From 0 to 3: Intermediate phases between normal and anomalous spreading of two-type branching Brownian motion*

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Abstract

The logarithmic correction for the order of the maximum of a two-type reducible branching Brownian motion on the real line exhibits a double jump when the parameters (the ratio of the diffusion coefficients of the two types of particles, and the ratio of the branching rates the two types of particles) cross the boundary of the anomalous spreading region identified by Biggins.

In this paper, we further examine this double jump phenomenon by studying a two-type reducible branching Brownian motion on the real line with its parameters depend on the time horizon t. We show that when the parameters approach the boundaries of the anomalous spreading region in an appropriate way, the order of the maximum can interpolate smoothly between different surrounding regimes. We also determine the asymptotic law of the maximum and characterize the extremal process.

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1 Introduction and main results

Branching Brownian motion (BBM) is a probabilistic model that describes the evolution of a population of individuals. This model has been intensively studied and continues to be the subject of many recent researches. A large literature focused on the link between BBM and the F-KPP reaction-diffusion equation, introduced in [21] and [25]. For results in this direction, see [10, 18, 19, 22, 26, 31, 32] and the references therein. Understanding the spatial spread of such a population, particularly the propagation of the front, is a fundamental question and has attracted significant interest, see e.g. [1, 3, 5, 11, 14, 17, 20, 27]. The insights and methods used in studying the extreme values of BBM are applicable to a large class of probabilistic models, including the two-dimensional discrete Gaussian free field, epsilon-cover time of the two-dimensional torus by Brownian motion, and characteristic polynomials of random matrices. For further details, we refer our reader to the lecture notes [24, 36] and the reviews [2, 6].

Multitype branching Brownian motion is a natural extension of BBM that can be used to describe the evolution a population composed of different types or species. In the irreducible case (where each type of individual can have descendants of all types), it behaves in some sense like an *effective* single-type BBM, see e.g. [23, 34]. This paper focuses on a two-type reducible case (where individuals of type 2 cannot have descendants of type 1), which is the simplest setting in which the maximum exhibits a phase transition not observed in the case of single-type BBM.

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Biggins [12, 13] gave a comprehensive description of the leading order of the maximum of the two-type reducible BBM. There are three cases $\mathscr{C}_I, \mathscr{C}_{II}$ and $\mathscr{C}_{III}$ for the maximum. In $\mathscr{C}_I$, the maximum is equal to the speed of individuals of type 1; in $\mathscr{C}_{II}$, the maximum is equal to the speed of individuals of type 2; and in $\mathscr{C}_{III}$, the spreading speed of the two-type process is strictly larger than the speeds of a single-type BBM of particles of type 1 or type 2. $\mathscr{C}_I$ and $\mathscr{C}_{II}$ are referred to as normal spreading regions, while $\mathscr{C}_{III}$ is referred to as the *anomalous spreading* region. The regime to which the process belongs depends on the ratio β of the branching rates and the ratio σ^2 of the diffusion coefficients for individuals of types 1 and 2. Belloum and Mallein [8] obtained the logarithmic correction of the maximum and the limiting extremal process when (β, σ^2) are interior points of $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$. Further studies on the boundary case by Belloum in [7] and by the authors in [28] completed the phase diagram for the maximum (see Figure 1). Notably, a double jump occurs in the logarithmic correction for the order of the maximum when the parameters (β, σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$.

A further interesting question is to make the logarithmic correction smoothly interpolates between normal spreading cases and anomalous spreading case. Similar problems for variable speed BBM were investigated in [15] and [35]: The logarithmic correction for the order of the maximum for two-speed BBM changes discontinuously when approaching slopes $\sigma_1^2 = \sigma_2^2 = 1$, which corresponds to standard BBM. Bovier and Huang [15] further studied this transition by choosing $\sigma_1^2 = 1 \pm t^{-\alpha}$ and $\sigma_2^2 = 1 \mp t^{-\alpha}$, and showed that the logarithmic correction for the order of the maximum smoothly interpolates between the correction in the i.i.d. case $\frac{1}{2\sqrt{2}} \log t$, standard BBM case $\frac{3}{2\sqrt{2}} \log t$, and $\frac{6}{2\sqrt{2}} \log t$ when $\alpha \in (0, 1/2)$. Inspired by these two papers, we study in this paper a two-type reducible BMM with parameters depending on the time horizon t. We assume that the parameters (β_t, σ_t^2) approach the boundary of $\mathscr{C}_{III}$ appropriately. We show that the logarithmic correction for the maximum smoothly interpolates between the normal spreading cases and anomalous spreading case. Moreover, we find the asymptotic law of the maximum and characterize the extremal process, which turns out to coincide (up to a constant) with that of a two-type reducible BBM with parameters $(\lim_{t\to\infty} \beta_t, \lim_{t\to\infty} \sigma_t^2)$.

1.1 Branching Brownian motions

A BBM on the real line can be described as follows: Initially, there is a particle which moves as a Brownian motion with diffusion coefficient σ^2 starting from the origin. At rate β, the initial particle splits into two particles. The offspring particles start moving from their place of birth independently, with same diffusion coefficient and obeying the same branching rule. We denote this process by $\{(\mathsf{X}_u^{\beta,\sigma^2}(t), u \in \mathsf{N}_t)_{t\geq 0}, \mathsf{P}\}$, where N_t is the set of all particles alive at time t and $\mathsf{X}_u^{\beta,\sigma^2}(t)$ is the position of an individual $u \in \mathsf{N}_t$. If $\beta = \sigma^2 = 1$, we call $\{\mathsf{X}_u^{1,1}(t)\}$ the standard BBM and write $\{\mathsf{X}_u(t)\}$ for short. The scaling property of Brownian motion implies that

$$\left(\mathsf{X}_u^{\beta,\sigma^2}(t) : u \in \mathsf{N}_t\right) \overset{\text{law}}{=} \left(\frac{\sigma}{\sqrt{\beta}} \mathsf{X}_u(\beta t) : t \in \mathsf{N}_{\beta t}\right).$$

Let $\mathsf{M}_t^{\beta,\sigma^2} = \max_{u\in\mathsf{N}_t} \mathsf{X}_u^{\beta,\sigma^2}(t)$ be the maximum of BBM at time t. It is well-known that the centered maximum $\mathsf{M}_t^{\beta,\sigma^2}$ converges in distribution to a randomly shifted Gumbel random variable (see [17, 18, 27]). More precisely, if

$$v = \sqrt{2\beta\sigma^2} \quad \text{and} \quad \theta = \sqrt{2\beta/\sigma^2},$$

then

$$\lim_{t\to\infty} \mathsf{P}\left(\mathsf{M}_t^{\beta,\sigma^2} - vt + \frac{3}{2\theta}\log t \leq x\right) = \mathsf{E}\left[\exp\left\{-C\mathsf{Z}_\infty^{\beta,\sigma^2} e^{-\theta x}\right\}\right]$$

for some constant C depending on β, σ^2, where $\mathsf{Z}_\infty^{\beta,\sigma^2}$ is the almost sure limit of the *derivative martingale* $(\mathsf{Z}_t^{\beta,\sigma^2})_{t>0}$ defined by $\mathsf{Z}_t^{\beta,\sigma^2} := \sum_{u\in\mathsf{N}_t}[vt - \mathsf{X}_u^{\beta,\sigma^2}(t)] \exp\{\theta \mathsf{X}_u^{\beta,\sigma^2}(t) - 2\beta t\}$. The name "derivative martingale" comes from the fact that $\mathsf{Z}_t^{\beta,\sigma^2} = -\frac{\partial}{\partial\lambda}|_{\lambda=\theta}\mathsf{W}_t^{\beta,\sigma^2}(\lambda)$, where $\mathsf{W}_t^{\beta,\sigma^2}(\lambda) := \sum_{u\in\mathsf{N}_t}\exp\{\lambda\mathsf{X}_u^{\beta,\sigma^2}(t)-(\beta+\frac{\lambda^2\sigma^2}{2})t\}$ are the *additive martingales* for BBM.

The construction of the limiting extremal process for BBM, obtained independently in [1] and [5], gives a deeper understanding of the extreme value statistics for BBM. Precisely,

$$\lim_{t\to\infty}\sum_{u\in\mathsf{N}_t}\delta_{\mathsf{X}_u(t)-\sqrt{2}t+\frac{3}{2\sqrt{2}}\log t} = \mathrm{DPPP}\left(\sqrt{2}C_\star\mathsf{Z}_\infty\mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x,\mathfrak{D}^{\sqrt{2}}\right)\ \text{in law}, \tag{1.1}$$

where DPPP $(\mu,\mathfrak{D})$ stands for a *decorated Poisson point process* with intensity measure μ and decoration law $\mathfrak{D}$. Given a (random) measure μ on $\mathbb{R}$ and a point process $\mathfrak{D}$ on $\mathbb{R}$, let $\sum_i\delta_{x_i}$ be a Poisson point process with intensity μ and let $\left(\sum_j\delta_{d_j^i}: i\geq 0\right)$ be an independent family of i.i.d. point processes with common law $\mathfrak{D}$, then $\sum_{i,j}\delta_{x_i+d_j^i}$ is a decorated Poisson point process with intensity measure μ and decoration law $\mathfrak{D}$. The decoration law $\mathfrak{D}^{\sqrt{2}}$ in (1.1) belongs to a family of "gap processes" $\left(\mathfrak{D}^\varrho,\varrho\geq\sqrt{2}\right)$ (see [9, 16]), defined as

$$\mathfrak{D}^\varrho(\cdot) := \lim_{t\to\infty}\mathsf{P}\left(\sum_{u\in\mathsf{N}_t}\delta_{\mathsf{X}_u(t)-\mathsf{M}_t}\in\cdot\mid\mathsf{M}_t\geq\varrho t\right). \tag{1.2}$$

1.2 Two-type reducible branching Brownian motions

In this paper, we study the following two-type reducible branching Brownian motion: Type 1 particles move according to a Brownian motion with diffusion coefficient σ^2, branch at rate β into two children of type 1 and give birth to particles of type 2 at rate 1; type 2 particles move as a standard Brownian motion and branch at rate 1 into 2 children of type 2, but cannot give birth to offspring of type 1. For $t\geq 0$, we use N_t to denote the total number of particles alive at time t. We can further categorize these particles into type 1 and 2, represented by N_t^1 and N_t^2 respectively. The position of an individual $u\in N_t$ is denoted by $X_u(t)$. The maximum position at time t is represented by $M_t=\max_{u\in N_t}X_u(t)$. Finally, the law that the two-type BBM starts with a type 1 particle at the origin is denoted by $\mathbb{P}^{\beta,\sigma^2}$.

The extremal value statistics of the two-type system behaves like that of the single-type BBM. The centered maximum $(M_t-m^{\beta,\sigma^2}(t),\mathbb{P}^{\beta,\sigma^2})$ converges in law to a random shifted Gumbel distribution, with a proper centering $m^{\beta,\sigma^2}(t)$ of the form $l(\beta,\sigma^2)\,t-s(\beta,\sigma^2)\log t$. Additionally, the extremal process $(\sum_{u\in N_t}\delta_{X_u(t)-m^{\beta,\sigma^2}(t)},\mathbb{P}^{\beta,\sigma^2})$ converges in law to a certain decorated Poisson point process. However, an intriguing phase transition occurs in the centering $m^{\beta,\sigma^2}(t)$ of the maximum of the two-type BBM (see Table 1 and Figure 1), but not in single type BBMs, due to the significant contribution of the added type 2 particles to the maximum in some situations.

Divide the parameter space $(\beta,\sigma^2)\in\mathbb{R}_+^2$ into three regions (see Figure 1)

$$\begin{aligned}
\mathscr{C}_I &= \left\{(\beta,\sigma^2):\sigma^2>\frac{1}{\beta}1_{\{\beta\leq 1\}}+\frac{\beta}{2\beta-1}1_{\{\beta>1\}}\right\},\\
\mathscr{C}_{II} &= \left\{(\beta,\sigma^2):\sigma^2<\frac{1}{\beta}1_{\{\beta\leq 1\}}+(2-\beta)1_{\{\beta>1\}}\right\},\\
\mathscr{C}_{III} &= \left\{(\beta,\sigma^2):\sigma^2+\beta>2\text{ and }\sigma^2<\frac{\beta}{2\beta-1}\right\};
\end{aligned}$$

and define $\mathscr{B}_{i,j}=\partial\mathscr{C}_i\cap\partial\mathscr{C}_j\backslash\{(1,1)\}$ for $i\neq j$ and $i,j\in\{I,II,III\}$. If $(\beta,\sigma^2)\in\mathscr{C}_I$, the order of the maximum of the two-type process is the same as that of particles of type 1 alone, and the asymptotic behavior of the extremal process is dominated by the long-time behavior of particles of type 1. If

$(\beta,\sigma^2) \in \mathscr{C}_{II} \cup \mathscr{B}_{I,II}$ the asymptotic behavior of particles of type 2 dominates the extremal process. If $(\beta,\sigma^2) \in \mathscr{C}_{III}$, the so-called anomalous spreading region, the speed of the two-type process is strictly larger than the speeds of both single type particle systems. Extreme values can only be achieved by descendants of first-generation type 2 particles born during a certain time interval and within a certain space interval.

To present the known results in a clear and accessible manner, we summarize the different regimes of the maximum and extremal process of the two-type BBM in a table. For cases $\mathscr{C}_I, \mathscr{C}_{II}, \mathscr{C}_{III}$, we refer to [8]. The case (1,1) was discussed in [7], and cases $\mathscr{B}_{I,II}, \mathscr{B}_{I,III}, \mathscr{B}_{II,III}$ were covered in [28]. Recall that the family of decoration laws $(\mathfrak{D}^\rho : \rho \geq \sqrt{2})$ are defined in (1.2).

Regime	Correct centering $m^{\beta,\sigma^2}(t)$	Limiting extremal process
$\mathscr{C}_I$	$\sqrt{2\beta\sigma^2}t - \frac{3}{2\theta}\log t$	$\mathrm{DPPP}(C\mathsf{Z}_\infty^{\beta,\sigma^2} e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}_{(I)})^1$
$\mathscr{C}_{II} \cup \mathscr{B}_{I,II}$	$\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$	$\mathrm{DPPP}(C\bar{Z}_\infty^{\beta,\sigma^2} e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}})^2$
$\mathscr{C}_{III}$	$v^* t = \frac{\beta-\sigma^2}{\sqrt{2(\beta-1)(1-\sigma^2)}}t$	$\mathrm{DPPP}(C\mathsf{W}_\infty^{\beta,\sigma^2}(\theta^*) e^{-\theta^* x}\,\mathrm{d}x, \mathfrak{D}^{\theta^*})$
$\mathscr{B}_{II,III}$	$\sqrt{2}t - \frac{1}{2\sqrt{2}}\log t$	$\mathrm{DPPP}(C\mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2}) e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}})$
$\mathscr{B}_{I,III} \cup \{(1,1)\}$	$\sqrt{2\beta\sigma^2}t - \frac{1}{2\theta}\log t$	$\mathrm{DPPP}(C\mathsf{Z}_\infty^{\beta,\sigma^2} e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}^{\theta})$

Table 1: Five regimes of limiting behavior of $(\sum_{u\in N_t} \delta_{X_u(t)-m^{\beta,\sigma^2}(t)}, \mathbb{P}^{\beta,\sigma^2})$.

To better visualize these results, we draw the phase diagrams for the maximum and the limiting extremal process in Figure 1. One can see that the leading coefficient $l(\beta,\sigma^2)$ is a continuous function of (β,σ^2). However the subleading coefficient $s(\beta,\sigma^2)$ exhibits discontinuity. Notably, a double jump in the maximum is observed when (β,σ^2) crosses the boundary of the anomalous spreading region $\mathscr{C}_{III}$.

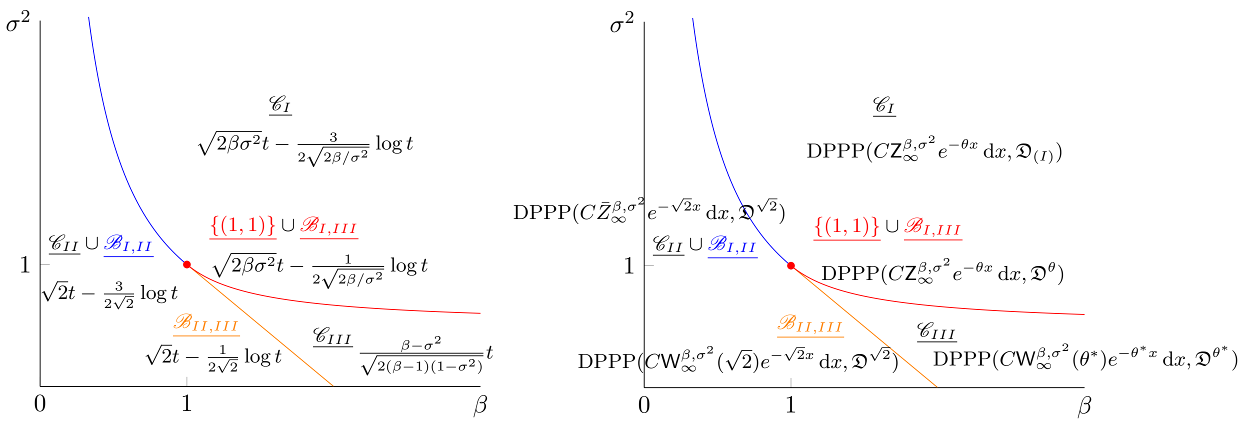

Figure 1: Phase diagram for the maximum and extremal process of two type reducible BBM

Inspired by papers [15] and [35], we further study the apparent discontinuities in the maximum that occur when the parameters (β,σ^2) cross the boundary of $\mathscr{C}_{III}$. For this, we assume that the parameters (β_t,σ_t^2) depend on the time horizon t in an explicit way and approach the boundary of $\mathscr{C}_{III}$ appropriately. Then we show that the logarithmic correction for the maximum now smoothly interpolates between the normal spreading case ($\mathscr{C}_I$ or $\mathscr{C}_{II}$), the boundary case ($\mathscr{B}_{I,III}$, $\{(1,1)\}$ or $\mathscr{B}_{I,III}$) and the anomalous spreading case $\mathscr{C}_{III}$.

