Law of iterated logarithm for supercritical symmetric branching Markov process

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Abstract

Let $\{(X_t)_{t\geq 0}, \mathbb{P}_x, x \in E\}$ be a supercritical symmetric branching Markov process on a locally compact metric measure space (E, μ) with spatially dependent local branching mechanism. Under some assumptions on the semigroup of the spatial motion, we first prove law of iterated logarithm type results for $\langle f, X_t\rangle$ under the second moment condition, where f is a linear combination of eigenfunctions of the mean semigroup $\{T_t, t \geq 0\}$ of X. Then we prove law of iterated logarithm type results for $\langle f, X_t\rangle$ under the fourth moment condition, where $f \in T_r(L^2(E, \mu))$ for some $r > 0$.

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1 Introduction

Let $\{Z_n : n \in \mathbb{N}\}$ be a supercritical Galton-Watson process with $Z_0 = 1$ and $\mathbb{E}(Z_1) = m \in (1, \infty)$. It is well-known that, under the assumption $\mathbb{E}(Z_1^2) < \infty$, the process $W_n := Z_n/m^n$ is a non-negative L^2 bounded martingale and thus converges almost surely and in $L^2(\mathbb{P})$ to a non-negative limit W_∞. Heyde [13, 15] found the rate at which $W_n - W_\infty$ converges to 0: $m^{n/2}(W_n - W_\infty)$ converges in distribution to $\sqrt{W_\infty}\mathcal{N}(0, \sigma^2)$, where $\mathcal{N}(0, \sigma^2)$ is a normal random variable, independent of W_∞, with variance $\sigma^2 := \frac{1}{m^2-m}(\mathbb{E}(Z_1^2) - m^2)$. The fluctuation in the almost sure sense of $W_n - W_\infty$ was established by Heyde [14]. Under the assumption $\mathbb{E}(Z_1^3) < \infty$, Heyde [14] proved that, on the event $\{W_\infty > 0\}$, it holds almost surely that

$$\limsup_{n\to\infty} \frac{m^{n/2}(W_n - W_\infty)}{\sqrt{2\log n}} = \sqrt{\sigma^2 W_\infty}, \quad \liminf_{n\to\infty} \frac{m^{n/2}(W_n - W_\infty)}{\sqrt{2\log n}} = -\sqrt{\sigma^2 W_\infty}. \tag{1.1}$$

Later, Heyde and Leslie [17] removed the assumption $\mathbb{E}(Z_1^3) < \infty$ and proved (1.1) under the second moment condition only. Since $\frac{\log\log Z_n}{\log n} = 1$ almost surely on $\{W_\infty > 0\}$, it follows from (1.1) that almost surely on $\{W_\infty > 0\}$,

$$\limsup_{n\to\infty} \frac{m^{n/2}(W_n - W_\infty)}{\sqrt{2\log\log Z_n}} = \sqrt{\sigma^2 W_\infty}, \quad \liminf_{n\to\infty} \frac{m^{n/2}(W_n - W_\infty)}{\sqrt{2\log\log Z_n}} = -\sqrt{\sigma^2 W_\infty}.$$

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Therefore, results like (1.1) are called "laws of iterated logarithm" (LIL) in the literature. See [18, Remark 1.3] and [19, Remark 2.4].

For supercritical (finite) multitype Galton-Watson processes $\{Z_n : n \in \mathbb{N}\}$, Kesten and Stigum [20, 21] established central limit theorems by using the Jordan canonical form of the expectation matrix M. Asmussen [2] extended (1.1) to $Z_n \cdot a$, where a is a vector satisfying certain conditions. In the continuous time setting, central limit type theorems were proved by Athreya [4, 5, 6] and an analog of (1.1) was given in [2, Theorem 2].

There are also some LIL type theorems for more general branching processes. For branching random walks, Iksanov and Kabluchko [18] proved an LIL type theorem for Biggins' martingale. For general Crump-Mode-Jagers branching processes, Iksanov et al [19] proved an LIL type theorem for Nerman's martingale. All known LIL type results for branching processes, including branching random walks and Crump-Mode-Jagers branching processes, are LIL for L^2 bounded martingales. For some related results for L^2 bounded martingale in the general case, see [16, 29].

In this paper, we are interested in supercritical branching Markov processes with spatially dependent (local) branching mechanism. We always assume that E is a locally compact separable metric space and that μ is a σ-finite Borel measure on E with full support. We assume that ∂ is a point not in E and put $E_\partial := E \cup \{\partial\}$. Any function f on E is automatically extended to E_∂ by defining $f(\partial) = 0$. We assume that $\xi = \{\xi_t, \mathbf{P}_x, x \in E\}$ is a μ-symmetric Hunt process on E and that $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. The semigroup of ξ is denoted by $\{P_t : t \geq 0\}$. Our assumption on ξ is as follow:

(H1) **(a)** There exists a family of continuous strictly positive symmetric functions $\{p_t(x, y) : t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p_t(x, y) f(y) \mu(\mathrm{d}y).$$

(b) For any $t > 0$, we have

$$\int_E p_t(x, x) \mu(\mathrm{d}x) < \infty.$$

(c) For any $t > 0$, $x \mapsto p_t(x, x)$ belongs to $L^2(E, \mu)$.

A branching Markov process can be described as follows: initially there is a particle located at $x \in E$ and it moves according to $\{\xi, \mathbf{P}_x\}$. When the particle is at site y, the branching rate is given by $\beta(y)$, where β is a non-negative Borel function, that is, each individual dies in $[t, t + \mathrm{d}t)$ with probability $\beta(\xi_t)\mathrm{d}t + o(\mathrm{d}t)$. When an individual dies at $y \in E$, it splits into k particles with probability $p_k(y)$. Once an individual reaches ∂, it disappears from the system. All the individuals, once born, evolve independently.

Our assumption on the branching particle system is as follow:

(H2) **(a)** $\beta(x)$ is a non-negative bounded Borel function on E.

(b) $\{p_k(x) : k = 0, 1, ...\}$ satisfies

$$\sup_{x \in E} \sum_{k=0}^{\infty} k^2 p_k(x) < \infty.$$

Let $\mathcal{M}_a(E)$ be the space of finite atomic measures on E and $\mathcal{B}_b(E)$ the set of bounded Borel functions on E. For $t \geq 0$ and $B \in \mathcal{B}(E)$, let $X_t(B)$ denote the number of particles alive at time t and located in B. Then $X = \{X_t : t \geq 0\}$ is an $\mathcal{M}_a(E)$- valued Markov process. For any $x \in E$, we denote by $\mathbb{P}_x$ the law of X with initial value $X_0 = \delta_x$. For any function f in E and $\nu \in \mathcal{M}_a(E)$, define $\langle f, \nu \rangle := \int_E f(y)\nu(\mathrm{d}y)$ and $\|f\|_2 := \sqrt{\int_E f^2(y)\mu(\mathrm{d}y)}$. Let

$$\omega(t,x) := \mathbb{E}_x\left(e^{-\langle f, X_t\rangle}\right),$$

then it is well-known that $\omega(t,x)$ is the unique positive solution to the equation

$$\omega(t,x) = \mathbf{E}_x\left(\int_0^t \psi\left(\xi_s, \omega(t-s,\xi_s)\right)\mathrm{d}s\right) + \mathbf{E}_x\left(e^{-f(\xi_t)}\right),$$

here $\psi(x,z) = \beta(x)\left(\sum_{k=0}^\infty p_k(x)z^k - z\right)$ if $x \in E, z \in [0,1]$, and $\psi(\partial, z) = 0, z \in [0,1]$. For $k = 1, 2, \ldots,$ define

$$A^{(k)}(x) := \frac{\partial^k}{\partial z^k}\psi(x,z)|_{z=1}. \tag{1.2}$$

In particular,

$$A^{(1)}(x) = \beta(x)\left(\sum_{k=0}^\infty k p_k(x) - 1\right), \quad A^{(2)}(x) = \beta(x)\sum_{k=0}^\infty k(k-1)p_k(x).$$

For any $f \in \mathcal{B}_b(E)$ and $(t,x) \in (0,\infty)\times E$, define

$$T_t f(x) := \mathbf{E}_x\left[e^{\int_0^t A^{(1)}(\xi_s)\mathrm{d}s} f(\xi_t)\right],$$

then it is well-known that for any $t \geq 0$ and $x \in E$, $T_t f(x) = \mathbb{E}_x\left(\langle f, X_t\rangle\right)$.

Under the assumption **(H1)** and **(H2)**, there exists a family of continuous strictly positive symmetric functions $\{q_t(x,y) : t \geq 0\}$ on $E \times E$ such that

$$T_t f(x) = \int_E q_t(x,y) f(y)\mu(\mathrm{d}y).$$

As summarized in [24], for any $t > 0$, $q_t(x,x) \in L^1(E,\mu) \cap L^2(E,\mu)$. Moreover, $(T_t)_{t\geq 0}$ is a strongly continuous semigroup and, for any $t > 0$, T_t is a Hilbert-Schmidt operator. Let $\mathcal{L}$ denote the infinitesimal generator of $\{T_t : t \geq 0\}$ in $L^2(E,\mu)$. It is well known that the spectrum of $\mathcal{L}$ is discrete. We list the eigenvalues $-\lambda_1 > -\lambda_2 > \ldots$ in decreasing order. The first eigenvalue $-\lambda_1$ is simple and the corresponding eigenfunction $\phi_1(x)$ can be chosen to be strictly positive everywhere and continuous. Without loss of generality, assume that $\|\phi_1\|_2 = 1$. For $k > 1$, let $\{\phi_j^{(k)}, j = 1, ..., n_k < \infty\}$ be an orthonormal basis of the eigenspace associated with $-\lambda_k$. We know that $\{\phi_j^{(k)}, j = 1, 2, ..., n_k; k = 1, 2, ...\}$ forms a complete orthonormal basis of $L^2(E,\mu)$ and all the are continuous, here $\phi_1^{(1)} := \phi_1$. Furthermore, all the eigenfunctions $\phi_j^{(k)}$ belong to $L^4(E,\mu)$.

For any $x, y \in E$ and $t > 0$, we have the following expansion for q:

$$q_t(x,y) = \sum_{k=1}^\infty e^{-\lambda_k t}\sum_{j=1}^{n_k}\phi_j^{(k)}(x)\phi_j^{(k)}(y). \tag{1.3}$$

We assume that the branching Markov process is supercritical, that is

(H3) $\lambda_1 < 0$.

We will also assume that

(H4) **(a)** $\{T_t : t \geq 0\}$ is intrinsically ultracontractive, that is, for any $t > 0$, there exists $c_t > 0$ such that for all $x \in E$,

$$\sqrt{q_t(x,x)} \leq c_t \phi_1(x).$$

(b) ϕ_1 is bounded on E.

The assumption **(H4)** is stronger than **(H1)(c)**. This stronger assumption will be used to control the fluctuation of the branching Markov process in small time. Indeed, in the proof of Lemma 4.4, we first need to give a suitable upper bound for the martingale difference in the small time interval $t \in [n\delta, (n+1)\delta)$ (see (4.32) below), and then we need to control the variance of the martingale in small time. For $Z_t^{k,j}$ defined in (4.42), although one can apply the inequality below (2.11) in [24] to show that $e^{(\lambda_1 - 2\lambda_k)t} Z_t^{k,j} \lesssim e^{\lambda_1 t} \langle T_{(n+1)\delta - t}\left((\phi_j^{(k)})^2\right), X_t\rangle$, we do not know how to show $\sup_{t>0} e^{\lambda_1 t}\langle T_{(n+1)\delta-t}\left((\phi_j^{(k)})^2\right), X_t\rangle < \infty$ without **(H4)**.

For a list of spatial processes satisfying **(H1)** and **(H4)**, see [25, Section 1.4]. Although branching OU processes do not satisfy these assumptions, parts of our argument still work for branching OU processes, see Remark 2.7 below.

For a brief introduction to intrinsically ultracontractive semigroups, one can refer to [8, Section 3]. For $k \geq 1$ and $1 \leq j \leq n_k$, define

$$W_t^{k,j} := e^{\lambda_k t}\langle \phi_j^{(k)}, X_t\rangle.$$

According to [24, Lemma 3.1], when $\lambda_1 > 2\lambda_k$, $W_\infty^{k,j} := \lim_{t\to\infty} W_t^{k,j}$ exists $\mathbb{P}_x$-a.s and in L^2. For simplicity, we set $W_t := W_t^{1,1}$ and $W_\infty := W_\infty^{1,1}$. Define $\mathcal{E} := \{W_\infty = 0\}$.

Some spatial central limit theorems were established in a framework a little more general than our framework, generalizing the corresponding results for branching OU process [1]. To state the main results of [24], we first introduce some notations. Denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(E,\mu)$. Every $f \in L^2(E,\mu)$ admits the following L^2 expansion

$$f = \sum_{k=1}^{\infty}\sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}, \tag{1.4}$$

where $a_j^k = \langle f, \phi_j^{(k)}\rangle$. Define

$$\gamma(f) := \inf\{k \geq 1 : \text{there exists } j \text{ with } 1 \leq j \leq n_k \text{ such that } a_j^k \neq 0\},$$

here we use the usual convention $\inf \emptyset = \infty$. Define

$$f_{la}(x) := \sum_{2\lambda_k < \lambda_1}\sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x), \quad f_{cr}(x) := \sum_{2\lambda_k = \lambda_1}\sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x), \tag{1.5}$$

$$f_{sm}(x) := f(x) - f_{la}(x) - f_{cr}(x), \tag{1.6}$$

$$f_1(x) := \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)} \phi_j^{(\gamma(f))}(x), \quad \tilde{f}(x) := f(x) - f_1(x). \tag{1.7}$$

Now for $f \in L^2(E,\mu) \cap L^4(E,\mu)$, we define $\sigma_{sm}^2(f), \sigma_{cr}^2(f)$ and $\sigma_{la}^2(f)$ by

$$\sigma_{sm}^2(f) := \int_0^\infty e^{\lambda_1 s}\langle A^{(2)} \cdot (T_s f)^2, \phi_1\rangle \mathrm{d}s + \langle f^2, \phi_1\rangle, \quad \sigma_{cr}^2(f) := \langle A^{(2)} \cdot f_1^2, \phi_1\rangle, \tag{1.8}$$

$$\sigma_{la}^2(f) := \int_0^\infty e^{-\lambda_1 s}\Big\langle A^{(2)}\cdot\Big(\sum_{2\lambda_k<\lambda_1} e^{\lambda_k s}\sum_{j=1}^{n_k} a_j^k\phi_j^{(k)}(x)\Big)^2, \phi_1\Big\rangle \mathrm{d}s - \langle (f_{la})^2, \phi_1\rangle. \tag{1.9}$$

The spatial central limit theorems, [24, Theorems 1.8–1.12], can be stated as follows.

Theorem 1.1 *(i) Small branching rate: If $f \in L^2(E,\mu)\cap L^4(E,\mu)$ with $\lambda_1 < 2\lambda_{\gamma(f)}$, then $\sigma_{sm}^2(f)\in(0,\infty)$ and as $t\to\infty$, under $\mathbb{P}_x(\cdot|\mathcal{E}^c)$,*

$$e^{\lambda_1 t/2}\langle f, X_t\rangle \xrightarrow{d} G_{sm}\sqrt{W^*}.$$

where W^* has the same law as W_∞ conditioned on $\mathcal{E}^c$, $G_{sm}\sim\mathcal{N}\left(0,\sigma_{sm}^2(f)\right)$ and W^* is independent of G_{sm}.

(ii) Critical branching rate: If $f \in L^2(E,\mu)\cap L^4(E,\mu)$ with $\lambda_1 = 2\lambda_{\gamma(f)}$, then $\sigma_{cr}^2(f)\in(0,\infty)$ and as $t\to\infty$, under $\mathbb{P}_x(\cdot|\mathcal{E}^c)$,

$$t^{-1/2}e^{\lambda_1 t/2}\langle f, X_t\rangle \xrightarrow{d} G_{cr}\sqrt{W^*}.$$

where W^* has the same law as W_∞ conditioned on $\mathcal{E}^c$, $G_{cr}\sim\mathcal{N}\left(0,\sigma_{cr}^2(f)\right)$ and W^* is independent of G_{cr}.

(iii) Large branching rate (I): If $f \in L^2(E,\mu)\cap L^4(E,\mu)$ with $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr}=0$, then $\sigma_{sm}^2(f_{sm})+\sigma_{la}^2(f)\in(0,\infty)$ and as $t\to\infty$, under $\mathbb{P}_x(\cdot|\mathcal{E}^c)$,

$$e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right) \xrightarrow{d} G_{la,1}\sqrt{W^*}.$$

where W^* has the same law as W_∞ conditioned on $\mathcal{E}^c$, $G_{la,1}\sim\mathcal{N}\left(0,\sigma_{sm}^2(f_{sm})+\sigma_{la}^2(f)\right)$ and W^* is independent of $G_{la,1}$.

(iv) Large branching rate (II): If $f \in L^2(E,\mu)\cap L^4(E,\mu)$ with $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr}\neq 0$, then as $t\to\infty$, under $\mathbb{P}_x(\cdot|\mathcal{E}^c)$,

$$t^{-1/2}e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right) \xrightarrow{d} G_{la,2}\sqrt{W^*}.$$

where W^* has the same law as W_∞ conditioned on $\mathcal{E}^c$, $G_{la,2}\sim\mathcal{N}\left(0,\sigma_{cr}^2(f_{cr})\right)$ and W^* is independent of $G_{la,2}$.

In this paper, we will complement the CLT type results above for $\langle f, X_t\rangle$ with law of iterated logarithm type results for $\langle f, X_t\rangle$.

2 Main results

Our first four results are LIL type results in the special case when $f(x)=\sum_{k=1}^m\sum_{j=1}^{n_k} a_j^k\phi_j^{(k)}(x)$ for some $m\in\mathbb{N}$ and $a_j^k\in\mathbb{R}$. In the next four theorems, we assume **(H1)**–**(H4)** hold and f is of the form above. Recall that $\mathcal{E}=\{W_\infty=0\}$.

Theorem 2.1 *Small branching rate case: If $\lambda_1 < 2\lambda_{\gamma(f)}$, then $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,*

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}} = \sqrt{\sigma_{sm}^2(f)W_\infty}, \quad \liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}} = -\sqrt{\sigma_{sm}^2(f)W_\infty}.$$

Remark 2.2 *Note that Theorem 2.1 is equivalent to that $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,*

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log\log\langle\phi_1, X_t\rangle}} = \sqrt{\sigma_{sm}^2(f)W_\infty},$$
$$\liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log\log\langle\phi_1, X_t\rangle}} = -\sqrt{\sigma_{sm}^2(f)W_\infty}.$$

Thus, the result above is a law of iterated logarithm in some sense. In this paper, we will call results like Theorem 2.1 "law of iterated logarithm" following the convention of [18, 19].

Theorem 2.3 *Critical branching rate case: If $\lambda_1 = 2\lambda_{\gamma(f)}$, then $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,*

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2t\log\log t}} = \sqrt{\sigma_{cr}^2(f)W_\infty}, \quad \liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2t\log\log t}} = -\sqrt{\sigma_{cr}^2(f)W_\infty}.$$

Remark 2.4 *For the special case where X is a (finite) multitype branching process and the mean matrix M is symmetric, our results are consistent with [2, Theorem 2]. More precisely, let $\rho_1 > \rho_2 > \cdots > \rho_K$ be the eigenvalues of M and, for $j = 1, \ldots, K$, let $\{v_j^k : 1 \le k \le n_j\}$ be a basis of the eigenspace corresponding to ρ_j. One can choose the eigenvectors v_j^k so that $\{v_j^k, 1 \le j \le K, 1 \le k \le n_j\}$ forms an orthonomal basis. For any vector a, define $\gamma(a) := \inf\left\{j : \exists k \le n_j \quad \text{such that} \quad a \cdot v_j^k \neq 0\right\}$ and $\lambda(a) := -\log\rho_{\gamma(a)}$, where $a \cdot b$ stands for the inner product. Asmussen [2, Theorem 2] proved that, if $2\lambda(a) \ge \lambda_1$ and $\mathbb{E}(|Z_1|^2) < \infty$, then there exists a deterministic function $C_t = C_t(a)$ and a non-negative random variable W such that, almost surely on the event $\{W > 0\}$,*

$$\limsup_{t\to\infty} \frac{X_t \cdot a}{C_t} = \sqrt{W}, \quad \text{and} \quad \liminf_{t\to\infty} \frac{X_t \cdot a}{C_t} = -\sqrt{W}.$$

Moreover, when $2\lambda(a) > \lambda_1$, $C_t = \sigma(a)e^{-\lambda_1 t/2}\log t$, and when $2\lambda(a) = \lambda_1$, $C_t = \sigma(a)e^{-\lambda_1 t/2}\log\log t$. Here $\sigma(a)$ is a positive constant and is given by (1.9).

The following two theorems give laws of iterated logarithm for $\langle f, X_t\rangle$ for the case that $\lambda_1 > 2\lambda_{\gamma(f)}$. As far as we know, there is no counterpart to these two results for multitype branching processes.

Theorem 2.5 *Large branching rate case (I): If $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr} = 0$, then $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,*

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right)}{\sqrt{2\log t}} = \sqrt{\left(\sigma_{sm}^2(f_{sm}) + \sigma_{la}^2(f)\right)W_\infty}, \tag{2.1}$$

and

$$\liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right)}{\sqrt{2\log t}} = -\sqrt{\left(\sigma_{sm}^2(f_{sm}) + \sigma_{la}^2(f)\right)W_\infty}. \tag{2.2}$$

As in Remark 2.2, we can replace $\log t$ by the asymptotically equivalent expression $\log\log\langle\phi_1, X_t\rangle$, thereby justifying the use of the term "law of the iterated logarithm".

