (2.15), we have

$$\mathbb{P}(B_t^c) \stackrel{t}{<} e^{\lambda t} e^{-c\log a(t)} e^{-\beta t} = e^{\lambda t} a(t)^{-c} e^{-\beta t}.$$

Since $c > 2\alpha$, we get

$$\lim_{t \to \infty} e^{-\lambda t} a(t)^{2\alpha} \mathbb{P}(B_t^c) = 0. \tag{2.16}$$

Step 3 Let $v' \in I_v$ be such that $|X_{v',t}| = \max_{u \in I_v} \{|X_{u,t}|\}$. We note that, on the event $A_t(\theta)$, $|X_{u,t}| \leq \theta a(t)/\log a(t)$ for any $u \in I_v \setminus \{v'\}$. Since $\xi_t^v = \sum_{u \in I_v} X_{u,t}$, on the event $A_t(\theta) \cap B_t$, we have that

$$|\xi_t^v - X_{v',t}| = \left| \sum_{u \in I_v \setminus \{v'\}} X_{u,t} \right| \le \frac{n^v \theta a(t)}{\log a(t)} \le c\theta a(t).$$
 (2.17)

For $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. Since φ is uniformly continuous, for any $\gamma > 0$, there exists $\eta > 0$ such that $|\varphi(x_1) - \varphi(x_2)| \leq \gamma$ whenever $|x_1 - x_2| < \eta$. We now fix an arbitrary $\gamma > 0$ and the corresponding η .

Recall $c > 2\alpha + e^2\beta/\epsilon$. Choose θ small enough so that $c\theta < \eta \land (\delta/2)$. We assume that t is sufficiently large so that $\log a(t) > 2\theta/\delta$. We note that, on the event $A_t(\theta)$, $|X_{u,t}| \le \theta a(t)/\log a(t) \le a(t)\delta/2$ for any $u \in I_v \setminus \{v'\}$, and thus $\varphi(X_{u,t}/a(t)) = 1$. It follows that on the event $A_t(\theta) \cap B_t$,

$$\left| \prod_{v \in \mathcal{L}_t} \varphi(\xi_t^v / a(t)) - \prod_{v \in \mathcal{L}_t} \prod_{u \in I_v} \varphi(X_{u,t} / a(t)) \right| = \left| \prod_{v \in \mathcal{L}_t} \varphi(\xi_t^v / a(t)) - \prod_{v \in \mathcal{L}_t} \varphi(X_{v',t} / a(t)) \right|$$

$$\leq \sum_{v \in \mathcal{L}_t} |\varphi(\xi_t^v / a(t)) - \varphi(X_{v',t} / a(t))|.$$
 (2.18)

By (2.17), on $A_t(\theta) \cap B_t$, we have

$$|\xi_t^v - X_{v',t}|/a(t) \le c\theta \le \eta \wedge (\delta/2).$$

Thus if $|X_{v',t}| \leq \delta a(t)/2$, then $|\xi_t^v|/a(t) < \delta$, which implies that $\varphi(\xi_t^v/a(t)) - \varphi(X_{v',t}/a(t)) = 0$. Hence by (2.18), on $A_t(\theta) \cap B_t$,

$$\begin{aligned} &|\mathcal{I}(\varphi, \mathbb{X}_t/a(t)) - \mathcal{I}(\varphi, \mathbb{Y}_t/a(t))| \le \gamma \sum_{v \in \mathcal{L}_t} \mathbf{1}_{\{|X_{v',t}| > a(t)\delta/2\}} \\ \le & \gamma \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > a(t)\delta/2\}} = \gamma \sum_{u \in \mathcal{D}_t} Z_t^u g_t(X_{u,t}), \end{aligned}$$

where $g_t(y) = \mathbf{1}_{|y| > a(t)\delta/2}$. Therefore, for any fixed $s \in (0, t)$, on $A_t(\theta) \cap B_t \cap \{M_{s,t} \leq a(t)\delta/2\}$, we have

$$|\mathcal{I}(\varphi, \mathbb{X}_t/a(t)) - \mathcal{I}(\varphi, \mathbb{Y}_t/a(t))| \le \gamma \sum_{u:t-s < b_u \le t} Z_t^u g_t(X_{u,t}) = \gamma \mathbb{Y}_{s,t}(g_t).$$

Since φ takes values in [0, 1], we have

$$\mathbb{E} \left| \mathcal{I}(\varphi, \mathbb{X}_t/a(t)) - \mathcal{I}(\varphi, \mathbb{Y}_t/a(t)) \right|$$

$$\leq \mathbb{P}(A_t(\theta)^c) + \mathbb{P}(B_t^c) + \mathbb{P}(M_{s,t} > a(t)\delta/2) + \gamma \mathbb{E}(\mathbb{Y}_{s,t}(g_t)).$$
(2.19)

By (2.7) and (2.2), we have that

$$\mathbb{E}(\mathbb{Y}_{s,t}(g_t)) = \mathbb{E}(Z_{t-s})\mathbb{E}(\mathbb{Y}'_s(g_t)) = e^{\lambda(t-s)}\mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u \mathbb{P}(|X_{u,s}| > a(t)\delta/2|\mathcal{F}_s^{\mathbb{T}})\right)$$

$$\stackrel{t}{\leq} c_0(\delta/2)^{-\alpha} a(t)^{-\alpha} L(\delta a(t)/2) e^{\lambda(t-s)} \mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u \tau_{u,s}\right)$$

$$< c_0(\delta/2)^{-\alpha} a(t)^{-\alpha} L(\delta a(t)/2) e^{\lambda t} s.$$

where in the last inequality we used the following inequality:

$$\mathbb{E}\left(\sum_{u\in\mathcal{D}_s'} Z_s^u \tau_{u,s}\right) \le \mathbb{E}\left(\sum_{u:b_u\le s} Z_s^u \tau_{u,s}\right) \le se^{\lambda s},$$

which follows from the display below (2.19) in the proof of [33, Proposition 2.1]. Thus combining (2.19), (2.11) and (2.16), we have

$$\limsup_{t \to \infty} e^{-\lambda t} a(t)^{\alpha} L(a(t))^{-1} \mathbb{E} \left| \mathcal{I}(\varphi, \mathbb{X}_t/a(t)) - \mathcal{I}(\varphi, \mathbb{Y}_t/a(t)) \right|$$

$$\leq \limsup_{t \to \infty} e^{-\lambda t} a(t)^{\alpha} L(a(t))^{-1} \mathbb{P}(M_{s,t} > a(t)\delta/2) + c_0(\delta/2)^{-\alpha} s \gamma.$$

Letting $\gamma \to 0$ first, and then letting $s \to \infty$ and applying Lemma 2.3, we arrive at the desired result.

2.2 Proof of Theorem 1.1

We emphasize here that the definition of $M_{s,t}$ in [33, Section 2.3] coincides with $\mathbb{Y}'_s/h(t)$.

Lemma 2.5. For any s > 0 and $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{x \to \infty} x^{\alpha} L(x)^{-1} \left[1 - \mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}'_s/x) \right) \right] = \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)} \right) \int_{\mathbb{R}} \mathbb{E} \left(1 - \varphi(y)^{Z_r} \right) v_{\alpha}(dy) \, \mathrm{d}r.$$

Proof: It has been proven in [33, (2.17) and the first display on page 636] that for $\varphi(x) = e^{-g(x)}$ with $g \in C_0^+(\bar{\mathbb{R}}_0)$,

$$\lim_{t \to \infty} h(t)^{\alpha} L(h(t))^{-1} \left[1 - \mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}'_s/h(t)) \right) \right]$$
$$= \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)} \right) \int_{\mathbb{R}} \mathbb{E} \left(1 - \varphi(y)^{Z_r} \right) v_{\alpha}(dy) \, \mathrm{d}r.$$

By examining the proof of the above limit, we observe that it holds for any $\varphi \in \mathcal{H}(\mathbb{R})$. The desired result now follows.

Suppose $\varphi \in \mathcal{B}_1(\mathbb{R})$. If $\{\varphi_n, n \geq 1\} \subset \mathcal{B}_1(\mathbb{R})$ are such that φ and φ_n are identically 1 on $[-\delta, \delta]$ for some $\delta > 0$, and that $\varphi_n \to \varphi$ almost everywhere, then by the dominated convergence theorem,

$$C(\varphi_n) \to C(\varphi)$$
.

Hence $C(\varphi)$ is continuous in φ . Recall M_t is defined in (2.6).

Proposition 2.6. If $\Lambda(t)$ is a positive function with $\lim_{t\to\infty}\frac{\Lambda(t)}{h(t)}=\infty$, then for any $\varphi\in\mathcal{H}(\mathbb{R})$,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))\right)\right) = C(\varphi), \tag{2.20}$$

where $C(\varphi)$ is defined in (1.7). Furthermore,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \mathbb{P}(M_t > \Lambda(t)) = \vartheta^*, \tag{2.21}$$

where ϑ^* is defined in (1.4).

Proof: For $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. It is easy to see that for any 0 < s < t, on the event $\{M_{s,t} \leq \delta \Lambda(t)\}$, it holds that $\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t)) = \mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t))$. Thus, for any 0 < s < t,

$$|\mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))\right) - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t))\right)| \le \mathbb{P}(M_{s,t} > \delta\Lambda(t)). \tag{2.22}$$

Using the Markov property and (2.7), we have that

$$1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t))|\mathcal{F}_{t-s}\right) = 1 - \prod_{v \in \mathcal{L}_{t-s}} \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{v}/\Lambda(t))|\mathcal{F}_{t-s}\right)$$
$$= 1 - \left[\mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s}^{\prime}/\Lambda(t))\right)\right]^{Z_{t-s}}.$$
 (2.23)

It follows from Lemma 2.5 that, as $t \to \infty$,

$$1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}'_s/\Lambda(t))\right) \sim C_s \Lambda(t)^{-\alpha} L(\Lambda(t)),$$

where
$$C_s = \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)} \right) \int_{\mathbb{R}} \mathbb{E} \left(1 - \varphi(y)^{Z_r} \right) v_{\alpha}(dy) \, dr$$
. Thus

$$-\log \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}'_s/\Lambda(t))\right) \sim 1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}'_s/\Lambda(t))\right) \sim C_s \Lambda(t)^{-\alpha} L(\Lambda(t))$$

and consequently

$$-Z_{t-s}\log \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}'_s/\Lambda(t))\right) \sim e^{\lambda(t-s)}C_s\Lambda(t)^{-\alpha}L(\Lambda(t))\cdot W,$$

which tends to 0 as $t \to \infty$ since $\lim_{t \to \infty} \frac{\Lambda(t)}{h(t)} = \infty$. Combining this with (2.23), we get that, as $t \to \infty$,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t)) | \mathcal{F}_{t-s}\right)\right)$$

$$= \lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} Z_{t-s} \left(-\log \mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}'_s/\Lambda(t))\right)\right)$$

$$= e^{-\lambda s} \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E} \left(1 - \varphi(y)^{Z_r}\right) v_{\alpha}(\mathrm{d}y) \, \mathrm{d}r \cdot W. \tag{2.24}$$

Moreover,

$$e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t))|\mathcal{F}_{t-s}\right)\right)$$

$$\leq e^{-\lambda t} Z_{t-s} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}'_{s}/\Lambda(t))\right)\right)$$

$$\stackrel{t}{\leq} 2e^{-\lambda s} \int_{0}^{s} \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_{r}}\right) v_{\alpha}(\mathrm{d}y) \,\mathrm{d}r \cdot e^{-\lambda(t-s)} Z_{t-s}. \tag{2.25}$$

In the first inequality we used the inequality $1-x^n \leq n(1-x), x \in (0,1)$. Note that

$$e^{-\lambda(t-s)}Z_{t-s} \to W$$
 a.s., and $\mathbb{E}\left(e^{-\lambda(t-s)}Z_{t-s}\right) \to \mathbb{E}W = 1, \quad t \to \infty.$ (2.26)

Combining (2.24), (2.25) and (2.26), and using the dominated convergence theorem we get

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}/\Lambda(t))\right)\right)$$

$$= e^{-\lambda s} \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_r}\right) v_{\alpha}(\mathrm{d}y) \,\mathrm{d}r. \tag{2.27}$$

Using (2.22), Lemma 2.3 and (2.27), we get that

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))\right)\right)$$

$$= \lim_{s \to \infty} e^{-\lambda s} \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_r}\right) v_{\alpha}(\mathrm{d}y) \, \mathrm{d}r$$

$$= \int_0^\infty e^{-\lambda r} \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_r}\right) v_{\alpha}(\mathrm{d}y) \, \mathrm{d}r = C(\varphi).$$

We now prove (2.21). For any $\epsilon \in (0,1)$, define $\tilde{I}_{\varepsilon}, I_{\varepsilon} \in \mathcal{H}(\mathbb{R})$ by

$$I_{\varepsilon}(y) := \begin{cases} 1, & y < 1, \\ \text{linear}, & 1 \leq y \leq 1 + \varepsilon, \\ 0, & y > 1 + \varepsilon \end{cases}$$

and

$$\tilde{I}_{\varepsilon}(y) := \begin{cases} 1, & y < 1 - \varepsilon, \\ \text{linear}, & 1 - \varepsilon \leq y \leq 1, \\ 0, & y > 1. \end{cases}$$

It is clear that $\tilde{I}_{\varepsilon}(y) \leq \mathbf{1}_{(-\infty,1]}(y) \leq I_{\varepsilon}(y)$. Applying (2.20) to \tilde{I}_{ε} and I_{ε} , and by the continuity of $C(\varphi)$, we can get (2.20) still holds for $\varphi(y) = \mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \mathbb{P}(M_t > \Lambda(t)) = C(\mathbf{1}_{(-\infty,1]})$$

$$= \int_0^{\infty} e^{-\lambda r} \int_1^{\infty} \mathbb{P}(Z_r > 0) v_{\alpha}(\mathrm{d}y) \mathrm{d}r = \frac{q_1}{\alpha} \vartheta.$$

The proof is now complete.

Proof of Theorem 1.1 Applying Lemma 2.4 and Proposition 2.6, we have that

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\mathcal{I}(\varphi, \mathbb{X}_t/\Lambda(t))\right) = C(\varphi), \quad \varphi \in \mathcal{H}(\mathbb{R}). \tag{2.28}$$

By the continuity of $C(\varphi)$, we can get (2.28) still holds for $\varphi(y) = \mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \mathbb{P}(R_t > \Lambda(t)) = C(\mathbf{1}_{(-\infty,1]}) = \vartheta^*.$$
 (2.29)

Proof of Corollary 1.2: By (2.29), we have for x > 1,

$$\lim_{t \to \infty} \mathbb{P}(R_t > x\Lambda(t)|R_t > \Lambda(t)) = x^{-\alpha} = P(R^* > x).$$

For any $g \in C_c^+(\overline{\mathbb{R}}_0)$ and $\theta > 0$, applying Theorem 1.1 with $\varphi(y) = e^{-\theta g(y)} \mathbf{1}_{(-\infty,1]}(y)$ and $\varphi(y) = e^{-\theta g(y)}$, we have that

$$\lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \mathbb{E}\left(e^{-\theta \frac{\mathbb{X}_{t}}{\Lambda(t)}(g)}, R_{t} > \Lambda(t)\right)$$

$$= \lim_{t \to \infty} e^{-\lambda t} \Lambda(t)^{\alpha} L(\Lambda(t))^{-1} \left[1 - \mathbb{E}\left(e^{-\theta \frac{\mathbb{X}_{t}}{\Lambda(t)}(g)}, R_{t} \leq \Lambda(t)\right) - \left(1 - \mathbb{E}\left(e^{-\theta \frac{\mathbb{X}_{t}}{\Lambda(t)}(g)}\right)\right)\right]$$

$$= \int_{0}^{\infty} e^{-\lambda r} \int_{1}^{\infty} \mathbb{E}\left[e^{-\theta g(y)Z_{r}}; Z_{r} > 0\right] v_{\alpha}(\mathrm{d}y) \mathrm{d}r.$$
(2.30)

By (2.29) and (2.30), we have

$$\lim_{t \to \infty} \mathbb{E}\left(e^{-\theta \frac{\mathbb{X}_t}{\Lambda(t)}(g)} | R_t > \Lambda(t)\right) = \frac{\alpha}{q_1 \vartheta} \int_0^\infty e^{-\lambda r} \int_1^\infty \mathbb{E}\left[e^{-\theta g(y)Z_r}; Z_r > 0\right] v_\alpha(\mathrm{d}y) \mathrm{d}r$$
$$= E\left(e^{-\theta T \cdot g(R^*)}\right).$$

The proof is now complete.

