ASYMPTOTIC BEHAVIORS OF SUBCRITICAL BRANCHING KILLED SPECTRALLY NEGATIVE LÉVY PROCESS

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ABSTRACT. In this paper, we investigate the asymptotic behaviors of the survival probability and maximal displacement of a subcritical branching killed spectrally negative Lévy process X in \mathbb{R} . Let ζ denote the extinction time, M_t be the maximal position of all the particles alive at time t, and $M := \sup_{t \geq 0} M_t$ be the all-time maximum. Under the assumption that the offspring distribution satisfies the $L \log L$ condition and some conditions on the spatial motion, we find the decay rate of the survival probability $\mathbb{P}_x(\zeta > t)$ and the tail behavior of M_t as $t \to \infty$. As a consequence, we establish a Yaglom-type theorem. We also find the asymptotic behavior of $\mathbb{P}_x(M > y)$ as $y \to \infty$.

1. Introduction

1.1. Background and motivation. A branching Lévy process on \mathbb{R} is defined as follows: at time 0, there is a particle at $x \in \mathbb{R}$ and it moves according to a Lévy process (ξ_t, \mathbf{P}_x) on \mathbb{R} . After an exponential time with parameter $\beta > 0$, independent of the spatial motion, this particle dies and is replaced by k offspring with probability p_k , $k \geq 0$. The offspring move independently according to the same Lévy process from their birthplace and obey the same branching mechanism as their parent. This procedure goes on. Let N_t be the set of particles alive at time t and for each t0, we denote by t1, we denote by t2, where t3 is a position of t3 at time t4. Also, for any t4 and t5 and t6 are t7, we use t8, we use t8, where t8 are position of t9 or its ancestor at time t8. Then the point process t9 defined by

$$Z_t := \sum_{u \in N_t} \delta_{X_u(t)}$$

is called a branching Lévy process. We shall denote by \mathbb{P}_x the law of this process when the initial particle starts from x and use \mathbb{E}_x to denote the corresponding expectation. Let

$$\widetilde{\zeta} := \inf\{t > 0 : Z_t(\mathbb{R}) = 0\}$$

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be the extinction time of Z. Note that $\widetilde{\zeta}$ is equal in law to that of the extinction time of a continuous-time Galton-Waston process with the same branching mechanism as the branching Lévy process. Let $m:=\sum_{k=0}^{\infty}kp_k$ be the mean number of offspring. It is well-known that Z will become extinct in finite time with probability 1 if and only if m<1 (subcritical) or m=1 and $p_1\neq 1$ (critical). Moreover, the process Z survives with positive probability when m>1 (supercritical).

In the critical case, i.e. m=1 and $p_1 \neq 1$, Sawyer and Fleischman [15] investigated the tail behavior of the all time maximal position of branching Brownian motion under the assumption that the offspring distribution has finite third moment. For a critical branching random walk with spatial motion having finite $(4 + \varepsilon)$ th moment, the tail behavior of the all time maximum was obtained by Lalley and Shao [12]. Hou et al. [8] studied the asymptotic behavior of the all time maximum of critical branching Lévy processes with offspring distribution belonging to the domain of attraction of an α -stable distribution with $\alpha \in (1, 2]$, under some assumptions on the spatial motion.

In the subcritical case, Profeta [14] gave the asymptotic behavior of the all time maximal position under the assumption that the offspring distribution has finite third moment. For related results about subcritical branching random walks, we refer the reader to [13].

The focus of this paper is on the asymptotic behaviors of a branching killed Lévy process, in which particles are killed upon entering the negative half-line. The point process $Z^0 = (Z_t^0)_{t>0}$ defined by

$$Z_t^0 := \sum_{u \in N_t} 1_{\{\inf_{s \le t} X_u(s) > 0\}} \delta_{X_u(t)}$$

is called a branching killed Lévy process. For any $t \geq 0$, let

$$M_t := \sup_{u \in N_t, \inf_{s \le t} X_u(s) > 0} X_u(t)$$

be the maximal position of all the particles alive at time t in the process Z^0 . We define the all-time maximum position and the extinction time of Z^0 by

$$M := \sup_{t \ge 0} M_t, \qquad \zeta := \inf\{t > 0 : Z_t^0((0, \infty)) = 0\}.$$

In the critical case, i.e., when m=1 and $p_1\neq 1$, the asymptotic behaviors of the tails of the extinction time and the maximal displacement of Z^0 were established in [9] under the assumption that the offspring distribution belongs to the domain of attraction of an α -stable distribution, $\alpha\in(1,2]$, and some moment assumptions on the spatial motion. They also proved that the scaling limit under $\mathbb{P}_{\sqrt{t}y}(\cdot|\zeta>t)$ can be represented in terms of a super killed Brownian motion. In the subcritical case, i.e., $m\in(0,1)$, under the assumption $\sum_{k=1}^{\infty}k(\log k)p_k<\infty$, the asymptotic behaviors of the survival probability and the all time maximal position of branching killed Brownian motion with drift were established in [10] recently.

The purpose of this paper is to extend the results of [10] to subcritical branching killed spectrally negative Lévy processes. This extension is quite challenging since properties of Brownian motion were used crucially in [10]. Fluctuation theory of spectrally negative

Lévy processes will play an important role in this paper. Another important tool is the conditioned limit theorem in Theorem 3.5 below.

1.2. Main results. Before we state our main results, we introduce some notation and some basic results on spectrally negative Lévy processes. We always assume that the offspring distribution is subcritical, i.e., $m \in (0,1)$. Let $\alpha := \beta(1-m)$ and let f be the generating function of the offspring distribution, i.e. $f(s) = \sum_{k=0}^{\infty} p_k s^k$, $s \in [0,1]$. Define

(1.1)
$$\Phi(u) := \beta \left(f(1-u) - (1-u) \right) =: (\alpha + \varphi(u)) u, \quad u \in [0,1],$$

where $\varphi(u) = \frac{\Phi(u) - \alpha u}{u}$ for $u \in (0, 1]$ and $\varphi(0) = \Phi'(0+) - \alpha = 0$. According to [10, Lemma 2.7], $\varphi(\cdot)$ is increasing on [0, 1] and under the condition

(1.2)
$$\sum_{k=1}^{\infty} k(\log k) p_k < \infty,$$

it holds that

(1.3)
$$\int_0^\infty \varphi\left(e^{-ct}\right) dt < \infty, \quad \text{for any } c > 0.$$

Moreover, it is well-known (see Theorem 2.4 in [2, p.121]) that

(1.4)
$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_0(\widetilde{\zeta} > t) = C_{sub} \in (0, \infty)$$

holds if and only if (1.2) holds. For any t > 0, define

$$g(t) := \mathbb{P}_0(\widetilde{\zeta} > t).$$

It is well-known that g(t) satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = -\Phi(g(t)) = -\left(\alpha + \varphi(g(t))\right)g(t),$$

thus

(1.5)
$$e^{\alpha t}g(t) = \exp\left\{-\int_0^t \varphi(g(s))ds\right\}.$$

It follows from (1.4) that

(1.6)
$$C_{sub} = \exp\left\{-\int_0^\infty \varphi(g(s)) ds\right\}.$$

Therefore, (1.2) is equivalent to

$$\int_0^\infty \varphi(g(s))\mathrm{d} s < \infty.$$

In this paper, we always assume that $\xi = ((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a Lévy process on \mathbb{R} and we will use \mathbf{E}_x to denote expectation with respect to \mathbf{P}_x . For any $z\in\mathbb{R}$, define

$$\tau_z^+ := \inf\{t > 0 : \xi_t \ge z\}$$
 and $\tau_z^- := \inf\{t > 0 : \xi_t < z\}.$

It follows from [9, Lemma 2.8] that, if $\mathbf{E}_0(\xi_1) = 0$ and $\mathbf{E}_0(\xi_1^2) \in (0, \infty)$, then $\mathbf{E}_x |\xi_{\tau_0^-}| < \infty$ and the function

(1.7)
$$R(x) := x - \mathbf{E}_x \left(\xi_{\tau_0^-} \right) = -\mathbf{E}_0 \left(\xi_{\tau_{-x}^-} \right), \quad x \ge 0,$$

satisfies the following:

- (1) $R(x) \ge x$ and R(x) is non-decreasing in x;
- (2) there exists a constant c > 0 such that $R(x) \le c(1+x)$ and

$$\lim_{x \to \infty} \frac{R(x)}{x} = 1 - \lim_{x \to \infty} \frac{\mathbf{E}_x(\xi_{\tau_0^-})}{x} = 1;$$

(3) $\left(R(\xi_s)1_{\{\tau_0^->s\}}\right)_{s>0}$ is a \mathbf{P}_x -martingale for any x>0.

In the case ξ is a Brownian motion with drift, it is obvious that

$$(1.8) R(x) = x, \quad x > 0.$$

Now we introduce some basics on spectrally negative Lévy processes. Let ξ be a spectrally negative Lévy process on \mathbb{R} . For any $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) \geq 0$, the Laplace exponent of ξ is given by

$$\Psi(\lambda) = \ln \mathbf{E}_0 \left[e^{\lambda \xi_1} \right] = a\lambda + \frac{\eta^2}{2} \lambda^2 + \int_{-\infty}^0 \left(e^{\lambda x} - 1 - \lambda x \mathbf{1}_{\{|x| < 1\}} \right) \Pi(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\eta \geq 0$ and the Lévy measure Π satisfies $\int_{-\infty}^{0} (1 \wedge x^2) \Pi(\mathrm{d}x) < +\infty$. It is well known that

$$\mathbf{E}_0[\xi_1] = \Psi'(0+) = a + \int_{(-\infty, -1]} x \Pi(\mathrm{d}x) \in [-\infty, \infty).$$

In some results, we will assume that ξ satisfies one or both of the following conditions:

- **(H1)** There exists $\delta \in (0,1)$ such that $\mathbf{E}_x(|\xi_1|^{2+\delta}) < \infty$.
- (H2) The law of ξ_1 is non-lattice.

For $q \geq 0$, let

$$\psi(q) := \sup\{\lambda \ge 0 : \Psi(\lambda) = q\}$$

be the right inverse of Ψ . By Kyprianou [11, Theorem 8.1], for any $q \geq 0$, there exists a scale function $W^{(q)}: \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for x < 0 and $W^{(q)}$ is a strictly increasing and continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-rx} W^{(q)}(x) \mathrm{d}x = \frac{1}{\Psi(r) - q}, \quad \text{for } r > \psi(q).$$

In the case when ξ is a standard Brownian motion with drift -b, by using tables of Laplace transforms, one can easily get that

(1.9)
$$W^{(q)}(x) = \frac{2e^{bx}}{\sqrt{b^2 + 2q}} \sinh(\sqrt{b^2 + 2q}x), \quad x \ge 0, \ q \ge 0.$$

For any $c \ge 0$ and $x \in \mathbb{R}$, since $\{e^{c(\xi_t - x) - \Psi(c)t} : t \ge 0\}$ is a \mathbf{P}_x -martingale, we can define the change of measure

(1.10)
$$\frac{\mathrm{d}\mathbf{P}_x^c}{\mathrm{d}\mathbf{P}_x}\Big|_{\mathcal{F}_t} = e^{c(\xi_t - x) - \Psi(c)t},$$

where $\mathcal{F}_t := \sigma\{\xi_s : s \leq t\}$, $t \geq 0$. According to [11, Theorem 3.9], $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ is also a spectrally negative Lévy process and its Laplace exponent $\Psi_c(\lambda)$ is given by $\Psi_c(\lambda) := \Psi(\lambda + c) - \Psi(c)$. We will use \mathbf{E}_x^c to denote expectation with respect to \mathbf{P}_x^c . Let $W_c^{(q)}$ be the scale function of $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$.

According to [11, Theorem 3.9], the Lévy measure of $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ is given by $1_{(-\infty,0)}e^{cx}\Pi(\mathrm{d}x)$. Thus, for any $p\geq 1$, if $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ has finite p-th moment, then $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ also has finite p-th moment. In case c>0, $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ will have finite exponential moment. It is also easy to see that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ is non-lattice if and only if $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is non-lattice. In particular, if $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ satisfies $(\mathbf{H1})$ (or $(\mathbf{H2})$), then $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ also satisfies $(\mathbf{H1})$ (or $(\mathbf{H2})$). It follows from [11, Theorem 3.9] that, if $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a Brownian motion with drift, $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$ is also a Brownian motion with drift.

When $\Psi'(0+) < 0$, we have $\psi(0) > 0$. In this case, Ψ is strictly convex and admits a unique minimum at a $\lambda_* > 0$ and $\Psi(\lambda_*) < 0$, $\Psi'(\lambda_*) = 0$ and $\Psi''(\lambda_*) > 0$. Consequently, the function ψ is well-defined on $[-q_*, +\infty)$, where $-q_* = \Psi(\lambda_*)$. Taking $c = \lambda_*$, we define the change of measure,

(1.11)
$$\frac{\mathrm{d}\mathbf{P}_x^{\lambda_*}}{\mathrm{d}\mathbf{P}_x}\Big|_{\mathcal{F}_t} = e^{\lambda_*(\xi_t - x) - \Psi(\lambda_*)t}.$$

Then $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^{\lambda_*})_{x\in\mathbb{R}})$ is a spectrally negative Lévy process and its Laplace exponent is given by $\Psi_{\lambda_*}(\lambda) := \Psi(\lambda + \lambda_*) - \Psi(\lambda_*)$. It is easy to see that $\Psi'_{\lambda_*}(0+) = \Psi'(\lambda_*) = 0$. Let $\mathbf{E}_x^{\lambda_*}$ be the expectation with respect to $\mathbf{P}_x^{\lambda_*}$.

Suppose that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ has finite second moment, then $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^{\lambda_*})_{x\in\mathbb{R}})$ also has finite second moment. Since $\mathbf{E}_0^{\lambda_*}(\xi_1) = \Psi'_{\lambda_*}(0+) = 0$, by [9, Lemma 2.8], we have $\mathbf{E}_x^{\lambda_*}|\xi_{\tau_0^-}| < \infty$. Define

(1.12)
$$R^*(x) = x - \mathbf{E}_x^{\lambda_*} \left(\xi_{\tau_0^-} \right), \quad x \ge 0.$$

Define the dual process of ξ by:

$$\widehat{\xi}_s := -\xi_s, \quad s \ge 0.$$

For any $z \in \mathbb{R}$, we define $\widehat{\tau}_z^- := \inf\{s > 0 : \widehat{\xi}_s < z\}$ and

(1.13)
$$\widehat{R}^*(x) := x - \mathbf{E}_x^{\lambda_*} \left(\widehat{\xi}_{\widehat{\tau}_0^-}\right), \quad x \ge 0.$$

Denote $\mathbb{R}_+ = [0, \infty)$. Our first main result is on the large-time asymptotics of the survival probability.

Theorem 1.1. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**. Let x>0.

(1) If $\mathbf{E}_0[\xi_1] = 0$, then

$$\lim_{t \to \infty} \sqrt{t} e^{\alpha t} \mathbb{P}_x(\zeta > t) = \frac{2C_{sub}R(x)}{\sqrt{2\pi\sigma^2}},$$

where C_{sub} is defined in (1.6) and R(x) in (1.7).

(2) If $\mathbf{E}_0[\xi_1] > 0$, then

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_x(\zeta > t) = C_{sub} \mathbf{E}_0[\xi_1] W^{(0)}(x).$$

(3) Assume further that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ satisfies **(H2)**. If $\mathbf{E}_0[\xi_1] < 0$, then

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} \mathbb{P}_x(\zeta > t) = \frac{2C_0 R^*(x) e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}},$$

where $C_0 := \lim_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(\zeta > N) e^{-\lambda_* z} \widehat{R}^*(z) dz \in (0, \infty), R^*$ is defined in (1.12) and \widehat{R}^* in (1.13).

Remark 1. Combining the result above with (1.8) and (1.9), we immediately recover the first two conclusions of [10, Theorem 1.1] as a corollary. Furthermore, when ξ is a standard Brownian motion with drift $-\rho$, we have $\Psi(\lambda) = -\rho\lambda + \frac{1}{2}\lambda^2$ and $\lambda_* = \rho$. When $\rho > 0$, a straightforward calculation yields that

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha + \frac{\rho^2}{2})t} \mathbb{P}_x(\zeta > t) = \frac{2C_0 x e^{\rho x}}{\sqrt{2\pi}}.$$

This result is consistent with [10, Theorem 1.1, (iii)], where $C_0(\rho) = C_0$.

Our second main result is on the asymptotic behavior of the tail probability of M_t .

Theorem 1.2. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1). Let x>0.

(1) If $\mathbf{E}_0[\xi_1] = 0$, then for any $y \geq 0$, we have

$$\lim_{t \to \infty} \sqrt{t} e^{\alpha t} \mathbb{P}_x \left(M_t > \sqrt{t} y \right) = \frac{2C_{sub} R(x)}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}.$$

(2) If $\mathbf{E}_0[\xi_1] > 0$, then for any $y \in \mathbb{R}$, we have

$$\lim_{t \to \infty} e^{\alpha t} \mathbb{P}_x \left(M_t > \sqrt{t} y + \mathbf{E}_0 \left[\xi_1 \right] t \right) = \frac{C_{sub} \mathbf{E}_0 \left[\xi_1 \right] W^{(0)}(x)}{\sqrt{2\pi}} \int_{\underline{u}}^{\infty} e^{-\frac{z^2}{2}} dz.$$

(3) Assume further that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ satisfies **(H2)**. If $\mathbf{E}_0[\xi_1] < 0$, then for any $y \geq 0$, we have

$$\lim_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} \mathbb{P}_x (M_t > y) = \frac{2C_1(y)R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}},$$

where $C_1(y) := \lim_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_* z} \widehat{R}^*(z) dz \in (0, \infty).$

Noticing that $\mathbb{P}_z(M_N > 0) = \mathbb{P}_z(\zeta > N)$ for z > 0, C_0 in Theorem 1.1 and $C_1(0)$ in Theorem 1.2 are the same. Combining the result above with (1.8) and (1.9), we immediately recover [10, Theorem 1.3] as a corollary.

Combining Theorems 1.1 and 1.2, we immediately get the following Yaglom-type conditional limit theorem.

Corollary 1.3. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1). Let x>0.

(1) If $\mathbf{E}_0[\xi_1] = 0$, then we have

$$\mathbb{P}_x\left(\frac{M_t}{\sqrt{t}}\in\cdot\Big|\zeta>t\right)\quad\overset{\mathrm{d}}{\Longrightarrow}\quad\mathcal{R}(\cdot),$$

where \mathcal{R} is the Rayleigh distribution with density $\rho(z) = ze^{-z^2/2} \mathbb{1}_{\{z>0\}}$.

(2) If $\mathbf{E}_0[\xi_1] > 0$, then we have

$$\mathbb{P}_x\left(\frac{M_t - \mathbf{E}_0\left[\xi_1\right]t}{\sqrt{t}} \in \cdot \middle| \zeta > t\right) \quad \stackrel{\mathrm{d}}{\Longrightarrow} \quad N(0, \sigma^2),$$

where $N(0, \sigma^2)$ is normal distribution with mean 0 and variance σ^2 .

(3) Assume further that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ satisfies **(H2)**. If $\mathbf{E}_0[\xi_1] < 0$, then there exists a random variable (X, \mathbb{P}) whose law is independent of x such that

$$\mathbb{P}_x\left(M_t \in \cdot \middle| \zeta > t\right) \quad \stackrel{\mathrm{d}}{\Longrightarrow} \quad \mathbb{P}(X \in \cdot).$$

Our third main result is on the asymptotic behavior of the all-time maximum M. Note that we do not need any assumption on the spectrally Lévy process in the following result.

Theorem 1.4. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process. There exits a constant $C_2(\alpha) \in (0,1]$ such that for any x>0,

$$\lim_{y \to \infty} e^{\psi(\alpha)y} \mathbb{P}_x(M > y) = C_2(\alpha) W^{(\alpha)}(x) \Psi'(\psi(\alpha)),$$

where $W^{(\alpha)}$ is the scale function of $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$.

Combining the result above with (1.9), we immediately recover [10, Theorem 1.2] as a corollary.

Organization of the paper: The rest of the paper is organized as follows. In Section 2, we give some results on spectrally negative Lévy processes which will be used in the proofs of our main results. We establish the conditioned limit theorem for Lévy process in Section 3. The proofs of Theorems 1.1 and 1.2 are given in Section 4, and the proof of Theorem 1.4 is given in Section 5.

In this paper, we use $\phi(\cdot)$ to denote the standard normal density, i.e., $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$, use $\rho(\cdot)$ to denote the Rayleigh density, i.e., $\rho(x) = xe^{-x^2/2}1_{\{x>0\}}$, and use $\mathcal{R}(x)$ to denote the Rayleigh distribution function, i.e., $\mathcal{R}(x) = (1 - e^{-x^2/2})1_{\{x\geq0\}}$. For v>0, we define $\phi_v(x) = \frac{1}{\sqrt{2\pi v}}e^{-x^2/(2v)}$ and $\rho_v(x) = (x/v)e^{-x^2/(2v)}1_{\{x>0\}}$. We use $F(x) \sim G(x)$ as $x \to \infty$ to denote $\lim_{x\to\infty} F(x)/G(x) = 1$. In this paper, capital letters C_i and T_i , $i=1,2,\ldots$, are used to denote constants in the statements of results and their value remain the same

throughout the paper. Lower case letters c_i , i = 1, 2, ..., are used for constants used in the proofs and their labeling starts anew in each proof. $c_i(\epsilon)$ and $C_i(\epsilon)$ mean that the constants c_i and C_i depend on ϵ .