Theorem 2.6 *Large branching rate case (II): If $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr} \neq 0$, then $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,*

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right)}{\sqrt{2t\log\log t}} = \sqrt{\sigma_{cr}^2(f_{cr})W_\infty},$$

and

$$\liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right)}{\sqrt{2t\log\log t}} = -\sqrt{\sigma_{cr}^2(f_{cr})W_\infty},$$

Remark 2.7 *In the case of branching OU process,* **(H4)** *is not satisfied and* **(H1)(c)** *should be replaced by "There exists $t_0 > 0$ such that $p_t(x, x) \in L^2(E, \mu)$ for any $t > t_0$". We mention here that for a branching OU process, there exists $\tilde{t} = \tilde{t}(t_0) > 0$ such that the proofs in the discrete-time setting $\{n\delta\}$ in Section 4.1 still work for any $\delta \geq \tilde{t}$. Thus for branching OU processes, one can get similar laws of iterated logarithm for $f = \sum_{k=1}^m \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}$ in discrete time $\{n\delta\}$ for $\delta \geq \tilde{t}$.*

If $f(x) = \phi_j^{(k)}(x)$ for some $k \in \mathbb{N}$ and $j \leq n_k$, then $e^{\lambda_k t}\langle f, X_t\rangle$ is a martingale. Therefore, Theorems 2.1, 2.3 and 2.5 can be regarded as LIL for martingales. In this case, the scenario dealt with in Theorem 2.6 does not occur. In the proofs, we use the fact that $e^{\lambda_k t}\langle f, X_t\rangle$ is a martingale to go from discrete-time to continuous-time. It is natural to ask whether the results of Theorems 2.1, 2.3, 2.5 and 2.6 remain valid for more general $f \in L^2(E, \mu) \cap L^4(E, \mu)$. However, for general $f = \sum_{k=1}^\infty \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}$, $\langle f, X_t\rangle$ is not well-approximated by martingales. The arguments in the proofs of Theorems 2.1, 2.3, 2.5 and 2.6 no longer work. We need the following stronger assumption and a different argument.

(H5) $\sup_{x\in E}\sum_{k=0}^\infty k^4 p_k(x) < \infty$

We will show that, if **(H5)** also holds, then the conclusions of Theorems 2.1, 2.3, 2.5 and 2.6 hold for functions in the following class:

$$\mathbb{T}(E) := \left\{f : \text{there exists } r > 0 \text{ and } g \in L^2(E, \mu) \text{ such that } f = T_r g\right\}.$$

Here is our law of iterated logarithm theorem for general $f \in \mathbb{T}(E)$.

Theorem 2.8 *If* **(H1)**–**(H5)** *hold, then the conclusions of Theorems 2.1, 2.3, 2.5 and 2.6 hold for any $f \in \mathbb{T}(E)$.*

The proof of Theorem 2.8 is different from that of Theorems 2.1, 2.3, 2.5 and 2.6. One of the key differences is that we choose a different discretization scheme.

We end this section with a brief description of the strategy and organization of this paper. In Section 3, we gather some useful results and give a general law of iterated logarithm for sequence of random variables. In Section 4, we prove Theorems 2.1, 2.3, 2.5 and 2.6. In Subsection 4.1 we give some general results and we prove Theorems 2.1, 2.3, 2.5 and 2.6 in Subsections 4.2–4.4 respectively. We first prove a law of iterated logarithm in the discrete-time $\{n\delta, n \in \mathbb{N}\}$ for any given $\delta > 0$. Then we prove our laws of iterated logarithm for continuous time t for linear combinations of the eigenfunctions under optimal second moment condition. The argument for discrete-time is inspired by [18] and the argument for continuous time is inspired by [3, Section 12] (for example, see the proof of [3, Theorem 12.4, p.340]) and [19, p.20–p.22].

In Section 5, we prove Theorem 2.8. In Subsection 5.1, we first give an upper bound, see Proposition 5.1, for the limsup of an expression involving $\langle f, X_t\rangle$ for any $f \in \mathbb{T}(E)$ with $\lambda_{\gamma(f)} > 0$, and then use this proposition to prove Theorem 2.8. The rest of the subsection is devoted to the proof of Proposition 5.1. To prove this proposition, first write f as the sum of $S_f^{(n)}$ and $\Gamma_f^{(n)}$, see (5.10), with $S_f^{(n)}$ being linear combinations of eigenfunctions (the number of eigenfunctions involved increases as $n \to \infty$). For $f \in \mathbb{T}(E)$, we can compare the small-time behavior of $\langle T_{t_{n+1}-t} S_f^{(n)}, X_t\rangle$ and $\langle f, X_t\rangle$ for $t \in [t_n, t_{n+1})$. The contribution of $\Gamma_f^{(n)}$ is negligible. Thus we get that $\langle T_{t_{n+1}-t} f, X_t\rangle \approx \langle f, X_t\rangle$ for any $t \in [t_n, t_{n+1})$ when n is large enough. The precise argument can be found in Lemmas 5.2, 5.3 and Corollary 5.4. Lemma 5.5 is a rough bound for the conditioned variance of $\langle T_{s_n} f, X_{t_n}\rangle$ where either $s_n = 0$ or $s_n = t_{n+1} - t_n$. Under the fourth moment condition, we give an upper bound for $\mathbb{E}_x\left(\langle f, X_t\rangle^4\right)$, see Lemma 5.6 (whose proof is postponed to Subsection 5.2), for any $f \in \mathbb{T}(E)$ with $\lambda_{\gamma(f)} > 0$. Using Lemma 5.6, we give an upper bound for the limit superior of the discrete-time version of the quantity in Proposition 5.1, see Lemma 5.7. Finally, we treat the continuous-time setting using an idea roughly similar to that used in Lemma 4.4

We believe that the general idea of this paper can be adapted to other branching Markov processes such as non-symmetric case [27], non-local branching Markov process [9] and superprocesses [12, 22, 26, 30]. We do not pursue this in this paper.

3 Preliminary

Throughout this paper, we always assume that **(H1)**–**(H4)** hold. We use $F(x) \lesssim_{r,f,\kappa,\dots} G(x), x \in E$ to denote that there exists some constant $C = C(r, f, \kappa, ...)$ such that $F(x) \le CG(x)$ for all $x \in E$.

Any function $f \in L^2(E, \mu)$ admits the expansion (1.4). We always use this expansion when dealing with $f \in L^2(E, \mu)$. It follows from [24, (2.1)] that for any $f \in L^2(E, \mu)$, $t > 0$ and $x \in E$,

$$T_t f(x) = \mathbb{E}_x(\langle f, X_t\rangle) = \sum_{k=\gamma(f)}^{\infty} e^{-\lambda_k t} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x). \tag{3.1}$$

By [24, (2.11)], we also have for any $f \in L^2(E, \mu) \cap L^4(E, \mu)$, $t > 0$ and $x \in E$ that

$$\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) = \int_0^t T_s\left[A^{(2)} \cdot (T_{t-s} f)^2\right](x)\mathrm{d}s + T_t(f^2)(x). \tag{3.2}$$

Define

$$\mathrm{Var}_x(Y|\mathcal{F}) := \mathbb{E}_x\left[Y^2\big|\mathcal{F}\right] - \left(\mathbb{E}_x\left[Y\big|\mathcal{F}\right]\right)^2.$$

Here and throughout the paper we use the notation $\mathrm{Var}_x(Y) = \mathbb{E}_x\left(Y^2\right) - (\mathbb{E}_x(Y))^2$.

Lemma 3.1 *If $f \in L^2(E, \mu)$, then for any $t_0 > 0$, we have*

$$e^{\lambda_{\gamma(f)} t}|T_t f(x)| + e^{\lambda_{\gamma(\tilde{f})} t}|T_t \tilde{f}(x)| \lesssim_{t_0} \|f\|_2 \phi_1(x), \quad t > t_0, x \in E,$$

where $\tilde{f}$ is defined in (1.7). *Consequently, $T_t f \in L^2(E, \mu) \cap L^4(E, \mu)$ for any $t > 0$ and $f \in L^2(E, \mu)$.*

Proof: The upper bound for $e^{\lambda_{\gamma(f)} t}|T_t f(x)|$ follows from [24, (2.10)] and **(H4)**. With f replaced by $\tilde{f}$, we see that

$$e^{\lambda_{\gamma(\tilde{f})} t}|T_t \tilde{f}(x)| \lesssim_{t_0} \|\tilde{f}\|_2 \phi_1(x), \quad t > t_0, x \in E.$$

Now the upper bound for $|T_t \tilde{f}(x)|$ follows from the fact that $\|\tilde{f}\|_2 \le \|f\|_2$. $\square$

Lemma 3.2 *Assume $f \in L^2(E,\mu)\cap L^4(E,\mu)$. Recall that $\sigma^2_{sm}(f), \sigma^2_{cr}(f)$ and $\sigma^2_{la}(f)$ are defined in* (1.8)–(1.9).
(1) *If $\lambda_1 < 2\lambda_{\gamma(f)}$, then for any $x \in E$,*

$$\lim_{t\to\infty} e^{\lambda_1 t/2}\mathbb{E}_x\left(\langle f, X_t\rangle\right) = 0, \quad \lim_{t\to\infty} e^{\lambda_1 t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) = \sigma^2_{sm}(f)\phi_1(x).$$

Moreover, for any $t_0 > 0$,

$$e^{\lambda_1 t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) \lesssim_{t_0} \left(\|f\|_2^2 + \|f\|_4^2\right)\phi_1(x), \quad t > t_0, x \in E. \tag{3.3}$$

(2) *If $\lambda_1 = 2\lambda_{\gamma(f)}$, then for any fixed $t_0 > 0$, it holds that*

$$\left|t^{-1}e^{\lambda_1 t}\mathrm{Var}_x\left(\langle f, X_t\rangle\right) - \sigma^2_{cr}(f)\phi_1(x)\right| \lesssim_{t_0,f} t^{-1}\phi_1(x), \quad t > t_0, x \in E.$$

(3) *If $\lambda_1 > 2\lambda_{\gamma(f)}$, then for any $x \in E$,*

$$\lim_{t\to\infty} e^{2\lambda_{\gamma(f)}t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) = \eta_f^2(x) := \int_0^\infty e^{2\lambda_{\gamma(f)}s}T_s\left[A^{(2)}\cdot f_1^2\right](x)\mathrm{d}s.$$

Moreover, for any $t_0 > 0$,

$$e^{2\lambda_{\gamma(f)}t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) \lesssim_{t_0,f} \phi_1(x), \quad t > t_0, x \in E.$$

Proof: All the assertions, except (3.3), follow from [24, Lemma 2.3] and **(H4)**. The proof of (3.3) is just a refinement of the proof of [24, (3.13)]. Combining (3.2) and Lemma 3.1, we get that for any $t > t_0$ and $x \in E$,

$$\begin{aligned}
&e^{\lambda_1 t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right) = e^{\lambda_1 t}\int_0^t T_{t-s}\left[A^{(2)}\cdot (T_s f)^2\right](x)\mathrm{d}s + e^{\lambda_1 t}T_t(f^2)(x)\\
&\lesssim_{t_0} e^{\lambda_1 t}\left(\int_{t_0/2}^t + \int_0^{t_0/2}\right)T_{t-s}\left[(T_s f)^2\right](x)\mathrm{d}s + \|f^2\|_2\phi_1(x)\\
&\lesssim_{t_0} e^{\lambda_1 t}\|f\|_2^2\int_{t_0/2}^t e^{-2\lambda_{\gamma(f)}s}T_{t-s}\left[\phi_1^2\right](x)\mathrm{d}s + e^{\lambda_1 t}\int_0^{t_0/2}T_{t-s}\left[(T_s f)^2\right](x)\mathrm{d}s + \|f^2\|_2\phi_1(x).
\end{aligned}$$

Let $k_0 \in \mathbb{N}$ be such that $2\lambda_{k_0-1} \le \lambda_1 < 2\lambda_{k_0}$. Then $\lambda_{\gamma(f)} \ge \lambda_{k_0}$. Therefore, combining the boundedness of ϕ_1 in **(H4)**, Lemma 3.1 and the fact that $(T_s f)^2 \lesssim_{t_0} T_s(f^2)$ for $(s,x) \in (0, t_0/2)\times E$, we get

$$\begin{aligned}
&e^{\lambda_1 t}\mathbb{E}_x\left(\langle f, X_t\rangle^2\right)\\
&\lesssim_{t_0} e^{\lambda_1 t}\|f\|_2^2\int_{t_0/2}^t e^{-2\lambda_{k_0}s}T_{t-s}\left[\phi_1\right](x)\mathrm{d}s + e^{\lambda_1 t}\int_0^{t_0/2}T_{t-s}\left[T_s(f^2)\right](x)\mathrm{d}s + \|f\|_4^2\phi_1(x)\\
&= \|f\|_2^2\int_{t_0/2}^t e^{\left(\lambda_1 - 2\lambda_{k_0}\right)s}\mathrm{d}s\phi_1(x) + \frac{t_0}{2}e^{\lambda_1 t}T_t(f^2)(x) + \|f\|_4^2\phi_1(x)\\
&\lesssim_{t_0} \left(\|f\|_2^2 + \|f\|_4^2\right)\phi_1(x),
\end{aligned}$$

which implies (3.3). □

Lemma 3.3 *Suppose that $f \in L^2(E,\mu)\cap L^4(E,\mu)$ with $\lambda_1 > 2\lambda_{\gamma(f)}$ and recall $\tilde{f}$ is defined in* (1.7). *Then there exists $c(f) > 0$ such that for any $t \ge 1$,*

$$e^{2\lambda_{\gamma(f)}t}\mathbb{E}_x\left[\langle \tilde{f}, X_t\rangle^2\right] \lesssim_f e^{-c(f)t}.$$

Proof: See the proof of [24, Theorem 1.6]. Moreover, one can choose $c(f) = 2(\lambda_{\gamma(\tilde{f})} - \lambda_{\gamma(f)})$ if $\lambda_1 > 2\lambda_{\gamma(\tilde{f})}$ and $c(f) = (\lambda_{\gamma(\tilde{f})} - \lambda_{\gamma(f)})$ if $\lambda_1 = 2\lambda_{\gamma(\tilde{f})}$ and $c(f) = (\lambda_1 - 2\lambda_{\gamma(f)})$ if $2\lambda_{\gamma(\tilde{f})} > \lambda_1 > 2\lambda_{\gamma(f)}$. □

As an application of Lemma 3.3, we have the following strong law of large numbers type result.

Lemma 3.4 *For any $f \in L^2(E,\mu) \cap L^4(E,\mu)$ and $\delta > 0$, we have*

$$\lim_{n\to\infty} e^{\lambda_1 n\delta}\langle f, X_{n\delta}\rangle = \langle f, \phi_1\rangle W_\infty, \quad \mathbb{P}_x\text{-}a.s.$$

Proof: By Lemma 3.3, we have for any $n \in \mathbb{N}$,

$$e^{2\lambda_1 n\delta}\mathbb{E}_x\left[\langle \tilde{f}, X_{n\delta}\rangle^2\right] \lesssim_f e^{-c(f)n\delta}.$$

Thus, for any $\varepsilon > 0$, by Markov's inequality,

$$\sum_{n\geq 0}\mathbb{P}_x\left(\left|e^{\lambda_1 n\delta}\langle \tilde{f}, X_{n\delta}\rangle\right| > \varepsilon\right) \lesssim_f \frac{1}{\varepsilon^2}\sum_{n\geq 0} e^{-c(f)n\delta} < \infty,$$

which implies that $e^{\lambda_1 n\delta}\langle \tilde{f}, X_{n\delta}\rangle$ converges to 0 $\mathbb{P}_x$-a.s. Since $e^{\lambda_1 n\delta}\langle \tilde{f}, X_{n\delta}\rangle = e^{\lambda_1 n\delta}\langle f, X_{n\delta}\rangle - \langle f, \phi_1\rangle W_{n\delta}$ and $\langle f, \phi_1\rangle W_{n\delta}$ converges to $\langle f, \phi_1\rangle W_\infty$ almost surely, the assertion of the lemma follows immediately. □

Lemma 3.5 *Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{E}X_i = 0$ and $\mathbb{E}|X_i|^3 < \infty, i = 1, 2, \ldots$. If $\sum_{i\geq 1}\mathbb{E}X_i^2 < \infty$, then there exists an absolute constant C_1 such that*

$$\sup_{y\in\mathbb{R}}\left|\mathbb{P}\left[\frac{\sum_{i\geq 1} X_i}{\sqrt{\sum_{i\geq 1}\mathbb{E}X_i^2}} \leq y\right] - \Phi(y)\right| \leq C_1\frac{\sum_{i\geq 1}\mathbb{E}|X_i|^3}{\sqrt{\left(\sum_{i\geq 1}\mathbb{E}X_i^2\right)^3}}.$$

Proof: See [18, Lemma A.2.]. □

Lemma 3.6 *For any $\delta > 0$ and any random variable Y such that $\mathbb{E}[Y^2] < \infty$, it holds that*

$$\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\mathbb{E}\left[|Y|^3 1_{\{|Y|\leq e^{-\lambda_1 n\delta/2}\}}\right] + \sum_{n\geq 0} e^{-\lambda_1 n\delta/2}\mathbb{E}\left[|Y| 1_{\{|Y|> e^{-\lambda_1 n\delta/2}\}}\right] \lesssim_\delta \mathbb{E}[Y^2].$$

Proof: Define $n_y := \inf\{n \in \mathbb{N} : e^{-\lambda_1 n\delta/2} \geq y\}$. Combining the inequalities $\mathbb{E}[|Y|^3 1_{\{|Y|\leq K\}}] \leq 3\int_0^K y^2\mathbb{P}(|Y| > y)\mathrm{d}y$ and $\sum_{n\geq 0} e^{-\lambda_1 n\delta/2}\mathbb{E}\left[|Y| 1_{\{|Y|> e^{-\lambda_1 n\delta/2}\}}\right] \leq \mathbb{E}\left[|Y|\sum_{n=0}^{n_Y} e^{-\lambda_1 n\delta/2}\right]$, we have

$$\begin{aligned}
&\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\mathbb{E}\left[|Y|^3 1_{\{|Y|\leq e^{-\lambda_1 n\delta/2}\}}\right] + \sum_{n\geq 0} e^{-\lambda_1 n\delta/2}\mathbb{E}\left[|Y| 1_{\{|Y|> e^{-\lambda_1 n\delta/2}\}}\right] \\
&\leq 3\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\int_0^{e^{-\lambda_1 n\delta/2}} y^2\mathbb{P}(|Y| > y)\mathrm{d}y + \mathbb{E}\left[|Y|\sum_{n=0}^{n_Y} e^{-\lambda_1 n\delta/2}\right] \\
&= 3\int_0^\infty y^2\mathbb{P}(|Y| > y)\mathrm{d}y\sum_{n\geq n_y} e^{\lambda_1 n\delta/2} + \frac{1}{e^{-\lambda_1\delta/2} - 1}\mathbb{E}\left[|Y|\left(e^{-\lambda_1(n_Y+1)\delta/2} - 1\right)\right] \\
&\leq 3\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\int_0^\infty y^2\mathbb{P}(|Y| > y)e^{\lambda_1 n_y\delta/2}\mathrm{d}y + \frac{1}{e^{-\lambda_1\delta/2} - 1}\mathbb{E}\left[|Y| e^{-\lambda_1(n_Y+1)\delta/2}\right].
\end{aligned}$$

Since $e^{-\lambda_1\delta/2}y \geq e^{-\lambda_1 n_y\delta/2} \geq y$ and $\mathbb{E}[Y^2] = \int_0^\infty 2y\mathbb{P}(|Y| > y)\mathrm{d}y$, we conclude that

$$\begin{aligned}
&\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\mathbb{E}\left[|Y|^3 1_{\{|Y|\leq e^{-\lambda_1 n\delta/2}\}}\right] + \sum_{n\geq 0} e^{-\lambda_1 n\delta/2}\mathbb{E}\left[|Y| 1_{\{|Y|\geq e^{-\lambda_1 n\delta/2}\}}\right] \\
&\leq 3\sum_{n\geq 0} e^{\lambda_1 n\delta/2}\int_0^\infty y\mathbb{P}(|Y| > y)\mathrm{d}y + \frac{e^{-\lambda_1\delta}}{e^{-\lambda_1\delta/2} - 1}\mathbb{E}\left[Y^2\right] \\
&= \frac{1}{2}\left(3\sum_{n\geq 0} e^{\lambda_1 n\delta/2} + \frac{2e^{-\lambda_1\delta}}{e^{-\lambda_1\delta/2} - 1}\right)\mathbb{E}(Y^2),
\end{aligned}$$

which implies the desired result. □

Lemma 3.7 *Let $\{\mathcal{G}_n : n = 0, 1, ...\}$ be an increasing sequence of σ-fields and B be an event with positive probability. Let $\{T_n : n = 0, 1, ...\}$ be a sequence of random variables such that*

$$1_B \sum_{n\geq 0} \sup_{y\in\mathbb{R}} |\mathbb{P}[T_n \leq y|\mathcal{G}_n] - \Phi(y)| < \infty \quad \mathbb{P}\text{-}a.s.,$$

where $\Phi(y) = (1/\sqrt{2\pi})\int_{-\infty}^y e^{-x^2/2}\mathrm{d}x, y \in \mathbb{R}$. Then

$$\limsup_{n\to\infty} \frac{T_n}{\sqrt{2\log n}} \leq 1 \quad \mathbb{P}(\cdot|B)\text{-}a.s.$$

If, furthermore, there exists a constant $k \geq 1$ such that T_n is $\mathcal{G}_{n+k}$-measurable for each $n = 0, 1, ...,$ then

$$\limsup_{n\to\infty} \frac{T_n}{\sqrt{2\log n}} = 1 \quad \mathbb{P}(\cdot|B)\text{-}a.s.$$

Proof: From [3, p. 430, 1.5], for any sequence B_n of events and any filtration $\mathcal{G}_n$,

$$\{B_n,\ \text{i.o.}\} \subset \left\{\sum_{n=1}^\infty \mathbb{P}\left(B_n \mid \mathcal{G}_n\right) = \infty\right\}$$

and the two events above are $\mathbb{P}$-a.s. equal if there exists a constant $k \geq 1$ such that $B_n \in \mathcal{G}_{n+k}$ for all n. Thus,

$$\begin{aligned}
&B \cap \{B_n,\ \text{i.o.}\} = \{B_n \cap B,\ \text{i.o.}\} \\
&\subset B \cap \left\{\sum_{n=1}^\infty \mathbb{P}\left(B_n \mid \mathcal{G}_n\right) = \infty\right\} = \left\{1_B\sum_{n=1}^\infty \mathbb{P}\left(B_n \mid \mathcal{G}_n\right) = \infty\right\}.
\end{aligned}$$

Applying this fact to $B_n = \{T_n > (1+\eta)\sqrt{2\log n}\}$ and noting that for any $\eta > 0$,

$$\sum_{n=1}^\infty \left(1 - \Phi((1+\eta)\sqrt{2\log n})\right) < \infty,$$

we conclude that $\mathbb{P}\left(B \cap \{B_n,\ \text{i.o.}\}\right) = 0$, which implies the first result. For the second result, let $B_n = \{T_n > (1-\eta)\sqrt{2\log n}\}$, then according to the fact that $B_n \in \mathcal{G}_{n+k}$, we have

$$\{B \cap B_n,\ \text{i.o.}\} = \left\{1_B\sum_{n=1}^\infty \mathbb{P}\left(B_n \mid \mathcal{G}_n\right) = \infty\right\}.$$

Noticing that $\sum_{n=1}^{\infty}\left(1-\Phi((1-\eta)\sqrt{2\log n})\right)=\infty$ for any $\eta>0$, we conclude that

$$B=\left\{1_B\sum_{n=1}^{\infty}\mathbb{P}\left(B_n\mid\mathcal{G}_n\right)=\infty\right\}=\left\{B\cap B_n,\ \text{i.o.}\right\},$$

which implies the desired result. □

4 Proof of Theorems 2.1, 2.3, 2.5 and 2.6

In this section, we always assume that **(H1)**—**(H4)** hold.

