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Large deviations and almost sure convergence for the extremes of branching Lévy processes with regularly varying tails*

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Abstract

In this paper, we study the asymptotic behaviors of supercritical branching Markov processes $\{\mathbb{X}_t, t \geq 0\}$ with spatial motions being Lévy processes having regularly varying tails. Recently, Ren et al. [J. Appl. Probab. 61 (2024)] proved that, when renormalized by some appropriate function $h(t)$, $\{\mathbb{X}_t, t \geq 0\}$ converges weakly to some random measure. In this paper, we first establish the large deviations of $\{\mathbb{X}_t, t \geq 0\}$. More precisely, we prove that, (i) when renormalized by a function $\Lambda(t)$ which grows faster than $h(t)$, $\{\mathbb{X}_t, t \geq 0\}$, conditioned on $\{R_t > \Lambda(t)\}$, converges weakly; (ii) when renormalized by a function $\Lambda(t)$ which grows slower than $h(t)$, $\{\mathbb{X}_t, t \geq 0\}$, conditioned on $\{R_t < \Lambda(t)\}$, converges weakly, where R_t is the maximum of $\mathbb{X}_t$. We also characterize the almost sure envelopes of R_t.

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1 Introduction

1.1 Model and notation

A branching Lévy process on $\mathbb{R}$ is a continuous-time particle system which can be described as follows. The system begins at time $t = 0$ with a single particle located at x which moves according to a Lévy process ξ. After an independent exponential time with parameter β, the initial particle dies and gives birth to k new particles with probability p_k, $k \geq 0$. Each new particle moves according to the Lévy process ξ starting from the position of its parent's death, and branches independently with the same branching rate β and offspring distribution $\{p_k, k \geq 0\}$. All particles, once born, evolve independently of one another. We use $\mathbf{P}_x$ to denote law of ξ starting from x and use $\mathbf{E}_x$ to denote expectation with respect to $\mathbf{P}_x$. We write $\mathbf{P} := \mathbf{P}_0$ and $\mathbf{E} := \mathbf{E}_0$.

We label each particle using the classical Ulam-Harris labeling system. We denote by $\mathbb{T}$ the set of all particles in the tree and use o represent the root of the tree. For any $u \in \mathbb{T}$, let N^u be the number of the offspring of u and τ_u denote the lifetime of u. Then $\{\tau_u, u \in \mathbb{T}\}$ are i.i.d. exponential random variables of parameter β.

We will use the notation $u < v$ to mean that u is an ancestor of v, and $u \leq v$ that either $u < v$ or $u = v$. For any $u \in \mathbb{T}$, let $I_u^0 := \{v \in \mathbb{T} : v < u\}$ and $I_u := \{v \in \mathbb{T} : v \leq u\}$. Let b_u and σ_u be the birth time and death time of u respectively. It is clear that

$$b_u = \sum_{v \in I_u^0} \tau_v, \qquad \sigma_u = b_u + \tau_u.$$

Let $\mathcal{L}_t$ be the set of all particles alive at time t. Then $u \in \mathcal{L}_t$ means $b_u \leq t < \sigma_u$. For $u \in \mathbb{T}$ with $b_u \leq t$, define $\tau_{u,t} = \sigma_u \wedge t - b_u$. For any $u \in \mathcal{L}_t$, let ξ_t^u be the position of u at time t. The measure-valued process defined by

$$\mathbb{X}_t := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u}$$

is called a branching Lévy process. For any $a > 0$, we define

$$\mathbb{X}_t^{(a)} := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u / a}. \tag{1.1}$$

We use $\mathbb{P}_x$ to denote the law of $\{\mathbb{X}_t : t \geq 0\}$ when the initial particle starts at position x. The expectation with respect to $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x$. We write $\mathbb{P} := \mathbb{P}_0$ and $\mathbb{E} := \mathbb{E}_0$. Let $\{\mathcal{F}_t\}$ be the natural filtration of $\mathbb{X}$ and

$$\mathcal{F}_t^{\mathbb{T}} := \sigma(\{N^u : u \in \mathbb{T} \text{ with } \sigma_u \leq t\} \cup \{\tau_{u,t} : u \in \mathbb{T} \text{ with } b_u \leq t\}).$$

In this paper, we assume the branching Lévy process $\mathbb{X}$ is supercritical, that is, $\mu := \sum_k k p_k > 1$. Let $\mathcal{S}$ be survival event. Then $\mathbb{P}_x(\mathcal{S}) > 0$ does not depend on the location x of the initial particle. The extinction probability $q := \mathbb{P}(\mathcal{S}^c)$ is the unique root in the interval $[0, 1)$ of the equation $f(s) = s$, where $f(s) := \sum_k p_k s^k$. For more details, see [5, Section III. 4]. For any $x \in \mathbb{R}$, we define $\mathbb{P}_x^*(\cdot) := \mathbb{P}_x(\cdot|\mathcal{S})$ and denote the corresponding expectation by $\mathbb{E}_x^*$. We write $\mathbb{P}^* := \mathbb{P}_0^*$ and $\mathbb{E}^* := \mathbb{E}_0^*$.

Let Z_t be the total number of the particles alive at time t. It is well known that $\{Z_t : t \geq 0\}$ is a continuous time Galton-Watson process. The following two quantities will play important roles in this paper:

$$\lambda := \beta(\mu - 1), \quad \vartheta := \int_0^\infty e^{-\lambda r} \mathbb{P}(Z_r > 0) \mathrm{d}r. \tag{1.2}$$

By the Markov property and the branching property, the process $\{e^{-\lambda t}Z_t, t \geq 0\}$ is a non-negative martingale with respect to $\{\mathcal{F}_t^{\mathbb{T}}\}$. Thus it has an almost sure limit

$$\lim_{t\to\infty} e^{-\lambda t}Z_t =: W. \tag{1.3}$$

It is well known that W is non-degenerate if and only if the following condition holds:

Assumption 1. $\sum_{k=1}^{\infty}(k\log kp_k) < \infty$.

Moreover, under **Assumption 1**, $\mathbb{P}(W > 0) = \mathbb{P}(\mathcal{S}) > 0$. For more details, see [5, Section III.7].

To state our results, we first introduce some notation. Put $\overline{\mathbb{R}}_0 = [-\infty, \infty] \setminus \{0\}$ with the topology generated by the set

$$\{(a, b), (-b, -a), (a, \infty], [-\infty, -a) : 0 < a < b \leq \infty\}.$$

Note that every compact subset $B \subset \overline{\mathbb{R}}_0$ is contained in $[-\infty, -\delta] \cup [\delta, \infty]$ for some $\delta > 0$. Let $C_c^+(\overline{\mathbb{R}}_0)$ be the family of all non-negative continuous functions on $\overline{R}_0$ with compact support, and $\mathcal{B}_b^+(\overline{\mathbb{R}}_0)$ denote the family of all non-negative, bounded Borel functions on $\overline{\mathbb{R}}_0$. Denote by $\mathcal{M}(\overline{\mathbb{R}}_0)$ the space of all Radon measures on $\overline{\mathbb{R}}_0$ endowed with the topology of vague convergence (denoted by $\xrightarrow{v}$), generated by the maps $\nu \to \int f\mathrm{d}\nu$ for all $f \in C_c^+(\overline{\mathbb{R}}_0)$. For any $g \in \mathcal{B}_b^+(\overline{\mathbb{R}}_0)$, $\nu \in \mathcal{M}(\overline{\mathbb{R}}_0)$, we write $\nu(g) := \int_{\overline{\mathbb{R}}_0} g(x)\nu(\mathrm{d}x)$. A sequence of random elements ν_n in $\mathcal{M}(\overline{\mathbb{R}}_0)$ converges weakly to ν, denoted as $\nu_n \xrightarrow{d} \nu$, if and only if for all $g \in C_c^+(\overline{\mathbb{R}}_0)$, $\nu_n(g)$ converges weakly to $\nu(g)$ (see [37, Proposition 3.19] or [35, Theorem 4.11], for instance.)

Recently, the extremes of branching Lévy processes defined by

$$R_t := \max_{u\in\mathcal{L}_t} \xi_t^u, \quad t > 0,$$

where we use the usual convention that $\max\emptyset = -\infty$, have been studied intensively. Among these, branching Brownian motions garnered the most attention. For branching Brownian motions, Bramson [12] (see also [13]) proved that, under some moment conditions on the offspring distribution, $\mathbb{P}(R_t - m(t) \leq x) \to 1 - w(x)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$ and $w(x)$ is a traveling wave solution. For the large deviation of R_t, Chauvin and Rouault [15, 16] studied the asymptotic behavior of $\mathbb{P}(R_t > \sqrt{2}\delta t)$ for $\delta \geq 1$. Derrida and Shi [19, 20] studied the lower large deviation of R_t, i.e., the asymptotic behavior of $\frac{1}{t}\log\mathbb{P}(R_t \leq \sqrt{2}\delta t)$ for $\delta < 1$, and found that the rate function has a phase transition at $1 - \sqrt{2}$. Subsequently, Chen, He and Mallein [17] studied the asymptotic behavior of $\mathbb{P}(R_t \leq \sqrt{2}\delta t)$ for $\delta < 1$. Recently, [2, 3, 4] studied the extremal processes of branching Brownian motions. For branching random walks, several authors have studied the convergence of the extremes under an exponential moment assumption on the displacements of the offspring from the parent, see Aïdékon [1], Hu and Shi [32], and Madaule [36].

In this paper we study the case when the spatial motion is a Lévy process with regularly varying Lévy measures. Now we state our assumption on the spatial motion ξ precisely. It is well-known that the characteristic exponent ψ has the following form:

$$\psi(\theta) = ia\theta - \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta y} - 1 - i\theta y\mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y),$$

where $a \in \mathbb{R}, b \geq 0$, and the Lévy measure $\mathbf{n}$ satisfying $\int_{\mathbb{R}/\{0\}}(1 \wedge y^2)\mathbf{n}(\mathrm{d}y) < \infty$. We will assume that the Lévy measure $\mathbf{n}$ satisfies

Assumption 2.

(i) There exist $\alpha > 0$, $q_1 > 0$, $q_2 \geq 0$ and a positive function $L(x) : \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that

$$\lim_{x\to+\infty} x^\alpha L(x)^{-1}\mathbf{n}(x,+\infty) = \frac{q_1}{\alpha}, \text{ and } \lim_{x\to+\infty} x^\alpha L(x)^{-1}\mathbf{n}(-\infty,-x) = \frac{q_2}{\alpha}.$$

(ii) if $\alpha > 1$, we assume that $\mathbf{E}(\xi_1) = 0$.

Remark 1.1. For $\alpha > 1$, if $a' := \mathbf{E}(\xi_1) \neq 0$, all our results remain valid if we replace $\mathbb{X}_t$ with the process $\mathbb{X}'_t = \sum_{u\in\mathcal{L}_t} \delta_{\xi^u_t - a't}$. We can translate results for $\mathbb{X}'_t$ to that of $\mathbb{X}_t$.

In Lemma 2.2 below, we will show that **Assumption 2** (i) implies that, for any $s > 0$, as $x \to \infty$, $\frac{\mathbf{P}(x^{-1}\xi_s\in\cdot)}{x^{-\alpha}L(x)} \xrightarrow{v} s\nu_\alpha(\cdot)$ on $\overline{\mathbb{R}}_0$, where

$$\nu_\alpha(\mathrm{d}x) = q_1 x^{-1-\alpha}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + q_2|x|^{-1-\alpha}\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x, \tag{1.4}$$

which is equivalent to ξ_s having regularly varying tails for each $s > 0$, i.e.,

$$\mathbf{P}(\xi_s \geq x) \sim \frac{q_1}{\alpha} s x^{-\alpha} L(x), \quad \mathbf{P}(\xi_s \leq -x) \sim \frac{q_2}{\alpha} s x^{-\alpha} L(x), \quad x \to \infty.$$

It is well known (see, [37, Proposition 0.8]) that there exists a continuous function $\tilde{L} : [0,\infty) \to (0,\infty)$ slowly varying at ∞ such that $\lim_{x\to\infty} \frac{\tilde{L}(x)}{L(x)} = 1$, and $x^{-\alpha}\tilde{L}(x)$ is strictly decreasing on $(0,\infty)$. In this paper, we always assume that L satisfies this property. Note that $\lim_{x\to 0} x^{-\alpha}L(x) = \infty$ and $\lim_{x\to\infty} x^{-\alpha}L(x) = 0$. Let $H(y) : (0,\infty) \to (0,\infty)$ be the inverse function of $x^{-\alpha}L(x)$. Then

$$H(y)^{-\alpha}L(H(y)) = y, \quad y > 0. \tag{1.5}$$

It is well known (see, [11, Theorem 1.5.12] for instance) that

$$H(y) = y^{-1/\alpha}\bar{L}(y^{-1}),$$

with $\bar{L}$ being slowly varying at ∞. From now on, $\bar{L}$ always stands for the function above.

Let $h(t) := H(e^{-\lambda t})$, that is,

$$h(t)^{-\alpha}L(h(t)) = e^{-\lambda t}. \tag{1.6}$$

h is strictly increasing. Recall the definition (1.1). In [39], we proved that $\mathbb{X}_t^{(h(t))}$ converges weakly to a random measure $\mathcal{N}_\infty$ under the following stronger condition:
(H) there exist $\alpha \in (0,2)$, a complex constant c_* with $\Re(c_*) > 0$ and a function $L(x) : \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that $\psi(\theta) \sim -c_*\theta^\alpha L(\theta^{-1})$ as $\theta \to 0_+$.

In this paper, we will show that the results in [39] still hold with **(H)** replaced by **Assumption 2**. Let $\mathcal{H}(\mathbb{R})$ denote the family of uniformly continuous functions $\varphi : \mathbb{R} \to [0,1]$ with $\varphi \equiv 1$ in some neighborhood of 0. Note that if $g \in C_c^+(\overline{\mathbb{R}}_0)$, then $e^{-g} \in \mathcal{H}(\mathbb{R})$.

Theorem 1.2. *Assume that **Assumption 1** and **Assumption 2** hold. Then $\mathbb{X}_t^{(h(t))}$ converges weakly to a random measure $\mathcal{N}_\infty$. More precisely, for any $g \in C_c^+(\overline{\mathbb{R}}_0)$,*

$$\lim_{t\to\infty} \mathbb{E}(e^{-\mathbb{X}_t^{(h(t))}(g)}) = E(e^{-\mathcal{N}_\infty(g)}) = \mathbb{E}\left(\exp\left\{-C(e^{-g})W\right\}\right),$$

where

$$C(\varphi) := \int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0} \mathbb{E}(1-\varphi(x)^{Z_r})\nu_\alpha(dx)dr, \quad \forall\varphi \in \mathcal{H}(\mathbb{R}). \tag{1.7}$$

Moreover, $\mathcal{N}_\infty = \sum_j T_j \delta_{e_j}$, *where given* W, $\sum_j \delta_{e_j}$ *is a Poisson random measure with intensity* $\vartheta W \nu_\alpha(dx)$, $\{T_j, j \geq 1\}$ *are i.i.d. copies of a random variable* T *with*

$$P(T = k) = \vartheta^{-1} \int_0^\infty e^{-\lambda r} \mathbb{P}(Z_r = k) dr, \quad k \geq 1, \tag{1.8}$$

where $\nu_\alpha(dx)$ *is given by* (1.4), ϑ *is defined in* (1.2), *and* $\sum_j \delta_{e_j}$ *and* $\{T_j, j \geq 1\}$ *are independent.*

As a consequence, we have that under $\mathbb{P}^*$, $\frac{R_t}{h(t)}$ converges weakly. More precisely,

$$\lim_{t\to\infty} \mathbb{P}^* \left(\frac{R_t}{h(t)} \leq x \right) = \begin{cases} \mathbb{E}^* \left(e^{-\vartheta^* W x^{-\alpha}} \right), & x > 0; \\ 0, & x \leq 0, \end{cases} \tag{1.9}$$

where

$$\vartheta^* := \frac{q_1}{\alpha} \vartheta. \tag{1.10}$$

Now consider a function $\Lambda(t)$ which grows faster than the function $h(t)$ in the sense that $\lim_{t\to\infty} \Lambda(t)/h(t) = \infty$. It is clear that $R_t/\Lambda(t) \to 0$ in probability. In this paper, we will find the rate that $\mathbb{P}(R_t > \Lambda(t))$ converges to 0 as $t \to \infty$, and study the weak convergence of $\mathbb{X}_t^{(\Lambda(t))}$, conditioned on $\{R_t > \Lambda(t)\}$, as $t \to \infty$. When $\Lambda(t)$ grows slower than $h(t)$ in the sense that $\lim_{t\to\infty} \Lambda(t)/h(t) = 0$, we study the convergence rate of the probability $\mathbb{P}^*(R_t \leq \Lambda(t))$ and the weak convergence of $\mathbb{X}_t^{(\Lambda(t))}$.

1.2 Main results

In this subsection, we state our main results.

1.2.1 Upper large deviations

Theorem 1.3. *Assume that* ***Assumption 1*** *and* ***Assumption 2*** *hold. If* $\Lambda : [0, \infty) \to (0, \infty)$ *satisfies* $\lim_{t\to\infty} \Lambda(t)/h(t) = \infty$, *then for any* $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\Big(\prod_{u\in\mathcal{L}_t} \varphi(\xi_t^u/\Lambda(t)) \Big) \right) = C(\varphi),$$

where $C(\varphi)$ *is defined in* (1.7). *In particular,*

$$\lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} \mathbb{P}(R_t > \Lambda(t)) = \vartheta^*,$$

where ϑ^* *is defined in* (1.10).

Theorem 1.4. *Assume that* ***Assumption 1*** *and* ***Assumption 2*** *hold. If* $\Lambda : [0, \infty) \to (0, \infty)$ *satisfies* $\lim_{t\to\infty} \Lambda(t)/h(t) = \infty$, *then conditioned on* $\{R_t > \Lambda(t)\}$,

(1) $R_t/\Lambda(t)$ *converges weakly to a random variable* R^* *with density* $\alpha x^{-1-\alpha} \mathbf{1}_{(1,\infty)}(x)$;

(2) $\mathbb{X}_t^{(\Lambda(t))}$ *converges weakly to* $T\delta_{R^*}$, *where the law of* T *is given in* (1.8), *and* T *and* R^* *are independent.*

In [41], Shiozawa studied the asymptotic behavior of the maximal displacement R_t of a branching symmetric stable process with spatially inhomogeneous branching structure. More precisely, he proved that $R_t/e^{\lambda t/\alpha}$ converges weakly, where λ is the first eigenvalue

of the mean semigroup. Moreover, for any function $\Lambda(t)$ growing faster than $e^{\lambda t/\alpha}$, he gave the rate at which $P(R_t > \Lambda(t))$ tends to 0.

Recently, related topics for branching random walks with regularly varying steps have received considerable attention, see [7, 9, 10, 22] for instance. Assume that the displacements of the offspring from the parents are i.i.d. with $P(|X| > x) = x^{-\alpha}L(x)$. It was proved in [9] that, the point random measures $\sum_{|v|=n} \delta_{b_n^{-1}S_v}$, where S_v is the position of v and $b_n = \inf\{x : x^{-\alpha}L(x) \le \mu^{-n}\}$, converges weakly to a Cox cluster process. Bhattacharya [7] studied the large deviations of the extremes of branching random walks. More precisely, he proved that there exists a non-null measure m^*_{iid} on $\mathcal{M}(\overline{\mathbb{R}}_0)$ such that for any γ_n with $\gamma_n/b_n \to \infty$ and any measurable subset $\mathbf{A}$ of $\mathcal{M}(\overline{\mathbb{R}}_0) \setminus \{\emptyset\}$, satisfying $m^*_{iid}(\partial A) = 0$, for which $\emptyset$ is not a limit point,

$$\frac{1}{\mu^n \gamma_n^{-\alpha} L(\gamma_n)} \mathbb{P}\left(\sum_{|v|=n} \delta_{S_v/\gamma_n} \in \mathbf{A}\right) \to m^*_{iid}(\mathbf{A}).$$

Analogous problems for the case where displacements from the same parent have jointly regularly varying tails have been studied in [7, 10]. For studies on the maximum M_n of a branching random walk with stretched exponential tails, i.e.,

$$P(X > x) \sim a e^{-L(x)x^r}, \quad x \to \infty,$$

where $a > 0$, L is slowly varying at ∞ and $r \in [0, 1)$, we refer the reader to [25, 26, 27, 29]. When $r \in (0, 1)$, the maximum M_n grows polynomially. For example, if L is a constant, then $M_n/n^{1/r}$ converges to a positive constant almost surely. Large deviation results for $P(M_n/n^{1/r} > x)$ were given in [27]. [8] investigated the extremes for the case when $r = 0$ and $\log t/L(t) \to 0$ (or $L(t) = o(\log t)$) as $t \to \infty$,

Remark 1.5. Our upper large deviation results are closely related to the paper [7], which used tools from [33]. In Theorem 1.4, we focus on characterizing the weak convergence properties of appropriate renormalizations of $\mathbb{X}_t$ under the condition that the rightmost position is large (a small-probability event), which differs slightly from the large deviations considered in [7] for branching random walks with i.i.d. displacements. We now give some results analogous to those in [7].

Let Λ satisfy the conditions in Theorem 1.4, and set $\mathbf{B_n} := \{\nu \in \mathcal{M}(\overline{\mathbb{R}}_0) : \nu(1, \infty) \ge n\}$. By Theorem 1.4, for any Borel measurable subset $\mathbf{A} \in \mathcal{M}(\overline{\mathbb{R}}_0)$ with $\mathbb{P}(T\delta_{R^*} \in \partial\mathbf{A}) = 0$,

$$\lim_{t\to\infty} e^{-\lambda t}\Lambda(t)^\alpha L(\Lambda(t))^{-1}\mathbb{P}(\mathbb{X}_t^{(\Lambda(t))} \in \mathbf{A} \cap \mathbf{B_1}) = \vartheta^*\mathbb{P}(T\delta_{R^*} \in \mathbf{A}).$$

In particular, for $n \ge 1$, by taking $\mathbf{A} = \mathbf{B_n}$, we get

$$\lim_{t\to\infty} e^{-\lambda t}\Lambda(t)^\alpha L(\Lambda(t))^{-1}\mathbb{P}(\mathbb{X}_t^{(\Lambda(t)} \in \mathbf{B_n}) = \vartheta^*\mathbb{P}(T\delta_{R^*} \in \mathbf{B_n}) = \frac{q_1}{\alpha}\int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r \ge n)\mathrm{d}r,$$

which gives the large deviations for the n-th order statistics. Using [34, Lemma 4.4], we can also obtain the joint distribution of the order statistics, and we omit the details.

1.2.2 Lower large deviations

In this subsection, we state our result for lower large deviations of $\mathbb{X}_t$. Note that if $\Lambda(t)$ grows slower than $h(t)$, then $R_t/\Lambda(t)$ goes to ∞ as $t \to \infty$. Thus we need to consider a class of functions vanishing in a neighborhood of ∞. Let $\mathcal{H}_0(\mathbb{R})$ denote the family of all the functions $\varphi \in \mathcal{H}(\mathbb{R})$ with $\varphi \equiv 0$ on (c, ∞) for some $c > 0$. To the best of our knowledge, lower large deviation results have not yet been established for branching random walks with heavy tails.

Recall that $f(s)=\sum_k p_k s^k$ and $q=\mathbb{P}(\mathcal{S}^c)$. Note that $f'(q)\in[0,1)$. Put

$$\rho:=\beta(1-f'(q)). \tag{1.11}$$

For any $s\in[0,1]$ and $t\geq 0$, define

$$F(s,t):=\mathbb{E}\left(s^{Z_t}\right).$$

It is well known (see, [5, Section III.8], for instance) that

$$\lim_{t\to\infty} e^{\rho t}[F(s,t)-q]=:A(s) \tag{1.12}$$

exists for $0\leq s<1$. Moreover, the convergence is uniform in $s\in[0,a]$ for any $0<a<1$.

Recall that W is defined in (1.3). For any $\theta\geq 0$, define

$$\phi(\theta)=\mathbb{E}(e^{-\theta W}). \tag{1.13}$$

In this subsection, we assume that the function Λ satisfies the following conditions:

(S1) $\lim_{t\to\infty}\Lambda(t)/h(t)=0$ and $\lim_{t\to\infty} t\psi(\Lambda(t)^{-1})=0$.

(S2) If $\alpha=1$ or $\alpha\geq 2$, $\sup_{t>1} tV_\delta(\Lambda(t))<\infty$ for some $\delta>0$, where V_δ is defined in (2.8) below.

(S3) If $p_k=0$ for all $k\geq 3$, we further assume that $\Lambda(t)>e^{\gamma t}$ with some $\gamma>0$ for t sufficiently large.