2. Preliminaries

In this section, we always assume that $\xi = ((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process with Laplace exponent Ψ . Recall that $\psi(q) := \sup\{\lambda \geq 0 : \Psi(\lambda) = q\}$ is the right inverse of Ψ . It is well known, see [11, Section 8], that for any $q \geq 0$,

$$\mathbf{E}_0\left(e^{-q\tau_x^+}1_{\{\tau_x^+<\infty\}}\right) = e^{-\psi(q)x}.$$

Moreover, since ξ is spectrally negative, we have

$$\mathbf{E}_0(\xi_{\tau_x^+} = x | \tau_x^+ < \infty) = 1.$$

The following result on exit probabilities is contained in [11, Theorem 8.1].

Theorem 2.1. (1) For any x > 0, if $\Psi'(0+) > 0$,

$$\mathbf{P}_x(\tau_0^- < \infty) = 1 - \Psi'(0+)W^{(0)}(x),$$

and if $\Psi'(0+) \leq 0$, $\mathbf{P}_x(\tau_0^- < \infty) = 1$.

(2) For any $0 < x \le y$ and $q \ge 0$,

$$\mathbf{E}_x \left(e^{-q\tau_y^+} 1_{\{\tau_0^- > \tau_y^+\}} \right) = \frac{W^{(q)}(x)}{W^{(q)}(y)}.$$

The following result, which can be found in [11, Lemma 8.4] and [16, Proposition 1], gives the relationship between $W_c^{(q)}$ for different values of q, c, and the asymptotic behavior of $W^{(q)}(x)$ as $x \to \infty$.

Lemma 2.2. For any $x \geq 0$, the function $q \mapsto W^{(q)}(x)$ may be analytically extended to $q \in \mathbb{C}$. Furthermore, for any $q \in \mathbb{C}$ and $c \in \mathbb{R}$ with $\Psi(c) < \infty$, we have

$$W^{(q)}(x) = e^{cx} W_c^{(q-\Psi(c))}(x), \quad x \ge 0,$$

where $W_c^{(q-\Psi(c))}$ is the scale function of $((\xi_t)_{t\geq 0}, (\mathbf{P}_x^c)_{x\in\mathbb{R}})$. Furthermore,

(2.1)
$$W^{(q)}(x) \sim \frac{e^{\psi(q)x}}{\Psi'(\psi(q))}, \quad as \ x \to \infty.$$

The following lemma gives the joint asymptotic behavior of the tail of τ_0^- and the Lévy process ξ when $\mathbf{E}_0[\xi_1] > 0$.

Lemma 2.3. Assume that $\mathbf{E}_0(\xi_1) > 0$ and $\sigma^2 := \mathbf{E}_0(\xi_1^2) < \infty$, then for any x > 0,

(2.2)
$$\lim_{t \to \infty} \mathbf{P}_x(\tau_0^- > t) = \mathbf{P}_x(\tau_0^- = \infty) = \Psi'(0+)W^{(0)}(x).$$

Moreover, for any $y \in \mathbb{R}$, we have

$$\lim_{t \to \infty} \mathbf{P}_x \left(\tau_0^- > t, \xi_t - \mathbf{E}_0 \left[\xi_1 \right] t > \sqrt{t} y \right) = \mathbf{P}_x (\tau_0^- = \infty) \int_{\frac{y}{z}}^{\infty} \phi(z) dz.$$

Proof. Note that (2.2) follows immediately from Theorem 2.1 (1). Fix t > 0, for $m \in (0, t)$, by the Markov property,

$$\mathbf{P}_{x}\left(\tau_{0}^{-} > t, \xi_{t} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right) \leq \mathbf{P}_{x}\left(\tau_{0}^{-} > m, \xi_{t} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right)$$

$$= \mathbf{E}_{x}\left[1_{\left\{\tau_{0}^{-} > m\right\}}\mathbf{P}_{\xi_{m}}\left(\xi_{t-m} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right)\right].$$

By the central limit theorem, for any z, as $t \to \infty$, we get

(2.3)
$$\mathbf{P}_{z}\left(\xi_{t-m} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right) \to \int_{y}^{\infty} \phi_{\sigma^{2}}(z)dz.$$

Letting $t \to \infty$ first, then $m \to \infty$, we get that

(2.4)
$$\limsup_{t \to \infty} \mathbf{P}_x \left(\tau_0^- > t, \xi_t - \mathbf{E}_0 \left[\xi_1 \right] t > \sqrt{t} y \right) \le \mathbf{P}_x \left(\tau_0^- = \infty \right) \int_u^\infty \phi_{\sigma^2}(z) \mathrm{d}z.$$

On the other hand, we have

$$\mathbf{P}_{x}\left(\tau_{0}^{-} > m, \xi_{t} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right) \leq \mathbf{P}_{x}\left(\tau_{0}^{-} > t, \xi_{t} - \mathbf{E}_{0}\left[\xi_{1}\right]t > \sqrt{t}y\right) + \mathbf{P}_{x}\left(\tau_{0}^{-} \in (m, t]\right).$$

It follows from (2.3) that

$$\lim_{m \to \infty} \lim_{t \to \infty} \mathbf{P}_x \left(\tau_0^- > m, \xi_t - \mathbf{E}_0 \left[\xi_1 \right] t > \sqrt{t} y \right) = \mathbf{P}_x \left(\tau_0^- = \infty \right) \int_y^\infty \phi_{\sigma^2}(z) dz,$$

this combined with

$$\lim_{m \to \infty} \lim_{t \to \infty} \mathbf{P}_x \left(\tau_0^- \in (m, t] \right) = \lim_{m \to \infty} \mathbf{P}_x \left(\tau_0^- \in (m, \infty) \right) = 0$$

yields that

$$\liminf_{t \to \infty} \mathbf{P}_x \left(\tau_0^- > t, \xi_t - \mathbf{E}_0 \left[\xi_1 \right] t > \sqrt{t} y \right) \ge \mathbf{P}_x \left(\tau_0^- = \infty \right) \int_y^\infty \phi_{\sigma^2}(z) \mathrm{d}z.$$

Combining this with (2.4), we get the desired result.

The following lemma is an important tool for proving Theorem 1.4.

Lemma 2.4. For any a > 0, $0 < x \le y$ and nonnegative Borel function h, we have

$$\mathbf{E}_{x} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-a\tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right] = e^{\psi(a)(x-y)} \mathbf{E}_{x}^{\psi(a)} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right].$$

Proof. By [3, Theorem 6], $\{\tau_y^+ < \tau_0^-\} \cap \{\tau_y^+ < t\} = \{\tau_y^+ \wedge t < \tau_0^-\} \cap \{\tau_y^+ \wedge t < t\}$ is $\mathcal{F}_{\tau_y^+ \wedge t}$ -measurable. For a > 0, since $e^{-a\tau_y^+} 1_{\{\tau_y^+ = \infty\}} = 0$, using (1.10) with $c = \psi(a)$, we have

$$(2.5) \quad \mathbf{E}_{x} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-a\tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right] = \lim_{t \to \infty} \mathbf{E}_{x} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}, \tau_{y}^{+} < t\}} e^{-a\tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right]$$
$$= \lim_{t \to \infty} \mathbf{E}_{x}^{\psi(a)} \left[e^{-\psi(a)(\xi_{t} - x) + at} e^{-a\tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} 1_{\{\tau_{y}^{+} < \tau_{0}^{-}, \tau_{y}^{+} < t\}} \right]$$

$$= \lim_{t \to \infty} \mathbf{E}_x^{\psi(a)} \left[e^{-a\tau_y^+ - \int_0^{\tau_y^+} h(\xi_s) \mathrm{d}s} 1_{\{\tau_y^+ < \tau_0^-, \tau_y^+ < t\}} \mathbf{E}_x^{\psi(a)} \left[e^{-\psi(a)(\xi_t - x) + at} \middle| \mathcal{F}_{\tau_y^+ \wedge t} \right] \right].$$

Note that $(e^{-\psi(a)(\xi_t-x)+at})_{t\geq 0}$ is a $\mathbf{P}_x^{\psi(a)}$ is a martingale with respect to \mathcal{F}_t . Using the optional stopping theorem and the absence of positive jumps, we get that, on $\{\tau_u^+ < t\}$,

$$\mathbf{E}_{x}^{\psi(a)} \left[e^{-\psi(a)(\xi_{t}-x)+at} \middle| \mathcal{F}_{\tau_{y}^{+} \wedge t} \right] = e^{-\psi(a)(\xi_{\tau_{y}^{+} \wedge t}-x)+a(\tau_{y}^{+} \wedge t)} = e^{-\psi(a)(y-x)+a\tau_{y}^{+}}.$$

Combining this with (2.5) and using the fact that $\mathbf{P}_x^{\psi(a)}(\tau_y^+ < \infty) = 1$, we get

$$\mathbf{E}_{x} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-a\tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right] = e^{\psi(a)(x-y)} \mathbf{E}_{x}^{\psi(a)} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} h(\xi_{s}) ds} \right].$$

This gives the desired result.

3. Conditioned limit theorems for Lévy processes

The purpose of this section is to prove Theorem 3.5, a conditioned limit theorem for spectrally negative Lévy processes. Theorem 3.5 will play an important role in this paper. We make some preparations first. The following result follows from [9, Lemmas 2.12 and 4.1].

Lemma 3.1. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$ and $(\mathbf{H1})$. Then for any x > 0 and $a \in (0, \infty]$, it holds that

$$\lim_{t \to \infty} \sqrt{t} \mathbf{P}_x \left(\xi_t \le a \sqrt{t}, \tau_0^- > t \right) = \frac{2R(x)}{\sqrt{2\pi\sigma^2}} \int_0^{\frac{a}{\sigma}} \rho(z) \mathrm{d}z,$$

where $\sigma^2 := \mathbf{E}_0(\xi_1^2)$. Furthermore, for any x > 0 and any bounded continuous function h on $(0, \infty)$, it holds that

$$\lim_{t \to \infty} \sqrt{t} \mathbf{E}_x \left[h \left(\frac{\xi_t}{\sigma \sqrt{t}} \right) \mathbf{1}_{\{\tau_0^- > t\}} \right] = \frac{2R(x)}{\sqrt{2\pi\sigma^2}} \int_0^\infty \rho(z) h(z) dz.$$

Recall that δ is the constant in **(H1)**. The following result is [9, Lemma 2.11].

Lemma 3.2. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$ and **(H1)**. Then there exists a Brownian motion W with diffusion coefficient σ^2 starting from the origin such that for any $\kappa \in (0, \frac{\delta}{2(2+\delta)})$, there exists a constant $C_3(\kappa) > 1$ such that for all $t \geq 1$,

$$\mathbf{P}_{x} \left(\sup_{s \in [0,1]} |\xi_{ts} - x - W_{ts}| > t^{\frac{1}{2} - \kappa} \right) \le \frac{C_{3}(\kappa)}{t^{(\frac{1}{2} - \kappa)(\delta + 2) - 1}}.$$

The following result is a conditioned limit theorem for Lévy processes with explicit rate of convergence. See [7, Theorem 2.7] for an analogous result for random walks.

Lemma 3.3. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a Lévy process satisfying $\mathbf{E}_0(\xi_1) = 0$, $\mathbf{E}_0(\xi_1^2) = \sigma^2$ and **(H1)**. Then one can find a constant $\varepsilon_0 \in (0, \frac{\delta}{4(2+\delta)})$ with the property

that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_0(\varepsilon)$ and $C_4(\varepsilon)$ such that for any x, y > 0 and $t > T_0(\varepsilon)$,

$$\left| \mathbf{P}_x \left(\frac{\xi_t}{\sigma \sqrt{t}} \le y, \tau_0^- > t \right) - \frac{2R(x)}{\sigma \sqrt{2\pi t}} \mathcal{R}(y) \right| \le \frac{C_4(\varepsilon)(1+x)}{t^{1/2+\varepsilon}}.$$

Proof. Let W be the Brownian motion in Lemma 3.2. For any r > 0 and $\epsilon \in (0, \delta/(2(5 + \epsilon)))$ (2δ)), define

$$A_r := \Big\{ \sup_{s \in [0,1]} |\xi_{sr} - \xi_0 - W_{sr}| > r^{\frac{1}{2} - 2\epsilon} \Big\}.$$

Let $(S_n)_{n\geq 0}$ be the random walk defined by $S_n:=\xi_n, n\in\mathbb{N}$. For any $b\in\mathbb{R}$, define

$$\tau_b^{S,+} := \inf\{j \in \mathbb{N}, |S_j| > b\}.$$

By the Markov property, we have the following decomposition:

$$\mathbf{P}_x \left(\frac{\xi_t}{\sigma \sqrt{t}} \le y, \tau_0^- > t \right) = \sum_{k=1}^4 I_k,$$

where

$$I_1 := \mathbf{P}_x \left(\frac{\xi_t}{\sigma \sqrt{t}} \le y, \tau_0^- > t, \tau_{t^{1/2 - \epsilon}}^{S,+} > [t^{1 - \epsilon}] \right),$$

$$I_2 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x \left(\mathbf{P}_{\xi_k} \left(\frac{\xi_{t-k}}{\sigma \sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k} \right); \tau_0^- > k, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right),$$

$$I_3 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x \left(\mathbf{P}_{\xi_k} \left(\frac{\xi_{t-k}}{\sigma \sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c \right); \tau_0^- > k, \xi_k > t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right),$$

$$I_4 := \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x \left(\mathbf{P}_{\xi_k} \left(\frac{\xi_{t-k}}{\sigma \sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c \right); \tau_0^- > k, \xi_k \le t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right).$$

We now deal with I_i , i=1,2,3,4, separately. (i) Upper bound of I_1 . Set $K:=[t^{\epsilon}-1]$ and $l:=[t^{1-2\epsilon}]$. Since $Kl \leq [t^{1-\epsilon}]$, we have

(3.1)
$$I_{1} \leq \mathbf{P}_{x} \left(\tau_{t^{1/2 - \epsilon}}^{S,+} > [t^{1 - \epsilon}] \right) \leq \mathbf{P}_{0} \left(\max_{1 \leq j \leq Kl} |x + S_{j}| \leq t^{1/2 - \epsilon} \right)$$
$$\leq \mathbf{P}_{0} \left(\max_{1 \leq j \leq K} |x + S_{lj}| \leq t^{1/2 - \epsilon} \right), \quad x > 0.$$

By the Markov property, we have

$$(3.2) \mathbf{P}_0 \left(\max_{1 \le j \le K} |x + S_{lj}| \le t^{1/2 - \epsilon} \right) \le \left(\sup_{x \in \mathbb{R}_+} \mathbf{P}_0 \left(|x + S_l| \le t^{1/2 - \epsilon} \right) \right)^K.$$

According to the display below [9, (4.6)], there exist positive constants $c_1 \in (0,1)$ and $t_1(\epsilon)$ such that for $t > t_1(\epsilon)$,

$$\mathbf{P}_0\left(|x+S_l| \le t^{1/2-\epsilon}\right) < c_1 \quad x \in \mathbb{R}_+.$$

Plugging this into (3.2), taking $c_2 = -\ln c_1$, and combining with (3.1), we get that for $t > t_1(\epsilon)$,

(3.3)
$$I_1 \le c_1^K = e^{-c_2[t^{\epsilon} - 1]} \le \frac{c_2}{t^{1/2 + \epsilon/4}}.$$

(ii) Upper bound of I_2 . By part (ii) of the proof of [9, Lemma 4.1], for any $\epsilon \in (0, \delta/(2(5+2\delta)))$, we have

(3.4)
$$I_2 \le \frac{C_3(2\epsilon)x}{t^{1+\delta/2 - (5+2\delta)\epsilon}} \le \frac{C_3(2\epsilon)x}{t^{\frac{1}{2} + \epsilon/4}},$$

where $C_3(2\epsilon)$ is the constant in Lemma 3.2.

(iii) Upper bound of I_3 . Repeating the argument in part (iii) of the proof of [9, Lemma 4.1] leading to [9, (4.8)] and using [6, Lemma 7.7], we can find $\epsilon_1 > 0$ with the property that for any $\epsilon \in (0, \epsilon_1 \wedge \delta/(2(5+2\delta)))$ there exists a positive constant $c_3(\epsilon)$ such that

(3.5)
$$I_{3} \leq \frac{1}{\sqrt{t}} \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_{x} \left(S_{k}; \tau_{0}^{S,-} > k, S_{k} > t^{(1-\epsilon)/2}, \tau_{t^{1/2-\epsilon}}^{S,+} = k \right)$$

$$\leq \frac{c_{3}(\epsilon)(1+x)}{t^{1+\delta/2-\epsilon(1+\epsilon+\delta/2)}} \leq \frac{c_{3}(\epsilon)(1+x)}{t^{\frac{1}{2}+\epsilon/4}}.$$

(iv) Upper bound of I_4 . For $k \leq [t^{1-\epsilon}]$ and x' > 0, define

$$K(k, x') := \mathbf{P}_{x'} \left(\frac{\xi_{t-k}}{\sigma \sqrt{t}} \le y, \tau_0^- > t - k, A_{t-k}^c \right).$$

Set

$$x^* := \frac{x' + (t-k)^{\frac{1}{2}-2\epsilon}}{\sigma}$$
 and $y^* := \frac{y\sqrt{t}}{\sqrt{t-k}} + \frac{2}{\sigma(t-k)^{2\epsilon}}$.

It follows from [9, (4.13)] that

(3.6)
$$K(k,x') \le \frac{2}{\sqrt{2\pi(t-k)}} \left(\frac{x'}{\sigma} + \frac{t^{\frac{1}{2}-2\epsilon}}{\sigma}\right) \int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} dz.$$

We claim that there exist positive constants $t_2(\epsilon)$ and $c_4(\epsilon)$ such that for $t > t_2(\epsilon)$,

(3.7)
$$K(k, x') \leq \frac{2}{\sigma \sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}} \right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}} \right) \left(x' + t^{\frac{1}{2} - 2\epsilon} \right).$$

To prove this claim, note that for any $k \leq [t^{1-\epsilon}]$ and $x' \leq t^{(1-\epsilon)/2}$, there exist positive constants $t_3(\epsilon)$, $c_5(\epsilon)$ and $c_6(\epsilon)$ such that for $t > t_3(\epsilon)$, the following holds:

$$\frac{x^*}{\sqrt{t-k}} \le c_5(\epsilon) \frac{t^{(1-\epsilon)/2} + t^{\frac{1}{2}-2\epsilon}}{\sqrt{t}} \le c_6(\epsilon) t^{-\epsilon/2}.$$

For $y \in [0, t^{\epsilon^p}]$ with $p \geq 2$ being a positive constant, and $z \leq y^*$, there exist positive constants $t_4(\epsilon)$ and $c_7(\epsilon)$ such that for $t > t_4(\epsilon)$,

$$z \le \frac{y\sqrt{t}}{\sqrt{t-k}} + \frac{2}{\sigma(t-k)^{2\epsilon}} \le c_7(\epsilon)t^{\epsilon^p},$$

and thus there exists a positive constant $c_8(\epsilon)$ such that for $t > t_3(\epsilon) \vee t_4(\epsilon)$,

$$e^{\frac{zx^*}{\sqrt{t-k}}} \le e^{c_6(\epsilon)c_7(\epsilon)t^{-\epsilon/2}t^{\epsilon^p}} \le 1 + \frac{c_8(\epsilon)}{t^{\epsilon/2-\epsilon^p}}, \quad \text{for } z \le y^*.$$

This implies that when $y \in [0, t^{e^p}]$, for $t > t_3(\epsilon) \vee t_4(\epsilon)$, it holds that

$$\int_0^{y^*} \rho(z) e^{\frac{zx^*}{\sqrt{t-k}}} dz \le \left(1 + \frac{c_8(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \int_0^{y^*} \rho(z) dz.$$

Moreover, by the definition of y^* , there exist positive constants $t_5(\epsilon)$ and $c_9(\epsilon)$ such that for $t > t_5(\epsilon)$,

$$y^* - y \le \frac{c_9(\epsilon)}{t^{\epsilon/2}}$$

Thus using the fact that $\rho(z) \leq 1$ for all $z \geq 0$, we get that for any $\epsilon \in (0, \epsilon_1 \wedge \delta/(2(5+2\delta)))$ and $t > \max\{t_i(\epsilon) : 3 \leq i \leq 5\}$,

(3.8)
$$\int_{0}^{y^{*}} \rho(z)e^{\frac{zx^{*}}{\sqrt{t-k}}} dz \leq \left(1 + \frac{c_{8}(\epsilon)}{t^{\epsilon/2 - \epsilon^{p}}}\right) (\mathcal{R}(y) + y^{*} - y)$$
$$\leq \left(1 + \frac{c_{8}(\epsilon)}{t^{\epsilon/2 - \epsilon^{p}}}\right) \left(\mathcal{R}(y) + \frac{c_{9}(\epsilon)}{t^{\epsilon/2}}\right).$$

For $y > t^{\epsilon^p}$, using [6, (7.31)] we get that there exist positive constants $t_6(\epsilon)$ and $c_{10}(\epsilon)$ such that for $t > t_6(\epsilon)$,

(3.9)
$$\int_{0}^{y^{*}} \rho(z)e^{\frac{zx^{*}}{\sqrt{t-k}}}dz \leq \left(1 + \frac{c_{10}(\epsilon)}{t^{\epsilon/2 - \epsilon^{p}}}\right) \int_{0}^{y} \rho(z)dz + c_{10}(\epsilon)e^{-c_{10}(\epsilon)t^{\epsilon^{p}}}$$
$$\leq \left(1 + \frac{c_{10}(\epsilon)}{t^{\epsilon/2 - \epsilon^{p}}}\right) \left(\mathcal{R}(y) + \frac{c_{10}(\epsilon)}{t^{\epsilon}}\right).$$

Combining (3.6), (3.8) and (3.9), we get that there exists a positive constant $c_{11}(\epsilon)$ such that for any y > 0 and $t > \max\{t_i(\epsilon) : 3 \le i \le 6\}$, we have

(3.10)
$$K(k, x') \le \frac{2x^*}{\sqrt{2\pi(t-k)}} \left(1 + \frac{c_{11}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_{11}(\epsilon)}{t^{\epsilon/2}}\right).$$

Since $k \leq [t^{1-\epsilon}]$, there exists a constant $t_7(\epsilon) > 0$ such that when $t > t_7(\epsilon)$,

$$\frac{1}{\sqrt{t-k}} \le \frac{1}{\sqrt{t}} \left(1 + \frac{c_{12}(\epsilon)}{t^{\epsilon}} \right),$$

and

(3.11)
$$\frac{x^*}{\sqrt{t-k}} \le \frac{1}{\sigma\sqrt{t}} \left(1 + \frac{c_{13}(\epsilon)}{t^{\epsilon}} \right) \left(x' + t^{\frac{1}{2} - 2\epsilon} \right),$$

for some positive constants $c_{12}(\epsilon)$ and $c_{13}(\epsilon)$. Taking $t_2(\epsilon) := \max\{t_i(\epsilon) : 3 \le i \le 7\}$, then the claim (3.7) follows from (3.10) and (3.11).