4.1 General theory

Suppose that $f\in L^2(E,\mu)\cap L^4(E,\mu)$. For any $t>0$, define

$$T_{-t}f(x):=\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}e^{\lambda_k t}a_j^k\phi_j^{(k)}(x)$$

if the right hand side is well-defined in L^2. Define $\mathcal{S}$ by

$$\mathcal{S}:=\left\{f\in L^2(E,\mu)\cap L^4(E,\mu)\ \Big|\ T_{-t}f\in L^2(E,\mu)\cap L^4(E,\mu)\text{ for all }t>0\right\}.$$

Obviously, for any $m\in\mathbb{N}$ and $a_j^k\in\mathbb{R}$, $\sum_{k=1}^{m}\sum_{j=1}^{n_k}a_j^k\phi_j^{(k)}\in\mathcal{S}$. One can also easily check that, for any $\epsilon_0>0$, $\sum_{k=1}^{\infty}\sum_{j=1}^{n_k}e^{-|\lambda_k|^{1+\epsilon_0}}\phi_j^{(k)}\in\mathcal{S}$. Note that $W_s^{k,j}=e^{\lambda_k s}\langle\phi_j^{(k)},X_s\rangle=\langle T_{-s}\phi_j^{(k)},X_s\rangle$.

Recall the definitions of f_{sm},f_{cr} and f_{la} in (1.5) and (1.6). By the branching property, for any $f\in\mathcal{S}$, $s\in(0,\infty]$ and $r\in(0,\infty)$,

$$\begin{aligned}&\langle f_{sm}+f_{cr},X_t\rangle+\langle T_{-r}f_{la},X_{t+r}\rangle-\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^kW_{t+s}^{k,j}-\langle T_{-r}(f_{sm}+f_{cr}),X_{t+r}\rangle\\&=\sum_{i=1}^{M_t}\left[\left(f_{sm}+f_{cr}\right)(X_t(i))+\langle T_{-r}f_{la},X_r^i\rangle-\sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}a_j^kW_s^{k,j,i}-\langle T_{-r}(f_{sm}+f_{cr}),X_r^i\rangle\right].\end{aligned}\tag{4.1}$$

Here M_t is the number of particles alive at time t. For $i=1,\dots,M_t$, $X_t(i)$ is the position of the i-th particle, and $\left(X_r^i,W_r^{k,j,i},W_s^{k,j,i}\right)$ has the same distribution as $\left(X_r,W_r^{k,j},W_s^{k,j}\right)$ under $\mathbb{P}_{X_t(i)}$. Furthermore, by the branching property, the random variables $W_r^{k,j,i}$ are independent conditioned on $\mathcal{F}_t:=\{X_s:s\le t\}$.

For $f\in\mathcal{S}$, $s\in(0,\infty]$ and $r\in(0,\infty)$, we define for $i=1,\dots,M_t$,

$$\begin{aligned}Y_t^{f,i}(s,r)&:=\left(f_{sm}+f_{cr}\right)(X_t(i))+\langle T_{-r}f_{la},X_r^i\rangle-\sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}a_j^kW_s^{k,j,i}-\langle T_{-r}(f_{sm}+f_{cr}),X_r^i\rangle,\\Z_t^{f,i}(s,r)&:=Y_t^{f,i}(s,r)1_{\{|Y_t^{f,i}(s,r)|\le e^{-\lambda_1 t/2}\}},\\U_t^f(s,r)&:=\sum_{i=1}^{M_t}\left(Z_t^{f,i}(s,r)-\mathbb{E}_x\left[Z_t^{f,i}(s,r)\big|\mathcal{F}_t\right]\right).\end{aligned}\tag{4.2}$$

Note that, for $i = 1, \cdots, M_t$, $Y_t^{f,i}(s,r)$, $Z_t^{f,i}(s,r)$ and $U_t^f(s,r)$ contain information about the branching Markov process after time t and therefore are not in $\mathcal{F}_t$. Note also that $\mathbb{E}_x\left[Y_t^{f,i}(s,r)\big|\mathcal{F}_t\right] = 0$.

For $f \in \mathcal{S}$, , $s \in (0,\infty]$ and $r \in (0,\infty)$, we define

$$\begin{aligned} Y^f(s,r) &:= (f_{sm} + f_{cr})(x) + \langle T_{-r}f_{la}, X_r\rangle - \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} a_j^k W_s^{k,j} - \langle T_{-r}(f_{sm}+f_{cr}), X_r\rangle, \\ V_{s,r}^f(x) &:= \mathrm{Var}_x\left(Y^f(s,r)\right) = \mathbb{E}_x\left((Y^f(s,r))^2\right). \end{aligned} \tag{4.3}$$

It follows from Lemma 3.2 that, for any $f \in \mathcal{S}$, $s \in (0,\infty]$ and $r \in (0,\infty)$,

$$V_{s,r}^f \in L^2(E,\mu) \cap L^4(E,\mu). \tag{4.4}$$

Note that for two random variables Y_1 and Y_2,

$$|\mathrm{Var}(Y_1+Y_2) - \mathrm{Var}(Y_1)| \le \mathrm{Var}(Y_2) + 2\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}. \tag{4.5}$$

Therefore, by the definition of $V_{s,r}^f$, we have

$$\lim_{s\to\infty} V_{s,r}^f = V_{\infty,r}^f, \quad \forall r \in (0,\infty), x \in E. \tag{4.6}$$

Lemma 4.1 *If $f \in \mathcal{S}$, then for any $s \in (0,\infty]$, $r \in (0,\infty)$ and $\delta > 0$,*

$$\lim_{n\to\infty} e^{\lambda_1 n\delta}\mathrm{Var}_x\left[U_{n\delta}^f(s,r)\big|\mathcal{F}_{n\delta}\right] = \langle V_{s,r}^f, \phi_1\rangle W_\infty, \quad \mathbb{P}_x\text{-a.s.} \tag{4.7}$$

Proof: We first prove that

$$\lim_{n\to\infty} e^{\lambda_1 n\delta}\mathrm{Var}_x\left[\sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r)\Big|\mathcal{F}_{n\delta}\right] = \langle V_{s,r}^f, \phi_1\rangle W_\infty, \quad \mathbb{P}_x\text{-a.s.} \tag{4.8}$$

Note that, conditioned on $\mathcal{F}_{n\delta}$, $Y_{n\delta}^{f,i}(s,r)$ are independent. Thus,

$$e^{\lambda_1 n\delta}\mathrm{Var}_x\left[\sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r)\Big|\mathcal{F}_{n\delta}\right] = e^{\lambda_1 n\delta}\sum_{i=1}^{M_{n\delta}} V_{s,r}^f\left(X_{n\delta}(i)\right) = e^{\lambda_1 n\delta}\langle V_{s,r}^f, X_{n\delta}\rangle.$$

Combining Lemma 3.4 and (4.4), we get (4.8).

Define $Y_1 = \sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r)$ and $Y_2 = U_{n\delta}^f(s,r) - \sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r)$. By (4.5), to get (4.7), it suffices to prove that

$$\lim_{n\to\infty} e^{\lambda_1 n\delta}\mathrm{Var}_x\left[\sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r) - U_{n\delta}^f(s,r)\Big|\mathcal{F}_{n\delta}\right] = 0 \quad \mathbb{P}_x\text{-a.s.} \tag{4.9}$$

Note that

$$\begin{aligned} e^{\lambda_1 n\delta}\mathrm{Var}_x\left[\sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f,i}(s,r) - U_{n\delta}^f(s,r)\Big|\mathcal{F}_{n\delta}\right] &= e^{\lambda_1 n\delta}\sum_{i=1}^{M_{n\delta}}\mathrm{Var}_x\left[Y_{n\delta}^{f,i}(s,r)1_{\{|Y_{n\delta}^{f,i}(s,r)|>e^{-\lambda_1 n\delta/2}\}}\Big|\mathcal{F}_{n\delta}\right] \\ &= e^{\lambda_1 n\delta}\langle V_{s,r}^{f,n\delta}, X_{n\delta}\rangle, \end{aligned} \tag{4.10}$$

where for $A>0$,

$$V_{s,r}^{f,A}(x) := \mathbb{E}_x\left[\left(Y^f(s,r)\right)^2 1_{\{|Y^f(s,r)|>e^{-\lambda_1 A/2}\}}\right] \le V_{s,r}^f(x).$$

For any fixed $A>0$, if $n>A/\delta$, then we have $V_{s,r}^{f,n\delta} \le V_{s,r}^{f,A}$. Applying Lemma 3.4 to $V_{s,r}^{f,A}$, we get that $e^{\lambda_1 n\delta}\langle V_{s,r}^{f,A}, X_{n\delta}\rangle$ converges to $\langle V_{s,r}^{f,A}, \phi_1\rangle W_\infty$ $\mathbb{P}_x$-a.s. Hence,

$$\limsup_{n\to\infty} e^{\lambda_1 n\delta}\langle V_{s,r}^{f,n\delta}, X_{n\delta}\rangle \le \limsup_{n\to\infty} e^{\lambda_1 n\delta}\langle V_{s,r}^{f,A}, X_{n\delta}\rangle = \langle V_{s,r}^{f,A}, \phi_1\rangle W_\infty, \quad \mathbb{P}_x\text{-a.s.}$$

Letting $A\to\infty$, together with (4.10), we get (4.9) and this completes the proof of the lemma. □

Lemma 4.2 *Let $f\in\mathcal{S}, s\in(0,\infty]$ and $r\in(0,\infty)$. For any $\delta>0$, define*

$$\Delta_{n\delta}^f(s,r) := \sup_{y\in\mathbb{R}}\left|\mathbb{P}_x\left[\frac{U_{n\delta}^f(s,r)}{\sqrt{\mathrm{Var}_x\left[U_{n\delta}^f(s,r)\big|\mathcal{F}_{n\delta}\right]}} \le y\Big|\mathcal{F}_{n\delta}\right] - \Phi(y)\right|.$$

Then $\mathbb{P}_x$-almost surely,

$$1_{\mathcal{E}^c}\sum_{n\ge 1}\Delta_{n\delta}^f(s,r) < \infty. \tag{4.11}$$

Proof: *Step 1*: The goal in this step is to prove that

$$\sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\sum_{i=1}^{M_{n\delta}}\mathbb{E}_x\left[\left|Z_{n\delta}^{f,i}(s,r)\right|^3\Big|\mathcal{F}_{n\delta}\right] < \infty, \quad \mathbb{P}_x\text{-a.s.} \tag{4.12}$$

It suffices to show that

$$\mathbb{E}_x\left(\sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\sum_{i=1}^{M_{n\delta}}\mathbb{E}_x\left[\left|Z_{n\delta}^{f,i}(s,r)\right|^3\Big|\mathcal{F}_{n\delta}\right]\right) < \infty. \tag{4.13}$$

Define

$$g_{s,r}^{f,n\delta}(x) := \mathbb{E}_x\left(\left|Y^f(s,r)\right|^3 1_{\{|Y^f(s,r)|\le e^{-\lambda_1 n\delta/2}\}}\right). \tag{4.14}$$

Then

$$\begin{aligned}
&\mathbb{E}_x\left(\sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\sum_{i=1}^{M_{n\delta}}\mathbb{E}_x\left[\left|Z_{n\delta}^{f,i}(s,r)\right|^3\Big|\mathcal{F}_{n\delta}\right]\right) = \sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\mathbb{E}_x\left(\sum_{i=1}^{M_{n\delta}}\mathbb{E}_x\left[\left|Z_{n\delta}^{f,i}(s,r)\right|^3\Big|\mathcal{F}_{n\delta}\right]\right)\\
&= \sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\mathbb{E}_x\left(\sum_{i=1}^{M_{n\delta}} g_{s,r}^{f,n\delta}\left(X_{n\delta}(i)\right)\right) = \sum_{n\ge 1} e^{3\lambda_1 n\delta/2}\left(T_{n\delta}g_{s,r}^{f,n\delta}\right)(x).
\end{aligned} \tag{4.15}$$

Fix a $t_0\in(0,\min\{\delta,s,r\})$. Note that $\gamma(\widetilde{g_{s,r}^{f,n\delta}})\ge 2$. By Lemma 3.1, we have

$$\left|\left(T_{n\delta}\widetilde{g_{s,r}^{f,n\delta}}\right)(x)\right| \lesssim_{t_0} e^{-\lambda_2 n\delta}\left\|g_{s,r}^{f,n\delta}\right\|_2\phi_1(x) \quad n\ge 1, x\in E. \tag{4.16}$$

Using the definition of $g^{f,n\delta}$, it is easy to see that

$$g^{f,n\delta}_{s,r}(x) \le e^{-\lambda_1 n\delta/2}\mathbb{E}_x\left(\left|Y^f(s,r)\right|^2\right) = e^{-\lambda_1 n\delta/2}V^f_{s,r}(x).$$

Plugging the inequality above into (4.16) and applying (4.4), we get that

$$\left|\left(T_{n\delta}\widetilde{g^{f,n\delta}_{s,r}}\right)(x)\right| \lesssim_{t_0} e^{-\lambda_2 n\delta}e^{-\lambda_1 n\delta/2}\|V^f_{s,r}\|_2\phi_1(x).$$

Therefore,

$$\begin{aligned}\sum_{n\ge1} e^{3\lambda_1 n\delta/2}\left|\left(T_{n\delta}\widetilde{g^{f,n\delta}_{s,r}}\right)(x)\right| &\lesssim_{t_0} \phi_1(x)\sum_{n\ge1} e^{3\lambda_1 n\delta/2}e^{-(\lambda_2+\lambda_1/2)n\delta}\\ &= \phi_1(x)\sum_{n\ge1} e^{(\lambda_1-\lambda_2)n\delta} < \infty. \qquad (4.17)\end{aligned}$$

We claim that

$$\sum_{n\ge1} e^{3\lambda_1 n\delta/2}\left|\left(T_{n\delta}\left(g^{f,n\delta}_{s,r} - \widetilde{g^{f,n\delta}_{s,r}}\right)\right)(x)\right| = \sum_{n\ge1} e^{\lambda_1 n\delta/2}\langle g^{f,n\delta}_{s,r},\phi_1\rangle\phi_1(x) < \infty. \qquad (4.18)$$

In fact, combining Lemma 3.6 (with $Y = Y^f(s,r)$) and the definition of $g^{f,n\delta}$ in (4.14), we get that

$$\sum_{n\ge1} e^{\lambda_1 n\delta/2}g^{f,n\delta}_{s,r}(x) \lesssim_\delta V^f_{s,r}(x).$$

Since $V^f_{s,r}(x)$ and $\phi_1(x)$ both belong to $L^2(E,\mu)$, we have $\langle V^f_{s,r},\phi_1\rangle < \infty$. Now (4.18) follows from Fubini's theorem. Combining (4.15), (4.17) and (4.18), we get (4.13).

Step 2: In this step, we prove the conclusion of the lemma. It is trivial that $\Delta^f_{n\delta}(s,r) \le 2$. Since $\{M_{n\delta} > 0\} \in \mathcal{F}_{n\delta}$, by Lemma 3.5, under $\mathbb{P}_x$, on the event $\{M_{n\delta} > 0\}$,

$$\begin{aligned}\Delta^f_{n\delta}(s,r) &\lesssim \frac{\sum_{i=1}^{M_{n\delta}} \mathbb{E}_x\left[\left|Z^{f,i}_{n\delta}(s,r) - \mathbb{E}_x\left[Z^{f,i}_{n\delta}(s,r)\middle|\mathcal{F}_{n\delta}\right]\right|^3\middle|\mathcal{F}_{n\delta}\right]}{\sqrt{\left(\mathrm{Var}_x\left[U^f_{n\delta}(s,r)\middle|\mathcal{F}_{n\delta}\right]\right)^3}}\\ &\lesssim \frac{\sum_{i=1}^{M_{n\delta}} \mathbb{E}_x\left[\left|Z^{f,i}_{n\delta}(s,r)\right|^3\middle|\mathcal{F}_{n\delta}\right]}{\sqrt{\left(\mathrm{Var}_x\left[U^f_{n\delta}(s,r)\middle|\mathcal{F}_{n\delta}\right]\right)^3}}, \qquad (4.19)\end{aligned}$$

where in the second inequality, we used the inequality $\mathbb{E}|Y-\mathbb{E}Y|^3 \le 8\mathbb{E}|Y|^3$ for any Y such that $\mathbb{E}|Y|^3 < \infty$. Since $\mathcal{E}^c \subset \{M_{n\delta} > 0\}$, (4.19) holds on the event $\mathcal{E}^c$ under $\mathbb{P}_x$. Now suppose that Ω_0 is an event with $\mathbb{P}(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, the assertion of Lemma 4.1, (4.12) and (4.19) hold. Then for each $\omega \in \Omega_0 \cap \mathcal{E}^c$, there exists a large integer $N = N(\omega)$ such that for $n \ge N$,

$$\mathrm{Var}_x\left[U^f_{n\delta}(s,r)\middle|\mathcal{F}_{n\delta}\right](\omega) \ge \frac{e^{-\lambda_1 n\delta}}{2}\langle V^f_{s,r},\phi_1\rangle W_\infty(\omega) > 0.$$

Therefore, on $\Omega_0 \cap \mathcal{E}^c$, by (4.19),

$$\sum_{n\ge1}\Delta^f_{n\delta}(s,r) \lesssim (1+N) + \frac{8}{\sqrt{\left[\langle V^f_{s,r},\phi_1\rangle W_\infty\right]^3}}\sum_{n\ge N} e^{3\lambda_1 n\delta/2}\sum_{i=1}^{M_{n\delta}}\mathbb{E}_x\left[\left|Z^{f,i}_{n\delta}(s,r)\right|^3\middle|\mathcal{F}_{n\delta}\right].$$

Then applying (4.12), we get that (4.11) holds $\mathbb{P}_x$-almost surely. □

Lemma 4.3 *If $f\in\mathcal{S}$, then for any $s\in(0,\infty]$, $r\in(0,\infty)$ and $\delta>0$,*

$$\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)\right)}{\sqrt{2\log(n\delta)}}=\sqrt{\langle V^f_{s,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}\tag{4.20}$$

and

$$\liminf_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)\right)}{\sqrt{2\log(n\delta)}}=-\sqrt{\langle V^f_{s,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}\tag{4.21}$$

Proof: *Step 1.* In this step, we prove that for any $s\in(0,\infty]$ and $r\in(0,\infty)$,

$$\lim_{n\to\infty}\left|e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)\right)-e^{\lambda_1 n\delta/2}U^f_{n\delta}(s,r)\right|=0,\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\ \text{a.s.}\tag{4.22}$$

Note that

$$\begin{aligned}&e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)\right)-e^{\lambda_1 n\delta/2}U^f_{n\delta}(s,r)\\&=e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)1_{\{|Y^{f,i}_{n\delta}(s,r)|>e^{-\lambda_1 n\delta/2}\}}+\mathbb{E}_x\left[Y^{f,i}_{n\delta}(s,r)1_{\{|Y^{f,i}_{n\delta}(s,r)|>e^{-\lambda_1 n\delta/2}\}}\Big|\mathcal{F}_{n\delta}\right]\right).\end{aligned}$$

Using the inequality $|\mathbb{E}[Y|\mathcal{F}]|\le\mathbb{E}[|Y|\big|\mathcal{F}]$, we get that

$$\mathbb{E}_x\left|e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(s,r)\right)-e^{\lambda_1 n\delta/2}U^f_{n\delta}(s,r)\right|\le 2e^{\lambda_1 n\delta/2}\mathbb{E}_x\left[\sum_{i=1}^{M_{n\delta}}\left|Y^{f,i}_{n\delta}(s,r)\right|1_{\{|Y^{f,i}_{n\delta}|(s,r)>e^{-\lambda_1 n\delta/2}\}}\right].$$

Therefore, to prove (4.22), we only need to show that $\mathbb{P}_x$-almost surely,

$$\sum_{n\ge1}e^{\lambda_1 n\delta/2}\mathbb{E}_x\left[\sum_{i=1}^{M_{n\delta}}\left|Y^{f,i}_{n\delta}(s,r)\right|1_{\{|Y^{f,i}_{n\delta}(s,r)|>e^{-\lambda_1 n\delta/2}\}}\right]<\infty.\tag{4.23}$$

Define

$$l^{f,n\delta}_{s,r}(x):=\mathbb{E}_x\left(\left|Y^f(s,r)\right|1_{\{|Y^f(s,r)|>e^{-\lambda_1 n\delta/2}\}}\right),$$

then $l^{f,n\delta}_{s,r}(x)\le e^{\lambda_1 n\delta/2}V^f_{s,r}(x)$ for any $n\in\mathbb{N}$ and $x\in E$. Combining Lemma 3.1 and the fact that $\lambda\left(\widetilde{l^{f,n\delta}_{s,r}}\right)\ge2$, we get that

$$\left|\left(T_{n\delta}\widetilde{l^{f,n\delta}_{s,r}}\right)\right|(x)\lesssim_\delta e^{-\lambda_2 n\delta}e^{\lambda_1 n\delta/2}\|V^f_{s,r}\|_2\phi_1(x),\quad n\ge1,x\in E.$$

Thus,

$$\sum_{n\ge1}e^{\lambda_1 n\delta/2}\left|\left(T_{n\delta}\widetilde{l^{f,n\delta}_{s,r}}\right)\right|(x)\lesssim_\delta\|V^f_{s,r}\|_2\phi_1(x)\sum_{n\ge1}e^{(\lambda_1-\lambda_2)n\delta}<\infty.\tag{4.24}$$

Since $\left(T_{n\delta}\left(l^{f,n\delta}_{s,r}-\widetilde{l^{f,n\delta}_{s,r}}\right)\right)(x) = e^{-\lambda_1 n\delta}\langle l^{f,n\delta}_{s,r},\phi_1\rangle\phi_1(x)$, by Lemma 3.6 (with $Y = Y^f(s,r)$), we get that

$$\sum_{n\geq 1} e^{\lambda_1 n\delta/2}e^{-\lambda_1 n\delta} l^{f,n\delta}_{s,r} \lesssim_\delta V^f_{s,r}(x) < \infty. \tag{4.25}$$

Combining (4.24) and (4.25), we obtain that

$$\begin{aligned}&\sum_{n\geq 1} e^{\lambda_1 n\delta/2}\mathbb{E}_x\left[\sum_{i=1}^{M_{n\delta}}\left|Y^{f,i}_{n\delta}(s,r)\right| 1_{\{|Y^{f,i}_{n\delta}(s,r)|>e^{-\lambda_1 n\delta/2}\}}\right] = \sum_{n\geq 1} e^{\lambda_1 n\delta/2}\left(T_{n\delta} l^{f,n\delta}_{s,r}\right)(x)\\ &\lesssim_\delta \|V^f_{s,r}\|_2\phi_1(x)\sum_{n\geq 1} e^{(\lambda_1-\lambda_2)n\delta} + V^f_{s,r}(x) < \infty,\end{aligned}$$

which implies (4.22).