[1]The decoration process $\mathfrak{D}_{(I)}$ was obtained implicitly in [8, Theorem 1.1]

[2]The random variable $\bar{Z}_\infty^{\beta,\sigma^2}$ is a composition of derivative martingale and additive martingale, see [8, Lemma 5.3]

Before presenting our results, we provide a very simple example to illustrate the idea as follows. Consider the function $f_t(x) = x^t$ for $x \geq 0$, $t > 0$. Clearly for fixed x, as $t \to \infty$, it holds that $f_t(x) \to 0$ if $x < 1$, $f_t(x) \to 1$ if $x = 1$ and $f_t(x) \to \infty$ if $x > 1$. Hence f has a double jump at $x = 1$. To get a continuous phase transition, we let x depend on t and approach the critical point 1 appropriately. We define $x_{t,h} = 1 - \frac{h}{t}$, where h stands for the proximity of $x_{t,h}$ and 1. Then $\lim_{t\to\infty} f_t(x_{t,h}) = e^{-h}$ which continuously interpolates between 1 and 0 as h runs over $(0, \infty]$. Similarly by letting $x_t = 1+\frac{h}{t}$, $\lim_{t\to\infty} f(x_{t,h})$ interpolates continuously between 1 and ∞. Our main results bear similarities to this example, but achieving the same goal in our problem is nontrivial and poses greater challenges.

1.3 Main results

As suggested by the previous simple example, we fix a time horizon $t > 0$ and run a two-type reducible BBM $\{X_u(s) : u \in N_s, s \leq t\}$ up to time t under the law $\mathbb{P}^{\beta_t,\sigma_t^2}$. That is, during time $s \in [0, t]$, type 1 particles have branching rate β_t and diffusion coefficient σ_t^2, while type 2 particles are standard. We need to find parameters (β_t, σ_t^2) that properly approximate a given point (β, σ^2) on the boundary of the anomalous spreading region $\mathscr{C}_{III}$. To do this, we introduce our choice for approximations.

Given parameters $(\beta, \sigma^2) \in \mathscr{B}_{I,III} = \{\frac{1}{\beta} + \frac{1}{\sigma^2} = 2, \beta > 1\}$ and $h \in (0, \infty]$, we say $(\beta_t, \sigma_t^2)_{t>0}$ is an h-admissible approximation for (β, σ^2) from $\mathscr{C}_{III}$, denoted by $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,+}_{(\beta,\sigma^2)}$, if

$$(\beta_t, \sigma_t^2) \in \mathscr{C}_{III} \ , \ \frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 + \frac{1}{t^h} \text{ for large } t \text{ and } (\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \text{ as } t \to \infty. \tag{1.3}$$

(Here $h = \infty$ we use the notation $\frac{1}{\infty} = 0$). Similarly, we say $(\beta_t, \sigma_t^2)_{t>0}$ is an h-admissible approximation for (β, σ^2) from $\mathscr{C}_I$, denoted by $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$, if

$$(\beta_t, \sigma_t^2) \in \mathscr{C}_{I} \ , \ \frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 - \frac{1}{t^h} \text{ for large } t \text{ and } (\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \text{ as } t \to \infty. \tag{1.4}$$

Throughout this paper, for given (β_t, σ_t^2), we set

$$\theta_t = \sqrt{2\beta_t/\sigma_t^2}, \quad v_t = \sqrt{2\beta_t\sigma_t^2} \quad \text{and} \quad v_t^* = \frac{\beta_t - \sigma_t^2}{\sqrt{2(\beta_t - 1)(1 - \sigma_t^2)}}.$$

Theorem 1.1. *Let $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$, $h > 0$. Define*

$$m^{1,3}_{h,+}(t) = v_t^* t - \frac{\min\{h, 1/2\}}{\theta_t} \log t \ ; \ m^{1,3}_{h,-}(t) = v_t t - \frac{3 - 4\min\{h, 1/2\}}{2\theta_t} \log t.$$

Then for $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,\pm}_{(\beta,\sigma^2)}$, defined in (1.3) and (1.4), we have

$$\lim_{t\to\infty} \left(\sum_{u\in N_t} \delta_{X_u(t) - m^{1,3}_{h,\pm}(t)}, \mathbb{P}^{\beta_t,\sigma_t^2} \right) = \text{DPPP}\left(C_{h,\pm}\theta \mathsf{Z}^{\beta,\sigma^2}_\infty e^{-\theta x}\, \mathrm{d}x, \mathfrak{D}^\theta \right) \text{ in law,}$$

for some constants $C_{h,\pm}$ depending only on h and (β, σ^2).

Remark 1.2. Recall Table 1. For $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$ and $(\beta_t, \sigma_t^2) \in \mathscr{A}^{h,\pm}_{(\beta,\sigma^2)}$, Theorem 1.1 shows that, for $h \geq \frac{1}{2}$, the perturbation is so small that the limiting behavior of the extreme values of BBM under $\mathbb{P}^{\beta_t,\sigma_t^2}$ and $\mathbb{P}^{\beta,\sigma^2}$ (i.e., no perturbation) are the same! For $(\beta_t, \sigma_t^2) \in \mathscr{A}^{h,+}_{(\beta,\sigma^2)}$, as h changes from $\frac{1}{2}$ to 0, the coefficient for the log correction for the maximum changes smoothly from 1 (corresponding to the regime $\mathscr{B}_{I,III}$) to 0 (corresponding to the regime $\mathscr{C}_{III}$). For $(\beta_t, \sigma_t^2) \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$, as h changes from $\frac{1}{2}$ to 0, the coefficient changes smoothly from 1 (corresponding to the regime $\mathscr{B}_{I,III}$) and 3 (corresponding to the regime $\mathscr{C}_I$).

Given $(\beta, \sigma^2) \in \mathscr{B}_{II,III} = \{\beta+\sigma^2 = 2, \beta > 1\}$ and $h \in (0, \infty]$, we say $(\beta_t, \sigma_t^2)_{t>0}$ is an h-admissible approximation for (β, σ^2) from $\mathscr{C}_{III}$, denoted by $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,+}_{(\beta,\sigma^2)}$, if

$$\beta_t + \sigma_t^2 = 2 + \frac{1}{t^h} \ , \ (\beta_t, \sigma_t^2) \in \mathscr{C}_{III} \text{ for large } t \text{ and } (\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \text{ as } t \to \infty. \tag{1.5}$$

We say $(\beta_t, \sigma_t^2)_{t>0}$ is an h-admissible approximation for (β, σ^2) from $\mathscr{C}_{II}$, denoted by $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$, if

$$\beta_t + \sigma_t^2 = 2 - \frac{1}{t^h} \ , \ (\beta_t, \sigma_t^2) \in \mathscr{C}_{II} \text{ for large } t \text{ and } (\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \text{ as } t \to \infty. \tag{1.6}$$

Theorem 1.3. *Let $(\beta, \sigma^2) \in \mathscr{B}_{II,III}$ and $h \in (0, \infty]$. Define*

$$m^{2,3}_{h,+}(t) := v_t^* t - \frac{\min\{h, 1/2\}}{\sqrt{2}} \log t \ ; \ m^{2,3}_{h,-}(t) := \sqrt{2}t - \frac{3 - 4\min\{h, 1/2\}}{2\sqrt{2}} \log t.$$

Then for $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,\pm}_{(\beta,\sigma^2)}$, defined in (1.5) and (1.6), we have

$$\lim_{t\to\infty} \left(\sum_{u\in N_t} \delta_{X_u(t) - m^{2,3}_{h,\pm}(t)}, \mathbb{P}^{\beta_t,\sigma_t^2} \right) = \mathrm{DPPP}\left(C_{h,\pm}\sqrt{2}\mathsf{W}^{\beta,\sigma^2}_\infty(\sqrt{2}) e^{-\sqrt{2}x} \,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}} \right) \text{ in law,}$$

for some constants $C_{h,\pm}$ depending only on h and (β, σ^2).

Theorem 1.3 has a similar explanation as in Remark 1.2.

Finally, we introduce the h-admissible approximation for $(1, 1)$ from $\mathscr{C}_I, \mathscr{C}_{II}$, and $\mathscr{C}_{III}$ as follows respectively.

- Let $\mathscr{A}^{h,1}_{(1,1)}$ be the collection of all $(\beta_t, \sigma_t^2)_{t>0}$ such that $\frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 - t^{-h}$, $\beta_t = \sigma_t^2$ for large t.
- Let $\mathscr{A}^{h,2}_{(1,1)}$ be the collection of all $(\beta_t, \sigma_t^2)_{t>0}$ such that $\beta_t + \sigma_t^2 = 2 - t^{-h}$, $\beta_t = \sigma_t^2$ for large t.
- Let $\mathscr{A}^{h,3}_{(1,1)}$ be the collection of all $(\beta_t, \sigma_t^2)_{t>0}$ such that $\beta_t + \sigma_t^2 = \frac{1}{\beta_t} + \frac{1}{\sigma_t^2} = 2 + t^{-h}$ for large t.

Our next theorem shows that the threshold for negligible perturbation is $h = 1$, which is twice as much as that in Theorem 1.1 and 1.3. Then as h changes from 1 to 0, the coefficient for the log correction changes smoothly from 1 (corresponding to the regime $(1, 1)$) to the target regime.

Theorem 1.4. *Let $h \in (0, \infty]$. Define*

$$m^{(1,1)}_{h,1}(t) = v_t t - \frac{3 - 2\min\{h, 1\}}{2\sqrt{2}} \log t \ , \ m^{(1,1)}_{h,2}(t) = \sqrt{2}t - \frac{3 - 2\min\{h, 1\}}{2\sqrt{2}} \log t, \text{ and}$$
$$m^{(1,1)}_{h,3}(t) = v_t^* t - \frac{\min\{h, 1\}}{2\sqrt{2}} \log t.$$

For $i = 1, 2, 3$ and $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,i}_{(1,1)}$, we have

$$\lim_{t\to\infty} \left(\sum_{u\in N_t} \delta_{X_u(t) - m^{(1,1)}_{h,i}(t)}, \mathbb{P}^{\beta_t,\sigma_t^2} \right) = \mathrm{DPPP}\left(C_{h,i}\sqrt{2}\mathsf{Z}_\infty e^{-\sqrt{2}x} \,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}} \right) \text{ in law,}$$

for some constants $C_{h,i}$ depending only on h.

Outline. The rest of the article is organized as follows. We discuss our results in the next subsection, offering some heuristics of our proof and giving relation to coupled F-KPP equations. In Section 2, we introduce several results on branching Brownian motions that will be needed in our proofs, in particular some estimates for the Laplace functional of the point process associated to BBM and central limit theorems for the Gibbs measures associated to BBM. In Sections 3, 4 and 5 we give the proofs of our theorems In section 3 we prove the case $\mathscr{A}^{h,-}_{(\beta,\sigma^2)}$ in Theorem 1.1 and the case $\mathscr{A}^{h,1}_{(1,1)}$ in Theorem 1.4. In section 4 we prove the case $\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$ in Theorem 1.1, 1.3 and the case $\mathscr{A}^{h,3}_{(1,1)}$ in Theorem 1.4. In section 5 we prove the case $\mathscr{A}^{h,-}_{(\beta,\sigma^2)}$ in Theorem 1.1 and the case $\mathscr{A}^{h,2}_{(1,1)}$ in Theorem 1.4.

Notation convention. Throughout this article C (also $C_{h,+}, C_{h,-}, \cdots$) are positive constants whose value may change from line to line. Let $\mathcal{T}$ be the set of functions $\varphi \in C_b^+(\mathbb{R})$ such that $\inf \operatorname{supp}(\varphi) > -\infty$ and for some $a \in \mathbb{R}$, $\varphi(x) \equiv$ some positive constant for all $x > a$. $\mathcal{T}$ will serve as test functions in the Laplace functional (see [9, Lemma 4.4]). For two quantities f and g, we write $f \sim g$ if $\lim f/g = 1$. We write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$. We write $f \lesssim_\lambda g$ to stress that the constant C depends on parameter λ. We use the standard notation $\Theta(f)$ to denote a non-negative quantity such that there exists constant $c_1, c_2 > 0$ such that $c_1 f \leq \Theta(f) \leq c_2 f$. When this is no ambiguity, we use $\mathbb{P}$ and $\mathbb{E}$ to denote $\mathbb{P}^{\beta_t,\sigma_t^2}$ and $\mathbb{E}^{\beta_t,\sigma_t^2}$, respectively. We always use the front `mathsf` to denote the probability or quantities related to single-type branching Brownian motion, like $\mathsf{P}, \mathsf{E}, \mathsf{X}_u, \mathsf{W}, \mathsf{Z}$ etc. The probability and expectation related to Brownian motion are denoted as $\mathbf{P}$ and $\mathbf{E}$.

1.4 Discussion of our results

1.4.1 Heuristics for localization of paths of extremal particles

For each type 2 particle $u \in N_t^2$, we define T_u as the time at which the oldest ancestor of type 2 of u was born. In other words, T_u is the "type transformation time" of u. For convenience, we set $T_u = t$ for $u \in N_t^1$.

We restate here the optimization problem introduced in [8, Section 2.1] (see also Biggins [12]). For $p \in [0,1]$, let $\mathcal{N}_{p,a,b}(t)$ be the expected number of particles at time t that have speed a before time $T_u \approx pt$ and speed b after time pt (under the law $\mathbb{P}^{\beta,\sigma^2}$). Note that these particles are at level $[pa+(1-p)b]t$. By first moment computations, $\mathcal{N}_{p,a,b}(t) = \exp\left\{[(\beta - \frac{a^2}{2\sigma^2})p + (1-\frac{b^2}{2})(1-p)]t + o(t)\right\}$. So the speed of the two-type BBM should be the maximum of $pa+(1-p)b$ among all the parameter p, a, b such that $\left(\beta - \frac{a^2}{2\sigma^2}\right)p \geq 0$ and $\mathcal{N}_{p,a,b}(t) \geq 1$. That is,

$$v^* = \max\left\{pa + (1-p)b : p \in [0,1], \left(\beta - \frac{a^2}{2\sigma^2}\right)p \geq 0, \left(\beta - \frac{a^2}{2\sigma^2}\right)p + \left(1 - \frac{b^2}{2}\right)(1-p) \geq 0\right\}. \tag{1.7}$$

Denote by (p^*, a^*, b^*) the maximizer of this optimization problem. If $(\beta, \sigma^2) \in \mathscr{C}_I$, then $p^* = 1$, $a^* = v$, and $v^* = v$; if $(\beta, \sigma^2) \in \mathscr{C}_{II}$, then $p^* = 0, b^* = \sqrt{2}$ and $v^* = \sqrt{2}$; if $(\beta, \sigma^2) \in \mathscr{C}_{III}$, then $p^* = \frac{\sigma^2+\beta-2}{2(1-\sigma^2)(\beta-1)}$, $b^* = \sqrt{2\frac{\beta-1}{1-\sigma^2}}$, $a^* = \sigma^2 b^*$, and $v^* = \frac{\beta-\sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}}$.

Inspired by the heuristics above, we are going to do some refined computations that provide more precise predictions for localization of extremal particles, under the law $\mathbb{P}^{\beta_t,\sigma_t^2}$. To avoid duplication, we only do this under the setting of Theorem 1.1, i.e., $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$ and $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,\pm}_{(\beta,\sigma^2)}$. Under the setting of Theorem 1.3, 1.4, one can use a similar argument.

The case $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,-}_{(\beta,\sigma^2)}$**.** According to the optimization problem (1.7), $M_t\approx v_t t$ and each individual $u\in N_t^2$ near the maximum should satisfy $T_u\approx t$. The expected number of type 1 particles that are at level $v_t s-\delta(t)$ (where $\delta(t)$ will be determined later) at time $s=t-o(t)$ is roughly $e^{\theta_t\delta(t)-\frac{\delta(t)^2}{2\sigma_t^2 s}+O(\log t)}$. The probability that a typical particle of type 2 has a descendant at level $v_t(t-s)+\delta(t)$ at time $t-s$ is $e^{-\left[(\frac{v_t^2}{2}-1)(t-s)+v_t\delta(t)+\frac{\delta(t)^2}{2(t-s)}\right]+O(\log t)}$. Hence there are around

$$\exp\left\{-\left[\left(\frac{v_t^2}{2}-1\right)(t-s)+\frac{\delta(t)^2}{2\sigma_t^2 s}+\frac{\delta(t)^2}{2(t-s)}\right]+(\theta_t-v_t)\delta(t)+O(\log t)\right\}\tag{1.8}$$

particles of type 2 at level $v_t t$ at time t. In order for the limit of the the quantity in (1.8) to be non-zero as $t\to\infty$, using the prior knowledge $s\sim t$, we first have to ensure that $\delta(t)$ has the same order as $t-s$ or $\frac{\delta(t)^2}{t-s}$. So we get $\delta(t)=\Theta(t-s)$. We also need to ensure that $\frac{\delta(t)^2}{2\sigma_t^2 s}=O(1)$, which implies $\delta(t)=O(\sqrt{t})$. Letting $\delta(t)=\lambda(t-s)$, we can rewrite (1.8) as

$$\exp\left\{\left[-\frac{1}{2}[\lambda-(\theta-v_t)]^2+\frac{(\theta_t-v_t)^2}{2}-\left(\frac{v_t^2}{2}-1\right)\right](t-s)+O(\log t)\right\}.$$

As $\frac{1}{\beta_t}+\frac{1}{\sigma_t^2}=2-t^{-h}$, we have $(\theta_t-v_t)^2-(v_t^2-2)=2\beta_t(\frac{1}{\sigma_t^2}-2)+2=-\frac{2\beta_t}{t^h}$. Then (1.8) becomes

$$\exp\left\{\left[-\frac{1}{2}[\lambda-(\theta-v_t)]^2+\frac{\beta_t}{t^h}\right](t-s)+O(\log t)\right\}.$$

To guarantee $\left[-\frac{1}{2}[\lambda-(\theta-v_t)]^2+\frac{\beta_t}{t^h}\right](t-s)\geq 0$, we need $\lambda=(\theta_t-v_t)$ and $t-s=O(t^h)$. In other words, the extremal particle $u\in N_t^2$ should satisfy

$$t-T_u=O(t^{h\wedge\frac{1}{2}})\quad\text{and}\quad X_u(T_u)\approx v_tT_u-(\theta_t-v_t)(t-T_u).$$