It is clear that the condition $\lim_{t\to\infty} t\psi(\Lambda(t)^{-1})=0$ is equivalent to $\xi_t/\Lambda(t)\xrightarrow{P}0$ as $t\to\infty$. In the Appendix, we provide the asymptotic of $\psi(\theta)$ as $\theta\to 0$. By the definition of V_δ, $V_\delta(x)$ is slightly bigger than $|\psi(1/x)|$.

Now we give an intuitive interpretation of the lower large deviations of R_t. Assume that $p_0=0$, then $q=0$. Let $r(t)$ be defined by $h(r(t))=\Lambda(t)$. Using the Markov property and the branching property of $\mathbb{X}_t$ at $t-r(t)$, we get that

$$\mathbb{P}\{R_t\leq\Lambda(t)\}=\mathbb{P}\left(\prod_{u\in\mathcal{L}_{t-r(t)}}\mathbb{P}_{\delta_{\xi^u_{t-r(t)}}}(R_{r(t)}\leq h(r(t)))\right).$$

Under **(S1)**, we have $\xi^u_{t-r(t)}=o(\Lambda(t))$, then $\mathbb{P}_{\delta_{\xi^u_{t-r(t)}}}(R_{r(t)}\leq h(r(t)))\approx\mathbb{P}(R_{r(t)}\leq h(r(t)))\to\phi(\vartheta^*)$. Thus,

$$\mathbb{P}\{R_t\leq\Lambda(t)\}\approx\mathbb{P}(\phi(\vartheta^*)^{Z_{t-r(t)}})\sim e^{-\rho(t-r(t))}A(\phi(\vartheta^*))$$

as $t\to\infty$, where the last relation follows from (1.12). Condition **(S1)** is a crucial condition, while **(S2)** is merely a technical condition. If $\Lambda(t)$ grows so slowly such that $\xi_t/\Lambda(t)\to\infty$, we conjecture that the $e^{-\rho(t-r(t))}$ is not the correct rate for $\mathbb{P}^*\{R_t\leq\Lambda(t)\}$. Now we state our main results.

Theorem 1.6. *Assume that **Assumption 1** and **Assumption 2** hold. Let $\Lambda:[0,\infty)\to(0,\infty)$ be a non-decreasing function satisfying **(S1)**, **(S2)** and **(S3)**, and let $r(t)$ be defined by $h(r(t))=\Lambda(t)$. Then for any $\varphi\in\mathcal{H}_0(\mathbb{R})$,*

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{E}^*\Big(\prod_{u\in\mathcal{L}_t}\varphi(\xi^u_t/\Lambda(t))\Big)=\frac{1}{1-q}A\left[\phi(C(\varphi))\right],$$

where $C(\varphi)$ is defined in (1.7) *and $A(s)$ is defined in* (1.12)*. In particular,*

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{P}^*\Big(R_t\leq\Lambda(t)\Big)=\frac{1}{1-q}A\left[\phi(\vartheta^*)\right].$$

In Lemma 4.3 below, we show that $\lambda \geq \rho$, and that the equality $\lambda = \rho$ holds if and only if $p_k = 0$ for all $k \geq 3$. Thus, when $p_k = 0$ for all $k \geq 3$, the additional stronger assumption **(S3)** on $\Lambda(t)$ is required. Currently we do not know how to remove this additional assumption in this case.

Using Theorem 1.6, we can get the following:

Theorem 1.7. *Assume that **Assumption 1** and **Assumption 2** hold. Let $\Lambda : [0, \infty) \to (0, \infty)$ be a non-decreasing function satisfying **(S1)**, **(S2)** and **(S3)**. Then under $\mathbb{P}^*$, conditioned on $\{R_t \leq \Lambda(t)\}$, $\mathbb{X}_t^{(\Lambda(t))}$ converges weakly to some random measure $\Xi = \sum_{k=1}^{\mathcal{K}} \bar{\mathcal{N}}_\infty^{(k)}$, where*

(i) $\{\bar{\mathcal{N}}_\infty^{(k)}, k \geq 1\}$ are i.i.d. with the same law as $P(\mathcal{N}_\infty \in \cdot | \mathcal{N}_\infty(\mathbb{R}) \neq 0, \mathcal{N}_\infty((1, \infty)) = 0)$;

(ii) $\mathcal{K}$ is a positive integer valued random variable with generating function

$$E\left(s^{\mathcal{K}}\right) = \frac{A((\phi(\vartheta^*) - q)s + q)}{A(\phi(\vartheta^*))};$$

(iii) $\{\bar{\mathcal{N}}_\infty^{(k)}, k \geq 1\}$ and $\mathcal{K}$ are independent.

1.2.3 Almost sure envelopes for the rightmost position

We state now our results on the almost sure convergence of R_t. Recall that $\lambda = \beta(\mu - 1)$, H is the inverse function of $x^{-\alpha}L(x)$ and $h(t) = H(e^{-\lambda t})$.

Theorem 1.8. *Assume that **Assumption 1** and **Assumption 2** hold. It holds that*

$$\liminf_{t\to\infty} R_t / H(e^{-\lambda t} \log t) = (\vartheta^* W)^{\frac{1}{\alpha}}, \qquad \mathbb{P}^*\text{-a.s.}$$

Theorem 1.9. *Assume that **Assumption 1** and **Assumption 2** hold. Suppose that $G : [0, \infty) \to (0, \infty)$ is a non-decreasing function satisfying $\lim_{t\to\infty} G(t)/h(t) = \infty$.*

(1) If $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, then

$$\limsup_{t\to\infty} R_t / G(t) = 0, \qquad \mathbb{P}^*\text{-a.s.}$$

(2) If $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$, then

$$\limsup_{t\to\infty} R_t / G(t) = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

This implies that, for a non-decreasing function G satisfying $\lim_{t\to\infty} G(t)/h(t) = \infty$, either $\limsup_{t\to\infty} R_t/G(t) = 0$ $\mathbb{P}^*$-a.s., or $\limsup_{t\to\infty} R_t/G(t) = \infty$ $\mathbb{P}^*$-a.s.. Similar result holds for a subordinator. If ξ is a subordinator with infinite mean, then for any increasing function $G : [0, \infty) \to (0, \infty)$ such that $G(t)/t$ increases, $\limsup_{t\to\infty} \xi_t/G(t) = 0$ or ∞ almost surely, see [6, Chapter III.4].

For $t \geq 0$, $u \in \mathbb{T}$, we set $X_{u,t} := \xi^u_{\sigma_u \wedge t} - \xi^u_{b_u \wedge t}$. Then we have

$$\mathbb{X}_t = \sum_{u\in\mathcal{L}_t} \delta_{\xi^u_t} = \sum_{u\in\mathcal{L}_t} \delta_{\sum_{v\in I_u} X_{v,t}}.$$

We will see in Proposition 3.1 that the asymptotic behavior of appropriate renormalization of $\mathbb{X}_t$ is governed by that of the same renormalization of

$$\mathbb{Y}_t := \sum_{u\in\mathcal{L}_t} \sum_{v\in I_u} \delta_{X_{v,t}}. \tag{1.14}$$

For the proofs of Theorem 1.3, 1.8 and 1.9, we first establish the corresponding results for $\mathbb{Y}_t$ and then use them to get the desired results for $\mathbb{X}_t$. This technique has been employed in [39] for branching Lévy processes and in [9, 10, 22] for branching random walks with heavy tails. However this technique (Proposition 3.1) does not work for the proof of the lower deviation result in Theorem 1.6. We establish lower large deviation results of $\mathbb{X}_t$ and $\mathbb{Y}_t$ separately. It turns out that the results are identical, which is somewhat surprising.

The rest of the paper is organized as follows. In Section 2 we derive several estimates for the tail probability of ξ_s. In Subsection 3.1, we give the one large jump principle, Proposition 3.1, and in Subsection 3.2 we prove Theorems 1.3 and 1.4. In Section 4, we prove the lower large deviation results for $\mathbb{X}_t$ and $\mathbb{Y}_t$. Finally, Section 5 contains the proofs of the almost sure convergence results of R_t.

For the reader's convenience, we summarize some notation and definitions used throughout this paper.

- $C(C_\theta)$ stands for a constant (depending only on θ), whose value may vary from line to line.
- $f(x) \lesssim_\theta g(x)$ means that there exists a constant C_θ such that $f(x) \le C_\theta g(x)$.
- $g_1(t) \overset{t}{\le} g_2(t)$ means that $g_1(t) \le g_2(t)$ for sufficiently large t.
- $\mathbb{R} := (-\infty, \infty)$ and $\overline{\mathbb{R}}_0 := [-\infty, \infty] \setminus \{0\}$.
  - $\mathcal{B}_b^+(\overline{\mathbb{R}}_0)$: the family of all non-negative, bounded Borel functions on $\overline{\mathbb{R}}_0$.
  - $C_c^+(\overline{\mathbb{R}}_0)$: the family of all non-negative continuous functions g on $\overline{\mathbb{R}}_0$ with $g \equiv 0$ on $(-\delta, 0) \cup (0, \delta)$ for some $\delta > 0$;
  - $C_b^0(\mathbb{R})$: the space of all bounded continuous functions vanishing in a neighborhood of 0.
  - $\mathcal{B}_1(\mathbb{R})$: the set of all the Borel functions $\varphi : \mathbb{R} \to [0, 1]$ with $\varphi \equiv 1$ in some neighborhood of 0.
  - $\mathcal{H}(\mathbb{R})$: the family of uniformly continuous functions $\varphi : \mathbb{R} \to [0, 1]$ with $\varphi \equiv 1$ in some neighborhood of 0.
  - $\mathcal{H}_0(\mathbb{R})$: the family of all the functions $\varphi \in \mathcal{H}(\mathbb{R})$ with $\varphi \equiv 0$ on (c, ∞) for some $c > 0$.
- $\mathbb{T}$ is a tree with root o
- For a particle $u \in \mathbb{T}$,
  - N^u is the number of the offspring of u, b_u and σ_u are the birth and death times of u, respectively.
  - ξ_t^u is the position of u at time t.
  - I_u^0 is the set of ancestors of u and $I_u := I_u^0 \cup \{u\}$.
  - $\mathcal{L}_t$ is the set of all the particles alive at time t, and Z_t is the cardinality of $\mathcal{L}_t$.
  - $D_t := \{u : b_u \le t\}$ and $|D_t|$ is cardinality of D_t.
  - Z_t^u is the number of offspring of u alive at time t, and

$$\mathcal{D}_t := \{u \in \mathbb{T} : b_u \le t, Z_t^u > 0\}, \quad \mathcal{D}_t' := \mathcal{D}_t \setminus \{o\}.$$

  - $X_{u,t} := \xi^u_{\sigma_u \wedge t} - \xi^u_{b_u \wedge t}$ and

$$M_t := \max_{u \in \mathcal{D}_t} X_{u,t}, \quad M_t' := \max_{u \in \mathcal{D}_t'} X_{u,t}.$$

  - For any $0 \le s < t$,

$$M_{s,t} := \max_{u \in D_{t-s}} |X_{u,t}|.$$

- For any nonnegative function g and measure $\nu = \sum_{k=1}^{n} \delta_{x_k}$,

$$\mathcal{I}(g, \nu) := \prod_{k=1}^{n} g(x_k).$$

- $\mathbb{X}_t := \sum_{u\in\mathcal{L}_t} \delta_{\xi_t^u}$; $\mathbb{Y}_t := \sum_{u\in\mathcal{L}_t} \sum_{v\in I_u} \delta_{X_{v,t}} = \sum_{u\in\mathcal{D}_t} Z_t^u \delta_{X_{u,t}}$.
- For any $a > 0$, $\mathbb{X}_t^{(a)} := \sum_{u\in\mathcal{L}_t} \delta_{\xi_t^u/a}$; $\mathbb{Y}_t^{(a)} := \sum_{u\in\mathcal{D}_t} Z_t^u \delta_{X_{u,t}/a}$.
- $R_t := \max_{u\in\mathcal{L}_t} \xi_t^u$, $t > 0$.

2 Preliminaries: Estimates on the tail probabilities of ξ

In this paper we always assume that L is a positive, continuous slowly varying function on $[0, \infty)$ such that $x^{-\alpha}L(x)$ is strictly decreasing on $(0, \infty)$. Recall that, for any $c > 0$, since $L(cx)/L(x) \to 1$ and L is locally bounded, we have $\sup_{x>0} L(cx)/L(x) < \infty$. It is well known (see [11, Theorem 1.5.6] for instance) that, for any $\epsilon > 0$, there exists $a_\epsilon > 0$ such that for any $x, y > a_\epsilon$,

$$\frac{L(y)}{L(x)} \le 2\max\left\{(y/x)^\epsilon, (y/x)^{-\epsilon}\right\}, \qquad \frac{\bar{L}(y)}{\bar{L}(x)} \le 2\max\left\{(y/x)^\epsilon, (y/x)^{-\epsilon}\right\}. \tag{2.1}$$

Estimates for the tail probability of ξ were obtained in [39] under certain stronger conditions. In this section, we prove several of these results under the weaker **Assumption 2**. First we give an elementary lemma, whose proof is omitted.

Lemma 2.1. *Let Y_1, Y_2 be two independent non-negative random variables. Assume the tail probability of Y_1 is regularly varying with index $-\alpha$, then*

$$\lim_{x\to\infty} \frac{P(Y_1 - Y_2 > x)}{P(Y_1 > x)} = 1.$$

Lemma 2.2. *If **Assumption 2** (i) holds, then as $x \to \infty$,*

$$\frac{\mathbf{P}(\frac{\xi_s}{x} \in \cdot)}{x^{-\alpha}L(x)} \xrightarrow{v} s\nu_\alpha(\cdot) \quad \text{on } \overline{\mathbb{R}}_0,$$

where ν_α is defined in (1.4).

Proof. We can write ξ as $\xi = \xi^1 + \xi^2$ with ξ^1 and ξ^2 being independent Lévy processes with generating triplet $(a, b, \mathbf{n}|_{\{|x|\le 1\}})$ and $(0, 0, \mathbf{n}|_{\{|x|>1\}})$ respectively. Since ξ^1 has exponential moments, for any $s > 0$, we have

$$\mathbf{P}(|\xi_s^1| > x) = o(x^{-\alpha}L(x)), \quad x \to \infty. \tag{2.2}$$

Now we deal with the tail probability of ξ_s^2. ξ^2 is a compound Poisson process which can be written as the difference of two independent positive compound Poisson processes $\xi^{2,1}$ and $\xi^{2,2}$, where $\xi^{2,1}$ and $-\xi^{2,2}$ are Lévy processes with generating triplet $(0, 0, \mathbf{n}|_{\{x>1\}})$ and $(0, 0, \mathbf{n}|_{\{x<-1\}})$. Since $\mathbf{n}(x, \infty)$ is regularly varying, by [24, Theorem 3] and Lemma 2.1, we have that

$$\mathbf{P}(\xi_s^2 > x) \sim \mathbf{P}(\xi_s^{2,1} > x) \sim s\mathbf{n}(x, \infty) \sim \frac{q_1}{\alpha} s x^{-\alpha} L(x), \quad x \to \infty. \tag{2.3}$$

If $q_2 > 0$, then using similar arguments, we can get

$$\frac{\mathbf{P}(\xi_s^2 < -x)}{s x^{-\alpha} L(x)} \to \frac{q_2}{\alpha}, \quad x \to \infty. \tag{2.4}$$

Now we consider the case $q_2 = 0$. Let $Y_k^-, k \geq 1$, be i.i.d. non-negative random variables with common law $P(Y^- > x) = \mathbf{n}(-\infty, -x)/\mathbf{n}(-\infty, -1)$. Since $q_2 = 0$, then for any $\epsilon > 0$, there exists x_0 such that $P(Y^- > x) < \epsilon x^{-\alpha} L(x)$. So for any $x > a_1$ and $n < x/(x_0 + a_1)$, we have by (2.1),

$$P\left(\sum_{k=1}^{n} Y_k^- > x\right) \leq nP(Y^- > x/n) \leq \epsilon n^{1+\alpha} x^{-\alpha} L(x/n) \leq 2\epsilon n^{2+\alpha} x^{-\alpha} L(x).$$

Since $-\xi^{2,2}$ is a compound process with Lévy process $\mathbf{n}|_{\{x<-1\}}$, $\xi_s^{2,2} \stackrel{d}{=} \sum_{k=1}^{N_s^-} Y_k^-$, where $\{N_s^-\}$ is a Poisson process with intensity $\mathbf{n}(-\infty, -1)$. Thus it follows that

$$\begin{aligned}\mathbf{P}(\xi_s^{2,2} > x) &\leq \mathbf{P}\left(\sum_{k=1}^{N_s^-} Y_k^- > x; N_s^- < x/(x_0 + a_1)\right) + \mathbf{P}(N_s^- > x/(x_0 + a_1)) \\ &\leq 2\epsilon x^{-\alpha} L(x) \mathbf{E}((N_s^-)^{2+\alpha}) + \mathbf{E}(e^{(x_0+a_1)N_s^-}) e^{-x}.\end{aligned}$$

Hence we have $\lim_{x\to\infty} \frac{\mathbf{P}(\xi_s^{2,2}>x)}{x^{-\alpha}L(x)} = 0$, which implies (2.4) also holds when $q_2 = 0$, since

$$\mathbf{P}(\xi_s^2 < -x) = \mathbf{P}(\xi_s^{2,2} - \xi_s^{2,1} > x) \leq \mathbf{P}(\xi_s^{2,2} > x).$$

Combining (2.2), (2.3) and (2.4), we get that for any $s > 0$,

$$\lim_{x\to\infty} \frac{\mathbf{P}(\xi_s > x)}{x^{-\alpha}L(x)} = \frac{q_1}{\alpha} s = s\nu_\alpha(1, \infty), \qquad \lim_{x\to\infty} \frac{\mathbf{P}(\xi_s < -x)}{x^{-\alpha}L(x)} = \frac{q_2}{\alpha} s = s\nu_\alpha(-\infty, -1).$$

Thus we get the desired result. □

Let $C_b^0(\mathbb{R})$ be the space of all bounded continuous functions vanishing in a neighborhood of 0. By Lemma 2.2, under **Assumption 2** (i), we have

$$\lim_{x\to\infty} x^\alpha L(x)^{-1} \mathbf{E}\left(g\left(\frac{\xi_s}{x}\right)\right) = s \int_{\mathbb{R}_0} g(x)\nu_\alpha(\mathrm{d}x), \quad \text{for all } g \in C_b^0(\mathbb{R}) \text{ and } s > 0.$$

It was shown in [42] that, under **Assumption 2**, for any $t > 0$,

$$\lim_{x\to\infty} x^\alpha L(x)^{-1} \mathbf{P}\left(\sup_{0\leq s\leq t} \xi_s > x\right) = \lim_{x\to\infty} x^\alpha L(x)^{-1} \mathbf{P}(\xi_t > x) = \frac{q_1}{\alpha} \tag{2.5}$$

and

$$\lim_{x\to\infty} x^\alpha L(x)^{-1} \mathbf{P}\left(\inf_{0\leq s\leq t} \xi_s < -x\right) = \lim_{x\to\infty} x^\alpha L(x)^{-1} \mathbf{P}(\xi_t < -x) = \frac{q_2}{\alpha}. \tag{2.6}$$

It was proved in [39, Lemma 2.2] that, under **(H)**, there exist $c_0 > 0$ and $x_0 > 0$ such that for any $s > 0$ and $x > x_0$,

$$\mathbf{P}(|\xi_s| > x) \leq c_0 s x^{-\alpha} L(x). \tag{2.7}$$

In the next lemma, we provide a similar upper bound for $\mathbf{P}(|\xi_s| > x)$ under **Assumption 2**. We first recall a useful result for the tail probability of random walks.

Let Y be a random variable with law $\frac{\mathbf{n}(\cdot)|_{\{|y|>1\}}}{\mathbf{n}(\{|y|>1\})}$. Let $Y_k, k \geq 1$, be i.i.d. random variables sharing the same law as Y if $\alpha \leq 1$ and sharing the same law with $Y - E(Y)$ otherwise. For any $\delta \in (0, 1)$, define

$$V_\delta(x) := \begin{cases} x^{-\alpha} L(x) & \alpha \in (0,1) \cup (1,2); \\ x^{-2}(\log x \vee 1)^{1+\delta}, & \alpha > 2; \\ x^{-1} \frac{(L^*(x))^{1+\delta})}{L(x)^\delta}, & \alpha = 1; \\ x^{-2}(\log x \vee 1)^2, & \alpha = 2, \mathbf{E}(\xi_1^2) < \infty; \\ x^{-2+\delta}, & \alpha = 2, \mathbf{E}(\xi_1^2) = \infty; \end{cases} \tag{2.8}$$

where $L^*(x) := \int_1^{x\vee 1} y^{-1}L(y)\,\mathrm{d}y$. Let $S_n = \sum_{k=1}^n Y_k$. It is known (see, [14, Corollary 2.2.4, Corollary 3.1.7, Remark 4.1.5] for instance) that for any $\theta > 0$, there exists $c(\theta) > 0$ such that for all (n, x) with $nV_\delta(x) \le \theta$,

$$P(S_n > x) \le c(\theta) n x^{-\alpha} L(x). \tag{2.9}$$

Lemma 2.3. *Assume **Assumption 2** holds.*

(1) For $\alpha \in (0,1)\cup(1,2)$, there exists a constant $c_0 > 0$ such that for all $s, x > 0$,

$$\mathbf{P}(|\xi_s| > x) \le c_0(1+s)x^{-\alpha}L(x). \tag{2.10}$$

(2) For $\alpha = 1$ or $\alpha \ge 2$, let V_δ be defined in (2.8) *with $\delta \in (0,1)$. Then for each $\theta > 0$, there exists $c_\theta > 0$ such that for all (s, x) with $0 < sV_\delta(x) < \theta$,*

$$\mathbf{P}(|\xi_s| > x) \le c_\theta(1+s)x^{-\alpha}L(x). \tag{2.11}$$

Proof. Let $a_0 > 0$. For all $s > 0, 0 < x < a_0$, we have

$$\mathbf{P}(|\xi_s| > x) \le 1 \le \frac{x^{-\alpha}L(x)}{a_0^{-\alpha}L(a_0)} \lesssim_{a_0} (1+s)x^{-\alpha}L(x). \tag{2.12}$$

In the remainder of the proof, we only need to consider the case when x is large enough. Let $\theta > 0$ and $\delta \in (0,1)$.

(1) If $\alpha \in (0,1)\cup(1,2)$, by Lemma A.2 in the Appendix, we know that condition **(H)** is satisfied. Thus by (2.7) and (2.12), we get (2.10) immediately.

(2) Let $\alpha = 1$. Let $a_0 := a_0(\theta, \delta)$ be a constant to be chosen later. Recall the definitions of ξ^1 and ξ^2 in the proof of Lemma 2.2. Since ξ^1_s has exponential moments, we have

$$\mathbf{P}(|\xi^1_s| > x) \le e^{-x}\left[\mathbf{E}(e^{\xi^1_s}) + \mathbf{E}(e^{-\xi^1_s})\right] \le 2e^{-x}e^{Cs} \le 2e^{-x}e^{C\theta/V_\delta(x)}$$

for $0 < sV_\delta(x) < \theta$, where $C = \max\{\log \mathbf{E}(e^{\xi^1_1}), \log \mathbf{E}(e^{-\xi^1_1})\} \in [0,\infty)$. Since $V_\delta(x)^{-1} = o(x)$, we have $e^{-x}e^{C\theta/V_\delta(x)} = o(x^{-\alpha}L(x))$. Choose a_0 large enough such that for all $x > a_0$, $e^{-x}e^{C\theta/V_\delta(x)} \le x^{-\alpha}L(x)$. It follows that, for all (s,x) with $0 < sV_\delta(x) < \theta$, and $x > a_0$,

$$\mathbf{P}(|\xi^1_s| > x) \le 2x^{-\alpha}L(x).$$

Note that ξ^2 is a compound Poisson process with $\xi^2_t \overset{d}{=} \sum_{k=1}^{N_t} Y_k$ where $\{N_t : t \ge 0\}$ is a Poisson processes with intensity $\mathbf{n}(\{|y| > 1\})$. It follows that for any $x > 0$ and $s > 0$, $A > 0$,

$$\begin{aligned}
\mathbf{P}(\xi^2_s > x) &\le \mathbf{E}\left[\sum_{k=1}^{N_s} Y_k > x, N_s \le A/V_\delta(x)\right] + \mathbf{P}(N_s > A/V_\delta(x)) \\
&\le c(A)\mathbf{E}(N_s)x^{-\alpha}L(x) + e^{-A/V_\delta(x)}\mathbf{E}(e^{N_s}) \\
&\le c(A)\mathbf{n}(\{|y|>1\})\, sx^{-\alpha}L(x) + e^{(e-1)\mathbf{n}(\{|y|>1\})s}e^{-A/V_\delta(x)}.
\end{aligned}$$

Now choose $A = \theta(e-1)\mathbf{n}(\{|y|>1\}) + 1$ and a_0 large enough such that $e^{-1/V_\delta(x)} \le x^{-\alpha}L(x)$ for all $x > a_0$. Thus by (2.9) we get that for any $0 < s < \theta/V_\delta(x)$ and $x > a_0$,

$$\mathbf{P}(\xi^2_s > x) \le c(A)\mathbf{n}(\{|y|>1\})\, sx^{-\alpha}L(x) + e^{-1/V_\delta(x)} \lesssim_\theta (1+s)x^{-\alpha}L(x). \tag{2.13}$$

Similarly, $\mathbf{P}(\xi^2_s < -x) \lesssim_\theta (1+s)x^{-\alpha}L(x)$. Combining above all, (2.11) is true for $\alpha = 1$.