Step 2: In this step, we prove the assertion of lemma for $s\in(0,\infty)$. Combining Lemma 3.7 (with $B=\mathcal{E}^c$) and Lemma 4.2, we get that, for $s\in(0,\infty)$,

$$\limsup_{n\to\infty}\frac{U^f_{n\delta}(s,r)}{\sqrt{2\log n \mathrm{Var}_x\left[U^f_{n\delta}(s,r)\middle|\mathcal{F}_{n\delta}\right]}} = 1, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Since $\log(n\delta)/\log n\to 1$ as $n\to\infty$, by Lemma 4.1, we have

$$\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}U^f_{n\delta}(s,r)}{\sqrt{2\log(n\delta)}} = \sqrt{\langle V^f_{s,r},\phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.26}$$

Since Lemma 4.2 also holds with $U^f_{n\delta}(s,r)$ replaced by $-U^f_{n\delta}(s,r)$, we have similarly,

$$\liminf_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}U^f_{n\delta}(s,r)}{\sqrt{2\log(n\delta)}} = -\sqrt{\langle V^f_{s,r},\phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.27}$$

Now combining (4.22), (4.26) and (4.27), we get the desired result for $s\in(0,\infty)$.

Step 3: In this step, we prove the assertion of the lemma for $s=\infty$. Combining Lemma 3.7 and Lemma 4.2, we get

$$\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}U^f_{n\delta}(\infty,r)}{\sqrt{2\log(n\delta)}} \leq \sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Together with (4.22), we get that

$$\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\infty,r)\right)}{\sqrt{2\log(n\delta)}} \leq \sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.28}$$

Similarly using the same argument with $U^f_{n\delta}(\infty,r)$ replaced to $-U^f_{n\delta}(\infty,r)$, we also see that

$$\liminf_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\infty,r)\right)}{\sqrt{2\log(n\delta)}} \geq -\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.29}$$

Note that for $s=\ell\delta+r,\ell\in\mathbb{N}$, it holds that

$$\begin{aligned}
&\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\infty,r)\right)}{\sqrt{2\log(n\delta)}}\\
&=\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\ell\delta+r,r)\right)}{\sqrt{2\log(n\delta)}}+\sum_{2\lambda_k<\lambda_1} e^{\lambda_k\ell\delta}\sum_{j=1}^{n_k} a_j^k\frac{e^{\lambda_1 n\delta/2}\sum_{i=1}^{M_{(n+\ell)\delta}}\left(W_r^{k,j,i}-W_\infty^{k,j,i}\right)}{\sqrt{2\log(n\delta)}}\\
&=\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\ell\delta+r,r)\right)}{\sqrt{2\log(n\delta)}}+\sum_{2\lambda_k<\lambda_1} e^{\lambda_k\ell\delta}\sum_{j=1}^{n_k} a_j^k\frac{e^{\lambda_1 n\delta/2}\sum_{i=1}^{M_{(n+\ell)\delta}} Y^{\phi_j^{(k)},i}_{(n+\ell)\delta}(\infty,r)}{\sqrt{2\log(n\delta)}}.
\end{aligned}$$

Using the inequality

$$\limsup_{n\to\infty}\left(\sum_{i=1}^p x_n^i\right)\ge\limsup_{n\to\infty}\left(x_n^1\right)+\sum_{i=2}^p\liminf_{n\to\infty}\left(x_n^i\right)$$

and applying (4.20) to $Y^{f,i}_{n\delta}(\ell\delta+r,r)$ and (4.29) to $Y^{\phi_j^{(k)},i}_{(n+\ell)\delta}(\infty,r)$, we conclude that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\infty,r)\right)}{\sqrt{2\log(n\delta)}}\\
&\ge\sqrt{\langle V^f_{\ell\delta+r,r},\phi_1\rangle W_\infty}-\sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}\left|a_j^k\right|e^{-(\lambda_1/2-\lambda_k)\ell\delta}\sqrt{\langle V^{\phi_j^{(k)}}_{\infty,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}
\end{aligned}$$

It follows from (4.6) that $V^f_{\ell\delta+r,r}(x)$ converges to $V^f_{\infty,r}(x)$. Letting $\ell\to\infty$ in the above inequality, we get that

$$\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}\left(\sum_{i=1}^{M_{n\delta}} Y^{f,i}_{n\delta}(\infty,r)\right)}{\sqrt{2\log(n\delta)}}\ge\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Combing the above with (4.28), we get that (4.20) holds for $s=\infty$. The same argument shows that (4.21) also holds for $s=\infty$. The proof is complete. □

Now we are ready to treat the continuous-time case.

Lemma 4.4 *If $f=\sum_{k=1}^m\sum_{j=1}^{n_k}a_j^k\phi_j^{(k)}(x)$ for some $m\in\mathbb{N}$, then for any $r\in(0,\infty)$,*

$$\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t} Y^{f,i}_t(\infty,r)\right)}{\sqrt{2\log t}}=\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}\tag{4.30}$$

and

$$\liminf_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t} Y^{f,i}_t(\infty,r)\right)}{\sqrt{2\log t}}=-\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty},\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}\tag{4.31}$$

Proof : By Lemma 4.3, to prove (4.30), it suffices to show that $\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t} Y^{f,i}_t(\infty,r)\right)}{\sqrt{2\log t}}\le\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty}$.

Step 1. In this step, we treat the special case where $f = \phi_j^{(k)}$ for some $k \in \mathbb{N}$ and $1 \le j \le n_k$. Our goal is to show that for any $\delta > 0$,

$$\limsup_{n\to\infty} \sup_{t\in[n\delta,(n+1)\delta)} \frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left|W_{n\delta}^{k,j} - W_t^{k,j}\right|}{\sqrt{2\log(n\delta)}} \le \sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta} V_{2\delta,\delta}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty}, \ \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.32}$$

By (4.1) and Lemma 4.3 (with $f = \phi_j^{(k)}$ and $s = 2\delta, r = \delta$), we get that if $2\lambda_k \ge \lambda_1$,

$$\limsup_{n\to\infty} \frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j}\right)}{\sqrt{2\log(n\delta)}} = \sqrt{\langle V_{2\delta,\delta}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

and if $2\lambda_k < \lambda_1$, it holds that

$$\limsup_{n\to\infty} \frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j}\right)}{\sqrt{2\log(n\delta)}} = \sqrt{\langle e^{(\lambda_1-2\lambda_k)\delta} V_{2\delta,\delta}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

In both cases, we have

$$\limsup_{n\to\infty} \frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j}\right)}{\sqrt{2\log(n\delta)}} \le \sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta} V_{2\delta,\delta}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.33}$$

For $\rho > 0$, define

$$\varepsilon_n(k,j,\delta) := (1+\rho)\sqrt{2e^{(2\lambda_k-\lambda_1)n\delta}\log(n\delta)\langle e^{|\lambda_1-2\lambda_k|\delta} V_{2\delta,\delta}^{\phi_j^{(k)}}, \phi_1\rangle W_{n\delta}}, \tag{4.34}$$

then by the second Borel-Cantalli lemma (see e.g. [10, Theorem 5.3.2]) and the definition of $W_t^{k,j}$,

$$\left\{W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j} > \varepsilon_n(k,j,\delta), \text{ i.o.}\right\} = \left\{\sum_{n=1}^{\infty} \mathbb{P}_x\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j} > \varepsilon_n(k,j,\delta) \middle| \mathcal{F}_{n\delta}\right) = +\infty\right\}.$$

Combining this with (4.33), we get that on $\mathcal{E}^c$, $\mathbb{P}_x$- almost surely,

$$\sum_{n=1}^{\infty} \mathbb{P}_x\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j} > \varepsilon_n(k,j,\delta) \middle| \mathcal{F}_{n\delta}\right) < +\infty. \tag{4.35}$$

For any $t \in [n\delta, (n+1)\delta)$, define

$$Z_t^{k,j} := \mathbb{E}_x\left[\left(W_{(n+1)\delta}^{k,j} - W_t^{k,j}\right)^2 \middle| \mathcal{F}_t\right], \quad B_n^{k,j} := \sup_{t\in[n\delta,(n+1)\delta)}\left[W_{n\delta}^{k,j} - W_t^{k,j} - \sqrt{2Z_t^{k,j}}\right],$$

$$T_n^{k,j} := \inf\left\{s \in [n\delta,(n+1)\delta) : W_{n\delta}^{k,j} - W_s^{k,j} - \sqrt{2Z_s^{k,j}} > \varepsilon_n(k,j,\delta)\right\} \wedge ((n+1)\delta).$$

We have

$$\mathbb{P}_x\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j} > \varepsilon_n(k,j,\delta) \middle| \mathcal{F}_{n\delta}\right) \ge \mathbb{P}_x\left(W_{n\delta}^{k,j} - W_{(n+1)\delta}^{k,j} > \varepsilon_n(k,j,\delta), T_n^{k,j} < (n+1)\delta \middle| \mathcal{F}_{n\delta}\right)$$

$$\geq \mathbb{P}_x\left(W^{k,j}_{T_n} - W^{k,j}_{(n+1)\delta} > -\sqrt{2Z^{k,j}_{T^{k,j}_n}}, T^{k,j}_n < (n+1)\delta\Big|\mathcal{F}_{n\delta}\right)$$
$$= \mathbb{E}_x\left(\mathbb{P}_x\left(W^{k,j}_{T^{k,j}_n} - W^{k,j}_{(n+1)\delta} > -\sqrt{2Z^{k,j}_{T^{k,j}_n}}\Big|\mathcal{F}_{T^{k,j}_n}\right)1_{\left\{T^{k,j}_n<(n+1)\delta\right\}}\Big|\mathcal{F}_{n\delta}\right). \tag{4.36}$$

By Markov's inequality and the strong Markov property, it is easy to see that

$$\mathbb{P}_x\left(W^{k,j}_{T_n} - W^{k,j}_{(n+1)\delta} > -\sqrt{2Z^{k,j}_{T^{k,j}_n}}\Big|\mathcal{F}_{T^{k,j}_n}\right) = 1 - \mathbb{P}_x\left(W^{k,j}_{(n+1)\delta} - W^{k,j}_{T^{k,j}_n} > \sqrt{2Z^{k,j}_{T^{k,j}_n}}\Big|\mathcal{F}_{T^{k,j}_n}\right)$$
$$\geq 1 - \mathbb{E}_x\left[\frac{\left(W^{k,j}_{(n+1)\delta} - W^{k,j}_{T^{k,j}_n}\right)^2}{2Z^{k,j}_{T^{k,j}_n}}\Big|\mathcal{F}_{T^{k,j}_n}\right] = \frac{1}{2}. \tag{4.37}$$

Therefore,

$$\mathbb{P}_x\left(W^{k,j}_{n\delta} - W^{k,j}_{(n+1)\delta} > \varepsilon_n(k,j,\delta)\Big|\mathcal{F}_{n\delta}\right) \geq \frac{1}{2}\mathbb{P}_x\left(T^{k,j}_n < (n+1)\delta\big|\mathcal{F}_{n\delta}\right)$$
$$= \frac{1}{2}\mathbb{P}_x\left(B^{k,j}_n > \varepsilon_n(k,j,\delta)\big|\mathcal{F}_{n\delta}\right). \tag{4.38}$$

Together with (4.35) and (4.38) we obtain that on $\mathcal{E}^c$, $\mathbb{P}_x$-almost surely,

$$\sum_{n=1}^{\infty}\mathbb{P}_x\left(B^{k,j}_n > \varepsilon_n(k,j,\delta)\big|\mathcal{F}_{n\delta}\right) < +\infty.$$

Since $\{B_n > \varepsilon_n(k,j,\delta)\} \in \mathcal{F}_{(n+1)\delta}$, using the second Borel-Cantelli lemma again, we get that for any $\rho > 0$ and $\delta > 0$,

$$\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left(W^{k,j}_{n\delta} - W^{k,j}_t\right)}{\sqrt{2\log(n\delta)}}$$
$$\leq (1+\rho)\sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta}V^{\phi^{(k)}_j}_{2\delta,\delta}, \phi_1\rangle W_\infty} + \limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{\sqrt{2e^{(\lambda_1-2\lambda_k)n\delta}Z^{k,j}_t}}{\sqrt{2\log(n\delta)}}. \tag{4.39}$$

By working with $\phi^{(k)}_j$ replaced by $-\phi^{(k)}_j$ in (4.39), we also have the following inequality for liminf:

$$\liminf_{n\to\infty}\inf_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left(W^{k,j}_{n\delta} - W^{k,j}_t\right)}{\sqrt{2\log(n\delta)}}$$
$$\geq -(1+\rho)\sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta}V^{\phi^{(k)}_j}_{2\delta,\delta}, \phi_1\rangle W_\infty} - \limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{\sqrt{2e^{(\lambda_1-2\lambda_k)n\delta}Z^{k,j}_t}}{\sqrt{2\log(n\delta)}}. \tag{4.40}$$

Now combining (4.39) and (4.40), we conclude that

$$\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left|W^{k,j}_{n\delta} - W^{k,j}_t\right|}{\sqrt{2\log(n\delta)}}$$
$$\leq (1+\rho)\sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta}V^{\phi^{(k)}_j}_{2\delta,\delta}, \phi_1\rangle W_\infty} + \limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{\sqrt{2e^{(\lambda_1-2\lambda_k)n\delta}Z^{k,j}_t}}{\sqrt{2\log(n\delta)}}. \tag{4.41}$$

By the definition of $Z_t^{k,j}$, we have

$$e^{(\lambda_1-2\lambda_k)t}Z_t^{k,j}=e^{\lambda_1 t}\langle \mathrm{Var.}\left(W^{k,j}_{(n+1)\delta-t}\right),X_t\rangle. \tag{4.42}$$

Taking $t=1,x=y$ in (1.3), then **(H4)(a)** implies that, for $k\in\mathbb{N}$ and $1\le y\le n_k$, there exists a constant $c(k,j)>0$ such that for all $x\in E$,

$$\left|\phi_j^{(k)}(x)\right|\le c(k,j)\phi_1(x). \tag{4.43}$$

Combining **(H4)**, (3.2) and (4.43), we get that for $\delta\in(0,1)$ and $t\in[n\delta,(n+1)\delta)$,

$$\begin{aligned}
&\mathrm{Var}_x\left(W^{k,j}_{(n+1)\delta-t}\right)\le e^{2\lambda_k((n+1)\delta-t)}\mathbb{E}_x\left(\langle\phi_j^{(k)},X_{(n+1)\delta-t}\rangle^2\right)\\
&=\int_0^{(n+1)\delta-t}e^{2\lambda_k s}T_s\left[A^{(2)}\cdot\left(\phi_j^{(k)}\right)^2\right](x)\mathrm{d}s+e^{2\lambda_k((n+1)\delta-t)}\left(T_{(n+1)\delta-t)}\left(\phi_j^{(k)}\right)^2\right)(x)\\
&\le c^2(k,j)\|A^{(2)}\|_\infty\|\phi_1\|_\infty\int_0^1 e^{(2\lambda_k-\lambda_1)s}\mathrm{d}s\phi_1(x)+c^2(k,j)\left(1+e^{2\lambda_k}\right)e^{-\lambda_1}\phi_1(x)=:C(k,j)\phi_1(x).
\end{aligned}$$

Therefore, we have

$$\limsup_{t\to\infty}\left(e^{(\lambda_1-2\lambda_k)t}Z_t^{k,j}\right)\le C(k,j)\limsup_{t\to\infty}e^{\lambda_1 t}\langle\phi_1,X_t\rangle=C(k,j)W_\infty,\quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.44}$$

Combining (4.41) and (4.44), we obtain that for any $\rho,\delta>0$,

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)n\delta/2}\left|W^{k,j}_{n\delta}-W^{k,j}_t\right|}{\sqrt{2\log(n\delta)}}\\
&\le(1+\rho)\sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta}V^{\phi_j^{(k)}}_{2\delta,\delta},\phi_1\rangle W_\infty},\ \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s..}
\end{aligned}$$

Letting $\rho\downarrow 0$, we arrive at (4.32)

Step 2:. In this step, we consider the general case $f=\sum_{k=1}^m\sum_{j=1}^{n_k}a_j^k\phi_j^{(k)}(x)$ for some $m\in\mathbb{N}$. We fix a $\delta=r/\ell$ for $\ell\in\mathbb{N}$. By the definition of $Y_t^{f,i}(\infty,r)$ in (4.1), we have for any $t\in[n\delta,(n+1)\delta)$,

$$\begin{aligned}
&\sum_{i=1}^{M_t}Y_t^{f,i}(\infty,r)\\
&=\langle f_{sm}+f_{cr},X_t\rangle+\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^kW^{k,j}_{t+r}-\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^kW^{k,j}_\infty-\sum_{2\lambda_k\ge\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^kW^{k,j}_{t+r}\\
&=\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_{t+r}-W^{k,j}_\infty\right)+\sum_{2\lambda_k\ge\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_t-W^{k,j}_{t+r}\right)\\
&\le\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_{n\delta+r}-W^{k,j}_\infty\right)+\sum_{2\lambda_k\ge\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_{n\delta}-W^{k,j}_{n\delta+r}\right)\\
&\quad+\sum_{k=1}^m e^{-\lambda_k t}\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{t+r}-W^{k,j}_{n\delta+r}\right|+\sum_{2\lambda_k\ge\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{n\delta}-W^{k,j}_t\right|\\
&\le|I_1|+I_2+I_3,
\end{aligned}$$

where

$$I_1=\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k n\delta}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_{n\delta+r}-W^{k,j}_{\infty}\right)+\sum_{2\lambda_k\geq\lambda_1}e^{-\lambda_k n\delta}\sum_{j=1}^{n_k}a_j^k\left(W^{k,j}_{n\delta}-W^{k,j}_{n\delta+r}\right)=\sum_{i=1}^{M_{n\delta}}Y^{f,i}_{n\delta}(\infty,r),$$

$$I_2=\sum_{2\lambda_k<\lambda_1}e^{-\lambda_k n\delta}\left|e^{-\lambda_k\delta}-1\right|\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{n\delta+r}-W^{k,j}_{\infty}\right|$$
$$+\sum_{2\lambda_k\geq\lambda_1}e^{-\lambda_k n\delta}\left|e^{-\lambda_k\delta}-1\right|\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{n\delta}-W^{k,j}_{n\delta+r}\right|=\sum_{k=1}^{m}\left|e^{-\lambda_k\delta}-1\right|\sum_{j=1}^{n_k}\left|a_j^k\right|\left|\sum_{i=1}^{M_{n\delta}}Y^{\phi_j^{(k)},i}_{n\delta}(\infty,r)\right|,$$

$$I_3=\sum_{k=1}^{m}e^{-\lambda_k t}\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{t+r}-W^{k,j}_{n\delta+r}\right|+\sum_{2\lambda_k\geq\lambda_1}e^{-\lambda_k t}\sum_{j=1}^{n_k}\left|a_j^k\right|\left|W^{k,j}_{n\delta}-W^{k,j}_{t}\right|.$$

Applying Lemma 4.3 with f for I_1, and with $f=\phi_j^{(k)}$ for I_2, we see that $\mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)$-almost surely,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t}Y_t^i(\infty,r)\right)}{\sqrt{2\log t}}=\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t}Y_t^i(\infty,r)\right)}{\sqrt{2\log t}}\\
&\leq\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}|I_1|}{\sqrt{2\log t}}+\limsup_{n\to\infty}\frac{e^{\lambda_1 n\delta/2}I_2}{\sqrt{2\log t}}+\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{\lambda_1 t/2}I_3}{\sqrt{2\log t}}\\
&=\sqrt{\langle V^f_{\infty,r},\phi_1\rangle W_\infty}+\sum_{k=1}^{m}\left|e^{-\lambda_k\delta}-1\right|\sum_{j=1}^{n_k}\left|a_j^k\right|\sqrt{\langle V^{\phi_j^{(k)}}_{\infty,r},\phi_1\rangle W_\infty}\\
&\quad+\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{\lambda_1 t/2}I_3}{\sqrt{2\log t}}.
\end{aligned}\tag{4.45}$$

Now we treat I_3. Recall that $r=\delta\ell$. By (4.32), we have

$$\begin{aligned}
&\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{\lambda_1 t/2}I_3}{\sqrt{2\log t}}\\
&\leq\sum_{k=1}^{m}\sum_{j=1}^{n_k}\left|a_j^k\right|\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)t/2}\left|W^{k,j}_{t+\delta\ell}-W^{k,j}_{(n+\ell)\delta}\right|}{\sqrt{2\log t}}\\
&\quad+\sum_{k=1}^{m}\sum_{j=1}^{n_k}\left|a_j^k\right|\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)t/2}\left|W^{k,j}_{n\delta}-W^{k,j}_{t}\right|}{\sqrt{2\log t}}\\
&=\sum_{k=1}^{m}\sum_{j=1}^{n_k}\left|a_j^k\right|\left(e^{-(\lambda_1-2\lambda_k)\delta\ell/2}+1\right)\limsup_{n\to\infty}\sup_{t\in[n\delta,(n+1)\delta)}\frac{e^{(\lambda_1-2\lambda_k)t/2}\left|W^{k,j}_{t}-W^{k,j}_{n\delta}\right|}{\sqrt{2\log t}}\\
&\leq\sum_{k=1}^{m}\sum_{j=1}^{n_k}\left|a_j^k\right|\left(e^{-(\lambda_1-2\lambda_k)\delta\ell/2}+1\right)\sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta}V^{\phi_j^{(k)}}_{2\delta,\delta},\phi_1\rangle W_\infty}
\end{aligned}\tag{4.46}$$

Plugging (4.46) into (4.45), we conclude that $\mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)$-almost surely,

$$\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{i=1}^{M_t}Y_t^i(\infty,r)\right)}{\sqrt{2\log t}}$$

$$\leq \sqrt{\langle V^f_{\infty,r}, \phi_1\rangle W_\infty} + \sum_{k=1}^m \left|e^{-\lambda_k\delta}-1\right| \sum_{j=1}^{n_k} \left|a^k_j\right| \sqrt{\langle V^{\phi_j^{(k)}}_{\infty,r}, \phi_1\rangle W_\infty}$$
$$+ \sum_{k=1}^m \sum_{j=1}^{n_k} \left|a^k_j\right| \left(e^{-(\lambda_1-2\lambda_k)\delta\ell/2}+1\right) \sqrt{\langle e^{|\lambda_1-2\lambda_k|\delta} V^{\phi_j^{(k)}}_{2\delta,\delta}, \phi_1\rangle W_\infty}.$$

Since $V^{\phi_j^{(k)}}_{2\delta,\delta} = \mathrm{Var}_x\left(W^{k,j}_{2\delta}\right)$ if $2\lambda_k < \lambda_1$ and $V^{\phi_j^{(k)}}_{2\delta,\delta} = \mathrm{Var}_x\left(W^{k,j}_{\delta}\right)$ if $2\lambda_k \geq \lambda_1$, it follows from (3.2) that $\lim_{\delta\to 0} V^{\phi_j^{(k)}}_{2\delta,\delta} = 0$. Now letting $\delta \to 0$ in the display above yields (4.30).