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3 Lower deviation of X_t and Y_t

We first give some results about the continuous time branching processes $\{Z_t : t \ge 0\}$. Recall that the extinction probability $q \in [0, 1)$. For any $s \in [0, 1]$ and $t \ge 0$, define

$$F(s,t) := \mathbb{E}\left(s^{Z_t}\right).$$

Recall the constant ρ defined in (1.10). It is well known (see, [5, Section III.8], for instance) that

$$\lim_{t \to \infty} e^{\rho t} [F(s, t) - q] =: A(s) \tag{3.1}$$

exists for $0 \le s < 1$. Moreover, the convergence is uniform in $s \in [0, a]$ for any 0 < a < 1. The function A(s) is the unique solution of

$$A(F(s,t)) = e^{-\rho t} A(s) \tag{3.2}$$

with A(q) = 0, A'(q) = 1. Since A(s) is the limit of power series, it is itself a power series

$$A(s) = \sum_{j=0}^{\infty} a_j s^j.$$

It is clear that $a_0 \le 0$ and $a_j \ge 0, j \ge 1$. For any 0 < c < 1 - q and $s \in [0, 1]$,

$$A(cs+q) = A(cs+q) - A(q) = \sum_{j=1}^{\infty} a_j [(cs+q)^j - q^j] = \sum_{k=1}^{\infty} \left[\sum_{j=k}^{\infty} a_j \binom{j}{k} q^{j-k} \right] c^k s^k.$$

Thus $\frac{A(cs+q)}{A(c+q)}$, $s \in [0,1]$, is a probability generating function.

Lemma 3.1. For any $s \in [0, 1)$,

$$A(s) = s - q + \int_0^\infty \beta e^{\rho t} V(F(s, r)) dr,$$

where

$$0 \le V(s) := f(s) - f'(q)s - q(1 - f'(q)) \le f''(s \lor q)(s - q)^2.$$
(3.3)

Proof: By the Markov property and the branching property, we have that

$$\begin{split} F(s,t) = & se^{-\beta t} + \int_0^t \beta e^{-\beta r} f(F(s,t-r)) \mathrm{d}r \\ = & se^{-\rho t} + \int_0^t \beta e^{-\rho r} f(F(s,t-r)) \mathrm{d}r - \int_0^t e^{-\rho r} \beta f'(q) F(s,t-r) \mathrm{d}r \\ = & se^{-\rho t} + \int_0^t \beta e^{-\rho r} V(F(s,t-r)) \mathrm{d}r + q\rho \int_0^t e^{-\rho r} \mathrm{d}r \\ = & e^{-\rho t} \left(s - q + \int_0^t \beta e^{\rho r} V(F(s,r)) \, \mathrm{d}r \right) + q. \end{split}$$

By (3.1), we have

$$A(s) = \lim_{t \to \infty} e^{\rho t} (F(s, t) - q) = s - q + \lim_{t \to \infty} \int_0^t \beta e^{\rho r} V(F(s, r)) dr$$
$$= s - q + \int_0^\infty \beta e^{\rho r} V(F(s, r)) dr.$$

Since q = f(q), we have V(s) = f(s) - f(q) - f'(q)(s - q). Now (3.3) follows immediately from Taylor's formula.

Recall that $\phi(\theta) = \mathbb{E}(e^{-\theta W})$.

Lemma 3.2. For any $\theta > 0$,

$$A\left(\phi(\theta)\right) = \int_{-\infty}^{\infty} \beta e^{\rho s} V\left[\phi\left(\theta e^{\lambda s}\right)\right] ds.$$

Proof: By the branching property and the Markov property, we have

$$Z_{t+s} = \sum_{u \in \mathcal{L}_s} Z_{t+s}^u.$$

Given Z_s , $\{Z_{t+s}^u, u \in \mathcal{L}_s\}$ are i.i.d. with the same law as Z_t . It follows that

$$W = \lim_{t \to \infty} e^{-\lambda s} \sum_{u \in \mathcal{L}_s} e^{-\lambda t} Z_{t+s}^u = e^{-\lambda s} \sum_{u \in \mathcal{L}_s} W^u,$$

where $\{W^u, u \in \mathcal{L}_s\}$ are i.i.d. with the same law as W. Thus we have

$$\phi(\theta e^{\lambda s}) = \mathbb{E}\left(\phi(\theta)^{Z_s}\right) = F(\phi(\theta), s). \tag{3.4}$$

Hence by Lemma 3.1 and (3.4), we have for any x > 0

$$\int_{-x}^{\infty} \beta e^{\rho s} V\left[\phi\left(\theta e^{\lambda s}\right)\right] ds = e^{-\rho x} \int_{0}^{\infty} \beta e^{\rho s} V\left[\left(\phi\left(\theta e^{\lambda s} e^{-\lambda x}\right)\right] ds$$

$$= e^{-\rho x} \int_{0}^{\infty} \beta e^{\rho s} V\left[F\left(\phi\left(\theta e^{-\lambda x}\right), s\right)\right] ds = e^{-\rho x} \left[A\left(\phi\left(\theta e^{-\lambda x}\right)\right) - \phi\left(\theta e^{-\lambda x}\right) + q\right]$$

$$= A\left(F\left(\phi\left(\theta e^{-\lambda x}\right), x\right)\right) - e^{-\rho x} \left(\phi\left(\theta e^{-\lambda x}\right) - q\right)$$

$$= A\left(\phi\left(\theta\right)\right) - e^{-\rho x} \left(\phi\left(\theta e^{-\lambda x}\right) - q\right),$$

where the fourth equality follows from (3.2). Letting $x \to \infty$, the desired result follows immediately.

Lemma 3.3. It holds that $\lambda \geq \rho$. Moreover, $\lambda = \rho$ if and only if $p_k = 0$ for all $k \geq 3$.

Proof: Note that $\lambda - \rho = \beta(f'(1) + f'(q) - 2)$.

We first consider the case when $p_0 = 0$. Then q = 0 and $f'(q) = f'(0) = p_1$. Hence we have

$$f'(1) + f'(0) - 2 = \left(\sum_{k=1}^{\infty} kp_k + p_1 - 2\sum_{k=1}^{\infty} p_k\right) = \sum_{k=2}^{\infty} (k-2)p_k \ge 0.$$

Furthermore, the left hand side is equal to 0 if and only if $p_k = f^{(k)}(0) = 0$, $k \ge 3$. If $p_0 > 0$, then q > 0. Define

$$\hat{f}(s) := \frac{f((1-q)s+q)-q}{1-q}, \text{ for } 0 \le s \le 1.$$

It is well known (see [5, Chapter 1, Section 12]) that \hat{f} is a probability generating function with $\hat{f}(0) = 0$. Note that $\hat{f}'(1) = f'(1)$ and $\hat{f}'(0) = f'(q)$. Applying the previous paragraph to \hat{f} , we get that

$$f'(1) + f'(q) - 2 = \hat{f}'(1) + \hat{f}'(0) - 2 \ge 0.$$

Moreover, the left hand side is equal to 0 if and only if $\hat{f}^{(k)}(0) = (1-q)^{k-1} f^{(k)}(q) = 0, k \ge 3$. It is easy to see that $f^{(3)}(q) = \sum_{k=3}^{\infty} k(k-1)(k-2)p_k q^{k-3} = 0$ if and only if $p_k = 0, k \ge 3$. The proof is now complete.

Note that the skeleton $\{Z_n, n=0,1,\cdots\}$ is a Galton-Watson process with offspring generating function $\tilde{f}(s):=F(s,1)$. It is clear that $\tilde{f}'(1)=\mathbb{E}(Z_1)=e^{\lambda}>1$ and $\mathbb{P}(Z_1=1)\geq \mathbb{P}(\tau_o>1)>0$. By [25, Theorem 4], we have that, for any nonnegative sequence a_n with $a_n\to\infty$ and $a_n=o(e^{\lambda n})$, there exists a constant c>0 such that

$$\mathbb{P}(0 < Z_n < a_n) \le c \mathbb{P}(0 < W < e^{-\lambda n} a_n), \qquad n \ge 1. \tag{3.5}$$

Furthermore, it was proven in [19] that there exist $c_1, c_2 > 0$ such that for any $x \in (0, 1)$,

$$c_1 x^{\rho/\lambda} < \mathbb{P}(0 < W < x) < c_2 x^{\rho/\lambda}. \tag{3.6}$$

By (3.5) and (3.6), there exists C > 0 such that

$$\mathbb{P}(0 < Z_n < a_n) \le C a_n^{\rho/\lambda} e^{-\rho n}, \quad n \ge 1. \tag{3.7}$$

3.1 Lower deviation of X_t

We have proved in [33, Proposition 2.1] that $\mathbb{Y}_t/h(t)$ converges weakly to \mathcal{N}_{∞} , that is for any $\varphi = e^{-g}$ with $g \in C_c^+(\overline{\mathbb{R}}_0)$,

$$\lim_{t \to \infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/h(t))) = \mathbb{E}((\mathcal{I}(\varphi, \mathcal{N}_{\infty}))) = \mathbb{E}(\exp\{-C(\varphi)W\}). \tag{3.8}$$

In fact, by examining the proofs in [33], (3.8) is valid for any $\varphi \in \mathcal{H}(\mathbb{R})$. By Lemma 2.4 we have for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t\to\infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{X}_t/h(t))) = \mathbb{E}((\mathcal{I}(\varphi, \mathcal{N}_\infty))) = \mathbb{E}(\exp\{-C(\varphi)W\}).$$

For any function g and $x \in \mathbb{R}$, we define the function $\mathfrak{m}_x g(\cdot)$ by the relation $\mathfrak{m}_x g(y) = g(y/x)$. By the definition of $C(\varphi)$ in (1.7), it is easy to see that

$$C(\mathfrak{m}_x\varphi) = x^{-\alpha}C(\varphi), \quad \varphi \in \mathcal{B}_1(\mathbb{R}).$$

Lemma 3.4. If a(t), b(t) are positive functions with $\frac{a(t)}{h(t)} \to a > 0$ and $\frac{b(t)}{a(t)} \to 0$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t\to\infty} \mathbb{E}_{b(t)}(\mathcal{I}(\varphi, \mathbb{X}_t/a(t))) = \mathbb{E}(\exp\{-a^{-\alpha}C(\varphi)W\}).$$

Proof: Note that

$$\mathbb{E}_{b(t)}(\mathcal{I}(\varphi, \mathbb{X}_t/a(t))) = \mathbb{E}\left(\prod_{u \in \mathcal{L}_t} \varphi\left(\frac{\xi_t^u + b(t)}{a(t)}\right)\right).$$

Since $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. Moreover for any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $|x - y| \le \eta$. It follows from the assumption that for any $\epsilon' > 0$, there exists t' such that for all t > t', $(1 - \epsilon')a(t) \le ah(t) \le (1 + \epsilon')a(t)$ and $|b(t)| \le \epsilon' a(t)$.

We now fix an arbitrary $\epsilon > 0$ and $0 < \epsilon' < \frac{\delta}{\delta + 2} \wedge \eta$. Note that if $\frac{|y|}{ah(t)} \le \delta/2$, then for t > t',

$$\left|\frac{y+b(t)}{a(t)}\right| \le \frac{(1+\epsilon')|y|}{ah(t)} + \epsilon' \le \left(1 + \frac{\delta}{\delta+2}\right) \frac{\delta}{2} + \frac{\delta}{\delta+2} = \delta,$$

and that if $\frac{|y|}{ah(t)} \leq \frac{\eta}{\epsilon'} - 1$, then for t > t',

$$\left| \frac{y + b(t)}{a(t)} - \frac{y}{ah(t)} \right| \le \epsilon' + \frac{|y|}{ah(t)} \epsilon' \le \eta.$$

Thus for any t > t',

$$\left|\varphi\Big(\frac{y+b(t)}{a(t)}\Big)-\varphi\Big(\frac{|y|}{ah(t)}\Big)\right| \leq \epsilon \mathbf{1}_{\{\frac{\delta}{2}<\frac{|y|}{ah(t)}\leq\frac{\eta}{\epsilon'}-1\}} + \mathbf{1}_{\{\frac{|y|}{ah(t)}>\frac{\eta}{\epsilon'}-1\}}.$$

Hence we have that

$$\left| \mathbb{E} \left(\prod_{u \in \mathcal{L}_{t}} \varphi \left(\frac{\xi_{t}^{u} + b(t)}{a(t)} \right) \right) - \mathbb{E} \left(\prod_{u \in \mathcal{L}_{t}} \varphi \left(\frac{\xi_{t}^{u}}{ah(t)} \right) \right) \right|$$

$$\leq \mathbb{E} \left[1 \wedge \sum_{u \in \mathcal{L}_{t}} \left| \varphi \left(\frac{\xi_{t}^{u} + b(t)}{a(t)} \right) - \varphi \left(\frac{\xi_{t}^{u}}{ah(t)} \right) \right| \right]$$

$$\leq \mathbb{E} \left[1 \wedge \left(\epsilon \sum_{u \in \mathcal{L}_{t}} \mathbf{1}_{\left\{ \frac{|\xi_{t}^{u}|}{h(t)} > a\delta/2 \right\}} \right) \right] + \mathbb{E} \left[1 \wedge \sum_{u \in \mathcal{L}_{t}} \mathbf{1}_{\left\{ \frac{|\xi_{t}^{u}|}{h(t)} > a(\eta/\epsilon' - 1) \right\}} \right]. \tag{3.9}$$

Since $\mathbb{X}_t/h(t)$ converges weakly to \mathcal{N}_{∞} , we have, as $t \to \infty$,

$$\epsilon \sum_{u \in \mathcal{L}_t} \mathbf{1}_{\{\frac{|\xi_t^u|}{h(t)} > a\delta/2\}} \xrightarrow{d} \epsilon \cdot \mathcal{N}_{\infty}(\{y \in \mathbb{R}; |y| > a\delta/2\})$$

and

$$\sum_{u \in \mathcal{L}_t} \mathbf{1}_{\left\{\frac{|\xi_t^u|}{h(t)} > a(\eta/\epsilon - 1)\right\}} \xrightarrow{d} \mathcal{N}_{\infty}(\{y \in \mathbb{R}; |y| > a(\eta/\epsilon' - 1)\}.$$

Thus letting $t \to \infty$ first and then $\epsilon' \to 0$, and finally $\epsilon \to 0$ in (3.9), applying the dominated convergence theorem, we get

$$\lim_{t \to \infty} \mathbb{E} \left(\prod_{u \in \mathcal{L}_t} \varphi \left(\frac{\xi_t^u + b(t)}{a(t)} \right) \right) = \lim_{t \to \infty} \mathbb{E} \left(\prod_{u \in \mathcal{L}_t} \varphi \left(\frac{\xi_t^u}{ah(t)} \right) \right)$$
$$= \mathbb{E} \left(\exp \left\{ -C(\mathfrak{m}_a \varphi) W \right\} \right) = \mathbb{E} \left(\exp \left\{ -a^{-\alpha} C(\varphi) W \right\} \right).$$

Lemma 3.5. If $a:[0,\infty)\to(0,\infty)$ is a non-decreasing positive function with

$$\sum_{n=1}^{\infty} na(n)^{-\alpha} L(a(n)) < \infty,$$

then

$$\lim_{t \to \infty} \frac{\xi_t}{a(t)} = 0.$$

Proof: For any positive integer n,

$$\sup_{t \in [n, n+1]} |\xi_t| \le |\xi_n| + \sup_{t \in [n, n+1]} |\xi_t - \xi_n|.$$

Since ξ is a Lévy process, $\{Y_n := \sup_{t \in [n,n+1]} |\xi_t - \xi_n|, n \ge 1\}$ are i.i.d. By (2.4), and (2.5), we have that as $x \to \infty$,

$$\mathbf{P}(Y_1 > x) \le \mathbf{P}\left(\sup_{t \in [0,1]} \xi_t > x\right) + \mathbf{P}\left(\sup_{t \in [0,1]} (-\xi_t) > x\right) \sim \mathbf{P}(|\xi_1| > x).$$

By (2.2), we have that for any c > 0 and n large enough,

$$\mathbf{P}\left(\sup_{t\in[n,n+1]}|\xi_t| > ca(n)\right) \leq \mathbf{P}(|\xi_n| > ca(n)/2) + \mathbf{P}(Y_n > ca(n)/2)$$
$$\leq c_0(c/2)^{-\alpha}(n+1)a(n)^{-\alpha}L(ca(n)/2)$$
$$\sim c_0(c/2)^{-\alpha}na(n)^{-\alpha}L(a(n)), \quad n \to \infty,$$

which implies that

$$\sum_{n} \mathbf{P} \left(\sup_{t \in [n, n+1]} |\xi_t| > ca(n) \right) < \infty.$$

Thus by the Borel-Cantelli lemma, we have

$$\lim_{n \to \infty} \frac{\sup_{t \in [n, n+1]} |\xi_t|}{a(n)} = 0, \quad a.s.$$

Since a(t) is non-decreasing, we have, for $t \in [n, n+1]$,

$$\frac{|\xi_t|}{a(t)} \le \frac{\sup_{t \in [n,n+1]} |\xi_t|}{a(n)} \to 0.$$

The proof is now complete.