(3) Finally, we consider the case when $\alpha \ge 2$. By **Assumption 2**, $\mathbf{E}(\xi_1) = 0$. We modify the definition of ξ^1 and ξ^2 slightly. Let ξ^1 and ξ^2 be two independent Lévy processes

with generating triplet $(0, b, \mathbf{n}|_{\{|x|\le 1\}})$ and $(a, 0, \mathbf{n}|_{\{|x|>1\}})$ such that $\xi = \xi^1 + \xi^2$. Note that (2.13) is still valid for $\alpha \ge 2$.

For ξ^1, we have $\mathbf{E}(e^{\theta\xi_1}) \le \exp\{D\theta^2\}$, where $D = b^2 + \int_{|y|<1} y^2\mathbf{n}(\mathrm{d}y)$. Thus by the Markov inequality, we have, for any $c > 0$,

$$\mathbf{P}(|\xi^1_s| > x) \le e^{-cx}\mathbf{E}(e^{c\xi^1_s} + e^{-c\xi^1_s}) \le 2e^{-cx}e^{Dc^2 s} \le e^{-\frac{x^2}{4Ds}} \le e^{-\frac{x^2 V_\delta(x)}{4D\theta}},$$

for $0 < sV_\delta(x) < \theta$. One can check that $e^{-\frac{x^2 V_\delta(x)}{4D\theta}} = o(x^{-\alpha}L(x))$. Letting a_0 be large enough so that $e^{-\frac{x^2 V_\delta(x)}{4D\theta}} \le x^{-\alpha}L(x)$, we get that for any $0 < s < \theta/V_\delta(x)$ and $x > a_0$

$$\mathbf{P}(|\xi^1_s| > x) \le x^{-\alpha}L(x).$$

The proof is now complete. □

3 Upper deviation of $\mathbb{X}_t$ and $\mathbb{Y}_t$

We first introduce some notations to be used throughout the paper. Let $D_t := \{u : b_u \le t\}$ be the set of all the particles born before or at t, and let $|D_t|$ denote the cardinality of D_t. We define

$$\mathcal{D}_t := \{u \in \mathbb{T} : b_u \le t, Z^u_t > 0\}, \quad \mathcal{D}'_t := \mathcal{D}_t \setminus \{o\},$$

where Z^u_t be the number of offspring of u alive at time t. Define the displacement of u up to time t as $X_{u,t} := \xi^u_{\sigma_u\wedge t} - \xi^u_{b_u\wedge t}$ and let

$$M_t := \max_{u\in\mathcal{D}_t} X_{u,t}, \quad M'_t := \sup_{u\in\mathcal{D}'_t} X_{u,t}. \tag{3.1}$$

For any $0 \le s < t$, we define

$$M_{s,t} := \max_{u\in D_{t-s}} |X_{u,t}|.$$

3.1 A key result on "one big jump"

For any nonnegative function g and measure $\nu = \sum_{k=1}^n \delta_{x_k}$, define

$$\mathcal{I}(g,\nu) = \prod_{k=1}^n g(x_k).$$

Here we use the convention that $\prod_{k=1}^0 g(x_k) = 1$. It is clear that $\mathcal{I}(g,\nu) = e^{\nu(\log g)}$. Recall the random measure $\mathbb{Y}_t$ defined in (1.14). It can be rewritten as

$$\mathbb{Y}_t = \sum_{u\in\mathcal{D}_t} Z^u_t \delta_{X_{u,t}}.$$

For any $a > 0$, similar to (1.1), we let $\mathbb{Y}^{(a)}_t := \sum_{u\in\mathcal{D}_t} Z^u_t \delta_{X_{u,t}/a}$. The following key lemma says that, under appropriate renormalizations, $\mathbb{X}_t$ and $\mathbb{Y}_t$ have similar asymptotic behaviors.

Proposition 3.1. *If $a(t)$ is a positive function such that $a(t) \overset{t}{>} e^{\epsilon t}$ for some $\epsilon \in (0,1)$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,*

$$\lim_{t\to\infty} e^{-\lambda t}a(t)^\alpha L(a(t))^{-1}\mathbb{E}\left(\left|\mathcal{I}(\varphi, \mathbb{X}^{(a(t))}_t) - \mathcal{I}(\varphi, \mathbb{Y}^{(a(t))}_t)\right|\right) = 0.$$

To prove the Proposition, we proceed as follows. (i) In Lemma 3.2 below, we prove that, with high probability, for all $v \in \mathcal{L}_t$, there exists at most one particle $v' \in I_v$ that

experiences a "large jump" of size larger than $\theta a(t)/\log a(t)$. (ii) In Lemma 3.3 below, we show that, with high probability, for all $v \in \mathcal{L}_t$, the number of ancestors of v is less than $c \log a(t)$. (iii) In Lemma 3.4 below, we prove that the event $\{M_{s,t} \le a(t)\}$ occurs with high probability. Using the results from (i) and (ii), we have that, with high probability, $|\frac{\xi_t^v}{a(t)} - \frac{X_{v',t}}{a(t)}|$ is small enough. By (iii), we find that $\mathbb{Y}_t^{(a(t))}$ is governed by the following random measure:

$$\mathbb{Y}_{s,t} := \sum_{u \in \mathcal{D}_t : t-s < b_u \le t} Z_t^u \delta_{X_{u,t}} = \sum_{u \in \mathbb{T} : t-s < b_u \le t} Z_t^u \delta_{X_{u,t}}. \tag{3.2}$$

Then, by dominating $\mathbb{Y}_{s,t}$ properly, we get the assertion of Proposition 3.1.

Let $a(t)$ be the same as in Proposition 3.1. For any $t > 0$ and $\theta \in (0, 1]$, define

$$A_t(\theta, a(t)) := \bigcap_{v \in \mathcal{L}_t} \left\{ \sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \le 1 \right\}. \tag{3.3}$$

Lemma 3.2. *For any $p \in (0, 2)$,*

$$\lim_{t \to \infty} e^{-\lambda t} a(t)^{p\alpha} \mathbb{P}(A_t(\theta, a(t))^c) = 0. \tag{3.4}$$

Proof. Note that

$$\mathbb{P}(A_t(\theta, a(t))^c | \mathcal{F}_t^{\mathbb{T}}) \le \sum_{v \in \mathcal{L}_t} \mathbb{P}\left(\left\{ \sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \ge 2 \right\} | \mathcal{F}_t^{\mathbb{T}} \right). \tag{3.5}$$

Since $tV_\delta(a(t)/\log a(t)) \to 0$, as $t \to \infty$, by Lemma 2.3 we have that

$$\begin{aligned} &\mathbb{P}\left(|X_{u,t}| > \theta a(t)/\log a(t) | \mathcal{F}_t^{\mathbb{T}} \right) = \mathbf{P}(|\xi_s| > \theta a(t)/\log a(t))|_{s=\tau_{u,t}} \\ &\overset{t}{\le} \left(c\theta^{-\alpha} t \cdot a(t)^{-\alpha} [\log a(t)]^\alpha L(\theta a(t)/\log a(t)) \right) \wedge 1 =: p_t. \end{aligned} \tag{3.6}$$

Recall that, for any $v \in \mathbb{T}$, n^v is the number of particles in $I_v \setminus \{o\}$. Thus $|I_v| = n^v + 1$. Since, conditioned on $\mathcal{F}_t^{\mathbb{T}}$, $\{X_{u,t}, u \in I_v\}$ are independent, we have

$$\begin{aligned} &\mathbb{P}\left(\sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}| > \theta a(t)/\log a(t)\}} \ge 2 | \mathcal{F}_t^{\mathbb{T}} \right) \le \sum_{m=2}^{n^v+1} \binom{n^v+1}{m} p_t^m \\ \le & p_t^2 \sum_{m=0}^{n^v-1} n^v(n^v+1) \binom{n^v-1}{m} p_t^m \\ = & p_t^2 n^v (n^v+1)(1+p_t)^{n^v-1}. \end{aligned}$$

Thus by (3.5) and the many-to-one formula, we have

$$\begin{aligned} \mathbb{P}(A_t(\theta, a(t))^c) &\le e^{\lambda t} p_t^2 E(n_t(n_t+1)(1+p_t)^{n_t-1}) = e^{\lambda t} p_t^2 (2 + (1+p_t)\beta t) \beta t e^{\beta t p_t} \\ &\sim (\beta t)^2 e^{\lambda t} p_t^2, \quad t \to \infty. \end{aligned} \tag{3.7}$$

Here we used the fact $tp_t \to 0$ as $t \to \infty$. Now (3.4) follows from (3.6) and (3.7) immediately. □

Let $a(t)$ be the same as in Proposition 3.1. For any $t > 0$ and $c > 0$, define

$$B_t(c, a(t)) := \bigcap_{v \in \mathcal{L}_t} \{n^v \le c \log a(t)\}. \tag{3.8}$$

Lemma 3.3. *For any $c > 2\alpha + e^2\beta/\epsilon$,*

$$\lim_{t\to\infty} e^{-\lambda t} a(t)^{2\alpha} \mathbb{P}(B_t(c, a(t))^c) = 0. \tag{3.9}$$

Proof. Using the many-to-one formula (see Lemma A.4 in the Appendix), we get

$$\begin{aligned}\mathbb{P}(B_t(c,a(t))^c) &= \mathbb{P}\left(\bigcup_{v\in\mathcal{L}_t}\{n^v > c\log a(t)\}\right) \le \mathbb{E}\left(\sum_{v\in\mathcal{L}_t}\mathbf{1}_{\{n^v>c\log a(t)\}}\right)\\ &= e^{\lambda t}P(n_t > c\log a(t)) \le e^{\lambda t}\inf_{r>0} e^{-rc\log a(t)}E(e^{rn_t})\\ &= e^{\lambda t}e^{-(\log\log a(t)+\log c-\log\beta t-1)c\log a(t)-\beta t}. \end{aligned}\tag{3.10}$$

Since $a(t) \overset{t}{>} e^{\epsilon t}$, we have

$$\log\log a(t) + \log c \overset{t}{>} \log(c\delta t) \ge \log(\beta t) + 2.$$

Thus by (3.10), we have

$$\mathbb{P}(B_t(c,a(t))^c) \overset{t}{<} e^{\lambda t}e^{-c\log a(t)}e^{-\beta t} = e^{\lambda t}a(t)^{-c}e^{-\beta t}.$$

Since $c > 2\alpha$, we get the desired result. □

Lemma 3.4. *If $a(t)$ is a positive function with $\sup_{t>1} tV_\delta(a(t)) < \infty$, then*

$$\lim_{s\to\infty}\limsup_{t\to\infty} e^{-\lambda t}a(t)^\alpha L(a(t))^{-1}\mathbb{P}(M_{s,t} > a(t)) = 0.$$

Proof. The proof is similar to that leading to [39, (2.14)]. To highlight the distinction between **Assumption 2** and Condition **(H)**, we give the short proof. Let $\theta := \sup_{t>1} tV_\delta(a(t)) < \infty$. Recall $G_s(x) = \mathbf{P}(|\xi_s| > x)$. By the definition of $M_{s,t}$, we have that for $t > 1$,

$$\begin{aligned}\mathbb{P}(M_{s,t} > a(t)) &= \mathbb{E}(\cup_{u:b_u\le t-s}\{X_{u,t} > a(t)\}) \le \mathbb{E}\left(\sum_{u,b_u<t-s} G_{\tau_{u,t}}(a(t))\right)\\ &\lesssim_\theta a(t)^{-\alpha}L(a(t))\mathbb{E}\left(\sum_{u,b_u<t-s}(1+\tau_{u,t})\right),\end{aligned}$$

where the last inequality follows from Lemma 2.3. In the proof of [39, lemma 2.6], it was shown that

$$\mathbb{E}\left(\sum_{u,b_u<t-s}\tau_{u,t}\right) \le e^{\lambda t}\Big(\lambda^{-1}e^{-\lambda s} + \frac{e^{-\beta s}-e^{-\lambda s}}{\lambda-\beta}\Big).$$

By Lemma A.5 in the Appendix, we have

$$\mathbb{E}\left(\sum_u \mathbf{1}_{\{b_u\le t-s\}}\right) = \mathbb{E}(|D_{t-s}|) = \frac{\beta}{\lambda}(\mu e^{\lambda(t-s)} - 1). \tag{3.11}$$

Combining the three displays above, we get the desired result immediately. □

Now we are ready to prove Proposition 3.1. Using the tree structure, we can rewrite the random measure $\mathbb{Y}_{s,t}$ defined in (3.2) as follows:

$$\mathbb{Y}_{s,t} = \sum_{v\in\mathcal{L}_{t-s}}\sum_{u:v<u,b_u\le t} Z_t^u\delta_{X_{u,t}} =: \sum_{v\in\mathcal{L}_{t-s}}\mathbb{Y}_{s,t}^v. \tag{3.12}$$

By the branching property and the Markov property, conditioned on $\mathcal{F}_{t-s}$, $\{\mathbb{Y}^v_{s,t}, v \in \mathcal{L}_{t-s}\}$ are i.i.d. with a common law equal to that of

$$\mathbb{Y}'_s := \sum_{u\in\mathcal{D}'_s} Z^u_s \delta_{X_{u,s}}. \tag{3.13}$$

Proof of Proposition 3.1. Let θ, c be the two constants in Lemma 3.2 and 3.3, and let $A_t(\theta, a(t))$ and $B_t(c, a(t))$ be the events defined in (3.3) and (3.8) respectively.

Let $v' \in I_v$ be such that $|X_{v',t}| = \max_{u\in I_v}\{|X_{u,t}|\}$. We note that, on the event $A_t(\theta, a(t))$, $|X_{u,t}| \le \theta a(t)/\log a(t)$ for any $u \in I_v \setminus \{v'\}$. Since $\xi^v_t = \sum_{u\in I_v} X_{u,t}$, on the event $A_t(\theta, a(t)) \cap B_t(c, a(t))$, we have that

$$|\xi^v_t - X_{v',t}| = \left|\sum_{u\in I_v\setminus\{v'\}} X_{u,t}\right| \le \frac{n^v \theta a(t)}{\log a(t)} \le c\theta a(t). \tag{3.14}$$

For $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. Since φ is uniformly continuous, for any $\gamma > 0$, there exists $\eta > 0$ such that $|\varphi(x_1) - \varphi(x_2)| \le \gamma$ whenever $|x_1 - x_2| < \eta$. We now fix an arbitrary $\gamma > 0$ and the corresponding η.

Recall $c > 2\alpha + e^2\beta/\epsilon$. Choose θ small enough so that $c\theta < \eta \wedge (\delta/2)$. We assume that t is sufficiently large so that $\log a(t) > 2\theta/\delta$. We note that, on the event $A_t(\theta, a(t))$, $|X_{u,t}| \le \theta a(t)/\log a(t) \le a(t)\delta/2$ for any $u \in I_v \setminus \{v'\}$, and thus $\varphi(X_{u,t}/a(t)) = 1$. It follows that on the event $A_t(\theta, a(t)) \cap B_t(c, a(t))$,

$$\begin{aligned}\left|\prod_{v\in\mathcal{L}_t} \varphi(\xi^v_t/a(t)) - \prod_{v\in\mathcal{L}_t}\prod_{u\in I_v} \varphi(X_{u,t}/a(t))\right| &= \left|\prod_{v\in\mathcal{L}_t} \varphi(\xi^v_t/a(t)) - \prod_{v\in\mathcal{L}_t} \varphi(X_{v',t}/a(t))\right| \\ &\le \sum_{v\in\mathcal{L}_t} |\varphi(\xi^v_t/a(t)) - \varphi(X_{v',t}/a(t))|. \end{aligned}\tag{3.15}$$

By (3.14), on $A_t(\theta, a(t)) \cap B_t(c, a(t))$, we have

$$|\xi^v_t - X_{v',t}|/a(t) \le c\theta \le \eta \wedge (\delta/2).$$

Thus if $|X_{v',t}| \le \delta a(t)/2$, then $|\xi^v_t|/a(t) < \delta$, which implies that $\varphi(\xi^v_t/a(t)) - \varphi(X_{v',t}/a(t)) = 0$. Hence by (3.15), on $A_t(\theta, a(t)) \cap B_t(c, a(t))$,

$$\begin{aligned}&\left|\mathcal{I}(\varphi, \mathbb{X}^{(a(t))}_t) - \mathcal{I}(\varphi, \mathbb{Y}^{(a(t))}_t)\right| \le \gamma \sum_{v\in\mathcal{L}_t} \mathbf{1}_{\{|X_{v',t}|>a(t)\delta/2\}} \\ \le&\gamma \sum_{v\in\mathcal{L}_t}\sum_{u\in I_v} \mathbf{1}_{\{|X_{u,t}|>a(t)\delta/2\}} = \gamma \sum_{u\in\mathcal{D}_t} Z^u_t g_t(X_{u,t}),\end{aligned}$$

where $g_t(y) = \mathbf{1}_{|y|>a(t)\delta/2}$. Therefore, for any fixed $s \in (0, t)$, on $A_t(\theta, a(t)) \cap B_t(c, a(t)) \cap \{M_{s,t} \le a(t)\delta/2\}$, we have

$$\left|\mathcal{I}(\varphi, \mathbb{X}^{(a(t))}_t) - \mathcal{I}(\varphi, \mathbb{Y}^{(a(t))}_t)\right| \le \gamma \sum_{u: t-s<b_u\le t} Z^u_t g_t(X_{u,t}) = \gamma \mathbb{Y}_{s,t}(g_t).$$

Since φ takes values in $[0, 1]$, we have

$$\begin{aligned}&\mathbb{E}\left|\mathcal{I}(\varphi, \mathbb{X}^{(a(t))}_t) - \mathcal{I}(\varphi, \mathbb{Y}^{(a(t))}_t)\right| \\ \le&\mathbb{P}(A_t(\theta, a(t))^c) + \mathbb{P}(B_t(c, a(t))^c) + \mathbb{P}(M_{s,t} > a(t)\delta/2) + \gamma\mathbb{E}(\mathbb{Y}_{s,t}(g_t)).\end{aligned}\tag{3.16}$$

Note that $tV_\delta(a(t)) \to 0$ for any $\delta \in (0,1)$. By (3.12) and Lemma 2.3, we have that for t large enough,

$$\begin{aligned}\mathbb{E}(\mathbb{Y}_{s,t}(g_t)) =&\mathbb{E}(Z_{t-s})\mathbb{E}(\mathbb{Y}'_s(g_t)) = e^{\lambda(t-s)}\mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u \mathbb{P}(|X_{u,s}| > a(t)\delta/2|\mathcal{F}_s^{\mathbb{T}})\right)\\ \lesssim_\delta& a(t)^{-\alpha}L(a(t))e^{\lambda(t-s)}\mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u(1+\tau_{u,s})\right).\end{aligned}$$

It follows from the display below (2.19) in the proof of [39, Proposition 2.1] that

$$\mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u\tau_{u,s}\right) \le \mathbb{E}\left(\sum_{u:b_u\le s} Z_s^u\tau_{u,s}\right) \le se^{\lambda s}.$$

Recall that n^u is the number of ancestors of u. By Lemma A.4 in the Appendix, we have

$$\mathbb{E}\left(\sum_{u\in\mathcal{D}'_s} Z_s^u\right) = \mathbb{E}\left(\sum_{u\in\mathcal{L}_s} n^u\right) = e^{\lambda s}E(n_s) = \beta s e^{\lambda s}. \tag{3.17}$$

Thus we get that, for t large enough,

$$\mathbb{E}(\mathbb{Y}_{s,t}(g_t)) \lesssim_\delta a(t)^{-\alpha}L(a(t))e^{\lambda t}(\beta+1)s. \tag{3.18}$$

Thus combining (3.16), (3.4), (3.9) and (3.18), we have

$$\begin{aligned}&\limsup_{t\to\infty} e^{-\lambda t}a(t)^\alpha L(a(t))^{-1}\mathbb{E}\left|\mathcal{I}(\varphi,\mathbb{X}_t^{(a(t))}) - \mathcal{I}(\varphi,\mathbb{Y}_t^{(a(t))})\right|\\ \lesssim_\delta& \limsup_{t\to\infty} e^{-\lambda t}a(t)^\alpha L(a(t))^{-1}\mathbb{P}(M_{s,t} > a(t)\delta/2) + (\beta+1)s\gamma.\end{aligned}$$

Letting $\gamma \to 0$ first, and then letting $s \to \infty$ and applying Lemma 3.4, we arrive at the desired result. □

3.2 Proofs of Theorems 1.3 and 1.4

In this subsection, we first study the asymptotic behavior of $\mathbb{Y}_t^{(\Lambda(t))}$ (see Proposition 3.6 below), and then using Proposition 3.1 to obtain the corresponding result for $\mathbb{X}_t^{(\Lambda(t))}$. According to Lemma 3.4, to get the asymptotic behavior of $\mathbb{Y}_t^{(\Lambda(t))}$, it suffices to study the property of $\mathbb{Y}_{s,t}^{(\Lambda(t))}$. The following result on $\mathbb{Y}'_s$ is due to [39]. We define $\mathbb{Y}'^{(a)}_s$ similarly as (1.1). We emphasize that the definition of $M_{s,t}$ in [39, Section 2.3] coincides with $\mathbb{Y}'^{(h(t))}_s$.

Lemma 3.5. *For any $s > 0$ and $\varphi \in \mathcal{H}(\mathbb{R})$,*

$$\lim_{x\to\infty} x^\alpha L(x)^{-1}\left[1 - \mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}'^{(x)}_s)\right)\right] = \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right)\int_{\mathbb{R}} \mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_\alpha(dy)\,dr.$$

Proof. Under **(H)**, it has been proven in [39, (2.17) and the first display on page 636] that for $\varphi(x) = e^{-g(x)}$ with $g \in C_0^+(\bar{\mathbb{R}}_0)$,

$$\begin{aligned}&\lim_{t\to\infty} h(t)^\alpha L(h(t))^{-1}\left[1 - \mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}'^{(h(t))}_s)\right)\right]\\ &= \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right)\int_{\mathbb{R}} \mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_\alpha(\mathrm{d}y)\,\mathrm{d}r.\end{aligned}$$

Under **Assumption 2**, we observe that, replacing (2.7) by Lemma 2.3, the argument above remains valid, and that the desired result holds for any $\varphi \in \mathcal{H}(\mathbb{R})$. □

Recall that $\mathcal{B}_1(\mathbb{R})$ be the set of all the Borel functions $\varphi : \mathbb{R} \to [0,1]$ with $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. We can extend the definition of $C(\varphi)$ in (1.7) to $\varphi \in \mathcal{B}_1(\mathbb{R})$. Note that for any $\varphi \in \mathcal{B}_1(\mathbb{R})$

$$C(\varphi) := \int_0^\infty e^{-\lambda r} \int_{\mathbb{R}_0} \mathbb{E}(1 - \varphi(x)^{Z_r}) \nu_\alpha(\mathrm{d}x)\mathrm{d}r < \infty.$$

If $\{\varphi_n, n \geq 1\} \subset \mathcal{B}_1(\mathbb{R})$ are such that φ and φ_n are identically 1 on $[-\delta, \delta]$ for some $\delta > 0$, and that $\varphi_n \to \varphi$ almost everywhere, then by the dominated convergence theorem,

$$C(\varphi_n) \to C(\varphi).$$

Hence $C(\varphi)$ is continuous in φ.