Note that on $\{\tau_{t^{1/2-\epsilon}}^{S,+}=k\}$, we have $\xi_k=S_k\geq t^{1/2-\epsilon}$. Thus, $t^{1/2-2\epsilon}\leq t^{-\epsilon}\xi_k$ on $\{\tau_{t^{1/2-\epsilon}}^{S,+}=k\}$. Also note that, by using that $(R(\xi_s)1_{\{\tau_0^->s\}})_{s\geq 0}$ is a \mathbf{P}_x -martingale for any x>0 and the optional stopping theorem,

(3.12)
$$R(x) = \mathbf{E}_x \left(R\left(\xi_{\tau_{t^{1/2-\epsilon}}^{S,+}}\right); \tau_0^- > \tau_{t^{1/2-\epsilon}}^{S,+} \right), \quad x > 0, \ t > 0.$$

Hence, by (3.7), for $t > t_2(\epsilon)$,

$$\begin{split} I_4 &\leq \frac{2}{\sigma\sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right) \\ &\times \sum_{k=1}^{[t^{1-\epsilon}]} \mathbf{E}_x \left(\xi_k + t^{\frac{1}{2} - 2\epsilon}; \tau_0^- > k, \xi_k \leq t^{(1-\epsilon)/2}, \tau_{t^{1/2 - \epsilon}}^{S,+} = k\right) \\ &\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right) \sum_{k=1}^{[t^{1-\epsilon}]} \\ &\times \mathbf{E}_x \left(\xi_k; \tau_0^- > k, \xi_k \leq t^{(1-\epsilon)/2}, \tau_{t^{1/2 - \epsilon}}^{S,+} = k\right) \\ &\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right) \mathbf{E}_x \left(\xi_{\tau_{t^{1/2 - \epsilon}}^{S,+}}; \tau_0^- > \tau_{t^{1/2 - \epsilon}}^{S,+}, \tau_{t^{1/2 - \epsilon}}^{S,+} \leq [t^{1-\epsilon}]\right) \\ &\leq \frac{2(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right) \mathbf{E}_x \left(\xi_{\tau_{t^{1/2 - \epsilon}}^{S,+}}; \tau_0^- > \tau_{t^{1/2 - \epsilon}}^{S,+}\right) \\ &= \frac{2R(x)(1 + t^{-\epsilon})}{\sigma\sqrt{2\pi t}} \left(1 + \frac{c_4(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) + \frac{c_4(\epsilon)}{t^{\epsilon/2}}\right), \end{split}$$

where in the last equality we used (3.12). Thus, there exist positive constants $t_8(\epsilon)$, $c_{14}(\epsilon)$ and $c_{15}(\epsilon)$ such that for $t > t_8(\epsilon)$,

$$(3.13) I_4 \le \frac{2R(x)\left(1 + \frac{c_{14}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right)\left(\mathcal{R}(y) + \frac{c_{14}(\epsilon)}{t^{\epsilon/2}}\right)}{\sigma\sqrt{2\pi t}} \le \frac{2R(x)\mathcal{R}(y)}{\sigma\sqrt{2\pi t}} + \frac{c_{15}(\epsilon)(1+x)}{t^{\frac{1}{2} + \epsilon/4}}.$$

(v) Lower bound of I_4 . Repeating the proof of [6, (7.40)], we get that there exist positive constants $t_9(\epsilon)$, $c_{16}(\epsilon)$ and $c_{17}(\epsilon)$ such that for $t > t_9(\epsilon)$,

$$(3.14) I_4 \ge \frac{2R(x)}{\sigma\sqrt{2\pi t}} \left(1 - \frac{c_{16}(\epsilon)}{t^{\epsilon/2 - \epsilon^p}}\right) \left(\mathcal{R}(y) - \frac{1}{t^{2\epsilon}}\right) - \frac{c_{16}(\epsilon)(1+x)}{t^{\delta/2 - \epsilon(1+\epsilon+\delta/2)}}$$
$$\ge \frac{2R(x)\mathcal{R}(y)}{\sigma\sqrt{2\pi t}} - \frac{c_{17}(\epsilon)(1+x)}{t^{\frac{1}{2}+\epsilon/4}}.$$

Set $\epsilon_0 := \min\{\delta/(2(5+2\delta)), \epsilon_1\}$, $\varepsilon := \epsilon/4$, $\varepsilon_0 := \epsilon_0/4$ and $T_0(\epsilon) := \max\{t_i(\epsilon) : i = 1, 2, 8, 9\}$. Using the fact that there exists $c_{18} > 0$ such that $R(x) \le c_{18}(1+x)$, and

combining (3.3), (3.4), (3.5), (3.13) and (3.14), we arrive at the conclusion of the lemma.

The duality relations in the following lemma, especially (3.15), are well known in probabilistic potential theory. We give an elementary proof here for the reader's convenience.

Lemma 3.4. For any t > 0 and any bounded Borel functions $g, h : \mathbb{R} \to \mathbb{R}_+$, we have

(3.15)
$$\int_{\mathbb{R}_{+}} h(x) \mathbf{E}_{x} \left[g(\xi_{t}) \mathbf{1}_{\{\tau_{0}^{-} > t\}} \right] dx = \int_{\mathbb{R}_{+}} g(y) \mathbf{E}_{y} \left[h(\widehat{\xi}_{t}) \mathbf{1}_{\{\widehat{\tau}_{0}^{-} > t\}} \right] dy$$

and

(3.16)
$$\int_{\mathbb{R}} h(x) \mathbf{E}_x \left[g(\xi_t) \mathbf{1}_{\{\tau_0^- \le t\}} \right] \mathrm{d}x = \int_{\mathbb{R}} g(y) \mathbf{E}_y \left[h(\widehat{\xi}_t) \mathbf{1}_{\{\widehat{\tau}_0^- \le t\}} \right] \mathrm{d}y.$$

Proof. For x > 0, by the change of variables $x + \xi_t = y$, we get

$$\int_{\mathbb{R}_{+}} h(x) \mathbf{E}_{x} \left[g(\xi_{t}) \mathbf{1}_{\{\tau_{0}^{-} > t\}} \right] dx = \int_{\mathbb{R}_{+}} h(x) \mathbf{E} \left[g(x + \xi_{t}) \mathbf{1}_{\{\tau_{-x}^{-} > t\}} \right] dx$$

$$= \int_{\mathbb{R}_{+}} h(x) \mathbf{E} \left[g(x + \xi_{t}), \inf_{s \le t} \xi_{s} > -x \right] dx = \int_{\mathbb{R}_{+}} h(x) \mathbf{E} \left[g(x + \xi_{t}), \inf_{s \le t} \xi_{t-s} > -x \right] dx$$

$$= \int_{\mathbb{R}_{+}} g(y) \mathbf{E} \left[h(y - \xi_{t}), \inf_{s \le t} (\xi_{t-s} - \xi_{t}) > -y \right] dy$$

$$= \int_{\mathbb{R}_{+}} g(y) \mathbf{E} \left[h(y + \widehat{\xi_{t}}), \inf_{s \le t} \widehat{\xi_{s}} > -y \right] dy = \int_{\mathbb{R}_{+}} g(y) \mathbf{E}_{y} \left[h(\widehat{\xi_{t}}) \mathbf{1}_{\{\widehat{\tau_{0}} > t\}} \right] dy,$$

which completes the proof of (3.15). Using the same argument, we can also get

(3.17)
$$\int_{\mathbb{R}} h(x) \mathbf{E}_{x}(g(\xi_{t})) dx = \int_{\mathbb{R}} h(x) \mathbf{E}(g(x+\xi_{t})) dx = \int_{\mathbb{R}} g(y) \mathbf{E}(h(y-\xi_{t})) dy$$
$$= \int_{\mathbb{R}} g(y) \mathbf{E}(h(y+\widehat{\xi_{t}})) dy = \int_{\mathbb{R}} g(y) \mathbf{E}_{y}(h(\widehat{\xi_{t}})) dy.$$

Note that for x < 0, $\mathbf{P}_x(\tau_0^- > t) = \mathbf{P}_x(\widehat{\tau}_0^- > t) = 0$. Therefore, (3.15) is equivalent to

$$\int_{\mathbb{R}} h(x) \mathbf{E}_x \left[g(\xi_t) \mathbf{1}_{\{\tau_0^- > t\}} \right] \mathrm{d}x = \int_{\mathbb{R}} g(y) \mathbf{E}_y \left[h(\widehat{\xi_t}) \mathbf{1}_{\{\widehat{\tau_0}^- > t\}} \right] \mathrm{d}y.$$

Combining this with (3.17), we get (3.16).

Before stating Theorem 3.5, we first introduce some necessary notation and definitions. Let $h_1, h_2 : \mathbb{R} \to \mathbb{R}_+$ be Borel functions and $\varepsilon > 0$. We say that $h_1 \varepsilon$ -dominates h_2 and write $h_2 \leq_{\varepsilon} h_1$ if

$$h_2(u) \le h_1(u+v), \quad \forall u \in \mathbb{R}, \ \forall \ v \in [-\varepsilon, \varepsilon].$$

For any a > 0 and Borel function $h : \mathbb{R} \to \mathbb{R}_+$, we define $I_{k,a} = [ka, (k+1)a]$ for $k \in \mathbb{Z}$ and

$$\bar{h}_a(u) := \sum_{k \in \mathbb{Z}} 1_{I_{k,a}}(u) \sup_{u' \in I_{k,a}} f(u'), \quad \underline{h}_a(u) := \sum_{k \in \mathbb{Z}} 1_{I_{k,a}}(u) \inf_{u' \in I_{k,a}} f(u'), \quad u \in \mathbb{R}.$$

The function h is called directly Riemann integrable if $\int_{\mathbb{R}} \bar{h}_a(u) du < \infty$ for any a > 0 small enough and

$$\lim_{a \to 0} \int_{\mathbb{R}} \left(\bar{h}_a(u) - \underline{h}_a(u) \right) du = 0.$$

Define

$$(3.18) \bar{h}_{a,\varepsilon}(u) := \sup_{[u-\varepsilon, u+\varepsilon]} \bar{h}_a(v), \quad \underline{h}_{a,-\varepsilon}(u) := \inf_{v \in [u-\varepsilon, u+\varepsilon]} \bar{h}_a(v), \quad u \in \mathbb{R},$$

then it holds that

$$\underline{h}_{a,-\varepsilon} \leq_{\varepsilon} \underline{h}_a \leq h \leq \overline{h}_a \leq_{\varepsilon} \overline{h}_{a,\varepsilon}$$
 on \mathbb{R} .

For more details about directly Riemann integrability, see [5, Section XI.1].

The following theorem will play an important role in this paper. We refer the reader to [7, Theorem 1.9] for an analogous result for random walks.

Theorem 3.5. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1), (H2) and $\mathbf{E}_0[\xi_1] < 0$. Let $f: \mathbb{R} \mapsto \mathbb{R}_+$ be a Borel function, which is not 0 almost everywhere on \mathbb{R}_+ , such that $f(x)e^{-\lambda_* x}(1+|x|)$ is directly Riemann integrable. Then for any x>0, it holds that

$$\lim_{t \to \infty} t^{3/2} e^{-\Psi(\lambda_*)t} \mathbf{E}_x \left(f(\xi_t), \tau_0^- > t \right) = \frac{2R^*(x) e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} f(z) e^{-\lambda_* z} \widehat{R}^*(z) dz,$$

where Ψ is the Laplace exponent of ξ .

Remark 2. Recall that, when $\mathbf{E}_0[\xi_1] < 0$, $((\xi_t)_{t \geq 0}, (\mathbf{P}_x^{\lambda_*})_{x \in \mathbb{R}})$ is a spectrally negative Lévy process with Laplace exponent $\Psi_{\lambda_*}(\lambda) = \Psi(\lambda + \lambda_*) - \Psi(\lambda_*)$ and that $\Psi'_{\lambda_*}(0+) = \Psi'(\lambda_*) = 0$. Using (1.11), we get that

(3.19)
$$\mathbf{E}_{x}\left(f(\xi_{t}), \tau_{0}^{-} > t\right) = e^{\Psi(\lambda_{*})t + \lambda_{*}x} \mathbf{E}_{x}^{\lambda_{*}}\left(f(\xi_{t})e^{-\lambda_{*}\xi_{t}}, \tau_{0}^{-} > t\right).$$

Therefore, to get the assertion of Theorem 3.5, we only need to consider the asymptotic behavior of

$$\mathbf{E}_x^{\lambda_*} \left(f(\xi_t) e^{-\lambda_* \xi_t}, \tau_0^- > t \right), \quad t \to \infty.$$

Theorem 3.6. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in \mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**, **(H2)** and $\mathbf{E}_0[\xi_1] < 0$. Let $h: \mathbb{R} \mapsto \mathbb{R}_+$ be a Borel function, which is not 0 almost everywhere on \mathbb{R}_+ , such that h(x)(1+|x|) is directly Riemann integrable. Then for any x > 0, it holds that

$$\lim_{t \to \infty} t^{3/2} \mathbf{E}_x^{\lambda_*} \left(h(\xi_t) 1_{\{\tau_0^- > t\}} \right) = \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) dz,$$

where Ψ is the Laplace exponent of ξ .

Remark 3. Taking $h(x) = f(x)e^{-\lambda_* x}$ in Theorem 3.6 and using (3.19), we immediately get the conclusion of Theorem 3.5.

In the next four lemmas, we provide some upper and lower bounds for $\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})\mathbf{1}_{\{\tau_{0}^{-}>t\}}\right)$. In the remainder of this section, ε_{0} will be the constant in Lemma 3.3. Recall that $\rho(\cdot)$ stands for the Rayleigh density.

Lemma 3.7. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**, **(H2)** and $\mathbf{E}_0[\xi_1] < 0$. Then one can find a constant $C_5 > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_1(\varepsilon)$ and $C_6(\varepsilon)$ such that for any x > 0, $t > T_1(\varepsilon)$ and any integrable functions $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_{\varepsilon} H$,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) \leq \frac{2(1+C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \int_{\mathbb{R}_{+}}^{\infty} H(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$

$$+ \frac{2C_{5}\sqrt{\varepsilon}R^{*}(x)}{\Psi''(\lambda_{*})t} \int_{-\varepsilon}^{\infty} H(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$

$$+ C_{6}(\varepsilon)(1+x)\|H1_{[-\varepsilon,\infty)}\|_{1}\left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right),$$

where Ψ is the Laplace exponent of ξ .

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H : \mathbb{R} \to \mathbb{R}_+$ be integrable functions satisfying $h \leq_{\varepsilon} H$. Fix $t \geq 1$ and set $m = [\varepsilon t]$. By the Markov property,

$$(3.20) \ \mathbf{E}_{x}^{\lambda_{*}} \left(h(\xi_{t}) \mathbf{1}_{\{\tau_{0}^{-} > t\}} \right) = \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}} \left(h(\xi_{m}) \mathbf{1}_{\{\tau_{0}^{-} > m\}} \right) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in \mathrm{d}y, \tau_{0}^{-} > t - m \right).$$

Define a random walk $(S_n)_{n\geq 0}$ by $S_n := \xi_n$, $n \in \mathbb{N}$. Since $h1_{[0,\infty)} \leq_{\varepsilon} H1_{[-\varepsilon,\infty)}$, it follows from [7, Theorem 2.7] that there exist constants c_1 (independent of ε) and $c_2(\varepsilon)$ such that for any $n \geq 1$,

$$(3.21) \qquad \mathbf{E}_{x}^{\lambda_{*}} \left(h(S_{n}) \mathbf{1}_{\{S_{n} \geq 0\}} \right) - \frac{1 + c_{1} \varepsilon}{\sqrt{\Psi''(\lambda_{*})n}} \int_{\mathbb{R}} H(z) \mathbf{1}_{\{z \geq -\varepsilon\}} \phi \left(\frac{z - x}{\sqrt{\Psi''(\lambda_{*})n}} \right) dz$$

$$\leq \frac{c_{2}(\varepsilon)}{n^{(1+\delta)/2}} \|H\mathbf{1}_{[-\varepsilon,\infty)}\|_{1}.$$

Thus, for any y > 0,

$$\mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\tau_{0}^{-}>m\}}\right) \leq \mathbf{E}_{y}^{\lambda_{*}}\left(h(S_{m})1_{\{S_{m}\geq0\}}\right)$$

$$\leq \frac{1+c_{1}\varepsilon}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\varepsilon\}}\phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_{*})m}}\right) dz + \frac{c_{2}(\varepsilon)}{m^{(1+\delta)/2}} \|H1_{[-\varepsilon,\infty)}\|_{1}.$$

Plugging this into (3.20) yields that

$$(3.22) \mathbf{E}_{x}^{\lambda_{*}} \left(h(\xi_{t}) \mathbf{1}_{\{\tau_{0}^{-} > t\}} \right) \leq \int_{\mathbb{R}_{+}} \left[\frac{1 + c_{1}\varepsilon}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} H(z) \mathbf{1}_{\{z \geq -\varepsilon\}} \phi \left(\frac{z - y}{\sqrt{\Psi''(\lambda_{*})m}} \right) dz \right] \times \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m \right)$$

$$+ \int_{\mathbb{R}_{+}} \frac{c_{2}(\varepsilon) \|H1_{[-\varepsilon,\infty)}\|_{1}}{m^{(1+\delta)/2}} \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m\right)$$

=: $A_{1}(x) + A_{2}(x)$.