The proof of (4.31) is similar and we omit the details. □

As a consequence of Lemma 4.4, we have the following useful collory:

Corollary 4.5 *If $2\lambda_k > \lambda_1$, then for each $1 \leq j \leq n_k$, it holds that*

$$\limsup_{t\to\infty} \frac{e^{\frac{\lambda_1}{2}t}\left|\langle \phi_j^{(k)}, X_t\rangle\right|}{\sqrt{\log t}} < \infty. \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Proof: Fix an arbitrary $L \in \mathbb{N}$. By (4.35) (with δ replaced by δ/L in the definition of $\varepsilon_\ell(k,j,\delta)$ in (4.34)), when n is large enough, it holds almost surely that

$$W^{k,j}_{n\delta/L} - W^{k,j}_{n\delta} = \sum_{\ell=n}^{nL-1}\left(W^{k,j}_{\ell\delta/L} - W^{k,j}_{(\ell+1)\delta/L}\right) \leq \sum_{\ell=n}^{nL-1} \varepsilon_\ell(k,j,\delta/L).$$

Since $\sup_t W_t < \infty$, by the definition of $\varepsilon_\ell(k,j,\delta/L)$, we see that for all $\rho\in(0,1)$,

$$\sum_{\ell=n}^{nL-1} \varepsilon_\ell(k,j,\delta/L) = \sum_{\ell=n}^{nL-1}(1+\rho)\sqrt{2e^{(2\lambda_k-\lambda_1)\ell\delta/L}\log(\ell\delta/L)\langle e^{|\lambda_1-2\lambda_k|\delta/L}V^{\phi_j^{(k)}}_{2\delta/L,\delta/L}, \phi_1\rangle W_{\ell\delta/L}}$$
$$\lesssim_{k,L} \sqrt{\log(n\delta)} \sum_{\ell=n}^{nL-1} e^{\frac{2\lambda_k-\lambda_1}{2}\ell\delta/L}$$
$$\lesssim_{k,L} e^{\frac{2\lambda_k-\lambda_1}{2}n\delta}\sqrt{\log(n\delta)},$$

where in the first inequality we also used the fact that $\log(\ell\delta/L) \lesssim \log(n\delta)$ uniformly for all $n \leq \ell \leq nL-1$ for any fixed L. Therefore, we have almost surely,

$$\limsup_{n\to\infty} \frac{e^{-\frac{2\lambda_k-\lambda_1}{2}n\delta}\left(W^{k,j}_{n\delta/L} - W^{k,j}_{n\delta}\right)}{\sqrt{\log(n\delta)}}$$
$$= -\liminf_{n\to\infty} \frac{e^{\frac{\lambda_1}{2}n\delta}\left(\langle\phi_j^{(k)}, X_{n\delta}\rangle - e^{-\lambda_k n\delta(L-1)/L}\langle\phi_j^{(k)}, X_{n\delta/L}\rangle\right)}{\sqrt{\log(n\delta)}} < \infty. \tag{4.47}$$

Using (4.43), the assumption $2\lambda_k > \lambda_1$ and taking $L > \frac{2(\lambda_k-\lambda_1)}{2\lambda_k-\lambda_1}$, we get that

$$e^{\frac{\lambda_1}{2}n\delta}\left|e^{-\lambda_k n\delta(L-1)/L}\langle\phi_j^{(k)}, X_{n\delta/L}\rangle\right| \leq e^{\frac{\lambda_1}{2}n\delta}e^{-\lambda_k n\delta(L-1)/L}\langle\left|\phi_j^{(k)}\right|, X_{n\delta/L}\rangle$$
$$\lesssim_{k,j} e^{\frac{\lambda_1}{2}n\delta}e^{-\lambda_k n\delta(L-1)/L}\langle\phi_1, X_{n\delta/L}\rangle \lesssim e^{\frac{\lambda_1}{2}n\delta}e^{-\lambda_k n\delta(L-1)/L}e^{-\lambda_1 n\delta/L}$$

$$= e^{-\frac{n\delta}{2L}((2\lambda_k-\lambda_1)L-2(\lambda_k-\lambda_1))} \stackrel{n\to\infty}{\longrightarrow} 0. \tag{4.48}$$

Combining (4.47) and (4.48), we finally conclude that almost surely,

$$\liminf_{n\to\infty} \frac{e^{\frac{\lambda_1}{2}n\delta}\langle \phi_j^{(k)}, X_{n\delta}\rangle}{\sqrt{\log(n\delta)}} = \liminf_{n\to\infty} \frac{e^{-\frac{2\lambda_k-\lambda_1}{2}n\delta}W_{n\delta}^{k,j}}{\sqrt{\log(n\delta)}} > -\infty.$$

Now it follows from (4.32) that

$$\liminf_{t\to\infty} \frac{e^{\frac{\lambda_1}{2}t}\langle \phi_j^{(k)}, X_t\rangle}{\sqrt{\log t}} > -\infty.$$

Using a similar argument to $-\phi_j^{(k)}$, we get

$$\limsup_{t\to\infty} \frac{e^{\frac{\lambda_1}{2}t}\langle \phi_j^{(k)}, X_t\rangle}{\sqrt{\log t}} < \infty.$$

□

4.2 Small branching rate case: $\lambda_1 < 2\lambda_{\gamma(f)}$

Proof of Theorem 2.1: We first deal with the special case $f = -\phi_j^{(k)}(x)$ with $2\lambda_k > \lambda_1$. In this case, we have $Y_t^{f,i}(\infty, r) = e^{\lambda_k r}\langle \phi_j^{(k)}, X_r^i\rangle - \phi_j^{(k)}(X_t(i))$ and

$$V_{\infty,r}^{-\phi_j^{(k)}}(x) = \mathrm{Var}_x\left(-\langle T_{-r}\phi_j^{(k)}, X_r\rangle\right) = e^{2\lambda_k r}\mathrm{Var}_x\left(\langle \phi_j^{(k)}, X_r\rangle\right) = V_{\infty,r}^{\phi_j^{(k)}}(x).$$

By Lemma 3.2 (1), we know that

$$\lim_{r\to\infty} e^{(\lambda_1-2\lambda_k)r}V_{\infty,r}^{-\phi_j^{(k)}}(x) = \sigma_{sm}^2\left(\phi_j^{(k)}\right)\phi_1(x). \tag{4.49}$$

By Lemma 4.4, we have $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,

$$\limsup_{t\to\infty} \frac{e^{\lambda_1(t+r)/2}\left(\langle \phi_j^{(k)}, X_{t+r}\rangle - e^{-\lambda_k r}\langle \phi_j^{(k)}, X_t\rangle\right)}{\sqrt{2\log t}} = \sqrt{\langle e^{(\lambda_1-2\lambda_k)r}V_{\infty,r}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty}. \tag{4.50}$$

Define

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle \phi_j^{(k)}, X_t\rangle}{\sqrt{2\log t}} =: D_+, \quad \liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle \phi_j^{(k)}, X_t\rangle}{\sqrt{2\log t}} =: D_-, \quad \mathbb{P}_x(\cdot|\mathcal{E}^c)\text{-a.s.}.$$

By Corollary 4.5, we have $|D_+|, |D_-| < \infty$. Using the inequality

$$\limsup_{t\to\infty} x_t + \liminf_{t\to\infty} y_t \le \limsup_{t\to\infty}(x_t + y_t) \le \limsup_{t\to\infty} x_t + \limsup_{t\to\infty} y_t, \tag{4.51}$$

we get from (4.50) that $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,

$$\left(1 - e^{(\lambda_1/2-\lambda_k)r}\right)D_+ \le \sqrt{\langle e^{(\lambda_1-2\lambda_k)r}V_{\infty,r}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty} \le D_+ - e^{(\lambda_1/2-\lambda_k)r}D_-. \tag{4.52}$$

Letting $r \to \infty$ in the first inequality in (4.52) and applying (4.49), we get

$$D_+ \le \sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.53}$$

By a similar argument, we have $D_- \ge -\sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty}$. Thus,

$$\sqrt{\langle e^{(\lambda_1-2\lambda_k)r} V_{\infty,r}^{\phi_j^{(k)}}, \phi_1\rangle W_\infty} \le D_+ - e^{(\lambda_1/2-\lambda_k)r} D_- \le D_+ + e^{(\lambda_1/2-\lambda_k)r}\sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty}.$$

Letting $r \to \infty$, we deduce that

$$D_+ \ge \sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.54}$$

Therefore, combining (4.53) and (4.54), we see that $\mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)$ -almost surely,

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle \phi_j^{(k)}, X_t\rangle}{\sqrt{2\log t}} = \sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty}. \tag{4.55}$$

Now we deal the general small branching case $f(x) = \sum_{k\le m: 2\lambda_k>\lambda_1} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x)$. In this case, for any $r \in (0,\infty)$, we have

$$Y_t^{T_r f, i}(\infty, r) = T_r f\left(X_t(i)\right) - \langle f, X_r^i\rangle, \quad V_{\infty,r}^{T_r f}(x) = \mathbb{E}_x\left(\langle f, X_r\rangle\right)^2 - \left(T_r f(x)\right)^2.$$

Note that Lemma 4.4 is equivalent to

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_{t+r}\rangle - \langle T_r f, X_t\rangle\right)}{\sqrt{2\log t}} = \sqrt{\langle V_{\infty,r}^{T_r f}, \phi_1\rangle W_\infty}.$$

Multiplying both sides of the equality above by $e^{\lambda_1 r/2}$, we get

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(\langle f, X_t\rangle - \langle T_r f, X_{t-r}\rangle\right)}{\sqrt{2\log t}} = \sqrt{\langle e^{\lambda_1 r} V_{\infty,r}^{T_r f}, \phi_1\rangle W_\infty}.$$

It follows from Lemma 3.2 (1) that $e^{\lambda_1 r} V_{\infty,r}^{T_r f}(x) \to \sigma_{sm}^2(f)\phi_1(x)$ as $r \to \infty$. Now combining (4.51) and (4.55), we deduce that

$$\begin{aligned}
\sqrt{\sigma_{sm}^2(f)\rangle W_\infty} &= \lim_{r\to\infty} \sqrt{\langle e^{\lambda_1 r} V_{\infty,r}^{T_r f}, \phi_1\rangle W_\infty} \\
&\ge \limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}} - \lim_{r\to\infty} \sum_{k\le m: 2\lambda_k>\lambda_1} \sum_{j=1}^{n_k} e^{(\lambda_1/2-\lambda_k)r} \left|a_j^k\right| \sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty} \\
&= \limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}}.
\end{aligned} \tag{4.56}$$

Similarly, we also have that

$$\begin{aligned}
\sqrt{\sigma_{sm}^2(f)\rangle W_\infty} &= \lim_{r\to\infty} \sqrt{\langle e^{\lambda_1 r} V_{\infty,r}^{T_r f}, \phi_1\rangle W_\infty} \le \limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}} \\
&+ \lim_{r\to\infty} \sum_{k\le m: 2\lambda_k>\lambda_1} \sum_{j=1}^{n_k} e^{(\lambda_1/2-\lambda_k)r} \left|a_j^k\right| \sqrt{\sigma_{sm}^2\left(\phi_j^{(k)}\right) W_\infty} = \limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}}.
\end{aligned} \tag{4.57}$$

Combining (4.56) and (4.57), we get the desired limsup result. The proof of the liminf is similar with f replaced by $-f$. □

4.3 Critical branching rate case: $\lambda_1 = 2\lambda_{\gamma(f)}$

Proof of Theorem 2.3: Since $f - f_{cr}$ belongs to the small branching case, by Theorem 2.1,

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2} \left|\langle f - f_{cr}, X_t\rangle\right|}{\sqrt{2t\log\log t}} = 0, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Therefore, we only need to prove the assertion of the theorem for $f = f_{cr}$. Combining (3.2) and the fact that $\mathbb{E}_x\left(\langle f_{cr}, X_r\rangle\right) = e^{-\lambda_{\gamma(f)} r} f_{cr}(x) = e^{-\lambda_1 r/2} f_{cr}(x)$, we get

$$\begin{aligned} V^{f_{cr}}_{\infty,r}(x) &= e^{\lambda_1 r}\mathrm{Var}_x\left(\langle f_{cr}, X_r\rangle\right) = e^{\lambda_1 r}\mathbb{E}_x\left(\langle f_{cr}, X_r\rangle^2\right) - \left(f_{cr}(x)\right)^2 \\ &= \int_0^r e^{\lambda_1 s} T_s\left[A^{(2)}\cdot (f_{cr})^2\right](x)\mathrm{d}s + e^{\lambda_1 r} T_r\left(f_{cr}\right)^2(x) - \left(f_{cr}(x)\right)^2. \end{aligned}$$

Since T_r is self-adjoint, we have

$$\begin{aligned} \langle V^{f_{cr}}_{\infty,r}, \phi_1\rangle &= \int_0^r e^{\lambda_1 s}\langle T_s\left[A^{(2)}\cdot (f_{cr})^2\right], \phi_1\rangle\mathrm{d}s + e^{\lambda_1 r}\langle T_r\left(f_{cr}\right)^2, \phi_1\rangle - \langle\left(f_{cr}\right)^2, \phi_1\rangle \\ &= r\sigma^2_{cr}(f_{cr}) = r\sigma^2_{cr}(f). \end{aligned}$$

We will apply [28, Theorem 3], so we first check the conditions of this theorem. In our case, W_∞ may take the value 0 with positive probability, so we introduce an independent random walk $S^\Upsilon_n := \sum_{i=1}^n \Upsilon_i$, where under $\mathbb{P}_x$, Υ_i are iid random variables with $\mathbb{P}_x(\Upsilon_i = \varepsilon) = \mathbb{P}_x(\Upsilon_i = -\varepsilon) = \frac{1}{2}$ for some small positive constant ε. Define $s^{f_{cr}}_n$ by

$$\begin{aligned} \left(s^{f_{cr}}_n\right)^2 &:= \sum_{\ell=1}^n \mathrm{Var}_x\left[e^{\lambda_1(\ell-1)\delta/2} U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta) + S^\Upsilon_\ell\middle|\mathcal{F}_{(\ell-1)\delta}, \Upsilon_1, ..., \Upsilon_{\ell-1}\right] \\ &= \sum_{\ell=1}^n e^{\lambda_1(\ell-1)\delta}\mathrm{Var}_x\left[U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)\middle|\mathcal{F}_{(\ell-1)\delta}\right] + n\varepsilon^2. \end{aligned}$$

Combining Lemma 4.1 and the fact that $\lambda_1 = 2\lambda_{\gamma(f)}$, we know that

$$\left(s^{f_{cr}}_n\right)^2/n \to \langle V^{f_{cr}}_{\infty,\delta}, \phi_1\rangle W_\infty + \varepsilon^2 = \delta\sigma^2_{cr}(f) W_\infty + \varepsilon^2, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.58}$$

Let $s_n := s^{f_{cr}}_n, u_n := \sqrt{2\log\log\left(s^{f_{cr}}_n\right)^2}, K^2_n := nu^2_n/(s^2_n\log n)$ in [28, Theorem 3]. By the definition of $U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)$ in (4.2), we see that conditioned on $\mathcal{F}_{(\ell-1)\delta}$, $e^{\lambda_k(\ell-1)\delta} U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)$ is the sum of finitely many independent centered random variables. For independent random variables $Y_1, ..., Y_n$ with $\mathbb{E}[Y_j] = 0, j = 1, 2, ..., n$, we have

$$\mathbb{E}\left[\sum_{j=1}^n Y_j\right]^4 = \sum_{j=1}^n \mathbb{E}\left[Y_j^4\right] + 6\sum_{i\neq j}\mathbb{E}[Y_i^2]\mathbb{E}[Y_j^2] \le \sum_{j=1}^n \mathbb{E}\left[Y_j^4\right] + 6\left(\sum_{j=1}^n \mathbb{E}\left[Y_j^2\right]\right)^2.$$

Therefore,

$$\begin{aligned} &\mathbb{E}_x\left[\left(e^{\lambda_k(\ell-1)\delta} U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)\right)^4\middle|\mathcal{F}_{(\ell-1)\delta}\right] \\ &\le 16 e^{2\lambda_1(\ell-1)\delta} \sum_{i=1}^{M_{(\ell-1)\delta}} \mathbb{E}_x\left[\left|Z^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^4\middle|\mathcal{F}_{(\ell-1)\delta}\right] \end{aligned}$$

$$+6e^{2\lambda_1(\ell-1)\delta}\left(\sum_{i=1}^{M_{(\ell-1)\delta}}\mathbb{E}_x\left[\left|Z^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^2\Big|\mathcal{F}_{(\ell-1)\delta}\right]\right)^2, \tag{4.59}$$

where in the inequality we also used the inequalities $\mathbb{E}\left[Y-\mathbb{E}[Y]\right]^4\le 16\mathbb{E}[Y^4]$ and $\mathbb{E}\left[Y-\mathbb{E}[Y]\right]^2\le\mathbb{E}[Y^2]$ for $Y=Z^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)$. Using the trivial upper bound

$$e^{\lambda_1(\ell-1)\delta}\left|Z^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^4\le\left|Z^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^2\le\left|Y^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^2,$$

we conclude from (4.59) that

$$\begin{aligned}
&\mathbb{E}_x\left[\left(e^{\lambda_k(\ell-1)\delta}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)\right)^4\Big|\mathcal{F}_{(\ell-1)\delta}\right]\\
&\le 16e^{\lambda_1(\ell-1)\delta}\sum_{j=1}^{M_{(\ell-1)\delta}}\mathbb{E}_x\left[\left|Y^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^2\Big|\mathcal{F}_{(\ell-1)\delta}\right]\\
&\qquad+6e^{2\lambda_1(\ell-1)\delta}\left(\sum_{j=1}^{M_{(\ell-1)\delta}}\mathbb{E}_x\left[\left|Y^{f_{cr},i}_{(\ell-1)\delta}(\infty,\delta)\right|^2\Big|\mathcal{F}_{(\ell-1)\delta}\right]\right)^2\\
&=16e^{\lambda_1(\ell-1)\delta}\langle V^{f_{cr}}_{\infty,\delta},X_{(\ell-1)\delta}\rangle+6\left(e^{\lambda_1(\ell-1)\delta}\langle V^{f_{cr}}_{\infty,\delta},X_{(\ell-1)\delta}\rangle\right)^2,
\end{aligned}\tag{4.60}$$

where the last equality follows from the Markov property and the definition of $V^f_{\infty,r}$ given as in (4.3). Therefore, combining Lemma 3.4 and (4.60), we conclude that $\mathbb{P}_x$-almost surely,

$$\sup_{\ell\in\mathbb{N}}\mathbb{E}_x\left[\left(e^{\lambda_k(\ell-1)\delta}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)\right)^4\Big|\mathcal{F}_{(\ell-1)\delta}\right]<\infty. \tag{4.61}$$