The case $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$**.** According to the optimization problem (1.7), $M_t\approx v_t^* t$ and each individual $u\in N_t^2$ near the maximum should satisfy $T_u\approx p_t^* t$. The expected number of type 1 particles that are at level $a_t^* s-\delta(t)$ at time $s=p_t^*t-o(t)$ is roughly $e^{-\frac{[a_t^*s-\delta(t)]^2}{2\sigma_t^2 s}+\beta_t s+O(\log t)}$. The probability that a typical particle of type 2 has a descendant at level $v_t^*t-a_t^*s+\delta(t)$ at time $t-s$ is $e^{-\frac{[v_t^*t-a_t^*s+\delta(t)]^2}{2(t-s)}+(t-s)+O(\log t)}$. Hence there are around

$$\exp\left\{\beta_t s-\frac{[a_t^*s-\delta(t)]^2}{2\sigma_t^2 s}+(t-s)-\frac{[v_t^*t-a_t^*s+\delta(t)]^2}{2(t-s)}+O(\log t)\right\}\tag{1.9}$$

particles of type 2 at level v_t^*t at time t. Let $s=p_t^*t-\varepsilon(t)$. We have $v_t^*t-a_t^*s=b_t^*(1-p_t^*)t+a_t^*\varepsilon(t)=b_t^*(t-s)-(a_t^*-b_t^*)\varepsilon(t)$. Hence

$$\frac{(v_t^*t-a_t^*s+\delta(t))^2}{2(t-s)}=\frac{(b_t^*)^2}{2}(t-s)+b_t^*\delta(t)-b_t^*(a_t^*-b_t^*)\varepsilon(t)+\frac{[\delta(t)-(a_t^*-b_t^*)\varepsilon(t)]^2}{2(t-s)}.$$

Applying the facts that $[\beta_t-\frac{(a_t^*)^2}{2\sigma_t^2}]p_t^*+[1-\frac{(b_t^*)^2}{2}](1-p_t^*)=0$ and $a_t^*=\sigma_t^2b_t^*$, we get

$$\begin{aligned}
&(\beta_t-\frac{(a_t^*)^2}{2\sigma_t^2})s+\frac{a_t^*}{\sigma_t^2}\delta(t)-\frac{\delta(t)^2}{2\sigma_t^2 s}+(t-s)-\frac{[v_t^*t-a_t^*s+\delta(t)]^2}{2(t-s)}\\
&=(\beta_t-\frac{(a_t^*)^2}{2\sigma_t^2})p_t^*t+(1-\frac{(b_t^*)^2}{2})(1-p_t^*)t-\left(\beta_t-1+\frac{\sigma_t^2-1}{2}(b_t^*)^2\right)\varepsilon(t)-\frac{[\delta(t)-(a_t^*-b_t^*)\varepsilon(t)]^2}{2(t-s)}\\
&=-\frac{[\delta(t)-(a_t^*-b_t^*)\varepsilon(t)]^2}{2(t-s)}-\frac{\delta(t)^2}{2\sigma_t^2 s}.
\end{aligned}$$

Hence (1.9) equals to $\exp\{-\frac{[\delta(t)-(a_t^*-b_t^*)\varepsilon(t)]^2}{2(t-s)} - \frac{\delta(t)^2}{2\sigma_t^2 s} + O(\log t)\}$. Thus we need $|\delta(t)| = O(\sqrt{t})$ and $|\delta(t) - (a_t^* - b_t^*)\varepsilon(t)| = O(\sqrt{t-s})$. Hence $|\varepsilon(t)| = O(\sqrt{t})$. In other words,

$$p_t^* t - T_u = O(\sqrt{t}) \quad \text{and} \quad X_u(T_u) \approx a_t^* T_u - (a_t^* - b_t^*)(t - T_u).$$

1.4.2 Application in F-KPP equations

A multitype BBM, like standard BBM, is associated to an F-KPP reaction diffusion equation. For more details, we refer to [8, Section 2.3]. Specifically, let $t > 0$, and $f, g : \mathbb{R} \to [0,1]$ be measurable functions. We define for all $x \in \mathbb{R}$ and $s \le t$:

$$u(s,x) = \mathbb{E}^{\beta_t,\sigma_t^2}\left(\prod_{u\in N_s^1} f\left(X_u(t)+x\right)\prod_{u\in N_s^2} g\left(X_u(t)+x\right)\right),$$

$$v(s,x) = \mathsf{E}\left(\prod_{u\in \mathsf{N}_s} g\left(\mathsf{X}_u(t)+x\right)\right), \text{ where } (\mathsf{X}_u(t), u\in \mathsf{N}_t, \mathsf{P}) \text{ is a standard BBM.}$$

Then (u,v) is a solution of the following coupled F-KPP equation

$$\begin{cases} \partial_s u = \dfrac{\sigma_t^2}{2}\Delta u - \beta_t u(1-u) - u(1-v), \quad 0 < s \le t, \\ \partial_s v = \dfrac{1}{2}\Delta v - v(1-v), \quad s > 0, \\ v(0,x) = g(x)\ ,\ u(0,x) = f(x). \end{cases} \tag{1.10}$$

Our main results give the the existence of a function m_t such that (with good initial functions f, g, e.g., $f = g = 1$ on $(-\infty, -A]$ and $f = g = 0$ on $[A, \infty)$) for all $x \in \mathbb{R}$,

$$\lim_{t\to\infty}\left(u\left(t, x-m_t\right), v\left(t, x-m_t\right)\right) = \left(w_1(x), w_2(x)\right),$$

where (w_1, w_2) is a solution of the the coupled ordinary differential equations (ODEs):

$$\begin{cases} \dfrac{\sigma^2}{2} w_1'' - \mathrm{c} w_1' - \beta w_1(1-w_1) - w_1(1-w_2) = 0, \\ \dfrac{1}{2} w_2'' - \mathrm{c} w_2' - w_2(1-w_2) = 0, \end{cases} \tag{1.11}$$

with $(\beta, \sigma^2) = \lim_{t\to\infty}(\beta_t, \sigma_t^2)$ and $\mathrm{c} = \lim_{t\to\infty} m_t/t$. In fact, m_t is defined as follows:

- if $\left(\beta,\sigma^2\right) \in \mathscr{B}_{I,III}$, $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$, then $m_t = \sqrt{2\beta_t\sigma_t^2}t - \frac{3-4\min\{h,1/2\}}{2\sqrt{2\beta_t/\sigma_t^2}}\log t$; if $(\beta,\sigma^2) = (1,1)$ and $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,1}_{(1,1)}$ then $m_t = \sqrt{2\beta_t\sigma_t^2}t - \frac{3-2\min\{h,1\}}{2\sqrt{2\beta_t/\sigma_t^2}}\log t$;
- if $\left(\beta,\sigma^2\right) \in \mathscr{B}_{II,III}$ and $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$, then $m_t = \sqrt{2}t - \frac{3-4\min\{h,1/2\}}{2\sqrt{2}}\log t$; if $(\beta,\sigma^2) = (1,1)$ and $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,2}_{(1,1)}$ then $m_t = \sqrt{2}t - \frac{3-2\min\{h,1\}}{2\sqrt{2}}\log t$;
- if $\left(\beta,\sigma^2\right) \in \mathscr{B}_{I,III} \cup \in \mathscr{B}_{II,III}$ and $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,+}_{(\beta,\sigma^2)}$, then $m_t = v_t^* t - \frac{\min\{h,1/2\}}{\sqrt{2}}\log t$; if $(\beta,\sigma^2) = (1,1)$ and $\left(\beta_t,\sigma_t^2\right)_{t>0} \in \mathscr{A}^{h,3}_{(1,1)}$ then $m_t = v_t^* t - \frac{\min\{h,1\}}{2\sqrt{2}}\log t$.

Now we show that (w_1, w_2) is a solution of (1.11). By Theorems 1.1, 1.3 and 1.4, given (β, σ^2), for all h-admissible approximation $(\beta_t, \sigma_t)_{t>0}$ with $h \in (0, \infty]$, the limit $(w_1(x), w_2(x))$ are the same (up to a translation depending on h). So it suffices to consider the case $h = \infty$, i.e., $(\beta_t, \sigma_t^2) \equiv (\beta, \sigma^2)$. Applying the branching property, we have

$$u(t, x - m_t) = \mathbb{E}^{\beta,\sigma^2} \left(\prod_{u \in N_s^1} u\left(t - s, X_u(s) + x - m_t\right) \prod_{u \in N_s^2} v\left(t - s, X_u(s) + x - m_t\right) \right).$$

Letting $t \to \infty$, since $m_t = \mathrm{c}s + m_{t-s} + o(1)$, we get

$$w_1(x + \mathrm{c}s) = \mathbb{E}^{\beta,\sigma^2} \left(\prod_{u \in N_s^1} w_1\left(X_u(s) + x\right) \prod_{u \in N_s^2} w_2\left(X_u(s) + x\right) \right), \quad s \geq 0;$$

and similarly $w_2(x + \mathrm{c}s) = \mathsf{E}\left(\prod_{u \in \mathsf{N}_s} w_2\left(\mathsf{X}_u(s) + x\right)\right)$. Then, as the derivation of (1.10), using again the argument in [8, Section 2.3], $(w_1(x + \mathrm{c}t), w_2(x + \mathrm{c}t))$ solves the coupled F-KPP equation:

$$\begin{cases} \partial_t u = \dfrac{\sigma^2}{2} \Delta u - \beta u(1 - u) - u(1 - v), \\ \partial_t v = \dfrac{1}{2} \Delta v - v(1 - v). \end{cases}$$

That is, (w_1, w_2) is a traveling wave solution of this coupled PDE; and (1.11) follows.

2 Preliminary results

2.1 Brownian motion estimates

The following lemma gives an upper bound for the probability that a Brownian bridge below a straight line.

Lemma 2.1 ([17, Lemma 2]). *Let $(\zeta_s^{[0,t]})_{s \in [0,t]}$ be a Brownian bridge from 0 to 0. Let $x_1, x_2 \geq 0$, then*

$$\mathbb{P}\left(\zeta_s \leq \frac{s}{t} x_1 + \frac{t - s}{t} x_2, \forall s \in [0, t] \right) = 1 - e^{-\frac{2x_1 x_2}{t}} \leq \frac{2x_1 x_2}{t}.$$

2.2 Branching Brownian motion estimates

Recall that $\{(\mathsf{X}_u^{\beta,\sigma^2}(t), u \in \mathsf{N}_t)_{t \geq 0}, \mathsf{P}\}$ is a BBM with branching rate β and diffusion coefficient σ^2. Let $v = \sqrt{2\beta\sigma^2}$ and $\theta = \sqrt{\frac{2\beta}{\sigma^2}}$. Then for some constant $C > 0$ there holds

$$\mathsf{P}\left(\exists s > 0, u \in \mathsf{N}_s : \mathsf{X}_u^{\beta,\sigma^2}(s) \geq vs + K \right) \leq C e^{-\theta K}. \tag{2.1}$$

In fact, this probability is comparable with respect to this upper bound, see [30, Lemma 3.4]. We state some fundamental results for the standard BBM (i.e., $\beta = \sigma^2 = 1$) that will be used later. The first one is the tail probability of the maximum of BBM. Applying the first moment method, we get a trivial upper bound: for every $y \geq 1$ and $t > 0$,

$$\mathsf{P}\left(\max_{u \in \mathsf{N}_t} \mathsf{X}_u(t) \geq y \right) \leq e^t \mathbb{P}(B_t \geq y) \leq \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t}}{y} e^{t - \frac{y^2}{2t}}. \tag{2.2}$$

A much better estimate, especially when y nears $\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$, was given in [4, Corollary 10] as follows. For every $x \geq 1$ and $t > 0$,

$$\mathsf{P}\left(\max_{u\in\mathsf{N_t}} \mathsf{X}_u(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t+1) + x\right) \leq Cx\exp\left(-\sqrt{2}x - \frac{x^2}{2t} + \frac{3}{2\sqrt{2}}\frac{x\log(t+1)}{t}\right).$$

We shall use a slight modification of this inequality as follows.

Lemma 2.2. *There exists some constant $C > 0$ such that for every $x \geq 1$ and $t > 0$,*

$$\mathsf{P}\left(\max_{u\in\mathsf{N_t}} \mathsf{X}_u(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t+1) + x\right) \leq Cx\exp\left(-\sqrt{2}x - \frac{1}{2t}\left[x - \frac{3}{2\sqrt{2}}\log(t+1)\right]^2\right).$$

Note that $\inf_{x\geq 1, t>0} \frac{1}{2t}[x - \frac{3}{2\sqrt{2}}\log(t+1)]^2 - \frac{x^2}{3t} > -\infty$. By enlarging the constant C, we also have, for $x \geq 1$ and $t > 0$,

$$\mathsf{P}\left(\max_{u\in\mathsf{N_t}} \mathsf{X}_u(t) \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log(t+1) + x\right) \leq Cx\exp\left(-\sqrt{2}x - \frac{x^2}{3t}\right). \tag{2.3}$$

Secondly, we need some estimates about the Laplace functional of the following point processes associated with BBM:

$$\sum_{u\in\mathsf{N}_t} \delta_{\mathsf{X}_u(t)-\rho t}\, t > 0, \qquad \text{for all } \rho \geq \sqrt{2}.$$

When looking at the long-time behavior of the Laplace functionals of these point processes, there are two distinct regimes: $\rho = \sqrt{2}$ and $\rho > \sqrt{2}$.

Lemma 2.3 ([7, Corollary 2.9],[8, Lemma 3.7])**.** *Let $\varphi \in \mathcal{T}$, $\epsilon > 0$ and $\rho \geq \sqrt{2}$. Define*

$$\Phi_\rho(t,x) := 1 - \mathsf{E}\left(e^{-\sum_{u\in\mathsf{N}_t}\varphi(x+\mathsf{X}_u(t)-\rho t)}\right). \tag{2.4}$$

(i) If $\rho = \sqrt{2}$, for $x \in [-t^{1-\epsilon}, -t^{\epsilon}]$ uniformly

$$\Phi_{\sqrt{2}}(t,x) = (1+o(1))\gamma_{\sqrt{2}}(\varphi)\frac{(-x)}{t^{3/2}}e^{\sqrt{2}x - \frac{x^2}{2t}}, \ \text{as } t\to\infty,$$

where $\gamma_{\sqrt{2}}(\varphi) = \sqrt{2}C_\star \int e^{-\sqrt{2}z}\left(1 - \mathbb{E}\left(e^{-\langle \mathfrak{D}^{\sqrt{2}}, \varphi(\cdot+z)\rangle}\right)\right)\mathrm{d}z.$

(ii) If $\rho > \sqrt{2}$, for $|x| \leq t^{1-\epsilon}$ uniformly

$$\Phi_\rho(t,x) = (1+o(1))\gamma_\rho(\varphi)\frac{e^{\left(1-\rho^2/2\right)t}}{\sqrt{t}}e^{\rho x - \frac{x^2}{2t}}, \ \text{as } t\to\infty,$$

where $\gamma_\rho(\varphi) = \frac{C(\rho)}{\sqrt{2\pi}}\int e^{-\rho z}\left(1 - \mathbf{E}(e^{-\langle \mathfrak{D}^\rho, \varphi(\cdot+z)\rangle})\right)\mathrm{d}z.$

In fact part (i) and part (ii) were proved for the case $x = -\Theta(\sqrt{t})$ in [7, Corollary 2.9] and for the case $|x| = O(\sqrt{t})$ in [8, Lemma 3.7] respectively. However their proofs still work in our setting. We omit the repetitive proofs here.

Thirdly, we introduce several central limit theorems about the Gibbs measures associated with standard BBM $\{(\mathsf{X}_u(t) : u \in \mathsf{N}_t), \mathsf{P}\}$. Conditioned on BBM at time t, we assign each particle $u \in \mathsf{N}_t$ with probability

$$\frac{e^{\lambda \mathsf{X}_u(t)}}{\sum_{u\in\mathsf{N}_t} e^{\lambda\mathsf{X}_u(t)}}.$$

Hence the additive martingale $\mathsf{W}_t(\lambda) = \sum_{u\in \mathsf{N}_t} e^{\lambda \mathsf{X}_u(t)-\left(\frac{\lambda^2}{2}+1\right)t}$ can be regarded as a normalized partition function of the Gibbs measure. The following law of large numbers is well-known: for $0\le\lambda<\sqrt{2}$, and bounded continuous function f,

$$\lim_{t\to\infty}\sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t)}{t}\right) e^{\lambda \mathsf{X}_u(t)-\left(\frac{\lambda^2}{2}+1\right)t} = \mathsf{W}_\infty(\lambda) f(\lambda) \quad \text{in probability,}$$

where $\mathsf{W}_\infty(\lambda)$ is the limit of the non-negative martingale W_t. (See [28, Proposition 2.5] for a proof). Furthermore, a central limit theorem holds (see [33, (1.14)]): for $\lambda\in(0,\sqrt{2})$ and bounded continuous function f,

$$\lim_{t\to\infty}\sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t)-\lambda t}{\sqrt{t}}\right) e^{\lambda \mathsf{X}_u(t)-t\left(\frac{\lambda^2}{2}+1\right)} = W_\infty(\lambda)\int_{\mathbb{R}} f(z)e^{-\frac{z^2}{2}}\frac{\mathrm{d}z}{\sqrt{2\pi}} \quad \text{in probability.}$$

In the following lemma, we generalize this central limit theorem to the case that the parameter λ and test function f both depend on t in a certain way. We postpone its proof to Appendix A.

Lemma 2.4. *Let G be a non-negative bounded measurable function with compact support. Suppose $F_t(z) = G\left(\frac{z-r_t}{h_t}\right)$ with r_t and h_t satisfying that for some $\epsilon>0$ and large t, $|r_t|\le\bar{r}<\infty$ and $|h_t|\le\bar{h}<\infty$. Let $\lambda_t=\sqrt{2}(1-\frac{1}{\alpha_t})$, where $\alpha_t\ge 1$ and $\sqrt{t}/\alpha_t\to\infty$. Define*

$$\mathsf{W}_t^{F_t}(\lambda_t) := \sum_{u\in\mathsf{N}_t} F_t\left(\frac{\lambda_t t-\mathsf{X}_u(t)}{\sqrt{t}}\right) e^{\lambda_t \mathsf{X}_u(t)-(\frac{\lambda_t^2}{2}+1)t}.$$

Set $\mu_{\mathrm{Gau}}(\mathrm{d}z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,\mathrm{d}z$. Write $\langle F_t,\mu_{\mathrm{Gau}}\rangle = \frac{1}{\sqrt{2\pi}}\int_0^\infty F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z$.

(i) If $\alpha_t\to\alpha\ge1$, and hence $\lambda_t\to\sqrt{2}(1-\frac{1}{\alpha})=:\lambda$, we have

$$\lim_{t\to\infty}\frac{\mathsf{W}_t^{F_t}(\lambda_t)}{\langle F_t,\mu_{\mathrm{Gau}}\rangle} = \mathsf{W}_\infty(\lambda) \quad \textit{in probability,}$$

(ii) If $\alpha_t\to\infty$ and hence $\lambda_t\to\sqrt{2}$, we have

$$\lim_{t\to\infty}\frac{1}{\langle F_t,\mu_{\mathrm{Gau}}\rangle}\frac{\mathsf{W}_t^{F_t}(\lambda_t)}{\sqrt{2}-\lambda_t} = 2\mathsf{Z}_\infty \quad \textit{in probability.}$$

The results in Lemma 2.4 do not include the case that $\lambda_t\equiv\sqrt{2}$, where the limiting distribution is no longer Gaussian. According to [29, Theorem 1.2], we know that for every bounded continuous function f,

$$\lim_{t\to\infty}\sqrt{t}\sum_{u\in\mathsf{N}_t} f\left(\frac{\sqrt{2}t-\mathsf{X}_u(t)}{\sqrt{t}}\right) e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))} = \sqrt{\frac{2}{\pi}}\,\mathsf{Z}_\infty\int_0^\infty f(z)ze^{-\frac{z^2}{2}}\,\mathrm{d}z \quad \text{in probability.}$$

The following lemma is a generalization of this central limit theorem.