By the definition of τ_0^- , we have

$$(3.23) \mathbf{P}_x^{\lambda_*} \left(\tau_0^- > s \right) = \mathbf{P}_x^{\lambda_*} \left(\inf_{l \le s} \xi_l > 0 \right) \le \mathbf{P}_x^{\lambda_*} \left(\inf_{j \le [s]} S_j > 0 \right) \le c_3 \frac{1+x}{\sqrt{s}},$$

for some positive constant c_3 (independent of ε), where in the last inequality we used [1, (2.7)]. Therefore, by (3.23) and the definition of m, there exists a positive constant $c_4(\varepsilon)$ such that

$$(3.24) \quad A_2(x) = \frac{c_2(\varepsilon) \|H1_{[-\varepsilon,\infty)}\|_1}{m^{(1+\delta)/2}} \mathbf{P}_x^{\lambda_*} \left(\tau_0^- > t - m\right) \le \frac{c_4(\varepsilon)(1+x)}{t^{1+\delta/2}} \|H1_{[-\varepsilon,\infty)}\|_1.$$

Now, by a change of variables, we get

$$A_{1}(x) = \int_{\mathbb{R}_{+}} \left[\frac{1 + c_{1}\varepsilon}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} H(z) 1_{\{z \geq -\varepsilon\}} \phi \left(\frac{z - \sqrt{\Psi''(\lambda_{*})(t - m)}u}{\sqrt{\Psi''(\lambda_{*})m}} \right) dz \right]$$

$$\times \mathbf{P}_{x}^{\lambda_{*}} \left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \in du, \tau_{0}^{-} > t - m \right)$$

$$= \int_{\mathbb{R}_{+}} \varphi_{t}(u) \mathbf{P}_{x}^{\lambda_{*}} \left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \in du, \tau_{0}^{-} > t - m \right),$$

where the function φ_t is defined by

$$\varphi_t(u) := \frac{1 + c_1 \varepsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}} H(z) 1_{\{z \ge -\varepsilon\}} \phi\left(\frac{z - \sqrt{\Psi''(\lambda_*)(t - m)}u}{\sqrt{\Psi''(\lambda_*)m}}\right) dz$$
$$= (1 + c_1 \varepsilon) \sqrt{\frac{t - m}{m}} \int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_*)(t - m)}w) 1_{\{\sqrt{\Psi''(\lambda_*)(t - m)}w \ge -\varepsilon\}} \phi\left(\frac{w - u}{\sqrt{\frac{m}{t - m}}}\right) dw.$$

Using integration by parts, we get that for any $x \in \mathbb{R}_+$,

$$(3.25) A_1(x) \le \int_{\mathbb{R}_+} \varphi_t'(u) \mathbf{P}_x^{\lambda_*} \left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_*)(t-m)}} > u, \tau_0^- > t - m \right) du.$$

It follows from Lemma 3.3 that for $t - m > T_0(\varepsilon)$,

$$\left| \mathbf{P}_{x}^{\lambda_{*}} \left(\frac{\xi_{t-m}}{\sqrt{\Psi''(\lambda_{*})(t-m)}} > u, \tau_{0}^{-} > t-m \right) - \frac{2R^{*}(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_{*})}} \int_{u}^{\infty} \rho(z) dz \right|$$

$$\leq \frac{C_{4}(\varepsilon)(1+x)}{(t-m)^{\frac{1}{2}+\varepsilon}},$$

which together with (3.25) implies that there exists a constant $c_5(\varepsilon)$ such that for $t-m > T_0(\varepsilon)$,

$$(3.26) \ A_1(x) - \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}} \int_{\mathbb{R}_+} \varphi_t'(u) e^{-\frac{u^2}{2}} du \le \frac{c_5(\varepsilon)(1+x)}{(t-m)^{\frac{1}{2}+\varepsilon}} \int_{\mathbb{R}_+} |\varphi_t'(u)| du.$$

By the definition of φ_t and a change of variables, we get that

$$\int_{\mathbb{R}_{+}} |\varphi'_{t}(u)| du$$

$$\leq (1 + c_{1}\varepsilon) \int_{\mathbb{R}_{+}} \int_{\mathbb{R}} \frac{t - m}{m} H(\sqrt{\Psi''(\lambda_{*})(t - m)}w) 1_{\{\sqrt{\Psi''(\lambda_{*})(t - m)}w \geq -\varepsilon\}} \Big| \phi'\left(\frac{w - u}{\sqrt{\frac{m}{t - m}}}\right) \Big| dw du$$

$$= (1 + c_{1}\varepsilon) \int_{\mathbb{R}_{+}} \int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_{*})m}u) 1_{\{\sqrt{\Psi''(\lambda_{*})m}u \geq -\varepsilon\}} |\phi'(u - y)| du dy$$

$$= (1 + c_{1}\varepsilon) \int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_{*})m}u) 1_{\{\sqrt{\Psi''(\lambda_{*})m}u \geq -\varepsilon\}} du \int_{\mathbb{R}_{+}} |\phi'(u - y)| dy.$$

Since there exists a constant $c_6 > 0$ such that $\int_{\mathbb{R}_+} |\phi'(u-y)| dy \le c_6$, a change of variables yields that

(3.27)
$$\int_{\mathbb{R}_+} |\varphi_t'(u)| \mathrm{d}u \le c_6 (1 + c_1 \varepsilon) \frac{\|H1_{[-\varepsilon,\infty)}\|_1}{\sqrt{\Psi''(\lambda_*)m}}.$$

Using integration by parts, we get

$$\int_{\mathbb{R}_{+}} \varphi'_{t}(y) e^{-\frac{y^{2}}{2}} dy = \int_{\mathbb{R}_{+}} \varphi_{t}(y) \rho(y) dy$$

$$= (1 + c_{1}\varepsilon) \int_{\mathbb{R}_{+}} \int_{\mathbb{R}} \sqrt{\frac{t - m}{m}} H(\sqrt{\Psi''(\lambda_{*})(t - m)} w) 1_{\{\sqrt{\Psi''(\lambda_{*})(t - m)}w \geq -\varepsilon\}} \phi\left(\frac{w - y}{\sqrt{\frac{m}{t - m}}}\right) dw \rho(y) dy$$

$$= (1 + c_{1}\varepsilon) \int_{\mathbb{R}_{+}} \int_{\mathbb{R}} \sqrt{\frac{t}{m}} H(\sqrt{\Psi''(\lambda_{*})t}u) 1_{\{\sqrt{\Psi''(\lambda_{*})t}u \geq -\varepsilon\}} \phi\left(\frac{u - z}{\sqrt{\frac{m}{t}}}\right) du \rho\left(\sqrt{\frac{t}{t - m}}z\right) \sqrt{\frac{t}{t - m}} dz.$$

According to [6, Lemma 3.3], for any $v \in (0, \frac{1}{2}]$ and $s \ge 0$, it holds that

(3.28)
$$\sqrt{1 - v}\rho(s) \le \phi_v * \rho_{1-v}(s) \le \sqrt{1 - v}\rho(s) + \sqrt{v}e^{-\frac{s^2}{2v}}.$$

Letting $v = \frac{m}{t}$, we get

$$\int_{\mathbb{R}_{+}} \varphi'_{t}(y) e^{-\frac{y^{2}}{2}} dy = (1 + c_{1}\varepsilon) \int_{\mathbb{R}} H(\sqrt{\Psi''(\lambda_{*})t}u) 1_{\{\sqrt{\Psi''(\lambda_{*})t}u \geq -\varepsilon\}} \phi_{\frac{m}{t}} * \rho_{\frac{t-m}{t}}(u) du$$

$$= \frac{(1 + c_{1}\varepsilon)}{\sqrt{\Psi''(\lambda_{*})t}} \int_{-\varepsilon}^{\infty} H(w) \phi_{\frac{m}{t}} * \rho_{\frac{t-m}{t}} \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$

$$\leq \frac{(1 + c_{1}\varepsilon)}{\sqrt{\Psi''(\lambda_{*})t}} \int_{-\varepsilon}^{\infty} H(w) \left[\sqrt{\frac{t-m}{t}} \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) + \sqrt{\frac{m}{t}} e^{-\frac{w^{2}}{2\Psi''(\lambda_{*})t}}\right] dw.$$

Combining this with (3.22), (3.24), (3.26) (3.27), and using the fact that $\rho(z) = 0$ for $z \leq 0$ and noticing that $m = [\varepsilon t]$, we get that there exist positive constants c_7 (independent of ε) and $t_1(\varepsilon)$, $t_2(\varepsilon)$ such that for $t > t_1(\varepsilon)$,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) \leq \frac{2(1+c_{7}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \int_{\mathbb{R}_{+}} H(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$

$$+ c_7 \sqrt{\varepsilon} \frac{2R^*(x)}{\Psi''(\lambda_*)t} \int_{-\varepsilon}^{\infty} H(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw + c_8(\varepsilon)(1+x) \|H1_{[-\varepsilon,\infty)}\|_1 \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right).$$

The proof is complete.

Lemma 3.8. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**, **(H2)** and $\mathbf{E}_0[\xi_1] < 0$. Then one can find a constant $C_7 > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_2(\varepsilon)$ and $C_8(\varepsilon)$ such that for any x > 0, $t > T_2(\varepsilon)$ and any Borel functions $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_{\varepsilon} H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z) dz < \infty$,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) \leq \left(1 + C_{7}t^{-1/2} + C_{7}\sqrt{\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2}} \int_{\mathbb{R}_{+}} H(z-\varepsilon)\widehat{R}^{*}(z)dz + \frac{C_{8}(\varepsilon)R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2+\varepsilon}} \int_{\mathbb{R}_{+}} H(z-\varepsilon)(1+z)dz + \frac{C_{8}(\varepsilon)(1+x)}{\sqrt{t}} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right) \int_{\mathbb{R}_{+}} H(z-\varepsilon)(1+z)dz.$$

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H : \mathbb{R} \to \mathbb{R}_+$ satisfying $h \leq_{\varepsilon} H$. For any $z \in \mathbb{R}$, we define

(3.29)
$$H_m(z) := \mathbf{E}_0^{\lambda^*} \left(H(\xi_m + z) \mathbf{1}_{\{\tau_{-z-\varepsilon}^- > m\}} \right) = \mathbf{E}_z^{\lambda^*} \left(H(\xi_m) \mathbf{1}_{\{\tau_{-\varepsilon}^- > m\}} \right).$$

Fix $t \ge 2$ and set $m = \lfloor t/2 \rfloor$. For any y > 0, we have

$$(3.30) I_m(y) := \mathbf{E}_y^{\lambda_*} \left(h(\xi_m) \mathbf{1}_{\{\tau_0^- > m\}} \right)$$

$$\leq \mathbf{E}_y^{\lambda_*} \left(H(\xi_m + v) \mathbf{1}_{\{\tau_{-v-\varepsilon}^- > m\}} \right) = H_m(y+v), \quad |v| \leq \varepsilon.$$

Consequently, $I_m \leq_{\varepsilon} H_m$. By the Markov property,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) = \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\tau_{0}^{-}>m\}}\right) \mathbf{P}_{x}^{\lambda_{*}}\left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m\right)$$

$$= \int_{\mathbb{R}_{+}} I_{m}(y) \mathbf{P}_{x}^{\lambda_{*}}\left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m\right) = \mathbf{E}_{x}^{\lambda_{*}}\left(I_{m}(\xi_{t-m}), \tau_{0}^{-} > t - m\right).$$

Now applying Lemma 3.7 with $h = I_m$, we get that for $t - m > T_1(\varepsilon)$,

(3.31)
$$\mathbf{E}_{x}^{\lambda_{*}} \left(h(\xi_{t}) 1_{\{\tau_{0}^{-} > t\}} \right) \leq J_{1} + J_{2} + J_{3},$$

where

$$J_{1} := \frac{2(1 + C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t - m)} \int_{\mathbb{R}_{+}} H_{m}(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t - m)}}\right) dw,$$

$$J_{2} := C_{5}\sqrt{\varepsilon} \frac{2R^{*}(x)}{\Psi''(\lambda_{*})(t - m)} \int_{-\varepsilon}^{\infty} H_{m}(w)\phi\left(\frac{w}{\Psi''(\lambda_{*})(t - m)}\right) dw,$$

$$J_{3} := C_{6}(\varepsilon) \left(\frac{1}{(t - m)^{1 + \varepsilon}} + \frac{1}{(t - m)^{1 + \delta/2}}\right) (1 + x) \|H_{m}1_{[-\varepsilon, \infty)}\|_{1}.$$

We will deal with the upper bounds of J_i separately. We first deal with J_1 . Note that

$$J_{1} = \frac{2(1 + C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t - m)} \int_{\mathbb{R}_{+}} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m}) \mathbf{1}_{\{\tau_{-\varepsilon}^{-} > m\}} \right) \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \right) dw$$

$$= \frac{2(1 + C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t - m)} \int_{\mathbb{R}} \mathbf{E}_{w+\varepsilon}^{\lambda_{*}} \left(H(\xi_{m} - \varepsilon) \mathbf{1}_{\{\tau_{0}^{-} > m\}} \right) \mathbf{1}_{\{w+\varepsilon \geq 0\}} \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \right) dw$$

$$= \frac{2(1 + C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t - m)} \int_{\mathbb{R}_{+}} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m} - \varepsilon) \mathbf{1}_{\{\tau_{0}^{-} > m\}} \right) \rho \left(\frac{w - \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \right) dw$$

$$= \frac{2(1 + C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t - m)} \int_{\mathbb{R}_{+}} H(w - \varepsilon) \mathbf{E}_{w}^{\lambda_{*}} \left(\rho \left(\frac{\widehat{\xi}_{m} - \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \right) \mathbf{1}_{\{\widehat{\tau_{0}}^{-} > m\}} \right) dw,$$

where in the last equality we used (3.15). Using integration by parts, we get for any $z \in \mathbb{R}_+$,

Set

$$u_{m,\varepsilon} := \frac{u\sqrt{\Psi(\lambda_*)(t-m)} + \varepsilon}{\sqrt{\Psi''(\lambda_*)m}}.$$

Applying Lemma 3.3 to $\hat{\xi}$, we get that for $m > T_0(\varepsilon)$,

$$\left| \mathbf{P}_{z}^{\lambda_{*}} \left(\frac{\widehat{\xi}_{m}}{\sqrt{\Psi''(\lambda_{*})m}} > u_{m,\varepsilon}, \widehat{\tau}_{0}^{-} > m \right) - \frac{2\widehat{R}^{*}(z)}{\sqrt{2\pi m \Psi''(\lambda_{*})}} \int_{u_{m,\varepsilon}}^{\infty} \rho(y) dy \right| \leq \frac{C_{4}(\varepsilon)(1+z)}{m^{1/2+\varepsilon}}.$$

Substituting this into (3.32) and using the fact that $\int_{\mathbb{R}_+} \rho'(u) du \leq c_1$ for some $c_1 > 0$, we get that

$$(3.33) \left| \mathbf{E}_{z}^{\lambda_{*}} \left(\rho \left(\frac{\widehat{\xi}_{m} - \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t - m)}} \right) 1_{\{\widehat{\tau_{0}} > m\}} \right) - \frac{2\widehat{R}^{*}(z)}{\sqrt{2\pi m \Psi''(\lambda_{*})}} \int_{\mathbb{R}_{+}} \rho'(u) e^{-\frac{u_{m,\varepsilon}^{2}}{2}} du \right) du$$

$$\leq \frac{C_{4}(\varepsilon)(1 + z)}{m^{1/2 + \varepsilon}} \int_{\mathbb{R}_{+}} \rho'(u) du \leq c_{1} \frac{C_{4}(\varepsilon)(1 + z)}{m^{1/2 + \varepsilon}}.$$

Using integration by parts again and the boundedness of ρ' , we get that there exists a positive constant c_2 (independent of ε) such that

(3.34)
$$\int_{\mathbb{R}_{+}} \rho'(u)e^{-\frac{u_{m,\varepsilon}^{2}}{2}} du = \sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_{+}} \rho(u)\rho(u_{m,\varepsilon}) du$$
$$\leq \sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_{+}} \rho(u)\rho\left(\frac{u\sqrt{t-m}}{\sqrt{m}}\right) du + \frac{c_{2}\varepsilon}{\sqrt{t}},$$

where in the last inequality we used the mean value theorem. By a change of variables, we see that

$$(3.35) \quad \sqrt{\frac{t-m}{m}} \int_{\mathbb{R}_{+}} \rho(u) \rho\left(\frac{u\sqrt{t-m}}{\sqrt{m}}\right) du = \frac{1}{\sqrt{m}} \int_{\mathbb{R}_{+}} \rho\left(\frac{y}{\sqrt{t-m}}\right) \rho\left(\frac{y}{\sqrt{m}}\right) dy$$

$$= \frac{1}{\sqrt{m}} \int_{\mathbb{R}_{+}} \frac{y^{2}}{\sqrt{(t-m)m}} e^{-\frac{ty^{2}}{2m(t-m)}} dy = \frac{\sqrt{m}(t-m)}{t^{3/2}} \int_{\mathbb{R}_{+}} y^{2} e^{-\frac{y^{2}}{2}} dy$$

$$= \frac{\sqrt{2\pi m}(t-m)}{2t^{3/2}}.$$

Combining this with (3.33), (3.34) and (3.35), we get that there exist positive constants c_4 (independent of ε) and $c_5(\varepsilon)$ such that

(3.36)
$$J_{1} \leq \left(1 + c_{4}t^{-\frac{1}{2}}\right) \frac{2\left(1 + c_{4}\varepsilon\right)R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2}} \int_{\mathbb{R}_{+}} H(z - \varepsilon)\widehat{R}^{*}(z)dz + \frac{c_{5}(\varepsilon)R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2+\varepsilon}} \int_{\mathbb{R}_{+}} H(z - \varepsilon)(1 + z)dz.$$

Next, we deal with J_2 . Note that

$$J_{2} = \frac{2C_{5}\sqrt{\varepsilon}R^{*}(x)}{\Psi''(\lambda_{*})(t-m)} \int_{-\varepsilon}^{\infty} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m}) \mathbf{1}_{\{\tau_{-\varepsilon}^{-} > m\}} \right) \phi \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \frac{2C_{5}\sqrt{\varepsilon}R^{*}(x)}{\Psi''(\lambda_{*})(t-m)} \int_{-\varepsilon}^{\infty} \mathbf{E}_{w+\varepsilon}^{\lambda_{*}} \left(H(\xi_{m} - \varepsilon) \mathbf{1}_{\{\tau_{0}^{-} > m\}} \right) \phi \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \frac{2C_{5}\sqrt{\varepsilon}R^{*}(x)}{\Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m} - \varepsilon) \mathbf{1}_{\{\tau_{0}^{-} > m\}} \right) \phi \left(\frac{w - \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \frac{2C_5\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)(t-m)} \int_{\mathbb{R}_+} H(w-\varepsilon)\mathbf{E}_w^{\lambda_*} \left(\phi\left(\frac{\widehat{\xi}_m - \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right) 1_{\{\widehat{\tau}_0^- > m\}}\right) dw,$$

where in the last equality we used (3.15). Now repeating the argument leading to (3.36), we get that there exist positive constants c_6 (independent of ε) and $c_7(\varepsilon)$ such that

$$(3.37) J_2 \leq \left(1 + c_6 \varepsilon t^{-\frac{1}{2}}\right) c_6 \sqrt{\varepsilon} \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2}} \int_{\mathbb{R}_+} H(z - \varepsilon) \widehat{R}^*(z) dz$$
$$+ \frac{c_7(\varepsilon) R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z - \varepsilon) (1 + z) dz.$$

Finally, we deal with J_3 . By the definition of H_m and (3.23), we have

where in the last equality we used (3.15) and c_8 is a positive constant independent of ε . Since m = [t/2], there exists a positive constant $c_9(\varepsilon)$ such that

(3.39)
$$J_3 \le \frac{c_9(\varepsilon)(1+x)}{\sqrt{t}} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right) \int_{\mathbb{R}_+} (1+z)H(z-\varepsilon)dz.$$

Combining (3.31), (3.36), (3.37) and (3.39), we complete the proof.

Lemma 3.9. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**, **(H2)** and $\mathbf{E}_0(\xi_1) < 0$. Then one can find positive constants C_9 and q with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_3(\varepsilon)$ and $C_{10}(\varepsilon)$ such that for any x > 0, $t > T_3(\varepsilon)$ and any integrable functions $h, H, g : \mathbb{R} \to \mathbb{R}_+$ satisfying $g \leq_{\varepsilon} h \leq_{\varepsilon} H$,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) \geq \frac{2R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \int_{\mathbb{R}_{+}} \left[g(w)1_{\{w\geq\varepsilon\}} - C_{9}\varepsilon h(w)\right] \rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$
$$-C_{9}\varepsilon^{1/12} \frac{2R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \int_{-\varepsilon}^{\infty} H(u)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$
$$-C_{10}(\varepsilon)(1+x) \|H1_{[-\varepsilon,\infty)}\|_{1} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}}\right),$$

where Ψ is the Laplace exponent of ξ .

Proof. Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H, g : \mathbb{R} \to \mathbb{R}_+$ be integrable functions satisfying $g \leq_{\varepsilon} h \leq_{\varepsilon} H$. Then $g1_{[\varepsilon,\infty)} \leq_{\varepsilon} h1_{[0,\infty)} \leq_{\varepsilon} H1_{[-\varepsilon,\infty)}$. Fix $t \geq 1$ and set $m = [\varepsilon t]$. By the Markov property,

$$\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) = \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\tau_{0}^{-}>m\}}\right) \mathbf{P}_{x}^{\lambda_{*}}\left(\xi_{t-m} \in dy, \tau_{0}^{-}>t-m\right)$$

$$= \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\xi_{m}\geq0\}}\right) \mathbf{P}_{x}^{\lambda_{*}}\left(\xi_{t-m} \in dy, \tau_{0}^{-}>t-m\right)$$

$$- \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\xi_{m}\geq0\}}1_{\{\tau_{0}^{-}\leq m\}}\right) \mathbf{P}_{x}^{\lambda_{*}}\left(\xi_{t-m} \in dy, \tau_{0}^{-}>t-m\right)$$

$$=: I_{1}(t) - I_{2}(t) =: I_{1}(t) - I_{2}^{1}(t) - I_{2}^{2}(t),$$

where

$$I_{1}(t) := \int_{\mathbb{R}_{+}} \mathbf{E}_{y}^{\lambda_{*}} \left(h(\xi_{m}) 1_{\{\xi_{m} \geq 0\}} \right) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m \right),$$

$$I_{2}^{1}(t) := \int_{0}^{\varepsilon^{1/6} \sqrt{[t]}} \mathbf{E}_{y}^{\lambda_{*}} \left(h(\xi_{m}) 1_{\{\xi_{m} \geq 0\}} 1_{\{\tau_{0}^{-} \leq m\}} \right) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m \right),$$

$$I_{2}^{2}(t) := \int_{\varepsilon^{1/6} \sqrt{[t]}}^{\infty} \mathbf{E}_{y}^{\lambda_{*}} \left(h(\xi_{m}) 1_{\{\xi_{m} \geq 0\}} 1_{\{\tau_{0}^{-} \leq m\}} \right) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m \right).$$

The proof of the lemma is divided into the following three steps.