Proposition 3.6. *If $\Lambda(t)$ is a positive function with $\lim_{t\to\infty} \Lambda(t)/h(t) = \infty$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,*

$$\lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})\right)\right) = C(\varphi), \tag{3.19}$$

where $C(\varphi)$ is defined in (1.7)*. Furthermore,*

$$\lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} \mathbb{P}(M_t > \Lambda(t)) = \vartheta^*, \tag{3.20}$$

where M_t is defined in (3.1)*, and ϑ^* is defined in* (1.10)*.*

Proof. For $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. It is easy to see that for any $0 < s < t$, on the event $\{M_{s,t} \leq \delta\Lambda(t)\}$, it holds that $\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))}) = \mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{(\Lambda(t))})$. Thus, for any $0 < s < t$,

$$\left|\mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})\right) - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{(\Lambda(t))})\right)\right| \leq \mathbb{P}(M_{s,t} > \delta\Lambda(t)). \tag{3.21}$$

Using the Markov property and (3.12), we have that

$$\begin{aligned} &1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{(\Lambda(t))}) | \mathcal{F}_{t-s}\right) = 1 - \prod_{v \in \mathcal{L}_{t-s}} \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{v,(\Lambda(t))}) | \mathcal{F}_{t-s}\right) \\ =&1 - \left[\mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right)\right]^{Z_{t-s}}. \end{aligned} \tag{3.22}$$

It follows from Lemma 3.5 that, as $t \to \infty$,

$$1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right) \sim C_s \Lambda(t)^{-\alpha} L(\Lambda(t)),$$

where $C_s = \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_r}\right) \nu_\alpha(\mathrm{d}y)\, \mathrm{d}r$. Thus

$$-\log \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right) \sim 1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right) \sim C_s \Lambda(t)^{-\alpha} L(\Lambda(t))$$

and consequently

$$-Z_{t-s} \log \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right) \sim e^{\lambda(t-s)} C_s \Lambda(t)^{-\alpha} L(\Lambda(t)) \cdot W,$$

which tends to 0 as $t \to \infty$ since $\lim_{t\to\infty} \Lambda(t)/h(t) = \infty$. Combining this with (3.22), we get that, as $t \to \infty$,

$$\begin{aligned} &\lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} \left(1 - \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_{s,t}^{(\Lambda(t))}) | \mathcal{F}_{t-s}\right)\right) \\ =& \lim_{t\to\infty} e^{-\lambda t} \Lambda(t)^\alpha L(\Lambda(t))^{-1} Z_{t-s}(-\log \mathbb{E}\left(\mathcal{I}(\varphi, \mathbb{Y}_s'^{(\Lambda(t))})\right)) \\ =&e^{-\lambda s} \int_0^s \left(e^{\lambda(s-r)} - e^{-\beta(s-r)}\right) \int_{\mathbb{R}} \mathbb{E}\left(1 - \varphi(y)^{Z_r}\right) \nu_\alpha(\mathrm{d}y)\, \mathrm{d}r \cdot W. \end{aligned} \tag{3.23}$$

Moreover,

$$\begin{aligned}
&e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\left(1-\mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}_{s,t}^{(\Lambda(t))})|\mathcal{F}_{t-s}\right)\right)\\
\leq&e^{-\lambda t}Z_{t-s}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\left(1-\mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}_{s}'^{(\Lambda(t))})\right)\right)\\
\overset{t}{\leq}&2e^{-\lambda s}\int_0^s\left(e^{\lambda(s-r)}-e^{-\beta(s-r)}\right)\int_{\mathbb{R}}\mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_{\alpha}(\mathrm{d}y)\,\mathrm{d}r\cdot e^{-\lambda(t-s)}Z_{t-s}. \qquad (3.24)
\end{aligned}$$

In the first inequality we used the inequality $1-x^n\leq n(1-x), x\in(0,1)$. Note that

$$e^{-\lambda(t-s)}Z_{t-s}\to W \quad \text{a.s.,} \quad \text{and } \mathbb{E}\left(e^{-\lambda(t-s)}Z_{t-s}\right)\to\mathbb{E}W=1, \quad t\to\infty, \qquad (3.25)$$

where we used **Assumption 1**. Combining (3.23), (3.24) and (3.25), and using the dominated convergence theorem we get

$$\begin{aligned}
&\lim_{t\to\infty}e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\left(1-\mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}_{s,t}^{(\Lambda(t))})\right)\right)\\
=&e^{-\lambda s}\int_0^s\left(e^{\lambda(s-r)}-e^{-\beta(s-r)}\right)\int_{\mathbb{R}}\mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_{\alpha}(\mathrm{d}y)\,\mathrm{d}r. \qquad (3.26)
\end{aligned}$$

Using (3.21), Lemma 3.4 and (3.26), we get that

$$\begin{aligned}
&\lim_{t\to\infty}e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\left(1-\mathbb{E}\left(\mathcal{I}(\varphi,\mathbb{Y}_{t}^{(\Lambda(t))})\right)\right)\\
=&\lim_{s\to\infty}e^{-\lambda s}\int_0^s\left(e^{\lambda(s-r)}-e^{-\beta(s-r)}\right)\int_{\mathbb{R}}\mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_{\alpha}(\mathrm{d}y)\,\mathrm{d}r\\
=&\int_0^{\infty}e^{-\lambda r}\int_{\mathbb{R}}\mathbb{E}\left(1-\varphi(y)^{Z_r}\right)\nu_{\alpha}(\mathrm{d}y)\,\mathrm{d}r=C(\varphi).
\end{aligned}$$

We now prove (3.20). For any $\epsilon\in(0,1)$, define $\tilde{I}_{\varepsilon},I_{\varepsilon}\in\mathcal{H}(\mathbb{R})$ by

$$I_{\varepsilon}(y):=\begin{cases}1, & y<1,\\ \text{linear}, & 1\leq y\leq 1+\varepsilon,\\ 0, & y>1+\varepsilon\end{cases}$$

and

$$\tilde{I}_{\varepsilon}(y):=\begin{cases}1, & y<1-\varepsilon,\\ \text{linear}, & 1-\varepsilon\leq y\leq 1,\\ 0, & y>1.\end{cases}$$

It is clear that $\tilde{I}_{\varepsilon}(y)\leq\mathbf{1}_{(-\infty,1]}(y)\leq I_{\varepsilon}(y)$. Applying (3.19) to $\tilde{I}_{\varepsilon}$ and I_{ε}, and by the continuity of $C(\varphi)$, we can get (3.19) still holds for $\varphi(y)=\mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\begin{aligned}
\lim_{t\to\infty}e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\mathbb{P}(M_t>\Lambda(t))&=C(\mathbf{1}_{(-\infty,1]})\\
&=\int_0^{\infty}e^{-\lambda r}\int_1^{\infty}\mathbb{P}(Z_r>0)\nu_{\alpha}(\mathrm{d}y)\mathrm{d}r=\frac{q_1}{\alpha}\vartheta.
\end{aligned}$$

The proof is now complete. □

Proof of Theorem 1.3. Applying Proposition 3.1 and Proposition 3.6, we have that

$$\lim_{t\to\infty}e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\left(1-\mathbb{E}\mathcal{I}(\varphi,\mathbb{X}_t^{(\Lambda(t))})\right)=C(\varphi), \quad \varphi\in\mathcal{H}(\mathbb{R}). \qquad (3.27)$$

By the continuity of $C(\varphi)$, we can get (3.27) still holds for $\varphi(y)=\mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\lim_{t\to\infty}e^{-\lambda t}\Lambda(t)^{\alpha}L(\Lambda(t))^{-1}\mathbb{P}(R_t>\Lambda(t))=C(\mathbf{1}_{(-\infty,1]})=\vartheta^*. \qquad (3.28)$$

□

Proof of Theorem 1.4. By (3.28), we have for $x > 1$,

$$\lim_{t\to\infty} \mathbb{P}(R_t > x\Lambda(t) | R_t > \Lambda(t)) = x^{-\alpha} = P(R^* > x).$$

For any $g \in C_c^+(\overline{\mathbb{R}}_0)$ and $\theta > 0$, applying Theorem 1.3 with $\varphi(y) = e^{-\theta g(y)}\mathbf{1}_{(-\infty,1]}(y)$ and $\varphi(y) = e^{-\theta g(y)}$, we have that

$$\begin{aligned}
&\lim_{t\to\infty} e^{-\lambda t}\Lambda(t)^\alpha L(\Lambda(t))^{-1}\mathbb{E}\Big(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)}, R_t > \Lambda(t)\Big)\\
&= \lim_{t\to\infty} e^{-\lambda t}\Lambda(t)^\alpha L(\Lambda(t))^{-1}\left[1 - \mathbb{E}\Big(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)}, R_t \le \Lambda(t)\Big) - \Big(1 - \mathbb{E}\Big(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)}\Big)\Big)\right]\\
&= \int_0^\infty e^{-\lambda r}\int_1^\infty \mathbb{E}[e^{-\theta g(y)Z_r}; Z_r > 0]\nu_\alpha(\mathrm{d}y)\mathrm{d}r. \qquad (3.29)
\end{aligned}$$

By (3.28) and (3.29), we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\Big(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)} | R_t > \Lambda(t)\Big) &= \frac{\alpha}{q_1\vartheta}\int_0^\infty e^{-\lambda r}\int_1^\infty \mathbb{E}[e^{-\theta g(y)Z_r}; Z_r > 0]\nu_\alpha(\mathrm{d}y)\mathrm{d}r\\
&= E\left(e^{-\theta T\cdot g(R^*)}\right).
\end{aligned}$$

The proof is now complete. □

4 Lower deviation of $\mathbb{X}_t$ and $\mathbb{Y}_t$

We first give some results about the continuous time branching processes $\{Z_t : t \ge 0\}$. Recall that the extinction probability $q \in [0,1)$, the constant ρ defined in (1.11), the function $A(s)$ defined in (1.12), and $F(s,t) = \mathbb{E}\left(s^{Z_t}\right)$. The function $A(s)$ is the unique solution of

$$A(F(s,t)) = e^{-\rho t}A(s) \qquad (4.1)$$

with $A(q) = 0, A'(q) = 1$. Since $A(s)$ is the limit of power series, it is itself a power series

$$A(s) = \sum_{j=0}^\infty a_j s^j.$$

It is clear that $a_0 \le 0$ and $a_j \ge 0, j \ge 1$. For any $0 < c < 1 - q$ and $s \in [0,1]$,

$$A(cs+q) = A(cs+q) - A(q) = \sum_{j=1}^\infty a_j[(cs+q)^j - q^j] = \sum_{k=1}^\infty\Big[\sum_{j=k}^\infty a_j\binom{j}{k}q^{j-k}\Big]c^k s^k.$$

Thus $\frac{A(cs+q)}{A(c+q)}$, $s \in [0,1]$, is a probability generating function.

Lemma 4.1. *For any $s \in [0,1)$,*

$$A(s) = s - q + \int_0^\infty \beta e^{\rho t}V(F(s,r))\,\mathrm{d}r,$$

where

$$0 \le V(s) := f(s) - f'(q)s - q(1-f'(q)) \le f''(s\vee q)(s-q)^2. \qquad (4.2)$$

Proof. By the Markov property and the branching property, we have that

$$\begin{aligned}F(s,t) =& se^{-\beta t} + \int_0^t \beta e^{-\beta r} f(F(s,t-r))\mathrm{d}r \\ =& se^{-\rho t} + \int_0^t \beta e^{-\rho r} f(F(s,t-r))\mathrm{d}r - \int_0^t e^{-\rho r}\beta f'(q)F(s,t-r)\mathrm{d}r \\ =& se^{-\rho t} + \int_0^t \beta e^{-\rho r} V(F(s,t-r))\mathrm{d}r + q\rho\int_0^t e^{-\rho r}\mathrm{d}r \\ =& e^{-\rho t}\left(s - q + \int_0^t \beta e^{\rho r} V(F(s,r))\,\mathrm{d}r\right) + q.\end{aligned}$$

By (1.12), we have

$$\begin{aligned}A(s) &= \lim_{t\to\infty} e^{\rho t}(F(s,t)-q) = s - q + \lim_{t\to\infty}\int_0^t \beta e^{\rho r} V(F(s,r))\,\mathrm{d}r \\ &= s - q + \int_0^\infty \beta e^{\rho r} V(F(s,r))\,\mathrm{d}r.\end{aligned}$$

Since $q = f(q)$, we have $V(s) = f(s) - f(q) - f'(q)(s-q)$. Now (4.2) follows immediately from Taylor's formula. □

Recall that $\phi(\theta) = \mathbb{E}(e^{-\theta W})$.

Lemma 4.2. *For any $\theta > 0$,*

$$A\left(\phi(\theta)\right) = \int_{-\infty}^{\infty} \beta e^{\rho s} V\left[\phi\left(\theta e^{\lambda s}\right)\right]\, \mathrm{d}s.$$

Proof. By the branching property and the Markov property, we have

$$Z_{t+s} = \sum_{u\in\mathcal{L}_s} Z^u_{t+s}.$$

Given Z_s, $\{Z^u_{t+s}, u\in\mathcal{L}_s\}$ are i.i.d. with the same law as Z_t. It follows that

$$W = \lim_{t\to\infty} e^{-\lambda s}\sum_{u\in\mathcal{L}_s} e^{-\lambda t} Z^u_{t+s} = e^{-\lambda s}\sum_{u\in\mathcal{L}_s} W^u,$$

where $\{W^u, u\in\mathcal{L}_s\}$ are i.i.d. with the same law as W. Thus we have

$$\phi(\theta e^{\lambda s}) = \mathbb{E}\left(\phi(\theta)^{Z_s}\right) = F(\phi(\theta), s). \tag{4.3}$$

Hence by Lemma 4.1 and (4.3), we have for any $x > 0$

$$\begin{aligned}&\int_{-x}^{\infty} \beta e^{\rho s} V\left[\phi\left(\theta e^{\lambda s}\right)\right]\,\mathrm{d}s = e^{-\rho x}\int_0^\infty \beta e^{\rho s} V\left[(\phi(\theta e^{\lambda s}e^{-\lambda x}\right]\,\mathrm{d}s \\ =& e^{-\rho x}\int_0^\infty \beta e^{\rho s} V\left[F(\phi(\theta e^{-\lambda x}), s)\right]\,\mathrm{d}s = e^{-\rho x}[A\left(\phi(\theta e^{-\lambda x})\right) - \phi(\theta e^{-\lambda x}) + q] \\ =& A(F(\phi(\theta e^{-\lambda x}), x)) - e^{-\rho x}(\phi(\theta e^{-\lambda x}) - q) \\ =& A(\phi(\theta)) - e^{-\rho x}(\phi(\theta e^{-\lambda x}) - q),\end{aligned}$$

where the fourth equality follows from (4.1). Letting $x\to\infty$, the desired result follows immediately. □

Lemma 4.3. *It holds that $\lambda \geq \rho$. Moreover, $\lambda = \rho$ if and only if $p_k = 0$ for all $k \geq 3$.*

Proof. Note that $\lambda - \rho = \beta(f'(1) + f'(q) - 2)$. We first consider the case when $p_0 = 0$. Then $q = 0$ and $f'(q) = f'(0) = p_1$. Hence we have

$$f'(1) + f'(0) - 2 = \left(\sum_{k=1}^{\infty} k p_k + p_1 - 2 \sum_{k=1}^{\infty} p_k \right) = \sum_{k=2}^{\infty} (k-2) p_k \geq 0.$$

Furthermore, the left hand side is equal to 0 if and only if $p_k = f^{(k)}(0) = 0$, $k \geq 3$.

If $p_0 > 0$, then $q > 0$. Define

$$\hat{f}(s) := \frac{f((1-q)s + q) - q}{1 - q}, \quad \text{for } 0 \leq s \leq 1.$$

It is well known (see [5, Chapter 1, Section 12]) that $\hat{f}$ is a probability generating function with $\hat{f}(0) = 0$. Note that $\hat{f}'(1) = f'(1)$ and $\hat{f}'(0) = f'(q)$. Applying the previous paragraph to $\hat{f}$, we get that

$$f'(1) + f'(q) - 2 = \hat{f}'(1) + \hat{f}'(0) - 2 \geq 0.$$

Moreover, the left hand side is equal to 0 if and only if $\hat{f}^{(k)}(0) = (1-q)^{k-1} f^{(k)}(q) = 0, k \geq 3$. It is easy to see that $f^{(3)}(q) = \sum_{k=3}^{\infty} k(k-1)(k-2) p_k q^{k-3} = 0$ if and only if $p_k = 0$, $k \geq 3$.

The proof is now complete. □

Note that the skeleton $\{Z_n, n = 0, 1, \cdots\}$ is a Galton-Watson process with offspring generating function $\tilde{f}(s) := F(s, 1)$. It is clear that $\tilde{f}'(1) = \mathbb{P}(Z_1) = e^{\lambda} > 1$ and $\mathbb{P}(Z_1 = 1) \geq \mathbb{P}(\tau_o > 1) > 0$. By [28, Theorem 4], we have that, for any nonnegative sequence a_n with $a_n \to \infty$ and $a_n = o(e^{\lambda n})$, there exists a constant $c > 0$ such that

$$\mathbb{P}(0 < Z_n < a_n) \leq c \mathbb{P}(0 < W < e^{-\lambda n} a_n), \qquad n \geq 1. \tag{4.4}$$

Furthermore, it was proven in [21] that there exist $c_1, c_2 > 0$ such that for any $x \in (0, 1)$,

$$c_1 x^{\rho/\lambda} < \mathbb{P}(0 < W < x) < c_2 x^{\rho/\lambda}. \tag{4.5}$$

By (4.4) and (4.5), there exists $C > 0$ such that

$$\mathbb{P}(0 < Z_n < a_n) \leq C a_n^{\rho/\lambda} e^{-\rho n}, \quad n \geq 1. \tag{4.6}$$

4.1 Lower deviation of $\mathbb{X}_t$

We first give the proof of Theorem 1.2.

Proof of Theorem 1.2. Under condition **(H)**, we have proved in [39, Proposition 2.1] that $\mathbb{Y}_t^{(h(t))}$ converges weakly to $\mathcal{N}_\infty$, that is for any $\varphi = e^{-g}$ with $g \in C_c^+(\overline{\mathbb{R}}_0)$,

$$\lim_{t \to \infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(h(t))})) = \mathbb{E}(\mathcal{I}(\varphi, \mathcal{N}_\infty)) = \mathbb{E}\left(\exp\left\{-C(\varphi) W\right\}\right). \tag{4.7}$$

Now we replace condition **(H)** with the weaker **Assumption 2**. By inspecting the arguments in [39], replacing (2.7) with Lemma 2.3, and using (3.11) and (3.17), we can show that (4.7) is valid for any $\varphi \in \mathcal{H}(\mathbb{R})$ under **Assumption 2**.

By Proposition 3.1 we have for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t \to \infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{X}_t^{(h(t))})) = \mathbb{E}(\mathcal{I}(\varphi, \mathcal{N}_\infty)) = \mathbb{E}\left(\exp\left\{-C(\varphi) W\right\}\right).$$

The proof is now complete. □

For any function g and $x \neq 0$, we define the function $\mathfrak{m}_x g(\cdot)$ by the relation $\mathfrak{m}_x g(y) = g(y/x)$. By the definition of $C(\varphi)$ in (1.7), it is easy to see that

$$C(\mathfrak{m}_x \varphi) = x^{-\alpha} C(\varphi), \quad \varphi \in \mathcal{B}_1(\mathbb{R}).$$

Lemma 4.4. *If $a(t), b(t)$ are functions with $a(t)/h(t) \to a > 0$ and $b(t)/a(t) \to 0$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,*

$$\lim_{t\to\infty} \mathbb{E}_{b(t)}(\mathcal{I}(\varphi, \mathbb{X}_t^{(a(t))})) = \mathbb{E}(\exp\{-a^{-\alpha} C(\varphi) W\}).$$

Proof. Note that

$$\mathbb{E}_{b(t)}(\mathcal{I}(\varphi, \mathbb{X}_t^{(a(t))})) = \mathbb{E}\left(\prod_{u\in\mathcal{L}_t} \varphi\left(\frac{\xi_t^u + b(t)}{a(t)}\right)\right).$$

Since $\varphi \in \mathcal{H}(\mathbb{R})$, we have $\varphi \equiv 1$ on $[-\delta, \delta]$ for some $\delta > 0$. Moreover for any $\epsilon > 0$, there exists $\eta = \eta(\epsilon) > 0$ such that $|\varphi(x) - \varphi(y)| < \epsilon$ whenever $|x - y| \leq \eta$. It follows from the assumption that for any $\epsilon' > 0$, there exists t' such that for all $t > t'$, $(1 - \epsilon')a(t) \leq ah(t) \leq (1 + \epsilon')a(t)$ and $|b(t)| \leq \epsilon' a(t)$.

We now fix an arbitrary $\epsilon > 0$ and $0 < \epsilon' < \frac{\delta}{\delta+2} \wedge \eta$. Note that if $\frac{|y|}{ah(t)} \leq \delta/2$, then for $t > t'$,

$$\left|\frac{y + b(t)}{a(t)}\right| \leq \frac{(1+\epsilon')|y|}{ah(t)} + \epsilon' \leq \left(1 + \frac{\delta}{\delta+2}\right)\frac{\delta}{2} + \frac{\delta}{\delta+2} = \delta,$$

and that if $\frac{|y|}{ah(t)} \leq \frac{\eta}{\epsilon'} - 1$, then for $t > t'$,

$$\left|\frac{y + b(t)}{a(t)} - \frac{y}{ah(t)}\right| \leq \epsilon' + \frac{|y|}{ah(t)}\epsilon' \leq \eta.$$

Thus for any $t > t'$,

$$\left|\varphi\Big(\frac{y + b(t)}{a(t)}\Big) - \varphi\Big(\frac{|y|}{ah(t)}\Big)\right| \leq \epsilon \mathbf{1}_{\{\frac{\delta}{2} < \frac{|y|}{ah(t)} \leq \frac{\eta}{\epsilon'} - 1\}} + \mathbf{1}_{\{\frac{|y|}{ah(t)} > \frac{\eta}{\epsilon'} - 1\}}.$$

Hence we have that

$$\begin{aligned}
&\left|\mathbb{E}\left(\prod_{u\in\mathcal{L}_t} \varphi\left(\frac{\xi_t^u + b(t)}{a(t)}\right)\right) - \mathbb{E}\left(\prod_{u\in\mathcal{L}_t} \varphi\left(\frac{\xi_t^u}{ah(t)}\right)\right)\right| \\
\leq &\mathbb{E}\left[1 \wedge \sum_{u\in\mathcal{L}_t} \left|\varphi\left(\frac{\xi_t^u + b(t)}{a(t)}\right) - \varphi\left(\frac{\xi_t^u}{ah(t)}\right)\right|\right] \\
\leq &\mathbb{E}\left[1 \wedge \left(\epsilon \sum_{u\in\mathcal{L}_t} \mathbf{1}_{\{\frac{|\xi_t^u|}{h(t)} > a\delta/2\}}\right)\right] + \mathbb{E}\left[1 \wedge \sum_{u\in\mathcal{L}_t} \mathbf{1}_{\{\frac{|\xi_t^u|}{h(t)} > a(\eta/\epsilon' - 1)\}}\right]. \qquad (4.8)
\end{aligned}$$

Since $\mathbb{X}_t^{(h(t))}$ converges weakly to $\mathcal{N}_\infty$, we have, as $t \to \infty$,

$$\epsilon \sum_{u\in\mathcal{L}_t} \mathbf{1}_{\{\frac{|\xi_t^u|}{h(t)} > a\delta/2\}} \xrightarrow{d} \epsilon \cdot \mathcal{N}_\infty(\{y \in \mathbb{R}; |y| > a\delta/2\})$$

and

$$\sum_{u\in\mathcal{L}_t} \mathbf{1}_{\{\frac{|\xi_t^u|}{h(t)} > a(\eta/\epsilon - 1)\}} \xrightarrow{d} \mathcal{N}_\infty(\{y \in \mathbb{R}; |y| > a(\eta/\epsilon' - 1)\}.$$

Thus letting $t \to \infty$ first and then $\epsilon' \to 0$, and finally $\epsilon \to 0$ in (4.8), applying the dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left(\prod_{u\in\mathcal{L}_t} \varphi\left(\frac{\xi_t^u + b(t)}{a(t)}\right)\right) = \lim_{t\to\infty} \mathbb{E}\left(\prod_{u\in\mathcal{L}_t} \varphi\left(\frac{\xi_t^u}{ah(t)}\right)\right) \\
&= \mathbb{E}\left(\exp\left\{-C(\mathfrak{m}_a \varphi) W\right\}\right) = \mathbb{E}(\exp\{-a^{-\alpha} C(\varphi) W\}).
\end{aligned}$$

□

For any $t>0$, $x\in\mathbb{R}$ and $\varphi\in\mathcal{B}_1(\mathbb{R})$, we define

$$U_\varphi(t,x):=\mathbb{E}_x(\mathcal{I}(\varphi,\mathbb{X}_t))=\mathbb{E}_x\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u)\right) \tag{4.9}$$

and

$$u(t,x):=U_{\mathbf{1}_{(-\infty,0]}}(t,x)=\mathbb{P}_x(R_t\le 0)=\mathbb{P}(R_t\le -x). \tag{4.10}$$

By the Markov property and the branching property, we have

$$U_\varphi(t,x)=e^{-\beta t}\mathbf{E}_x(\varphi(\xi_t))+\mathbf{E}_x\int_0^t\beta e^{-\beta s}f(U_\varphi(t-s,\xi_s))\mathrm{d}s,$$

which implies that

$$\begin{aligned}U_\varphi(t,x)&=e^{\beta(f'(q)-1)t}\mathbf{E}_x(\varphi(\xi_t))+\mathbf{E}_x\int_0^t\beta e^{\beta(f'(q)-1)s}f(U_\varphi(t-s,\xi_s))\,\mathrm{d}s\\&\quad-\mathbf{E}_x\int_0^t e^{\beta(f'(q)-1)s}\beta f'(q)U_\varphi(t-s,\xi_s)\,\mathrm{d}s\\&=e^{-\rho t}\mathbf{E}_x(\varphi(\xi_t))+\mathbf{E}_x\int_0^t\beta e^{-\rho s}V(U_\varphi(t-s,\xi_s))\,\mathrm{d}s+q(1-e^{-\rho t}),\end{aligned}$$

where ρ is defined in (1.11) and V is defined in (4.2). Thus

$$U_\varphi(t,x)-q=e^{-\rho t}(\mathbf{E}(\varphi(\xi_t+x))-q)+\mathbf{E}\int_0^t\beta e^{-\rho s}V(U_\varphi(t-s,x+\xi_s))\,\mathrm{d}s. \tag{4.11}$$

The following lemma will used in the proof of Theorem 1.6.