Lemma 2.5 ([28, Proposition 2.6])**.** *Let G be a non-negative bounded measurable function with compact support. Suppose $F_t(z)=G\left(\frac{z-r_t}{h_t}\right)$ with r_t and h_t satisfying that for some $\epsilon>0$ and large t, $t^{-\frac{1}{2}+\epsilon}\le r_t\le\bar{r}<\infty$, and $r_t+yh_t=\Theta(r_t)$ uniformly for $y\in\operatorname{supp}(G)$. Define*

$$\mathsf{W}_t^{F_t}(\sqrt{2}) := \sum_{u\in\mathsf{N}_t}\mathsf{F}_t\left(\frac{\sqrt{2}t-\mathsf{X}_u(t)}{\sqrt{t}}\right)e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}.$$

Put $\mu_{\mathrm{Mea}}(\mathrm{d}z)=ze^{-\frac{z^2}{2}}1_{\{z>0\}}\,\mathrm{d}z$. Write $\langle F_t,\mu_{\mathrm{Mea}}\rangle=\int_0^\infty F_t(z)ze^{-\frac{z^2}{2}}\,\mathrm{d}z$. Then we have

$$\lim_{t\to\infty}\frac{\sqrt{t}}{\langle\mathsf{F}_t,\mu_{\mathrm{Mea}}\rangle}\mathsf{W}_t^{F_t}(\sqrt{2}) = \sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty \quad \textit{in probability.}$$

2.3 Many-to-one lemmas

Recall that the type transformation time T_u of some particle $u \in N_t^2$, is the time at which the oldest ancestor of type 2 of u was born. We write

$$\mathcal{B} = \left\{ u \in \bigcup_{t \geq 0} N_t^2, T_u = b_u \right\} \tag{2.5}$$

for the set of particles of type 2 that are born from a particle of type 1. We write $u' \succcurlyeq u$ if u' is a descendant of u.

Lemma 2.6 (Many-to-one lemmas [8, Section 4])**.** *Let f be a non-negative measurable function.*

(i) $\mathbb{E}^{\beta,\sigma^2}\left(\sum_{u\in\mathcal{B}} f\left(X_u(s), s \leq T_u\right)\right) = \int_0^\infty e^{\beta t}\mathbf{E}\left(f\left(\sigma B_s, s \leq t\right)\right) \mathrm{d}t.$

(ii) $\mathbb{E}^{\beta,\sigma^2}\left(\exp\left(-\sum_{u\in\mathcal{B}} f\left(X_u(s), s \leq T_u\right)\right)\right) = \mathbb{E}^{\beta,\sigma^2}\left(\exp\left(-\int_0^\infty \sum_{u\in N_t^1} 1 - e^{-f(X_u(s),s\leq t)}\, \mathrm{d}t\right)\right).$

The many-to-one lemma is a fundamental tool to compute the first moment or the Laplacian transform of the functionals of our two type BBM. In the rest of this paper, to simplify notation, when there is no ambiguity, we use $\mathbb{P}$ and $\mathbb{E}$ to denote $\mathbb{P}^{\beta_t,\sigma_t^2}$ and $\mathbb{E}^{\beta_t,\sigma_t^2}$, respectively.

Corollary 2.7. *Let $m(t)$ be a function on $\mathbb{R}_+$. For each $R > 0$, $t > 0$, take $\Omega_t^R \subset [0,t] \times \mathbb{R}$.*

(i) For $A > 0$, $0 \leq r \leq t$ and $x \in \mathbb{R}$ define

$$\mathsf{F}_t(r,x) = \mathsf{F}_t(r,x;m(\cdot)) := \mathsf{P}\left(x + \max_{u\in\mathsf{N}_r} \mathsf{X}_u(r) \geq m(t) - A\right),$$

and for $K > 0$, define

$$I(t,R) = I(t,R;A,K) := \int_0^t e^{\beta_t s}\mathbf{E}\left[\mathsf{F}_t\left(t-s,\sigma_t B_s\right) 1_{\{\sigma_t B_r \leq v_t t + \sigma_t K, \forall r\leq s\}} 1_{\left\{(s,\sigma_t B_s)\notin\Omega_t^R\right\}}\right] \mathrm{d}s. \tag{2.6}$$

If for each fixed A, K, we have $\lim_{R\to\infty} \limsup_{t\to\infty} I(t,R) = 0$, then for each $A > 0$,

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(\exists u \in N_t^2 : X_u(t) > m(t) - A, (T_u, X_u(T_u)) \notin \Omega_t^R\right) = 0. \tag{2.7}$$

(ii) Let $\widehat{\mathcal{E}}_t := \sum_{u\in N_t^2} \delta_{X_u(t)-m(t)}$ and $\widehat{\mathcal{E}}_t^R := \sum_{u\in N_t^2} 1_{\{(T_u,X_u(T_u))\in\Omega_t^R\}}\delta_{X_u(t)-m(t)}$. For any $\rho \geq \sqrt{2}$ and $\varphi \in \mathcal{T}$,

$$\mathbb{E}\left(e^{-\langle\varphi,\widehat{\mathcal{E}}_t^R\rangle}\right) = \mathbb{E}\left[\exp\left(-\int_0^\infty \sum_{u\in N_s^1} \Phi_\rho\big(t-s, X_u(s) + \rho(t-s) - m(t)\big) 1_{\{(s,X_u(s))\in\Omega_t^R\}}\, \mathrm{d}s\right)\right],$$

where Φ_ρ is defined by (2.4). Moreover if (2.7) holds, then

$$\lim_{R\to\infty} \limsup_{t\to\infty} \left|\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) - \mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right)\right| = 0\,, \text{ for all } \varphi \in \mathcal{T}.$$

Proof. (i). Fix $A, K > 0$. For $R, t > 0$, define

$$Y_t(R) = Y_t(R; A, K) := \sum_{u\in\mathcal{B}} 1_{\{X_u(r)\le v_t r+\sigma_t K, \forall r\le T_u\}} 1_{\{(T_u, X_u(T_u))\notin\Omega_t^R\}} 1_{\{M_t^u\ge m(t)-A\}},$$

where M_t^u is the position of the rightmost descendant of the individual u at time t. Then the probability in (2.7) is less than

$$\mathbb{P}(\exists s \le t, u \in \mathsf{N}_s : \mathsf{X}_u(s) \ge v_t s + K) + \mathbb{P}(Y_t(R) \ge 1).$$

Applying the Markov inequality, $\mathbb{P}(Y_t(R) \ge 1)$ is bounded above by $\mathbb{E}[Y_t(R)]$. The branching property implies that

$$\mathbb{E}[Y_t(R)] = \mathbb{E}\left[\sum_{u\in\mathcal{B}} \mathsf{F}_t(t - T_u, X_u(T_u)) 1_{\{X_u(r)\le v_t r+\sigma_t K, \forall r\le T_u\}}\right].$$

Applying Lemma 2.6 (i), we get

$$\mathbb{E}[Y_t(R)] = \int_0^t e^{\beta_t s}\mathbf{E}\left[\mathsf{F}_t(t-s, \sigma_t B_s) 1_{\{\sigma_t B_r\le v_t t+\sigma_t K, \forall r\le s\}} 1_{\{(s,\sigma_t B_s)\notin\Omega_t^R\}}\right] \mathrm{d}s.$$

Then by the assumption, $\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}(Y_t(R) \ge 1) = 0$. Now the desired result (2.7) follows from (2.1).

(ii). Notice that $\langle\widehat{\mathcal{E}}_t^R, \varphi\rangle$ can be rewritten as

$$\sum_{\substack{u\in\mathcal{B}\\ (T_u, X_u(T_u))\in\Omega_t^R}} \sum_{\substack{u'\in N_t^2\\ u'\succcurlyeq u}} \varphi\left(X_{u'}(t) - X_u(T_u) - \rho(t - T_u) + X_u(T_u) + \rho(t - T_u) - m(t)\right)$$

First using the branching property, and then applying Lemma 2.6 (ii), we get

$$\begin{aligned}
\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) &= \mathbb{E}\Bigg(\prod_{\substack{u\in\mathcal{B}\\ (T_u, X_u(T_u))\in\Omega_t^R}} \left[1 - \Phi_\rho\big(t - T_u, X_u(T_u) + \rho(t - T_u) - m(t)\big)\right]\Bigg)\\
&= \mathbb{E}\left(\exp\left(-\int_0^\infty \sum_{u\in N_s^1} \Phi_\rho\big(t - s, X_u(s) + \rho(t - s) - m(t)\big) 1_{\{(s,X_u(s))\in\Omega_t^R\}}\,\mathrm{d}s\right)\right)
\end{aligned}$$

as desired. Taking $A > 0$ such that $\mathrm{supp}(\varphi) \subset [-A, \infty)$, we have

$$\left|\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) - \mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right)\right| \le \mathbb{P}\left(\exists u \in N_t^2 : X_u(t) > m(t) - A, (T_u, X_u(T_u)) \notin \Omega_t^R\right).$$

Then the result assertion follows. $\square$

3 Approximation From $\mathscr{C}_I$

3.1 From $\mathscr{C}_I$ to $\mathscr{B}_{I,III}$

In this subsection, we are going to prove Theorem 1.1 for the case $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$ and $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}_{(\beta,\sigma^2)}^{h,-}$. For simplicity, in this subsection, we set

$$h' := \min\{h, 1/2\}.$$

Then $m_{h,-}^{1,3}(t) = v_t t - \frac{3-4h'}{2\theta_t}\log t$. Define

$$\delta(x; s, t) := x - v_t s + (\theta_t - v_t)(t - s) \tag{3.1}$$

and

$$\Omega_{t,h}^R := \left\{(s, x) : t - s \in [\frac{1}{R}t^{h'}, Rt^{h'}],\ |\delta(x; s, t)| \le R\sqrt{t - s}\right\}. \tag{3.2}$$

Lemma 3.1. *For any $A > 0$ and $h \in (0, \infty]$, we have*

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(\exists u \in N_t^2 : X_u(t) \geq m_{h,-}^{1,3}(t) - A, (T_u, X_u(T_u)) \notin \Omega_{t,h}^R\right) = 0.$$

Proof. Applying Corollary 2.7 with $m(t) = m_{h,-}^{1,3}(t)$ and $\Omega_t^R = \Omega_{t,h}^R$ defined in (3.2), it suffices to show that for each $A, K > 0$, $I(t,R) = I(t,R;A,K)$ defined in (2.6) vanishes first as $t \to \infty$ and then $R \to \infty$. By conditioning on "$B_s = \sqrt{2\beta_t}s - x$" in (2.6), we have

$$\begin{aligned}I(t,R) = &\int_0^t \mathrm{d}s \int_{-K}^{\infty} \mathbf{P}\left(B_r \leq \sqrt{2\beta_t}r + K, \forall r \leq s | B_s = \sqrt{2\beta_t}s - x\right) \\ &\qquad e^{\beta_t s} \mathsf{F}_t\left(t-s, v_t s - \sigma_t x\right) 1_{\{(s, v_t s - \sigma_t x) \notin \Omega_{t,h}^R\}} e^{-\frac{(\sqrt{2\beta_t}s - x)^2}{2s}} \frac{\mathrm{d}x}{\sqrt{2\pi s}} \qquad (3.3)\\ \lesssim_K &\int_0^t \mathrm{d}s \int_{-K}^{\infty} \frac{K+x}{s^{3/2}} e^{\sqrt{2\beta_t}x - \frac{x^2}{2s}} \mathsf{F}_t\left(t-s, v_t s - \sigma_t x\right) 1_{\{(s, v_t s - \sigma_t x) \notin \Omega_{t,h}^R\}} \,\mathrm{d}x.\end{aligned}$$

In the inequality above we used $\mathbf{P}\left(B_r \leq \sqrt{2\beta_t}t + K, \forall r \leq s | B_s = \sqrt{2\beta_t}s - x\right) \lesssim_K \frac{K+x}{s}$, which holds by Lemma 2.1 since $(B_r - \frac{r}{s}B_s)_{r \leq s}$ is a Brownian bridge independent of B_s. Put $\mathrm{w} := \frac{3-4h'}{2\theta_t}\log t + A$.

- If $v_t(t-s) + \sigma_t x - \mathrm{w} \geq 1$, by (2.2), we have

$$\begin{aligned}\mathsf{F}_t\left(t-s, v_t s - \sigma_t x\right) &= \mathsf{P}\left(\max_{u \in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) > v_t(t-s) + \sigma_t x - \mathrm{w}\right) \\ &\lesssim \frac{\sqrt{t-s}}{v_t(t-s) + \sigma_t x - \mathrm{w}} \exp\left\{-(\frac{v_t^2}{2} - 1)(t-s) - v_t\sigma_t x + v_t \mathrm{w} - \frac{(\sigma_t x - \mathrm{w})^2}{2(t-s)}\right\}. \qquad (3.4)\end{aligned}$$

- If $v_t(t-s) + \sigma_t x - \mathrm{w} \leq 1$, we simply upper bound $\mathsf{F}_t\left(t-s, v_t s - \sigma_t x\right)$ by 1. Note that, provided t is large, we can deduce from $v_t(t-s) + \sigma_t x - \mathrm{w} \leq 1$ that $s \geq t - (\log t)^2$ and $\sigma_t x \leq \mathrm{w} + 1$ and hence $\sqrt{2\beta_t}x \leq \frac{3-4h'}{2}\log t + (A+1)\theta_t$. Therefore,

$$\begin{aligned}&\int_{t-(\log t)^2}^{t} \mathrm{d}s \int_{-K}^{O(\log t)} \frac{K+x}{s^{3/2}} e^{\sqrt{2\beta_t}x - \frac{x^2}{2s}} 1_{\{v_t(t-s) + \sigma_t x - \mathrm{w} \leq 1\}} \,\mathrm{d}x \\ \lesssim &\int_{t-(\log t)^2}^{t} \mathrm{d}s \int_{-K}^{O(\log t)} \frac{K + O(\log t)}{t^{3/2}} e^{\sqrt{2\beta_t}x} \,\mathrm{d}x \lesssim (\log t)^4 t^{-3/2} t^{\frac{3-4h'}{2}} = o(1). \qquad (3.5)\end{aligned}$$

Combining (3.4) and (3.5) and letting $J = \frac{K+x}{s^{3/2}} \frac{\sqrt{t-s}}{v_t(t-s) + \sigma_t x - \mathrm{w}} e^{v_t \mathrm{w}}$, we get

$$I(t,R) \lesssim \int_0^t \mathrm{d}s \int_{-K}^{\infty} J e^{\sqrt{2\beta_t}(1-\sigma_t^2)x - (\frac{v_t^2}{2} - 1)(t-s) - \frac{(\sigma_t x - \mathrm{w})^2}{2(t-s)}} e^{-\frac{x^2}{2s}} 1_{\{(s, v_t s + \sigma_t x) \notin \Omega_{t,h}^R\}} \,\mathrm{d}x + o(1).$$

Make a change of variable $x = \frac{\theta_t - v_t}{\sigma_t}(t-s) + y = \sqrt{2\beta_t}(\sigma_t^{-2} - 1)(t-s) + y$. Note that $(s, v_t s + \sigma_t x) \in \Omega_{t,h}^R$ if and only if $(t-s)/t^{h'} \in \Gamma_R := [R^{-1}, R]$ and $|\sigma_t y| \leq R\sqrt{s}$. Moreover, we compute that

$$\begin{aligned}&\sqrt{2\beta_t}(1-\sigma_t^2)x - (\frac{v_t^2}{2} - 1)(t-s) - \frac{(\sigma_t x - \mathrm{w})^2}{2(t-s)} \\ &= [2\beta_t(1-\sigma_t^2)(\sigma_t^{-2} - 1) - (v_t^2/2 - 1) - \sigma_t^2\beta_t(\sigma_t^{-2} - 1)^2](t-s) \\ &\qquad + [\sqrt{2\beta_t}(1-\sigma_t^2) - (\theta_t - v_t)\sigma_t]y + (\theta_t - v_t)\mathrm{w} - \frac{(\sigma_t y - \mathrm{w})^2}{2(t-s)} \\ &= -\left[\frac{2\beta_t - 1}{\sigma_t^2 t^h} - \frac{\beta_t}{\sigma_t^2 t^{2h}}\right](t-s) + (\theta_t - v_t)\mathrm{w} - \frac{(\sigma_t y - \mathrm{w})^2}{2(t-s)}.\end{aligned}$$

Hence we get

$$I(t,R) \lesssim \int_0^t \mathrm{d}s \int_{-K}^{\infty} e^{(\theta_t - v_t)\mathrm{w}} J e^{-[\frac{2\beta_t - 1}{\sigma_t^2} + o(1)]\frac{t-s}{t^h}} e^{-\frac{(\sigma_t y - \mathrm{w})^2}{2(t-s)}} e^{-\frac{x^2}{2s}} 1_{\left\{ \substack{(t-s)/t^{h'} \notin \Gamma^R \\ \text{or } |\sigma_t y| > R\sqrt{t-s}} \right\}} \mathrm{d}y + o(1).$$

Now make change of variables again $t - s = \xi t^{h'}$, $y = \eta\sqrt{t-s}$. Denote $\tilde{J} := t^{h'}\sqrt{t-s}e^{(\theta_t - v_t)\mathrm{w}}J$. We have

$$I(t,R) \lesssim \int_0^{t^{1-h'}} \mathrm{d}\xi \int_{\mathbb{R}} \tilde{J} e^{-\left[\frac{2\beta_t - 1}{\sigma_t^2} + o(1)\right]\xi t^{h'-h}} e^{-\frac{1}{2}\left(\sigma_t \eta - \frac{o(1)}{\sqrt{\xi}}\right)^2} e^{-\frac{[(\theta_t - v_t)\xi t^{h'} + \sigma_t \eta\sqrt{\xi t^{h'}}]^2}{2\sigma_t^2 (t - \xi t^{h'})}} 1_{\left\{ \substack{\xi \notin \Gamma^R, \text{or} \\ |\sigma_t \eta| > R} \right\}} \mathrm{d}\eta + o(1).$$

Notice that for fixed $\xi > 0$ and $\eta > 0$,

$$\lim_{t\to\infty} \tilde{J} = \lim_{t\to\infty} t^{h'} \frac{(\theta_t - v_t)\xi t^{h'} + o(t^{h'})}{(t - \xi t^{h'})^{3/2}} \frac{\xi t^{h'}}{\theta_t \xi t^{h'} + o(t^{h'})} t^{\frac{3-4h'}{2}} e^{A\theta_t} = \frac{\theta - v}{\theta}\xi e^{A\theta}.$$

Letting $t \to \infty$, applying the dominated convergence theorem twice, we finally get

$$\limsup_{t\to\infty} I(t,R) \lesssim_{A,K} \int_0^{\infty} \mathrm{d}\xi \int_{-K}^{\infty} \xi e^{-\frac{2\beta - 1}{\sigma^2}\xi 1_{\{h \le \frac{1}{2}\}}} e^{-\frac{\sigma^2\eta^2}{2}} e^{-\frac{(\theta - v)^2}{2\sigma^2}\xi^2 1_{\{h \ge \frac{1}{2}\}}} 1_{\left\{ \substack{\xi \notin \Gamma^R, \text{or} \\ |\sigma\eta| > R} \right\}} \mathrm{d}\eta \overset{R\to\infty}{\longrightarrow} 0.$$

This completes the proof. □

Now we are ready to show Theorem 1.1 for the case that $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$.