Step 1. In this step, we give a lower bound for $I_1(t)$. By [7, Theorem 2.7], there exist positive constants c_1 (independent of ε) and $c_2(\varepsilon)$ such that for any $m \geq 1$,

$$\mathbf{E}_{y}^{\lambda_{*}}\left(h(\xi_{m})1_{\{\xi_{m}\geq 0\}}\right) \geq \frac{1}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} \left[g(z)1_{\{z\geq \varepsilon\}} - c_{1}\varepsilon h(z)1_{\{z\geq 0\}}\right] \phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_{*})m}}\right) dz - \frac{c_{2}(\varepsilon)}{m^{(1+\delta)/2}} \|h1_{[0,\infty)}\|_{1}.$$

Note that $||h1_{[0,\infty)}||_1 \leq ||H1_{[0,\infty)}||_1$. Following the analysis of $A_1(x)$ in Lemma 3.7 and using the lower bound in (3.28), we see that there exist positive constants $t_1(\varepsilon)$ and $c_3(\varepsilon)$ such that for $t > t_1(\varepsilon)$,

$$\frac{1}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}} g(z) 1_{\{z \ge \varepsilon\}} \phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_*)m}}\right) dz \right] \mathbf{P}_x^{\lambda_*} \left(\xi_{t-m} \in dy, \tau_0^- > t-m\right) \\
\ge \frac{2R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} g(w) 1_{\{w \ge \varepsilon\}} \rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw - \frac{c_3(\varepsilon)(1+x) \|H1_{[0,\infty)}\|_1}{t^{1+\varepsilon}}.$$

and using the upper bound in (3.28), we have

$$\frac{c_1 \varepsilon}{\sqrt{\Psi''(\lambda_*)m}} \int_{\mathbb{R}_+} \left[\int_{\mathbb{R}} h(z) 1_{\{z \ge 0\}} \phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_*)m}}\right) dz \right] \mathbf{P}_x^{\lambda_*} \left(\xi_{t-m} \in dy, \tau_0^- > t-m\right)$$

$$\leq \frac{2c_4\varepsilon R^*(x)}{\sqrt{2\pi}\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} h(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw + \frac{2c_4\sqrt{\varepsilon}R^*(x)}{\Psi''(\lambda_*)t} \int_{\mathbb{R}_+} h(w)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw + \frac{c_3(\varepsilon)(1+x)\|H1_{[0,\infty)}\|_1}{t^{1+\varepsilon}},$$

where c_4 is a positive constant independent of ε . Thus there exists a positive constant $c_5(\varepsilon)$ such that for $t > t_1(\varepsilon)$,

$$(3.40) I_{1}(t) \geq \frac{2R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \int_{\mathbb{R}_{+}} \left[g(w)1_{\{w\geq\varepsilon\}} - c_{4}\varepsilon h(w)\right] \rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$
$$-c_{4}\sqrt{\varepsilon} \frac{2R^{*}(x)}{\Psi''(\lambda_{*})t} \int_{\mathbb{R}_{+}} h(w) \phi\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw$$
$$-c_{5}(\varepsilon)(1+x) \|H1_{[0,\infty)}\|_{1} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right).$$

Step 2. Next, we give an upper bound for $I_2^1(t)$. Combining (3.21) and (3.23), we get that there exist positive constants c_6 (independent of ε) and $c_7(\varepsilon)$ such that

$$I_{2}^{1}(t) \leq \int_{0}^{\varepsilon^{1/6}\sqrt{[t]}} \mathbf{E}_{y}^{\lambda_{*}} \left(h(\xi_{m})1_{\{\xi_{m}\geq0\}}\right) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m\right)$$

$$\leq \int_{0}^{\varepsilon^{1/6}\sqrt{[t]}} \frac{1 + c_{6}\varepsilon}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} H(z)1_{\{z\geq-\varepsilon\}} \phi\left(\frac{z - y}{\sqrt{\Psi''(\lambda_{*})m}}\right) dz \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t - m\right)$$

$$+ \frac{c_{7}(\varepsilon)(1 + x) \|H1_{[-\varepsilon,\infty)}\|_{1}}{m^{(1+\delta)/2}\sqrt{t - m}}.$$

For any $u \in \mathbb{R}$, define

$$J(u) := \int_0^{\varepsilon^{1/6}\sqrt{[t]}} \left[\frac{1}{\sqrt{\Psi''(\lambda_*)m}} \phi\left(\frac{u-y}{\sqrt{\Psi''(\lambda_*)m}}\right) \right] \mathbf{P}_x^{\lambda_*} \left(\xi_{t-m} \in \mathrm{d}y, \tau_0^- > t-m\right).$$

Then by Fubini's theorem, we have

$$\int_{0}^{\varepsilon^{1/6}\sqrt{[t]}} \frac{1}{\sqrt{\Psi''(\lambda_{*})m}} \int_{\mathbb{R}} H(z) 1_{\{z \geq -\varepsilon\}} \phi\left(\frac{z-y}{\sqrt{\Psi''(\lambda_{*})m}}\right) dz \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \tau_{0}^{-} > t-m\right)$$

$$= \int_{\mathbb{R}} H(u) 1_{\{u \geq -\varepsilon\}} J(u) du.$$

For any $u \in \mathbb{R}_+$, define

$$F_u(y) := \frac{1}{\sqrt{\Psi''(\lambda_*)m}} \phi\left(\frac{u-y}{\sqrt{\Psi''(\lambda_*)m}}\right).$$

Using the definition of J(u) and integration by parts, we get

$$J(u) = \int_{0}^{\infty} F_{u}(y) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in dy, \xi_{t-m} \leq \varepsilon^{1/6} \sqrt{[t]}, \tau_{0}^{-} > t - m \right)$$

$$\leq \int_{0}^{\infty} F'_{u}(y) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} > y, \xi_{t-m} \in (0, \varepsilon^{1/6} \sqrt{[t]}], \tau_{0}^{-} > t - m \right) dy$$

$$= \int_{0}^{\varepsilon^{1/6} \sqrt{[t]}} F'_{u}(y) \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in (y, \varepsilon^{1/6} \sqrt{[t]}], \tau_{0}^{-} > t - m \right) dy.$$

Since $m = [\varepsilon t]$, using Lemma 3.3, it holds that for $t - m > T_0(\varepsilon)$,

$$\left| \mathbf{P}_{x}^{\lambda_{*}} \left(\xi_{t-m} \in (y, \varepsilon^{1/6} \sqrt{[t]}], \tau_{0}^{-} > t - m \right) - \frac{2R^{*}(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_{*})}} \int_{\frac{y}{\sqrt{\Psi''(\lambda_{*})(t-m)}}}^{\frac{\varepsilon^{1/6} \sqrt{[t]}}{\sqrt{[t]}}} \rho(z) dz \right| \leq \frac{C_{4}(\varepsilon)(1+x)}{(t-m)^{1/2+\varepsilon}}.$$

Now, using the fact

$$F_u(\varepsilon^{1/6}\sqrt{[t]}) - F_u(0) \le \frac{c_8(\varepsilon)}{\sqrt{t}},$$

for some $c_8(\varepsilon) > 0$, we get that there exists a positive constant $c_9(\varepsilon)$ such that for $t - m > T_0(\varepsilon)$,

$$J(u) \leq \frac{c_9(\varepsilon)(1+x)}{(t-m)^{1/2+\varepsilon}} \frac{1}{\sqrt{t}} + \frac{2R^*(x)}{\sqrt{2\pi(t-m)\Psi''(\lambda_*)}} \int_0^{\varepsilon^{1/6}\sqrt{[t]}} F_u'(y) \times \left[\mathcal{R}\left(\frac{\varepsilon^{1/6}\sqrt{[t]}}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right) - \mathcal{R}\left(\frac{y}{\sqrt{\Psi''(\lambda_*)(t-m)}}\right) \right] dy.$$

Using integration by parts and the fact $F_u(0) \ge 0$ (see [7, (3.33)]), we get that there exists a constant c_{10} (independent of ε) such that

$$\int_{0}^{\varepsilon^{1/6}\sqrt{[t]}} F'_{u}(y) \left[\mathcal{R} \left(\frac{\varepsilon^{1/6}\sqrt{[t]}}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) - \mathcal{R} \left(\frac{y}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) \right] dy$$

$$\leq \frac{c_{10}\varepsilon^{1/12}}{\sqrt{t-m}} \phi \left(\frac{u}{\sqrt{\Psi''(\lambda_{*})t}} \right).$$

It follows that there exist positive constants c_{11} (independent of ε), and $t_2(\varepsilon)$, $c_{12}(\varepsilon)$ such that for $t > t_2(\varepsilon)$,

$$(3.41) I_2^1(t) \le c_{11} \varepsilon^{1/12} \frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)t}} \int_{-\varepsilon}^{\infty} H(u)\phi\left(\frac{u}{\sqrt{\Psi''(\lambda_*)t}}\right) du + c_{12}(\varepsilon)(1+x) \|H1_{[-\varepsilon,\infty)}\|_1 \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}}\right).$$

Step 3. Finally, we study the upper bound for $I_2^2(t)$. By the definition of $I_2^2(t)$, we have

$$I_2^2(t) = \int_{\mathbb{R}} J_m(y) \mathbf{P}_x^{\lambda_*} \left(\xi_{t-m} \in dy, \tau_0^- > t - m \right),$$

where $J_m(y) := \mathbf{E}_y^{\lambda_*} \left(h(\xi_m) \mathbf{1}_{\{\xi_m \ge 0\}} \mathbf{1}_{\{\tau_0^- \le m\}} \right) \mathbf{1}_{\{y > \varepsilon^{1/6} \sqrt{|t|}\}}$. For any $z \in \mathbb{R}$, define

$$M_m(z) := \mathbf{E}_z^{\lambda_*} \left(H(\xi_m) \mathbf{1}_{\{\xi_m \ge -\varepsilon\}} \mathbf{1}_{\{\tau_\varepsilon^- \le m\}} \right) \mathbf{1}_{\{z+\varepsilon > \varepsilon^{1/6} \sqrt{[t]}\}}.$$

Consequently, $J_m \leq_{\varepsilon} M_m$. Applying Lemma 3.7 with h and H instead of J_m and M_m , we get that for $t-m > T_1(\varepsilon)$,

$$(3.42) I_{2}^{2}(t) \leq \frac{2(1+C_{5}\varepsilon)R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} M_{m}(w)\rho\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}}\right) dw + \frac{2C_{5}\sqrt{\varepsilon}R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t-m)} \int_{-\varepsilon}^{\infty} M_{m}(w)e^{-\frac{w^{2}}{2\Psi''(\lambda_{*})t}} dw + C_{6}(\varepsilon)(1+x)\|M_{m}1_{[-\varepsilon,\infty)}\|_{1}\left(\frac{1}{(t-m)^{1+\varepsilon}} + \frac{1}{(t-m)^{1+\delta/2}}\right).$$

Now we bound the three terms on the right-hand side of (3.42) from above. Using (3.17),

$$(3.43) \quad \|M_{m}1_{[-\varepsilon,\infty)}\|_{1} = \int_{\mathbb{R}} \mathbf{E}_{z}^{\lambda_{*}} \left(H(\xi_{m})1_{\{\xi_{m}\geq -\varepsilon\}}1_{\{\tau_{\varepsilon}^{-}\leq m\}}\right) 1_{\{z+\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}} 1_{\{z\geq -\varepsilon\}} dz$$

$$\leq \int_{\mathbb{R}} \mathbf{E}_{z}^{\lambda_{*}} \left(H(\xi_{m})1_{\{\xi_{m}\geq -\varepsilon\}}\right) 1_{\{z+\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}} 1_{\{z\geq -\varepsilon\}} dz$$

$$= \int_{\mathbb{R}} H(z)1_{\{z\geq -\varepsilon\}} \mathbf{P}_{z}^{\lambda_{*}} \left(\widehat{\xi}_{m} + \varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\xi}_{m} \geq -\varepsilon\right) dz$$

$$\leq \int_{-\varepsilon}^{\infty} H(z) dz \leq \|H1_{[-\varepsilon,\infty)}\|_{1}.$$

Moreover, using the definition of M_m and (3.16), we get that

$$\int_{\mathbb{R}_{+}} M_{m}(w) \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \int_{\mathbb{R}_{+}} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m}) \mathbf{1}_{\{\xi_{m} \geq -\varepsilon\}} \mathbf{1}_{\{\tau_{\varepsilon}^{-} \leq m\}} \right) \mathbf{1}_{\{w+\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}} \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \int_{\mathbb{R}} \mathbf{E}_{w+\varepsilon}^{\lambda_{*}} \left(H(\xi_{m}) \mathbf{1}_{\{\xi_{m} \geq -\varepsilon\}} \mathbf{1}_{\{\tau_{\varepsilon}^{-} \leq m\}} \right) \mathbf{1}_{\{w+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}} \rho \left(\frac{w+\varepsilon}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \int_{\mathbb{R}} \mathbf{E}_{w}^{\lambda_{*}} \left(H(\xi_{m}+\varepsilon) \mathbf{1}_{\{\xi_{m}+\varepsilon \geq -\varepsilon\}} \mathbf{1}_{\{\tau_{0}^{-} \leq m\}} \right) \mathbf{1}_{\{w+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]}\}} \rho \left(\frac{w+\varepsilon}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$= \int_{\mathbb{R}} H(w+\varepsilon) 1_{\{w \geq -2\varepsilon\}} \mathbf{E}_{w}^{\lambda_{*}} \left(\rho \left(\frac{\widehat{\xi}_{m} + \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) 1_{\{\widehat{\xi}_{m} + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\tau}_{0}^{-} \leq m\}} \right) dw$$

$$=: J_{1}(t) + J_{2}(t),$$

where

$$J_1(t) := \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon) \mathbf{E}_w^{\lambda_*} \left(\rho \left(\frac{\widehat{\xi}_m + \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right) 1_{\{\widehat{\xi}_m + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\tau}_0^- \le m\}} \right) dw,$$

$$J_2(t) := \int_{\varepsilon^{1/4}\sqrt{[t]}}^{\infty} H(w+\varepsilon) \mathbf{E}_w^{\lambda_*} \left(\rho \left(\frac{\widehat{\xi}_m + \varepsilon}{\sqrt{\Psi''(\lambda_*)(t-m)}} \right) 1_{\{\widehat{\xi}_m + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}, \widehat{\tau}_0^- \le m\}} \right) \mathrm{d}w.$$

Next, we consider the upper bounds of $J_1(t)$ and $J_2(t)$ separately. We claim that there exist positive constants c_{13} and q (both independent of ε), and $c_{14}(\varepsilon)$ such that

(3.44)
$$J_1(t) \le c_{13} \varepsilon^{1/6} \int_{-2\varepsilon}^{\varepsilon^{1/4} \sqrt{[t]}} H(w + \varepsilon) dw,$$

and

$$(3.45) \ J_2(t) \le c_{13} \varepsilon^{1/12} \int_{\varepsilon^{1/4} \sqrt{[t]}}^{\infty} H(w + \varepsilon) \phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw + \frac{c_{14}(\varepsilon)}{t^q} \|H1_{[-\varepsilon,\infty)}\|_1.$$

Using (3.44), (3.45) and the fact that ϕ is bounded, we immediately get there exists a positive constant c_{15} (independent of ε) such that

$$\int_{\mathbb{R}_{+}} M_{m}(w) \rho \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dw$$

$$\leq c_{15} \varepsilon^{1/12} \int_{-2\varepsilon}^{\infty} H(w+\varepsilon) \phi \left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}} \right) dw + \frac{c_{14}(\varepsilon)}{t^{q}} ||H1_{[-\varepsilon,\infty)}||_{1}.$$

Similarly, there exist constants c_{16} (independent of ε) and $c_{17}(\varepsilon)$ such that

$$\int_{-\varepsilon}^{\infty} M_m(w) e^{-\frac{w^2}{2\Psi''(\lambda_*)t}} dw \le c_{16} \varepsilon^{1/12} \int_{-2\varepsilon}^{\infty} H(w+\varepsilon) \phi\left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}}\right) dw + \frac{c_{17}(\varepsilon)}{t^q} \|H1_{[-\varepsilon,\infty)}\|_1.$$

Combining the last two displays with (3.42) and (3.43), we get that there exist positive constants c_{18} (independent of ε), $t_3(\varepsilon)$ and $c_{19}(\varepsilon)$ such that for $t > t_3(\varepsilon)$,

(3.46)
$$I_{2}^{2}(t) \leq \frac{c_{18}R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})t} \varepsilon^{1/12} \int_{-2\varepsilon}^{\infty} H(w+\varepsilon)\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right) dw + c_{19}(\varepsilon)(1+x) \|H1_{[-\varepsilon,\infty)}\|_{1} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}}\right).$$

Combining (3.41) and (3.46), and using the fact that there exists $c_{20} > 0$ such that $R^*(x) \leq c_{20}(1+x)$, we get that there exist positive constants c_{21} (independent of ε), $c_{22}(\varepsilon)$ and $t_4(\varepsilon)$, such that for $t > t_4(\varepsilon)$,

$$I_{2}(t) \leq c_{20} \varepsilon^{1/12} \frac{2R^{*}(x)}{\sqrt{2\pi} \Psi''(\lambda_{*})t} \int_{-\varepsilon}^{\infty} H(u) \phi\left(\frac{u}{\sqrt{\Psi''(\lambda_{*})t}}\right) du + c_{22}(\varepsilon) (1+x) \|H1_{[-\varepsilon,\infty)}\|_{1} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}}\right).$$

Combining this with (3.40) gives the desired result.

Now we prove the claims (3.44) and (3.45). Using the boundedness of ρ and the fact that there exists a positive constant t_5 independent of ε such that

(3.47)
$$\varepsilon^{1/6}\sqrt{[t]} - \varepsilon^{1/4}\sqrt{[t]} - 2\varepsilon > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}, \quad t > t_5.$$

Therefore, we get that there exists a positive constant c_{23} such that for $t > t_5$,

$$(3.48) J_{1}(t) \leq c_{23} \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon) \mathbf{P}_{w}^{\lambda_{*}} \left(\widehat{\xi}_{m} + 2\varepsilon > \varepsilon^{1/6}\sqrt{[t]}\right) dw$$
$$\leq c_{23} \mathbf{P}_{0}^{\lambda_{*}} \left(\widehat{\xi}_{m} > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \int_{-2\varepsilon}^{\varepsilon^{1/4}\sqrt{[t]}} H(w+\varepsilon) dw.$$

Using [7, Lemma 3.4] with $u = v = \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}$, since $m = [\varepsilon t]$, we get that there exist positive constants c_{24} and c_{25} both independent of ε such that

$$\mathbf{P}_{0}^{\lambda_{*}}\left(\widehat{\xi}_{m} > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right) \leq 2\exp\left\{\left(1 + \frac{4m}{\varepsilon^{1/3}[t]}\right)\right\} + m\mathbf{P}_{0}^{\lambda_{*}}\left(|\widehat{\xi}_{1}| > \frac{1}{2}\varepsilon^{1/6}\sqrt{[t]}\right)$$
$$\leq c_{24}\varepsilon^{2/3} + [\varepsilon t]\frac{4\mathbf{E}_{0}^{\lambda_{*}}(\widehat{\xi}_{1}^{2})}{\varepsilon^{1/3}[t]} \leq c_{25}\varepsilon^{1/6},$$

where in the second inequality we used Chebyshev's inequality and (H1). Combining this with (3.48), we complete the proof of (3.44).