Using the fact $|\Upsilon_i|=\varepsilon$, and the independence of $\{\Upsilon_\ell,\ell\in\mathbb{N}\}$ and $\left\{U^{f_{cr}}_{\ell\delta}(\infty,\delta),\ell\in\mathbb{N}\right\}$, we conclude from (4.61) that $\mathbb{P}_x$-almost surely,

$$\sup_{\ell\in\mathbb{N}}\mathbb{E}_x\left[\left(e^{\lambda_k(\ell-1)\delta}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)+\Upsilon_\ell\right)^4\Big|\mathcal{F}_{(\ell-1)\delta},\Upsilon_1,...,\Upsilon_{\ell-1}\right]<\infty,$$

which implies that

$$\sum_{\ell=1}^\infty\left\{\frac{\log^2\ell}{\ell^2}\mathbb{E}_x\left[\left(e^{\lambda_1(\ell-1)\delta/2}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)+\Upsilon_\ell\right)^4\Big|\mathcal{F}_{(\ell-1)\delta},\Upsilon_1,...,\Upsilon_{\ell-1}\right]\right\}<\infty.$$

Combining the above with the trivial inequality $M\mathbb{E}[|Y|1_{\{|Y|>M\}}]\le\mathbb{E}[Y^2]$, we get that $\mathbb{P}_x$- almost surely,

$$\begin{aligned}
&\sum_{\ell=1}^\infty\left\{\frac{\log\ell}{\ell}\mathbb{E}_x\Bigg[\left(e^{\lambda_1(\ell-1)\delta/2}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)+\Upsilon_\ell\right)^2\right.\\
&\qquad\left.\times 1_{\left\{\left(e^{\lambda_1(\ell-1)\delta/2}U^{f_{cr}}_{(\ell-1)\delta}(\infty,\delta)+\Upsilon_\ell\right)^2>\ell/\log\ell\right\}}\Big|\mathcal{F}_{(\ell-1)\delta},\Upsilon_1,...,\Upsilon_{\ell-1}\Bigg]\right\}<\infty.
\end{aligned}\tag{4.62}$$

Note that $K_n \to 0$ as $n \to \infty$. Combining [28, Theorem 3], (4.58) and (4.62), we get

$$\limsup_{n\to\infty} \frac{S_n^{\Upsilon} + \sum_{\ell=1}^{n} e^{\lambda_1(\ell-1)\delta/2} U_{(\ell-1)\delta}^{f_{cr}}(\infty,\delta)}{\sqrt{2n\log\log n}} = \sqrt{\delta\sigma_{cr}^2(f)W_\infty + \varepsilon^2}, \quad \mathbb{P}_x\text{-a.s.} \tag{4.63}$$

It is well-known that almost surely,

$$\limsup_{n\to\infty} \frac{|S_n^{\Upsilon}|}{\sqrt{2n\log\log n}} = \varepsilon. \tag{4.64}$$

Combining (4.63) and (4.64), and letting $\varepsilon \downarrow 0$, we get that

$$\limsup_{n\to\infty} \frac{\sum_{\ell=1}^{n} e^{\lambda_1(\ell-1)\delta/2} U_{(\ell-1)\delta}^{f_{cr}}(\infty,\delta)}{\sqrt{2n\log\log n}} = \sqrt{\delta\sigma_{cr}^2(f)W_\infty}, \quad \mathbb{P}_x\text{-a.s.} \tag{4.65}$$

We have proved in (4.23) that

$$\begin{aligned} &\sum_{n\geq 0} \mathbb{E}_x \left| e^{\lambda_1 n\delta/2} \left(\sum_{i=1}^{M_{n\delta}} Y_{n\delta}^{f_{cr},i}(\infty,\delta) \right) - e^{\lambda_1 n\delta/2} U_{n\delta}^{f_{cr}}(\infty,\delta) \right| \\ &\leq 2\sum_{n\geq 0} e^{\lambda_1 n\delta/2} \mathbb{E}_x \left[\sum_{i=1}^{M_{n\delta}} \left| Y_{n\delta}^{f_{cr},i}(\infty,\delta) \right| 1_{\{|Y_{n\delta}^{f_{cr},i}(\infty,\delta)| > e^{-\lambda_1 n\delta/2}\}} \right] < \infty. \end{aligned} \tag{4.66}$$

Combining the definition of $Y_{n\delta}^{f_{cr},i}(\infty,\delta)$, (4.65), (4.66) and recalling that $\lambda_1 = 2\lambda_{\gamma(f)}$, we conclude that $\mathbb{P}_x$-almost surely,

$$\limsup_{n\to\infty} \frac{f_{cr}(x) - e^{\lambda_1 n\delta/2}\langle f_{cr}, X_{n\delta}\rangle}{\sqrt{2n\log\log n}} = \limsup_{n\to\infty} \frac{\sum_{\ell=0}^{n-1} \left(e^{\lambda_1 \ell\delta/2} \left(\sum_{i=1}^{M_{\ell\delta}} Y_{\ell\delta}^{f_{cr},i}(\infty,\delta) \right) \right)}{\sqrt{2n\log\log n}} = \sqrt{\delta\sigma_{cr}^2(f)W_\infty}.$$

Noticing that $\mathbb{P}_x(\mathcal{E}^c) > 0$, the above limit implies that

$$\liminf_{n\to\infty} \frac{e^{\lambda_1 n\delta/2}\langle f_{cr}, X_{n\delta}\rangle}{\sqrt{2(n\delta)\log\log(n\delta)}} = -\sqrt{\sigma_{cr}^2(f)W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.67}$$

The same argument for $U_{(\ell-1)\delta}^{f_{cr}}(\infty,\delta)$ replaced by $-U_{(\ell-1)\delta}^{f_{cr}}(\infty,\delta)$ implies that

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 n\delta/2}\langle f_{cr}, X_{n\delta}\rangle}{\sqrt{2(n\delta)\log\log(n\delta)}} = \sqrt{\sigma_{cr}^2(f)W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Let k_0 be the unique integer such that $\lambda_1 = 2\lambda_{k_0}$, then we see that $f_{cr} = \sum_{j=1}^{n_{k_0}} a_j^{k_0}\phi_j^{(k_0)}$. Recall that by (4.32), we have for any $\delta > 0$,

$$\limsup_{n\to\infty} \sup_{t\in[n\delta,(n+1)\delta)} \frac{\left| W_{n\delta}^{k_0,j} - W_t^{k_0,j} \right|}{\sqrt{2(n\delta)\log\log(n\delta)}} = 0, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{4.68}$$

Therefore, combining (4.67) and (4.68), we conclude that

$$\begin{aligned} \liminf_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f_{cr}, X_t\rangle}{\sqrt{2t\log\log t}} &= \liminf_{n\to\infty} \inf_{t\in[n\delta,(n+1)\delta)} \frac{\sum_{j=1}^{n_k} a_j^{k_0} W_t^{k_0,j}}{\sqrt{2t\log\log t}} \\ &= \liminf_{n\to\infty} \frac{\sum_{j=1}^{n_k} a_j^{k_0} W_{n\delta}^{k_0,j}}{\sqrt{2(n\delta)\log\log(n\delta)}} = -\sqrt{\sigma_{cr}^2(f)W_\infty} \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.,} \end{aligned}$$

which implies the desired liminf result.

The proof of the limsup is similar with f replaced by $-f$. The proof is complete. □

4.4 Large branching rate case

Proof of Theorem 2.5: Suppose $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr} = 0$. Let $f = \sum_{k=1}^{m}\sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x)$. For any $r \in (0,\infty)$, define

$$f^{(r)} := \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \phi_j^{(k)} - \sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \phi_j^{(k)} = T_r f_{la} - T_r f_{sm}.$$

Then by direct calculation, we get

$$\begin{aligned}\sum_{i=1}^{M_t} Y_t^{f^{(r)},i}(\infty,r) &= \langle f_{sm}^{(r)}, X_t\rangle + \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_{t+r}^{k,j}\\ &\quad - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_{\infty}^{k,j} + \sum_{2\lambda_k>\lambda_1} e^{-\lambda_k t}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_{t+r}^{k,j}\\ &= \langle f, X_{t+r}\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k (t+r)}\sum_{j=1}^{n_k} a_j^k W_{\infty}^{k,j} - \sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \langle \phi_j^{(k)}, X_t\rangle\\ &=: M_{t+r} - \sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \langle \phi_j^{(k)}, X_t\rangle. \end{aligned} \tag{4.69}$$

Applying Lemma 4.4 to $Y_t^{f^{(r)},i}(\infty,r)$, we get that $\mathbb{P}_x(\cdot|\mathcal{E}^c)$ -almost surely,

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\left(M_{t+r} - \sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \langle \phi_j^{(k)}, X_t\rangle\right)}{\sqrt{2\log t}} = \sqrt{\langle V_{\infty,r}^{f^{(r)}}, \phi_1\rangle W_\infty}. \tag{4.70}$$

By Corollary 4.5, we have

$$\begin{aligned}&\lim_{r\to\infty} e^{\lambda_1 r/2}\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left(\sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r}\left|a_j^k\right|\left|\langle \phi_j^{(k)}, X_t\rangle\right|\right)}{\sqrt{2\log t}}\\ &\le \sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k}\left|a_j^k\right| \lim_{r\to\infty} e^{(\lambda_1/2-\lambda_k)r}\limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}\left|\langle \phi_j^{(k)}, X_t\rangle\right|}{\sqrt{2\log t}} = 0, \quad \mathbb{P}_x(\cdot|\mathcal{E}^c)\text{-a.s.}\end{aligned} \tag{4.71}$$

Combining (4.69), (4.70) and (4.71), we conclude that

$$\begin{aligned}&\limsup_{t\to\infty}\frac{e^{\lambda_1 (t+r)/2}\left(\langle f, X_{t+r}\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k(t+r)}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}\right)}{\sqrt{2\log t}} = \limsup_{t\to\infty}\frac{e^{\lambda_1 t/2}M_t}{\sqrt{2\log t}}\\ &= \lim_{r\to\infty}\sqrt{\langle e^{\lambda_1 r} V_{\infty,r}^{f^{(r)}}, \phi_1\rangle W_\infty}, \quad \mathbb{P}_x(\cdot|\mathcal{E}^c)\text{-a.s.}\end{aligned}$$

Thus, to prove (2.1), it suffices to show that

$$\lim_{r\to\infty} e^{\lambda_1 r} V_{\infty,r}^{f^{(r)}}(x) = \left(\sigma_{sm}^2(f_{sm}) + \sigma_{la}^2(f)\right)\phi_1(x). \tag{4.72}$$

Recall that

$$V_{\infty,r}^{f^{(r)}}(x) = \mathbb{E}_x\left[\left(-\sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \phi_j^{(k)}(x) + \langle f, X_r\rangle - \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_\infty^{k,j}\right)^2\right].$$

Define $H_\infty^f := \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} a_j^k W_\infty^{k,j}$ and $Q(x) := \mathbb{E}_x\left(f(x)-H_\infty^f\right)^2$. Then we have that

$$\begin{aligned}
&\mathbb{E}_x\left[\left(\langle f, X_r\rangle - \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_\infty^{k,j}\right)^2\right] = \mathbb{E}_x\left[\left(\sum_{i=1}^{M_r}\left(f\left(X_r(i)\right)-H_\infty^{f,i}\right)\right)^2\right]\\
&= \mathbb{E}_x\left(\sum_{i=1}^{M_r}\left(f\left(X_r(i)\right)-H_\infty^{f,i}\right)^2\right) + \mathbb{E}_x\left(\sum_{i\neq j}\left(f\left(X_r(i)\right)-H_\infty^{f,i}\right)\left(f\left(X_r(j)\right)-H_\infty^{f,j}\right)\right)\\
&= \mathbb{E}_x\left(\langle Q, X_r\rangle\right) + \mathbb{E}_x\left(\sum_{i\neq j} f_{sm}\left(X_r(i)\right) f_{sm}\left(X_r(j)\right)\right)\\
&= \mathbb{E}_x\left(\langle Q, X_r\rangle\right) + \mathbb{E}_x\left(\langle f_{sm}, X_r\rangle^2\right) - \mathbb{E}_x\left(\langle f_{sm}^2, X_r\rangle\right).
\end{aligned}$$

Since $Q(x) = f_{sm}^2(x) + \mathrm{Var}_x\left(H_\infty^f\right) = f_{sm}^2(x) + \sigma_{la}^2(f)\phi_1(x)$, we conclude that

$$\lim_{r\to\infty} e^{\lambda_1 r}\mathbb{E}_x\left[\left(\langle f, X_r\rangle - \sum_{2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k W_\infty^{k,j}\right)^2\right] = \left(\sigma_{la}^2(f)+\sigma_{sm}^2(f_{sm})\right)\phi_1(x).$$

Since $\lim_{r\to\infty} e^{\lambda_1 r}\left(-\sum_{2\lambda_k>\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k r} a_j^k \phi_j^{(k)}(x)\right)^2 = 0$, we get (4.72) and this completes the proof of (2.1). The proof of (2.2) is similar. □

Proof of Theorem 2.6: Suppose $\lambda_1 > 2\lambda_{\gamma(f)}$ and $f_{cr} \neq 0$. Applying Theorem 2.5 to $f - f_{cr}$ and Theorem 2.3 to f_{cr}, we immediately get the desired result. □

5 Proof of Theorem 2.8

In this section, we always assume that **(H1)**–**(H4)** hold.

For any $r > 0$ and $g = \sum_{k=0}^\infty\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)} \in L^2(E,\mu)$, we have

$$f := T_r g = \sum_{k=\gamma(g)}^{\infty}\sum_{j=1}^{n_k} a_j^k\phi_j^{(k)}, \tag{5.1}$$

with $a_j^k = e^{-\lambda_k r} b_j^k$. Combining (1.3) and **(H4)**, we get

$$\sum_{k=1}^{\infty} e^{-\lambda_k r}\sum_{j=1}^{n_k}\left(\phi_j^{(k)}\right)^2 \le c_r^2\phi_1^2, \tag{5.2}$$

which implies that

$$\sum_{k=1}^{\infty} e^{-\lambda_k r}\sum_{j=1}^{n_k} 1 \le c_r^2. \tag{5.3}$$

Noticing that $|b_j^k| \le \sqrt{\|g\|_2}$ and using (5.2), we get that, for any $n > m$,

$$\left|\sum_{k=m}^{n} e^{-\lambda_k r}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}(x)\right| \le \sum_{k=m}^{n} e^{-\lambda_k r}\sum_{j=1}^{n_k}\left|b_j^k\phi_j^{(k)}(x)\right|$$

$$\leq \sqrt{\sum_{k=m}^{n} e^{-\lambda_k r} \sum_{j=1}^{n_k} 1} \sqrt{\sum_{k=m}^{n} e^{-\lambda_k r} \sum_{j=1}^{n_k} \left| b_j^k \phi_j^{(k)}(x) \right|^2} \leq \sqrt{\|g\|_2} c_r \phi_1(x) \sqrt{\sum_{k=m}^{n} e^{-\lambda_k r} \sum_{j=1}^{n_k} 1}, \quad (5.4)$$

which implies that the series in (5.1) also converges pointwisely.

From (5.2) with r replaced by $r/4$, we see that for any $r > 0$ and $k \geq 0$,

$$\sup_{1 \leq j \leq n_k} \left| \phi_j^{(k)} \right| \lesssim_r e^{\lambda_k r/4} \phi_1 \lesssim e^{\lambda_k r/4}. \quad (5.5)$$

5.1 Proof of Theorem 2.8

In this section, we will first use the following proposition to prove Theorem 2.8 and then present the proof of the proposition.

Proposition 5.1 *Suppose in addition that* **(H5)** *holds. If $f \in \mathbb{T}(E)$ satisfies $\lambda_{\gamma(f)} > 0$, then*

$$\limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} |\langle f, X_t \rangle|}{\sqrt{2 \log t}} \leq 18 \sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x \left(\cdot | \mathcal{E}^c \right) \text{-a.s.}$$

Note that $\lambda_{\gamma(f)} > 0$ implies that $\lambda_1 < 2\lambda_{\gamma(f)}$, which is the small branching rate case.

Proof of Theorem 2.8: We only give the proof of the small branching case here, the proofs for the critical and the large branching cases are similar. By Lemma 3.2 (1), we have

$$\sigma_{sm}^2(f) \lesssim \left(\|f\|_2^2 + \|f\|_4^2 \right) \lesssim \left(\|f\|_2 + \|f\|_4 \right)^2.$$

Therefore, by Proposition 5.1, for any $f \in \mathbb{T}(E)$ with $\lambda_{\gamma(f)} > 0$, there exists a constant C independent of f such that

$$\limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} |\langle f, X_t \rangle|}{\sqrt{2 \log t}} \leq C \sqrt{W_\infty} \left(\|f\|_2 + \|f\|_4 \right), \quad \mathbb{P}_x \left(\cdot | \mathcal{E}^c \right) \text{-a.s.} \quad (5.6)$$

For any $f \in \mathbb{T}(E)$ with $\lambda_1 < 2\lambda_{\gamma(f)}$, we write $f = f_{main} + f_{rest}$, where

$$f_{main} = \sum_{k=\gamma(f)}^{N} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}, \quad f_{rest} = \sum_{k=N}^{\infty} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}$$

and N is a large integer such that $\lambda_N > 0$. Applying Theorem 2.1 to f_{main} and (5.6) to f_{rest}, we see that $\mathbb{P}_x \left(\cdot | \mathcal{E}^c \right)$ almost surely,

$$\begin{aligned} &\limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} \langle f, X_t \rangle}{\sqrt{2 \log t}} \leq \limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} \langle f_{main}, X_t \rangle}{\sqrt{2 \log t}} + \limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} |\langle f_{rest}, X_t \rangle|}{\sqrt{2 \log t}} \\ &\leq \sqrt{\sigma_{sm}^2(f_{main}) W_\infty} + C \sqrt{W_\infty} \left(\|f_{rest}\|_2 + \|f_{rest}\|_4 \right), \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} &\limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} \langle f, X_t \rangle}{\sqrt{2 \log t}} \geq \limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} \langle f_{main}, X_t \rangle}{\sqrt{2 \log t}} - \limsup_{t \to \infty} \frac{e^{\lambda_1 t/2} |\langle f_{rest}, X_t \rangle|}{\sqrt{2 \log t}} \\ &\geq \sqrt{\sigma_{sm}^2(f_{main}) W_\infty} - C \sqrt{W_\infty} \left(\|f_{rest}\|_2 + \|f_{rest}\|_4 \right). \end{aligned} \quad (5.8)$$

Combining the dominated convergence theorem and (5.4), we get that

$$\sigma_s(f_{main}) \to \sigma_s(f), \quad \|f_{rest}\|_2 + \|f_{rest}\|_4 \to 0.$$

Therefore, letting $N \to \infty$ in (5.7) and (5.8), we get

$$\limsup_{t\to\infty} \frac{e^{\lambda_1 t/2}\langle f, X_t\rangle}{\sqrt{2\log t}} = \sqrt{\sigma_{sm}^2(f)W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

The proof for the liminf is similar. The proof is complete. □

Now we are going to prove Proposition 5.1. In the remainder of this section, we always assume that $f \in \mathbb{T}(E)$ is given by (5.1) for some $r > 0$. Note that for $f \in \mathbb{T}(E)$ given by (5.1), we have $\gamma(g) = \gamma(f)$ and $\lambda_{\gamma(g)} = \lambda_{\gamma(f)}$.

We will use a different discretization scheme. Let $k_0 \in \mathbb{N}$ be the unique integer such that $\lambda_{k_0} > 0 \geq \lambda_{k_0-1}$. For any $n \in \mathbb{N}$, define

$$t_n := n^{1/10}, \quad \kappa(n) := \sup\left\{k > 0 : \lambda_k \leq n^{1/5}\right\}, \quad N_r := \inf\left\{n > 0 : n^{1/10}r > -4\lambda_1\right\} \tag{5.9}$$

and

$$S_f^{(n)} := \sum_{k=k_0}^{\kappa(n)} e^{-\lambda_k r}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}, \quad \Gamma_f^{(n)} := \sum_{k=\kappa(n)+1}^{\infty} e^{-\lambda_k r}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}. \tag{5.10}$$

Then $f = S_f^{(n)} + \Gamma_f^{(n)}$.