Lemma 4.5. *Let $c>0$, and let $\Lambda(t)$ be a non-decreasing function satisfying **(S1)** and **(S2)**. Let $r(t)$ be defined by $\Lambda(t)=h(r(t))$. For any $\epsilon\in(0,1)$, there exist $c(\epsilon)>0$ and $t_\epsilon>0$ such that for any $t_\epsilon<s<t-r(t)$,*

$$\mathbf{E}[(u(r(t)+s,-c\,h(r(t))+\xi_{t-r(t)-s})-q)^2]\le c(\epsilon)s^{6\rho/\lambda}\left[e^{-2\rho(1-\epsilon)s}+te^{-\lambda r(t)}e^{-(\lambda\wedge(2\rho))(1-\epsilon)s}\right].$$

Proof. We always assume $0<s<t-r(t)$. Let $l=t-r(t)-s$, and we sometimes write r for $r(t)$ for short. Define $\tilde{u}(t,x):=\mathbb{P}(-\infty<R_t\le -x)=u(t,x)-\mathbb{P}(Z_t=0)$. Then

$$\begin{aligned}&\mathbf{E}[(u(r+s,-c\,h(r)+\xi_l)-q)^2]\le 2\mathbf{E}[(\tilde{u}(r+s,-c\,h(r)+\xi_l))^2]+2(q-\mathbb{P}(Z_{r+s}=0))^2\\&\le 2\mathbf{E}[(\tilde{u}(r+s,-c\,h(r)+\xi_l)^2]+2(q-\mathbb{P}(Z_s=0))^2.\end{aligned} \tag{4.12}$$

By (1.12) with $s=0$, we have

$$\lim_{s\to\infty}e^{\rho s}(q-\mathbb{P}(Z_s=0))=-A(0)\in[0,\infty). \tag{4.13}$$

For $t>0$, define

$$T_t:=\inf\{s>0:Z_s>t^3\}.$$

Here we use the convention that $\inf\emptyset=\infty$. Let $K>3$ be an integer. Note that

$$\begin{aligned}&\tilde{u}(s+r,-c\,h(r)+x)=\mathbb{P}(-\infty<R_{s+r}\le c\,h(r)-x)\\&\le\mathbb{P}\left(T_s>(K-1)s/K,Z_{(K-1)s/K}>0\right)\\&\quad+\sum_{k=1}^{K-1}\mathbb{P}\left((k-1)s/K<T_s\le ks/K,R_{s+r}\le c\,h(r)-x\right).\end{aligned} \tag{4.14}$$

By (4.6) with $a_n = (2K)^3 n^3$, there exists a constant $C_K > 0$ such that

$$\mathbb{P}(0 < Z_n \le (2K)^3 n^3) \le C_K (2Kn)^{3\rho/\lambda} e^{-\rho n}, \quad n \ge 1.$$

It follows that, for $s > K$ and $k = 1, 2, \cdots, K-1$,

$$\begin{aligned}
&\mathbb{P}\left(T_s > ks/K, Z_{ks/K} > 0\right) \le \mathbb{P}\left(0 < Z_{\lfloor ks/K \rfloor} \le s^3\right) \le \mathbb{P}\left(0 < Z_{\lfloor ks/K \rfloor} \le (2K)^3 \lfloor ks/K \rfloor^3\right) \\
&\le C_K \left(2Ks\right)^{3\rho/\lambda} e^{-\rho \lfloor ks/K \rfloor} \le C_K (2K)^{3\rho/\lambda} e^{\rho} \cdot s^{3\rho/\lambda} e^{-\rho ks/K}. \qquad (4.15)
\end{aligned}$$

Choose $b > 1$ such that $e^{-\lambda b/\alpha} < 1$. Note that $\lim_{t\to\infty} h(t)/h(t+b) = e^{-\lambda b/\alpha} < 1$. By (1.9), we have $\lim_{t\to\infty} \mathbb{P}(R_t \le h(t)) = \mathbb{E}(e^{-\vartheta^* W}) < 1$. Thus there exist $c_1, c_2 \in (0,1)$ and $t_1 > 0$ such that for all $t > t_1$, and $t' > b$,

$$\mathbb{P}(R_t \le h(t)) \le c_1, \quad \text{and } h(t) \le c_2 h(t+b) \le c_2 h(t+t').$$

Note that

$$R_{s+r} = \max_{u \in \mathcal{L}_{ks/K}} (\xi^u_{ks/K} + R^u_{s+r}),$$

where, given $\mathcal{F}^{\mathbb{T}}_{sk/K}$, $\{R^u_{s+r}, u \in \mathcal{L}_{ks/K}\}$ are i.i.d. with the same law as $R_{s+r-ks/K}$. Moreover, given $\mathcal{F}^{\mathbb{T}}_{ks/K}$, $\{\xi^u_{ks/K}, u \in \mathcal{L}_{ks/K}\}$ have the same law as $\xi_{ks/K}$, and $\{\xi^u_{ks/K}, u \in \mathcal{L}_{ks/K}\}$ and $\{R^u_{s+r}, u \in \mathcal{L}_{ks/K}\}$ are independent. Applying [30, Lemma 5.1] in the first inequality below we get that, for $k = 1, 2, \cdots, K-1$, $r > t_1$ and $s > bK$,

$$\begin{aligned}
&\mathbb{P}(R_{s+r} \le c\, h(r) - x | \mathcal{F}^{\mathbb{T}}_{ks/K}) \le \mathbb{P}(\xi_{ks/K} + \max_{u \in \mathcal{L}_{ks/K}} R^u_{s+r} \le c\, h(r) - x | \mathcal{F}^{\mathbb{T}}_{ks/K}) \\
\le &\mathbf{P}(x + \xi_{ks/K} \le c\, h(r) - h(s+r-ks/K)) + [\mathbb{P}(R_{s+r-ks/K} \le h(s+r-ks/K))]^{Z_{ks/K}} \\
\le &\mathbf{P}(x + \xi_{ks/K} \le -(1 - c\, c_2) h(s+r-ks/K)) + c_1^{Z_{ks/K}}.
\end{aligned}$$

Thus we have that for $k = 1, 2, \cdots, K-1$, $r > t_1$ and $s > bK$,

$$\begin{aligned}
&\mathbb{P}\left((k-1)s/K < T_s \le ks/K, R_{s+r} \le c\, h(r) - x\right) \\
\le &\mathbf{P}(x + \xi_{ks/K} \le -(1 - c\, c_2) h(s+r-ks/K)) \mathbb{P}\left(T_s > (k-1)s/K, Z_{(k-1)s/K} > 0\right) \\
&+ \mathbb{E}\left(c_1^{Z_{ks/K}}; (k-1)s/K < T_s \le ks/K\right). \qquad (4.16)
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E}\left(c_1^{Z_{ks/K}}; (k-1)s/K < T_s \le ks/K\right) \\
\le &c_1^{s^2} + \mathbb{P}\left((k-1)s/K < T_s \le ks/K, Z_{ks/K} < s^2\right) \\
\le &c_1^{s^2} + \mathbb{P}(\exists t \ge 0 : Z_t \le \lfloor s^2 \rfloor | Z_0 = \lfloor s^3 \rfloor) \\
\le &c_1^{s^2} + \binom{\lfloor s^3 \rfloor}{\lfloor s^2 \rfloor} q^{\lfloor s^3 \rfloor - \lfloor s^2 \rfloor} \le c_1^{s^2} + (s^3+1)^{s^2+1} q^{s^3 - s^2 - 2} = o(e^{-\rho s}). \qquad (4.17)
\end{aligned}$$

Combining (4.14), (4.15), (4.16) and (4.17), we get that there exist C'_k and $t_2 > 0$ such that for any $r > t_2, s > t_2$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&[\mathbb{P}(-\infty < R_{s+r} \le c\, h(r) - x)]^2 \le C'_K \Big[s^{6\rho/\lambda} e^{-2\rho(K-1)s/K} \\
&+ \sum_{k=1}^{K-1} s^{6\rho/\lambda} e^{-2\rho(k-1)s/K} \mathbf{P}(x + \xi_{ks/K} \le -(1 - c\, c_2) h(s+r-ks/K)) \Big]. \qquad (4.18)
\end{aligned}$$

Now we consider $x = \xi_l$. Note that

$$(l + ks/K) V_\delta((1 - c\, c_2) h(s+r-ks/K)) \le t V_\delta((1 - c\, c_2) h(r(t))) = t V_\delta((1 - c\, c_2) \Lambda(t)),$$

and $tV_\delta((1-c\,c_2)\Lambda(t))$ is bounded by **(S2)**.

By the Markov property of ξ and Lemma 2.3, we get that

$$\begin{aligned}&\mathbf{E}[\mathbf{P}(x+\xi_{ks/K}\le -(1-c\,c_2)h(s+r-ks/K))|_{x=\xi_l}]\\ =&\mathbf{P}(\xi_{l+ks/K}\le -(1-c\,c_2)h(s+r-ks/K))\\ \le& c_3te^{-\lambda(r+s-ks/K)},\end{aligned}$$

for some constant $c_3>0$. It follows from (4.18) that

$$\begin{aligned}&\mathbf{E}[\tilde{u}(s+r,-c\,h(r)+\xi_l)]^2\\ \le& C''_K\Big[s^{6\rho/\lambda}e^{-2\rho(K-1)s/K}+s^{6\rho/\lambda}t\sum_{k=1}^{K-1}e^{-2\rho(k-1)s/K}e^{-\lambda(r+s-ks/K)}\Big]\\ \le& C''_K\Big[s^{6\rho/\lambda}e^{-2\rho(K-1)s/K}+Ks^{6\rho/\lambda}te^{-\lambda r}e^{-(\lambda\wedge(2\rho))(K-1)s/K}\Big],\end{aligned}$$

where $C''_K=C'_k(1+c_3)$ and in the last inequality we used

$$2\rho(k-1)+\lambda(K-k)\ge(\lambda\wedge 2\rho)(K-1),\quad k=1,\cdots,K.$$

For any $\epsilon>0$, we choose K such that $1/K<\epsilon$, we get that there exists $c(\epsilon)>0$ such that

$$2\mathbf{E}[(\tilde{u}(r+s,-c\,h(r)+\xi_l)^2]\le c(\epsilon)s^{6\rho/\lambda}\Big[e^{-2\rho(1-\epsilon)s}+te^{-\lambda r}e^{-(\lambda\wedge(2\rho))(1-\epsilon)s}\Big].$$

Combining this with (4.12) and (4.13), we immediately get the desired result. The proof is now complete. □

Proof of Theorem 1.6. Since $\varphi\in\mathcal{H}_0(\mathbb{R})$, there exists $c>0$ such that $\varphi(y)=0$ for all $y>c$. For any $t>0$, define $r=r(t)$ by $h(r(t))=\Lambda(t)$, then $e^{-\lambda r(t)}=\Lambda(t)^{-\alpha}L(\Lambda(t))$. Since $\lim_{t\to\infty}\Lambda(t)/h(t)=0$ and $\Lambda(t)\to\infty$, we have $t-r(t)\to\infty$ and $r(t)\to\infty$ as $t\to\infty$. Thus for any $T>0$, there exists $t_0>0$ such that $t-r(t)>T$ and $r(t)>T$ for all $t\ge t_0$. In the remainder of this proof, we assume $t\ge t_0$. Note that

$$\begin{aligned}&(1-q)\mathbb{E}^*\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t))\right)\\ =&\mathbb{E}\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t))\right)-q+q-\mathbb{E}\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t));\mathcal{S}^c\right).\end{aligned}\tag{4.19}$$

By (1.12) with $s=0$, we have

$$0\le q-\mathbb{E}\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t));\mathcal{S}^c\right)\le q-\mathbb{P}(Z_t=0)\sim -A(0)e^{-\rho t}.\tag{4.20}$$

By (4.11), we have that

$$\mathbb{E}\left(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t))\right)-q=e^{-\rho t}(\mathbf{E}(\varphi(\xi_t/\Lambda(t)))-q)+J(t),$$

where

$$\begin{aligned}J(t)=&\left(\int_0^{t-r(t)-T}+\int_{t-r(t)-T}^{t-r(t)+T}+\int_{t-r(t)+T}^{t}\right)\beta e^{-\rho s}\mathbf{E}[V(U_{\varphi_t}(t-s,\xi_s))]\mathrm{d}s\\ =:&J_1(t,T)+J_2(t,T)+J_3(t,T),\end{aligned}$$

$\varphi_t(y) = \varphi(y/\Lambda(t))$ and U_φ is defined in (4.9). It is easy to see that

$$e^{-\rho t}(\mathbf{E}(\varphi(\xi_t/\Lambda(t))) - q) = o(e^{-\rho(t-r(t))}).$$

We now deal with $J_1(t,T)$, $J_2(t,T)$ and $J_3(t,T)$ separately.

(1) Since $0 \le V(s) \le f(s) \le 1$, we have

$$e^{\rho(t-r(t))}J_3(t,T) \le e^{\rho(t-r(t))}\int_{t-r(t)+T}^{\infty} \beta e^{-\rho s}\, ds = \frac{1}{1-f'(q)}e^{-\rho T}. \tag{4.21}$$

Hence $\lim_{T\to\infty} \limsup_{t\to\infty} e^{\rho(t-r(t))}J_3(t,T) = 0$.

(2) Note that

$$e^{\rho(t-r(t))}J_2(t,T) = \int_{-T}^{T} \beta e^{\rho s}\mathbf{E}V(U_{\varphi_t}(s+r(t)), \xi_{t-r(t)-s})\, \mathrm{d}s. \tag{4.22}$$

Since $\Lambda(t) = h(r(t))$, we have

$$U_{\varphi_t}(s+r(t), \xi_{t-r(t)-s}) = \mathbb{E}_{\xi_{t-r(t)-s}}\left(\prod_{u\in\mathcal{L}_{s+r(t)}} \varphi(\xi^u_{s+r(t)}/h(r(t)))\right).$$

By **(S1)** and the fact that Λ is non-decreasing, we have that for any $s \in [-T,T]$,

$$\lim_{t\to\infty}\frac{|\xi_{t-r(t)-s}|}{h(s+r(t))} = \lim_{t\to\infty}\frac{|\xi_{t-r(t)-s}|}{e^{\lambda s/\alpha}\Lambda(t)} \le \lim_{t\to\infty} e^{-\frac{\lambda}{\alpha}s}\frac{|\xi_{t-r(t)-s}|}{\Lambda(t-r(t)-s)} = 0,$$

in probability. Note that $h(r(t))/h(s+r(t)) \to e^{-\frac{\lambda}{\alpha}s}$. By Lemma 4.4, we have that for $s \in [-T,T]$,

$$\lim_{t\to\infty} U_{\varphi_t}(s+r(t), \xi_{t-r(t)-s}) = \mathbb{E}(\exp\{-e^{\lambda s}C(\varphi)W\}) = \phi(e^{\lambda s}C(\varphi)),$$

in probability. Since $V(s)$ is bounded and continuous, by (4.22) and the dominated convergence theorem we have

$$\lim_{t\to\infty} e^{\rho(t-r(t))}J_2(t,T) = \int_{-T}^{T}\beta e^{\rho s}V\left[\phi(e^{\lambda s}C(\varphi))\right]\mathrm{d}s. \tag{4.23}$$

(3) We now deal with $J_1(t,T)$. Recall u is defined in (4.10). Note that

$$\begin{aligned} e^{\rho(t-r(t))}J_1(t,T) &= \int_{T}^{t-r(t)} \beta e^{\rho s}\mathbf{E}V(U_{\varphi_t}(s+r(t), \xi_{t-r(t)-s}))\, \mathrm{d}s \\ &\le \int_{T}^{t-r(t)} \beta e^{\rho s}\mathbf{E}[V(u(s+r(t), -ch(r(t)) + \xi_{t-r(t)-s}))]\mathrm{d}s, \end{aligned} \tag{4.24}$$

where in the last inequality we used the fact that

$$U_{\varphi_t}(s+r(t), x) \le \mathbb{P}_x(R_{s+r(t)} \le c\,h(r(t))) = u(s+r(t), -c\,h(r(t)) + x).$$

(a) We first consider the case $\lambda > \rho$. By Lemma 2.3 we have

$$\begin{aligned} I_1(t,s) :=& \mathbf{E}[V(u(s+r(t), -c\,h(r(t)) + \xi_{t-r(t)-s}); |\xi_{t-r(t)-s}| > h(s+r(t)))] \\ \le& \mathbf{E}[\xi_{t-r(t)-s}| > h(s+r(t))] \\ \le& Cte^{-\lambda(s+r(t))}, \end{aligned}$$

where in the last inequality we used the fact $(t-r(t)-s)V_\delta(h(s+r(t))) \le tV_\delta(h(r(t))) = tV_\delta(\Lambda(t))$, which is bounded by **(S2)**. It follows by **(S1)** that

$$\begin{aligned}\int_T^{t-r(t)} \beta e^{\rho s} I_1(t,s)\,\mathrm{d}s &\le Cte^{-\lambda r(t)} \int_T^{t-r(t)} \beta e^{(\rho-\lambda)s}\,\mathrm{d}s \\ &\le \frac{C\beta}{\lambda-\rho} t\Lambda(t)^{-\alpha} L(\Lambda(t)) \to 0.\end{aligned} \tag{4.25}$$

By (1.9), we have $\mathbb{P}(R_t \le (1+c)h(t)) \to \mathbb{E}(e^{-\vartheta^*(1+c)^{-\alpha}W}) \in (q,1)$. Thus there exists $a \in (q,1)$ such that when T is large enough,

$$\mathbb{P}(R_t \le (1+c)h(t)) \le a, \quad t \ge T.$$

If $|\xi_{t-r(t)-s}| \le h(s+r(t))$, then for any $t > t_0$ and $s \in (T, t-r(t))$,

$$\begin{aligned}u(r(t)+s, -ch(r(t)) + \xi_{t-r(t)-s}) &\le \mathbb{P}(R_{s+r(t)} \le ch(r(t)) + h(s+r(t))) \\ &\le \mathbb{P}(R_{s+r(t)} \le (1+c)h(s+r(t))) \le a.\end{aligned}$$

Thus by (4.2), we have

$$V(u(r(t)+s, -ch(r(t)) + \xi_{t-r(t)-s})) \le f''(a)(u(r(t)+s, -ch(r(t)) + \xi_{t-r(t)-s}) - q)^2.$$

Choose $\epsilon \in (0, \rho/2 \wedge (\lambda-\rho))$. By Lemma 4.5, we have for any $T > t_\epsilon$ and $s \in (T, t-r(t))$,

$$\begin{aligned}I_2(t,s) &:= \mathbf{E}(V(u(s+r(t), -c\,h(r(t)) + \xi_{t-r(t)-s})); |\xi_{t-r(t)-s}| \le h(s+r(t))) \\ &\le f''(a)\mathbf{E}(u(s+r(t), -c\,h(r(t)) + \xi_{t-r(t)-s}) - q)^2 \\ &\le f''(a)c(\epsilon)s^{6\rho/\lambda}[e^{-2\rho(1-\epsilon)s} + te^{-\lambda r(t)}e^{-(2\rho\wedge\lambda)(1-\epsilon)s}] \\ &\le f''(a)c(\epsilon)s^{6\rho/\lambda}[(1+te^{-\lambda r(t)})e^{-2\rho(1-\epsilon)s} + te^{-\lambda r(t)}e^{-\lambda(1-\epsilon)s}].\end{aligned}$$

It follows that

$$\begin{aligned}&\int_T^{t-r(t)} \beta e^{\rho s} I_2(t,s)\,\mathrm{d}s \\ \le& f''(a)c(\epsilon)\left[(1+te^{-\lambda r(t)})\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s + te^{-\lambda r(t)}\int_T^\infty s^{6\rho/\lambda}e^{-(\lambda-\rho-\epsilon)s}\,\mathrm{d}s\right] \\ \to& f''(a)c(\epsilon)\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s, \quad t\to\infty.\end{aligned} \tag{4.26}$$

By (4.25) and (4.26), we have

$$\limsup_{t\to\infty} e^{\rho(t-r(t))} J_1(t,T) \le f''(a)c(\epsilon)\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s. \tag{4.27}$$

(b) We now assume $\lambda = \rho$. Since $\Lambda(t) \overset{t}{>} e^{\gamma t}$, we have

$$e^{-\lambda r(t)} = \Lambda(t)^{-\alpha}L(\Lambda(t)) \overset{t}{\le} \Lambda(t)^{-\alpha/2} \le e^{-\alpha\gamma t/2},$$

which implies that $r(t) \overset{t}{\ge} \frac{\alpha\gamma}{2\lambda}t$. Let $\epsilon \in (0, \frac{\alpha\gamma}{2\lambda} \wedge \frac{\rho}{2})$, and let $T > t_\epsilon$. By Lemma 4.3, we get $p_k = 0$ for $k \ge 3$. Thus $f''(1) < \infty$. By (4.2), we have, for any $t \ge t_0$ and $s \in (T, t-r(t))$,

$$V(u(r(t)+s, -ch(r(t)) + \xi_{t-r(t)-s})) \le f''(1)(u(r(t)+s, -ch(r(t)) + \xi_{t-r(t)-s}) - q)^2.$$

Applying Lemma 4.5, we have for any $t \ge t_0$ and $s \in (T, t-r(t))$,

$$\mathbf{E}(V(u(s+r(t), -c\,h(r) + \xi_{t-r(t)-s}))) \le f''(1)c(\epsilon)s^{6\rho/\lambda}[e^{-2\rho(1-\epsilon)s} + te^{-\lambda r(t)}e^{-\rho(1-\epsilon)s}].$$

By (4.24), it follows that

$$\begin{aligned}
&e^{\rho(t-r(t))}J_1(t,T) \le f''(1)c(\epsilon)\left[\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s + te^{-\lambda r(t)}\int_T^{t-r(t)} s^{6\rho/\lambda}e^{\rho\epsilon s}\,\mathrm{d}s\right]\\
\le& f''(1)c(\epsilon)\left[\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s + (\rho\epsilon)^{-1}t^{1+6\rho/\lambda}e^{-\alpha\gamma t/2}e^{\lambda\epsilon t}\right]\\
\to& f''(1)c(\epsilon)\int_T^\infty s^{6\rho/\lambda}e^{-(\rho-2\epsilon)s}\,\mathrm{d}s, \quad t\to\infty. \qquad (4.28)
\end{aligned}$$

Combining (4.21), (4.23), (4.27) and (4.28), and letting $T\to\infty$, we obtain that

$$\lim_{t\to\infty} e^{\rho(t-r(t))}J(t) = \int_{-\infty}^\infty \beta e^{\rho s}V\left[\phi(C(\varphi)e^{\lambda s})\right]\,\mathrm{d}s = A(\phi(C(\varphi))). \qquad (4.29)$$

Thus by (4.19), (4.20) and (4.29), we have that

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{E}^*\Big(\prod_{u\in\mathcal{L}_t}\varphi(\xi_t^u/\Lambda(t))\Big) = \frac{1}{1-q}A\left[\phi(C(\varphi))\right].$$

Since $C(\varphi)$ is continuous in φ, the above limit is also valid for $\varphi(y)=\mathbf{1}_{(-\infty,1]}(y)$, that is,

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{E}^*\Big(R_t\le\Lambda(t)\Big) = \frac{1}{1-q}A(\phi(q_1\vartheta/\alpha)).$$

The proof is now complete. □

Proof of Theorem 1.7. For any $g\in C_c^+(\overline{\mathbb{R}}_0)$ and $\theta>0$, applying Theorem 1.6 with $\varphi(y)=e^{-\theta g(y)}\mathbf{1}_{(-\infty,1]}(y)$ we get

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}^*\left(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)}|R_t\le\Lambda(t)\right) &= \lim_{t\to\infty}\frac{e^{\rho(t-r(t))}\mathbb{E}^*\left(e^{-\theta\mathbb{X}_t^{(\Lambda(t))}(g)};R_t\le\Lambda(t)\right)}{e^{\rho(t-r(t))}\mathbb{P}^*(R_t\le\Lambda(t))}\\
&= \frac{A(\phi(C(\varphi)))}{A(\phi(\vartheta^*))}.
\end{aligned}$$

By the definition of Ξ, we have

$$\mathbb{E}(e^{-\theta\Xi(g)}) = \sum_{j=1}^\infty P(\mathcal{K}=j)\left[\mathbb{E}(e^{-\theta\bar{\mathcal{N}}_\infty(g)})\right]^j = \frac{A\left((\phi(\vartheta^*)-q)\mathbb{E}(e^{-\theta\bar{\mathcal{N}}_\infty(g)})+q\right)}{A(\phi(\vartheta^*))}. \qquad (4.30)$$

Note that $\{\mathcal{N}_\infty(\mathbb{R})>0\}=\{W>0\}$. Thus

$$\begin{aligned}
E(e^{-\theta\bar{\mathcal{N}}_\infty(g)}) &= E(e^{-\theta\mathcal{N}_\infty(g)}|\mathcal{N}_\infty(\mathbb{R})>0,\mathcal{N}_\infty(1,\infty)=0)\\
&= \frac{E(e^{-\theta\mathcal{N}_\infty(g)};\mathcal{N}_\infty(\mathbb{R})>0,\mathcal{N}_\infty(1,\infty)=0)}{P(\mathcal{N}_\infty(\mathbb{R})>0,\mathcal{N}_\infty(1,\infty)=0)}\\
&= \frac{E(e^{-\theta\mathcal{N}_\infty(g)};\mathcal{N}_\infty(1,\infty)=0)-P(\mathcal{N}_\infty(\mathbb{R})=0)}{P(\mathcal{N}_\infty(1,\infty)=0)-P(\mathcal{N}_\infty(\mathbb{R})=0)}\\
&= \frac{\mathbb{E}(\exp\{-C(\varphi)W\})-\mathbb{P}(W=0)}{\mathbb{E}(\exp\{-\vartheta^*W\})-\mathbb{P}(W=0)} = \frac{\phi(C(\varphi))-q}{\phi(\vartheta^*)-q}.
\end{aligned}$$

By (4.30), we have that

$$E(e^{-\theta\Xi(g)}) = \frac{A(\phi(C(\varphi)))}{A(\phi(\vartheta^*))}.$$

The proof is complete. □

4.2 Lower deviation of $\mathbb{Y}_t$

For the proof of Theorem 1.3, we first established the upper deviation for $\mathbb{Y}_t$, and then used it to get the corresponding result for $\mathbb{X}_t$. However we cannot get the lower deviation of $\mathbb{X}_t$ from that of $\mathbb{Y}_t$. In Section 4.1, we established the lower large deviation of $\mathbb{X}_t$, see Theorem 1.7. In this subsection, we establish the lower deviation of $\mathbb{Y}_t$. Recall that ϕ is defined by (1.13).