Next we prove (3.45). Using (3.47) and Hölder's inequality, we get that for all $t > t_5$ and w > 0, we have

$$(3.49) \mathbf{P}_{w}^{\lambda_{*}} \left(\widehat{\xi}_{m} + 2\varepsilon > \varepsilon^{1/6} \sqrt{[t]}, \widehat{\tau}_{0}^{-} \leq m\right)$$

$$\leq \mathbf{P}_{w}^{\lambda_{*}} \left(\max_{s \in [0,m]} |\widehat{\xi}_{s}| > \frac{1}{2} \varepsilon^{1/6} \sqrt{[t]}\right)^{1/2} \mathbf{P}_{w}^{\lambda_{*}} \left(\widehat{\tau}_{0}^{-} \leq m\right)^{1/2}$$

$$= \mathbf{P}_{w}^{\lambda_{*}} \left(\max_{s \in [0,m]} |\widehat{\xi}_{s}| > \frac{1}{2} \varepsilon^{1/6} \sqrt{[t]}\right)^{1/2} \mathbf{P}_{0}^{\lambda_{*}} \left(\widehat{\tau}_{-w}^{-} \leq m\right)^{1/2}.$$

By Lemma 3.2, there exists a Brownian motion W with diffusion coefficient $\Psi''(\lambda_*)$, starting from the origin, such that for any $t \geq 1$ and x > 0,

$$\mathbf{P}_0^{\lambda_*}(\widehat{A}_t) \le \frac{C_3(2\varepsilon)}{t^{(\frac{1}{2} - 2\varepsilon)(\delta + 2) - 1}},$$

where \widehat{A}_t is defined by

$$\widehat{A}_t := \Big\{ \sup_{s \in [0,1]} |\widehat{\xi}_{ts} - \widehat{W}_{ts}| > t^{\frac{1}{2} - 2\varepsilon} \Big\}.$$

Therefore, there exists positive constant q (independent of ε) and $c_{26}(\varepsilon)$ such that

$$(3.50) \mathbf{P}_0^{\lambda_*} \left(\widehat{\tau}_{-w}^- \le m, \widehat{A}_m \right) \le \frac{C_3(2\varepsilon)}{m^{(\frac{1}{2} - 2\varepsilon)(\delta + 2) - 1}} \le \frac{c_{26}(\varepsilon)}{t^{2q}}.$$

Moreover, for $w > \varepsilon^{1/4} \sqrt{[t]}$, we have there exists a positive constant c_{27} such that

$$(3.51) \quad \mathbf{P}_{0}^{\lambda_{*}} \left(\widehat{\tau}_{-w}^{-} \leq m, A_{m}^{c} \right) = \mathbf{P}_{0}^{\lambda_{*}} \left(\inf_{s \in [0,m]} \widehat{\xi}_{s} < -w, \widehat{A}_{m}^{c} \right)$$

$$\leq \mathbf{P}_{0}^{\lambda_{*}} \left(\inf_{s \in [0,m]} \widehat{W}_{s} < m^{\frac{1}{2} - 2\varepsilon} - w \right) = \frac{2}{\sqrt{2\pi \Psi''(\lambda_{*})m}} \int_{w-m^{\frac{1}{2} - 2\varepsilon}}^{\infty} e^{-\frac{s^{2}}{2\Psi''(\lambda_{*})m}} ds$$

$$\leq c_{27} \int_{\frac{2\sqrt{\psi'''(\lambda_{*})m}}{2\sqrt{\Psi'''(\lambda_{*})m}}}^{\infty} e^{-\frac{s^{2}}{2}} ds \leq \frac{2c_{27}\sqrt{\Psi'''(\lambda_{*})m}}{w} e^{-\frac{w^{2}}{8\Psi'''(\lambda_{*})m}},$$

where in the last inequality we used the fact that $\int_a^\infty e^{-\frac{s^2}{2}} ds \leq \frac{1}{a} e^{-\frac{a^2}{2}}$ for any a > 0. Combining (3.51) and (3.50), for $w > \varepsilon^{1/4} \sqrt{[t]}$, since $\frac{\sqrt{m}}{w} \leq 1$, it holds that

$$\mathbf{P}_0^{\lambda_*} \left(\widehat{\tau}_{-w}^- \le m \right)^{1/2} \le c_{28} \phi \left(\frac{w}{\sqrt{\Psi''(\lambda_*)t}} \right) + \frac{c_{29}(\varepsilon)}{t^q},$$

for some positive constants c_{28} and $c_{29}(\varepsilon)$. Similarly, we can get that

$$\mathbf{P}_w^{\lambda_*} \left(\max_{s \in [0,m]} |\widehat{\xi_s}| > \frac{1}{2} \varepsilon^{1/6} \sqrt{[t]} \right) \le c_{30} \varepsilon^{1/6} + \frac{c_{31}(\varepsilon)}{t^{2q}},$$

for some positive constants c_{30} and $c_{31}(\varepsilon)$. Combining this with (3.49), we get there exist positive constants c_{32} and $c_{33}(\varepsilon)$ such that

$$\mathbf{P}_{w}^{\lambda_{*}}\left(\widehat{\xi}_{m}+2\varepsilon>\varepsilon^{1/6}\sqrt{[t]},\widehat{\tau}_{0}^{-}\leq m\right)\leq c_{32}\varepsilon^{1/12}\phi\left(\frac{w}{\sqrt{\Psi''(\lambda_{*})t}}\right)+\frac{c_{33}(\varepsilon)}{t^{q}},\quad w>\varepsilon^{1/4}\sqrt{[t]}.$$

This completes the proof of (3.45).

Lemma 3.10. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)**, **(H2)** and $\mathbf{E}_0[\xi_1] < 0$. Then one can find positive constants C_{11} and q with the property that for any $\varepsilon \in (0, \varepsilon_0)$ there exist positive constants $T_4(\varepsilon)$ and $C_{12}(\varepsilon)$ such that for any x > 0, $t > T_4(\varepsilon)$ and any Borel functions $h, H, g : \mathbb{R} \to \mathbb{R}_+$ satisfying $g \leq_{\varepsilon} h \leq_{\varepsilon} H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z) dz < \infty$,

$$\mathbf{E}_{x}^{\lambda^{*}}\left(h(\xi_{t})1_{\{\tau_{0}^{-}>t\}}\right) \geq \left(1 - C_{11}t^{-1/2} - C_{12}(\varepsilon)t^{-\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2}} \int_{\mathbb{R}_{+}} g(z+\varepsilon)\widehat{R}^{*}(z)dz$$

$$-C_{12}(\varepsilon) \left(1 + C_{11}\varepsilon t^{-1/2} + t^{-\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2}} \int_{\mathbb{R}_{+}} H(z - \varepsilon)\widehat{R}^{*}(z)dz$$
$$-\frac{C_{12}(\varepsilon)(1+x)}{\sqrt{t}} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}}\right) \int_{\mathbb{R}_{+}} H(z - \varepsilon)(1+z)dz.$$

Proof. Recall that the functions H_m and I_m are defined in (3.29) and (3.30). Fix $\varepsilon \in (0, \varepsilon_0)$ and let $h, H, g : \mathbb{R} \mapsto \mathbb{R}_+$ be Borel functions satisfying $g \leq_{\varepsilon} h \leq_{\varepsilon} H$ and $\int_{\mathbb{R}_+} H(z - \varepsilon)(1 + z) dz < \infty$. For any $g \in \mathbb{R}$, define

$$N_m(y) := \mathbf{E}_y^{\lambda_*} \left(g(\xi_m) 1_{\{\xi_m \ge \varepsilon\}} 1_{\{\tau_\varepsilon^- > m\}} \right).$$

Then for any y > 0 and $|v| \le \varepsilon$,

$$N_m(y) \le \mathbf{E}_y^{\lambda_*} \left(h(\xi_m + v) \mathbf{1}_{\{\xi_m \ge \varepsilon\}} \mathbf{1}_{\{\tau_{\varepsilon}^- > m\}} \right) \le \mathbf{E}_y^{\lambda_*} \left(h(\xi_m + v) \mathbf{1}_{\{\tau_{-v}^- > m\}} \right) = I_m(y + v).$$

Therefore, $N_m \leq_{\varepsilon} I_m \leq_{\varepsilon} H_m$. Applying Lemma 3.9 with $h = I_m$, we get that for $t - m > T_3(\varepsilon)$,

$$\begin{split} \mathbf{E}_{x}^{\lambda_{*}} \left(h(\xi_{t}) \mathbf{1}_{\{\tau_{0}^{-} > t\}} \right) &= \mathbf{E}_{x}^{\lambda_{*}} \left(I_{m}(\xi_{t-m}), \tau_{0}^{-} > t - m \right) \\ &\geq \frac{2R^{*}(x)}{\sqrt{2\pi} \Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} N_{m}(z) \mathbf{1}_{\{z \geq \varepsilon\}} \rho \left(\frac{z}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) \mathrm{d}z \\ &- \frac{2C_{9} \varepsilon R^{*}(x)}{\sqrt{2\pi} \Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} I_{m}(z) \rho \left(\frac{z}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) \mathrm{d}z \\ &- \frac{C_{9} \varepsilon^{1/12} R^{*}(x)}{\sqrt{2\pi} \Psi''(\lambda_{*})(t-m)} \int_{-\varepsilon}^{\infty} H_{m}(z) \phi \left(\frac{z}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) \mathrm{d}z \\ &- C_{10}(\varepsilon) (1+x) \|H_{m} \mathbf{1}_{[-\varepsilon,\infty)}\|_{1} \left(\frac{1}{(t-m)^{1+\delta/2}} + \frac{1}{(t-m)^{1+\varepsilon}} + \frac{1}{(t-m)^{1+q}} \right) =: \sum_{i=1}^{4} K_{i}, \end{split}$$

where q is the constant in Lemma 3.9. By (3.15), we have

$$K_{1} = \frac{2R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} \mathbf{E}_{z}^{\lambda_{*}} \left(g(\xi_{m}) \mathbf{1}_{\{\xi_{m} \geq \varepsilon\}} \mathbf{1}_{\{\tau_{\varepsilon}^{-} > m\}} \right) \mathbf{1}_{\{z \geq \varepsilon\}} \rho \left(\frac{z}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) dz$$
$$= \frac{2R^{*}(x)}{\sqrt{2\pi}\Psi''(\lambda_{*})(t-m)} \int_{\mathbb{R}_{+}} g(z+\varepsilon) \mathbf{E}_{z}^{\lambda_{*}} \left(\rho \left(\frac{\widehat{\xi}_{m} + \varepsilon}{\sqrt{\Psi''(\lambda_{*})(t-m)}} \right) \mathbf{1}_{\{\widehat{\tau}_{0}^{-} > m\}} \right) dz.$$

Repeating the argument leading to (3.36), we get that there exist positive constants c_1 (independent of ε) and $c_2(\varepsilon)$ such that

(3.52)
$$K_{1} \geq \left(1 - c_{1} t^{-1/2}\right) \frac{2R^{*}(x)}{\sqrt{2\pi \Psi''(\lambda_{*})^{3}} t^{3/2}} \int_{\mathbb{R}_{+}} g(z + \varepsilon) \widehat{R}^{*}(z) dz - \frac{2c_{2}(\varepsilon)R^{*}(x)}{\sqrt{2\pi \Psi''(\lambda_{*})^{3}} t^{3/2+\varepsilon}} \int_{\mathbb{R}_{+}} g(z + \varepsilon) (1 + z) dz.$$

Using an argument similar to that leading to (3.36), we get that there exist positive constants c_3 independent of ε and $c_4(\varepsilon)$ such that

(3.53)
$$K_{2} \geq -c_{3}\varepsilon(1 + c_{3}\varepsilon t^{-1/2}) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2}} \int_{\mathbb{R}_{+}} H(z - \varepsilon)\widehat{R}^{*}(z)dz$$
$$-c_{4}(\varepsilon) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{3/2+\varepsilon}} \int_{\mathbb{R}_{+}} H(z - \varepsilon)(1 + z)dz,$$

and

$$(3.54) K_3 \ge -c_3 \varepsilon^{1/12} (1 + c_3 \varepsilon t^{-1/2}) \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2}} \int_{\mathbb{R}_+} H(z - \varepsilon) \widehat{R}^*(z) dz - c_4(\varepsilon) \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3} t^{3/2+\varepsilon}} \int_{\mathbb{R}_+} H(z - \varepsilon) (1 + z) dz.$$

Moreover, by (3.38), we get that

$$(3.55) K_4 \ge -\frac{c_5(\varepsilon)(1+x)}{\sqrt{t}} \left(\frac{1}{t^{1+\varepsilon}} + \frac{1}{t^{1+\delta/2}} + \frac{1}{t^{1+q}} \right) \int_{\mathbb{R}_+} (1+z)H(z-\varepsilon) dz,$$

for some positive constant $c_5(\varepsilon)$. Combining (3.52), (3.53), (3.54) and (3.55), we get the desired result.

Proof of Theorem 3.6: Since $h: \mathbb{R} \to \mathbb{R}_+$ is a Borel function and $z \mapsto h(z)(1+|z|)$ is directly Riemann integrable, by [7, Lemma 2.3], there exists $a \in (0,1)$ such that $\int_{\mathbb{R}} \bar{h}_{a,\varepsilon}(1+|z|) dz < \infty$, for any $\varepsilon \in (0,a)$, where $\bar{h}_{a,\varepsilon}$ is defined in (3.18). Applying Lemma 3.8 to h, we have for $t > T_2(\varepsilon)$,

$$t^{3/2}\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})\mathbf{1}_{\{\tau_{0}^{-}>t\}}\right) \leq \left(1 + C_{7}t^{-1/2} + C_{7}\sqrt{\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}} \int_{\mathbb{R}_{+}} \bar{h}_{a_{m},\varepsilon}(z-\varepsilon)\widehat{R}^{*}(z)\mathrm{d}z$$
$$+ \frac{2C_{7}R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}t^{\varepsilon}} \int_{\mathbb{R}_{+}} \bar{h}_{a_{m},\varepsilon}(z-\varepsilon)(1+z)\mathrm{d}z$$
$$+ C_{8}(\varepsilon)(1+x)\left(\frac{1}{t^{\varepsilon}} + \frac{1}{t^{\delta/2}}\right) \int_{\mathbb{R}_{+}} \bar{h}_{a_{m},\varepsilon}(z-\varepsilon)(1+z)\mathrm{d}z,$$

where $a_m = 2^{-m}a$, $m \ge 0$. On the other hand, by Lemma 3.10, we have for $t > T_4(\varepsilon)$,

$$t^{3/2}\mathbf{E}_{x}^{\lambda_{*}}\left(h(\xi_{t})\mathbf{1}_{\{\tau_{0}^{-}>t\}}\right) \geq \left(1 - C_{11}t^{-1/2} - C_{12}(\varepsilon)t^{-\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}} \int_{\mathbb{R}_{+}} \underline{h}_{a_{m},\varepsilon}(z+\varepsilon)\widehat{R}^{*}(z)\mathrm{d}z$$
$$-C_{11}\varepsilon\left(1 + C_{11}\varepsilon t^{-1/2} + t^{-\varepsilon}\right) \frac{2R^{*}(x)}{\sqrt{2\pi\Psi''(\lambda_{*})^{3}}} \int_{\mathbb{R}_{+}} \bar{h}_{a_{m},\varepsilon}(z-\varepsilon)\widehat{R}^{*}(z)\mathrm{d}z$$
$$-C_{12}(\varepsilon)(1+x)\left(\frac{1}{t^{\varepsilon}} + \frac{1}{t^{\delta/2}} + \frac{1}{t^{q}}\right) \int_{\mathbb{R}_{+}} (1+z)\bar{h}_{a_{m},\varepsilon}(z-\varepsilon)\mathrm{d}z.$$

Since h is not almost everywhere 0 on $(0, \infty)$, we have

$$\int_{\mathbb{R}_+} h(z)\widehat{R}^*(z)dz \ge \widehat{R}^*(0) \int_{\mathbb{R}_+} h(z)dz > 0.$$

Thus,

$$(3.56) \qquad \limsup_{t \to \infty} \frac{t^{3/2} \mathbf{E}_x^{\lambda_*} \left(h(\xi_t) \mathbf{1}_{\{\tau_0^- > t\}} \right)}{\frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) dz} \le \left(1 + C_7 \sqrt{\varepsilon} \right) \limsup_{t \to \infty} I(\varepsilon, m),$$

and

$$\limsup_{t\to\infty} \frac{t^{3/2} \mathbf{E}_x^{\lambda_*} \left(h(\xi_t) \mathbf{1}_{\{\tau_0^- > t\}} \right)}{\frac{2R^*(x)}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) \mathrm{d}z} \ge \liminf_{t\to\infty} \left[J(\varepsilon, m) - C_{11} \varepsilon I(\varepsilon, m) \right],$$

where

$$I(\varepsilon,m) := \frac{\int_{\mathbb{R}_+} \bar{h}_{a_m,\varepsilon}(z-\varepsilon) \widehat{R}^*(z) dz}{\int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) dz}, \quad J(\varepsilon,m) := \frac{\int_{\mathbb{R}_+} \underline{h}_{a_m,\varepsilon}(z+\varepsilon) (1+z) dz}{\int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) dz}.$$

Repeating the argument for $I(y, \varepsilon, m)$ on [7, pp. 40–41], we get

$$\lim_{\varepsilon \to \infty} \limsup_{t \to \infty} I(\varepsilon, m) = 1.$$

This combined with (3.56) yields that

$$\limsup_{t \to \infty} t^{3/2} \mathbf{E}_x^{\lambda_*} \left(h(\xi_t) \mathbf{1}_{\{\tau_0^- > t\}} \right) \le \frac{2R^*(x)}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} h(z) \widehat{R}^*(z) dz.$$

The lower bound can be obtained in a similar way and this gives the desired result.

4. Proof of Theorem 1.1 and Theorem 1.2

In this section, we give the proofs of Theorems 1.1 and 1.2. For any x, t > 0 and $y \ge 0$, define

$$u(x,t) := \mathbb{P}_x(\zeta > t),$$

and

$$Q_y(x,t) := \mathbb{P}_x(M_t > y).$$

It is easy to see that

$$Q_0(x,t) := \mathbb{P}_x(M_t > 0) = \mathbb{P}_x(\zeta > t) = u(x,t).$$

The following result is [9, Lemma 2.1] which is true for any branching killed Lévy process.

Lemma 4.1. For any $h \in B_b^+(\mathbb{R}_+)$, the function

$$u_h(x,t) := \mathbb{E}_x \left[e^{-\int_{\mathbb{R}_+} h(y) X_t(dy)} \right], \quad t > 0, \ x > 0,$$

solves the equation

$$u_h(x,t) = \mathbf{E}_x \left[e^{-h(\xi_{t \wedge \tau_0^-})} \right] + \beta \mathbf{E}_x \left[\int_0^t \left(\sum_{k=0}^\infty p_k u_h(\xi_{s \wedge \tau_0^-}, t - s)^k - u_h(\xi_{s \wedge \tau_0^-}, t - s) \right) ds \right].$$

Consequently, $v_h(x,t) = 1 - u_h(x,t)$ satisfies

$$v_h(x,t) = \mathbf{E}_x \left[1 - e^{-h(\xi_{t \wedge \tau_0^-})} \right] - \mathbf{E}_x \left[\int_0^t \Phi(v_h(\xi_{s \wedge \tau_0^-}, t - s)) ds \right].$$

The next result is also valid for any branching killed Lévy process.

Lemma 4.2. For any x, t > 0 and $y \ge 0$, it holds that

$$Q_y(x,t) = e^{-\alpha t} \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_0^t \varphi(Q_y(\xi_s, t-s)) ds} \right].$$

Proof. For any x, t > 0 and $y \ge 0$, by the dominated convergence we have

$$1 - Q_y(x,t) = \mathbb{P}_x(M_t \le y) = \mathbb{P}_x(Z_t^0(y,\infty) = 0) = \lim_{\theta \to \infty} \mathbb{E}_x \left(e^{-\theta Z_t^0(y,\infty)} \right)$$
$$= \lim_{\theta \to \infty} \mathbb{E}_x \left(e^{-\int_{\mathbb{R}_+} \theta 1_{(y,\infty)}(z) Z_t^0(\mathrm{d}z)} \right) =: \lim_{\theta \to \infty} u_{\theta 1_{(y,\infty)}}(x,t).$$

Now applying Lemma 4.1 we get

$$Q_{y}(x,t) = \lim_{\theta \to \infty} \mathbf{E}_{x} \left[1 - e^{-\theta \mathbf{1}_{(y,\infty)}(\xi_{t \wedge \tau_{0}^{-}})} \right] - \mathbf{E}_{x} \left[\int_{0}^{t} \Phi(Q_{y}(\xi_{s \wedge \tau_{0}^{-}}, t - s)) ds \right]$$
$$= \mathbf{P}_{x} \left(\xi_{t \wedge \tau_{0}^{-}} > y \right) - \mathbf{E}_{x} \left[\int_{0}^{t} \Phi(Q_{y}(\xi_{s \wedge \tau_{0}^{-}}, t - s)) ds \right].$$

This combined with the Feynman-Kac formula gives the desired result.

The next lemma will be used to prove the lower bounds in Theorems 1.1 and 1.2.

Lemma 4.3. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1). Let x>0.

(1) If $\mathbf{E}_0[\xi_1] = 0$, then for any $y \geq 0$, we have

$$\liminf_{t \to \infty} \sqrt{t} e^{\alpha t} Q_{\sqrt{t}y}(x, t) \ge 2C_{sub} R(x) \phi_{\sigma^2}(y).$$

(2) If $\mathbf{E}_0[\xi_1] > 0$, we have

$$\liminf_{t \to \infty} e^{\alpha t} u(x, t) \ge C_{sub} \mathbf{E}_0 \left[\xi_1 \right] W^{(0)}(x).$$

Moreover, for any $y \in \mathbb{R}$, we have

$$\liminf_{t \to \infty} e^{\alpha t} Q_{\sqrt{t}y + \mathbf{E}_0[\xi_1]t}(x, t) \ge C_{sub} \mathbf{E}_0[\xi_1] W^{(0)}(x) \int_{\frac{y}{\sigma}}^{\infty} \phi(z) dz.$$

(3) Assume further that (H2) holds. If $\mathbf{E}_0[\xi_1] < 0$, then for any $y \ge 0$, we have

$$\liminf_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} Q_y(x, t) \ge \frac{2C_{sub} R^*(x) e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_y^{\infty} e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

Proof. For any $y \geq 0$, by the definition of Q, we have

$$Q_y(x,t) \le \mathbb{P}_x(\zeta > t) \le \mathbb{P}_x(\widetilde{\zeta} > t) = g(t).$$

It follows from Lemma 4.2 that

$$(4.1) Q_{y}(x,t) = e^{-\alpha t} \mathbf{E}_{x} \left[1_{\{\tau_{0}^{-} > t, \xi_{t} > y\}} e^{-\int_{0}^{t} \varphi(Q_{y}(\xi_{s}, t-s)) ds} \right]$$

$$\geq e^{-\alpha t} e^{-\int_{0}^{t} \varphi(g(t-s)) ds} \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > y \right) \geq C_{sub} e^{-\alpha t} \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > y \right).$$

Applying Lemma 3.1, we immediately get the assertion of (1). Using the fact that $u(x,t) = Q_0(x,t)$ and applying Theorem 2.1, we get the first result of (2). Applying Lemma 2.3 with y replaced by $\sqrt{t}y + \mathbf{E}_0[\xi_1]t$, we get the second result of (2). Applying Theorem 3.5 with $f(x) = 1_{(y,\infty)}(x)$, we get the assertion of (3).