For any $\varphi \in \mathcal{B}_1(\mathbb{R})$, we define

$$U_{\varphi}(t,x) := \mathbb{E}_{x}(\mathcal{I}(\varphi, \mathbb{X}_{t})) = \mathbb{E}_{x}\left(\prod_{u \in \mathcal{L}_{t}} \varphi(\xi_{t}^{u})\right), \quad t \geq 0, x \in \mathbb{R}.$$
 (3.10)

By the Markov property and the branching property, we have

$$U_{\varphi}(t,x) = e^{-\beta t} \mathbf{E}_{x}(\varphi(\xi_{t})) + \mathbf{E}_{x} \int_{0}^{t} \beta e^{-\beta s} f(U_{\varphi}(t-s,\xi_{s})) ds,$$

which implies that

$$U_{\varphi}(t,x) = e^{\beta(f'(q)-1)t} \mathbf{E}_{x}(\varphi(\xi_{t})) + \mathbf{E}_{x} \int_{0}^{t} \beta e^{\beta(f'(q)-1)s} f(U_{\varphi}(t-s,\xi_{s})) ds$$
$$-\mathbf{E}_{x} \int_{0}^{t} e^{\beta(f'(q)-1)s} \beta f'(q) U_{\varphi}(t-s,\xi_{s}) ds$$
$$= e^{-\rho t} \mathbf{E}_{x}(\varphi(\xi_{t})) + \mathbf{E}_{x} \int_{0}^{t} \beta e^{-\rho s} V(U_{\varphi}(t-s,\xi_{s})) ds + q(1-e^{-\rho t}),$$

where ρ is defined in (1.10) and V is defined in (3.3). Thus

$$U_{\varphi}(t,x) - q = e^{-\rho t} (\mathbf{E}(\varphi(\xi_t + x)) - q) + \mathbf{E} \int_0^t \beta e^{-\rho s} V(U_{\varphi}(t - s, x + \xi_s)) ds. \quad (3.11)$$

For any t > 0 and $x \in \mathbb{R}$, we define

$$u(t,x) := U_{\mathbf{1}_{(-\infty,0]}}(t,x) = \mathbb{P}_x(R_t \le 0) = \mathbb{P}(R_t \le -x).$$
 (3.12)

Lemma 3.6. Let c > 0. For any $\epsilon \in (0,1)$, there exist $C = C(\epsilon) > 0$ and $t_{\epsilon} > 0$ such that for any $r > t_{\epsilon}$, $s > t_{\epsilon}$, l > 0,

$$\mathbf{E}[(u(r+s,-c\,h(r)+\xi_l)-q)^2] \le c(\epsilon)s^{6\rho/\lambda} \Big[e^{-2\rho(1-\epsilon)s} + (l+s)e^{-\lambda r}e^{-(\lambda\wedge(2\rho))(1-\epsilon)s}\Big].$$

Proof: Define $\tilde{u}(t,x) := \mathbb{P}(-\infty < R_t \le -x) = u(t,x) - \mathbb{P}(Z_t = 0)$. Then for any s > 0,

$$\mathbf{E}[(u(r+s, -ch(r) + \xi_l) - q)^2] \le 2\mathbf{E}[(\tilde{u}(r+s, -ch(r) + \xi_l))^2] + 2(q - \mathbb{P}(Z_{r+s} = 0))^2$$

$$\le 2\mathbf{E}[(\tilde{u}(r+s, -ch(r) + \xi_l)^2] + 2(q - \mathbb{P}(Z_s = 0))^2.$$
(3.13)

By (3.1) with s = 0, we have

$$\lim_{s \to \infty} e^{\rho s} (q - \mathbb{P}(Z_s = 0)) = -A(0) \in [0, \infty). \tag{3.14}$$

For t > 0, define

$$T_t := \inf\{s > 0 : Z_s > t^3\}.$$

Here we use the convention that $\inf \emptyset = \infty$. Let K > 3 be an integer. For any r > 0, s > 1 and $x \in \mathbb{R}$, we have

$$\tilde{u}(s+r, -ch(r)+x) = \mathbb{P}(-\infty < R_{s+r} \le ch(r) - x)$$

$$\le \mathbb{P}\left(T_s > \frac{K-1}{K}s, Z_{(K-1)s/K} > 0\right)$$

$$+\sum_{k=1}^{K-1} \mathbb{P}\left(\frac{(k-1)s}{K} < T_s \le \frac{ks}{K}, R_{s+r} \le c h(r) - x\right). \tag{3.15}$$

By (3.7) with $a_n = (2K)^3 n^3$, there exists a constant $C_K > 0$ such that

$$\mathbb{P}(0 < Z_n \le (2K)^3 n^3) \le C_K (2Kn)^{3\rho/\lambda} e^{-\rho n}, \quad n \ge 1.$$

It follows that, for s > K and $k = 1, 2, \dots, K - 1$,

$$\mathbb{P}\left(T_s > \frac{k}{K}s, Z_{ks/K} > 0\right) \leq \mathbb{P}\left(0 < Z_{\lfloor \frac{k}{K}s \rfloor} \leq s^3\right) \leq \mathbb{P}\left(0 < Z_{\lfloor \frac{k}{K}s \rfloor} \leq (2K)^3 \lfloor \frac{k}{K}s \rfloor^3\right) \\
\leq C_K (2Ks)^{3\rho/\lambda} e^{-\rho \lfloor \frac{k}{K}s \rfloor} \leq C_K (2K)^{3\rho/\lambda} e^{\rho} \cdot s^{3\rho/\lambda} e^{-\rho \frac{k}{K}s}. \quad (3.16)$$

Choose b > 1 such that $e^{-\lambda b/\alpha} < 1/c$. Note that $\lim_{t\to\infty} \frac{h(t)}{h(t+b)} = e^{-\lambda b/\alpha} < 1/c$. By (1.9), we have $\lim_{t\to\infty} \mathbb{P}(R_t \le h(t)) = \mathbb{E}(e^{-\vartheta^*W}) < 1$. Thus there exist $c_1 \in (0,1)$, $c_2 \in (0,1/c)$ and $t_1 > 0$ such that for all $t > t_1$, and t' > b,

$$\mathbb{P}(R_t \le h(t)) \le c_1$$
, and $h(t) \le c_2 h(t+b) \le c_2 h(t+t')$.

Note that

$$R_{s+r} = \max_{u \in \mathcal{L}_{ks/K}} (\xi_{ks/K}^u + R_{s+r}^u),$$

where, given $\mathcal{F}_{sk/K}^{\mathbb{T}}$, $\{R_{s+r}^u, u \in \mathcal{L}_{ks/K}\}$ are i.i.d. with the same law as $R_{s+r-ks/K}$. Moreover, given $\mathcal{F}_{ks/K}^{\mathbb{T}}$, $\{\xi_{ks/K}^u, u \in \mathcal{L}_{ks/K}\}$ have the same law as $\xi_{ks/K}$, and $\{\xi_{ks/K}^u, u \in \mathcal{L}_{ks/K}\}$ and $\{R_{s+r}^u, u \in \mathcal{L}_{ks/K}\}$ are independent. Applying [27, Lemma 5.1] (in the first inequality below) we get that, for $k = 1, 2, \dots, K-1, r > t_1$ and s > bK,

$$\mathbb{P}(R_{s+r} \leq c \, h(r) - x | \mathcal{F}_{ks/K}^{\mathbb{T}}) \leq \mathbb{P}(\xi_{ks/K} + \max_{u \in \mathcal{L}_{ks/K}} R_{s+r}^{u} \leq c \, h(r) - x | \mathcal{F}_{ks/K}^{\mathbb{T}}) \\
\leq \mathbf{P}(x + \xi_{ks/K} \leq c \, h(r) - h(s + r - ks/K)) + [\mathbb{P}(R_{s+r-ks/K} \leq h(s + r - ks/K))]^{Z_{ks/K}} \\
\leq \mathbf{P}(x + \xi_{ks/K} \leq -(1 - cc_2)h(s + r - ks/K)) + c_1^{Z_{ks/K}}.$$

Thus we have that for $k = 1, 2, \dots, K - 1, r > t_1$ and s > bK,

$$\mathbb{P}\left(\frac{(k-1)s}{K} < T_s \le \frac{ks}{K}, R_{s+r} \le c h(r) - x\right)$$

$$\le \mathbf{P}(x + \xi_{ks/K} \le -(1 - cc_2)h(s + r - ks/K))\mathbb{P}\left(T_s > \frac{(k-1)s}{K}, Z_{(k-1)s/K} > 0\right)$$

$$+ \mathbb{E}\left(c_1^{Z_{ks/K}}; \frac{(k-1)s}{K} < T_s \le \frac{ks}{K}\right).$$
(3.17)

Note that

$$\mathbb{E}\left(c_{1}^{Z_{ks/K}}; \frac{(k-1)s}{K} < T_{s} \leq \frac{ks}{K}\right) \\
\leq c_{1}^{s^{2}} + \mathbb{P}\left(\frac{(k-1)s}{K} < T_{s} \leq \frac{ks}{K}, Z_{ks/K} < s^{2}\right) \\
\leq c_{1}^{s^{2}} + \mathbb{P}(\exists t \geq 0 : Z_{t} \leq \lfloor s^{2} \rfloor | Z_{0} = \lfloor s^{3} \rfloor) \\
\leq c_{1}^{s^{2}} + \left(\frac{\lfloor s^{3} \rfloor}{\lfloor s^{2} \rfloor}\right) q^{\lfloor s^{3} \rfloor - \lfloor s^{2} \rfloor} \leq c_{1}^{s^{2}} + (s^{3} + 1)^{s^{2} + 1} q^{s^{3} - s^{2} - 2} = o(e^{-\rho s}). \tag{3.18}$$

Combining (3.15), (3.16), (3.17) and (3.18), we get that there exist C'_k and $t_2 > 0$ such that for any $r > t_2$, $s > t_2$ and $x \in \mathbb{R}$,

$$\left[\mathbb{P}(-\infty < R_{s+r} \le h(r) - x)\right]^{2} \le C_{K}' \left[s^{6\rho/\lambda} e^{-2\rho \frac{K-1}{K}s} + \sum_{k=1}^{K-1} s^{6\rho/\lambda} e^{-2\rho(k-1)s/K} \mathbf{P}(x + \xi_{ks/K} \le -(1 - cc_{2})h(s + r - ks/K))\right].$$
(3.19)

By the Markov property of ξ and (2.3), we get that there exists $t_3 > t_2$ such that for $r > t_3$,

$$\mathbf{E}[\mathbf{P}(x + \xi_{ks/K} \le -(1 - cc_2)h(s + r - ks/K))|_{x=\xi_l}]$$

$$= \mathbf{P}(\xi_{l+ks/K} \le -(1 - cc_2)h(s + r - ks/K))$$

$$\le 2c_0(1 - cc_2)^{-\alpha - 1}(l + s)e^{-\lambda(r + s - ks/K)}.$$

It follows from (3.19) that for any $s > t_3$, $r > t_3$ and l > 0,

$$\begin{split} &\mathbf{E}[\tilde{u}(s+r,-h(r)+\xi_{l})]^{2} \\ \leq &C_{K}'' \left[s^{6\rho/\lambda} e^{-2\rho \frac{K-1}{K}s} + s^{6\rho/\lambda} (l+s) \sum_{k=1}^{K-1} e^{-2\rho(k-1)s/K} e^{-\lambda(r+s-ks/K)} \right] \\ \leq &C_{K}'' \left[s^{6\rho/\lambda} e^{-2\rho \frac{K-1}{K}s} + K s^{6\rho/\lambda} (l+s) e^{-\lambda r} e^{-(\lambda \wedge (2\rho)) \frac{K-1}{K}s} \right], \end{split}$$

where $C_K'' = C_k'(1 + 2c_0(1 - cc_2)^{-\alpha - 1})$ and in the last inequality we used

$$2\rho(k-1) + \lambda(K-k) \ge (\lambda \wedge 2\rho)(K-1), \quad k = 1, \dots, K.$$

For any $\epsilon > 0$, we choose K such that $1/K < \epsilon$, we get that there exists $c(\epsilon) > 0$ such that

$$2\mathbf{E}[(\tilde{u}(r+s,-c\,h(r)+\xi_l)^2] \le c(\epsilon)s^{6\rho/\lambda} \Big[e^{-2\rho(1-\epsilon)s} + (l+s)e^{-\lambda r}e^{-(\lambda\wedge(2\rho))(1-\epsilon)s}\Big].$$

Combining this with (3.13) and (3.14), we immediately get the desired result. The proof is now complete.

Proof of Theorem 1.3: Since $\varphi \in \mathcal{H}_0(\mathbb{R})$, there exists c > 0 such that $\varphi(y) = 0$ for all y > c. For any t > 0, define r = r(t) by $h(r(t)) = \Lambda(t)$, that is,

$$e^{-\lambda r(t)} = \Lambda(t)^{-\alpha} L(\Lambda(t)).$$

Since $\lim_{t\to\infty} \frac{\Lambda(t)}{h(t)} = 0$ and $\Lambda(t) \to \infty$, we have $t - r(t) \to \infty$ and $r(t) \to \infty$ as $t \to \infty$. Thus for any T > 0, there exists $t_0 > 0$ such that t - r(t) > T and r(t) > T for all $t \ge t_0$. In the remainder of this proof, we assume $t \ge t_0$. Note that

$$(1-q)\mathbb{E}^* \left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u/\Lambda(t)) \right)$$

$$= \mathbb{E} \left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u/\Lambda(t)) \right) - q + q - \mathbb{E} \left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u/\Lambda(t)); \mathcal{S}^c \right). \tag{3.20}$$

By (3.1) with s = 0, we have

$$0 \le q - \mathbb{E}\left(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u/\Lambda(t)); \mathcal{S}^c\right) \le q - \mathbb{P}(Z_t = 0) \sim -A(0)e^{-\rho t}. \tag{3.21}$$

By (3.11), we have that

$$\mathbb{E}\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t))\right)-q=e^{-\rho t}(\mathbf{E}(\varphi(\xi_t/\Lambda(t)))-q)+J(t),$$

where

$$J(t) = \left(\int_0^{t-r(t)-T} + \int_{t-r(t)-T}^{t-r(t)+T} + \int_{t-r(t)+T}^t \right) \beta e^{-\rho s} \mathbf{E}[V(U_{\varphi_t}(t-s,\xi_s))] ds$$

=: $J_1(t,T) + J_2(t,T) + J_3(t,T),$

 $\varphi_t(y) = \varphi(y/\Lambda(t))$ and U_{φ} is defined in (3.10). It is easy to see that

$$e^{-\rho t}(\mathbf{E}(\varphi(\xi_t/\Lambda(t))) - q) = o(e^{-\rho(t-r(t))}).$$

We now deal with the three components of J(t) separately. We will show that

$$\lim_{T \to \infty} \limsup_{t \to \infty} e^{\rho(t-r(t))} J_1(t,T) = \lim_{T \to \infty} \limsup_{t \to \infty} e^{\rho(t-r(t))} J_3(t,T) = 0.$$

(1) Since $0 \le V(s) \le f(s) \le 1$, we have

$$e^{\rho(t-r(t))}J_3(t,T) \le e^{\rho(t-r(t))} \int_{t-r(t)+T}^{\infty} \beta e^{-\rho s} ds = \frac{1}{1-f'(q)} e^{-\rho T}.$$
 (3.22)

Hence $\lim_{T\to\infty} \limsup_{t\to\infty} e^{\rho(t-r(t))} J_3(t,T) = 0.$

(2) Note that

$$e^{\rho(t-r(t))}J_2(t,T) = \int_{-T}^{T} \beta e^{\rho s} \mathbf{E} V(U_{\varphi_t}(s+r(t)), \xi_{t-r(t)-s}) ds.$$
 (3.23)

Since $\Lambda(t) = h(r(t))$, we have

$$U_{\varphi_t}(s+r(t),\xi_{t-r(t)-s}) = \mathbb{E}_{\xi_{t-r(t)-s}} \left(\prod_{u \in \mathcal{L}_{s+r(t)}} \varphi(\xi_{s+r(t)}^u/h(r(t))) \right).$$

By Lemma 3.5 and the fact that Λ is non-decreasing, we have that for any $s \in [-T, T]$,

$$\lim_{t \to \infty} \frac{|\xi_{t-r(t)-s}|}{h(s+r(t))} = \lim_{t \to \infty} \frac{|\xi_{t-r(t)-s}|}{e^{\lambda s/\alpha} \Lambda(t)} \le \lim_{t \to \infty} e^{-\frac{\lambda}{\alpha}s} \frac{|\xi_{t-r(t)-s}|}{\Lambda(t-r(t)-s)} = 0.$$