Proposition 4.6. *If $\Lambda : [0,\infty) \to (0,\infty)$ satisfies $\Lambda(t) \to \infty$ and*

$$t^2\Lambda(t)^{-\alpha}L(\Lambda(t)) \to 0, \quad \Lambda(t)/h(t) \to 0,$$

as $t \to \infty$, then for any $\varphi \in \mathcal{H}(\mathbb{R})$,

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{E}^*(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})) = A\left(\phi\left(C(\varphi)\right)\right),$$

where $r = r(t)$ is defined by $\Lambda(t) = h(r(t))$. In particular,

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\mathbb{P}^*\Big(M_t \le \Lambda(t)\Big) = A\left[\phi(\vartheta^*)\right].$$

Proof. Since $\varphi \in \mathcal{H}(\mathbb{R})$, there exists $\delta > 0$ such that $\varphi \equiv 1$ on $[-\delta, \delta]$. Note that

$$(1-q)\mathbb{E}^*(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})) = \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})) - q + q - \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))}); \mathcal{S}^c).$$

On the event $\{Z_t = 0\}$, we have $\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))}) = 1$, thus

$$0 \le q - \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))}); \mathcal{S}^c) \le q - \mathbb{P}(Z_t = 0) \sim -A(0)e^{-\rho t} = o(e^{-\rho(t-r(t))}).$$

By the definition of $\mathbb{Y}_{r,t}$ given by (3.12), we have

$$\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})) \le \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t),t}^{(\Lambda(t))})) = \mathbb{E}\left(\left(\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))}))\right)^{Z_{t-r(t)}}\right), \tag{4.31}$$

where $\mathbb{Y}'_t$ is defined in (3.13). If $Z_{r(t)} = 0$, then $\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{(h(r(t)))}) = \mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))}) = 1$; if $Z_{r(t)} > 0$ then

$$\begin{aligned}\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{(h(r(t)))}) =& \mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))}) \cdot \varphi(X_{o,r(t)}/h(r(t)))^{Z_{r(t)}} \\ \ge& \mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))}) \cdot \mathbf{1}_{\{|X_{o,r(t)}| \le \delta h(r(t))\}}.\end{aligned}$$

By Lemma 2.3 and (1.6) we have for t large enough,

$$\mathbb{P}(|X_{o,r(t)}| > \delta h(r(t))) \lesssim_\delta r(t)(h(r(t)))^{-\alpha}L(h(r(t))) = r(t)e^{-\lambda r(t)} \to 0.$$

Thus by (4.7) we have

$$\lim_{t\to\infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))})) = \lim_{t\to\infty} \mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{(h(r(t)))})) = \phi(C(\varphi)) < 1. \tag{4.32}$$

Thus, by (1.12), we have that

$$\lim_{t\to\infty} e^{\rho(t-r(t))}\left(\mathbb{E}\left(\left[\mathbb{E}\mathcal{I}(\varphi, \mathbb{Y}_{r(t)}^{\prime(h(r(t)))})\right]^{Z_{t-r(t)}}\right) - q\right) = A\left(\phi\left(C(\varphi)\right)\right).$$

Hence by (4.31) we have

$$\limsup_{t\to\infty} e^{\rho(t-r(t))}(\mathbb{E}(\mathcal{I}(\varphi, \mathbb{Y}_t^{(\Lambda(t))})) - q) \le A\left(\phi\left(C(\varphi)\right)\right).$$

On the other hand,

$$\begin{aligned}&\mathbb{E}(\mathcal{I}(\varphi,\mathbb{Y}_t^{(\Lambda(t))})|\mathcal{F}_t^{\mathbb{T}})\\ \geq&\mathbb{P}(\max_{u\in\mathcal{D}_t:b_u\leq t-r(t)}|X_{u,t}|\leq\delta h(r(t))|\mathcal{F}_t^{\mathbb{T}})\mathbb{E}(I(\varphi,\mathbb{Y}_{r(t),t}^{(h(r(t)))})|\mathcal{F}_t^{\mathbb{T}}).\end{aligned}\tag{4.33}$$

Note that $\{u\in\mathcal{D}_t:b_u\leq t-r(t)\}\subset\mathcal{D}_{t-r(t)}$. Thus by Lemma 2.3 we have that

$$\begin{aligned}\mathbb{P}(\max_{u\in\mathcal{D}_t:b_u\leq t-r(t)}|X_{u,t}|\leq\delta h(r(t))|\mathcal{F}_t^{\mathbb{T}})\geq&\prod_{u\in\mathcal{D}_{t-r(t)}}\mathbb{P}(|X_{u,t}|\leq\delta h(r(t)))|\mathcal{F}_t^{\mathbb{T}})\\ \overset{t}{\geq}&(1-C_\delta te^{-\lambda r(t)})^{|\mathcal{D}_{t-r(t)}|}.\end{aligned}$$

By (4.33), we have that

$$\begin{aligned}&\mathbb{E}(\mathcal{I}(\varphi,\mathbb{Y}_t^{(\Lambda(t))}))\overset{t}{\geq}\mathbb{E}\left((1-C_\delta te^{-\lambda r(t)})^{|\mathcal{D}_{t-r(t)}|}\cdot\mathcal{I}(\varphi,\mathbb{Y}_{r(t),t}^{(\Lambda(t))})\right)\\ =&\mathbb{E}\left((1-C_\delta te^{-\lambda r(t)})^{|\mathcal{D}_{t-r(t)}|}\cdot[\mathbb{E}\mathcal{I}(\varphi,\mathbb{Y}_{r(t)}^{\prime(\Lambda(t))})]^{Z_{t-r(t)}}\right)\\ \geq&\mathbb{E}\Big((1-C_\delta te^{-\lambda r(t)})^{c\cdot(t-r(t))Z_{t-r(t)}}[\mathbb{E}\mathcal{I}(\varphi,\mathbb{Y}_{r(t)}^{\prime(\Lambda(t))})]^{Z_{t-r(t)}};\\ &\qquad\qquad|\mathcal{D}_{t-r(t)}|\leq c\cdot(t-r(t))Z_{t-r(t)}\Big)\\ \geq&\mathbb{E}\left(C(t)^{Z_{t-r(t)}}\right)-\mathbb{P}(|\mathcal{D}_{t-r(t)}|>c\cdot(t-r(t))Z_{t-r(t)}),\end{aligned}$$

where $C(t)=(1-C_\delta te^{-\lambda r(t)})^{c\cdot(t-r(t))}\mathbb{E}\mathcal{I}(\varphi,\mathbb{Y}_{r(t)}^{\prime(\Lambda(t))})$ and $c>0$ is a constant. Since $t^2e^{-\lambda r(t)}=t^2\Lambda(t)^{-\alpha}L(\Lambda(t))\to0$, by (4.32), we have

$$\lim_{t\to\infty}C(t)=\lim_{t\to\infty}\mathbb{E}\mathcal{I}(\varphi,\mathbb{Y}_{r(t)}^{\prime(\Lambda(t))})=\phi\left(C(\varphi)\right)<1.$$

By (1.12), we have that

$$\lim_{t\to\infty}e^{\rho(t-r(t))}(\mathbb{E}\left(C(t)^{Z_{t-r(t)}}\right)-q)=A\left(\phi\left(C(\varphi)\right)\right).$$

Note that $|\mathcal{D}_{t-r(t)}|\leq\sum_{u\in\mathcal{L}_{t-r(t)}}|I_u|$. Thus by the many-to-one formula,

$$\begin{aligned}&\mathbb{P}(|\mathcal{D}_{t-r(t)}|>c\cdot(t-r(t))Z_{t-r(t)})\leq\mathbb{P}\left(\sum_{u\in\mathcal{L}_{t-r(t)}}|I_u|>c\cdot(t-r(t))Z_{t-r(t)}\right)\\ &\leq\mathbb{P}\left(\sum_{u\in\mathcal{L}_{t-r(t)}}\mathbf{1}_{\{|I_u|>c\cdot(t-r(t))\}}\geq1\right)\leq\mathbb{E}\left(\sum_{u\in\mathcal{L}_{t-r(t)}}\mathbf{1}_{\{|I_u|>c\cdot(t-r(t))\}}\right)\\ &\leq e^{\lambda(t-r(t))}P(1+n_{t-r(t)}>c\cdot(t-r(t)))\\ &\leq e^{\lambda(t-r(t))}e^{-c\cdot(t-r(t))}E(e^{(1+n_{t-r(t)})})=e\cdot e^{\lambda(t-r(t))}e^{-c\cdot(t-r(t))}e^{(e-1)\beta(t-r(t))}\\ &=e\cdot e^{-(c-\lambda-(e-1)\beta)(t-r(t))}.\end{aligned}$$

Now we choose c such that $c>\lambda+(e-1)\beta-\rho$. Then

$$\lim_{t\to\infty}e^{\rho(t-r(t))}\mathbb{P}(|\mathcal{D}_{t-r(t)}|>c(t-r(t))Z_{t-r(t)})=0.$$

Hence we have

$$\liminf_{t\to\infty}e^{\rho(t-r(t))}(\mathbb{E}(\mathcal{I}(\varphi,\mathbb{Y}_t^{(\Lambda(t))}))-q)\geq A\left(\phi\left(C(\varphi)\right)\right).$$

The proof is complete now. □

5 Almost sure convergence results

In this section we will give the proofs of Theorems 1.8 and 1.9. We first prove the corresponding results for $\mathbb{Y}$. Note that for any $a > 0$,

$$\{R_t \le a\} = \mathcal{I}(\mathbf{1}_{(-\infty,a]}, \mathbb{X}_t), \quad \{M_t \le a\} = \mathcal{I}(\mathbf{1}_{(-\infty,a]}, \mathbb{Y}_t).$$

By the continuity of $C(\varphi)$, the results established in the previous sections for $\mathcal{H}(\mathbb{R})$ are also valid for $\mathbf{1}_{(-\infty,a]}$.

5.1 Almost sure convergence results for M_t

Lemma 5.1. *Suppose that $G : [0,\infty) \to (0,\infty)$ satisfies $\lim_{t\to\infty} G(t)/h(t) = \infty$.*

(1) If $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, then

$$\limsup_{n\to\infty} M_n/G(n) \le 0, \qquad \mathbb{P}\text{-a.s.}$$

(2) If $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$, then

$$\limsup_{n\to\infty} M_n/G(n) = \infty, \qquad \mathbb{P}^*\text{-a.s.}$$

Proof. (1) Assume that $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$. By Proposition 3.6, we have for any $c > 0$,

$$\lim_{t\to\infty} e^{-\lambda t} G(t)^{\alpha} L(G(t))^{-1} \mathbb{P}(M_t > cG(t)) = c^{-\alpha}\vartheta^*,$$

where ϑ^* is given in (1.10). Combining this with $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) < \infty$, we get that for any $c > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n > cG(n)) < \infty.$$

By the Borel-Cantelli lemma, we get that

$$\limsup_{n\to\infty} M_n/G(n) \le c, \qquad \mathbb{P}\text{-a.s.}$$

Letting $c \to 0$, we get the desired result.

(2) Assume that $\sum_n e^{\lambda n} G(n)^{-\alpha} L(G(n)) = \infty$. For $0 < s < t$, define

$$M_{s,t}^{(2)} := \max_{u\in\mathcal{D}_t : t-s<b_u\le t} X_{u,t}.$$

For any $c > 0$, we have that

$$\mathbb{P}(M_n > cG(n)|\mathcal{F}_{n-1}) \ge \mathbb{P}(M_{1,n}^{(2)} > cG(n)|\mathcal{F}_{n-1}). \tag{5.1}$$

By (3.23) with $s = 1$, $t = n$, $\Lambda(t) = cG(t)$ and $\varphi = \mathbf{1}_{(-\infty,1]}$, we get that, as $n \to \infty$,

$$\begin{aligned} &e^{-\lambda n} G(n)^{\alpha} L^{-1}(G(n)) \mathbb{P}(M_{1,n}^{(2)} > cG(n)|\mathcal{F}_{n-1}) \\ \to &c^{-\alpha}\frac{q_1}{\alpha} e^{-\lambda} \int_0^1 \left(e^{\lambda r} - e^{-\beta r}\right) \mathbb{P}(Z_{1-r} > 0)\mathrm{d}r \cdot W \quad \text{a.s.} \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), we obtain that, for any $c > 0$, on the event $\{W > 0\}$,

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n > cG(n)|\mathcal{F}_{n-1}) = \infty, \quad \mathbb{P}\text{-a.s.}$$

Applying the conditional Borel-Cantelli lemma, we get that for any $c > 0$,

$$\mathbb{P}^*(M_n > cG(n), i.o.) = 1.$$

Since c is arbitrary, desired result follows immediately. □

Lemma 5.2. *It holds that*

$$\liminf_{n\to\infty} \frac{M_n}{H(e^{-\lambda n}\log n)} \le (\vartheta^* W)^{\frac{1}{\alpha}}, \qquad \mathbb{P}^*\text{-a.s.}$$

Proof. For any $\epsilon \in (0,1)$, choose $\epsilon' > 0$ small enough such that $(1+\epsilon')(1+\epsilon/2)^{-\alpha} < 1$. Let $n_k := \lfloor k^{1+\epsilon'} \rfloor$, $k \ge 1$, where $\lfloor x \rfloor$ is the integer part of x. It is easy to see $n_k < n_{k+1}$ for any $k \ge 1$. Let

$$U_k := H\left(\frac{\log n_k}{\vartheta^* Z_{n_{k-1}} e^{\lambda(n_k - n_{k-1})}}\right) = H\left(\frac{e^{-\lambda n_k}\log n_k}{\vartheta^* W_{n_{k-1}}}\right).$$

Since $H(y) = y^{-1/\alpha}\bar{L}(y^{-1})$, we have

$$\lim_{k\to\infty} \frac{U_k}{H(e^{-\lambda n_k}\log n_k)} = \lim_{k\to\infty} (\vartheta^* W_{n_{k-1}})^{1/\alpha} = (\vartheta^* W)^{1/\alpha}, \qquad \mathbb{P}^*\text{-a.s.} \tag{5.3}$$

Applying the conditioned Borel-Cantelli lemma and (5.3), to prove the desired result, it suffices to show that for any $\epsilon > 0$,

$$\sum_k \mathbb{P}\left(M_{n_k} \le (1+\epsilon)U_k | \mathcal{F}_{n_{k-1}}\right) = \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}. \tag{5.4}$$

Note that $U_k \in \mathcal{F}_{n_{k-1}}$. Thus we have

$$\mathbb{P}\left(M_{n_k} \le (1+\epsilon)U_k | \mathcal{F}_{n_{k-1}}\right) \ge \mathbf{1}_{\{M_{n_{k-1}} \le (1+\epsilon/2)U_k\}} J_k, \tag{5.5}$$

where

$$J_k := \mathbb{P}\left(\forall v \in \mathcal{L}_{n_{k-1}}, X_{v,n_k} - X_{v,n_{k-1}} \le \epsilon U_k/2, M^v_{n_k} \le (1+\epsilon/2)U_k | \mathcal{F}_{n_{k-1}}\right)$$

and $M^v_t = \max_{u\in\mathcal{D}_t : v<u} X_{u,t}$.

We first show that, on the survival event $\mathcal{S}$, $\{M_{n_{k-1}} > (1+\epsilon/2)U_k\}$ can occur only finitely many times. By (4.6), we have for $p > \lambda/\rho$,

$$\sum_{n=1}^{\infty} \mathbb{P}(0 < e^{-\lambda n} Z_n < n^{-p}) \le C \sum_{n=1}^{\infty} n^{-p\rho/\lambda} < \infty. \tag{5.6}$$

Note that

$$\begin{aligned}
&\mathbb{P}\left(M_{n_{k-1}} > (1+\epsilon/2)U_k, \mathcal{S}\right) \\
&\le \mathbb{P}\left(e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) + \mathbb{P}\left(M_{n_{k-1}} > (1+\epsilon/2)U_k, e^{-\lambda n_{k-1}} Z_{n_{k-1}} \ge n_{k-1}^{-p}\right) \\
&\le \mathbb{P}\left(e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) + \mathbb{P}\left(M_{n_{k-1}} > (1+\epsilon/2) H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)\right).
\end{aligned}$$

By (5.6), we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}, \mathcal{S}\right) \le \sum_{k=1}^{\infty} \mathbb{P}\left(0 < e^{-\lambda n_{k-1}} Z_{n_{k-1}} < n_{k-1}^{-p}\right) < \infty. \tag{5.7}$$

Observe that $n_k - n_{k-1} \sim (1+\epsilon')k^{\epsilon'}$. Thus

$$\lim_{k\to\infty} \frac{H\left(\dfrac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)}{h(n_{k-1})} = \lim_{k\to\infty} \left(\frac{\dfrac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}}{e^{-\lambda n_{k-1}}}\right)^{-1/\alpha} = \infty.$$

Hence by Proposition 3.6, we have that as $k \to \infty$,

$$\begin{aligned}
&\mathbb{P}\left(M_{n_{k-1}} > (1+\epsilon/2)H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)\right)\\
&\sim \vartheta^* e^{\lambda n_{k-1}}(1+\epsilon/2)^{-\alpha} H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)^{-\alpha} L\left(H\left(\frac{\log n_k}{\vartheta^* n_{k-1}^{-p} e^{\lambda n_k}}\right)\right)\\
&= (1+\epsilon/2)^{-\alpha}\frac{n_{k-1}^p \log n_k}{e^{\lambda(n_k - n_{k-1})}}, \qquad (5.8)
\end{aligned}$$

where in the last equality we used (1.5). Since $n_k = \lfloor k^{1+\epsilon'} \rfloor$, we have $\sum_k \frac{n_{k-1}^p \log n_k}{e^{\lambda(n_k-n_{k-1})}} < \infty$. Combining (5.7) and (5.8), we get that

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\{M_{n_{k-1}} > (1+\epsilon/2)U_k\} \cap \mathcal{S}\right) < \infty.$$

Using the Borel-Cantelli lemma, we get that

$$\sum_{k=1}^{\infty} \mathbf{1}_{\{M_{n_{k-1}} > (1+\epsilon/2)U_k\}} < \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}. \qquad (5.9)$$

Now we consider J_k. Recall that M'_t is defined in (3.1). By the Markov property and the branching property, we have that

$$\begin{aligned}
J_k =& \left[\mathbb{P}\left(X_{o,n_k-n_{k-1}} \le \epsilon x/2, M'_{n_k-n_{k-1}} \le (1+\epsilon/2)x\right)\right]^{Z_{n_{k-1}}}|_{x=U_k}\\
=& e^{Z_{n_{k-1}} \log \mathbb{P}\left(X_{o,n_k-n_{k-1}} \le \epsilon x/2, M'_{n_k-n_{k-1}} \le (1+\epsilon/2)x\right)}|_{x=U_k}\\
\ge& e^{Z_{n_{k-1}} \log\left(1-\mathbb{P}\left(X_{o,n_k-n_{k-1}} > \epsilon x/2\right) - \mathbb{P}\left(M'_{n_k-n_{k-1}} > (1+\epsilon/2)x\right)\right)}|_{x=U_k}. \qquad (5.10)
\end{aligned}$$

Note that, on the survival event $\mathcal{S}$, $U_k \to \infty$ and $n_k V_\delta(U_k) \to 0$. Then by Lemma 2.3 we have for k large enough,

$$\begin{aligned}
\mathbb{P}\left(X_{o,n_k-n_{k-1}} > \epsilon x/2\right)|_{x=U_k} \le& C_\epsilon(n_k - n_{k-1})U_k^{-\alpha} L(U_k)\\
=& C_\epsilon(n_k - n_{k-1})\frac{\log n_k}{\vartheta^* Z_{n_{k-1}} e^{\lambda(n_k-n_{k-1})}}\\
\to& 0, \quad \mathbb{P}\text{-a.s. on } \mathcal{S} \qquad (5.11)
\end{aligned}$$

as $k \to \infty$. By Proposition 3.6 we have

$$\begin{aligned}
\mathbb{P}\left(M'_{n_k-n_{k-1}} > (1+\epsilon/2)x\right)|_{x=U_k} &\le \mathbb{P}\left(M_{n_k-n_{k-1}} > (1+\epsilon/2)x\right)|_{x=U_k}\\
&\sim \vartheta^* e^{\lambda(n_k-n_{k-1})}(1+\epsilon/2)^{-\alpha} U_k^{-\alpha} L(U_k)\\
&= (1+\epsilon/2)^{-\alpha}\frac{\log n_k}{Z_{n_{k-1}}} \to 0, \quad \mathbb{P}\text{-a.s. on } \mathcal{S} \qquad (5.12)
\end{aligned}$$

as $k \to \infty$. Since $n_k = \lfloor k^{1+\epsilon'} \rfloor$, using (5.11) and (5.12), we obtain that

$$\begin{aligned}
&\limsup_{k\to\infty} -\frac{1}{\log k} Z_{n_{k-1}} \log\left(1 - \mathbb{P}\left(X_{o,n_k-n_{k-1}} > \epsilon x/2\right) - \mathbb{P}\left(M'_{n_k-n_{k-1}} > (1+\epsilon/2)x\right)\right)_{x=U_k}\\
=&\limsup_{k\to\infty} \frac{1}{\log k} Z_{n_{k-1}} \left(\mathbb{P}\left(X_{o,n_k-n_{k-1}} > \epsilon x/2\right) + \mathbb{P}\left(M'_{n_k-n_{k-1}} > (1+\epsilon/2)x\right)\right)_{x=U_k}\\
\le&(1+\epsilon')(1+\epsilon/2)^{-\alpha} < 1, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}. \qquad (5.13)
\end{aligned}$$

Combining (5.10) and (5.13), we get that

$$\sum_{k=1}^{\infty} J_k = \infty, \quad \mathbb{P}\text{-a.s. on } \mathcal{S}. \tag{5.14}$$

Combining (5.5), (5.9) and (5.14), we get (5.4). The proof is now complete. □

We now modify the definition of $\mathcal{D}_t$ and $\mathcal{L}_t$ slightly. For any $\delta \geq 0$, and $t > 0$, define

$$\mathcal{L}_{t,\delta} := \{u \in \mathcal{L}_t : Z^u_{t+\delta} > 0\},$$

$$\mathcal{D}_{t,\delta} := \{u \in \mathbb{T} : b_u < t, Z^u_{t+\delta} > 0\} = \cup_{u\in\mathcal{L}_{t,\delta}} I_u,$$

$$R_{t,\delta} := \max_{u\in\mathcal{L}_{t,\delta}} \xi^u_t, \quad M_{t,\delta} := \max_{u\in\mathcal{D}_{t,\delta}} X_{u,t}, \text{ and } M'_{t,\delta} := \max_{u\in\mathcal{D}_{t,\delta}\setminus\{o\}} X_{u,t}.$$

Using argument similar to that in the proof of Lemma 3.5 with $\varphi = \mathbf{1}_{(-\infty,1]}$, we get the following result. We omit the proof.