Proof of Theorem 1.1 for $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$. Take $\varphi \in \mathcal{T}$. Applying Corollary 2.7 (ii) with $m(t) = m^{1,3}_{h,-}(t)$, $\rho = \theta_t$, and $\Omega^R_{t,h}$ defined in (3.2), it suffices to study the asymptotic behavior of

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}^R_t, \varphi\rangle}\right) = \mathbb{E}\left(\exp\left\{ -\int_0^t \sum_{u\in N^1_s} \Phi_{\theta_t}\left(t-s, \delta(X_u(s); s,t) + \frac{3-4h'}{2\theta_t}\log t\right) 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}} \mathrm{d}s \right\}\right),$$

where we used the fact that $X_u(s) + \theta_t(t-s) - m^{1,3}_{h,-}(t) = \delta(X_u(s); s,t) + \frac{3-4h'}{2\theta_t}\log t$ with δ defined in (3.1). Since $\theta_t \to \theta > \sqrt{2}$, applying Lemma 2.3, we have, uniformly for $(s, X_u(s)) \in \Omega^R_{t,h}$,

$$\begin{aligned}
&\Phi_{\theta_t}\big(t-s, \delta(X_u(s); s,t) + \frac{3-4h'}{2\theta_t}\log t\big) \\
&\sim \frac{\gamma_\theta(\varphi)}{\sqrt{t-s}} e^{-(\frac{\theta_t^2}{2}-1)(t-s)} e^{\theta_t[X_u(s) + \theta_t(t-s) - v_t t] + \frac{3-4h'}{2}\log t} e^{-\frac{1}{2(t-s)}\delta(X_u(s);s,t)^2} \\
&\sim \gamma_\theta(\varphi)\frac{t^{3/2}}{t^{2h'}(t-s)^{1/2}} e^{-\beta_t\frac{t-s}{t^h}} e^{\theta_t X_u(s) - 2\beta_t s} e^{-\frac{1}{2(t-s)}\delta(X_u(s);s,t)^2}.
\end{aligned}$$

where we used the fact that $-(\frac{\theta_t^2}{2} - 1)(t-s) + \theta_t^2(t-s) - \theta_t v_t t = -\beta_t\frac{t-s}{t^h} - 2\beta_t s$. Thus substituting this asymptotic equality into the integral, we get

$$\begin{aligned}
&\int_0^t \sum_{u\in N^1_s} \Phi_{\theta_t}\left(t-s, \delta(X_u(s); s,t) + \frac{3-4h'}{2\theta_t}\log t\right) 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}} \mathrm{d}s \\
&= [1+o(1)] \int_{t-Rt^{h'}}^{t-\frac{1}{R}t^{h'}} \sum_{u\in N^1_s} \frac{\gamma_\theta(\varphi)t^{3/2}}{t^{2h'}(t-s)^{1/2}} e^{-\beta_t\frac{t-s}{t^h}} e^{\theta_t X_u(s) - 2\beta_t s} e^{-\frac{1}{2(t-s)}\delta(X_u(s);s,t)^2} 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}} \mathrm{d}s \\
&= [1+o(1)]\gamma_\theta(\varphi) \int_{\frac{1}{R}}^{R} \lambda e^{-\beta_t \lambda t^{h'-h}} \frac{t^{3/2}}{r^{3/2}} \sum_{u\in N^1_s} e^{\theta_t X_u(s) - 2\beta_t s} e^{-\frac{\delta(X_u(s);s,t)^2}{2r}} 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}} \mathrm{d}\lambda, \qquad (3.6)
\end{aligned}$$

where in the last equality we made change of variables $t-s=r$ and $r=\lambda t^{h'}$. Let $G(x)=G_R(x)=e^{-\frac{x^2}{2}}1_{\{|x|\le R\}}$, and define

$$W^G(s,r;t) := \sum_{u\in N_s^1} e^{-\theta_t[v_t s - X_u(s)]} G\left(\frac{v_t s - X_u(s) - (\theta_t - v_t)r}{\sqrt{r}}\right) \text{ for } s>0, r>0.$$

Then the integral in (3.6) can be rewritten as $\int_{\frac{1}{R}}^R \lambda e^{-\beta_t \lambda t^{h'-h}} \left(\frac{t}{r}\right)^{3/2} W^G(t-r,r;t)\,\mathrm{d}\lambda$. Hence

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathbb{E}\left(\exp\left\{-[1+o(1)]\gamma_\theta(\varphi)\int_{\frac{1}{R}}^R \lambda e^{-\beta_t\lambda t^{h'-h}}\left(\frac{t}{r}\right)^{3/2} W^G(t-r,r;t)\,\mathrm{d}\lambda\right\}\right). \tag{3.7}$$

By the scaling property of Brownian motion, $\left\{X_u(s): u\in N_s^1\right\} \overset{law}{=} \left\{\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s'): u\in \mathsf{N}_{s'}\right\}$, where $s'=\beta_t s$. So $W^G(s,r;t)$ has the same distribution as

$$\mathsf{W}^G(s',r;t) := \sum_{u\in\mathsf{N}_{s'}} e^{\sqrt{2}\mathsf{X}_u(s')-2s'} G\left(\frac{\frac{\sqrt{2}s'-\mathsf{X}_u(s')}{\sqrt{s'}} - \frac{\sqrt{\beta_t}}{\sigma_t}(\theta_t - v_t)\frac{r}{\sqrt{s'}}}{\frac{\sqrt{\beta_t}}{\sigma_t}\frac{\sqrt{r}}{\sqrt{s'}}}\right).$$

Here we remind that W^G is for single type BBM and W^G is for two-type BBM. For each fixed $\lambda>0$ and $r=\lambda t^{h'}$, applying Lemma 2.5 with $r_t = \frac{\sqrt{\beta_t}}{\sigma_t}(\theta_t - v_t)\frac{r}{\sqrt{\beta_t(t-r)}}$ and $h_t = \frac{\sqrt{\beta_t}}{\sigma_t}\frac{\sqrt{r}}{\sqrt{\beta_t(t-r)}}$ (noticing that $r_t = \Theta(t^{h'-\frac{1}{2}})$ and $h_t \ll r_t$, which implies the conditions in Lemma 2.5 are satisfied), we have

$$\begin{aligned}
&\lim_{t\to\infty}\left(\frac{t}{r}\right)^{3/2} W^G(t-r,r;t) = \lim_{t\to\infty}\left(\frac{t}{r}\right)^{3/2}\mathsf{W}^G(\beta_t(t-r),r;t)\\
&= \mathsf{Z}_\infty\sqrt{\frac{2}{\pi}}\lim_{t\to\infty}\left(\frac{t}{r}\right)^{3/2}\frac{1}{\sqrt{\beta_t t}}\int_0^\infty z e^{-\frac{z^2}{2}} G_R\left(\frac{z - \frac{\sqrt{\beta_t}}{\sigma_t}(\theta_t - v_t)\frac{r}{\sqrt{\beta_t(t-r)}}}{\frac{\sqrt{\beta_t}}{\sigma_t}\frac{\sqrt{r}}{\sqrt{\beta_t(t-r)}}}\right)\mathrm{d}z\\
&= \mathsf{Z}_\infty^{\beta,\sigma^2}\frac{(\theta-v)}{\sigma^3}e^{-\frac{(\theta-v)^2}{2\sigma^2}\lambda^2 1_{\{h\ge 1/2\}}}\sqrt{\frac{2}{\pi}}\int G_R(y)\,\mathrm{d}y \quad \text{ in law,}
\end{aligned}$$

where in the last equality we used the fact that $\mathsf{Z}_\infty \overset{law}{=} \frac{\sqrt{\beta}}{\sigma}\mathsf{Z}_\infty^{\beta,\sigma^2}$. Letting $t\to\infty$ in (3.7) and applying the dominated convergence theorem, we finally get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\left(\exp\left\{-C_{h,R}\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\right\}\right),$$

where

$$\begin{aligned}
C_{h,R} &= \int_{\frac{1}{R}}^R e^{-\beta\lambda 1_{\{h\le 1/2\}}}\frac{(\theta-v)\lambda}{\sigma^3}e^{-\frac{(\theta-v)^2}{2\sigma^2}\lambda^2 1_{\{h\ge 1/2\}}}\,\mathrm{d}\lambda\sqrt{\frac{2}{\pi}}\int G_R(y)\,\mathrm{d}y\\
&\overset{R\to\infty}{\longrightarrow} C_h := \frac{2(\theta-v)}{\beta^2\sigma^3}1_{\{h<1/2\}} + \frac{2}{\sigma(\theta-v)}1_{\{h>1/2\}} + 2\int_0^\infty \frac{(\theta-v)\lambda}{\sigma^3}e^{-\beta\lambda-\frac{(\theta-v)^2}{2\sigma^2}\lambda^2}\,\mathrm{d}\lambda 1_{\{h=1/2\}}.
\end{aligned}$$

Then by part (ii) of Corollary 2.7, letting $R\to\infty$ we get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \mathsf{E}\left(\exp\left\{-C_h\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\right\}\right).$$

Recalling the definition of $\gamma_\theta(\varphi)$ in Lemma 2.3, the right hand side is the Laplace functional of DPPP $\left(C_{h,-}\mathsf{Z}_\infty^{\beta,\sigma^2}\theta e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}^\theta\right)$ with $C_{h,-} := \frac{C_h C(\theta)}{\theta\sqrt{2\pi}}$. By [9, Lemma 4.4], we complete the proof. □

3.2 From $\mathcal{C}_I$ to $(1,1)$

In this subsection, we are going to prove Theorem 1.4 for the case $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,1}_{(1,1)}$. We set, in this subsection,

$$h' := \min\{h,1\}.$$

Then $m^{(1,1)}_{h,1}(t) = v_t t - \frac{3-2h'}{2\sqrt{2}}\log t$. Define

$$\Omega^R_{t,h} = \begin{cases} \{(s,x) : t-s\in[\frac{1}{R}t^h, Rt^h],\ v_t s - x \in [\frac{1}{R}\sqrt{t-s}, R\sqrt{t-s}]\} & \text{for } h\in(0,1); \\ \{(s,x) : t-s\in[\frac{1}{R}t, (1-\frac{1}{R})t],\ v_t s - x \in [\frac{1}{R}\sqrt{t}, R\sqrt{t}]\} & \text{for } h\in[1,\infty]. \end{cases} \tag{3.8}$$

Lemma 3.2. *For each $A>0$,*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\exists u\in N_t^2 : X_u(t)\geq m^{(1,1)}_{h,1}(t)-A, (T_u, X_u(T_u))\notin\Omega^R_{t,h}\right) = 0.$$

Proof. Applying Corollary 2.7 with $m(t)=m^{(1,1)}_{h,1}(t)$ and $\Omega^R_{t,h}$ defined in (3.8), it suffices to show that for each $A,K>0$, $I(t,R)=I(t,R;A,K)$ defined in (2.6) vanishes as first $t\to\infty$ and then $R\to\infty$. As in (3.3) we have

$$I(t,R)\lesssim_K \int_0^t \mathrm{d}s\int_{-K}^{\infty}\frac{K+x}{s^{3/2}}e^{\sqrt{2\beta_t}x-\frac{x^2}{2s}}\mathsf{F}_t\left(t-s, v_t s-\sigma_t x\right)1_{\left\{(s,v_t s-\sigma_t x)\notin\Omega^R_{t,h}\right\}}\,\mathrm{d}x. \tag{3.9}$$

Now we need an finer upper bound for F_t, which are given below. Let $L_{s,t} := \sqrt{2}(\sigma_t^2-1)(t-s) - \frac{3}{2\sqrt{2}}\log(\frac{t}{t-s+1}) + \frac{h'}{\sqrt{2}}\log t - A$.

- If $L_{s,t}+\sigma_t x>1$, noticing that $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,1}_{(1,1)}$ implies that $\theta_t=\sqrt{2}$ and $v_t=\sqrt{2}\sigma_t^2$, we have

$$\begin{aligned}\mathsf{F}_t(t-s, v_t s-\sigma_t x) &= \mathsf{P}\left(\max_{u\in\mathsf{N}_{t-s}}\mathsf{X}_u(t-s)\geq v_t(t-s)+\sigma_t x - \frac{3-2h'}{2\sqrt{2}}\log t - A\right)\\ &= \mathsf{P}\left(\max_{u\in\mathsf{N}_{t-s}}\mathsf{X}_u(t-s)\geq\sqrt{2}(t-s)-\frac{3}{2\sqrt{2}}\log(t-s+1)+L_{s,t}+\sigma_t x\right)\\ &\lesssim_A (\sigma_t x+L_{s,t})\frac{t^{3/2}}{(t-s+1)^{3/2}t^{h'}}e^{-\sqrt{2}\sigma_t x-2(\sigma_t^2-1)(t-s)}e^{-\frac{(\sigma_t x+L_{s,t})^2}{3(t-s)}},\end{aligned}$$

  where in that last inequality we used (2.3).

- If $L_{s,t}+\sigma_t x\leq 1$, we simply upper bound $\mathsf{F}_t(t-s,v_t s-\sigma_t x)$ by 1. Note that $L_{s,t}+\sigma_t x\leq 1$ implies that when t is large, $s\geq t/2$ and $\sigma_t x\leq 1-L_{s,t}\leq\frac{3}{2\sqrt{2}}\log(\frac{t}{t-s+1})-\frac{h'}{\sqrt{2}}\log t+1$. Moreover as $\beta_t=\sigma_t^2$,

$$\int_{t/2}^t\mathrm{d}s\int_{-K}^{O(\log t)}\frac{K+x}{s^{3/2}}e^{\sqrt{2\beta_t}x}1_{\{L_{s,t}+\sigma_t x\leq 1\}}\,\mathrm{d}x\lesssim\frac{O(\log t)^2}{t^{h'}}\int_0^{t/2}\frac{1}{(u+1)^{3/2}}\,\mathrm{d}u = o(1).$$

Therefore we have

$$I(t,R)\lesssim\int_0^t\int_{-K}^{\infty}\frac{K+x}{s^{3/2}}|\sigma_t x+L_{s,t}|\frac{t^{3/2}}{(t-s+1)^{3/2}t^{h'}}e^{-2(\sigma_t^2-1)(t-s)}e^{-\frac{x^2}{2s}-\frac{(\sigma_t x+L_{s,t})^2}{3(t-s)}}\,\mathrm{d}x\,\mathrm{d}s+o(1). \tag{3.10}$$

For the case $h\in(0,1)$, make change of variables $t-s=\xi t^h$ and $x=\eta\sqrt{t-s}$. Note that $(s,v_t s+\sigma_t x)\in\Omega^R_{t,h}$ if and only if $\xi\in\Gamma_R := [R^{-1},R]$ and $\sigma_t\eta\in\Gamma_R$. Use $\ell_{\xi,t}$ to denote $L(t-\xi t^h,t)$.