Lemma 5.2 *Let $f \in \mathbb{T}(E)$ be given by (5.1) for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$. Then under $\mathbb{P}(\cdot|\mathcal{E})$, almost surely,*

$$\lim_{n\to\infty} \sup_{t_n\leq t<t_{n+1}} e^{\lambda_1 t/2}\left(\left|\langle\Gamma_f^{(n)}, X_t\rangle\right| + \left|\langle T_{t_{n+1}-t}\Gamma_f^{(n)}, X_t\rangle\right|\right) = 0.$$

Proof: Recall the definitions of $\kappa(n)$ and N_r in (5.9), then $\lambda_k r/4 \geq n^{1/5}r/4 > -\lambda_1 t_n$ for all $k \geq \kappa(n)+1$. Therefore, using $|b_j^k| \leq \sqrt{\|g\|_2}$, we conclude that

$$\begin{aligned}\left|\Gamma_f^{(n)}\right| &\lesssim_r \sqrt{\|g\|_2}\phi_1 \sum_{k=\kappa(n)+1}^{\infty} e^{-3\lambda_k r/4}\sum_{j=1}^{n_k} 1\\ &\leq \sqrt{\|g\|_2}e^{\lambda_1 t_n}\phi_1 \sum_{k=\kappa(n)+1}^{\infty} e^{-\lambda_k r/2}\sum_{j=1}^{n_k} 1 \leq c_{r/2}^2\sqrt{\|g\|_2}e^{\lambda_1 t_n}\phi_1,\end{aligned}$$

where in the first inequality we used (5.5) and in the last inequality we used (5.3). Since $t_{n+1}-t_n \leq 1$, we conclude that for any $n > N_r$ and $t \in [t_n, t_{n+1})$,

$$\left|\langle\Gamma_f^{(n)}, X_t\rangle\right| + \left|\langle T_{t_{n+1}-t}\Gamma_f^{(n)}, X_t\rangle\right| \lesssim_{r,g} e^{\lambda_1 t_n}\langle\phi_1, X_t\rangle \lesssim W_t.$$

Since $\lambda_1 < 0$, the desired assertion follows immediately. □

Lemma 5.3 *Let $f \in \mathbb{T}(E)$ be given by (5.1) for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$. Then under $\mathbb{P}(\cdot|\mathcal{E})$, almost surely,*

$$\lim_{n\to\infty} \sup_{t_n\leq t<t_{n+1}} e^{\lambda_1 t/2}\left|\langle T_{t_{n+1}-t}S_f^{(n)} - S_f^{(n)}, X_t\rangle\right| = 0.$$

Proof: Since $|b_j^k| \le \sqrt{\|g\|_2}$, we have for any $t \in [t_n, t_{n+1})$,

$$\sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} S_f^{(n)} - S_f^{(n)}, X_t \rangle \right| = \sup_{t_n \le t < t_{n+1}} \left| \sum_{k=k_0}^{\kappa(n)} e^{-\lambda_k r} \left(e^{-\lambda_k (t_{n+1}-t)} - 1 \right) \sum_{j=1}^{n_k} b_j^k \langle \phi_j^{(k)}, X_t \rangle \right|$$
$$\le \sqrt{\|g\|_2} \sum_{k=k_0}^{\kappa(n)} e^{-\lambda_k r} \left(1 - e^{-\lambda_k (t_{n+1}-t)} \right) \sum_{j=1}^{n_k} \sup_{t_n \le t < t_{n+1}} \left| \langle \phi_j^{(k)}, X_t \rangle \right|$$
$$\le \sqrt{\|g\|_2} \sum_{k=k_0}^{\kappa(n)} \lambda_k (t_{n+1} - t) e^{-\lambda_k r} \sum_{j=1}^{n_k} \sup_{t_n \le t < t_{n+1}} \left| \langle \phi_j^{(k)}, X_t \rangle \right|.$$

Since $t_{n+1} - t_n \lesssim n^{-9/10}$ and that $\lambda_k \le \lambda_{\kappa(n)} \le n^{1/5}$ for all $k \le \kappa(n)$, we get that $1 \lesssim e^{-\lambda_k (t_{n+1}-t)}$ for all $t \in [t_n, t_{n+1})$ and $k \le \kappa(n)$. Therefore, using the inequality $e^{\lambda_1 t/2} \lesssim e^{\lambda_1 t_{n+1}/2}$, we get that

$$\sup_{t_n \le t < t_{n+1}} e^{\lambda_1 t/2} \left| \langle T_{t_{n+1}-t} S_f^{(n)} - S_f^{(n)}, X_t \rangle \right|$$
$$\le \sqrt{\|g\|_2} n^{-7/10} \sum_{k=k_0}^{\kappa(n)} e^{-\lambda_k r} \sum_{j=1}^{n_k} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} e^{-\lambda_k (t_{n+1}-t)} \left| \langle \phi_j^{(k)}, X_t \rangle \right|$$
$$= \sqrt{\|g\|_2} n^{-7/10} \sum_{k=k_0}^{\kappa(n)} e^{-\lambda_k r} \sum_{j=1}^{n_k} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right|$$
$$\le \sqrt{\|g\|_2} n^{-7/10} \sum_{k=k_0}^{\infty} e^{-\lambda_k r} \sum_{j=1}^{n_k} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right|.$$

Define $h(r) := \sqrt{\|g\|_2} \sum_{k=k_0}^{\infty} e^{-\lambda_k r/2} \sum_{j=1}^{n_k} 1$, which is finite by (5.3). Then for any $\varepsilon > 0$ and $n > N_r$,

$$\mathbb{P}_x \left(\sup_{t_n \le t < t_{n+1}} e^{\lambda_1 t/2} \left| \langle T_{t_{n+1}-t} S_f^{(n)} - S_f^{(n)}, X_t \rangle \right| > h(r)\varepsilon \right)$$
$$\le \mathbb{P}_x \left(\sqrt{\|g\|_2} n^{-7/10} \sum_{k=k_0}^{\infty} e^{-\lambda_k r/2} \sum_{j=1}^{n_k} e^{-\lambda_k r/2} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right| > h(r)\varepsilon \right)$$
$$\le \mathbb{P}_x \left(\exists\, k \ge k_0,\ 1 \le j \le n_k \text{ such that } n^{-7/10} e^{-\lambda_k r/2} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right| > \varepsilon \right)$$
$$\le \sum_{k=k_0}^{\infty} \sum_{k=1}^{n_k} \mathbb{P}_x \left(n^{-7/10} e^{-\lambda_k r/2} e^{\lambda_1 t_{n+1}/2} \sup_{t_n \le t < t_{n+1}} \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right| > \varepsilon \right).$$

Since $\left\{ \left| \langle T_{t_{n+1}-t} \phi_j^{(k)}, X_t \rangle \right| : t \in [t_n, t_{n+1}) \right\}$ is a submartingale, we get from the L^2 maximal inequality that

$$\mathbb{P}_x \left(\sup_{t_n \le t < t_{n+1}} e^{\lambda_1 t/2} \left| \langle T_{t_{n+1}-t} S_f^{(n)} - S_f^{(n)}, X_t \rangle \right| > h(r)\varepsilon \right)$$
$$\le \frac{4n^{-7/5}}{\varepsilon^2} \sum_{k=k_0}^{\infty} e^{-\lambda_k r} \sum_{k=1}^{n_k} e^{\lambda_1 t_{n+1}} \mathbb{E}_x \left(\langle \phi_j^{(k)}, X_{t_{n+1}} \rangle^2 \right)$$

$$\lesssim_r \frac{n^{-7/5}}{\varepsilon^2}\phi_1 \sum_{k=k_0}^{\infty} e^{-\lambda_k r/2} \sum_{k=1}^{n_k} \left(\|\phi_j^{(k)}\|_2^2 + \|\phi_j^{(k)}\|_4^2 \right),$$

where in the last inequality we used (3.3). Combining $\|\phi_j^{(k)}\|_2 = 1$, (5.3) and (5.5), we get

$$\begin{aligned}
&\mathbb{P}_x \left(\sup_{t_n \le t < t_{n+1}} e^{\lambda_1 t/2} \left| \langle T_{t_{n+1}-t} S_f^{(n)} - S_f^{(n)}, X_t \rangle \right| > h(r)\varepsilon \right) \\
&\lesssim_r \frac{n^{-7/5}}{\varepsilon^2}\phi_1 \sum_{k=k_0}^{\infty} e^{-\lambda_k r/2} \sum_{k=1}^{n_k} \left(\|\phi_j^{(k)}\|_2^2 + e^{\lambda_k r/4} \|\phi_j^{(k)}\|_2 \right) \lesssim \frac{n^{-7/5}}{\varepsilon^2}\phi_1 \sum_{k=k_0}^{\infty} e^{-\lambda_k r/4} \sum_{k=1}^{n_k} 1 \\
&\lesssim_r \frac{n^{-7/5}}{\varepsilon^2}\phi_1.
\end{aligned}$$

Since $n^{-7/5}$ is summable, we get the desired result by the Borel-Cantelli lemma. □

Combining Lemmas 5.2 and 5.3, we immediately get the following corollary.

Corollary 5.4 *Let $f \in \mathbb{T}(E)$ be given by (5.1) for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$. Then under $\mathbb{P}(\cdot|\mathcal{E})$, almost surely,*

$$\lim_{n\to\infty} \sup_{t_n \le t < t_{n+1}} e^{\lambda_1 t/2} \left| \langle T_{t_{n+1}-t} f - f, X_t \rangle \right| = 0.$$

In the small branching case, we have the following decomposition analogous to (4.1):

$$\langle T_{t_n/2} f, X_{t_n/2} \rangle - \langle f, X_{t_n} \rangle = \sum_{i=1}^{M_{t_n/2}} \left[T_{t_n/2} f \left(X_{t_n/2}(i) \right) - \langle f, X^i_{t_n/2} \rangle \right].$$

Define

$$J_t^f := T_t f(x) - \langle f, X_t \rangle, \quad R_t^f(x) := \mathbb{E}_x \left(\left(J_t^f \right)^2 \right). \tag{5.11}$$

By Lemma 3.2(1), we have

$$\lim_{t\to\infty} e^{\lambda_1 t} R_t^f(x) = \sigma_{sm}^2(f)\phi_1(x), \quad \sup_{t>1} e^{\lambda_1 t} R_t^f(x) \lesssim_f \phi_1(x). \tag{5.12}$$

For any $s \ge 0$, combining $J_t^{T_s f} = \mathbb{E}_x \left(J_{t+s}^f \middle| \mathcal{F}_t \right)$, Jensen's inequality and (5.12), we see that

$$\sup_{t>1} e^{\lambda_1 t} R_t^{T_s f}(x) \le \sup_{t>1} e^{\lambda_1 t} \mathbb{E}_x \left(\left(J_{t+s}^f \right)^2 \right) \lesssim_f e^{-\lambda_1 s}\phi_1. \tag{5.13}$$

The following lemma is a modification of Lemma 4.1.

Lemma 5.5 *Let $f \in \mathbb{T}(E)$ be given by (5.1) for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$. Assume either $s_n = 0$ or $s_n = t_{n+1} - t_n$ for all $n \in \mathbb{N}$. Then it holds that*

$$\liminf_{n\to\infty} e^{\lambda_1 t_n} \mathrm{Var}_x \left[\langle T_{s_n} f, X_{t_n} \rangle \middle| \mathcal{F}_{t_n/2} \right] \ge \langle f^2, \phi_1 \rangle W_\infty, \quad \mathbb{P}_x\text{-a.s.} \tag{5.14}$$

and

$$\limsup_{n\to\infty} e^{\lambda_1 t_n} \mathrm{Var}_x \left[\langle T_{s_n} f, X_{t_n} \rangle \middle| \mathcal{F}_{t_n/2} \right] \le \sigma_{sm}^2(f) W_\infty, \quad \mathbb{P}_x\text{-a.s.}$$

Proof: Using conditional independence, we get

$$e^{\lambda_1 t_n}\mathrm{Var}_x\left[\langle T_{s_n}f, X_{t_n}\rangle\Big|\mathcal{F}_{t_n/2}\right] = e^{\lambda_1 t_n}\sum_{i=1}^{M_{t_n/2}} R^{T_{s_n}f}_{t_n/2}\left(X_{t_n/2}(i)\right) = e^{\lambda_1 t_n}\langle R^{T_{s_n}f}_{t_n/2}, X_{t_n/2}\rangle. \quad (5.15)$$

By (3.2), we have

$$R^{T_{s_n}f}_t = \int_0^t T_s\left[A^{(2)}\cdot (T_{t-s+s_n}f)^2\right]\mathrm{d}s + T_t((T_{s_n}f)^2) - (T_{t+s_n}f)^2 \geq T_t((T_{s_n}f)^2) - (T_{t+s_n}f)^2. (5.16)$$

Since $\gamma(\widetilde{(T_{s_n}f)^2}) \geq 2$ and $s_n \in [0,1]$, combining Lemma 3.1 and the fact that $|f| \lesssim_r \phi_1$ for all $f \in \mathbb{T}(E)$, we get that for any $t > 1$,

$$e^{\lambda_2 t}\left|T_t\left(\widetilde{(T_{s_n}f)^2}\right)\right| \lesssim \|(T_{s_n}f)^2\|_2\phi_1 \lesssim_r \phi_1. \quad (5.17)$$

Therefore, by (5.16) and (5.17), applying Lemma 3.1 for $T_{t+s_n}f$ and noticing that ϕ_1 is bounded, we get that there exists a constant $C(f) > 0$ such that for any $t > 1$ and $x \in E$,

$$\begin{aligned}
e^{\lambda_1 t}R^{T_{s_n}f}_t &\geq e^{\lambda_1 t}T_t((T_{s_n}f)^2) - e^{\lambda_1 t}(T_{t+s_n}f)^2\\
&\geq \langle (T_{s_n}f)^2, \phi_1\rangle\phi_1 - e^{\lambda_1 t}\left|T_t\left(\widetilde{(T_{s_n}f)^2}\right)\right| - e^{\lambda_1 t}(T_{t+s_n}f)^2\\
&\geq \left(\langle (T_{s_n}f)^2, \phi_1\rangle - C(f)e^{(\lambda_1-\lambda_2)t} - C(f)e^{-\left(2\lambda_{\gamma(f)}-\lambda_1\right)(t+s_n)}\right)\phi_1,
\end{aligned}$$

which together with (5.15) implies that

$$\begin{aligned}
&e^{\lambda_1 t_n}\mathrm{Var}_x\left[\langle T_{s_n}f, X_{t_n}\rangle\Big|\mathcal{F}_{t_n/2}\right]\\
&\geq \left(\langle (T_{s_n}f)^2, \phi_1\rangle - C(f)e^{(\lambda_1-\lambda_2)t_n/2} - C(f)e^{-\left(2\lambda_{\gamma(f)}-\lambda_1\right)(t_n+s_n)/2}\right)W_{t_n/2}.
\end{aligned}$$

Letting $n \to \infty$ in the inequality above yields (5.14).

For the upper bound, combining (3.2) and (5.13), we get

$$\begin{aligned}
e^{\lambda_1 t}R^{T_{s_n}f}_t(x) &\leq e^{\lambda_1 t}R^f_{t+s_n}(x) \leq e^{\lambda_1 t}\mathbb{E}_x\left(\langle f, X_{t+s_n}\rangle^2\right)\\
&= e^{\lambda_1 t}\int_0^{t+s_n} T_{t+s_n-s}\left[A^{(2)}\cdot (T_s f)^2\right](x)\mathrm{d}s + e^{\lambda_1 t}T_{t+s_n}(f^2)(x).
\end{aligned}$$

Using the fact $s_n \in [0,1]$ and the spectral decomposition similar to (5.17), we get that there exists a constant $C(f)$ such that

$$e^{\lambda_1 t}T_{t+s_n}(f^2)(x) \leq e^{-\lambda_1 s_n}\langle f^2, \phi_1\rangle\phi_1(x) + C(f)e^{-(\lambda_2-\lambda_1)t}\phi_1(x). \quad (5.18)$$

Thus for any $N < t/2$,

$$\begin{aligned}
&e^{\lambda_1 t}\int_N^{t+s_n} T_{t+s_n-s}\left[A^{(2)}\cdot (T_s f)^2\right](x)\mathrm{d}s\\
&\lesssim e^{\lambda_1 t}\int_N^{t+s_n} e^{-2\lambda_{\gamma(f)}s}T_{t+s_n-s}\phi_1(x)\mathrm{d}s \leq e^{-\lambda_1 s_n}\int_N^{\infty} e^{-(2\lambda_{\gamma(f)}-\lambda_1)s}\mathrm{d}s\phi_1(x). \quad (5.19)
\end{aligned}$$

By Lemma 3.1, there exists a constant $C > 0$ such that

$$\begin{aligned}
& e^{\lambda_1 t}\int_0^N T_{t+s_n-s}\left[A^{(2)}\cdot (T_s f)^2\right](x)\mathrm{d}s \\
& \le e^{\lambda_1 t}\int_0^N e^{-\lambda_1(t+s_n-s)}\langle A^{(2)}\cdot (T_s f)^2,\phi_1\rangle \mathrm{d}s\phi_1(x) + Ce^{\lambda_1 t}\int_0^N e^{-\lambda_2(t+s_n-s)}\|A^{(2)}\cdot (T_s f)^2\|_2\mathrm{d}s\phi_1(x) \\
& = \phi_1(x)\left(e^{-\lambda_1 s_n}\int_0^N e^{\lambda_1 s}\langle A^{(2)}\cdot (T_s f)^2,\phi_1\rangle \mathrm{d}s + Ce^{-\lambda_2 s_n}e^{(\lambda_1-\lambda_2)t}\int_0^N e^{\lambda_2 s}\|A^{(2)}\cdot (T_s f)^2\|_2\mathrm{d}s\right).
\end{aligned} \tag{5.20}$$

Therefore, combining (5.18), (5.19) and (5.20), there exists a constant $C'(f) > 0$ such that for all $t > 1, x \in E$,

$$\begin{aligned}
e^{\lambda_1 t}R_t^{T_{s_n} f} \le & \left(e^{-\lambda_1 s_n}\langle f^2,\phi_1\rangle + C'(f)e^{-(\lambda_2-\lambda_1)t} + C'(f)\int_N^\infty e^{-(2\lambda_{\gamma(f)}-\lambda_1)s}\mathrm{d}s\right. \\
& \left. + e^{-\lambda_1 s_n}\int_0^N e^{\lambda_1 s}\langle A^{(2)}\cdot (T_s f)^2,\phi_1\rangle \mathrm{d}s + C'(f)e^{(\lambda_1-\lambda_2)t}\int_0^N e^{\lambda_2 s}\|A^{(2)}\cdot (T_s f)^2\|_2\mathrm{d}s\right)\phi_1,
\end{aligned}$$

which implies that $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,

$$\begin{aligned}
& \limsup_{n\to\infty} e^{\lambda_1 t_n}\mathrm{Var}_x\left[\langle T_{s_n} f, X_{t_n}\rangle\Big|\mathcal{F}_{t_n/2}\right] \\
& \le W_\infty\left(C'(f)\int_N^\infty e^{-(2\lambda_{\gamma(f)}-\lambda_1)s}\mathrm{d}s + \langle f^2,\phi_1\rangle + \int_0^N e^{\lambda_1 s}\langle A^{(2)}\cdot (T_s f)^2,\phi_1\rangle \mathrm{d}s\right) \\
& \overset{N\uparrow+\infty}{\longrightarrow} \sigma_{sm}^2(f)W_\infty.
\end{aligned}$$

The proof is complete. □

Under the Assumption **(H5)**, we have the following useful lemma whose proof is postponed to Section 5.2.

Lemma 5.6 *Suppose in addition that* **(H5)** *holds. If $f \in \mathbb{T}(E)$ is given by* (5.1) *for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$, then*

$$e^{2\lambda_1 t}\mathbb{E}_x\left(\langle f, X_t\rangle^4\right) \lesssim_f \phi_1, \quad t > 0, x \in E.$$

Recall the definition of J_t^f in (5.11). Combining Lemma 5.6, (5.12) and inequalities $x^3 \lesssim x^2 + x^4$ (for $e^{\lambda_1 t/2}|J_t^f|$) and $\mathbb{E}(|X - \mathbb{E}X|^4) \lesssim \mathbb{E}\left(X^4\right)$ (for $X = \langle f, X_t\rangle$), it is easy to get that, for any $t > 1$ and $x \in E$,

$$e^{3\lambda_1 t/2}\mathbb{E}_x\left(|J_t^f|^3\right) = \mathbb{E}_x\left(|e^{\lambda_1 t/2}J_t^f|^3\right) \lesssim e^{\lambda_1 t}R_t^f(x) + e^{2\lambda_1 t}\mathbb{E}_x\left(|J_t^f|^4\right) \lesssim_f \phi_1(x).$$

By Jensen's inequality, we get from the inequality above that

$$e^{3\lambda_1 t/2}\mathbb{E}_x\left(\sup_{s\in[0,1]}|J_t^{T_s f}|^3\right) \le \sup_{s\in[0,1]} e^{3\lambda_1 t/2}\mathbb{E}_x\left(|J_{t+s}^f|^3\right) \lesssim_f \phi_1(x). \tag{5.21}$$

The following result is a modification of Lemmas 4.2 and 4.3.

Lemma 5.7 *Suppose in addition that* **(H5)** *holds. If $f \in \mathbb{T}(E)$ is given by* (5.1) *for some $r > 0$ such that $\lambda_{\gamma(g)} > 0$, then*

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 t_n/2}\left(|\langle f, X_{t_n}\rangle| + \left|\langle T_{t_{n+1}-t_n} f, X_{t_n}\rangle\right|\right)}{\sqrt{2\log(t_n)}} \le 8\sqrt{\sigma_{sm}^2(f)W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Proof: In the following $s_n = 0$ or $s_n = t_{n+1} - t_n$ for all $n \in \mathbb{N}$. Define

$$\Delta_n^{T_{s_n} f} := \sup_{y\in\mathbb{R}} \left| \mathbb{P}_x \left[\frac{\langle T_{s_n} f, X_{t_n}\rangle - \langle T_{t_n/2+s_n} f, X_{t_n/2}\rangle}{\sqrt{\mathrm{Var}_x\left[\langle T_{s_n} f, X_{t_n}\rangle \middle| \mathcal{F}_{t_n/2}\right]}} \le y \middle| \mathcal{F}_{t_n/2}\right] - \Phi(y)\right|.$$

We claim that, $\mathbb{P}_x$-almost surely,

$$1_{\mathcal{E}^c} \sum_{n\ge 0} \Delta_n^{T_{s_n} f} < \infty. \tag{5.22}$$

Combining the branching property and (5.21), we get that, almost surely,

$$\begin{aligned}
&\sum_{n\ge1} e^{3\lambda_1 t_n/2} \sum_{i=1}^{M_{t_n/2}} \mathbb{E}_x\left[\left|T_{t_n/2+s_n} f\left(X_{t_n/2}(i)\right) - \langle T_{s_n} f, X^i_{t_n/2}\rangle\right|^3 \middle| \mathcal{F}_{t_n/2}\right] \\
&= \sum_{n\ge1} e^{3\lambda_1 t_n/2} \langle \mathbb{E}_\cdot\left(\left|J^{T_{s_n} f}_{t_n/2}\right|^3\right), X_{t_n/2}\rangle \\
&\lesssim \sum_{n\ge1} e^{3\lambda_1 t_n/2} e^{-3\lambda_1 t_n/4}\langle \phi_1, X_{t_n/2}\rangle \lesssim \sum_{n\ge1} e^{\lambda_1 t_n/4} = \sum_{n\ge1} e^{\lambda_1 n^{1/10}/4} < \infty,
\end{aligned} \tag{5.23}$$

where in the second inequality we also used the fact that $\sup_{t>0} W_t < \infty$.