Lemma 5.3. *For any $\delta \geq 0$, it holds that*

$$\lim_{x\to\infty} x^\alpha L(x)^{-1}\mathbb{P}(M'_{s,\delta} > x) = \frac{q_1}{\alpha}\int_0^s \left(e^{\lambda r} - e^{-\beta r}\right)\mathbb{P}(Z_{s+\delta-r} > 0)dr =: c(s,\delta).$$

Lemma 5.4. *For any $\delta \geq 0$, it holds that*

$$\liminf_{n\to\infty} \frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)} \geq (\vartheta_\delta W)^{\frac{1}{\alpha}} \qquad \mathbb{P}^*\text{-a.s.},$$

where $\vartheta_\delta = \frac{q_1}{\alpha}\int_0^\infty e^{-\lambda r}\mathbb{P}(Z_{r+\delta} > 0)\, dr$.

Proof. For $1 \leq m < n$, let $U_{n,m} = H\left(\dfrac{\log n}{Z_{(n-m)\delta}c(m\delta,\delta)}\right)$ when $Z_{(n-m)\delta} > 0$ and $U_{n,m} = 0$ when $Z_{(n-m)\delta} = 0$. Note that

$$M_{n\delta,\delta} \geq \max_{u\in\mathcal{L}_{(n-m)\delta}} \max_{v\in\mathcal{D}^u_{n\delta,\delta}} X_{v,n\delta},$$

where $\mathcal{D}^u_{t,\delta} = \{v \in \mathcal{D}_{t,\delta:} : v > u\}$. By the Markov property and the branching property, we have that, conditioned on $\mathcal{F}_{(n-m)\delta}$, $\{\max_{v\in\mathcal{D}^u_{n\delta,\delta}} X_{v,n\delta}, u \in \mathcal{L}_{(n-m)\delta}\}$ are i.i.d. with the same law as $M'_{m\delta,\delta}$. Let $G_{m,\delta}(x) := \mathbb{P}(M^{'}_{m\delta,\delta} > x)$. Note that $U_{n,m} \in \mathcal{F}_{(n-m)\delta}$. Thus for any $\epsilon \in (0,1)$, we have

$$\begin{aligned}&\mathbb{P}\left(M_{n\delta,\delta} < (1-\epsilon)U_{n,m}|\mathcal{F}_{(n-m)\delta}\right) \leq \left(1 - G_{m,\delta}((1-\epsilon)U_{n,m})\right)^{Z_{(n-m)\delta}}\\ \leq& e^{-Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m})}.\end{aligned}$$

Note that on the survival event $\mathcal{S}$, $\lim_{n\to\infty} U_{n,m} = \infty$. Thus by Lemma 5.3, we have that, on the survival event $\mathcal{S}$, as $n \to \infty$,

$$Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m}) \sim Z_{(n-m)\delta}c(m\delta,\delta)(1-\epsilon)^{-\alpha}U^{-\alpha}_{n,m}L(U_{n,m}) = (1-\epsilon)^{-\alpha}\log n, \quad \mathbb{P}\text{-a.s.}$$

where in the last equality we used (1.5). Since $(1-\epsilon)^{-\alpha} > 1$, we have that, on the survival event $\mathcal{S}$, for any $m \geq 1$,

$$\sum_{n=m+1}^{\infty} \mathbb{P}\left(M_{n\delta,\delta} < (1-\epsilon)U_{n,m}|\mathcal{F}_{(n-m)\delta}\right) \leq \sum_{n=m+1}^{\infty} e^{-Z_{(n-m)\delta}G_{m,\delta}((1-\epsilon)U_{n,m})} < \infty, \quad \mathbb{P}\text{-a.s..}$$

By the second Borel-Cantelli lemma, we have that, on the survival event $\mathcal{S}$,

$$\sum_{n=m+1}^{\infty} \mathbf{1}_{\{M_{n\delta,\delta}<(1-\epsilon)U_{n,m}\}} < \infty, \quad \mathbb{P}\text{-a.s.}$$

Thus, by the definition of H, we obtain that under $\mathbb{P}^*$, for any $\epsilon \in (0,1)$ and integer $m \geq 1$,

$$\begin{aligned}\liminf_{n\to\infty} \frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)} &\geq \liminf_{n\to\infty} \frac{(1-\epsilon)U_{n,m}}{H(e^{-\lambda n\delta}\log n)} = (1-\epsilon)\lim_{n\to\infty} \left(\frac{\log n}{\frac{Z_{(n-m)\delta}c(m\delta,\delta)}{e^{-\lambda n\delta}\log n}}\right)^{-1/\alpha} \\ &= (1-\epsilon)\left(We^{-\lambda m\delta}c(m\delta,\delta)\right)^{1/\alpha}, \quad \mathbb{P}^*\text{-a.s.} \end{aligned} \tag{5.15}$$

Note that $\lim_{m\to\infty} e^{-\lambda m\delta}c(m\delta,\delta) = \vartheta_\delta$. Letting $m \to \infty$ and $\epsilon \to 0$ in (5.15), we get the desired result. The proof is now complete. □

5.2 Proofs of Theorems 1.8 and 1.9

Lemma 5.5. *If $\{a_n : n \geq 1\}$ is a sequence of positive numbers satisfying $\sum_n e^{\lambda n}a_n^{-p\alpha} < \infty$ for some $p \in (0,2)$, then for any $\delta \geq 0$,*

$$\limsup_{n\to\infty} (R_{n,\delta} - M_{n,\delta})/a_n \leq 0, \quad \mathbb{P}^*\text{-a.s.} \tag{5.16}$$

and

$$\limsup_{n\to\infty} (M_{n,\delta} - R^+_{n,\delta})/a_n \leq 0, \quad \mathbb{P}^*\text{-a.s.} \tag{5.17}$$

Proof. Let $a(t)$ be a positive function satisfying $a(t) \overset{t}{\geq} e^{\epsilon t}$ for some $\epsilon \in (0,1)$. Let $c > 2\alpha + e^2\beta/\epsilon^2$ be a constant. We choose $\theta \in (0,1)$ so that $\theta c < 1$. Let $A_t(\theta, a(t))$ and $B_t(c, a(t))$ be the events defined in (3.3) amd (3.8). We claim that for n sufficiently large, on the event $A_t(\theta, a(t)) \cap B_t(c, a(t)) \cap \{Z_t \geq 1\}$,

$$\{R_{t,\delta} - M_{t,\delta} > a(t)\} \cup \{M_{t,\delta} - R^+_{t,\delta} > a(t)\} = \emptyset.$$

In fact, by (3.14), we have that on the event $A_t(\theta, a(t)) \cap B_t(c, a(t)) \cap \{Z_t \geq 1\}$,

$$R_{t,\delta} = \max_{v\in\mathcal{L}_{t,\delta}} \xi^v_t \leq \max_{v\in\mathcal{L}_{t,\delta}} X_{v',t} + a(t) \leq M_{t,\delta} + a(t) \tag{5.18}$$

and

$$\begin{aligned} M_{t,\delta} &= \max_{v\in\mathcal{L}_{t,\delta}} \max_{u\in I_v} X_{u,t} = \left(\max_{v\in\mathcal{L}_{t,\delta}} X_{v',t}\right) \vee \left(\max_{v\in\mathcal{L}_{t,\delta}, u\in I_v\setminus\{v'\}} X_{u,t}\right) \\ &\leq \left(\max_{v\in\mathcal{L}_{t,\delta}} \xi^v_t + a(t)\right) \vee (\theta a(t)/\log a(t)) \leq R^+_{t,\delta} + a(t) \end{aligned} \tag{5.19}$$

for t large enough so that $\log a(t) > \theta$. Hence, by (5.18) and (5.19), the claim is true.

Combining (3.4), (3.9), we get that for any $p \in (0,2)$,

$$\begin{aligned} &\limsup_{t\to\infty} e^{-\lambda t}a(t)^{p\alpha}\mathbb{P}\left(\left(\{R_{t,\delta} - M_{t,\delta} > a(t)\} \cup \{M_{t,\delta} - R^+_{t,\delta} > a(t)\}\right) \cap \{Z_t \geq 1\}\right) \\ \leq &\limsup_{t\to\infty} e^{-\lambda t}a(t)^{p\alpha}\{\mathbb{P}(A_t(\theta, a(t))^c) + \mathbb{P}(B_t(c, a(t))^c)\} = 0. \end{aligned}$$

Thus we have

$$\lim_{t\to\infty} e^{-\lambda t}a(t)^{p\alpha}\mathbb{P}\left(\left(\{R_{t,\delta}-M_{t,\delta}>a(t)\}\cup\{M_{t,\delta}-R^+_{t,\delta}>a(t)\}\right)\cap\{Z_t\geq 1\}\right)=0.$$

Since $\sum_n e^{\lambda n}a_n^{-p\alpha}<\infty$, we have $a_n \overset{n}{\geq} e^{\frac{\lambda n}{p\alpha}}$. It follows that for any $\epsilon\in(0,1)$ and $p\in(0,2)$,

$$\mathbb{P}^*\left(R_{n,\delta}-M_{n,\delta}>\epsilon a_n\right)\leq\frac{1}{\mathbb{P}(\mathcal{S})}\mathbb{P}\left(R_{n,\delta}-M_{n,\delta}>\epsilon a_n, Z_n\geq 1\right)=o(e^{\lambda n}a_n^{-p\alpha}).$$

So we have that

$$\sum_{n=1}^{\infty}\mathbb{P}^*\left(R_{n,\delta}-M_{n,\delta}>\epsilon a_n\right)<\infty.$$

Applying the Borel-Cantelli lemma and then letting $\epsilon\to 0$, we get (5.16). Similarly, we can get (5.17) as well. □

Proof of Theorem 1.8. Note that

$$H(e^{-\lambda n\delta}\log n)=e^{\lambda n\delta/\alpha}(\log n)^{-1/\alpha}\bar{L}(e^{\lambda n\delta}/\log n).$$

Applying Lemma 5.5 with $\delta=0$ and $\delta\geq 0$, we have

$$\limsup_{n\to\infty}\frac{R_n-M_n}{H(e^{-\lambda n}\log n)}\leq 0,\qquad \limsup_{n\to\infty}\frac{M_{n\delta,\delta}-R^+_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)}\leq 0,\qquad \mathbb{P}^*\text{-a.s.}$$

By Lemma 5.2 we have

$$\liminf_{t\to\infty}\frac{R(t)}{H(e^{-\lambda t}\log t)}\leq\liminf_{n\to\infty}\frac{M_n}{H(e^{-\lambda n}\log n)}+\limsup_{n\to\infty}\frac{R_n-M_n}{H(e^{-\lambda n}\log n)}\leq(\vartheta^*W)^{1/\alpha},\quad\mathbb{P}^*\text{-a.s.}$$

We now consider the lower bound of $\liminf_{t\to\infty}\frac{R(t)}{H(e^{-\lambda t}\log t)}$. By Lemma 5.4 we have that for any $\delta>0$,

$$\begin{aligned}\liminf_{n\to\infty}\frac{R^+_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)}&\geq\liminf_{n\to\infty}\frac{M_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)}-\limsup_{n\to\infty}\frac{M_{n\delta,\delta}-R^+_{n\delta,\delta}}{H(e^{-\lambda n\delta}\log n)}\\&\geq(\vartheta_\delta W)^{1/\alpha},\quad\mathbb{P}^*\text{-a.s.}\end{aligned}\tag{5.20}$$

For any $n\geq 1$, let $Q_{n,\delta}=\inf_{n\delta\leq t\leq(n+1)\delta}R_t$. We claim that for any $\epsilon\in(0,1)$ and $\delta>0$,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(Q_{n,\delta}<(1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_\delta W_{n\delta}\right)^{\frac{1}{\alpha}}-H(n^{-2})|\mathcal{F}_{n\delta}\right)<\infty,\qquad\mathbb{P}^*\text{-a.s.}\tag{5.21}$$

Let $B_n:=\{R_{n\delta,\delta}<(1-\epsilon)H(e^{-\lambda\delta n}\log n)\left(\vartheta_\delta W_{n\delta}\right)^{\frac{1}{\alpha}}\}$. Note that

$$\begin{aligned}&\mathbb{P}\left(Q_{n,\delta}<(1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_\delta W_{n\delta}\right)^{\frac{1}{\alpha}}-H(n^{-2})|\mathcal{F}_{n\delta}\right)\\\leq&\mathbb{P}(B_n|\mathcal{F}_{n\delta})+\mathbb{P}\left(B_n^c,Q_{n,\delta}<(1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_\delta W_{n\delta}\right)^{\frac{1}{\alpha}}-H(n^{-2})|\mathcal{F}_{n\delta}\right).\end{aligned}\tag{5.22}$$

By (5.20) and the Borel-Cantelli lemma, we know that on $\mathcal{S}$,

$$\sum_{n=1}^{\infty}\mathbb{P}(B_n|\mathcal{F}_{n\delta})<\infty,\qquad\mathbb{P}\text{-a.s.}\tag{5.23}$$

Now we consider the second term of the right hand side of (5.22). Take $Z_0=\nu=\sum_{k=1}^K\delta_{a_k}$ with $a_1\geq a_2\geq\cdots\geq a_K$. We use $Z_t^{(j)}$ to denote the number of descendants at

time t of the j-th particle, and $\mathcal{L}_t^{(j)}$ the set of all the descendants of j alive at time t. For any x, y we have that

$$\begin{aligned}
&\mathbb{P}_\nu(R_{0,\delta} > x, Q_{0,\delta} < x - y)\\
&= \sum_{k=1}^{K} \mathbb{P}(Z_\delta^{(j)} = 0, 1 \le j \le k-1, Z_\delta^{(k)} > 0, a_k > x, Q_{0,\delta} < x - y)\\
&\le \sum_{k=1}^{K} \mathbb{P}(Z_\delta^{(j)} = 0, 1 \le j \le k-1, Z_\delta^{(k)} > 0, \inf_{0\le t\le\delta} \max_{u\in\mathcal{L}_t^{(k)}} (\xi_t^u - a_k) < -y)\\
&= \sum_{k=1}^{K} \mathbb{P}(Z_\delta = 0)^{k-1} \mathbb{P}(Z_\delta > 0, \inf_{0\le t\le\delta} R_t < -y)\\
&\le \frac{1}{\mathbb{P}(Z_\delta > 0)} \mathbb{P}(Z_\delta > 0, \inf_{0\le t\le\delta} R_t < -y). \qquad (5.24)
\end{aligned}$$

By the many-to-one formula, we have

$$\begin{aligned}
\mathbb{P}(Z_\delta > 0, \inf_{0\le t\le\delta} R_t < -y) &\le \mathbb{E}\left(\sum_{u\in\mathcal{L}_\delta} \mathbf{1}_{\{\inf_{0\le t\le\delta} \xi_t^u < -y\}} \ge 1\right) \le \mathbb{E}\left(\sum_{u\in\mathcal{L}_\delta} \mathbf{1}_{\{\inf_{0\le t\le\delta} \xi_t^u < -y\}}\right)\\
&= e^{\lambda\delta}\mathbf{P}\left(\inf_{0\le t\le\delta} \xi_t < -y\right) \sim e^{\lambda\delta}\frac{q_2}{\alpha}\delta y^{-\alpha}L(y), \qquad y\to\infty,
\end{aligned}$$

where the last limit follows from (2.6). Combining the Markov property and (5.24) with $\nu = X_{n\delta}$ and $y = H(n^{-2})$, we get

$$\begin{aligned}
&\mathbb{P}\left(B_n^c, Q_{n,\delta} < (1-\epsilon)H(e^{-\lambda n\delta}\log n)\left(\vartheta_\delta W_{n\delta}\right)^{\frac{1}{\alpha}} - H(n^{-2}) | \mathcal{F}_{n\delta}\right)\\
\le &\frac{1}{\mathbb{P}(Z_\delta > 0)} \mathbb{P}(Z_\delta > 0, \inf_{0\le t\le\delta} R_t < -H(n^{-2})) \le \frac{2\delta e^{\lambda\delta}}{\mathbb{P}(Z_\delta > 0)}\frac{q_2}{\alpha} n^{-2}, \qquad (5.25)
\end{aligned}$$

where the last inequality holds for n large enough. Combining (5.22), (5.23) and (5.25), we arrive at (5.21).

Using the Borel-Cantelli lemma and (5.21), we have that for any $\delta > 0$,

$$\liminf_{n\to\infty} \frac{Q_{n,\delta}}{H(e^{-\lambda n\delta}\log n)} \ge \left(\vartheta_\delta W\right)^{\frac{1}{\alpha}} \qquad \mathbb{P}^*\text{-a.s.}$$

Since $H(e^{-\lambda t}\log t)$ is increasing in t for t large, for any $\delta > 0$, we have

$$\liminf_{t\to\infty} \frac{R_t \vee 0}{H(e^{-\lambda t}\log t)} \ge \liminf_{n\to\infty} \frac{Q_{n,\delta} \vee 0}{H(e^{-\lambda n\delta}\log n)} \frac{H(e^{-\lambda n\delta}\log n)}{H(e^{-\lambda(n+1)\delta}\log(n+1)\delta)} \ge e^{-\lambda\delta/\alpha}\left(\vartheta_\delta W\right)^{\frac{1}{\alpha}},$$

$\mathbb{P}^*$-a.s. Letting $\delta \to 0$, we get the desired result. The proof is now complete. □

It follows from Lemma 5.5 that Lemma 5.1 also holds with M_n replaced by R_n. To get the limit of R_t as $t\to\infty$, we need to deal with $\sup_{0\le s\le t} R_s$. If $p_0 = 0$, then $\sup_{0\le s\le t} R_s = \max_{v\in\mathcal{L}_t} \sup_{0\le s\le t} \xi_s^v$. Thus by the many-to-one formula, we have that

$$\mathbb{P}\left(\sup_{0\le s\le t} R_s \ge x\right) \le \mathbb{E}\left(\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\{\sup_{0\le s\le t}\xi_s^v \ge x\}}\right) = e^{\lambda t}\mathbf{P}\left(\sup_{0\le s\le t}\xi_s \ge x\right).$$

In the following lemma, we show that this assertion is still valid when $p_0 > 0$.

Lemma 5.6. *For any $x > 0$ and $t > 0$, it holds that*

$$\mathbb{P}\left(\sup_{0\le s\le t} R_s \ge x\right) \le e^{\lambda t}\mathbf{P}\left(\sup_{0\le s\le t}\xi_s \ge x\right).$$

Proof. For $r \geq 0$ and $x \in \mathbb{R}$, let $\mathbb{P}_{r,x}$ denote the law of $\mathbb{X}$ starting with an individual at position x at time r, and $\mathbf{P}_{r,x}$ stands for the law of ξ condition on $\{\xi_r = x\}$. We still use σ_o to denote the death time of the initial particle. Fix $t > 0$. For any $0 \leq r < t$ and $x < 0$, we have that

$$\begin{aligned}\omega(r,x) := \mathbb{P}_{r,x}\left(\sup_{r\leq s\leq t} R_s < 0\right) &= \mathbb{P}_{r,x}\left(\sup_{r\leq s\leq t} R_s < 0, \sigma_o > t\right) \\ &\quad + \mathbb{P}_{r,x}\left(\sup_{r\leq s\leq t} R_s < 0, r \leq \sigma_o \leq t\right) \\ &=: I_1 + I_2.\end{aligned}$$

We first consider I_1. Let $T_0 := \inf\{u \geq r : \xi_u \geq 0\}$. Since $\sigma_o - r \sim \mathcal{E}(\beta)$ under $\mathbb{P}_{r,x}$, we have

$$\begin{aligned}I_1 &= \mathbf{P}_{r,x}\left(\sup_{r\leq s\leq t} \xi_s < 0\right)\mathbb{P}_{r,x}(\sigma_o > t) \\ &= \mathbf{P}_{r,x}(T_0 > t)e^{-\beta(t-r)} = \mathbf{P}_{r,x}\left(T_0 \geq t, e^{-\int_r^{T_0\wedge t}\beta ds}\right).\end{aligned}$$

By the branching property and the strong Markov property, we have

$$\begin{aligned}I_2 &= \mathbb{E}_{r,x}\left(\mathbb{P}_{r,x}\left(\sup_{r\leq s\leq t} R_s < 0, r \leq \sigma_o \leq t | \mathcal{F}_{\sigma_o}\right)\right) \\ &= \mathbf{E}_{r,x}\left(\int_r^t \mathbf{1}_{\{T_0>u\}} \sum_{k=0}^{\infty} p_k \left(\mathbb{P}_{u,\xi_u}\left(\sup_{u\leq s\leq t} R_s < 0\right)\right)^k \beta e^{-\beta(u-r)} du\right) \\ &= \mathbf{E}_{r,x}\left(\int_r^{t\wedge T_0} f(\omega(u,\xi_u))\beta e^{-\beta(u-r)} du\right),\end{aligned}$$

where $f(s) = \sum_{k=0}^{\infty} p_k s^k$. Consequently, $\omega(r,x)$ satisfies the equation

$$\omega(r,x) = \mathbf{P}_{r,x}\left(T_0 > t, e^{-\int_r^{T_0\wedge t}\beta ds}\right) + \mathbf{E}_{r,x}\left(\int_r^{t\wedge T_0} f(\omega(u,\xi_u))\beta e^{-\beta(u-r)} du\right).$$

Then, $1 - \omega(r,x)$ satisfies the following equation

$$1 - \omega(r,x) = \mathbf{P}_{r,x}\left(T_0 \leq t, e^{-\int_r^{T_0\wedge t}\beta ds}\right) + \mathbf{E}_{r,x}\left(\int_r^{t\wedge T_0} (1 - f(\omega(u,\xi_u)))\beta e^{-\beta(u-r)} du\right).$$

Using [23, Lemma 1.5], we get that $g(r,x) := 1 - \omega(r,x)$ also satisfies the following equation

$$\begin{aligned}g(r,x) &= \mathbf{P}_{r,x}\left(T_0 \leq t, e^{\int_r^{T_0\wedge t}\lambda ds}\right) + \mathbf{E}_{r,x}\left(\int_r^{t\wedge T_0} (1 - f(\omega(u,\xi_u)))\beta e^{\lambda(u-r)} du\right) \\ &\quad - \mathbf{E}_{r,x}\left(\int_r^{t\wedge T_0} \mu\beta g(u,\xi_u) e^{\lambda(u-r)} du\right), \quad 0 \leq r < t, x < 0.\end{aligned}$$

Note that $1 - f(1-g) - \mu g \leq 0$. So we have that for $0 \leq r < t$, and $x < 0$,

$$g(r,x) \leq \mathbf{P}_{r,x}\left(T_0 \leq t, e^{\int_r^{T_0\wedge t}\lambda ds}\right) \leq e^{\lambda t}\mathbf{P}_{r,x}(T_0 \leq t).$$

Using the time-homogeneity and space-homogeneity of branching Lévy processes and Lévy processes, we get that for any $t>0$ and $x>0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{0\le s\le t} R_s \ge x\right) &= 1-\mathbb{P}\left(\sup_{0\le s\le t} R_s < x\right) = g(0,-x)\\ &\le e^{\lambda t}\mathbf{P}_{0,-x}(T_0\le t) = e^{\lambda t}\mathbf{P}\left(\sup_{0\le s\le t}\xi_s \ge x\right).\end{aligned}$$

The proof is now complete. □

Proof of Theorem 1.9. Since $\lim_{t\to\infty} G(t)/h(t)=\infty$, we have that $G(t)\overset{t}{\ge} e^{\frac{2\lambda}{3\alpha}t}$. By Lemma 5.5 with $3/2<p<2$, we get that

$$\limsup_{n\to\infty}(R_n-M_n)/G(n)\le 0,\quad \limsup_{n\to\infty}(M_n-R_n^+)/G(n)\le 0,\quad \mathbb{P}^*\text{-a.s.} \tag{5.26}$$

(1) Assume that $\sum_n e^{\lambda n}G(n)^{-\alpha}L(G(n))<\infty$. By Theorem 1.8, we have $\mathbb{P}^*(\exists T>0,\forall t>T, R_t>0)=1$, which implies that

$$\limsup_{t\to\infty} R_t/G(t)=\limsup_{t\to\infty} R_t^+/G(t)\ge 0,\quad \mathbb{P}^*\text{-a.s.}.$$

Let $V_n:=\sup_{n\le t\le(n+1)} R_t, n\ge 0$. Since G is non-decreasing, by Lemma 5.1 and (5.26), we have, under $\mathbb{P}^*$,

$$\limsup_{t\to\infty} R_t/G(t)\le\limsup_{n\to\infty} V_n/G(n)\le\limsup_{n\to\infty}(V_n-R_n)/G(n).$$

By the Borel-Cantelli lemma, to get the desired result, it suffices to prove that, for any $c>0$,

$$\sum_{n=1}^{\infty}\mathbb{P}^*(V_n-R_n>cG(n))<\infty. \tag{5.27}$$

Observe that on $\{Z_n>0\}$, $R_n\in\mathbb{R}$ and

$$V_n=\max_{u\in\mathcal{L}_n}(\xi_n^u+V_n^u)\le R_n+\max_{u\in\mathcal{L}_n}V_n^u,$$

where

$$V_n^u=\sup_{n\le t\le n+1}\max_{v\in\mathcal{L}_t,u\le v}(\xi_t^v-\xi_n^u).$$

It is clear that, conditioned on $\mathcal{F}_n$, $\{V_n^u,u\in\mathcal{L}_n\}$ are i.i.d. with the same law as $(V_0,\mathbb{P})$. Thus

$$\begin{aligned}(1-q)\mathbb{P}^*(V_n-R_n>cG(n)) &\le \mathbb{P}(V_n-R_n>cG(n),Z_n>0)\le\mathbb{P}(\max_{u\in\mathcal{L}_n}V_n^u>cG(n),Z_n>0)\\ &\le\mathbb{E}(Z_n)\mathbb{P}(V_0>cG(n))=e^{\lambda n}\mathbb{P}(V_0>cG(n))\\ &\le e^{\lambda n}e^{\lambda}\mathbf{P}(\sup_{0\le t\le 1}\xi_t>cG(n)),\end{aligned}$$

where the last inequality follows from Lemma 5.6. By (2.5), we have that

$$e^{\lambda n}\mathbf{P}\left(\sup_{0\le t\le 1}\xi_t\ge cG(n)\right)\sim c^{-\alpha}\frac{q_1}{\alpha}e^{\lambda n}G(n)^{-\alpha}L(G(n)).$$

Thus (5.27) follows immediately.