In the following three lemmas, we prove the upper bounds in Theorems 1.1 and 1.2.

Lemma 4.4. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1). If $\mathbf{E}_0[\xi_1] = 0$, then for any x > 0 and $y \geq 0$, we have

$$\limsup_{t \to \infty} \sqrt{t} e^{\alpha t} Q_{\sqrt{t}y}(x, t) \le 2C_{sub} R(x) \phi_{\sigma^2}(y).$$

Proof. Fix an N > 0. For t > N, by Lemma 4.2, we have

$$Q_{\sqrt{t}y}(x,t) \leq e^{-\alpha t} \mathbf{E}_{x} \left[1_{\{\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y\}} e^{-\int_{t-N}^{t} \varphi(Q_{\sqrt{t}y}(\xi_{s},t-s)) ds} \right]$$

$$\leq e^{-\alpha t} \mathbf{E}_{x} \left[1_{\{\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y\}} e^{-\int_{0}^{N} \varphi(Q_{\sqrt{t}y}(\inf_{r \in [t-N,t]} \xi_{r},s)) ds} \right].$$

Take a $\gamma \in (0, \frac{1}{2})$ and define

$$J_1(t) := \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y, \inf_{r \in [t-N,t]} \xi_r \ge \sqrt{t}y + t^{\gamma}\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right],$$

$$J_2(t) := \mathbf{E}_x \left[\mathbf{1}_{\{\tau_0^- > t, \xi_t > \sqrt{t}y, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + t^\gamma\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\inf_{r \in [t-N,t]} \xi_r, s)) \mathrm{d}s} \right].$$

Then $Q_{\sqrt{ty}}(x,t) \leq e^{-\alpha t}(J_1(t) + J_2(t))$. Since $Q_{\sqrt{ty}}(x,t)$ is increasing in x, we have

$$(4.2) J_1(t) \le e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\sqrt{t}y+t^{\gamma},s))ds} \mathbf{P}_x \left(\tau_0^- > t, \xi_t > \sqrt{t}y\right).$$

By (4.1) and (1.5), we have

$$Q_{\sqrt{t}y}(x,t) \ge g(t)\mathbf{P}_x\left(\tau_0^- > t, \xi_t > \sqrt{t}y\right).$$

Thus,

$$e^{-\int_0^N \varphi(Q_{\sqrt{t}y}(\sqrt{t}y+t^{\gamma},s))\mathrm{d}s} \le \exp\Big\{-\int_0^N \varphi\left(g(s)\mathbf{P}_{\sqrt{t}y+t^{\gamma}}\left(\tau_0^- > s, \xi_s > \sqrt{t}y\right)\right)\Big\}.$$

Plugging this into (4.2) and applying the dominated convergence theorem, we get

$$\limsup_{N \to \infty} \limsup_{t \to \infty} \frac{J_1(t)}{\mathbf{P}_x \left(\tau_0^- > t, \xi_t > \sqrt{ty}\right)} \le \limsup_{N \to \infty} e^{-\int_0^N \varphi(g(s)) ds} = C_{sub}.$$

Therefore, by Lemma 3.1, we have

$$\limsup_{N \to \infty} \limsup_{t \to \infty} \sqrt{t} J_1(t) \le 2C_{sub} R(x) \phi_{\sigma^2}(y).$$

Now we show that

$$\lim_{t \to \infty} \sqrt{t} J_2(t) = 0.$$

For any $\epsilon > 0$ and t > N, it holds that

$$J_{2}(t) \leq \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y, \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + t^{\gamma} \right)$$

$$\leq \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \sqrt{t}y < \xi_{t} \leq \sqrt{t}(y+\epsilon) \right)$$

$$+ \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}(y+\epsilon), \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + t^{\gamma} \right).$$

By Lemma 3.1, we have

$$\lim_{t \to \infty} \sqrt{t} \mathbf{P}_x \left(\tau_0^- > t, \sqrt{t} y < \xi_t \le \sqrt{t} (y + \epsilon) \right) = \frac{2R(x)}{\sqrt{2\pi\sigma^2}} \int_{\frac{y}{\sigma}}^{\frac{y+\epsilon}{\sigma}} \rho(z) dz \xrightarrow{\epsilon \to 0} 0.$$

For any t > 0 and $\kappa \in (0, \frac{\delta}{2(2+\delta)})$, define

(4.4)
$$A_t := \left\{ \sup_{s \in [0,1]} |\xi_{ts} - \xi_0 - W_{ts}| > t^{\frac{1}{2} - \kappa} \right\},$$

where W is the Brownian motion in Lemma 3.2. Then by the Markov property of ξ , for k < t - N,

$$\mathbf{P}_{x}\left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}(y+\epsilon), \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + t^{\gamma}\right)$$

$$= \mathbf{E}_{x}\left(1_{\{\tau_{0}^{-} > k\}} \mathbf{P}_{\xi_{k}}\left(\tau_{0}^{-} > t - k, \xi_{t-k} > \sqrt{t}(y+\epsilon), \inf_{r \in [t-k-N,t-k]} \xi_{r} < \sqrt{t}y + t^{\gamma}\right)\right)$$

$$\leq H_{1}(t) + H_{2}(t),$$

where

$$H_1(t) := \mathbf{E}_x \left(1_{\{\tau_0^- > k\}} \mathbf{P}_{\xi_k} \left(\tau_0^- > t - k, \xi_{t-k} > \sqrt{t}(y + \epsilon), A_{t-k} \right) \right),$$

$$H_2(t) := \mathbf{E}_x \left(\mathbf{P}_{\xi_k} \left(\xi_{t-k} > \sqrt{t}(y + \epsilon), \inf_{r \in [t-k-N, t-k]} \xi_r < \sqrt{t}y + t^{\gamma}, A_{t-k}^c \right) \right).$$

To prove (4.3), we only need to prove

(4.5)
$$\limsup_{t \to \infty} \sqrt{t} H_1(t) = 0, \quad \text{and} \quad \limsup_{t \to \infty} \sqrt{t} H_2(t) = 0.$$

Using (3.23) and Lemma 3.2, we get that for any k < t,

$$H_1(t) \le \frac{CN_*(\kappa)}{(t-k)^{(\frac{1}{2}-\kappa)(\delta+2)-1}} \frac{1+x}{\sqrt{k}},$$

where C > 0 is a constant. Taking $k = \frac{t}{2}$, we get that

$$\limsup_{t \to \infty} \sqrt{t} H_1(t) = 0.$$

For any z > 0, we have

$$\mathbf{P}_{z}\left(\xi_{t-k} > \sqrt{t}(y+\epsilon), \inf_{r \in [t-k-N, t-k]} \xi_{r} < \sqrt{t}y + t^{\gamma}, A_{t-k}^{c}\right)$$

$$\leq \mathbf{Q}_{z}\left(W_{t-k} > \sqrt{t}(y+\epsilon) - t^{\frac{1}{2}-\kappa}, \inf_{r \in [t-k-N, t-k]} W_{r} < \sqrt{t}y + t^{\gamma} + t^{\frac{1}{2}-\kappa}\right),$$

where (W_t, \mathbf{Q}_z) is a mean 0 Brownian motion with diffusion coefficient σ^2 , starting from z. Therefore, for any z > 0, using the reflection principle for Brownian motion, we get

$$\lim_{t \to \infty} \sqrt{t} \mathbf{P}_{z} \left(\xi_{t-k} > \sqrt{t} (y+\epsilon), \inf_{r \in [t-k-N, t-k]} \xi_{r} < \sqrt{t} y + t^{\gamma}, A_{t-k}^{c} \right)$$

$$\leq \lim_{t \to \infty} \sqrt{t} \mathbf{Q}_{0} \left(\inf_{r \in [0,N]} W_{r} < -\epsilon \sqrt{t} + t^{\gamma} + 2t^{\frac{1}{2}-\kappa} \right)$$

$$= \lim_{t \to \infty} \sqrt{t} \mathbf{Q}_{0} \left(\max_{r \in [0,N]} W_{r} > \epsilon \sqrt{t} - t^{\gamma} - 2t^{\frac{1}{2}-\kappa} \right) = 0,$$

which implies that

$$\lim_{t \to \infty} \sqrt{t} H_2(t) = 0.$$

Then (4.5) follows from (4.6) and (4.7), and we complete the proof.

Lemma 4.5. Assume that (1.2) holds and that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process satisfying (H1). If $\mathbf{E}_0[\xi_1] > 0$, then for any x > 0, we have

$$\limsup_{t \to \infty} e^{\alpha t} u(x, t) \le C_{sub} \mathbf{E}_0 \left[\xi_1 \right] W^{(0)}(x).$$

Moreover, for any $y \in \mathbb{R}$, we have

$$\limsup_{t \to \infty} e^{\alpha t} Q_{\sqrt{t}y + \mathbf{E}_0[\xi_1]t}(x, t) \le C_{sub} \mathbf{E}_0[\xi_1] W^{(0)}(x) \int_{\frac{y}{z}}^{\infty} \phi(z) dz.$$

Proof. Take $\gamma \in (0, \frac{1}{2})$ and fix an N > 0. For t > N, we have

$$Q_{\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t}(x,t) \leq e^{-\alpha t}\mathbf{E}_{x} \left[1_{\{\tau_{0}^{-}>t,\xi_{t}>\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t\}} e^{-\int_{t-N}^{t} \varphi(Q_{\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t}(\xi_{s},t-s))\mathrm{d}s} \right]$$

$$\leq e^{-\alpha t}\mathbf{E}_{x} \left[1_{\{\tau_{0}^{-}>t,\xi_{t}>\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t\}} e^{-\int_{0}^{N} \varphi(Q_{\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t}(\inf_{r\in[t-N,t]}\xi_{r},s))\mathrm{d}s} \right]$$

$$=: e^{-\alpha t} (J_{1}(t) + J_{2}(t)),$$

where

$$J_1(t) := \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0[\xi_1]t, \inf_{r \in [t-N,t]} \xi_r \ge \sqrt{t}y + \mathbf{E}_0[\xi_1]t + t^{\gamma}\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y + \mathbf{E}_0[\xi_1]t}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right],$$

$$J_2(t) := \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0[\xi_1]t, \inf_{r \in [t-N,t]} \xi_r < \sqrt{t}y + \mathbf{E}_0[\xi_1]t + t^{\gamma}\}} e^{-\int_0^N \varphi(Q_{\sqrt{t}y + \mathbf{E}_0[\xi_1]t}(\inf_{r \in [t-N,t]} \xi_r, s)) ds} \right].$$

Since $Q_{\sqrt{t}y+\mathbf{E}_0[\xi_1]t}(x,t)$ is increasing in x, we have

$$(4.8) \quad J_{1}(t) \leq e^{-\int_{0}^{N} \varphi(Q_{\sqrt{t}y + \mathbf{E}_{0}[\xi_{1}]t}(\sqrt{t}y + \mathbf{E}_{0}[\xi_{1}]t + t^{\gamma}, s)) ds} \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y + \mathbf{E}_{0}[\xi_{1}]t \right).$$

By (4.1) and (1.5),

$$Q_{\sqrt{t}y+\mathbf{E}_{0}\left[\xi_{1}\right]t}(x,t) \geq g(t)\mathbf{P}_{x}\left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y + \mathbf{E}_{0}\left[\xi_{1}\right]t\right).$$

Thus,

$$e^{-\int_0^N \varphi(Q_{\sqrt{t}y+\mathbf{E}_0[\xi_1]t}(\sqrt{t}y+\mathbf{E}_0[\xi_1]t+t^{\gamma},s))\mathrm{d}s}$$

$$\leq \exp\left\{-\int_{0}^{N} \varphi\left(g(s)\mathbf{P}_{\sqrt{t}y+\mathbf{E}_{0}[\xi_{1}]t+t^{\gamma}}\left(\tau_{0}^{-}>s,\xi_{s}>\sqrt{t}y+\mathbf{E}_{0}\left[\xi_{1}\right]t\right)\right)\right\}.$$

Plugging this into (4.8) and applying the dominated convergence theorem, we get

$$(4.9) \limsup_{N \to \infty} \limsup_{t \to \infty} \frac{J_1(t)}{\mathbf{P}_x\left(\tau_0^- > t, \xi_t > \sqrt{t}y + \mathbf{E}_0\left[\xi_1\right]t\right)} \le \limsup_{N \to \infty} e^{-\int_0^N \varphi(g(s)) \mathrm{d}s} = C_{sub}.$$

Therefore, by Lemma 2.3, we have

$$\limsup_{N \to \infty} \limsup_{t \to \infty} J_1(t) \le C_{sub} \mathbf{E}_0 \left[\xi_1 \right] W^{(0)}(x) \int_{\frac{y}{\sigma}}^{\infty} \phi(z) dz.$$

Next, we show that $\lim_{t\to\infty} J_2(t) = 0$. For $\epsilon > 0$, it holds that

$$J_{2}(t) \leq \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \xi_{t} > \sqrt{t}y + \mathbf{E}_{0} \left[\xi_{1} \right] t, \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + \mathbf{E}_{0} \left[\xi_{1} \right] t + t^{\gamma} \right)$$

$$\leq \mathbf{P}_{x} \left(\tau_{0}^{-} > t, \sqrt{t}y + \mathbf{E}_{0} \left[\xi_{1} \right] t < \xi_{t} \leq \sqrt{t} (y + \epsilon) + \mathbf{E}_{0} \left[\xi_{1} \right] t \right)$$

$$+ \mathbf{P}_{x} \left(\xi_{t} > \sqrt{t} (y + \epsilon) + \mathbf{E}_{0} \left[\xi_{1} \right] t, \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + \mathbf{E}_{0} \left[\xi_{1} \right] t + t^{\gamma} \right).$$

By Lemma 2.3, we have

(4.10)
$$\lim_{t \to \infty} \mathbf{P}_x \left(\tau_0^- > t, \sqrt{t}y + \mathbf{E}_0 \left[\xi_1 \right] t < \xi_t \le \sqrt{t} (y + \epsilon) + \mathbf{E}_0 \left[\xi_1 \right] t \right)$$
$$= \frac{\mathbf{E}_0 \left[\xi_1 \right] W^{(0)}(x)}{\sqrt{2\pi}} \int_{\frac{y}{\sigma}}^{\frac{y+\epsilon}{\sigma}} e^{-\frac{z^2}{2}} dz \xrightarrow{\epsilon \to 0} 0.$$

Recall the event A_t defined in (4.4). Then we have

$$\mathbf{P}_{x}\left(\xi_{t} > \sqrt{t}(y+\epsilon) + \mathbf{E}_{0}\left[\xi_{1}\right]t, \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + \mathbf{E}_{0}\left[\xi_{1}\right]t + t^{\gamma}\right)$$

$$\leq \mathbf{P}_{x}\left(A_{t}\right) + \mathbf{P}_{x}\left(\xi_{t} > \sqrt{t}(y+\epsilon) + \mathbf{E}_{0}\left[\xi_{1}\right]t, \inf_{r \in [t-N,t]} \xi_{r} < \sqrt{t}y + \mathbf{E}_{0}\left[\xi_{1}\right]t + t^{\gamma}, A_{t}^{c}\right)$$

By Lemma 3.2, we have $\lim_{t\to\infty} \mathbf{P}_x(A_t) = 0$. Furthermore, using the reflection principle for Brownian motion, we get that, as $t\to\infty$,

$$\mathbf{P}_{x}\left(\xi_{t} > \sqrt{t}(y+\epsilon) + \mathbf{E}_{0}\left[\xi_{1}\right]t, \inf_{r\in\left[t-N,t\right]}\xi_{r} < \sqrt{t}y + \mathbf{E}_{0}\left[\xi_{1}\right]t + t^{\gamma}, A_{t}^{c}\right)$$

$$\leq \mathbf{Q}_{x}\left(W_{t} > \sqrt{t}(y+\epsilon) + \mathbf{E}_{0}\left[\xi_{1}\right]t - t^{\frac{1}{2}-\kappa}, \inf_{r\in\left[t-N,t\right]}W_{r} < \sqrt{t}y + \mathbf{E}_{0}\left[\xi_{1}\right]t + t^{\gamma} + t^{\frac{1}{2}-\kappa}\right)$$

$$\leq \mathbf{Q}_{0}\left(\inf_{r\in\left[0,N\right]}W_{r} < -\varepsilon\sqrt{t} + t^{\gamma} + 2t^{\frac{1}{2}-\kappa}\right) = \mathbf{Q}_{0}\left(\max_{r\in\left[0,N\right]}W_{r} > \varepsilon\sqrt{t} - t^{\gamma} - 2t^{\frac{1}{2}-\kappa}\right) \to 0,$$

where (W_t, \mathbf{Q}_x) is a mean 0 Brownian motion with diffusion coefficient σ^2 , starting from x. This combined with (4.9) and (4.10) gives the desired result.

Lemma 4.6. Fix an N > 0. Assume that (1.2) holds and that $((\xi_t)_{t \geq 0}, (\mathbf{P}_x)_{x \in \mathbb{R}})$ is a spectrally negative Lévy process satisfying **(H1)** and **(H2)**. If $\mathbf{E}_0[\xi_1] < 0$, then we have

$$\lim_{t \to \infty} \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_{t-N}^t \varphi(Q_y(\xi_s, t-s)) ds} \right]$$

$$= e^{(\alpha - \Psi(\lambda_*))N} \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y)e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

Proof. By the Markov Property,

$$\begin{split} & \mathbf{E}_{x} \left[\mathbf{1}_{\{\tau_{0}^{-} > t, \xi_{t} > y\}} e^{-\int_{t-N}^{t} \varphi(Q_{y}(\xi_{s}, t-s)) \mathrm{d}s} \right] \\ & = \mathbf{E}_{x} \left[\mathbf{1}_{\{\tau_{0}^{-} > t-N\}} \mathbf{E}_{\xi_{t-N}} \left[\mathbf{1}_{\{\tau_{0}^{-} > N, \xi_{N} > y\}} e^{-\int_{0}^{N} \varphi(Q_{y}(\xi_{s}, N-s)) \mathrm{d}s} \right] \right] \\ & =: \mathbf{E}_{x} \left[\mathbf{1}_{\{\tau_{0}^{-} > t-N\}} f_{N}^{y}(\xi_{t-N}) \right], \end{split}$$

where for any $z \geq 0$, f_N^y is defined by

$$f_N^y(z) := \mathbf{E}_z \left[1_{\{\tau_0^- > N, \xi_N > y\}} e^{-\int_0^N \varphi(Q_y(\xi_s, N - s)) ds} \right].$$

By Lemma 4.2,

$$(4.11) f_N^y(z) = e^{\alpha N} Q_y(z, N),$$

which implies that $f_N^y(z)$ is bounded and increasing with respect to z. Then f_N^y is a.e. continuous. Applying Theorem 3.5 to f_N^y , we get that

$$\lim_{t \to \infty} t^{3/2} e^{-\Psi(\lambda_*)t} \mathbf{E}_x \left[\mathbf{1}_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_{t-N}^t \varphi(Q_y(\xi_s, t-s)) ds} \right]$$

$$= e^{-\Psi(\lambda_*)N} \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_{\mathbb{R}_+} f_N^y(z) e^{-\lambda_* z} \widehat{R}^*(z) dz,$$

which gives the desired result together with (4.11).

Proofs of Theorems 1.1 and 1.2: Combining Lemmas 4.3, 4.4 and 4.5, we get parts (1) and (2) of both Theorem 1.1 and Theorem 1.2 immediately. Next, we prove part (3) of both theorems. For $\mathbf{E}_0[\xi_1] < 0$, fix N > 0 and $y \ge 0$. By Lemma 4.2, we have for $t \ge N$,

(4.12)
$$Q_y(x,t) \le e^{-\alpha t} \mathbf{E}_x \left[1_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_{t-N}^t \varphi(Q_y(\xi_s, t-s)) ds} \right].$$

Combining this with Lemma 4.6, we get that

$$(4.13) \qquad \limsup_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} Q_y(x, t)$$

$$\leq \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi \Psi''(\lambda_*)^3}} \liminf_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

Moreover, using the fact that $Q_y(x,t) \leq g(t)$ and Lemma 4.2, we get

$$Q_y(x,t) \ge e^{-\alpha t} \mathbf{E}_x \left[\mathbf{1}_{\{\tau_0^- > t, \xi_t > y\}} e^{-\int_{t-N}^t \varphi(Q_y(\xi_s, t-s)) ds} \right] e^{-\int_0^{t-N} \varphi(g(t-s)) ds}.$$

Using Lemma 4.6 again, we have

(4.14)
$$\lim_{t \to \infty} \inf t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} Q_y(x, t)$$

$$\geq \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \lim_{N \to \infty} \sup e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y)e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

Combining (4.13) and (4.14), we obtain that

$$\begin{split} &\lim_{t\to\infty} t^{3/2} e^{(\alpha-\Psi(\lambda_*))t} Q_y(x,t) \\ &= \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \lim_{N\to\infty} e^{(\alpha-\Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_*z} \widehat{R}^*(z) \mathrm{d}z := \frac{2R^*(x)e^{\lambda_*x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} C_y, \end{split}$$

where $C_y := \lim_{N \to \infty} e^{(\alpha - \Psi(\lambda_*))N} \int_{\mathbb{R}_+} \mathbb{P}_z(M_N > y) e^{-\lambda_* z} \widehat{R}^*(z) dz$. Next, we show that $C_y \in (0, \infty)$. First, applying Lemma 4.3 (3), we get

$$C_y \ge C_{sub} \int_y^\infty e^{-\lambda_* z} \widehat{R}^*(z) \mathrm{d}z > 0.$$

Using (4.12) and taking $f(x) = 1_{(y,\infty)}(x)$ in Theorem 3.5, we get

$$\limsup_{t \to \infty} t^{3/2} e^{(\alpha - \Psi(\lambda_*))t} Q_y(x, t) \leq \lim_{t \to \infty} t^{3/2} e^{-\Psi(\lambda_*)t} \mathbf{P}_x \left(\tau_0^- > t, \xi_t > y\right)$$

$$= \frac{2R^*(x)e^{\lambda_* x}}{\sqrt{2\pi\Psi''(\lambda_*)^3}} \int_y^\infty e^{-\lambda_* z} \widehat{R}^*(z) dz.$$

Therefore, $C_y \leq \int_u^\infty e^{-\lambda_* z} \widehat{R}^*(z) dz < \infty$. This completes the proof.