Noting that $|\ell_{\xi,t}| \le \sqrt{2}(\sigma_t^2-1)t^h\xi + \frac{3}{2\sqrt{2}}\log(\xi t^h) + \frac{3-4h'}{2\sqrt{2}}\log t + A = \Theta(\xi) + O(\log t)$, applying the dominated convergence theorem we have

$$\begin{aligned}
&I(t,R)\\
&\lesssim \int_0^{t^{1-h}} \mathrm{d}\xi \int_{-\frac{K}{\sqrt{\xi t^h}}}^{\infty} 1_{\left\{\substack{\xi\notin\Gamma_R \text{ or}\\ \sigma_t\eta\notin\Gamma_R}\right\}} \frac{K+\eta\sqrt{\xi t^h}}{(t-\xi t^h+1)^{3/2}}[\sigma_t\eta\sqrt{\xi t^h}+\ell_{\xi,t}]\frac{t^{3/2}}{\xi t^h} e^{-2(\sigma_t^2-1)t^h\xi-\frac{1}{3}(\sigma_t\eta+\frac{\ell_{\xi,t}}{\sqrt{\xi t^h}})^2}\,\mathrm{d}\eta + o(1)\\
&\overset{t\to\infty}{\longrightarrow} \int_0^\infty \mathrm{d}\xi \int_0^\infty 1_{\left\{\substack{\xi\notin\Gamma_R\\ \text{or } \eta\notin\Gamma_R}\right\}} \eta^2 e^{-\xi} e^{-\frac{\eta^2}{3}}\,\mathrm{d}\eta \overset{R\to\infty}{\longrightarrow} 0.
\end{aligned}$$

For the case $h\ge 1$, make change of variables $s=\xi t$ and $x=\eta\sqrt{t}$ to the integral in (3.10). Now $(s, v_t s+\sigma_t x)\in\Omega_{t,h}^R$ if and only if $\xi\in[R^{-1},1-R^{-1}]$ and $\sigma_t\eta\in\Gamma_R$. Similarly letting $\ell_{\xi,t} := L(\xi t,t)$ and noting that $|\ell_{\xi,t}| = \Theta(\xi)+O(\log t)$, applying the dominated convergence theorem

$$\begin{aligned}
I(t,R) &\lesssim \int_0^1 \mathrm{d}\xi \int_{-\frac{K}{\sqrt{t}}}^{\infty} 1_{\left\{\substack{\xi\notin[R^{-1},1-R^{-1}]\\ \text{or } \sigma_t\eta\in\Gamma_R}\right\}} \frac{K+\eta\sqrt{t}}{(\xi t)^{3/2}}[\sigma_t\eta\sqrt{t}+\ell_{\xi,t}]\frac{t^{3/2}\sqrt{t}}{(t-\xi t+1)^{3/2}} e^{-\frac{\eta^2}{2\xi}-\frac{(\sigma_t\eta+\ell_{\xi,t}/\sqrt{t})^2}{3(1-\xi)}}\,\mathrm{d}\eta + o(1)\\
&\overset{t\to\infty}{\longrightarrow} \int_0^1 \mathrm{d}\xi \int_0^\infty 1_{\left\{\substack{\xi\notin[R^{-1},1-R^{-1}]\\ \text{or } \eta\in[R^{-1},R]}\right\}} \eta^2\frac{1}{\xi^{3/2}}\frac{1}{(1-\xi)^{3/2}} e^{-\frac{\eta^2}{2\xi}\frac{\eta^2}{3(1-\xi)}}\,\mathrm{d}\eta \overset{R\to\infty}{\to} 0,
\end{aligned}$$

where we used the fact that $\int_0^1 \frac{1}{\xi^{3/2}}\frac{1}{(1-\xi)^{3/2}}\mathrm{d}\xi \int_0^\infty \eta^2 e^{-\frac{\eta^2}{2\xi}-\frac{\eta^2}{3(1-\xi)}}\mathrm{d}\eta < \infty$. We now complete the proof. $\square$

Proof of Theorem 1.4 for $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}_{(1,1)}^{h,1}$. Take $\varphi\in\mathcal{T}$. Applying Corollary 2.7 with $m(t) = m_{h,1}^{(1,1)}(t)$, $\rho=\sqrt{2}$, and $\Omega_{t,h}^R$ defined in (3.8), it suffices to study the asymptotic behavior of

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathbb{E}\Bigg(\exp\Bigg\{-\int_0^t \sum_{u\in N_s^1}\Phi_{\sqrt{2}}\big(t-s, X_u(s)+\sqrt{2}(t-s)-m_{h,1}^{(1,1)}(t)\big)1_{\{(s,X_u(s))\in\Omega_{t,h}^R\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

Rewrite $\sqrt{2}(t-s)-m_{h,1}^{(1,1)}(t)$ as $-v_t s-\mathrm{y}$, where $\mathrm{y} := (v_t-\sqrt{2})(t-s)-\frac{3-2h'}{2\sqrt{2}}\log t$. For $(s,X_u(s))\in\Omega_{t,h}^R$, we have $t-s=\Theta(t^{h'})$, $v_t s - X_u(s) = \Theta(\sqrt{t-s})$ and $\mathrm{y}=O(\log t)$. Then part (i) of Lemma 2.3 yields that as $t\to\infty$ uniformly in $(s,X_u(s))\in\Omega_{t,h}^R$,

$$\begin{aligned}
&\Phi_{\sqrt{2}}(t-s, X_u(s)-v_t s-\mathrm{y})\\
&= [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{v_t s - X_u(s)+\mathrm{y}}{(t-s)^{3/2}} e^{\sqrt{2}(X_u(s)-v_t s)-\sqrt{2}(v_t-\sqrt{2})(t-s)} t^{\frac{3-2h'}{2}} e^{-\frac{(X_u(s)-v_t s-\mathrm{y})^2}{2(t-s)}}\\
&= [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{t^{3/2}}{(t-s)^{3/2}t^{h'}} e^{-2(\beta_t-1)(t-s)}[\sqrt{2}\beta_t s - X_u(s)] e^{\sqrt{2}X_u(s)-2\beta_t s} e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-s)}},
\end{aligned}$$

where we used that $v_t=\sqrt{2}\beta_t$ as $\beta_t=\sigma_t^2$. Thus

$$\begin{aligned}
&\int_0^t \sum_{u\in N_s^1}\Phi_{\sqrt{2}}\big(t-s, X_u(s)+\sqrt{2}(t-s)-m_{h,1}^{(1,1)}(t)\big)1_{\{(s,X_u(s))\in\Omega_{t,h}^R\}}\,\mathrm{d}s = [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\\
&\quad\times\int_0^t \frac{t^{3/2}e^{-2(\beta_t-1)(t-s)}}{(t-s)^{3/2}t^{h'}}\sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}[\sqrt{2}\beta_t s - X_u(s)] e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-s)}} 1_{\{(s,X_u(s))\in\Omega_{t,h}^R\}}\,\mathrm{d}s.
\end{aligned} \tag{3.11}$$

Case 1: $h \in (0,1)$. By the Brownian scaling $\big(X_u(s) : u \in N_s^1\big) \overset{law}{=} \Big(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s') : u \in \mathsf{N}_{s'}\Big)$ where $s' = \beta_t s$. Let $G(x) = G_R(x) = xe^{-\frac{x^2}{2}} 1_{\{x\in\Gamma_R\}}$ (Recall that $\Gamma_R := [R^{-1}, R]$). Hence for $(t-s)/t^h \in \Gamma_R$ we have

$$\begin{aligned}
&\sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}\frac{\sqrt{2}\beta_t s - X_u(s)}{\sqrt{t-s}} e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-s)}} 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}} \\
&\sim \sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}\frac{\sqrt{2}\beta_t s - X_u(s)}{\sqrt{t-\beta_t s}} e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-\beta_t s)}} 1_{\left\{\frac{|X_u(s)-\sqrt{2}\beta_t s|}{\sqrt{t-s'}}\in\Gamma_R\right\}} \\
&= \sum_{u\in\mathsf{N}_{s'}} e^{\sqrt{2}\mathsf{X}_u(s')-2s'} G\left(\frac{\sqrt{2}s' - \mathsf{X}_u(s')}{\sqrt{t-s'}}\right) =: \mathsf{W}^G(s', t).
\end{aligned}$$

Making a change of variable $s = t - \lambda t^h$, we get

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right) = \mathsf{E}\bigg(\exp\bigg\{ - [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}}^R e^{-2(\beta_t-1)t^h\lambda}\frac{t^{3/2}}{\lambda t^h}\mathsf{W}^G(\beta_t(t-\lambda t^h), t)\,\mathrm{d}\lambda\bigg\}\bigg).$$

Let $h_t = \frac{\sqrt{t-s'}}{\sqrt{s'}}$ where $s' = \beta_t s = \beta_t(t-\lambda t^h)$. Applying Lemma 2.5 we have

$$\lim_{t\to\infty}\sqrt{t}\,\frac{\mathsf{W}^G(t-\lambda\beta_t t^h, t)}{\int_0^\infty G(z/h_t)ze^{-\frac{z^2}{2}}\,\mathrm{d}z} = \sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty \Longrightarrow \lim_{t\to\infty}\frac{t^{3/2}}{\lambda t^h}\mathsf{W}^G(t-\lambda\beta_t t^h, t) = \sqrt{\frac{2}{\pi}}\,\mathsf{Z}_\infty\int_{\frac{1}{R}}^R y^2 e^{-\frac{y^2}{2}}\,\mathrm{d}y.$$

Letting $t\to\infty$ then $R\to\infty$, applying the dominated convergence theorem and part (ii) of Corollary 2.7, we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right) \\
&= \lim_{R\to\infty}\mathsf{E}\bigg(\exp\bigg\{ - \gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\int_{\frac{1}{R}}^R e^{-\lambda}\,\mathrm{d}\lambda\sqrt{\frac{2}{\pi}}\int_{\frac{1}{R}}^R z^2 e^{-\frac{z^2}{2}}\,\mathrm{d}z\bigg\}\bigg) = \mathsf{E}\Big(\exp\big\{ - \gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\big\}\Big),
\end{aligned}$$

which is the Laplace functional of DPPP $\left(\sqrt{2}C_\star\mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$ (by the definition of $\gamma_{\sqrt{2}}(\varphi)$ in Lemma 2.3). By [9, Lemma 4.4], we complete the proof for the case $h \in (0,1)$.

Case 2: $h \in [1,\infty]$**.** Making change of variable $s = \lambda t$, the integral in (3.11) equals

$$\int_{\frac{1}{R}}^{1-\frac{1}{R}}\frac{e^{-2(\beta_t-1)(1-\lambda)t}}{(1-\lambda)^{3/2}}\sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}[\sqrt{2}\beta_t s - X_u(s)]e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-s)}} 1_{\{v_t s - X_u(s)\in[\frac{1}{R}\sqrt{t}, R\sqrt{t}]\}}\,\mathrm{d}\lambda.$$

Let $G_\lambda(x) = xe^{-\frac{\lambda}{2(1-\lambda)}x^2} 1_{\{\sqrt{\lambda}x\in\Gamma_R\}}$. For $\lambda \in [\frac{1}{R}, (1-\frac{1}{R})]$, $s' := \beta_t s = \beta_t\lambda t$, we have

$$\begin{aligned}
&\sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}\frac{\sqrt{2}\beta_t s - X_u(s)}{\sqrt{\beta_t s}} e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2(t-s)}} 1_{\{v_t s - X_u(s)\in[\frac{1}{R}\sqrt{t}, R\sqrt{t}]\}} \\
&\sim \sum_{u\in N_s^1} e^{\sqrt{2}X_u(s)-2\beta_t s}\frac{\sqrt{2}\beta_t s - X_u(s)}{\sqrt{\beta_t s}} e^{-\frac{\lambda}{1-\lambda}\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2\beta_t s}} 1_{\left\{\frac{\sqrt{2}\beta_t s - X_u(s)}{\sqrt{\beta_t s}}\in[\frac{1}{R}\frac{\sqrt{t}}{\sqrt{s}}, R\frac{\sqrt{t}}{\sqrt{s}}]\right\}} \\
&= \sum_{u\in\mathsf{N}_{s'}} e^{\sqrt{2}\mathsf{X}_u(s')-2s'} G_\lambda\left(\frac{\sqrt{2}s' - \mathsf{X}_u(s')}{\sqrt{s'}}\right) =: \mathsf{W}^{G_\lambda}(s'; t).
\end{aligned}$$

Therefore we have

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\Bigg\{ - [1+o(1)]\gamma_{\sqrt{2}}(\varphi) \int_{\frac{1}{R}}^{1-\frac{1}{R}} \frac{e^{-2(\beta_t-1)(1-\lambda)t}}{(1-\lambda)^{3/2}} \sqrt{\lambda\beta_t t}\mathsf{W}^{G_\lambda}(\lambda\beta_t t,t)\,\mathrm{d}\lambda \Bigg\}\Bigg).$$

Applying Lemma 2.5 we get

$$\lim_{t\to\infty} \sqrt{\lambda\beta_t t}\,\mathsf{W}^{G_\lambda}(\lambda\beta_t t,t) = \mathsf{Z}_\infty\sqrt{\frac{2}{\pi}}\int_0^\infty G_\lambda(z)ze^{-\frac{z^2}{2}}\,\mathrm{d}z = \mathsf{Z}_\infty\sqrt{\frac{2}{\pi}}\int_{\frac{1}{R\sqrt{\lambda}}}^{\frac{R}{\sqrt{\lambda}}} z^2 e^{-\frac{z^2}{2(1-\lambda)}}\,\mathrm{d}z$$

in probability. Let $C_{h,1} = 1_{\{h>1\}} + (1-e^{-1})1_{\{h=1\}}$. Letting $t\to\infty$ then $R\to\infty$ and by the dominated convergence theorem and Corollary 2.7 we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)\\
&= \lim_{R\to\infty}\mathsf{E}\left(\exp\left\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\int_{\frac{1}{R}}^{1-\frac{1}{R}}\frac{e^{-\lambda 1_{\{h=1\}}}}{(1-\lambda)^{3/2}}\,\mathrm{d}\lambda\sqrt{\frac{2}{\pi}}\int_{\frac{1}{R\sqrt{\lambda}}}^{\frac{R}{\sqrt{\lambda}}} z^2e^{-\frac{z^2}{2(1-\lambda)}}\mathrm{d}z\right\}\right) = \mathsf{E}\left(e^{-C_{h,1}\gamma_{\sqrt{2}}(\varphi)Z_\infty}\right),
\end{aligned}$$

which is the Laplace functional of DPPP $\left(C_{h,1}\sqrt{2}C_\star\mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$. By [9, Lemma 4.4], we complete the proof for the case $h\in[1,\infty]$. □

4 Approximation From $\mathscr{C}_{III}$

We first introduce several important constants introduced in [8]. For $(\beta,\sigma^2)\in\mathscr{C}_{III}$, we set

$$\begin{aligned}
b^*(\beta,\sigma^2) &:= \sqrt{2\frac{\beta-1}{1-\sigma^2}},\ a^*(\beta,\sigma^2) := \sigma^2 b(\beta,\sigma^2),\ p^*(\beta,\sigma^2) := \frac{\sigma^2+\beta-2}{2(\beta-1)(1-\sigma^2)};\\
v^*(\beta,\sigma^2) &:= a^*p^* + b^*(1-p^*) = \frac{\beta-\sigma^2}{\sqrt{2(\beta-1)(1-\sigma^2)}}.
\end{aligned}\tag{4.1}$$

Moreover we have

$$\left(\beta - \frac{(a^*)^2}{2\sigma^2}\right)p^* + \left(1-\frac{(b^*)^2}{2}\right)(1-p^*) = 0;\tag{4.2}$$

$$b^*v^* - \beta - \frac{\sigma^2(b^*)^2}{2} = 0.\tag{4.3}$$

For the sake of simplicity, we will write

$$b_t = b^*(\beta_t,\sigma_t^2),\quad a_t := a^*(\beta_t,\sigma_t^2),\quad p_t = p^*(\beta_t,\sigma_t^2),\quad v_t^* = v^*(\beta_t,\sigma_t^2).\tag{4.4}$$

Lemma 4.1. *Let $s = p_t t + u\in(0,t)$, $y := (\sqrt{2}-a_t)s + (v_t^*-\sqrt{2})$ and*

$$L(u,t) = (\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}y - \frac{y^2}{2(t-s)}.\tag{4.5}$$

(i) For $(\beta,\sigma^2)\in\mathscr{B}_{II,III}$ and $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$. We have

$$L(\xi\sqrt{t},t) = -(1-\sigma^2)^2\xi^2 - R(\xi,t),$$

where for each fixed ξ, $R(\xi,t)\to 0$ as $t\to\infty$; and there is some $c>0$ such that $L(\xi\sqrt{t},t)\le -c\xi^2$ for all ξ satisfying that $p_t t + \xi\sqrt{t}\in(0,t)$.

(ii) For $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,3}_{(1,1)}$ and fixed $h\in(0,1)$,

$$L(\xi t^{\frac{1+h}{2}},t)=-(\sqrt{2}+1)\xi^2-R(\xi,t),$$

where for each fixed ξ, $R(\xi,t)\to 0$ as $t\to\infty$; and there is some $c>0$ such that $L(\xi t^{\frac{1+h}{2}},t)\le -c\xi^2$ for all ξ satisfying that $p_t t+\xi\sqrt{t}\in(0,t)$.

The proof of Lemma 4.1 is postponed to Appendix B.

4.1 From $\mathscr{C}_{III}$ to $\mathscr{B}_{II,III}$

Assume that $(\beta,\sigma^2)\in\mathscr{B}_{II,III}$, and $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$. Combining with (4.1) we have

$$p_t\sim\frac{1}{2(1-\sigma^2)^2t^h},\ \frac{b_t}{\sqrt{2}}-1\sim\frac{1}{2(1-\sigma^2)t^h}\text{ and }v_t^*=\sqrt{2}+\frac{1}{b_t}\left(\frac{b_t}{\sqrt{2}}-1\right)^2. \tag{4.6}$$

In fact, as $t\to\infty$, $p_t=\frac{\sigma_t^2+\beta_t-2}{2(\beta_t-1)(1-\sigma_t^2)}=\frac{1}{2(\beta_t-1)(1-\sigma_t^2)t^h}\sim\frac{1}{2(\beta-1)(1-\sigma^2)t^h}=\frac{1}{2(1-\sigma^2)^2t^h}$, $\frac{b_t}{\sqrt{2}}-1=\sqrt{\frac{\beta_t-1}{1-\sigma_t^2}}-1=\sqrt{\frac{1-\sigma_t^2+t^{-h}}{1-\sigma_t^2}}-1\sim\frac{1}{2(1-\sigma_t^2)t^h}\sim\frac{1}{2(1-\sigma^2)t^h}$ and $v_t^*=\frac{\beta_t-1+1-\sigma_t^2}{\sqrt{2(\beta_t-1)(1-\sigma_t^2)}}=\frac{1}{\sqrt{2}}(\frac{b_t}{\sqrt{2}}+\frac{\sqrt{2}}{b_t})=\sqrt{2}+\frac{1}{b_t}\left(\frac{b_t}{\sqrt{2}}-1\right)^2$.