Note that $h(r(t))/h(s+r(t)) \to e^{-\frac{\lambda}{\alpha}s}$. By Lemma 3.4, we have that for $s \in [-T, T]$,

$$\lim_{t \to \infty} U_{\varphi_t}(s + r(t), \xi_{t-r(t)-s}) = \mathbb{E}(\exp\{-e^{\lambda s}C(\varphi)W\}) = \phi(e^{\lambda s}C(\varphi)).$$

Since V(s) is bounded and continuous, by (3.23) and the dominated convergence theorem we have

$$\lim_{t \to \infty} e^{\rho(t-r(t))} J_2(t,T) = \int_{-T}^{T} \beta e^{\rho s} V\left[\phi(e^{\lambda s} C(\varphi))\right] ds.$$
 (3.24)

(3) We now deal with $J_1(t,T)$. Recall u is defined in (3.12). Note that

$$e^{\rho(t-r(t))}J_{1}(t,T) = \int_{T}^{t-r(t)} \beta e^{\rho s} \mathbf{E}V(U_{\varphi_{t}}(s+r(t),\xi_{t-r(t)-s})) ds$$

$$\leq \int_{T}^{t-r(t)} \beta e^{\rho s} \mathbf{E}[V(u(s+r(t),-ch(r(t))+\xi_{t-r(t)-s}))] ds, \qquad (3.25)$$

where in the last inequality we used the fact that

$$U_{\varphi_t}(s+r(t),x) \le \mathbb{P}_x(R_{s+r(t)} \le c \, h(r(t))) = u(s+r(t),-c \, h(r(t))+x).$$

By Taylor's formula and the fact that f'(s) is increasing, we have

$$0 \le V(s) = f(s) - f(q) - f'(q)(s - q) \le |f'(s) - f'(q)||s - q| \le f'(1)|s - q|. \quad (3.26)$$

(a) We first consider the case $\lambda > \rho$. For any $\epsilon \in (0, (\lambda - \rho) \land (\rho/2) \land (\lambda/2\rho))$, let t_{ϵ} be the constant in Lemma 3.6. Let $T > t_{\epsilon}$ be large enough so that $h(T) > x_0$. By Hölder's inequality and (3.26), we have for $t \geq t_0$ and $s \in (T, t - r(t))$,

$$\begin{split} I_{1}(t,s) &:= \mathbf{E}V(u(s+r(t),-c\,h(r(t))+\xi_{t-r(t)-s});|\xi_{t-r(t)-s}|>h(s+r(t)))\\ &\leq f'(1)\mathbf{E}(|u(s+r(t),-ch(r(t))+\xi_{t-r(t)-s})-q|;|\xi_{t-r(t)-s}|>h(s+r(t)))\\ &\leq f'(1)\sqrt{\mathbf{E}(|u(s+r(t),-c\,h(r(t))+\xi_{t-r(t)-s})-q|^{2})\mathbf{P}(|\xi_{t-r(t)-s}|>h(s+r(t)))}\\ &\leq f'(1)\sqrt{c_{0}c(\epsilon)s^{6\rho/\lambda}[e^{-2\rho(1-\epsilon)s}+te^{-\lambda r(t)}e^{-(2\rho\wedge\lambda)(1-\epsilon)s}]te^{-\lambda(s+r(t))}}\\ &\leq f'(1)\sqrt{c_{0}c(\epsilon)te^{-\lambda r(t)}}s^{3\rho/\lambda}\left[\sqrt{1+te^{-\lambda r(t)}}e^{-\rho(1-\epsilon)s}e^{-\lambda s/2}+\sqrt{te^{-\lambda r(t)}}e^{-\lambda(1-\epsilon/2)s}\right], \end{split}$$

where in the third inequality we used Lemma 3.6 and (2.2), and in the fourth inequality we used the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $a,b \geq 0$. Since $\sum_{n} n\Lambda(n)^{-\alpha} L(\Lambda(n)) = \sum_{n} ne^{-\lambda r(n)} < \infty$ and Λ is non-decreasing, we know that $te^{-\lambda r(t)} \to 0$ as $t \to \infty$. It follows that

$$\int_{T}^{t-r(t)} \beta e^{\rho s} I_{1}(t,s) ds \leq f'(1) \sqrt{c_{0} c(\epsilon) t e^{-\lambda r(t)}} \\
\times \left[\sqrt{1 + t e^{-\lambda r(t)}} \int_{T}^{\infty} \beta s^{3\rho/\lambda} e^{-(\lambda/2 - \rho \epsilon)s} ds + \sqrt{t e^{-\lambda r(t)}} \int_{T}^{\infty} \beta s^{3\rho/\lambda} e^{-(\lambda - \rho - \epsilon/2)s} ds \right] \\
\to 0, \quad t \to \infty. \tag{3.27}$$

By (1.9), we have $\mathbb{P}(R_t \leq 2h(t)) \to \mathbb{E}(e^{-\vartheta^* 2^{-\alpha}W}) \in (q,1)$. Thus there exists $a \in (q,1)$ such that when T is large enough,

$$\mathbb{P}(R_t \le 2h(t)) \le a, \quad t \ge T.$$

If $|\xi_{t-r(t)-s}| \leq h(s+r(t))$, then for any $t > t_0$ and $s \in (T, t-r(t))$,

$$u(r(t) + s, -h(r(t)) + \xi_{t-r(t)-s}) \le \mathbb{P}(R_{s+r(t)} \le h(r(t)) + h(s+r(t)))$$

 $\le \mathbb{P}(R_{s+r(t)} \le 2h(s+r(t))) \le a.$

Thus by (3.3), we have

$$V(u(r(t)+s,-h(r(t))+\xi_{t-r(t)-s})) \le f''(a)(u(r(t)+s,-h(r(t))+\xi_{t-r(t)-s})-q)^2.$$

By Lemma 3.6, we have for any $t > t_0$ and $s \in (T, t - r(t))$.

$$I_{2}(t,s) := \mathbf{E}(V(u(s+r(t),-ch(r(t))+\xi_{t-r(t)-s})); |\xi_{t-r(t)-s}| \leq h(s+r(t)))$$

$$\leq f''(a)\mathbf{E}(u(s+r(t),-ch(r(t))+\xi_{t-r(t)-s})-q)^{2}$$

$$\leq f''(a)c(\epsilon)s^{6\rho/\lambda}[e^{-2\rho(1-\epsilon)s}+te^{-\lambda r(t)}e^{-(2\rho\wedge\lambda)(1-\epsilon)s}]$$

$$\leq f''(a)c(\epsilon)s^{6\rho/\lambda}[(1+te^{-\lambda r(t)})e^{-2\rho(1-\epsilon)s}+te^{-\lambda r(t)}e^{-\lambda(1-\epsilon)s}].$$

It follows that

$$\int_{T}^{t-r(t)} \beta e^{\rho s} I_{2}(t,s) ds$$

$$\leq f''(a) c(\epsilon) \left[(1 + t e^{-\lambda r(t)}) \int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\rho - 2\epsilon)s} ds + t e^{-\lambda r(t)} \int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\lambda - \rho - \epsilon)s} ds \right]$$

$$\to f''(a) c(\epsilon) \int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\rho - 2\epsilon)s} ds, \quad t \to \infty. \tag{3.28}$$

By (3.27) and (3.28), we have

$$\limsup_{t \to \infty} e^{\rho(t-r(t))} J_1(t,T) \le f''(a)c(\epsilon) \int_T^\infty s^{6\rho/\lambda} e^{-(\rho-2\epsilon)s} ds.$$
 (3.29)

(b) We now assume $\lambda = \rho$. Since $\Lambda(t) \stackrel{t}{>} e^{\gamma t}$, we have

$$e^{-\lambda r(t)} = \Lambda(t)^{-\alpha} L(\Lambda(t)) \stackrel{t}{\leq} \Lambda(t)^{-\alpha/2} \leq e^{-\alpha \gamma t/2},$$

which implies that $r(t) \stackrel{t}{\geq} \frac{\alpha \gamma}{2\lambda} t$. Let $\epsilon \in (0, \frac{\alpha \gamma}{2\lambda} \wedge \frac{\rho}{2})$, and let $T > t_{\epsilon}$. By Lemma 3.3, $p_k = 0$ for $k \geq 3$. Thus $f''(1) < \infty$. By (3.3), we have, for any $t \geq t_0$ and $s \in (T, t - r(t)),$

$$V(u(r(t)+s,-h(r(t))+\xi_{t-r(t)-s})) \le f''(1)(u(r(t)+s,-h(r(t))+\xi_{t-r(t)-s})-q)^2.$$

Applying Lemma 3.6, we have for any $t \ge t_0$ and $s \in (T, t - r(t))$,

$$\mathbf{E}(V(u(s+r(t), -c\,h(r) + \xi_{t-r(t)-s}))) \le f''(1)c(\epsilon)s^{6\rho/\lambda}[e^{-2\rho(1-\epsilon)s} + te^{-\lambda r(t)}e^{-\rho(1-\epsilon)s}].$$

By (3.25), it follows that

$$e^{\rho(t-r(t))}J_{1}(t,T) \leq f''(1)c(\epsilon) \left[\int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\rho-2\epsilon)s} \, ds + t e^{-\lambda r(t)} \int_{T}^{t-r(t)} s^{6\rho/\lambda} e^{\rho\epsilon s} \, ds \right]$$

$$\leq f''(1)c(\epsilon) \left[\int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\rho-2\epsilon)s} \, ds + (\rho\epsilon)^{-1} t^{1+6\rho/\lambda} e^{-\alpha\gamma t/2} e^{\lambda\epsilon t} \right]$$

$$\to f''(1)c(\epsilon) \int_{T}^{\infty} s^{6\rho/\lambda} e^{-(\rho-2\epsilon)s} \, ds, \quad t \to \infty.$$

$$(3.30)$$

Combining (3.22),(3.24),(3.29) and (3.30), and letting $T \to \infty$, we obtain that

$$\lim_{t \to \infty} e^{\rho(t-r(t))} J(t) = \int_{-\infty}^{\infty} \beta e^{\rho s} V\left[\phi(C(\varphi)e^{\lambda s})\right] ds = A(\phi(C(\varphi))). \tag{3.31}$$

Thus by (3.20), (3.21) and (3.31), we have that

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{E}^* \Big(\prod_{u \in \mathcal{L}_t} \varphi(\xi_t^u / \Lambda(t)) \Big) = \frac{1}{1-q} A \left[\phi(C(\varphi)) \right].$$

Since $C(\varphi)$ is continuous in φ , the above limit also valid for $\varphi(y) = \mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{E}^* \Big(R_t \le \Lambda(t) \Big) = \frac{1}{1-q} A(\phi(q_1 \vartheta/\alpha)).$$

The proof is now complete.

Proof of Corollary 1.4: For any $g \in C_c^+(\overline{\mathbb{R}}_0)$ and $\theta > 0$, applying Theorem 1.3 with $\varphi(y) = e^{-\theta g(y)} \mathbf{1}_{(-\infty,1]}(y)$ we get

$$\lim_{t\to\infty} \mathbb{E}^* \left(e^{-\theta \frac{\mathbb{X}_t}{\Lambda(t)}(g)} | R_t \le \Lambda(t) \right) = \lim_{t\to\infty} \frac{e^{\rho(t-r(t))} \mathbb{E}^* \left(e^{-\theta \frac{\mathbb{X}_t}{\Lambda(t)}(g)}; R_t \le \Lambda(t) \right)}{e^{\rho(t-r(t))} \mathbb{P}^* (R_t \le \Lambda(t))} = \frac{A(\phi(C(\varphi)))}{A(\phi(\vartheta^*))}.$$

By the definition of Ξ , we have

$$E(e^{-\theta\Xi(g)}) = \sum_{j=1}^{\infty} P(\mathcal{K} = j) \left[E(e^{-\theta\bar{\mathcal{N}}_{\infty}(g)}) \right]^{j} = \frac{A\left((\phi(\vartheta^*) - q)E(e^{-\theta\bar{\mathcal{N}}_{\infty}(g)}) + q \right)}{A(\phi(\vartheta^*))}. \quad (3.32)$$

Note that $\{\mathcal{N}_{\infty}(\mathbb{R}) > 0\} = \{W > 0\}$. Thus

$$\begin{split} E(e^{-\theta \bar{\mathcal{N}}_{\infty}(g)}) &= E(e^{-\theta \mathcal{N}_{\infty}(g)} | \mathcal{N}_{\infty}(\mathbb{R}) > 0, \mathcal{N}_{\infty}(1, \infty) = 0) \\ &= \frac{E(e^{-\theta \mathcal{N}_{\infty}(g)}; \mathcal{N}_{\infty}(\mathbb{R}) > 0, \mathcal{N}_{\infty}(1, \infty) = 0)}{P(\mathcal{N}_{\infty}(\mathbb{R}) > 0, \mathcal{N}_{\infty}(1, \infty) = 0)} \\ &= \frac{E(e^{-\theta \mathcal{N}_{\infty}(g)}; \mathcal{N}_{\infty}(1, \infty) = 0) - P(\mathcal{N}_{\infty}(\mathbb{R}) = 0)}{P(\mathcal{N}_{\infty}(1, \infty) = 0) - P(\mathcal{N}_{\infty}(\mathbb{R}) = 0)} \\ &= \frac{\mathbb{E}(\exp\{-C(\varphi)W\}) - \mathbb{P}(W = 0)}{\mathbb{P}(\exp\{-\vartheta^*W\}) - \mathbb{P}(W = 0)} = \frac{\phi(C(\varphi)) - q}{\phi(\vartheta^*) - q}. \end{split}$$

By (3.32), we have that

$$E(e^{-\theta\Xi(g)}) = \frac{A(\phi(C(\varphi)))}{A(\phi(\vartheta^*))}.$$

The proof is complete.

3.2 Lower deviation of \mathbb{Y}_t

For the proof of Theorem 1.1, we first established the upper deviation for \mathbb{Y}_t , and then used it to get the corresponding result for \mathbb{X}_t . However we can not get the lower deviation of \mathbb{X}_t from that of \mathbb{Y}_t . In Section 3.1, we proved Theorem 1.6, the lower large deviation result of \mathbb{X}_t . In this subsection, we establish the lower deviation result of \mathbb{Y}_t . Recall that ϕ is defined by (1.11).

Proposition 3.7. If $\Lambda:[0,\infty)\to(0,\infty)$ satisfies $\Lambda(t)\to\infty$ and

$$t^2 \Lambda(t)^{-\alpha} L(\Lambda(t)) \to 0, \quad \Lambda(t)/h(t) \to 0,$$

as $t \to \infty$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{E}^* \left(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t)) \right) = A \left(\phi \left(C(\varphi) \right) \right),$$

where r = r(t) is defined by $\Lambda(t) = h(r(t))$. In particular,

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{P}^* \Big(M_t \le \Lambda(t) \Big) = A \left[\phi(\vartheta^*) \right].$$

Proof: Since $\varphi \in \mathcal{H}(\mathbb{R})$, there exists $\delta > 0$ such that $\varphi \equiv 1$ on $[-\delta, \delta]$. Note that

$$(1-q)\mathbb{E}^*(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))) = \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))) - q + q - \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t)); \mathcal{S}^c).$$

On the event $\{Z_t = 0\}$, we have $\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t)) = 1$, thus

$$0 \le q - \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t)); \mathcal{S}^c) \le q - \mathbb{P}(Z_t = 0) \sim -A(0)e^{-\rho t} = o(e^{-\rho(t-r(t))}).$$

By the definition of $\mathbb{Y}_{r,t}$ given by (2.7), we have

$$\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))) \leq \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t),t}/\Lambda(t))) = \mathbb{E}\left(\left(\mathbb{E}\mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t)))\right)^{Z_{t-r(t)}}\right), \quad (3.33)$$

where \mathbb{Y}'_t is defined in (2.8). If $Z_{r(t)} = 0$, then $\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}/h(r(t))) = \mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t))) = 1$; if $Z_{r(t)} > 0$ then

$$\begin{split} \mathcal{I}(\varphi, \mathbb{Y}_{r(t)}/h(r(t))) = & \mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t))) \cdot \varphi(X_{o,r(t)}/h(r(t)))^{Z_{r(t)}} \\ \geq & \mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t))) \cdot \mathbf{1}_{\{|X_{o,r(t)}| \leq \delta h(r(t))\}}. \end{split}$$

By (2.2) and (1.6) we have

$$\mathbb{P}(|X_{o,r(t)}| > \delta h(r(t))) \stackrel{t}{\leq} c_0 r(t) (\delta h(r(t)))^{-\alpha} L(\delta h(r(t))) \stackrel{t}{\leq} 2c_0 \delta^{-\alpha} \cdot r(t) e^{-\lambda r(t)} \to 0.$$

Thus by (3.8) we have

$$\lim_{t \to \infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t)))) = \lim_{t \to \infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}/h(r(t)))) = \phi(C(\varphi)) < 1.$$
 (3.34)