Proof of Corollary 1.3: We only prove (3). Combining Theorem 1.1 and 1.2, for any 0 < a < b, we get that

$$\lim_{t \to \infty} \mathbb{P}_x \left(M_t \in (a, b] | \zeta > t \right) = \lim_{t \to \infty} \frac{Q_a(x, t) - Q_b(x, t)}{u(x, t)} = \frac{\int_a^b e^{-\lambda_* z} \widehat{R}^*(z) dz}{\int_0^\infty e^{-\lambda_* z} \widehat{R}^*(z) dz}.$$

Therefore, there exists a random variable (X, \mathbb{P}) such that $\mathbb{P}_x(M_t \in \cdot | \zeta > t)$ vaguely converge to $\mathbb{P}(X \in \cdot)$. Moreover, by (4.1), we have

$$\mathbb{P}_{x}\left(M_{t} > y | \zeta > t\right) = \frac{Q_{y}(x, t)}{u(x, t)}$$

$$\leq \frac{e^{-\alpha t} \mathbf{P}_{x}\left(\tau_{0}^{-} > t, \xi_{t} > y\right)}{C_{sub}e^{-\alpha t} \mathbf{P}_{x}\left(\tau_{0}^{-} > t\right)} = \frac{1}{C_{sub}} \mathbf{P}_{x}\left(\xi_{t} > y | \tau_{0}^{-} > t\right).$$

Thus by Theorem 3.5, the tightness of M_t under $\mathbb{P}_x(\cdot|\zeta>t)$ follows from the tightness of ξ_t under $\mathbf{P}_x(\cdot|\tau_0^->t)$. This gives the desired result.

5. Proof of Theorem 1.4

For any 0 < x < y, define

$$v(x,y) := \mathbb{P}_x(M > y).$$

The following result is valid for any branching killed Lévy processes.

Lemma 5.1. For any 0 < x < y, it holds that

$$v(x,y) = \mathbf{E}_x \left[1_{\{\tau_y^+ < \tau_0^-\}} e^{-\alpha \tau_y^+ - \int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right],$$

where φ is defined by (1.1). Consequently, for 0 < x < z < y, by the strong Markov property, we have

$$v(x,y) = \mathbf{E}_x \left[1_{\{\tau_z^+ < \tau_0^-\}} v(\xi_{\tau_z^+}, y) e^{-\alpha \tau_z^+ - \int_0^{\tau_z^+} \varphi(v(\xi_s, y)) ds} \right].$$

Proof. For 0 < x < y, comparing the first branching time with τ_y^+ , we have

$$\begin{split} v(x,y) &= \int_{0}^{\infty} \beta e^{-\beta s} \mathbf{P}_{x}(\tau_{y}^{+} < \tau_{0}^{-}, \tau_{y}^{+} \leq s) \mathrm{d}s \\ &+ \int_{0}^{\infty} \beta e^{-\beta s} \mathbf{E}_{x} \left(\left(1 - \sum_{k=0}^{\infty} p_{k} \left(1 - v(\xi_{s}, y) \right)^{k} \right) \mathbf{1}_{\{\tau_{y}^{+} \wedge \tau_{0}^{-} > s\}} \right) \\ &= \mathbf{E}_{x} \left(e^{-\beta \tau_{y}^{+}} \mathbf{1}_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} \right) + \int_{0}^{\infty} \beta e^{-\beta s} \mathbf{E}_{x} \left(\left(1 - \sum_{k=0}^{\infty} p_{k} \left(1 - v(\xi_{s}, y) \right)^{k} \right) \mathbf{1}_{\{\tau_{y}^{+} \wedge \tau_{0}^{-} > s\}} \right) \mathrm{d}s. \end{split}$$

By [4, Lemma 4.1], the above equation is equivalent to

$$v(x,y) + \beta \int_0^\infty \mathbf{E}_x \left(v(\xi_s, y) \mathbf{1}_{\{\tau_y^+ \wedge \tau_0^- > s\}} \right) ds$$

$$= \mathbf{P}_x \left(\tau_y^+ < \tau_0^- \right) + \beta \int_0^\infty \mathbf{E}_x \left(\left(1 - \sum_{k=0}^\infty p_k \left(1 - v(\xi_s, y) \right)^k \right) \mathbf{1}_{\{\tau_y^+ \wedge \tau_0^- > s\}} \right) ds,$$

which is also equivalent to

$$v(x,y) = \mathbf{P}_x \left(\tau_y^+ < \tau_0^- \right) - \mathbf{E}_x \left(\int_0^{\tau_y^+ \wedge \tau_0^-} \Phi(v(\xi_s, y)) \mathrm{d}s \right),$$

where Φ is defined in (1.1). Using the Feynman-Kac formula, we get that

$$v(x,y) = \mathbf{E}_x \left(1_{\{\tau_y^+ < \tau_0^-\}} e^{-\alpha \tau_y^+ - \int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right).$$

This gives the desired result.

Lemma 5.2. Assume that $((\xi_t)_{t\geq 0}, (\mathbf{P}_x)_{x\in\mathbb{R}})$ is a spectrally negative Lévy process. For any 0 < x < y, we have

(5.1)
$$v(x,y) \le \frac{e^{x\psi(\alpha)} W_{\psi(\alpha)}^{(0)}(x)}{e^{y\psi(\alpha)} W_{\psi(\alpha)}^{(0)}(y)} \le e^{(x-y)\psi(\alpha)}.$$

Proof. Since the function φ is non-negative, combining Lemma 5.1 and Theorem 2.1 (2), we get

$$v(x,y) \le \mathbf{E}_x \left[e^{-\alpha \tau_y^+} 1_{\{\tau_y^+ < \tau_0^-\}} \right] = \frac{W^{(\alpha)}(x)}{W^{(\alpha)}(y)}, \quad x < y.$$

This combined with Lemma 2.2 yields that

$$v(x,y) \le \frac{e^{x\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(x)}{e^{y\psi(\alpha)}W_{\psi(\alpha)}^{(0)}(y)} \le e^{(x-y)\psi(\alpha)}.$$

This gives the desired result.

Proof of Theorem 1.4: By Lemmas 5.1 and 2.4, we have

$$v(x,y) = \mathbf{E}_{x} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\alpha \tau_{y}^{+} - \int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s},y)) ds} \right]$$
$$= e^{(x-y)\psi(\alpha)} \mathbf{E}_{x}^{\psi(\alpha)} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s},y)) ds} \right].$$

Fix a $\gamma \in (0,1)$, by the Markov property of $(\xi_t, \mathbf{P}_x^{\psi(\alpha)})$, we have

$$(5.2) \quad v(x,y) := e^{(x-y)\psi(\alpha)} \mathbf{E}_{x}^{\psi(\alpha)} \left[1_{\{\tau_{y-y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y-y}^{+}} \varphi(v(\xi_{s},y)) ds} \right]$$

$$\times \mathbf{E}_{y-y}^{\psi(\alpha)} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s},y)) ds} \right] =: e^{(x-y)\psi(\alpha)} A_{1}(x,y) A_{2}(y),$$

where

$$A_1(x,y) := \mathbf{E}_x^{\psi(\alpha)} \left[1_{\{\tau_{y-y^{\gamma}}^+ < \tau_0^-\}} e^{-\int_0^{\tau_{y-y^{\gamma}}^+} \varphi(v(\xi_s,y)) \mathrm{d}s} \right],$$

$$A_2(y) := \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[1_{\{\tau_y^+ < \tau_0^-\}} e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right].$$

We first consider the asymptotic behavior of $A_1(x,y)$ as $y\to\infty$. We claim that

(5.3)
$$\lim_{y \to \infty} \left[\mathbf{P}_x^{\psi(\alpha)} (\tau_{y-y^{\gamma}}^+ < \tau_0^-) - A_1(x,y) \right] = 0.$$

Indeed, using the inequality $1 - e^{-|x|} \le |x|$, we get

$$(5.4) \quad 0 \leq \mathbf{P}_{x}^{\psi(\alpha)}(\tau_{y-y^{\gamma}}^{+} < \tau_{0}^{-}) - A_{1}(x,y)$$

$$= \mathbf{E}_{x}^{\psi(\alpha)} \left[1_{\{\tau_{y-y^{\gamma}}^{+} < \tau_{0}^{-}\}} \left(1 - e^{-\int_{0}^{\tau_{y-y^{\gamma}}^{+}} \varphi(v(\xi_{s},y)) \mathrm{d}s} \right) \right]$$

$$\leq \mathbf{E}_{x}^{\psi(\alpha)} \left[1_{\{\tau_{y-y^{\gamma}}^{+} < \tau_{0}^{-}\}} \int_{0}^{\tau_{y-y^{\gamma}}^{+}} \varphi(v(\xi_{s},y)) \mathrm{d}s \right] \leq \mathbf{E}_{x}^{\psi(\alpha)} \left[\int_{0}^{\tau_{y-y^{\gamma}}^{+}} \varphi(v(\xi_{s},y)) \mathrm{d}s \right].$$

Set $y_*(x) := \inf\{t \geq y - y^{\gamma}, t - x \in \mathbb{N}\}$. By (5.1), we have

$$\mathbf{E}_{x}^{\psi(\alpha)} \left[\int_{0}^{\tau_{y-y^{\gamma}}^{+}} \varphi(v(\xi_{s}, y)) \mathrm{d}s \right] \leq \mathbf{E}_{x}^{\psi(\alpha)} \left[\int_{0}^{\tau_{y_{*}(x)}^{+}} \varphi\left(e^{(\xi_{s}-y)\psi(\alpha)}\right) \mathrm{d}s \right]$$

$$= \sum_{k=0}^{y_{*}(x)-x-1} \mathbf{E}_{x}^{\psi(\alpha)} \left[\int_{\tau_{x+k}^{+}}^{\tau_{x+k+1}^{+}} \varphi\left(e^{(\xi_{s}-y)\psi(\alpha)}\right) \mathrm{d}s \right]$$

$$\leq \sum_{k=0}^{y_{*}(x)-x-1} \mathbf{E}_{x}^{\psi(\alpha)} \left(\tau_{x+k+1}^{+} - \tau_{x+k}^{+}\right) \varphi\left(e^{\psi(\alpha)(x+k+1-y)}\right)$$

$$= \mathbf{E}_{0}^{\psi(\alpha)} \left(\tau_{1}^{+}\right) \sum_{k=0}^{y_{*}(x)} \varphi\left(e^{-\psi(\alpha)(y-x-1-y_{*}(x)+k)}\right).$$

By the definition of $y_*(x)$, we have that for y large enough,

$$y-x-1-y_*(x) \ge y-x-1-(y-y^{\gamma}+1)=y^{\gamma}-x-2.$$

Therefore, when y is sufficient large so that $y^{\gamma} - x - 2 \ge y^{\gamma/2}$, by (1.3), we have

$$\mathbf{E}_{x}^{\psi(\alpha)} \left[\int_{0}^{\tau_{y-y^{\gamma}}^{+}} \varphi(v(\xi_{s}, y)) \mathrm{d}s \right]$$

$$\leq \mathbf{E}_{0}^{\psi(\alpha)} (\tau_{1}) \sum_{k=1}^{\infty} \varphi\left(e^{-\psi(\alpha)(y^{\gamma/2}+k)}\right) \leq \mathbf{E}_{0}^{\psi(\alpha)} (\tau_{1}) \int_{0}^{\infty} \varphi\left(e^{-\psi(\alpha)(y^{\gamma/2}+z)}\right) \mathrm{d}z$$

$$= \mathbf{E}_{0}^{\psi(\alpha)} (\tau_{1}) \int_{y^{\gamma/2}}^{\infty} \varphi\left(e^{-\psi(\alpha)z}\right) \mathrm{d}z \xrightarrow{y \to \infty} 0.$$

This combined with (5.4) yields (5.3). Using Lemma 2.4 and Theorem 2.1(2), we get

$$\lim_{y \to \infty} A_1(x, y) = \lim_{y \to \infty} \mathbf{P}_x^{\psi(\alpha)} (\tau_{y-y^{\gamma}}^+ < \tau_0^-)$$

$$= \lim_{y \to \infty} e^{(y - y^{\gamma} - x)\psi(\alpha)} \mathbf{E}_x \left[e^{-\alpha \tau_{y - y^{\gamma}}^+} 1_{\{\tau_{y - y^{\gamma}}^+ < \tau_0^-\}} \right] = \lim_{y \to \infty} e^{(y - y^{\gamma} - x)\psi(\alpha)} \frac{W^{(\alpha)}(x)}{W^{(\alpha)}(y - y^{\gamma})}.$$

Using (2.1), we get that

$$W^{(\alpha)}(y-y^{\gamma}) \sim \frac{e^{\psi(\alpha)(y-y^{\gamma})}}{\Psi'(\psi(\alpha))}.$$

Therefore,

(5.5)
$$\lim_{y \to \infty} A_1(x, y) = e^{-\psi(\alpha)x} \Psi'(\psi(\alpha)) W^{(\alpha)}(x).$$

Next, we consider the asymptotic behavior of $A_2(y)$ as $y \to \infty$. Recall that

$$A_{2}(y) = \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right]$$

$$= \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right] - \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[1_{\{\tau_{y}^{+} \ge \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right].$$

We claim that

(5.6)
$$\lim_{y \to \infty} \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right] = C_*(\alpha) \in (0, 1],$$

and

(5.7)
$$\lim_{y \to \infty} \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[1_{\{\tau_y^+ \ge \tau_0^-\}} e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right] = 0.$$

Then we get

$$\lim_{y \to \infty} A_2(y) = C_*(\alpha).$$

Combining (5.2), (5.5) and (5.8) gives that

$$\lim_{y \to \infty} e^{y^{\gamma} \psi(\alpha)} v(x, y) = C_*(\alpha) \Psi'(\psi(\alpha)) W^{(\alpha)}(x),$$

which gives the desired result. Now we are left to prove (5.6) and (5.7). By Lemma 2.4 and Theorem 2.1, we have

$$\mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[1_{\{\tau_{y}^{+} \geq \tau_{0}^{-}\}} e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right] \leq \mathbf{P}_{y-y^{\gamma}}^{\psi(\alpha)} \left(\tau_{y}^{+} \geq \tau_{0}^{-} \right) = 1 - \mathbf{P}_{y-y^{\gamma}}^{\psi(\alpha)} \left(\tau_{y}^{+} < \tau_{0}^{-} \right)$$

$$= 1 - e^{y^{\gamma} \psi(\alpha)} \mathbf{E}_{y-y^{\gamma}} \left[e^{-\alpha \tau_{y}^{+}} 1_{\{\tau_{y}^{+} < \tau_{0}^{-}\}} \right] = 1 - e^{y^{\gamma} \psi(\alpha)} \frac{W^{(\alpha)}(y - y^{\gamma})}{W^{(\alpha)}(y)}$$

which tends to 0 as $y \to \infty$ by (2.1). Thus (5.7) is valid. To prove (5.6), for any y > 0, define

$$G(y) := \mathbf{E}_{y-y^{\gamma}}^{\psi(\alpha)} \left[e^{-\int_0^{\tau_y^+} \varphi(v(\xi_s, y)) ds} \right].$$

For any z > y, by the translation invariance and the strong Markov property of ξ , we have

$$G(z) = \mathbf{E}_{z-z^{\gamma}}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{z}^{+}} \varphi(v(\xi_{s},z)) ds} \right] = \mathbf{E}_{0}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{z}^{+}} \varphi(v(\xi_{s}+z-z^{\gamma},z)) ds} \right]$$

$$= \mathbf{E}_{0}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{z}^{+} - y^{\gamma}} \varphi(v(\xi_{s}+z-z^{\gamma},z)) ds} \right] \mathbf{E}_{z^{\gamma} - y^{\gamma}}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{z}^{+}} \varphi(v(\xi_{s}+z-z^{\gamma},z)) ds} \right],$$

where the first term of the above display is dominated by 1 from above and the second term is equal to $\mathbf{E}_0^{\psi(\alpha)} \left[e^{-\int_0^{\tau_y^{+\gamma}} \varphi(v(\xi_s + z - y^{\gamma}, z)) \mathrm{d}s} \right]$. It follows that

(5.9)
$$G(z) \le \mathbf{E}_0^{\psi(\alpha)} \left[e^{-\int_0^{\tau_{y^{\gamma}}^+} \varphi(v(\xi_s + z - y + y - y^{\gamma}, z - y + y)) \mathrm{d}s} \right].$$

Note that for any w > 0, it holds that

$$v(x+w,y+w) = \mathbb{P}_{x+w} \left(\exists \ t > 0, u \in N_t \ s.t. \ \min_{s \le t} X_u(s) > 0, X_u(t) > y + w \right)$$
$$\ge \mathbb{P}_{x+w} \left(\exists \ t > 0, u \in N_t \ s.t. \ \min_{s \le t} X_u(s) > w, X_u(t) > y + w \right) = v(x,y).$$

This combined with (5.9) gives that for z > y,

$$G(z) \le \mathbf{E}_0^{\psi(\alpha)} \left[e^{-\int_0^{\tau_y^{+\gamma}} \varphi(v(\xi_s + y - y^{\gamma}, y)) ds} \right] = G(y).$$

Thus, the limit $C_*(\alpha) := \lim_{y\to\infty} G(y)$ exists. It is obvious that $C_*(\alpha) \leq 1$. Next, we only need to show $C_*(\alpha) > 0$. We assume without loss of generality that y is an integer. By the strong Markov property and Jensen's inequality,

$$G(y) = \frac{\mathbf{E}_{0}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right]}{\mathbf{E}_{0}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right]} \ge \mathbf{E}_{0}^{\psi(\alpha)} \left[e^{-\int_{0}^{\tau_{y}^{+}} \varphi(v(\xi_{s}, y)) ds} \right]$$
$$\ge \exp \left\{ -\sum_{n=1}^{y} \mathbf{E}_{0}^{\psi(\alpha)} \left[\int_{\tau_{n-1}^{+}}^{\tau_{n}^{+}} \varphi(v(\xi_{s}, y)) ds \right] \right\}.$$

By (1.3) and (5.1), we get

$$\int_{\tau_{n-1}^+}^{\tau_n^+} \varphi(v(\xi_s, y)) ds \le (\tau_n^+ - \tau_{n-1}^+) \varphi(v(n, y)) \le (\tau_n^+ - \tau_{n-1}^+) \varphi\left(e^{(n-y)\psi(\alpha)}\right).$$

Note that under $\mathbf{P}_0^{\psi(\alpha)}$, $\{\tau_n^+ - \tau_{n-1}^+\}$ are i.i.d. random variables with finite first moment. Therefore,

$$G(y) \ge \exp\left\{-\sum_{n=1}^{y} \varphi\left(e^{(n-y)\psi(\alpha)}\right) \mathbf{E}_{0}^{\psi(\alpha)} \left[\left(\tau_{n}^{+} - \tau_{n-1}^{+}\right)\right]\right\}$$

$$= \exp\left\{-\mathbf{E}_{0}^{\psi(\alpha)}\left(\tau_{1}^{+}\right) \sum_{n=0}^{y-1} \varphi\left(e^{-n\psi(\alpha)}\right)\right\} \ge \exp\left\{-\mathbf{E}_{0}^{\psi(\alpha)}\left(\tau_{1}^{+}\right) \sum_{n=0}^{\infty} \varphi\left(e^{-n\psi(\alpha)}\right)\right\},$$

which implies that

$$C_*(\alpha) \ge \exp\left\{-\mathbf{E}_0^{\psi(\alpha)}(\tau_1^+) \sum_{n=0}^{\infty} \varphi\left(e^{-n\psi(\alpha)}\right)\right\}.$$

According to (1.3), we have

$$\sum_{n=0}^{\infty} \varphi\left(e^{-n\psi(\alpha)}\right) \le \varphi(1) + \int_{0}^{\infty} \varphi\left(e^{-z\psi(\alpha)}\right) dz < \infty,$$

which implies that $C_*(\alpha) > 0$. This gives the desired result.

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