Let $m^{2,3}_{h,+}(t):=v_t^*t-\frac{h'}{\sqrt{2}}\log t$, where $h'=\min\{h,1/2\}$, Define

$$\Omega^R_{t,h}=\begin{cases}\{(s,x):|s-p_tt|\le R\sqrt{t},\ |x-a_ts|\le R\sqrt{s}\}\text{ if }h\in(0,\frac{1}{2}),\\ \{(s,x):s\in[\frac{1}{R}\sqrt{t},R\sqrt{t}],|x-a_ts|\le R\sqrt{s}\}\text{ if }h\in[\frac{1}{2},\infty].\end{cases} \tag{4.7}$$

Lemma 4.2. *For all $A>0$,*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\exists u\in N_t^2:X_u(t)\ge m^{2,3}_{h,+}(t)-A,(T_u,X_u(T_u))\notin\Omega^R_{t,h}\right)=0.$$

Proof. Applying Corollary 2.7 with $m(t)=m^{2,3}_{h,+}(t)$, and $\Omega^R_{t,h}$ defined in (4.7), it suffices to show that for each $A,K>0$, $I(t,R)=I(t,R;A,K)$ defined in (2.6) vanishes as first $t\to\infty$ and then $R\to\infty$. Conditioned on the Brownian motion B_s in (2.6) equals $\frac{a_t}{\sigma_t}s+x$, we have

$$\begin{aligned}I(t,R)=&\int_0^t\mathrm{d}s\int_{-\infty}^{\frac{v_t-a_t}{\sigma_t}s+K}\mathbf{P}\left(B_r\le\sqrt{2\beta_t}r+K,\forall r\le s|B_s=\frac{a_t}{\sigma_t}s+x\right)\\&\qquad e^{\beta_ts}\mathsf{F}_t\left(t-s,v_ts-\sigma_tx\right)1_{\{(s,v_ts-\sigma_tx)\notin\Omega^R_{t,h}\}}e^{-\frac{(a_ts/\sigma_t+x)^2}{2s}}\frac{\mathrm{d}x}{\sqrt{2\pi s}}\\ \lesssim&\int_0^t\mathrm{d}s\int_{-\infty}^{\frac{v_t-a_t}{\sigma_t}s+K}\mathsf{F}_t\left(t-s,a_ts+\sigma_tx\right)1_{\{(s,a_ts+\sigma_tx)\notin\Omega^R_{t,h}\}}e^{(\beta_t-\frac{a_t^2}{2\sigma_t^2})s-\frac{a_t}{\sigma_t}x-\frac{x^2}{2s}}\frac{\mathrm{d}x}{\sqrt{s}}.\end{aligned} \tag{4.8}$$

Now we aim to get an upper bound for $\mathsf{F}_t\left(t-s,a_ts+\sigma_tx\right)$. Let $\mathrm{y}:=(\sqrt{2}-a_t)s+(v_t^*-\sqrt{2})t$ and $\mathrm{w}:=\frac{h'}{\sqrt{2}}\log t-\frac{3}{2\sqrt{2}}\log(t-s+1)+A$. By Lemma 2.2, provided that $\mathrm{y}-\sigma_tx-\mathrm{w}>1$, we have

$$\begin{aligned}\mathsf{F}_t\left(t-s,a_ts+\sigma_tx\right)&=\mathsf{P}\left(\max_{u\in\mathsf{N}_{t-s}}\mathsf{X}_u(t-s)\ge v_t^*t-a_ts-\sigma_tx-\frac{h'}{\sqrt{2}}\log t-A\right)\\&=\mathsf{P}\left(\max_{u\in\mathsf{N}_{t-s}}\mathsf{X}_u(t-s)>\sqrt{2}(t-s)-\frac{3}{2\sqrt{2}}\log(t-s+1)+\mathrm{y}-\sigma_tx-\mathrm{w}\right)\\&\lesssim_A(\mathrm{y}-\sigma_tx-\mathrm{w})\frac{t^{h'}}{(t-s+1)^{3/2}}\exp\left\{-\sqrt{2}\mathrm{y}+\sqrt{2}\sigma_tx-\frac{1}{2(t-s)}[\mathrm{y}-\sigma_tx-\widetilde{\mathrm{w}}]^2\right\},\end{aligned} \tag{4.9}$$

where $\widetilde{\mathrm{w}} := \mathrm{w} - \frac{3}{2\sqrt{2}}\log(t-s+1)$. We claim that for large t we always have $\mathrm{y} - \sigma_t x - \mathrm{w} > 1$ for $s \in (0,t)$ and $\sigma_t x \le (v_t - a_t)s + \sigma_t K$. In fact, $\mathrm{y} - \sigma_t x - \mathrm{w} > (\sqrt{2} - v_t)s - \frac{h'}{\sqrt{2}}\log t + \frac{3}{2\sqrt{2}}\log(t-s+1) - O(1)$. By our assumption $(\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \in \mathscr{B}_{II,III}$, we have $\sqrt{2} - v_t > \delta > 0$ for large t. Then for each $\delta s > 2\log t$, $\mathrm{y} - \sigma_t x - \mathrm{w} > 2\log t - \frac{h'}{\sqrt{2}}\log t - O(1) > 1$; for each $\delta s \le 2\log t$, we have $t - s + 1 \ge t/2$, and hence $\mathrm{y} - \sigma_t x - \mathrm{w} > \frac{3}{2\sqrt{2}}\log(\frac{t}{2}) - \frac{h'}{\sqrt{2}}\log t - O(1) > 1$.

Substituting (4.9) into (4.8), we get

$$\begin{aligned} I(t,R) \lesssim \int_0^t \frac{t^{h'}}{(t-s+1)^{3/2}s^{1/2}} \exp\left\{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}\mathrm{y} - \frac{\mathrm{y}^2}{2(t-s)}\right\} \mathrm{d}s \\ \int_{\mathbb{R}} |\mathrm{y} - \sigma_t x - \mathrm{w}| \exp\left\{(\sqrt{2}\sigma_t - \frac{a_t}{\sigma_t} + \frac{\mathrm{y}\sigma_t}{t-s})x\right\} e^{\frac{-\widetilde{\mathrm{w}}\mathrm{y}}{t-s}} e^{-\frac{[\sigma_t x + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{x^2}{2s}} 1_{\left\{(s, a_t s + \sigma_t x) \notin \Omega_{t,h}^R\right\}} \mathrm{d}x \end{aligned} \tag{4.10}$$

Making a change of variable $s = p_t t + \xi\sqrt{t}$, by (4.1), we have $\mathrm{y} = (b_t - \sqrt{2})(1-p_t)t + (\sqrt{2} - a_t)\xi\sqrt{t}$. Thanks to Lemma 4.1,

$$(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}\mathrm{y} - \frac{\mathrm{y}^2}{2(t-s)} = L(\xi\sqrt{t}, t) = -(1-\sigma^2)^2\xi^2 - R(\xi, t). \tag{4.11}$$

Moreover, since $a_t/\sigma_t = \sigma_t b_t$ (see (4.1)), the coefficient for the term x in (4.10) is

$$\sqrt{2}\sigma_t - \frac{a_t}{\sigma_t} + \frac{\mathrm{y}\sigma_t}{t-s} = \frac{\sigma_t(b_t - a_t)}{t-s}\xi\sqrt{t}. \tag{4.12}$$

Let $\Gamma_h^R = [-R, R]$ if $h \in (0, 1/2)$ and $\Gamma_h^R = [R^{-1}, R]$ if $h \in [1/2, \infty]$. Note that $(s, a_t s + \sigma_t x) \in \Omega_{t,h}^R$ if and only if $\xi + p_t\sqrt{t}1_{\{h \ge 1/2\}} \in \Gamma_h^R$ and $|\sigma_t x| \le R\sqrt{s}$. Combining equalities (4.11) and (4.12), we have

$$\begin{aligned} I_1(t,R) = \int_{-p_t\sqrt{t}}^{(1-p_t)\sqrt{t}} \frac{t^{h'}t^{1/2}}{(t-s)^{3/2}s^{1/2}} e^{-(1-\sigma^2)^2\xi^2 - R(\xi,t)} \mathrm{d}\xi \\ \int_{\mathbb{R}} 1_{\left\{\substack{p_t\sqrt{t}1_{\{h\ge 1/2\}} + \xi \notin \Gamma_h^R \\ \text{or } |x| > R\sqrt{s}}\right\}} |\mathrm{y} - \sigma_t x - \mathrm{w}| \exp\left\{\frac{\sigma_t(b_t - a_t)\sqrt{t}}{t-s}\xi x\right\} e^{\frac{-\widetilde{\mathrm{w}}\mathrm{y}}{t-s}} e^{-\frac{[\sigma_t x + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{x^2}{2s}} \mathrm{d}x. \end{aligned}$$

Now again making a change of variable $x = \eta\sqrt{s}$ (where $s = p_t t + \xi\sqrt{t}$), we get

$$\begin{aligned} I_1(t,R) \lesssim \int_{-p_t\sqrt{t}}^{(1-p_t)\sqrt{t}} \frac{t^{h'+1/2}}{(t-s)^{3/2}} e^{-(1-\sigma^2)^2\xi^2 - R(\xi,t)} \mathrm{d}\xi \\ \int_{\mathbb{R}} 1_{\left\{\substack{p_t\sqrt{t}1_{\{h\ge 1/2\}} + \xi \notin \Gamma_h^R \\ \text{or } |\eta| > R}\right\}} |\mathrm{y} - \sigma_t\eta\sqrt{s} - \mathrm{w}| \exp\left\{\frac{\sigma_t(b_t - a_t)\sqrt{ts}}{t-s}\xi\eta\right\} e^{\frac{-\widetilde{\mathrm{w}}\mathrm{y}}{t-s}} e^{-\frac{[\sigma_t\eta\sqrt{s} + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{\eta^2}{2}} \mathrm{d}\eta. \end{aligned}$$

For fixed ξ, by (4.6), $p_t = \Theta(t^{-h})$ and $b_t - \sqrt{2} = \Theta(t^{-h})$, and then $s = O(t^{1-h}) + O(\sqrt{t}) = O(t^{1-h'})$, $\mathrm{y} = (b_t - \sqrt{2})(1-p_t)t + (\sqrt{2} - a_t)\xi\sqrt{t} = O(t^{1-h'})$ and $\mathrm{w} = O(\log t) = \widetilde{\mathrm{w}}$. Let $\xi_0(h) = 1_{\{h \ge 1/2\}} \lim_{t\to\infty} p_t\sqrt{t} = \frac{1}{2(1-\sigma^2)^2} 1_{\{h = 1/2\}}$. Then the dominated convergence theorem yields that

$$\limsup_{t\to\infty} I_1(t,R) \lesssim \int_{-\lim_t p_t\sqrt{t}}^{\infty} e^{-(1-\sigma^2)^2\xi^2} \mathrm{d}\xi \int_{\mathbb{R}} C(h,\xi) e^{-\frac{\eta^2}{2}} 1_{\left\{\substack{\xi \notin \Gamma_h^R - \xi_0(h) \\ \text{or } |\eta| > R}\right\}} \mathrm{d}\eta,$$

where for fixed ξ, by (4.6), $C(h,\xi) := \lim_t \frac{|\mathrm{y} - \sigma_t\eta\sqrt{s} - \mathrm{w}|}{t^{1-h'}} = \lim_t \frac{\mathrm{y}}{t^{1-h'}} = \frac{1_{\{h < 1/2\}}}{\sqrt{2}(1-\sigma^2)} + \sqrt{2}(1-\sigma^2)\xi 1_{\{h \ge 1/2\}}$. Finally, letting $R \to \infty$, the desired result follows. $\square$

We now show Theorem 1.3 for the case that for $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}_{(\beta,\sigma^2)}^{h,+}$.

Proof of Theorem 1.3 for $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$. Take $\varphi\in\mathcal{T}$. Applying Corollary 2.7 with $m(t)=m^{2,3}_{h,+}(t)$, $\rho=\sqrt{2}$, and $\Omega^R_{t,h}$ defined in (4.7), it suffices to study the asymptotic behavior of

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right)=\mathbb{E}\Bigg(\exp\Bigg\{-\int_0^t\sum_{u\in N^1_s}\Phi_{\sqrt{2}}\big(t-s,X_u(s)-a_ts-\mathrm{y}+\frac{h'}{\sqrt{2}}\log t\big)1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

where we used the fact that $\sqrt{2}(t-s)-m^{2,3}_{h,+}(t)=-a_ts-\mathrm{y}+\frac{h'}{\sqrt{2}}\log t$ and $\mathrm{y}:=(\sqrt{2}-a_t)s+(v^*_t-\sqrt{2})t$. Moreover, by Lemma 2.3, uniformly for $(s,X_u(s))\in\Omega^R_{t,h}$, $s=\Theta(t^{1-h'})$ and $|X_u(s)-a_ts|=\Theta(\sqrt{s})$, we have

$$\begin{aligned}
&\Phi_{\sqrt{2}}\left(t-s,X_u(s)-a_ts+\frac{h'}{\sqrt{2}}\log t-\mathrm{y}\right)\\
&=(1+o(1))\gamma_{\sqrt{2}}(\varphi)\frac{\mathrm{y}}{(t-s)^{3/2}}t^{h'}\exp\left\{\sqrt{2}X_u(s)-\sqrt{2}a_ts-\sqrt{2}\mathrm{y}-\frac{[X_u(s)-a_ts-\mathrm{y}-\Theta(\log t)]^2}{2(t-s)}\right\}\\
&=(1+o(1))\gamma_{\sqrt{2}}(\varphi)\frac{t^{h'}\mathrm{y}}{t^{3/2}}\exp\left\{\sqrt{2}(X_u(s)-a_ts)-\sqrt{2}\mathrm{y}-\frac{\mathrm{y}^2}{2(t-s)}-\frac{\mathrm{y}(X_u(s)-a_ts)}{t-s}\right\}.
\end{aligned}\tag{4.13}$$

Make a change of variable $s=p_tt+\xi\sqrt{t}$. On the one hand,

$$\begin{aligned}
\frac{\mathrm{y}(X_u(s)-a_ts)}{t-s}&=(b_t-\sqrt{2})(X_u(s)-a_ts)+\frac{(b_t-a_t)\xi\sqrt{t}}{t-s}(X_u(s)-a_ts)\\
&=(b_t-\sqrt{2})(X_u(s)-a_ts)+o(1).
\end{aligned}\tag{4.14}$$

On the other hand, by Lemma 4.1, we have

$$-\sqrt{2}\mathrm{y}-\frac{\mathrm{y}^2}{2(t-s)}=-(\beta_t-\frac{a_t^2}{2\sigma_t^2})s-(1-\sigma^2)^2\xi^2+o(1).\tag{4.15}$$

So combining (4.13), (4.14) and (4.14), we have, uniformly for $(s,X_u(s))\in\Omega^R_{t,h}$,

$$\begin{aligned}
&\Phi_{\sqrt{2}}\big(t-s,X_u(s)-a_ts+\frac{h'}{\sqrt{2}}\log t-\mathrm{y}\big)\\
&=[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{t^{h'}\mathrm{y}}{t^{3/2}}e^{b_tX_u(s)-b_ta_ts-(\beta_t-\frac{a_t^2}{2\sigma_t^2})s}e^{-(1-\sigma^2)^2\xi^2}\\
&=[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{t^{h'}\mathrm{y}}{t^{3/2}}e^{b_tX_u(s)-(\beta_t+\frac{\sigma_t^2b_t^2}{2})s}e^{-(1-\sigma^2)^2\xi^2},
\end{aligned}$$

where we used the fact $a_t=\sigma^2b_t$ (see (4.1)). We compute that

$$\begin{aligned}
&\int_0^t\sum_{u\in N^1_s}\Phi_{\sqrt{2}}\big(t-s,X_u(s)-a_ts-\mathrm{y}+\frac{h'}{\sqrt{2}}\log t\big)1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\\
&=[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\Gamma^R_h-p_t\sqrt{t}1_{\{h\geq1/2\}}}\frac{\mathrm{y}}{t^{1-h'}}\sum_{u\in N^1_s}e^{b_tX_u(s)-(\beta_t+\frac{\sigma_t^2b_t^2}{2})s}1_{\{|X_u(s)-\sigma_t^2b_ts|\leq R\sqrt{s}\}}e^{-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi\\
&=[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\xi\in\Gamma^R_h-p_t\sqrt{t}1_{\{h\geq1/2\}}}\frac{\mathrm{y}}{t^{1-h'}}W(p_tt+\xi\sqrt{t};t)e^{-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi,
\end{aligned}$$

where $W(s;t):=\sum_{u\in N^1_s}e^{b_tX_u(s)-(\beta_t+\frac{\sigma_t^2b_t^2}{2})s}1_{\{|X_u(s)-\sigma_t^2b_ts|\leq R\sqrt{s}\}}$. By the Brownian scaling, $(X_u(s):u\in N^1_s)\overset{law}{=}(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s'):u\in\mathsf{N}_{s'})$, where $s'=\beta_ts$. Let $\lambda_t=b_t\sigma_t/\sqrt{\beta_t}$. We have

$$W(s,t)\overset{law}{=}\sum_{u\in\mathsf{N}_{s'}}e^{\lambda_t\mathsf{X}_u(s')-(1+\lambda_t^2/2)s'}1_{\{|\frac{\mathsf{X}_u(s')-\lambda_ts'}{\sqrt{s'}}|\leq\frac{R}{\sigma_t}\}}.$$

Since $\lambda_t \to \sqrt{2}\sigma/\sqrt{\beta} < \sqrt{2}$, by part(i) of Lemma 2.4, we have

$$\lim_{t\to\infty} W(p_t t + \xi\sqrt{t}, t) = \mathsf{W}_\infty\left(\frac{\sqrt{2}\sigma}{\sqrt{\beta}}\right)\int_{[-\frac{R}{\sigma},\frac{R}{\sigma}]} e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}} = \mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2})\int_{[-\frac{R}{\sigma},\frac{R}{\sigma}]} e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}} \quad \text{in law.}$$

Also we have $\lim_{t\to\infty}\frac{\mathrm{y}}{t^{1-h'}} = C(h,\xi) = \frac{1_{\{h\leq 1/2\}}}{\sqrt{2}(1-\sigma^2)} + \sqrt{2}(1-\sigma^2)\xi 1_{\{h\geq 1/2\}}$. Therefore

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\left\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2})\int_{\xi\in\Gamma_h^R-\xi_0(h)} C(h,\xi)e^{-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi\int_{|\sigma x|\leq R} e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}}\right\}\Bigg),$$

where $\xi_0(h) = 1_{\{h\geq 1/2\}}\lim_{t\to\infty} p_t\sqrt{t} = \frac{1}{2(1-\sigma^2)^2}1_{\{h=1/2\}}$. Applying Corollary 2.7 we finally get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Big(\exp\big\{-C_{h,+}\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\big\}\Big),$$

which is the Laplace functional of DPPP $\left(C_{h,+}\sqrt{2}C_\star\mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$, and where

$$\begin{aligned} C_{h,+} :=& \lim_{R\to\infty}\int_{\Gamma_h^R-\xi_0(h)} C(h,\xi)e^{-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi \\ =& \frac{\sqrt{\pi/2}}{(1-\sigma^2)^2}1_{\{h<1/2\}} + \frac{1}{\sqrt{2}(1-\sigma^2)}1_{\{h>1/2\}} \\ &+ \int_{-\frac{1}{2(1-\sigma^2)^2}}^{\infty}\left(\frac{1}{\sqrt{2}(1-\sigma^2)} + \sqrt{2}(1-\sigma^2)\xi\right)e^{-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi 1_{\{h=1/2\}}. \end{aligned}$$

We now complete the proof. □

4.2 From $\mathcal{C}_{III}$ to $(1,1)$

Assume that $(\beta_t,\sigma_t^2)_{t>0} \in \mathscr{A}_{(1,1)}^{h,3}$. Let $m_{h,3}^{(1,1)}(t) = v_t^* t - \frac{h'}{2\sqrt{2}}\log t$ where $h' = \min\{h,1\}$. By the assumption $\beta_t + \sigma_t^2 = \frac{1}{\beta_t} + \frac{1}{\sigma_t^2}$, we have $\beta_t\sigma_t^2 = 1$. So $v_t = \sqrt{2}$ and $\theta_t = \sqrt{2}/\sigma_t^2$. Moreover, we have

$$\sigma_t = 1 - \frac{1}{2t^{h/2}} + \Theta(\frac{1}{t^h}),\ p_t \equiv \frac{1}{2}\ ,\ b_t = \frac{\sqrt{2}}{\sigma_t}\ ,\ \sqrt{2} - a_t \sim \frac{1}{\sqrt{2}t^{h/2}}\ ,\ v_t^* - \sqrt{2} \sim \frac{1}{4\sqrt{2}t^h}\,. \tag{4.16}$$

In fact, (4.16) follows from the following computations. Firstly, $\beta_t + \sigma_t^2 = \sigma_t^2 + \sigma_t^{-2} = 2 + t^{-h} \Rightarrow (\sigma_t^2 - 1)^2 = \sigma_t^2 t^{-h} \sim t^{-h} \Rightarrow \sigma_t^2 = 1 - t^{-h/2} + \Theta(t^{-h})$. Secondly, $p_t = \frac{\beta_t + \sigma_t^2 - 2}{2(1-\sigma_t^2)(\beta_t - 1)} = \frac{t^{-h}}{2[\beta_t + \sigma_t^2 - 1 - \beta_t\sigma_t^2]} = \frac{1}{2}$. Thirdly, we have $(\beta_t - 1)^2 = \beta_t t^{-h}$. So $b_t = \sqrt{2}\sqrt{\frac{\beta_t - 1}{1-\sigma_t^2}} = \sqrt{2}(\frac{\beta_t}{\sigma_t^2})^{1/4} = \frac{\sqrt{2}}{\sigma_t}$. Besides $\sqrt{2} - a_t = \sqrt{2}(1-\sigma_t) \sim \frac{1}{\sqrt{2}t^{h/2}}$. Finally $v_t^* - \sqrt{2} = \frac{1}{\sqrt{2}}(\sigma_t + \frac{1}{\sigma_t} - 2) = \frac{(\sigma_t - 1)^2}{\sqrt{2}\sigma_t} \sim \frac{1}{4\sqrt{2}t^h}$.