Thus, by (3.1), we have that

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \left(\mathbb{E} \left(\left[\mathbb{E} \mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/h(r(t))) \right]^{Z_{t-r(t)}} \right) - q \right) = A \left(\phi(C(\varphi)) \right).$$

Hence by (3.33) we have

$$\lim \sup_{t \to \infty} e^{\rho(t-r(t))} \left(\mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}_t / \Lambda(t)) \right) - q \right) \le A \left(\phi \left(C(\varphi) \right) \right).$$

On the other hand,

$$\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t/\Lambda(t))|\mathcal{F}_t^{\mathbb{T}})$$

$$\geq \mathbb{P}(\max_{u \in \mathcal{D}_t: b_u < t - r(t)} |X_{u,t}| \leq \delta h(r(t))|\mathcal{F}_t^{\mathbb{T}}) \mathbb{E}(I(\varphi, \mathbb{Y}_{r(t),t}/h(r(t)))|\mathcal{F}_t^{\mathbb{T}}). \tag{3.35}$$

Note that $\{u \in \mathcal{D}_t : b_u \leq t - r(t)\} \subset \mathcal{D}_{t-r(t)}$. Thus by (2.3) we have that

$$\mathbb{P}\left(\max_{u\in\mathcal{D}_t:b_u\leq t-r(t)}|X_{u,t}|\leq \delta h(r(t))|\mathcal{F}_t^{\mathbb{T}}\right) \geq \prod_{u\in\mathcal{D}_{t-r(t)}} \mathbb{P}(|X_{u,t}|\leq \delta h(r(t)))|\mathcal{F}_t^{\mathbb{T}}) \\
\geq \left(1-2c_0\delta^{-\alpha-1}te^{-\lambda r(t)}\right)^{|\mathcal{D}_{t-r(t)}|}.$$

By (3.35), we have that

$$\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{t}/\Lambda(t))) \stackrel{t}{\geq} \mathbb{E}\left((1 - 2c_{0}\delta^{-\alpha}te^{-\lambda r(t)})^{|\mathcal{D}_{t-r(t)}|} \cdot \mathcal{I}(\varphi, \mathbb{Y}_{r(t),t}/\Lambda(t))\right) \\
= \mathbb{E}\left((1 - 2c_{0}\delta^{-\alpha}te^{-\lambda r(t)})^{|\mathcal{D}_{t-r(t)}|} \cdot \left[\mathbb{E}\mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/\Lambda(t))\right]^{Z_{t-r(t)}}\right) \\
\geq \mathbb{E}\left((1 - 2c_{0}\delta^{-\alpha}te^{-\lambda r(t)})^{c \cdot (t-r(t))Z_{t-r(t)}} \left[\mathbb{E}\mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/\Lambda(t))\right]^{Z_{t-r(t)}}; |\mathcal{D}_{t-r(t)}| \leq c \cdot (t-r(t))Z_{t-r(t)}\right) \\
\geq \mathbb{E}\left(C(t)^{Z_{t-r(t)}}\right) - \mathbb{P}(|\mathcal{D}_{t-r(t)}| > c \cdot (t-r(t))Z_{t-r(t)}),$$

where $C(t) = (1 - 2c_0\delta^{-\alpha}te^{-\lambda r(t)})^{c\cdot(t-r(t))}\mathbb{E}\mathcal{I}(\varphi, \mathbb{Y}'_{r(t)}/\Lambda(t))$ and c > 0 is a constant. Since $t^2e^{-\lambda r(t)} = t^2\Lambda(t)^{-\alpha}L(\Lambda(t)) \to 0$, by (3.34), we have

$$\lim_{t\to\infty}C(t)=\lim_{t\to\infty}\mathbb{E}\mathcal{I}(\varphi,\mathbb{Y}_r'/\Lambda(t))=\phi\left(C(\varphi)\right)<1.$$

By (3.1), we have that

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \left(\mathbb{E}\left(C(t)^{Z_{t-r(t)}} \right) - q \right) = A\left(\phi\left(C(\varphi) \right) \right).$$

Note that $|\mathcal{D}_{t-r(t)}| \leq \sum_{u \in \mathcal{L}_{t-r(t)}} |I_u|$. Thus by the many-to-one formula,

$$\mathbb{P}(|\mathcal{D}_{t-r(t)}| > c \cdot (t - r(t))Z_{t-r(t)}) \leq \mathbb{P}\left(\sum_{u \in \mathcal{L}_{t-r(t)}} |I_u| > c \cdot (t - r(t))Z_{t-r(t)}\right) \\
\leq \mathbb{P}\left(\sum_{u \in \mathcal{L}_{t-r(t)}} \mathbf{1}_{\{|I_u| > c \cdot (t-r(t))\}} \geq 1\right) \leq \mathbb{E}\left(\sum_{u \in \mathcal{L}_{t-r(t)}} \mathbf{1}_{\{|I_u| > c \cdot (t-r(t))\}}\right) \\
\leq e^{\lambda(t-r(t))}P(1 + n_{t-r(t)} > c \cdot (t - r(t))) \\
\leq e^{\lambda(t-r(t))}e^{-c \cdot (t-r(t))}E(e^{(1+n_{t-r(t)})}) = e \cdot e^{\lambda(t-r(t))}e^{-c \cdot (t-r(t))}e^{(e-1)\beta(t-r(t))} \\
= e \cdot e^{-(c-\lambda - (e-1)\beta)(t-r(t))}$$

Now we choose c such that $c > \lambda + (e-1)\beta - \rho$. Then

$$\lim_{t \to \infty} e^{\rho(t-r(t))} \mathbb{P}(|\mathcal{D}_{t-r(t)}| > c(t-r(t)) Z_{t-r(t)}) = 0.$$

Hence we have

$$\liminf_{t \to \infty} e^{\rho(t-r(t))} \left(\mathbb{E} \left(\mathcal{I}(\varphi, \mathbb{Y}_t / \Lambda(t)) \right) - q \right) \ge A \left(\phi \left(C(\varphi) \right) \right).$$

The proof is complete now.

4 Almost sure convergence results

In this section we will give the proofs of Theorems 1.6 and 1.7. We first prove the corresponding results for \mathbb{Y} . Note that for any a > 0,

$$\{R_t \leq a\} = \mathcal{I}(\mathbf{1}_{(-\infty,a]}, \mathbb{X}_t), \quad \{M_t \leq a\} = \mathcal{I}(\mathbf{1}_{(-\infty,a]}, \mathbb{Y}_t).$$

By the continuity of $C(\varphi)$, the results established in the previous sections for $\mathcal{H}(\mathbb{R})$ are also valid for $\mathbf{1}_{(-\infty,a]}$.

4.1 Almost sure convergence results for M_t

Lemma 4.1. Suppose that $G:[0,\infty)\to(0,\infty)$ satisfies $\lim_{t\to\infty}\frac{G(t)}{h(t)}=\infty$.

(1) If $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, then

$$\limsup_{t \to \infty} \frac{M_n}{G(n)} \le 0, \qquad \mathbb{P}\text{-a.s.}$$

(2) If $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$, then

$$\limsup_{t \to \infty} \frac{M_n}{G(n)} = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Proof: (1) Assume that $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$. By Proposition 2.6, we have for any c > 0,

$$\lim_{t \to \infty} e^{-\lambda t} G(t)^{\alpha} L(G(t))^{-1} \mathbb{P}(M_t > cG(t)) = c^{-\alpha} \vartheta^*,$$

where ϑ^* is given in (1.4). Combining this with $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, we get that for any c > 0,

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n > cG(n)) < \infty.$$

By the Borel-Cantelli lemma, we get that

$$\limsup_{n \to \infty} \frac{M_n}{G(n)} \le c, \qquad \mathbb{P}\text{-a.s.}$$

Letting $c \to 0$, we get the desired result.

(2) Assume that $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$. For 0 < s < t, define

$$M_{s,t}^{(2)} := \max_{u \in \mathcal{D}_t: t-s < b_u < t} X_{u,t}.$$

For any c > 0, we have that

$$\mathbb{P}(M_n > cG(n)|\mathcal{F}_{n-1}) \ge \mathbb{P}(M_{1,n}^{(2)} > cG(n)|\mathcal{F}_{n-1}). \tag{4.1}$$

By (2.24) with $s=1,\,t=n,\,\Lambda(t)=cG(t)$ and $\varphi=\mathbf{1}_{(-\infty,1]},$ we get that, as $n\to\infty,$

$$e^{-\lambda n}G(n)^{\alpha}L^{-1}(G(n))\mathbb{P}(M_{1,n}^{(2)} > cG(n)|\mathcal{F}_{n-1})$$

$$\to c^{-\alpha}\frac{q_1}{\alpha}e^{-\lambda}\int_0^1 \left(e^{\lambda r} - e^{-\beta r}\right)\mathbb{P}(Z_{1-r} > 0)\mathrm{d}r \cdot W \quad \text{a.s.}.$$
(4.2)

Combining (4.1) and (4.2), we obtain that, for any c > 0, on the event $\{W > 0\}$,

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n > cG(n)|\mathcal{F}_{n-1}) = \infty, \quad \mathbb{P}\text{-a.s.}$$

Applying the conditional Borel-Cantelli lemma, we get that for any c > 0,

$$\mathbb{P}^*(M_n > cG(n), i.o.) = 1.$$

Since c is arbitrary, desired result follows immediately.

Lemma 4.2. It holds that

$$\liminf_{t \to \infty} \frac{M_n}{H(e^{-\lambda n} \log n)} \le (\vartheta^* W)^{\frac{1}{\alpha}}, \qquad \mathbb{P}^* \text{-a.s.}$$

Proof: For any $\epsilon \in (0,1)$, choose $\epsilon' > 0$ small enough such that $(1+\epsilon')(1+\epsilon/2)^{-\alpha} < 1$. Let $n_k := \lfloor k^{1+\epsilon'} \rfloor$, $k \ge 1$, where $\lfloor x \rfloor$ is the integer part of x. It is easy to see $n_k < n_{k+1}$ for any $k \ge 1$. Let

$$U_k := H\left(\frac{\log n_k}{\vartheta^* Z_{n_{k-1}} e^{\lambda(n_k - n_{k-1})}}\right).$$

Since $H(y) = y^{-1/\alpha} \bar{L}(y^{-1})$, we have

$$\lim_{k \to \infty} \frac{U_k}{H(e^{-\lambda n_k} \log n_k)} = \lim_{k \to \infty} (\vartheta^* W_{n_{k-1}})^{1/\alpha} = (\vartheta^* W)^{1/\alpha}, \qquad \mathbb{P}^* \text{-a.s.}$$
 (4.3)

Note that $U_k \in \mathcal{F}_{n_{k-1}}$. Thus we have

$$\mathbb{P}\left(M_{n_k} \le (1+\epsilon)U_k | \mathcal{F}_{n_{k-1}}\right) \ge \mathbf{1}_{\{M_{n_{k-1}} \le (1+\epsilon/2)U_k\}} J_k \tag{4.4}$$

where

$$J_k := \mathbb{P}\left(\forall v \in \mathcal{L}_{n_{k-1}}, X_{v,n_k} - X_{v,n_{k-1}} \le \epsilon U_k/2, M_{n_k}^v \le (1 + \epsilon/2)U_k | \mathcal{F}_{n_{k-1}}\right)$$

and $M_t^v = \max_{u \in \mathcal{D}_t: v < u} X_{u,t}$.

We first show that, on the survival event S, $\{M_{n_{k-1}} > (1+\epsilon/2)U_k\}$ can occur only finitely many times. By (3.7), we have for $p > \lambda/\rho$,

$$\sum_{n=1}^{\infty} \mathbb{P}(0 < e^{-\lambda n} Z_n < n^{-p}) \le C \sum_{n=1}^{\infty} n^{-p\rho/\lambda} < \infty.$$

$$(4.5)$$

Note that

$$\mathbb{P}\left(M_{n_{k-1}} > (1 + \epsilon/2)U_{k}, \mathcal{S}\right)
\leq \mathbb{P}\left(e^{-\lambda n_{k-1}}Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) + \mathbb{P}\left(M_{n_{k-1}} > (1 + \epsilon/2)U_{k}, e^{-\lambda n_{k-1}}Z_{n_{k-1}} \ge n_{k-1}^{-p}\right)
\leq \mathbb{P}\left(e^{-\lambda n_{k-1}}Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) + \mathbb{P}\left(M_{n_{k-1}} > (1 + \epsilon/2)H\left(\frac{\log n_{k}}{\vartheta^{*}n_{k-1}^{-p}e^{\lambda n_{k}}}\right)\right).$$

By (4.5), we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) \le \sum_{k=1}^{\infty} \mathbb{P}\left(0 < e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}\right) < \infty. \tag{4.6}$$

Observe that $n_k - n_{k-1} \sim (1 + \epsilon')k^{\epsilon'}$. Thus

$$\lim_{k\to\infty}\frac{H\left(\frac{\log n_k}{\vartheta^*n_{k-1}^{-p}e^{\lambda n_k}}\right)}{h(n_{k-1})}=\lim_{k\to\infty}\left(\frac{\frac{\log n_k}{\vartheta^*n_{k-1}^{-p}e^{\lambda n_k}}}{e^{-\lambda n_{k-1}}}\right)^{-1/\alpha}=\infty.$$

Hence by Proposition 2.6, we have that as $k \to \infty$,

$$\mathbb{P}\left(M_{n_{k-1}} > (1 + \epsilon/2)H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)\right)
\sim \vartheta^* e^{\lambda n_{k-1}} (1 + \epsilon/2)^{-\alpha} H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)^{-\alpha} L\left(H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)\right)
= (1 + \epsilon/2)^{-\alpha} \frac{n_{k-1}^p \log n_k}{e^{\lambda (n_k - n_{k-1})}},$$
(4.7)

where in the last equality we used (1.5). Since $n_k = \lfloor k^{1+\epsilon'} \rfloor$, we have $\sum_k \frac{n_{k-1}^p \log n_k}{e^{\lambda(n_k - n_{k-1})}} < \infty$. Combining (4.6) and (4.7), we get that

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\left\{M_{n_{k-1}} > (1 + \epsilon/2)U_k\right\} \cap \mathcal{S}\right) < \infty.$$

Using the Borel-Cantelli lemma, we get that

$$\sum_{k=1}^{\infty} \mathbf{1}_{\{M_{n_{k-1}} > (1+\epsilon/2)U_k\}} < \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}.$$

$$\tag{4.8}$$

Now we consider J_k . Recall that M'_t is defined in (2.9). By the Markov property and the branching property, we have that

$$J_{k} = \left[\mathbb{P} \left(X_{o,n_{k}-n_{k-1}} \le \epsilon x/2, M'_{n_{k}-n_{k-1}} \le (1+\epsilon/2)x \right) \right]^{Z_{n_{k-1}}} |_{x=U_{k}}$$

$$= e^{Z_{n_{k-1}}} \log \mathbb{P} \left(X_{o,n_{k}-n_{k-1}} \le \epsilon x/2, M'_{n_{k}-n_{k-1}} \le (1+\epsilon/2)x \right) |_{x=U_{k}}$$

$$\ge e^{Z_{n_{k-1}}} \log \left(1 - \mathbb{P} \left(X_{o,n_{k}-n_{k-1}} > \epsilon x/2 \right) - \mathbb{P} \left(M'_{n_{k}-n_{k-1}} > (1+\epsilon/2)x \right) \right) |_{x=U_{k}}. \tag{4.9}$$

Note that, on the survival event S, $U_k \to \infty$. Then by (2.3), we have for k large enough,

$$\mathbb{P}\left(X_{o,n_{k}-n_{k-1}} > \epsilon x/2\right)|_{x=U_{k}} \leq 2c_{0}(\epsilon/2)^{-\alpha-1}(n_{k}-n_{k-1})U_{k}^{-\alpha}L(U_{k})
= 2c_{0}(\epsilon/2)^{-\alpha-1}(n_{k}-n_{k-1})\frac{\log n_{k}}{\vartheta^{*}Z_{n_{k-1}}e^{\lambda(n_{k}-n_{k-1})}}
\to 0, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}$$
(4.10)

as $k \to \infty$. By Proposition 2.6 we have

$$\mathbb{P}\left(M'_{n_{k}-n_{k-1}} > (1+\epsilon/2)x\right)|_{x=U_{k}} \leq \mathbb{P}\left(M_{n_{k}-n_{k-1}} > (1+\epsilon/2)x\right)|_{x=U_{k}}
\sim \vartheta^{*}e^{\lambda(n_{k}-n_{k-1})}(1+\epsilon/2)^{-\alpha}U_{k}^{-\alpha}L(U_{k})
= (1+\epsilon/2)^{-\alpha}\frac{\log n_{k}}{Z_{n_{k-1}}} \to 0, \quad \mathbb{P}\text{-a.s. on } \mathcal{S} \tag{4.11}$$

as $k \to \infty$. Since $n_k = \lfloor k^{1+\epsilon'} \rfloor$, using (4.10) and (4.11), we obtain that

$$\limsup_{k \to \infty} -\frac{1}{\log k} Z_{n_{k-1}} \log \left(1 - \mathbb{P} \left(X_{o,n_k - n_{k-1}} > \epsilon x/2 \right) - \mathbb{P} \left(M'_{n_k - n_{k-1}} > (1 + \epsilon/2) x \right) \right)_{x = U_k}$$

$$= \limsup_{k \to \infty} \frac{1}{\log k} Z_{n_{k-1}} \left(\mathbb{P} \left(X_{o,n_k - n_{k-1}} > \epsilon x/2 \right) + \mathbb{P} \left(M'_{n_k - n_{k-1}} > (1 + \epsilon/2) x \right) \right)_{x = U_k}$$

$$\leq (1 + \epsilon') (1 + \epsilon/2)^{-\alpha} < 1, \quad \mathbb{P} \text{-a.s. on } \mathcal{S}. \tag{4.12}$$

Combining (4.9) and (4.12), we get that,

$$\sum_{k=1}^{\infty} J_k = \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}. \tag{4.13}$$

Combining (4.4),(4.8) and (4.13), we get that

$$\sum_{k} \mathbb{P}\left(M_{n_{k}} \leq (1+\epsilon)U_{k}|\mathcal{F}_{n_{k-1}}\right) = \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}.$$

Applying the Borel-Cantelli lemma and then letting $\epsilon \to 0$, we obtain that

$$\liminf_{k \to \infty} \frac{M_{n_k}}{U_k} \le 1 \quad \mathbb{P}^* \text{-a.s.}$$

Hence the desired result follows from (4.3) immediately.