It is trivial that $\Delta_n^{T_{s_n} f} \le 2$. Since $\{M_{t_n/2} > 0\} \in \mathcal{F}_{t_n/2}$, by Lemma 3.5, there exists a constant C_1 such that under $\mathbb{P}_x$, on the event $\{M_{t_n/2} > 0\}$,

$$\Delta_n^{T_{s_n} f} \le C_1 \frac{\sum_{i=1}^{M_{t_n/2}} \mathbb{E}_x\left[\left|T_{t_n/2+s_n} f\left(X_{t_n/2}(i)\right) - \langle T_{s_n} f, X^i_{t_n/2}\rangle\right|^3 \middle| \mathcal{F}_{t_n/2}\right]}{\sqrt{\left(\mathrm{Var}_x\left[\langle T_{s_n} f, X_{t_n}\rangle \middle| \mathcal{F}_{t_n/2}\right]\right)^3}}. \tag{5.24}$$

Since $\mathcal{E}^c \subset \{M_{t_n/2} > 0\}$, we see that (5.24) holds on the event $\mathcal{E}^c$. Now suppose Ω_0 is an event with $\mathbb{P}_x(\Omega_0) = 1$ such that, for any $\omega \in \Omega_0$, the conclusion of Lemma 5.5, (5.23) and (5.24) hold. Then for $\omega \in \Omega_0 \cap \mathcal{E}^c$, there exists a large $N = N(\omega)$ such that for $n \ge N$,

$$\mathrm{Var}_x\left[\langle T_{s_n} f, X_{t_n}\rangle \middle| \mathcal{F}_{t_n/2}\right](\omega) \ge \frac{e^{-\lambda_1 t_n}}{2}\langle f^2, \phi_1\rangle W_\infty(\omega) > 0.$$

Together with (5.24), we have that on $\Omega_0 \cap \mathcal{E}^c$,

$$\begin{aligned}
&\sum_{n\ge0} \Delta_n^{T_{s_n} f} \le 2(1+N) \\
&\quad + \frac{C_1\sqrt{8}}{\sqrt{\left[\langle f^2, \phi_1\rangle W_\infty\right]^3}} \sum_{n\ge N} e^{3\lambda_1 t_n/2} \sum_{i=1}^{M_{t_n/2}} \mathbb{E}_x\left[\left|T_{t_n/2+s_n} f\left(X_{t_n/2}(i)\right) - \langle T_{s_n} f, X^i_{t_n/2}\rangle\right|^3 \middle| \mathcal{F}_{t_n/2}\right].
\end{aligned}$$

Combining (5.23) with the inequality above, we arrive at (5.22).

Combining Lemma 3.7 (with $B = \mathcal{E}^c$) and (5.22), we get

$$\limsup_{n\to\infty} \frac{\langle T_{s_n} f, X_{t_n}\rangle - \langle T_{t_n/2+s_n} f, X_{t_n/2}\rangle}{\sqrt{2\log n \mathrm{Var}_x\left[\langle T_{s_n} f, X_{t_n}\rangle \middle| \mathcal{F}_{t_n/2}\right]}} \leq 1, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Noticing that $t_n = n^{1/10}$, it follows from Lemma 5.5 and $\sqrt{10} < 4$ that

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 t_n/2}\left(\langle T_{s_n} f, X_{t_n}\rangle - \langle T_{t_n/2+s_n} f, X_{t_n/2}\rangle\right)}{\sqrt{2\log(t_n)}} \leq 4\sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Since $\lambda_{\gamma(f)} > 0$, by Lemma 3.1, we have that, $\mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)$-almost surely,

$$e^{\lambda_1 t_n/2}\left|\langle T_{t_n/2+s_n} f, X_{t_n/2}\rangle\right| \lesssim_f e^{\lambda_1 t_n/2 - \lambda_{\gamma(f)}(t_n/2+s_n)}\langle \phi_1, X_{t_n/2}\rangle = e^{-\lambda_{\gamma(f)}(t_n/2+s_n)} W_{t_n/2} \to 0,$$

which implies that

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 t_n/2}\langle T_{s_n} f, X_{t_n}\rangle}{\sqrt{2\log(t_n)}} \leq 4\sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Repeating the argument above with f replaced by $-f$, we get

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 t_n/2}\langle -T_{s_n} f, X_{t_n}\rangle}{\sqrt{2\log(t_n)}} \leq 4\sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Combining the two displays above, we arrive at the desired assertion. □

Lemma 5.8 *Suppose in addition that* **(H5)** *holds. If* $f \in \mathbb{T}(E)$ *is given by* (5.1) *for some* $r > 0$ *such that* $\lambda_{\gamma(g)} > 0$*, then*

$$\limsup_{n\to\infty} \sup_{t\in[t_n, t_{n+1})} \frac{e^{\lambda_1 t_n/2}\left|\langle T_{t_{n+1}-t} f, X_t\rangle\right|}{\sqrt{2\log t_n}} \leq 18\sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.}$$

Proof: From Lemma 5.7, we see that

$$\limsup_{n\to\infty} \frac{e^{\lambda_1 t_n/2}\left|\langle T_{t_{n+1}-t_n} f, X_{t_n}\rangle - \langle f, X_{t_{n+1}}\rangle\right|}{\sqrt{2\log(t_n)}} \leq 8\sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x\left(\cdot|\mathcal{E}^c\right)\text{-a.s.} \tag{5.25}$$

Define

$$\varepsilon_n(f) := 10\sqrt{2\sigma_{sm}^2(f) e^{-\lambda_1 t_n}\log(t_n) W_{t_n}}.$$

Set $\mathcal{G}_n = \mathcal{F}_{t_n}$ and $B_n := \left\{\langle T_{t_{n+1}-t_n} f, X_{t_n}\rangle - \langle f, X_{t_{n+1}}\rangle > \varepsilon_n(f)\right\}$, then $B_n \in \mathcal{G}_{n+1}$ for all n. From the second Borel-Cantalli lemma, we get that

$$\begin{aligned}&\left\{\langle T_{t_{n+1}-t_n} f, X_{t_n}\rangle - \langle f, X_{t_{n+1}}\rangle > \varepsilon_n(f), \text{ i.o. }\right\}\\ &= \left\{\sum_{n=1}^\infty \mathbb{P}_x\left(\langle T_{t_{n+1}-t_n} f, X_{t_n}\rangle - \langle f, X_{t_{n+1}}\rangle > \varepsilon_n(f) \middle| \mathcal{F}_{t_n}\right) = \infty\right\}.\end{aligned}$$

By (5.25), on $\mathcal{E}^c$, $\mathbb{P}_x$-almost surely,

$$\sum_{n=1}^{\infty} \mathbb{P}_x \left(\langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle f, X_{t_{n+1}} \rangle > \varepsilon_n(f) \middle| \mathcal{F}_{t_n} \right) < \infty. \tag{5.26}$$

Define

$$\begin{aligned} Z_t(f) &:= \mathbb{E}_x \left[\left(\langle f, X_{t_{n+1}} \rangle - \langle T_{t_{n+1}-t} f, X_t \rangle \right)^2 \middle| \mathcal{F}_t \right], \quad t \in [t_n, t_{n+1}), \\ B_n(f) &:= \sup_{t \in [t_n, t_{n+1})} \left[\langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle T_{t_{n+1}-t} f, X_t \rangle - \sqrt{2 Z_t(f)} \right], \\ T_n(f) &:= \inf \left\{ s \in [t_n, t_{n+1}) : \langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle T_{t_{n+1}-s} f, X_s \rangle - \sqrt{2 Z_s(f)} > \varepsilon_n(f) \right\}. \end{aligned}$$

Similar to (4.36) and (4.37), by the strong Markov property and Markov's inequality, we have

$$\begin{aligned} &\mathbb{P}_x \left(\langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle f, X_{t_{n+1}} \rangle > \varepsilon_n(f) \middle| \mathcal{F}_{t_n} \right) \\ &\geq \mathbb{P}_x \left(\langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle f, X_{t_{n+1}} \rangle > \varepsilon_n(f), T_n(f) < t_{n+1} \middle| \mathcal{F}_{t_n} \right) \\ &\geq \mathbb{P}_x \left(\langle T_{t_{n+1}-T_n(f)} f, X_{T_n(f)} \rangle - \langle f, X_{t_{n+1}} \rangle > -\sqrt{2 Z_{T_n(f)}}, T_n(f) < t_{n+1} \middle| \mathcal{F}_{t_n} \right) \\ &\geq \frac{1}{2} \mathbb{P}_x \left(T_n(f) < t_{n+1} \middle| \mathcal{F}_{t_n} \right) = \frac{1}{2} \mathbb{P}_x \left(B_n(f) > \varepsilon_n(f) \middle| \mathcal{F}_{t_n} \right), \end{aligned} \tag{5.27}$$

where the second inequality follows by the argument of (4.36), and the last inequality follows by the argument of (4.37). Combining (5.26) and (5.27), we get that $\mathbb{P}_x$-almost surely on $\mathcal{E}^c$,

$$\sum_{n=1}^{\infty} \mathbb{P}_x \left(B_n(f) > \varepsilon_n(f) \middle| \mathcal{F}_{t_n} \right) < +\infty.$$

Applying again the second Borel-Cantelli lemma, we get that $\mathbb{P}_x(\cdot|\mathcal{E}^c)$-almost surely,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{t \in [t_n, t_{n+1})} \frac{e^{\lambda_1 t_n/2} \left(\langle T_{t_{n+1}-t_n} f, X_{t_n} \rangle - \langle T_{t_{n+1}-t} f, X_t \rangle \right)}{\sqrt{2 \log(t_n)}} \\ &\leq \limsup_{n\to\infty} \sup_{t \in [t_n, t_{n+1})} \frac{\sqrt{e^{\lambda_1 t_n} Z_t(f)}}{\sqrt{2 \log(t_n)}} + 10 \sqrt{\sigma_{sm}^2(f) W_\infty}. \end{aligned} \tag{5.28}$$

By the display below [24, (2.11)] and the fact that $|f| \lesssim \phi_1$ (see (5.4)), we have

$$\sup_{t>0} e^{\lambda_1 t} Z_t(f) \leq \sup_{t>0} e^{\lambda_1 t} \langle \mathbb{E}_\cdot \left(\langle f, X_{t_{n+1}-t} \rangle^2 \right), X_t \rangle \lesssim \sup_{t>0} e^{\lambda_1 t} \langle \phi_1, X_t \rangle < \infty. \tag{5.29}$$

Combining Lemma 5.7, (5.28) and (5.29), we conclude that

$$-\liminf_{n\to\infty} \inf_{t \in [t_n, t_{n+1})} \frac{e^{\lambda_1 t_n/2} \langle T_{t_{n+1}-t} f, X_t \rangle}{\sqrt{2 \log t_n}} \leq 18 \sqrt{\sigma_{sm}^2(f) W_\infty}, \quad \mathbb{P}_x(\cdot|\mathcal{E}^c)\text{-a.s.}$$

Using a similar argument with f replaced by $-f$, we complete the proof of the lemma. □

Proof of Proposition 5.1: Proposition 5.1 follows from Corollary 5.4 and Lemma 5.8. □

5.2 Proof of Lemma 5.6

Proof of Lemma 5.6: Define $T_t^{(k)} f := \mathbb{E}_x\left(\langle f, X_t\rangle^k\right)$. Then $T_t^{(1)} f = T_t f$ is given by (3.1), and $T_t^{(2)} f = \mathbb{E}_x\left(\langle f, X_t\rangle^2\right)$ is given by (3.2). Recall that for any $\theta > 0$, $\omega_\theta(t,x) := \mathbb{E}_x\left(e^{-\theta\langle f, X_t\rangle}\right)$ solves the equation

$$\omega_\theta(t,x) = \mathbf{E}_x \int_0^t \psi\left(\xi_s, \omega_\theta(t-s,\xi_s)\right) \mathrm{d}s + \mathbf{E}_x\left(e^{-\theta f(\xi_t)}\right).$$

Since $|\psi(x,z+\Delta z) - \psi(x,z)| \lesssim |\Delta z|$ by **(H2) (b)** and that $\left|\frac{\omega_{\theta+\Delta\theta}(t,x)-\omega_\theta(t,x)}{\Delta\theta}\right| \lesssim \mathbb{E}_x\left(\langle f, X_t\rangle e^{-\frac{\theta}{2}\langle f, X_t\rangle}\right)$ for all $t>0, x\in E$ and $0<|\Delta\theta|\le\theta/2$, by the dominated convergence theorem, we can change the order of differentiation and integration and see that $\partial_\theta\omega_\theta$ solves the equation

$$\partial_\theta\omega_\theta(t,x) = \mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta\omega_\theta(t-s,\xi_s)\mathrm{d}s - \mathbf{E}_x\left(f(\xi_t)e^{-\theta f(\xi_t)}\right).$$

Repeating the above procedure and taking derivative with respect to θ in the above equation, we see that $\partial_\theta^2\omega_\theta$ solves the equation

$$\begin{aligned}\partial_\theta^2\omega_\theta(t,x) &= \mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\left(\partial_\theta\omega_\theta(t-s,\xi_s)\right)^2\mathrm{d}s \\ &\quad + \mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta^2\omega_\theta(t-s,\xi_s)\mathrm{d}s + \mathbf{E}_x\left(f(\xi_t)^2e^{-\theta f(\xi_t)}\right).\end{aligned}$$

Again, taking derivative with respect to θ in the above equation, we obtain that

$$\begin{aligned}\partial_\theta^3\omega_\theta(t,x) &= \mathbf{E}_x\int_0^t \partial_z^3\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\left(\partial_\theta\omega_\theta(t-s,\xi_s)\right)^3\mathrm{d}s \\ &\quad + 3\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta^2\omega_\theta(t-s,\xi_s)\partial_\theta\omega_\theta(t-s,\xi_s)\mathrm{d}s \\ &\quad + \mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta^3\omega_\theta(t-s,\xi_s)\mathrm{d}s - \mathbf{E}_x\left(f(\xi_t)^3e^{-\theta f(\xi_t)}\right),\end{aligned} \tag{5.30}$$

and similarly, for $\partial_\theta^4\omega_\theta$, we also have

$$\begin{aligned}\partial_\theta^4\omega_\theta(t,x) &= \mathbf{E}_x\int_0^t \partial_z^4\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\left(\partial_\theta\omega_\theta(t-s,\xi_s)\right)^4\mathrm{d}s \\ &\quad + 6\mathbf{E}_x\int_0^t \partial_z^3\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\left(\partial_\theta\omega_\theta(t-s,\xi_s)\right)^2\partial_\theta^2\omega_\theta(t-s,\xi_s)\mathrm{d}s \\ &\quad + 4\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta^3\omega_\theta(t-s,\xi_s)\partial_\theta\omega_\theta(t-s,\xi_s)\mathrm{d}s \\ &\quad + 3\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\left(\partial_\theta^2\omega_\theta(t-s,\xi_s)\right)^2\mathrm{d}s \\ &\quad + \mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,\omega_\theta(t-s,\xi_s)\right)\partial_\theta^4\omega_\theta(t-s,\xi_s)\mathrm{d}s + \mathbf{E}_x\left(f(\xi_t)^4e^{-\theta f(\xi_t)}\right).\end{aligned} \tag{5.31}$$

Taking $\theta\downarrow 0$ in (5.30) and (5.31), we get that

$$T_t^{(3)}f(x) = \mathbf{E}_x\int_0^t \partial_z^3\psi\left(\xi_s,1\right)\left(T_{t-s}f(\xi_s)\right)^3\mathrm{d}s + 3\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,1\right)T_{t-s}^{(2)}f(\xi_s)T_{t-s}f(\xi_s)\mathrm{d}s$$

$$+\mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,1\right)T^{(3)}_{t-s}f(\xi_s)\mathrm{d}s+\mathbf{E}_x\left(f(\xi_t)^3\right) \tag{5.32}$$

and that

$$\begin{aligned}T^{(4)}_t f(x)&=\mathbf{E}_x\int_0^t \partial_z^4\psi\left(\xi_s,1\right)\left(T_{t-s}f(\xi_s)\right)^4\mathrm{d}s+6\mathbf{E}_x\int_0^t \partial_z^3\psi\left(\xi_s,1\right)\left(T_{t-s}f(\xi_s)\right)^2 T^{(2)}_{t-s}f(\xi_s)\mathrm{d}s\\&+4\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,1\right)T^{(3)}_{t-s}f(\xi_s)T_{t-s}f(\xi_s)\mathrm{d}s+3\mathbf{E}_x\int_0^t \partial_z^2\psi\left(\xi_s,1\right)\left(T^{(2)}_{t-s}f(\xi_s)\right)^2\mathrm{d}s\\&+\mathbf{E}_x\int_0^t \partial_z\psi\left(\xi_s,1\right)T^{(4)}_{t-s}f(\xi_s)\mathrm{d}s+\mathbf{E}_x\left(f(\xi_t)^4\right).\end{aligned} \tag{5.33}$$

Recalling that $A^{(k)}(x)=\partial_z^k\psi(x,1)$ defined in (1.2), we get that (5.32) and (5.33) are respectively equivalent to

$$T^{(3)}_t f=\int_0^t T_{t-s}\left(A^{(3)}\cdot(T_sf)^3\right)\mathrm{d}s+3\int_0^t T_{t-s}\left(A^{(2)}\cdot\left(T^{(2)}_sf\right)T_sf\right)\mathrm{d}s+T_t(f^3)$$

and

$$\begin{aligned}T^{(4)}_t f&=\int_0^t T_{t-s}\left(A^{(4)}\cdot(T_sf)^4\right)\mathrm{d}s+6\int_0^t T_{t-s}\left(A^{(3)}\cdot(T_sf)^2\,T^{(2)}_sf\right)\mathrm{d}s\\&+4\int_0^t T_{t-s}\left(A^{(2)}\cdot T^{(3)}_{t-s}fT_sf\right)\mathrm{d}s+3\int_0^t T_{t-s}\left(A^{(2)}\cdot\left(T^{(2)}_sf\right)^2\right)\mathrm{d}s+T_t(f^4).\end{aligned} \tag{5.34}$$

Using the fact that $\lambda_{\gamma(f)}>0,|f|\lesssim\phi_1$ and the fact that $A^{(k)}$ is bounded for all $1\le k\le 4$ under **(H5)**, we see that for any $t>1$ and $x\in E$,

$$\left|T^{(3)}_t f\right|\lesssim\int_0^t T_{t-s}\left(|T_sf|^3\right)\mathrm{d}s+\int_0^t T_{t-s}\left(\left(T^{(2)}_sf\right)|T_sf|\right)\mathrm{d}s+T_t(\phi_1). \tag{5.35}$$

We claim that the following inequalities hold.

$$|T_sf|\lesssim_f e^{-\lambda_{\gamma(f)}s}\phi_1,\quad T^{(2)}_sf\lesssim_f e^{-\lambda_1 s}\phi_1,\quad s>0,x\in E. \tag{5.36}$$

We treat T_sf first. For $s<1$, using the fact that $|f|\lesssim_f\phi_1$, we have $|T_sf|\lesssim_f T_s\phi_1\lesssim_f e^{-\lambda_{\gamma(f)}s}\phi_1$. For $s>1$, then $|T_sf|\lesssim_f e^{-\lambda_{\gamma(f)}s}\phi_1$ follows from Lemma 3.1. For $T^{(2)}_sf$, similarly, for $s>1$, (5.36) follows from Lemma 3.2(1); for $s<1$, using the fact that W_t is an L^2 bounded martingale, we also have that $T^{(2)}_sf\lesssim_f T^{(2)}_s\phi_1\lesssim\phi_1^2\lesssim e^{-\lambda_1 s}\phi_1$. Therefore, (5.36) holds.

Combining (5.35), (5.36) and the fact that ϕ_1 is bounded, we get that for all $t>0,x\in E$.

$$\begin{aligned}&\left|T^{(3)}_t f\right|\lesssim_f\int_0^t e^{-3\lambda_{\gamma(f)}s}T_{t-s}\left((\phi_1)^3\right)\mathrm{d}s+\int_0^t e^{-\lambda_{\gamma(f)}s}e^{-\lambda_1 s}T_{t-s}\left(\phi_1^2\right)\mathrm{d}s+T_t(\phi_1)\\&\lesssim\left(\int_0^t e^{-3\lambda_{\gamma(f)}s}e^{-\lambda_1(t-s)}\mathrm{d}s+\int_0^t e^{-\lambda_{\gamma(f)}s}e^{-\lambda_1 s}e^{-\lambda_1(t-s)}\mathrm{d}s+e^{-\lambda_1 t}\right)\phi_1\\&\le e^{-\lambda_1 t}\phi_1\left(1+\int_0^\infty e^{-(3\lambda_{\gamma(f)}-\lambda_1)s}\mathrm{d}s+\int_0^\infty e^{-\lambda_{\gamma(f)}s}\mathrm{d}s\right)\lesssim_f e^{-\lambda_1 t}\phi_1.\end{aligned} \tag{5.37}$$

Now we bound $T^{(4)}_t$ from above. Similarly to $T^{(3)}_tf$, combining (5.34), (5.36) and and (5.37),

$$\left|T^{(4)}_t f\right|\lesssim\int_0^t T_{t-s}\left((T_sf)^4\right)\mathrm{d}s+\int_0^t T_{t-s}\left((T_sf)^2\,T^{(2)}_sf\right)\mathrm{d}s$$

$$
\begin{aligned}
&+\int_0^t T_{t-s}\left(\left|T_s^{(3)} f T_s f\right|\right) \mathrm{d}s+\int_0^t T_{t-s}\left(\left(T_s^{(2)} f\right)^2\right) \mathrm{d}s+T_t(f^4) \\
\lesssim_f & \int_0^t e^{-4\lambda_{\gamma(f)} s} T_{t-s}\left(\phi_1\right) \mathrm{d}s+\int_0^t e^{-\left(2\lambda_{\gamma(f)}+\lambda_1\right) s} T_{t-s}\left(\phi_1\right) \mathrm{d}s \\
&+\int_0^t e^{-\left(\lambda_1+\lambda_{\gamma(f)}\right) s} T_{t-s}\left(\phi_1\right) \mathrm{d}s+\int_0^t e^{-2\lambda_1} T_{t-s}\left(\phi_1\right) \mathrm{d}s+T_t(\phi_1) \\
\lesssim & \phi_1 e^{-\lambda_1 t}\left(\int_0^{\infty} e^{-\left(4\lambda_{\gamma(f)}-\lambda_1\right) s} \mathrm{d}s+\int_0^{\infty} e^{-2\lambda_{\gamma(f)} s} \mathrm{d}s+\int_0^{\infty} e^{-\lambda_{\gamma(f)} s} \mathrm{d}s+\int_0^t e^{-\lambda_1 s} \mathrm{d}s+1\right) \\
\lesssim & \phi_1 e^{-2\lambda_1 t},
\end{aligned}
$$

which completes the proof of the lemma. □

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