(2) Assume that $\sum_n e^{\lambda n}G(n)^{-\alpha}L(G(n))=\infty$. By Lemma 5.1 and (5.26), we have

$$\limsup_{t\to\infty} R_t^+/G(t)\ge\limsup_{n\to\infty} R_n^+/G(n)\ge\limsup_{n\to\infty} M_n/G(n)-\limsup_{n\to\infty}(M_n-R_n^+)/G(n)=\infty,$$

$\mathbb{P}^*$-a.s., which implies that $\limsup_{t\to\infty} R_t/G(t)=\infty$.

The proof is now complete. □

As a consequence of Theorem 1.9 and Theorem 1.8, we have the following result.

Corollary 5.7. *(1)*

$$\limsup_{t\to\infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log\log t} = \frac{1}{\alpha}, \qquad \mathbb{P}^*\text{-a.s.} \tag{5.28}$$

(2)

$$\lim_{t\to\infty} \frac{\log R_t^+}{t} = \frac{\lambda}{\alpha}, \qquad \mathbb{P}^*\text{-a.s.}$$

Proof. (1) For any $\epsilon > 0$, by Theorem 1.9 (1) with $G(t) = H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}$,

$$\limsup_{t\to\infty} \frac{R_t}{H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}} = 0, \quad \mathbb{P}^*\text{-a.s.}$$

which implies that for any $\delta > 0$, $\frac{R_t^+}{H(t^{-1}e^{-\lambda t})(\log t)^{(1+\epsilon)/\alpha}} \le \delta$ when t is large enough. Thus for t large enough,

$$\log R_t^+ - \log H(t^{-1}e^{-\lambda t}) \le \log\delta + \frac{1+\epsilon}{\alpha}\log\log t, \quad \mathbb{P}^*\text{-a.s.}$$

Therefore

$$\limsup_{t\to\infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log\log t} \le \frac{1+\epsilon}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}$$

Letting $\epsilon \to 0$, we get

$$\limsup_{t\to\infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log\log t} \le \frac{1}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}$$

For any $\epsilon \in (0,1)$, by Theorem 1.9 (2) with $G(t) = H(t^{-1}e^{-\lambda t})(\log t)^{(1-\epsilon)/\alpha}$, we have for any $\delta > 0$, $\limsup_{t\to\infty} \frac{R_t^+}{H(t^{-1}e^{-\lambda t})(\log t)^{(1-\epsilon)/\alpha}} \ge \delta$, which implies that

$$\limsup_{t\to\infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log\log t} \ge \limsup_{t\to\infty} \frac{\log\delta}{\log\log t} + \frac{1-\epsilon}{\alpha} = \frac{1-\epsilon}{\alpha}.$$

Therefore letting $\epsilon \to 0$, we get

$$\limsup_{t\to\infty} \frac{\log R_t^+ - \log H(t^{-1}e^{-\lambda t})}{\log\log t} \ge \frac{1}{\alpha}, \quad \mathbb{P}^*\text{-a.s.}$$

Hence we have (5.28).

(2) Note that

$$\log H(t^{-1}e^{-\lambda t}) = \frac{\lambda}{\alpha}t + \frac{1}{\alpha}\log t + \log\bar{L}(te^{\lambda t}) \sim \frac{\lambda}{\alpha}t.$$

By (5.28), we have

$$\limsup_{t\to\infty} \frac{\log R_t^+}{t} = \frac{\lambda}{\alpha}, \qquad \mathbb{P}^*\text{-a.s.}$$

On the other hand, by Theorem 1.8, we have

$$\liminf_{t\to\infty} \frac{\log R_t^+}{t} - \frac{\log H(e^{-\lambda t}\log t)}{t} = 0, \qquad \mathbb{P}^*\text{-a.s.}$$

Note that

$$\log H(e^{-\lambda t}\log t) = \frac{\lambda}{\alpha}t - \frac{1}{\alpha}\log\log t + \log\bar{L}(e^{\lambda t}/\log t) \sim \frac{\lambda}{\alpha}t.$$

The proof is now complete. □

Remark 5.8. Since $H(y) = y^{-1/\alpha}\bar{L}(y^{-1})$, we have

$$\lim_{y\to 0} \frac{\log H(y)}{\log y^{-1}} = 1/\alpha. \tag{5.29}$$

Assume that $L(x) = (\log x)^r$, where $r \in \mathbb{R}$. By (5.29), we have that

$$H(y) = y^{-1/\alpha}(\log H(y))^{r/\alpha} \sim \alpha^{-r/\alpha} y^{-1/\alpha}(\log(y^{-1}))^{r/\alpha}, \quad y \to 0.$$

Hence we have

$$\log H(t^{-1}e^{-\lambda t}) - \frac{\lambda}{\alpha}t - \frac{1+r}{\alpha}\log t \to \frac{r}{\alpha}\log(\lambda/\alpha), \quad t \to \infty,$$

and

$$H(e^{-\lambda t}\log t) \sim (\lambda/\alpha)^{r/\alpha} t^{r/\alpha}(\log t)^{-1/\alpha} e^{\lambda t/\alpha}, \quad t \to \infty.$$

A Appendix

Recall that ξ is a Lévy process with characteristic exponent given by:

$$\psi(\theta) = ia\theta - \frac{1}{2}b^2\theta^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y).$$

In this section, we study the behavior of $\psi(\theta)$ as $\theta \to 0$ under **Assumption 2**. We will show that as $\theta \to 0+$,

$$\psi(\theta) \sim \begin{cases} C\theta^\alpha L(\theta^{-1}); & \alpha \in (0,1)\cup(1,2); \\ C\theta, & \alpha = 1, \mathbf{E}(|\xi_1|) < \infty; \\ C\theta L^*(\theta^{-1}), & \alpha = 1, \mathbf{E}(|\xi_1|) = \infty; \\ C\theta^2 L^*(\theta^{-1}), & \alpha = 2, \mathbf{E}(|\xi_1|) = \infty; \\ C\theta^2, & \alpha \geq 2, \mathbf{E}(|\xi_1|^2) < \infty. \end{cases}$$

Here $L^*(x) = \int_1^x t^{-1}L(t)\mathrm{d}t$, and C stands different complex constants which satisfies $\Re(C) < 0$ when $\alpha \neq 1$ and might be 0 when $\alpha = 1$. By Lemma A.1(1) below with $r = -1$, we have $L(x) = o(L^*(x))$.

If $\alpha > 2$ or $\alpha = 2$ with $\int_{\{|y|>1\}} y^2\mathbf{n}(\mathrm{d}y) < \infty$, then since $\mathbf{E}(\xi_1) = 0$, it follows immediately that

$$\lim_{\theta\to 0} \theta^{-2}\psi(\theta) = -\frac{1}{2}\left(b^2 + \int_{\mathbb{R}} y^2\mathbf{n}(\mathrm{d}y)\right).$$

If $\alpha = 1$ and $\int_{\{|y|>1\}} y\mathbf{n}(\mathrm{d}y) < \infty$, then

$$\lim_{\theta\to 0} \theta^{-1}\psi(\theta) = \mathbf{E}(\xi_1) = i\left(a + \int_{\{|y|>1\}} y\mathbf{n}(\mathrm{d}y)\right).$$

Now we consider other cases.

We first recall Karamata's theorem ([11, Theorem 1.5.11] or [37, Proposition 0.8]).

Lemma A.1. *Let L be slowly varying at ∞ and locally bounded in $[a, +\infty)$ for some $a > 0$.*

(1) If $r \geq -1$, then

$$\lim_{x\to\infty} \frac{x^{r+1}L(x)}{\int_a^x t^r L(t)\mathrm{d}t} = r + 1.$$

(2) If $r > 1$, then

$$\lim_{x\to\infty} \frac{x^{-r+1}L(x)}{\int_x^{+\infty} t^{-r}L(t)dt} = r-1.$$

For any $\theta > 0$, we define a measure $\mathbf{n}_\theta(\mathrm{d}y)$ as follows: for any positive function g,

$$\int_{\mathbb{R}/\{0\}} g(y)\mathbf{n}_\theta(\mathrm{d}y) = \int_{\mathbb{R}/\{0\}} g(\theta y)\mathbf{n}(\mathrm{d}y).$$

Recall that

$$\nu_\alpha(\mathrm{d}x) = q_1 x^{-1-\alpha}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + q_2|x|^{-1-\alpha}\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x.$$

Lemma A.2. *Assume that **Assumption 2** holds. Then as $\theta \to 0+$,*

$$\theta^{-\alpha}L(\theta^{-1})^{-1}\psi(\theta) \to \begin{cases} \int_{-\infty}^{\infty}(e^{iy}-1)\nu_\alpha(dy), & \alpha \in (0,1); \\ \int_{-\infty}^{\infty}(e^{iy}-1-iy)\nu_\alpha(dy), & \alpha \in (1,2). \end{cases}$$

If $\alpha = 1$ and $\int_{\{|y|>1\}} y\mathbf{n}(dy) = \infty$, then as $\theta \to 0+$,

$$\theta^{-1}(L^*(\theta^{-1}))^{-1}\psi(\theta) \to i\frac{q_1-q_2}{\alpha}.$$

Proof. For any $\theta > 0$, let $\tilde{\mathbf{n}}_\theta(\mathrm{d}y) := \theta^{-\alpha}L(\theta^{-1})^{-1}\mathbf{n}_\theta(\mathrm{d}y)$ and $\pi(x) = \mathbf{n}(x,\infty)$. By **Assumption 2**, we have that for any $x > 0$,

$$\tilde{\mathbf{n}}_\theta(x,\infty) = \theta^{-\alpha}L(\theta^{-1})^{-1}\mathbf{n}(x/\theta,\infty) \to \frac{q_1}{\alpha}x^{-\alpha} = \nu_\alpha(x,\infty), \quad \theta \to 0+,$$

and

$$\tilde{\mathbf{n}}_\theta(-\infty,-x) = \theta^{-\alpha}L(\theta^{-1})^{-1}\mathbf{n}(-\infty,-x/\theta) \to \frac{q_2}{\alpha}x^{-\alpha} = \nu_\alpha(-\infty,-x), \quad \theta \to 0+.$$

Thus $\tilde{\mathbf{n}}_\theta \xrightarrow{v} \nu_\alpha$ on $\overline{\mathbb{R}}_0$. Since $\nu_\alpha(\{-1,1\}) = 0$, thus

$$\begin{aligned}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_{|y|>\theta^{-1}}(e^{i\theta y}-1)\mathbf{n}(\mathrm{d}y) &= \int_{\{|y|>1\}}[e^{iy}-1]\tilde{\mathbf{n}}_\theta(\mathrm{d}y) \\ &\to \int_{\{|y|>1\}}[e^{iy}-1]\nu_\alpha(\mathrm{d}y). \end{aligned} \tag{A.1}$$

Now we consider the integral over $(0,\theta^{-1})$. By Fubini's theorem

$$\begin{aligned}&\int_0^{\theta^{-1}}(e^{i\theta y}-1-i\theta y)\mathbf{n}(\mathrm{d}y) = i\theta\int_0^{\theta^{-1}}(e^{i\theta y}-1)(\pi(y)-\pi(\theta^{-1}))\,\mathrm{d}y \\ &= \int_0^1 i(e^{iy}-1)(\pi(y/\theta)-\pi(\theta^{-1}))\mathrm{d}y.\end{aligned}$$

If $\alpha < 2$, then by the dominated convergence theorem we have, as $\theta \to 0+$

$$\begin{aligned}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_0^{\theta^{-1}}(e^{i\theta y}-1-i\theta y)\mathbf{n}(\mathrm{d}y) &\to \frac{q_1}{\alpha}\int_0^1 i(e^{iy}-1)(y^{-\alpha}-1)\mathrm{d}y \\ &= \int_0^1(e^{iy}-1-iy)\nu_\alpha(\mathrm{d}y).\end{aligned} \tag{A.2}$$

Note that for $\theta < 1$,

$$\begin{aligned}\int_1^{\theta^{-1}} y\mathbf{n}(\mathrm{d}y) &= \int_1^{\theta^{-1}}\left(\int_0^y \mathrm{d}x\right)\mathbf{n}(\mathrm{d}y) = \int_0^1 \mathrm{d}x\int_1^{\theta^{-1}}\mathbf{n}(\,dy) + \int_1^{\theta^{-1}}\mathrm{d}x\int_x^{1/\theta}\mathbf{n}(\mathrm{d}y) \\ &= \pi(1) + \int_1^{\theta^{-1}}\pi(x)\mathrm{d}x - \theta^{-1}\pi(\theta^{-1}).\end{aligned} \tag{A.3}$$

(1) If $0<\alpha<1$, by Lemma A.1, we have that $\int_1^{\theta^{-1}}\pi(x)\mathrm{d}x\sim\frac{q_1}{\alpha(1-\alpha)}\theta^{-1+\alpha}L(\theta^{-1})$. Thus we have

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_1^{\theta^{-1}}\theta y\mathbf{n}(\mathrm{d}y)=\frac{1}{1-\alpha}\frac{q_1}{\alpha}-\frac{q_1}{\alpha}=\int_0^1 y\nu_\alpha(\mathrm{d}y). \tag{A.4}$$

By (A.1), (A.2) and (A.4), we obtain that

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_0^{\infty}(e^{i\theta y}-1-i\theta y\mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y)=\int_0^{\infty}(e^{iy}-1)\nu_\alpha(\mathrm{d}y).$$

Similarly, we can deal with the integral over $(-\infty,0)$, i.e.

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_{-\infty}^{0}(e^{i\theta y}-1-i\theta y\mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y)=\int_{-\infty}^{0}(e^{iy}-1)\nu_\alpha(\mathrm{d}y).$$

Thus the desired result follows for the case when $\alpha\in(0,1)$.

(2) If $\alpha=1$ and $\int_{\{|y|>1\}}y\mathbf{n}(\mathrm{d}y)=\infty$, then $\mathbf{E}(|\xi_1|)=\infty$. Since $\pi(x)$ is regularly varying with index -1, thus

$$\lim_{\theta\to0+}\frac{\theta^{-1}\pi(\theta^{-1})}{L^*(\theta^{-1})}=\frac{q_1}{\alpha}\lim_{\theta\to0+}\frac{L(\theta^{-1})}{L^*(\theta^{-1})}=0.$$

Then we have by (A.3) that

$$\lim_{\theta\to0+}(L^*(\theta^{-1}))^{-1}\int_1^{\theta^{-1}}y\mathbf{n}(\mathrm{d}y)=\lim_{\theta\to0+}(L^*(\theta^{-1}))^{-1}\int_1^{\theta^{-1}}\pi(x)\mathrm{d}x=\frac{q_1}{\alpha}. \tag{A.5}$$

Combining (A.1), (A.2) and (A.5), we obtain that

$$\lim_{\theta\to0+}\theta^{-1}L^*(\theta^{-1})^{-1}\int_0^{\infty}(e^{i\theta y}-1-i\theta y\mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y)=i\frac{q_1}{\alpha}.$$

By similar arguments, we obtain the corresponding result for the integral over $(-\infty,0)$, i.e.

$$\lim_{\theta\to0+}\theta^{-1}L^*(\theta^{-1})^{-1}\int_{-\infty}^{0}(e^{i\theta y}-1-i\theta y\mathbf{1}_{\{|y|<1\}})\mathbf{n}(\mathrm{d}y)=-i\frac{q_2}{\alpha}.$$

Thus the proof for the case when $\alpha=1$ is complete.

(3) Now we consider the case when $\alpha\in(1,2)$. By Fubini's theorem, we have that for $\alpha>1$

$$\begin{aligned}\int_{\theta^{-1}}^{+\infty}y\mathbf{n}(\mathrm{d}y)&=\int_{\theta^{-1}}^{+\infty}\left(\int_0^y\mathrm{d}x\right)\mathbf{n}(\mathrm{d}y)=\int_0^{\theta^{-1}}\mathrm{d}x\int_{\theta^{-1}}^{\infty}\mathbf{n}(\mathrm{d}y)+\int_{\theta^{-1}}^{+\infty}\mathrm{d}x\int_x^{+\infty}\mathbf{n}(\mathrm{d}y)\\&=\theta^{-1}\pi(\theta^{-1})+\int_{\theta^{-1}}^{+\infty}\pi(x)\mathrm{d}x.\end{aligned}$$

Thus by Lemma A.1 we have for $\alpha>1$

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_{\theta^{-1}}^{+\infty}\theta y\mathbf{n}(\mathrm{d}y)=\frac{q_1}{\alpha}+\frac{q_1}{\alpha(\alpha-1)}=\int_1^{+\infty}y\nu_\alpha(\mathrm{d}y). \tag{A.6}$$

Thus by (A.1), (A.2) and (A.6), we obtain that for $\alpha\in(1,2)$,

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_0^{\infty}(e^{i\theta y}-1-i\theta y)\mathbf{n}(\mathrm{d}y)=\int_0^{\infty}(e^{iy}-1-iy)\nu_\alpha(\mathrm{d}y).$$

Using similar arguments, we can get

$$\lim_{\theta\to0+}\theta^{-\alpha}L(\theta^{-1})^{-1}\int_{-\infty}^{0}(e^{i\theta y}-1-i\theta y)\mathbf{n}(\mathrm{d}y)=\int_{-\infty}^{0}(e^{iy}-1-iy)\nu_\alpha(\mathrm{d}y).$$

The proof is now complete. □

Lemma A.3. *Assume that **Assumption 2** holds with $\alpha = 2$ and $\int_{\{|y|>1\}} y^2 \mathbf{n}(dy) = \infty$. Then as $\theta \to 0+$,*

$$\lim_{\theta \to 0+} \theta^{-2} L^*(\theta^{-1})^{-1} \psi(\theta) = -\frac{q_1 + q_2}{2}.$$

Proof. Note that $\int_{\{|y|>1\}} y^2 \mathbf{n}(\mathrm{d}y) = \infty$ is equivalent to $\int_1^\infty y^{-1} L(y)\, \mathrm{d}y = \infty$. By (A.1) and (A.6), we only need to prove

$$\lim_{\theta \to 0+} \theta^{-2} L^*(\theta^{-1})^{-1} \int_{-\theta^{-1}}^{\theta^{-1}} (e^{i\theta y} - 1 - i\theta y) \mathbf{n}(\mathrm{d}y) = -\frac{q_1 + q_2}{2}.$$

In the following, we only deal with the integral over $(0, \theta^{-1})$. By Fubini's theorem we get that

$$\begin{aligned}
&\int_0^{\theta^{-1}} (e^{i\theta y} - 1 - i\theta y) \mathbf{n}(\mathrm{d}y) = i\theta \int_0^{\theta^{-1}} (e^{i\theta y} - 1)(\pi(y) - \pi(\theta^{-1}))\, \mathrm{d}y \\
&= i\theta \int_0^1 (e^{i\theta y} - 1)\pi(y)\, \mathrm{d}y + i\theta \int_1^{\theta^{-1}} (e^{i\theta y} - 1 - i\theta y)\pi(y)\, \mathrm{d}y - \theta^2 \int_1^{\theta^{-1}} y\pi(y)\, \mathrm{d}y \\
&\quad - i \int_0^1 (e^{iy} - 1) \mathrm{d}y \pi(\theta^{-1}).
\end{aligned}$$

By **Assumption 2**, we have

$$\theta^2 \int_1^{\theta^{-1}} y\pi(y)\, \mathrm{d}y \sim \frac{q_1}{2} \theta^2 L^*(1/\theta), \quad \theta \to 0+.$$

Note that

$$\left| \theta \int_0^1 (e^{i\theta y} - 1)\pi(y)\, \mathrm{d}y \right| \le \theta^2 \int_0^1 y\pi(y)\, \mathrm{d}y.$$

By Lemma A.1 (a) with $r = 0$, we have

$$\left| \theta \int_1^{\theta^{-1}} (e^{i\theta y} - 1 - i\theta y)\pi(y)\, \mathrm{d}y \right| \le \theta^3 \int_1^{\theta^{-1}} y^2 \pi(y)\, \mathrm{d}y \sim \frac{q_1}{\alpha} \theta^2 L(1/\theta).$$

Combining above all, we get that

$$\int_0^{\theta^{-1}} (e^{i\theta y} - 1 - i\theta y) \mathbf{n}(\mathrm{d}y) \sim -\frac{q_1}{2} \theta^2 L^*(1/\theta).$$

Similarly, we have that

$$\int_{-\theta^{-1}}^0 (e^{i\theta y} - 1 - i\theta y) \mathbf{n}(\mathrm{d}y) \sim -\frac{q_2}{2} \theta^2 L^*(1/\theta).$$

Now we finish the proof. □

For the reader's convenience, we provide a special many-to-one formula. For more general many-to-one formulas, see [31, Theorem 8.5]. For any $u \in \mathbb{T}$, let n^u be the number of particles in $I_u \setminus \{o\}$.

Lemma A.4 (Many-to-one formula)**.** *Let $\{n_t\}$ be a Poisson process with parameter β on some probability space $(\Omega, \mathcal{G}, P)$. Then for any $g \in \mathcal{B}_b^+(\mathbb{R})$,*

$$\mathbb{E}\left(\sum_{v \in \mathcal{L}_t} g(n^v) \right) = e^{\lambda t} E\left(g(n_t) \right).$$

Lemma A.5. *For any $t > 0$,*

$$\mathbb{E}(|D_t|) = \frac{\beta}{\lambda}(\mu e^{\lambda t} - 1).$$

Proof. Applying Markov property and branching property at the first split time, we have

$$\begin{aligned} g(t) := \mathbb{E}(|D_t|) &= e^{-\beta t} + \int_0^t \beta e^{-\beta s} \sum_k p_k (1 + kg(t-s)) \mathrm{d}s \\ &= e^{-\beta t} + \int_0^t \beta e^{-\beta s} (1 + \mu g(t-s)) \mathrm{d}s. \end{aligned}$$

Using Feynman-Kac formula, we get that

$$g(t) = e^{\lambda t} + \int_0^t \beta e^{\lambda s} \mathrm{d}s = \frac{\beta}{\lambda}(\mu e^{\lambda t} - 1).$$

□

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