We now modify the definition of \mathcal{D}_t and \mathcal{L}_t slightly. For any $\delta \geq 0$, and t > 0, define

$$\mathcal{L}_{t,\delta} := \{ u \in \mathcal{L}_t : Z_{t+\delta}^u > 0 \},$$

$$\mathcal{D}_{t,\delta} := \{ u \in \mathbb{T} : b_u < t, Z_{t+\delta}^u > 0 \} = \bigcup_{u \in \mathcal{L}_{t,\delta}} I_u,$$

$$R_{t,\delta} := \max_{u \in \mathcal{L}_{t,\delta}} \xi_t^u, \quad M_{t,\delta} := \max_{u \in \mathcal{D}_{t,\delta}} X_{u,t}, \text{ and } M'_{t,\delta} := \max_{u \in \mathcal{D}_{t,\delta} \setminus \{o\}} X_{u,t}.$$

Using argument similar to that in the proof of Lemma 2.5 with $\varphi = \mathbf{1}_{(-\infty,1]}$, we get the following result. We omit the proof.

Lemma 4.3. For any $\delta \geq 0$, it holds that

$$\lim_{x \to \infty} x^{\alpha} L(x)^{-1} \mathbb{P}(M'_{s,\delta} > x) = \frac{q_1}{\alpha} \int_0^s \left(e^{\lambda r} - e^{-\beta r} \right) \mathbb{P}(Z_{s+\delta-r} > 0) dr =: c(s,\delta).$$

Lemma 4.4. For any $\delta \geq 0$, it holds that

$$\liminf_{n \to \infty} \frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta} \log n)} \ge (\vartheta_{\delta} W)^{\frac{1}{\alpha}} \qquad \mathbb{P}^* \text{-a.s.},$$

where $\vartheta_{\delta} = \frac{q_1}{\alpha} \int_0^{\infty} e^{-\lambda r} \mathbb{P}(Z_{r+\delta} > 0) dr$.

Proof: For $1 \le m < n$, let $U_{n,m} = H\left(\frac{\log n}{Z_{(n-m)\delta}c(m\delta,\delta)}\right)$ when $Z_{(n-m)\delta} > 0$ and $U_{n,m} = 0$ when $Z_{(n-m)\delta} = 0$. Note that

$$M_{n\delta,\delta} \ge \max_{u \in \mathcal{L}_{(n-m)\delta}} \max_{v \in \mathcal{D}_{n\delta,\delta}^u} X_{v,n\delta},$$

where $\mathcal{D}_{t,\delta}^u = \{v \in \mathcal{D}_{t,\delta}: v > u\}$. By the Markov property and the branching property, we have that, conditioned on $\mathcal{F}_{(n-m)\delta}$, $\{\max_{v \in \mathcal{D}_{n\delta,\delta}^u} X_{v,n\delta}, u \in \mathcal{L}_{(n-m)\delta}\}$ are i.i.d. with the same law as $M'_{m\delta,\delta}$. Let $G_{m,\delta}(x) := \mathbb{P}(M'_{m\delta,\delta} > x)$. Note that $U_{n,m} \in \mathcal{F}_{(n-m)\delta}$. Thus for any $\epsilon \in (0,1)$, we have

$$\mathbb{P}\left(M_{n\delta,\delta} < (1-\epsilon)U_{n,m}|\mathcal{F}_{(n-m)\delta}\right) \le (1-G_{m,\delta}((1-\epsilon)U_{n,m}))^{Z_{(n-m)\delta}} < e^{-Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m})}.$$

Note that on the survival event S, $\lim_{n\to\infty} U_{n,m} = \infty$. Thus by Lemma 4.3, we have that, on the survival event S, as $n\to\infty$,

$$Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m}) \sim Z_{(n-m)\delta}c(m\delta,\delta)(1-\epsilon)^{-\alpha}U_{n,m}^{-\alpha}L(U_{n,m}) = (1-\epsilon)^{-\alpha}\log n, \quad \mathbb{P}\text{-a.s}$$

where in the last equality we used (1.5). Since $(1 - \epsilon)^{-\alpha} > 1$, we have that, on the survival event S, for any $m \ge 1$,

$$\sum_{n=m+1}^{\infty} \mathbb{P}\left(M_{n\delta,\delta} < (1-\epsilon)U_{n,m}|\mathcal{F}_{(n-m)\delta}\right) \le \sum_{n=m+1}^{\infty} e^{-Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m})} < \infty, \quad \mathbb{P}\text{-a.s.}.$$

By the second Borel-Cantelli lemma, we have that, on the survival event \mathcal{S} ,

$$\sum_{n=m+1}^{\infty} \mathbf{1}_{\{M_{n\delta,\delta} < (1-\epsilon)U_{n,m}\}} < \infty, \quad \mathbb{P}\text{-a.s.}$$

Thus, by the definition of H, we obtain that under \mathbb{P}^* , for any $\epsilon \in (0,1)$ and integer $m \geq 1$,

$$\liminf_{n \to \infty} \frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta} \log n)} \ge \liminf_{n \to \infty} \frac{(1-\epsilon)U_{n,m}}{H(e^{-\lambda n\delta} \log n)} = (1-\epsilon) \lim_{n \to \infty} \left(\frac{\frac{\log n}{Z_{(n-m)\delta}c(m\delta,\delta)}}{e^{-\lambda n\delta} \log n}\right)^{-1/\alpha}$$

$$= (1-\epsilon) \left(We^{-\lambda m\delta}c(m\delta,\delta)\right)^{1/\alpha}, \quad \mathbb{P}^*-\text{a.s.} \tag{4.14}$$

Note that $\lim_{m\to\infty} e^{-\lambda m\delta}c(m\delta,\delta) = \vartheta_{\delta}$. Letting $m\to\infty$ and $\epsilon\to 0$ in (4.14), we get the desired result. The proof is now complete.

4.2 Proofs of Theorems 1.6 and 1.7

Lemma 4.5. If $\{a_n : n \geq 1\}$ is a sequence of positive numbers satisfying $\sum_n e^{\lambda n} a_n^{-p\alpha} < \infty$ for some $p \in (0,2)$, then for any $\delta \geq 0$,

$$\limsup_{n \to \infty} \frac{R_{n,\delta} - M_{n,\delta}}{a_n} \le 0, \quad \mathbb{P}^* - a.s. \tag{4.15}$$

and

$$\limsup_{n \to \infty} \frac{M_{n,\delta} - R_{n,\delta}^+}{a_n} \le 0, \quad \mathbb{P}^* - a.s. \tag{4.16}$$

Proof: Let a(t) be a positive function satisfying $a(t) \stackrel{t}{\geq} e^{\epsilon t}$ for some $\epsilon \in (0,1)$. Let $c > 2\alpha + e^2\beta/\epsilon^2$ be a constant. We choose $\theta \in (0,1)$ so that $\theta c < 1$. We define the events $A_t(\theta)$ and B_t as in the proof of Lemma 2.4. We claim that for n sufficiently large, on the event $A_t(\theta) \cap B_t \cap \{Z_t \geq 1\}$,

$$\{R_{t,\delta} - M_{t,\delta} > a(t)\} \cup \{M_{t,\delta} - R_{t,\delta}^+ > a(t)\} = \emptyset.$$

In fact, by (2.17), we have that on the event $A_t(\theta) \cap B_t \cap \{Z_t \geq 1\}$,

$$R_{t,\delta} = \max_{v \in \mathcal{L}_{t,\delta}} \xi_t^v \le \max_{v \in \mathcal{L}_{t,\delta}} X_{v',t} + a(t) \le M_{t,\delta} + a(t), \tag{4.17}$$

and

$$M_{t,\delta} = \max_{v \in \mathcal{L}_{t,\delta}} \max_{u \in I_v} X_{u,t} = \left(\max_{v \in \mathcal{L}_{t,\delta}} X_{v',t}\right) \vee \left(\max_{v \in \mathcal{L}_{t,\delta}, u \in I_v \setminus \{v'\}} X_{u,t}\right)$$

$$\leq \left(\max_{v \in \mathcal{L}_{t,\delta}} \xi_t^v + a(t)\right) \vee (\theta a(t)/\log a(t)) \leq R_{t,\delta}^+ + a(t), \tag{4.18}$$

for t large enough so that $\log a(t) > \theta$. Hence, by (4.17) and (4.18), the claim is true. Combining (2.11), (2.16), we get that for any $p \in (0, 2)$,

$$\limsup_{t \to \infty} e^{-\lambda t} a(t)^{p\alpha} \mathbb{P}\left(\left(\left\{R_{t,\delta} - M_{t,\delta} > a(t)\right\} \cup \left\{M_{t,\delta} - R_{t,\delta}^+ > a(t)\right\}\right) \cap \left\{Z_t \ge 1\right\}\right)$$

$$\le \limsup_{t \to \infty} e^{-\lambda t} a(t)^{p\alpha} \left\{\mathbb{P}(A_t(\theta)^c) + \mathbb{P}(B_t^c)\right\} = 0.$$

Thus we have

$$\lim_{t \to \infty} e^{-\lambda t} a(t)^{p\alpha} \mathbb{P}\left(\left\{R_{t,\delta} - M_{t,\delta} > a(t)\right\} \cup \left\{M_{t,\delta} - R_{t,\delta}^+ > a(t)\right\}\right) \cap \left\{Z_t \ge 1\right\}\right) = 0.$$

Since $\sum_n e^{\lambda n} a_n^{-p\alpha} < \infty$, we have $a_n \stackrel{n}{\geq} e^{\frac{\lambda n}{p\alpha}}$. It follows that for any $\epsilon \in (0,1)$ and $p \in (0,2)$,

$$\mathbb{P}^* \left(R_{n,\delta} - M_{n,\delta} > \epsilon a_n \right) \le \frac{1}{\mathbb{P}(\mathcal{S})} \mathbb{P} \left(R_{n,\delta} - M_{n,\delta} > \epsilon a_n, Z_n \ge 1 \right) = o(e^{\lambda n} a_n^{-p\alpha}).$$

So we have that

$$\sum_{n=1}^{\infty} \mathbb{P}^* \left(R_{n,\delta} - M_{n,\delta} > \epsilon a_n \right) < \infty.$$

Applying the Borel-Cantelli lemma and then letting $\epsilon \to 0$, we get (4.15). Similarly, we can get (4.16) as well.

Proof of Theorem 1.6: Note that

$$H(e^{-\lambda n\delta}\log n) = e^{\lambda n\delta/\alpha}(\log n)^{-1/\alpha}\bar{L}(e^{\lambda n\delta}/\log n).$$

Applying Lemma 4.5 with $\delta = 0$ and $\delta \geq 0$, we have

$$\limsup_{n \to \infty} \frac{R_n - M_n}{H(e^{-\lambda n} \log n)} \le 0, \qquad \limsup_{n \to \infty} \frac{M_{n\delta, \delta} - R_{n\delta, \delta}^+}{H(e^{-\lambda n\delta} \log n)} \le 0, \qquad \mathbb{P}^* \text{-a.s.}$$

By Lemma 4.2 we have

$$\liminf_{t \to \infty} \frac{R(t)}{H(e^{-\lambda t} \log t)} \le \liminf_{t \to \infty} \frac{M_n}{H(e^{-\lambda n} \log n)} + \limsup_{n \to \infty} \frac{R_n - M_n}{H(e^{-\lambda n} \log n)} \le (\vartheta^* W)^{1/\alpha}, \quad \mathbb{P}^* \text{-a.s.}$$

We now consider the lower bound of $\liminf_{t\to\infty} \frac{R(t)}{H(e^{-\lambda t}\log t)}$. By Lemma 4.4 we have that for any $\delta>0$,

$$\liminf_{n \to \infty} \frac{R_{n\delta,\delta}^+}{H(e^{-\lambda n\delta} \log n)} \ge \liminf_{n \to \infty} \frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta} \log n)} - \limsup_{n \to \infty} \frac{M_{n\delta,\delta} - R_{n\delta,\delta}^+}{H(e^{-\lambda n\delta} \log n)} \ge (\vartheta_{\delta} W)^{1/\alpha}, \quad \mathbb{P}^* \text{-a.s.}$$
(4.19)

For any $n \geq 1$, let $Q_{n,\delta} = \inf_{n\delta \leq t \leq (n+1)\delta} R_t$. We claim that for any $\epsilon \in (0,1)$ and $\delta > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(Q_{n,\delta} < (1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_{\delta}W_{n\delta}\right)^{\frac{1}{\alpha}} - H(n^{-2})|\mathcal{F}_{n\delta}\right) < \infty, \qquad \mathbb{P}^*\text{-a.s. } (4.20)$$

Let $B_n := \{R_{n\delta,\delta} < (1 - \epsilon)H(e^{-\lambda \delta n} \log n) (\vartheta_{\delta} W_{n\delta})^{\frac{1}{\alpha}} \}$. Note that

$$\mathbb{P}\left(Q_{n,\delta} < (1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_{\delta}W_{n\delta}\right)^{\frac{1}{\alpha}} - H(n^{-2})|\mathcal{F}_{n\delta}\right) \\
\leq \mathbb{P}(B_n|\mathcal{F}_{n\delta}) + \mathbb{P}\left(B_n^c, Q_{n,\delta} < (1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_{\delta}W_{n\delta}\right)^{\frac{1}{\alpha}} - H(n^{-2})|\mathcal{F}_{n\delta}\right). (4.21)$$

By (4.19) and the Borel-Cantelli lemma, we know that on S,

$$\sum_{n=1}^{\infty} \mathbb{P}(B_n | \mathcal{F}_{n\delta}) < \infty, \qquad \mathbb{P}\text{-a.s.}$$
(4.22)

Now we consider the second term of the right hand side of (4.21). Take $Z_0 = \nu = \sum_{k=1}^K \delta_{a_k}$ with $a_1 \geq a_2 \geq \cdots \geq a_K$. We use $Z_t^{(j)}$ to denote the number of descendants at time t of

the j-th particle, and $\mathcal{L}_t^{(j)}$ the set of all the descendants of j alive at time t. For any x, y we have that

$$\mathbb{P}_{\nu}(R_{0,\delta} > x, Q_{0,\delta} < x - y)
= \sum_{k=1}^{K} \mathbb{P}(Z_{\delta}^{(j)} = 0, 1 \le j \le k - 1, Z_{\delta}^{(k)} > 0, a_{k} > x, Q_{0,\delta} < x - y)
\le \sum_{k=1}^{K} \mathbb{P}(Z_{\delta}^{(j)} = 0, 1 \le j \le k - 1, Z_{\delta}^{(k)} > 0, \inf_{0 \le t \le \delta} \max_{u \in \mathcal{L}_{t}^{(k)}} (\xi_{t}^{u} - a_{k}) < -y)
= \sum_{k=1}^{K} \mathbb{P}(Z_{\delta} = 0)^{k-1} \mathbb{P}(Z_{\delta} > 0, \inf_{0 \le t \le \delta} R_{t} < -y)
\le \frac{1}{\mathbb{P}(Z_{\delta} > 0)} \mathbb{P}(Z_{\delta} > 0, \inf_{0 \le t \le \delta} R_{t} < -y).$$
(4.23)

By the many-to-one formula, we have

$$\mathbb{P}(Z_{\delta} > 0, \inf_{0 \le t \le \delta} R_{t} < -y) \le \mathbb{E}\left(\sum_{u \in \mathcal{L}_{\delta}} \mathbf{1}_{\{\inf_{0 \le t \le \delta} \xi_{t}^{u} < -y\}} \ge 1\right) \le \mathbb{E}\left(\sum_{u \in \mathcal{L}_{\delta}} \mathbf{1}_{\{\inf_{0 \le t \le \delta} \xi_{t}^{u} < -y\}}\right)$$

$$= e^{\lambda \delta} \mathbf{P}\left(\inf_{0 \le t \le \delta} \xi_{t} < -y\right) \sim e^{\lambda \delta} \frac{q_{2}}{\alpha} \delta y^{-\alpha} L(y), \qquad y \to \infty,$$

where the last limit follows from (2.5). Combining the Markov property and (4.23) with $\nu = X_{n\delta}$ and $y = H(n^{-2})$, we get

$$\mathbb{P}\left(B_{n}^{c}, Q_{n,\delta} < (1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_{\delta}W_{n\delta}\right)^{\frac{1}{\alpha}} - H(n^{-2})|\mathcal{F}_{n\delta}\right)
\leq \frac{1}{\mathbb{P}(Z_{\delta} > 0)}\mathbb{P}(Z_{\delta} > 0, \inf_{0 \leq t \leq \delta} R_{t} < -H(n^{-2})) \leq \frac{2\delta e^{\lambda\delta}}{\mathbb{P}(Z_{\delta} > 0)} \frac{q_{2}}{\alpha}n^{-2},$$
(4.24)

where the last inequality holds for n large enough. Combining (4.21), (4.22) and (4.24), we arrive at (4.20).

Using the Borel-Cantelli lemma and (4.20), we have that for any $\delta > 0$,

$$\liminf_{n \to \infty} \frac{Q_{n,\delta}}{H(e^{-\lambda n\delta} \log n)} \ge (\vartheta_{\delta} W)^{\frac{1}{\alpha}} \qquad \mathbb{P}^* \text{-a.s}$$

Since $H(e^{-\lambda t} \log t)$ is increasing in t for t large, for any $\delta > 0$, we have

$$\liminf_{t \to \infty} \frac{R_t \vee 0}{H(e^{-\lambda t} \log t)} \ge \liminf_{n \to \infty} \frac{Q_{n,\delta} \vee 0}{H(e^{-\lambda n\delta} \log n)} \frac{H(e^{-\lambda n\delta} \log n)}{H(e^{-\lambda(n+1)\delta} \log(n+1)\delta)} \ge e^{-\lambda \delta/\alpha} \left(\vartheta_{\delta} W\right)^{\frac{1}{\alpha}},$$

 \mathbb{P}^* -a.s. Letting $\delta \to 0$, we get the desired result. The proof is now complete.

It follows from Lemma 4.5 that Lemma 4.1 also holds with M_n replaced by R_n . To get the limit of R_t as $t \to \infty$, we need to deal with $\sup_{0 \le s \le t} R_s$. If $p_0 = 0$, then $\sup_{0 \le s \le t} R_s = 0$

 \Box .

 $\max_{v \in \mathcal{L}_t} \sup_{0 \le s \le t} \xi_s^v$. Thus by the many-to-one formula, we have that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} R_s \geq x\right) \leq \mathbb{E}\left(\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\{\sup_{0\leq s\leq t} \xi_s^v \geq x\}}\right) = e^{\lambda t} \mathbf{P}\left(\sup_{0\leq s\leq t} \xi_s \geq x\right).$$

In the following lemma, we show that this assertion is still valid when $p_0 > 0$.

Lemma 4.6. For any x > 0 and t > 0, it holds that

$$\mathbb{P}\left(\sup_{0\leq s\leq t} R_s \geq x\right) \leq e^{\lambda t} \mathbf{P}\left(\sup_{0\leq s\leq t} \xi_s \geq x\right).$$

Proof For $r \geq 0$ and $x \in \mathbb{R}$, let $\mathbb{P}_{r,x}$ denote the law of \mathbb{X} starting with an individual at position x at time r, and $\mathbf{P}_{r,x}$ stands for the law of ξ condition on $\{\xi_r = x\}$. We still use σ_o to denote the death time of the initial particle. Fix t > 0. For any $0 \leq r < t$ and x < 0, we have that

$$\omega(r,x) := \mathbb{P}_{r,x} \left(\sup_{r \le s \le t} R_s < 0 \right) = \mathbb{P}_{r,x} \left(\sup_{r \le s \le t} R_s < 0, \sigma_o > t \right) + \mathbb{P}_{r,x} \left(\sup_{r \le s \le t} R_s < 0, r \le \sigma_o \le t \right)$$

$$=: I_1 + I_2.$$

We first consider I_1 . Let $T_0 := \inf\{u \geq r : \xi_u \geq 0\}$. Since $\sigma_o - r \sim \mathcal{E}(\beta)$ under $\mathbb{P}_{r,x}$, we have

$$I_1 = \mathbf{P}_{r,x} \left(\sup_{r \le s \le t} \xi_s < 0 \right) \mathbb{P}_{r,x} (\sigma_o > t)$$
$$= \mathbf{P}_{r,x} (T_0 > t) e^{-\beta(t-r)} = \mathbf{P}_{r,x} \left(T_0 \ge t, e^{-\int_r^{T_0 \wedge t} \beta ds} \right).$$

By the branching property and the strong Markov property, we have

$$I_{2} = \mathbb{E}_{r,x} \left(\mathbb{P}_{r,x} \left(\sup_{r \leq s \leq t} R_{s} < 0, r \leq \sigma_{o} \leq t | \mathcal{F}_{\sigma_{o}} \right) \right)$$

$$= \mathbf{E}_{r,x} \left(\int_{r}^{t} \mathbf{1}_{\{T_{0} > u\}} \sum_{k=0}^{\infty} p_{k} \left(\mathbb{P}_{u,\xi_{u}} \left(\sup_{u \leq s \leq t} R_{s} < 0 \right) \right)^{k} \beta e^{-\beta(u-r)} du \right)$$

$$= \mathbf{E}_{r,x} \left(\int_{r}^{t \wedge T_{0}} f(\omega(u,\xi_{u})) \beta e^{-\beta(u-r)} du \right),$$

where $f(s) = \sum_{k=0}^{\infty} p_k s^k$. Consequently, $\omega(r, x)$ satisfies the equation

$$\omega(r,x) = \mathbf{P}_{r,x} \left(T_0 > t, e^{-\int_r^{T_0 \wedge t} \beta ds} \right) + \mathbf{E}_{r,x} \left(\int_r^{t \wedge T_0} f(\omega(u,\xi_u)) \beta e^{-\beta(u-r)} du \right).$$

Then, $1 - \omega(r, x)$ satisfies the following equation

$$1 - \omega(r, x) = \mathbf{P}_{r, x} \left(T_0 \le t, e^{-\int_r^{T_0 \wedge t} \beta ds} \right) + \mathbf{E}_{r, x} \left(\int_r^{t \wedge T_0} (1 - f(\omega(u, \xi_u))) \beta e^{-\beta(u-r)} du \right).$$

Using [21, Lemma 1.5], we get that $g(r,x) := 1 - \omega(r,x)$ also satisfies the following equation

$$g(r,x) = \mathbf{P}_{r,x} \left(T_0 \le t, e^{\int_r^{T_0 \wedge t} \lambda ds} \right) + \mathbf{E}_{r,x} \left(\int_r^{t \wedge T_0} (1 - f(\omega(u, \xi_u))) \beta e^{\lambda(u-r)} du \right)$$
$$- \mathbf{E}_{r,x} \left(\int_r^{t \wedge T_0} \mu \beta g(u, \xi_u) e^{\lambda(u-r)} du \right), \quad 0 \le r < t, x < 0.$$

Note that $1 - f(1 - g) - \mu g \le 0$. So we have that for $0 \le r < t$, and x < 0,

$$g(r,x) \le \mathbf{P}_{r,x} \left(T_0 \le t, e^{\int_r^{T_0 \wedge t} \lambda ds} \right) \le e^{\lambda t} \mathbf{P}_{r,x} (T_0 \le t).$$

Using the time-homogeneity and space-homogeneity of branching Lévy processes and Lévy processes, we get that for any t > 0 and x > 0,

$$\mathbb{P}\left(\sup_{0 \le s \le t} R_s \ge x\right) = 1 - \mathbb{P}\left(\sup_{0 \le s \le t} R_s < x\right) = g(0, -x)$$
$$\le e^{\lambda t} \mathbf{P}_{0, -x}(T_0 \le t) = e^{\lambda t} \mathbf{P}\left(\sup_{0 \le s \le t} \xi_s \ge x\right).$$

The proof is now complete.

Proof of Theorem 1.7: Since $\lim_{t\to\infty} \frac{G(t)}{h(t)} = \infty$, we have that $G(t) \stackrel{t}{\geq} e^{\frac{2\lambda}{3\alpha}t}$. By Lemma 4.5 with 3/2 , we get that

$$\limsup_{t \to \infty} \frac{R_n - M_n}{G(n)} \le 0, \quad \limsup_{t \to \infty} \frac{M_n - R_n^+}{G(n)} \le 0, \quad \mathbb{P}^* \text{-a.s.}$$
 (4.25)

(1) Assume that $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$. By Theorem 1.6, we have $\mathbb{P}^*(\exists T > 0, \forall t > T, R_t > 0) = 1$, which implies that

$$\limsup_{t \to \infty} \frac{R_t}{G(t)} = \limsup_{t \to \infty} \frac{R_t^+}{G(t)} \ge 0, \quad \mathbb{P}^*\text{-a.s.}.$$

Let $V_n := \sup_{n \le t \le (n+1)} R_t, n \ge 0$. Since G is non-decreasing, by Lemma 4.1 and (4.25), we have, under \mathbb{P}^* ,

$$\limsup_{t \to \infty} \frac{R_t}{G(t)} \le \limsup_{n \to \infty} \frac{V_n}{G(n)} \le \limsup_{n \to \infty} \frac{V_n - R_n}{G(n)} + \limsup_{n \to \infty} \frac{R_n - M_n}{G(n)} + \limsup_{n \to \infty} \frac{M_n}{G(n)}$$
$$\le \limsup_{n \to \infty} \frac{V_n - R_n}{G(n)}.$$

By the Borel-Cantelli lemma, to get the desired result, it suffices to prove that, for any c > 0,

$$\sum_{n=1}^{\infty} \mathbb{P}^*(V_n - R_n > cG(n)) < \infty. \tag{4.26}$$

Observe that on $\{Z_n > 0\}$, $R_n \in \mathbb{R}$ and

$$V_n = \max_{u \in \mathcal{L}_n} (\xi_n^u + V_n^u) \le R_n + \max_{u \in \mathcal{L}_n} V_n^u,$$

where

$$V_n^u = \sup_{n \le t \le n+1} \max_{v \in \mathcal{L}_t, u \le v} (\xi_t^v - \xi_n^u).$$

It is clear that, conditioned on \mathcal{F}_n , $\{V_n^u, u \in \mathcal{L}_n\}$ are i.i.d. with the same law as (V_0, \mathbb{P}) . Thus

$$(1-q)\mathbb{P}^*(V_n - R_n > cG(n)) \leq \mathbb{P}(V_n - R_n > cG(n), Z_n > 0) \leq \mathbb{P}(\max_{u \in \mathcal{L}_n} V_n^u > cG(n), Z_n > 0)$$

$$\leq \mathbb{E}(Z_n)\mathbb{P}(V_0 > cG(n)) = e^{\lambda n}\mathbb{P}(V_0 > cG(n))$$

$$\leq e^{\lambda n}e^{\lambda}\mathbf{P}(\sup_{0 \leq t \leq 1} \xi_t > cG(n)),$$

where the last inequality follows from Lemma 4.6. By (2.4), we have that

$$e^{\lambda n} \mathbf{P} \left(\sup_{0 \le t \le 1} \xi_t \ge cG(n) \right) \sim c^{-\alpha} \frac{q_1}{\alpha} e^{\lambda n} G(n)^{-\alpha} L(G(n)).$$

Thus (4.26) follows immediately.

(2) Assume that $\sum_{n} e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$. By Lemma 4.1 and (4.25), we have

$$\limsup_{t \to \infty} \frac{R_t^+}{G(t)} \ge \limsup_{n \to \infty} \frac{R_n^+}{G(n)} \ge \limsup_{n \to \infty} \frac{M_n}{G(n)} - \limsup_{n \to \infty} \frac{M_n - R_n^+}{G(n)} = \infty \quad \mathbb{P}^*\text{-a.s.},$$

which implies that $\limsup_{t\to\infty} \frac{R_t}{G(t)} = \infty$.

The proof is now complete.

As a consequence of Theorem 1.7 and Theorem 1.6, we have the following result.

Corollary 4.7. (1)

$$\limsup_{t \to \infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log \log t} = \frac{1}{\alpha}, \qquad \mathbb{P}^*\text{-a.s.}.$$
 (4.27)

(2)

$$\lim_{t \to \infty} \frac{\log R_t^+}{t} = \frac{\lambda}{\alpha}, \qquad \mathbb{P}^* \text{-a.s.}$$

Proof: (1) For any $\epsilon > 0$, by Theorem 1.7 (1) with $G(t) = H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}$,

$$\limsup_{t \to \infty} \frac{R_t}{H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}} = 0, \quad \mathbb{P}^*\text{-a.s.}.$$

which implies that for any $\delta > 0$, $\frac{R_t^+}{H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}} \leq \delta$ when t is large enough. Thus for t large enough,

$$\log R_t^+ - \log H(t^{-1}e^{-\lambda t}) \le \log \delta + \frac{1+\epsilon}{\alpha} \log \log t, \quad \mathbb{P}^*$$
-a.s..

Therefore

$$\limsup_{t \to \infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log \log t} \le \frac{1+\epsilon}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}.$$

Letting $\epsilon \to 0$, we get

$$\limsup_{t \to \infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log \log t} \le \frac{1}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}.$$

For any $\epsilon \in (0,1)$, by Theorem 1.7 (2) with $G(t) = H(t^{-1}e^{-\lambda t})(\log t)^{(1-\epsilon)/\alpha}$, we have for any $\delta > 0$, $\limsup_{t \to \infty} \frac{R_t^+}{H(t^{-1}e^{-\lambda t})(\log t)^{(1-\epsilon)/\alpha}} \ge \delta$, which implies that

$$\limsup_{t \to \infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log \log t} \ge \limsup_{t \to \infty} \frac{\log \delta}{\log \log t} + \frac{1 - \epsilon}{\alpha} = \frac{1 - \epsilon}{\alpha}.$$

Therefore letting $\epsilon \to 0$, we get

$$\limsup_{t \to \infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log \log t} \ge \frac{1}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}.$$

Hence we have (4.27).

(2) Note that

$$\log H(t^{-1}e^{-\lambda t}) = \frac{\lambda}{\alpha}t + \frac{1}{\alpha}\log t + \log \bar{L}(te^{\lambda t}) \sim \frac{\lambda}{\alpha}t.$$

By (4.27), we have

$$\limsup_{t \to \infty} \frac{\log R_t^+}{t} = \frac{\lambda}{\alpha}, \qquad \mathbb{P}^* \text{-a.s.}$$

On the other hand, by Theorem 1.6, we have

$$\liminf_{t \to \infty} \frac{\log R_t^+}{t} - \frac{\log H(e^{-\lambda t} \log t)}{t} = 0, \qquad \mathbb{P}^*\text{-a.s.}$$

Note that

$$\log H(e^{-\lambda t} \log t) = \frac{\lambda}{\alpha} t - \frac{1}{\alpha} \log \log t + \log \bar{L}(e^{\lambda t} / \log t) \sim \frac{\lambda}{\alpha} t.$$

The proof is now complete.

Remark 4.8. Since $H(y) = y^{-1/\alpha} \bar{L}(y^{-1})$, we have

$$\lim_{y \to 0} \frac{\log H(y)}{\log y^{-1}} = 1/\alpha. \tag{4.28}$$

Assume that $L(x) = (\log x)^r$, where $r \in \mathbb{R}$. By (4.28), we have that

$$H(y) = y^{-1/\alpha} (\log H(y))^{r/\alpha} \sim \alpha^{-r/\alpha} y^{-1/\alpha} (\log(y^{-1}))^{r/\alpha}, \quad y \to 0.$$

Hence we have

$$\log H(t^{-1}e^{-\lambda t}) - \frac{\lambda}{\alpha}t - \frac{1+r}{\alpha}\log t \to \frac{r}{\alpha}\log(\lambda/\alpha), \quad t \to \infty,$$

and

$$H(e^{-\lambda t} \log t) \sim (\lambda/\alpha)^{r/\alpha} t^{r/\alpha} (\log t)^{-1/\alpha} e^{\lambda t/\alpha}, \quad t \to \infty.$$

5 Appendix

In this section, we give some further discussion of **Assumption 2** We first recall Karamata's theorem ([11, Theorem 1.5.11]).

Lemma 5.1. (1) If L is slowly varying at ∞ and locally bounded in $[a, +\infty)$ for some a > 0, then for r > -1,

$$\int_{a}^{x} t^{r} L(t)dt \sim \frac{1}{r+1} x^{r+1} L(x), \quad x \to +\infty.$$

(2) If L is slowly varying at infinity, then for r > 1,

$$\int_{r}^{+\infty} t^{-r} L(t)dt \sim \frac{1}{r-1} x^{1-r} L(x), \quad x \to +\infty.$$

Example 5.2. (Strictly Stable process.) Let ξ be a strictly α -stable process, $\alpha \in (0, 2)$, on \mathbb{R} with Lévy measure

$$n(dy) = c_1 x^{-(1+\alpha)} \mathbf{1}_{(0,\infty)}(x) dx + c_2 |x|^{-(1+\alpha)} \mathbf{1}_{(-\infty,0)}(x) dx,$$

where $c_1, c_2 \ge 0$, $c_1 + c_2 \ge 0$, and if $\alpha = 1$, $c_1 = c_2 = c$. The Lévy exponent of ξ is given by, for $\theta > 0$,

$$\psi(\theta) = \begin{cases} \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y), & \alpha \in (1,2); \\ \int_{\mathbb{R}} (e^{i\theta y} - 1) n(\mathrm{d}y), & \alpha \in (0,1); \\ \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y| \le 1}) n(\mathrm{d}y) + ia\theta, & \alpha = 1 \end{cases}$$
$$= \begin{cases} -\alpha \Gamma(1 - \alpha) (c_1 e^{-i\pi\alpha/2} + c_2 e^{i\pi\alpha/2}) \theta^{\alpha}, & \alpha \in (1,2); \\ -\alpha \Gamma(1 - \alpha) (c_1 e^{-i\pi\alpha/2} + c_2 e^{i\pi\alpha/2}) \theta^{\alpha}, & \alpha \in (0,1); \\ -c\pi\theta + ia\theta, & \alpha = 1, \end{cases}$$

where $a \in \mathbb{R}$ is a constant. It is clear that ψ satisfies **Assumption 2**. For more details on stable processes, we refer the readers to [34, Section 14].

Note that for $\alpha \neq 1$, $e^{-c|\theta|^{\alpha}}$, $\theta \in \mathbb{R}$, is the characteristic function of a strictly α -stable random variable if and only if $|\tan(\pi\alpha/2)|\Re(c) > |\Im(c)|$.

Let $\{(\xi_t)_{t\geq 0}\}$ be a Lévy process with the generating triplet (a,b,n), that is,

$$\psi(\theta) = \log E(e^{i\theta\xi_1})$$

= $ia\theta - \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}/\{0\}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|y|<1\}}) n(dy),$

where $a \in \mathbb{R}$, $b \ge 0$, and $\int_{\mathbb{R}/\{0\}} (1 \wedge y^2) n(dy) < \infty$. Let $r(x) : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function, satisfying

$$r(x) = \begin{cases} 1 + o(x), & |x| \to 0; \\ O(\frac{1}{|x|}), & |x| \to \infty. \end{cases}$$
 (5.1)

Then ψ can be rewritten as

$$\psi(\theta) = ia_r \theta - \frac{1}{2}b^2 \theta^2 + \int_{\mathbb{R}/\{0\}} (e^{i\theta y} - 1 - i\theta y r(y)) n(dy),$$

where $a_r = a + \int_{\mathbb{R}/\{0\}} y(r(y) - \mathbf{1}_{\{|y|<1\}}) n(dy)$. The triplet of ξ is written as $(a_r, b, n)_r$. For any t > 0, we define a measure $n_t(dy)$ as follows: for any positive function g,

$$\int_{\mathbb{R}/\{0\}} g(y) n_t(dy) = \int_{\mathbb{R}/\{0\}} g(yt^{-1}) n(dy).$$

Lemma 5.3. If there exist $q_1, q_2 \ge 0$ such that

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} n(t, +\infty) = \frac{q_1}{\alpha}, \text{ and } \lim_{t \to +\infty} t^{\alpha} L(t)^{-1} n(-\infty, -t) = \frac{q_2}{\alpha}, \tag{5.2}$$

then as $\theta \to 0^+$,

$$\theta^{-\alpha} L(\theta^{-1}) \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|\theta y| < 1\}}) n(dy) \to \int_{\mathbb{R}} (e^{iy} - 1 - iy \mathbf{1}_{\{|y| < 1\}}) \nu_{\alpha}(dy),$$

where $\nu_{\alpha}(dx) := q_1 x^{-1-\alpha} \mathbf{1}_{(0,+\infty)}(x) dx + q_2 |x|^{-1-\alpha} \mathbf{1}_{(-\infty,0)}(x) dx$. Furthermore, if $0 < \alpha < 1$, then as $\theta \to 0^+$,

$$\theta^{-\alpha} L(\theta^{-1}) \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|y| < 1\}}) n(dy) \to -\alpha \Gamma(1 - \alpha) (q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2}); \quad (5.3)$$

if $1 < \alpha < 2$, then as $\theta \to 0^+$,

$$\theta^{-\alpha}L(\theta^{-1})\int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y)n(dy) \to -\alpha\Gamma(1 - \alpha)(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2}). \tag{5.4}$$

Proof: For any t > 0, let $\tilde{n}_t(dy) := t^{\alpha}L(t)^{-1}n_t(dy)$. By (5.2), we have that for any x > 0,

$$\tilde{n}_t(x,\infty) = t^{\alpha}L(t)^{-1}n(tx,\infty) \to \frac{q_1}{\alpha}x^{-\alpha} = \nu_{\alpha}(x,\infty)$$

and

$$\tilde{n}_t(-\infty, -x) = t^{\alpha}L(t)^{-1}n(-\infty, -tx) \to \frac{q_2}{\alpha}x^{-\alpha} = \nu_{\alpha}(-\infty, -x).$$

Thus, for any $g \in C_b^0(\mathbb{R})$,

$$\int_0^\infty g(y)\tilde{n}_t(dy) \to \int_0^\infty g(y)\nu_\alpha(dy). \tag{5.5}$$

We claim that

$$\lim_{\varepsilon \downarrow 0} \limsup_{t \to +\infty} \int_{0 < |y| < \varepsilon} y^2 \widetilde{n}_t(dy) = 0.$$
 (5.6)

In fact, by Fubini's theorem, we have that

$$\int_{0 < y < \varepsilon} y^2 \widetilde{n}_t(dy) = t^{\alpha} L(t)^{-1} \int_0^{\varepsilon} y^2 n_t(dy) = t^{\alpha} L(t)^{-1} \int_0^{\varepsilon} \left(\int_0^y 2x dx \right) n_t(dy)$$

$$= t^{\alpha} L(t)^{-1} \int_0^{\varepsilon} 2x dx \int_x^{\varepsilon} n_t(dy) \le t^{\alpha} L(t)^{-1} \int_0^{\varepsilon} 2x n_t(x, \infty) dx$$

$$= t^{\alpha - 2} L(t)^{-1} \int_0^{t\varepsilon} 2x n(x, \infty) dx.$$

Note that

$$t^{\alpha-2}L(t)^{-1} \int_0^1 2x n(x,\infty) dx = t^{\alpha-2}L(t)^{-1} \int (1 \wedge y^2) n(dy) \to 0, \quad t \to \infty.$$

By (5.2) and Karamata's theorem, we have, as $t \to \infty$,

$$\int_{1}^{t\epsilon} x n(x,\infty) dx \sim \frac{q_1}{\alpha} \frac{(t\epsilon)^{2-\alpha} L(t)}{2-\alpha}.$$

Thus we have

$$\limsup_{t \to +\infty} \int_0^{\varepsilon} y^2 \widetilde{n}_t(dy) \le \frac{q_1}{\alpha} \frac{\varepsilon^{2-\alpha}}{2-\alpha}.$$

Similarly, we have

$$\limsup_{t \to +\infty} \int_{-\varepsilon}^{0} y^{2} \widetilde{n}_{t}(dy) \leq \frac{q_{2}}{\alpha} \frac{\varepsilon^{2-\alpha}}{2-\alpha}.$$

Letting $\epsilon \to 0$, we get (5.6).

By (5.5) and (5.6), and applying [34, Theorem 8.7], we obtain that, for any bounded continuous function r satisfying (5.1),

$$\lim_{t \to +\infty} \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y r(y)) \widetilde{n}_t(dy) = \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y r(y)) \nu_{\alpha}(dy). \tag{5.7}$$

Since $v_{\alpha}(\{y\}) = 0$ for all $y \in \mathbb{R}$, thus (5.7) holds for $r(y) = \mathbf{1}_{\{|y| \leq 1\}}$. Thus we have that

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} \int_{\mathbb{R}} (e^{it^{-1}y} - 1 - it^{-1}y \mathbf{1}_{\{|y| < t\}}) n(dy) = \int_{\mathbb{R}} (e^{iy} - 1 - i \ y \mathbf{1}_{\{|y| < 1\}}) \nu_{\alpha}(dy). (5.8)$$

(1) Now we assume that $0 < \alpha < 1$. Note that for t > 1,

$$\int_{1}^{t} y n(dy) = \int_{1}^{t} \left(\int_{0}^{y} dx \right) n(dy) = \int_{0}^{1} dx \int_{1}^{t} n(dy) + \int_{1}^{t} dx \int_{x}^{t} n(dy)$$
$$= n(1, t) + \int_{1}^{t} n(x, t) dx$$
$$= n(1, t) + \int_{1}^{t} n(x, +\infty) dx - (t - 1)n(t, +\infty).$$

By Lemma 5.1, we have that $\int_1^t n(x,+\infty)dx \sim \frac{q_1}{\alpha(1-\alpha)}t^{1-\alpha}L(t)$. Thus we have

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} t^{-1} \int_{1}^{t} y n(dy) = \frac{1}{1-\alpha} \frac{q_1}{\alpha} - \frac{q_1}{\alpha} = \int_{0}^{1} y \nu_{\alpha}(dy). \tag{5.9}$$

Similarly, we have

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} t^{-1} \int_{-t}^{-1} y n(dy) = \frac{1}{1 - \alpha} \frac{q_2}{\alpha} - \frac{q_2}{\alpha} = \int_{-1}^{0} y \nu_{\alpha}(dy).$$
 (5.10)

By (5.8), (5.9) and (5.10), we obtain that (5.3) holds.

(2) Now we consider the case when $\alpha \in (1,2)$. By Fubini's theorem and lemma 5.1, we have that

$$\begin{split} \int_t^{+\infty} y n(dy) &= \int_t^{+\infty} \left(\int_0^y dx \right) n(dy) = \int_0^t dx \int_t^{\infty} n(dy) + \int_t^{+\infty} dx \int_x^{+\infty} n(dy) \\ &= t n(t, +\infty) + \int_t^{+\infty} n(x, +\infty) dx \\ &\sim \frac{q_1}{\alpha} t^{\alpha-1} L(t) + \frac{q_1}{\alpha(\alpha-1)} t^{\alpha-1} L(t) = \frac{q_1}{\alpha-1} t^{\alpha-1} L(t). \end{split}$$

Thus

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} \int_{t}^{+\infty} t^{-1} y n(dy) = \int_{1}^{+\infty} y \nu_{\alpha}(dy).$$
 (5.11)

Similarly, we have

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} \int_{-\infty}^{-t} t^{-1} y n(dy) = \int_{-\infty}^{-1} y \nu_{\alpha}(dy).$$
 (5.12)

Thus by (5.8), (5.11) and (5.12), we obtain that (5.4) holds.

The proof is now complete.

Theorem 5.4. There exist a constant c_* , $\alpha \in (0,2)$ and a function L slowly varying at ∞ such that

$$\psi(\theta) \sim -c_* \theta^{\alpha} L(\theta^{-1}), \theta \to 0^+, \tag{5.13}$$

if and only if the following two conditions hold:

(1) there exists nonnegative numbers q_1, q_2 such that

$$\lim_{t\to +\infty}t^{\alpha}L(t)^{-1}n(t,+\infty)=\frac{q_1}{\alpha},\quad and\lim_{t\to +\infty}t^{\alpha}L(t)^{-1}n(-\infty,-t)=\frac{q_2}{\alpha};$$

(2) if $\alpha \in (1,2)$, $a + \int_{\{|y|>1\}} yn(dy) = 0$; if $\alpha = 1$, $q_1 = q_2$ and

$$\lim_{t \to \infty} L(t)^{-1} \left(a + \int_{1 < |y| < t} y n(dy) \right) =: c_0$$

exists.

Moreover, the relationship between c_* and (q_1, q_2, c_0) is as follows: if $\alpha \neq 1$,

$$c_* = \alpha \Gamma(1 - \alpha) \left(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2} \right),$$

and if $\alpha = 1$, $c_0 = -\Im(c_*)$ and $q_1 = q_2 = \Re(c_*)/\pi$.

Proof: First, we prove the sufficiency. Assume the two conditions hold. If $\alpha \in (0,1)$, then by Lemma 5.3, we have that as $\theta \to 0^+$,

$$\theta^{-\alpha}L(\theta^{-1})^{-1}\psi(\theta) \to -\alpha\Gamma(1-\alpha)\left(q_1e^{-i\pi\alpha/2} + q_2e^{i\pi\alpha/2}\right).$$

If $\alpha \in (1,2)$, then as $\theta \to 0^+$,

$$\psi(\theta) = -\frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y)n(dy) \sim -\alpha\Gamma(1 - \alpha) \left(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2}\right) \theta^{\alpha} L(\theta^{-1}).$$

If $\alpha = 1$, then as $\theta \to 0^+$,

$$\psi(\theta) = i \left(a + \int_{1 < |y| < \theta^{-1}} y n(dy) \right) \theta - \frac{1}{2} b^2 \theta^2 + \int_{\mathbb{R}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|y| \le \theta^{-1}\}}) n(dy)$$
$$\sim (ic_0 - q_1 \pi) \theta L(\theta^{-1}).$$

Now we assume that (5.13) holds. It is clear that (5.13) is equivalent

$$\lim_{t \to +\infty} e^{t^{\alpha}L(t)^{-1}\psi(\theta t^{-1})} = e^{\widetilde{\psi}(\theta)}, \quad \theta > 0, \tag{5.14}$$

where $\widetilde{\psi}(\theta) = -c_*\theta^{\alpha}$.

Note that the left side of (5.14) is the characteristic function of an infinitely divisible random variable Y_t with Lévy measure $\tilde{n}_t(dy)$. By [34, Theorem 8.7 (1)], if (5.14) holds, then $e^{\tilde{\psi}(\theta)}$ is the characteristic function of an infinitely divisible random variable. By the expression of $\tilde{\psi}(\theta)$, $e^{\tilde{\psi}(\theta)}$ must be the characteristic function of a strictly α -stable random variable Y. Thus if $\alpha \neq 1$, then $|\tan(\pi\alpha/2)|\Re(c_*) \geq |\Im(c_*)|$. Consequently, the Lévy measure of Y is given by

$$\nu_{\alpha}(dx) := q_1 x^{-1-\alpha} \mathbf{1}_{(0,+\infty)}(x) dx + q_2 |x|^{-1-\alpha} \mathbf{1}_{(-\infty,0)}(x) dx,$$

where $q_1 \geq 0$ and $q_2 \geq 0$ satisfy the following equation: if $\alpha \neq 1$,

$$c_* = \alpha \Gamma(1 - \alpha) \left(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2} \right),$$

and if $\alpha = 1$, $q_1 = q_2 = \Re(c_*)/\pi$.

By [34, Theorem 8.7 (1)], we get that for any $g \in C_b^0(\mathbb{R})$,

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} \int_{\mathbb{R}/\{0\}} g(t^{-1}y) n(dy) = \int_{\mathbb{R}/\{0\}} g(y) \nu_{\alpha}(dy).$$
 (5.15)

Because $\nu_{\alpha}(\{x\}) = 0, \forall x \in \mathbb{R}$, then (5.15) holds for $g(x) = \mathbf{1}_{(1,\infty)}(x)$ and $g(x) = \mathbf{1}_{(-\infty,-1)}(x)$, i.e.,

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} n(t, +\infty) = \nu_{\alpha}(1, +\infty) = \frac{q_1}{\alpha};$$

$$\lim_{t \to +\infty} t^{\alpha} L(t)^{-1} n(-\infty, -t) = \nu_{\alpha}(-\infty, -1) = \frac{q_2}{\alpha}.$$

Now using Lemma 5.3, we can get the second condition holds.

The proof is now complete.

Remark 5.5. For $\alpha = 1$, we can assume that Lévy measure n is symmetric and a = 0.

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