Define

$$\Omega_{t,h}^R = \begin{cases} \{(s,x) : |s - \frac{t}{2}| \leq Rt^{\frac{1+h}{2}},\ |x - a_t s| \leq R\sqrt{s}\} & \text{for } h \in (0,1); \\ \{(s,x) : s \in [\frac{1}{R}t, (1-\frac{1}{R})t],\ \sqrt{2}s - x \in [\frac{1}{R}\sqrt{t}, R\sqrt{t}]\} & \text{for } h \in [1,\infty] \end{cases} \tag{4.17}$$

Lemma 4.3. *For all $A > 0$, and $h \in (0,\infty]$*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\exists u \in N_t^2 : X_u(t) \geq m_{h,3}^{(1,1)}(t) - A, (T_u, X_u(T_u)) \notin \Omega_t^{R,h}\right) = 0.$$

Proof. Applying Corollary 2.7 with $m(t) = m_{h,3}^{(1,1)}(t)$, and $\Omega_{t,h}^R$ defined in (4.17), it suffices to show that for each $A, K > 0$, $I(t,R) = I(t,R;A,K)$ defined in (2.6) vanishes as first $t \to \infty$ and then $R \to \infty$.

Case 1: $h \in (0,1)$**.** Conditioned on the Brownian motion B_s in (2.6) equals $\frac{a_t}{\sigma_t}s + x$, we have

$$
\begin{aligned}
I(t,R) \lesssim \int_0^t \mathrm{d}s \int_{-\infty}^{\frac{\sqrt{2}-a_t}{\sigma_t}s+K} &\frac{(\sqrt{2}-a_t)s + \sigma_t K - \sigma_t x}{s^{3/2}} \mathsf{F}_t\left(t-s, a_t s + \sigma_t x\right) e^{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \frac{a_t}{\sigma_t}x - \frac{x^2}{2s}} \\
&\times 1_{\{(s, a_t s + \sigma_t x) \notin \Omega_{t,h}^R\}} \,\mathrm{d}x.
\end{aligned} \tag{4.18}
$$

where we use that $\mathbf{P}(B_u \leq \frac{\sqrt{2}}{\sigma_t}u + K, \forall u \leq s | B_s = \frac{a_t}{\sigma_t} + x) \lesssim \frac{(\sqrt{2}-a_t)s + \sigma_t K - \sigma_t x}{s}$ by Lemma 2.1.

We still denote $\mathrm{y} := (\sqrt{2}-a_t)s + (v_t^* - \sqrt{2})t$. Let $\mathrm{w} := \frac{h}{2\sqrt{2}}\log t - \frac{3}{2\sqrt{2}}\log(t-s+1) + A$. As in (4.9), provided that $\mathrm{y} - \sigma_t x - \mathrm{w} > 1$, we have

$$
\mathsf{F}_t\left(t-s, a_t s + \sigma_t x\right) \lesssim_A \frac{(\mathrm{y} - \sigma_t x - \mathrm{w})t^{h/2}}{(t-s+1)^{3/2}} \exp\left\{-\sqrt{2}\mathrm{y} + \sqrt{2}\sigma_t x - \frac{1}{2(t-s)}[\mathrm{y} - \sigma_t x - \widetilde{\mathrm{w}}]^2\right\}. \tag{4.19}
$$

where $\widetilde{\mathrm{w}} := \mathrm{w} - \frac{3}{2\sqrt{2}}\log(t-s+1)$. Indeed for large t and for all $a_t s + \sigma_t x \leq \sqrt{2}s + \sigma_t K$ we have $\mathrm{y} - \sigma_t x - \mathrm{w} \geq (v_t^* - \sqrt{2})t - O(\log t) = \Theta(t^{1-h}) > 1$ by (4.16).

Let $J = J_{x,s,t} := \frac{|(\sqrt{2}-a_t)s + \sigma_t K - \sigma_t x|}{s^{3/2}} \frac{|\mathrm{y} - \sigma_t x - \mathrm{w}| t^{h/2}}{(t-s+1)^{3/2}}$. Substituting (4.19) into (4.18) we get

$$
\begin{aligned}
I(t,R) \lesssim &\int_0^t \exp\left\{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}\mathrm{y} - \frac{\mathrm{y}^2}{2(t-s)}\right\} \mathrm{d}s \\
&\int_{\mathbb{R}} J \exp\left\{(\sqrt{2}\sigma_t - \frac{a_t}{\sigma_t} + \frac{\mathrm{y}\sigma_t}{t-s})x\right\} e^{\frac{-\widetilde{\mathrm{w}}\mathrm{y}}{t-s}} e^{-\frac{[\sigma_t x + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{x^2}{2s}} 1_{\{(s, a_t s + \sigma_t x) \notin \Omega_{t,h}^R\}} \,\mathrm{d}x.
\end{aligned}
$$

Making a change of variable $s = p_t t + \xi t^{\frac{1+h}{2}} = \frac{1}{2}t + \xi t^{\frac{1+h}{2}}$, by Lemma 4.1, we have

$$
\begin{aligned}
(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}\mathrm{y} - \frac{\mathrm{y}^2}{2(t-s)} &= L(\xi t^{\frac{1+h}{2}}, t) = -(\sqrt{2}+1)\xi^2 - R(\xi, t), \\
(\sqrt{2}\sigma_t - \frac{a_t}{\sigma_t} + \frac{\mathrm{y}\sigma_t}{t-s}) &= \frac{\sigma_t(b_t - a_t)}{t-s}\xi t^{\frac{1+h}{2}}.
\end{aligned}
$$

Note that $(s, a_t s + \sigma_t x) \in \Omega_{t,h}^R$ if and only if $|\xi| \leq R$ and $|\sigma_t x| \leq R\sqrt{s}$. Then we have

$$
\begin{aligned}
I(t,R) \lesssim &\int_{-\frac{1}{2}t^{\frac{1-h}{2}}}^{\frac{1}{2}t^{\frac{1-h}{2}}} e^{-(\sqrt{2}+1)\xi^2 - R(\xi,t)} \,\mathrm{d}\xi \\
&\int_{\mathbb{R}} t^{\frac{1+h}{2}} J \exp\left\{\frac{\sigma_t(b_t - a_t)}{t-s}\xi t^{\frac{1+h}{2}} x - \frac{\widetilde{\mathrm{w}}\mathrm{y}}{t-s} - \frac{[\sigma_t x + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{x^2}{2s}\right\} 1_{\left\{\substack{|\xi| > R, \text{or} \\ |\sigma_t x| > R\sqrt{s}}\right\}} \,\mathrm{d}x.
\end{aligned}
$$

Again making a change of variable $x = \eta\sqrt{s}$, we get

$$
\begin{aligned}
I(t,R) \lesssim &\int_{-\frac{1}{2}t^{\frac{1-h}{2}}}^{\frac{1}{2}t^{\frac{1-h}{2}}} e^{-(\sqrt{2}+1)\xi^2 - R(\xi,t)} \,\mathrm{d}\xi \\
&\int_{\mathbb{R}} \sqrt{s}\, t^{\frac{1+h}{2}} J \exp\left\{\frac{\sigma_t(b_t - a_t)}{t-s}\sqrt{s}\, t^{\frac{1+h}{2}}\xi\eta - \frac{\widetilde{\mathrm{w}}\mathrm{y}}{t-s} - \frac{[\sigma_t \eta\sqrt{s} + \widetilde{\mathrm{w}}]^2}{2(t-s)} - \frac{\eta^2}{2}\right\} 1_{\left\{\substack{|\xi| > R, \text{or} \\ |\sigma_t \eta| > R}\right\}} \,\mathrm{d}\eta.
\end{aligned}
$$

By (4.16), $\sqrt{2}-a_t \sim \frac{1}{\sqrt{2}t^{h/2}}$, $\mathrm{y} \sim (\sqrt{2}-a_t)\frac{t}{2}$ and $\mathrm{w} = O(\log t)$. For fixed ξ, η we have

$$\lim_{t\to\infty} \sqrt{s}\, t^{\frac{1+h}{2}} J = \lim_{t\to\infty} t^{\frac{1}{2}+h} \frac{|(\sqrt{2}-a_t)s + \sigma_t K - \sigma_t \eta \sqrt{s}|}{s} \frac{|\mathrm{y} - \sigma_t \eta \sqrt{s} - \mathrm{w}|}{(t-s+1)^{3/2}}$$
$$\leq \lim_{t\to\infty} t^{\frac{1}{2}+h} [(\sqrt{2}-a_t) + \frac{\eta}{\sqrt{t}}] \frac{(\sqrt{2}-a_t)\frac{t}{2}}{(t/2)^{3/2}} = 2.$$

Similarly, $\lim_{t\to\infty} \frac{\sigma_t(b_t-a_t)}{t-s} \sqrt{s}\, t^{\frac{1+h}{2}} = 2$. Then the dominated convergence theorem yields that

$$\limsup_{t\to\infty} I(t,R) \lesssim \iint_{\mathbb{R}^2 \setminus [-R,R]^2} \exp\{-(\sqrt{2}+1)\xi^2 + 2\xi\eta - \eta^2\} \, \mathrm{d}\eta \, \mathrm{d}\xi \overset{R\to\infty}{\longrightarrow} 0.$$

Case 2: $h \in [1,\infty]$**.** Now conditioned on the Brownian motion B_s in (2.6) equals $\sqrt{2\beta_t}s - x$, we get

$$\begin{aligned} I(t,R) &= \int_0^t \mathrm{d}s \int_{-K}^{\infty} \mathbf{P}\left(B_r \leq \sqrt{2\beta_t}r + K, \forall r \leq s \mid B_s = \sqrt{2\beta_t}s - x\right) \\ &\quad e^{\beta_t s} \mathsf{F}_t\left(t-s, \sqrt{2}s - \sigma_t x\right) 1_{\{(s,\sqrt{2}s-\sigma_t x)\notin \Omega^R_{t,h}\}} e^{-\frac{-(\sqrt{2\beta_t}s-x)^2}{2s}} \frac{\mathrm{d}x}{\sqrt{2\pi s}} \\ &\lesssim \int_0^t \mathrm{d}s \int_{-K}^{\infty} \frac{K+x}{s^{3/2}} \mathsf{F}_t\left(t-s, \sqrt{2}s - \sigma_t x\right) 1_{\{(s,\sqrt{2}s-\sigma_t x)\notin \Omega^R_{t,h}\}} e^{-\sqrt{2\beta_t}x - \frac{x^2}{2s}} \, \mathrm{d}x. \end{aligned} \tag{4.20}$$

Let $\mathrm{w} := \frac{1}{2\sqrt{2}} \log t - \frac{3}{2\sqrt{2}} \log(t-s+1) + A$. By Lemma 2.2, if $\sigma_t x - \mathrm{w} > 1$, we have

$$\begin{aligned} \mathsf{F}_t\left(t-s, \sqrt{2}s - \sigma_t x\right) &= \mathsf{P}(\max_{u\in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) > \sqrt{2}(t-s) - \frac{3}{2\sqrt{2}} \log(t-s+1) + \sigma_t x - \mathrm{w}) \\ &\lesssim_A (\sigma_t x - \mathrm{w}) \frac{t^{1/2}}{(t-s+1)^{3/2}} \exp\left\{\sqrt{2}\sigma_t x - \frac{[\sigma_t x - \widetilde{\mathrm{w}}]^2}{2(t-s)}\right\}, \end{aligned} \tag{4.21}$$

where $\widetilde{\mathrm{w}} := \mathrm{w} - \frac{3}{2\sqrt{2}} \log(t-s+1)$. Note that $\sigma_t x - \mathrm{w} \geq \frac{3}{2\sqrt{2}} \log(t-s+1) - \frac{1}{2\sqrt{2}} \log t - O(1)$. So if $\sigma_t x - \mathrm{w} \leq 1$ it must be $s \geq t/2$ and $-\sqrt{2\beta_t}x \leq -\sqrt{2\beta_t}\mathrm{w} = -\sqrt{2}\mathrm{w} + o(1)$. We upper bound $\mathsf{F}_t\left(t-s, \sqrt{2}s - \sigma_t x\right)$ by 1. Futhermore, there holds

$$\int_{t/2}^t \mathrm{d}s \int_{-K}^{O(\log t)} \frac{K+x}{s^{3/2}} e^{-\sqrt{2\beta_t}x - \frac{x^2}{2s}} 1_{\{\sigma_t x - \mathrm{w} \leq 1\}} \, \mathrm{d}x \lesssim \frac{O(\log t)^2}{t^{3/2}} \int_{t/2}^t \frac{t^{1/2}}{(t-s+1)^{3/2}} \, \mathrm{d}s = o(1).$$

In summary, we have

$$I(t,R) \lesssim \int_0^t \int_{-K}^{\infty} \frac{|K+x|\, t^{1/2}\, |\sigma_t x - \mathrm{w}|}{s^{3/2}(t-s+1)^{3/2}} e^{\sqrt{2}(\sqrt{\beta_t}-\sigma_t)x} e^{-\frac{x^2}{2s} - \frac{(\sigma_t x - \widetilde{\mathrm{w}})^2}{2(t-s)}} 1_{\{(s,\sqrt{2}s-\sigma_t x)\notin \Omega^R_{t,h}\}} \, \mathrm{d}x \, \mathrm{d}s + o(1).$$

Make change of variables $s = \xi t$ and $x = \eta\sqrt{t}$. Then $(s, \sqrt{s} - \sigma_t x) \in \Omega^R_{t,h}$ if and only if $\xi \in [R^{-1}, 1-R^{-1}]$ and $\sigma_t \eta \in [R^{-1}, R]$. Hence

$$I(t,R) \lesssim \int_0^1 \mathrm{d}\xi \int_{-\frac{K}{\sqrt{t}}}^{\infty} \frac{(|\eta| + \frac{K}{\sqrt{t}})(|\eta| + \frac{\mathrm{w}}{\sqrt{t}})}{\xi^{3/2}(1-\xi)^{3/2}} e^{\sqrt{2}(\sqrt{\beta_t}-\sigma_t)\sqrt{t}\eta} e^{-\frac{\eta^2}{2\xi} - \frac{(\sigma_t \eta - \widetilde{\mathrm{w}}/\sqrt{t})^2}{2(1-\xi)}} 1_{\left\{\substack{\xi \notin [R^{-1}, 1-R^{-1}] \\ \text{or } \sigma_t \eta \notin [R^{-1}, R]}\right\}} \, \mathrm{d}\eta.$$

By (4.16), we have $\lim_{t\to\infty} (\sqrt{\beta_t} - \sigma_t)\sqrt{t} = \lim_{t\to\infty} (\sigma_t^{-1} - \sigma_t)\sqrt{t} = 1_{\{h=1/2\}}$. Applying the dominated convergence theorem, we get

$$\limsup_{t\to\infty} I(t,R) \lesssim \int_0^1 \mathrm{d}\xi \int_0^{\infty} \frac{\eta^2}{\xi^{3/2}(1-\xi)^{3/2}} e^{\sqrt{2}\eta} e^{-\frac{\eta^2}{2\xi} - \frac{\eta^2}{2(1-\xi)}} 1_{\left\{\substack{\xi \notin [R^{-1}, 1-R^{-1}] \\ \text{or } \eta \notin [R^{-1}, R]}\right\}} \, \mathrm{d}\eta \overset{R\to\infty}{\longrightarrow} 0,$$

where we use the integratibility of $\frac{\eta^2}{\xi^{3/2}(1-\xi)^{3/2}}e^{\sqrt{2}\eta-\frac{\eta^2}{2\xi}-\frac{\eta^2}{2(1-\xi)}}$ on $(0,1)\times[0,\infty]$. We now compelte the proof. $\square$

Finally we give the proof of Theorem 1.4 for the case that $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,3}_{(1,1)}$.

Proof of Theorem 1.4 for $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,3}_{(1,1)}$. Take $\varphi\in\mathcal{T}$. Applying Corollary 2.7 with $m(t)=m^{(1,1)}_{h,3}(t)$, $\rho=\sqrt{2}$, and $\Omega^R_{t,h}$ defined in (4.17), it suffices to study the asymptotic of

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right)=\mathbb{E}\Bigg(\exp\Bigg\{-\int_0^t\sum_{u\in N^1_s}\Phi_{\sqrt{2}}\big(t-s,X_u(s)+\sqrt{2}(t-s)-m^{(1,1)}_{h,3}(t)\big)1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

Case 1: $h\in(0,1)$. Let $\mathrm{y}:=(v_t^*-\sqrt{2})t+(\sqrt{2}-a_t)s$, then $\sqrt{2}(t-s)-m^{(1,1)}_{h,3}(t)=-a_ts-\mathrm{y}+\frac{h}{2\sqrt{2}}\log t$. Making a change of variable $s=\frac{t}{2}+\xi t^{\frac{1+h}{2}}$, we can rewrite $\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right)$ as

$$\mathbb{E}\Bigg(\exp\Bigg\{-t^{\frac{1+h}{2}}\int_{-R}^R\sum_{u\in N^1_s}\Phi_{\sqrt{2}}\left(t-s,X_u(s)-a_ts-\mathrm{y}+\frac{h}{2\sqrt{2}}\log t\right)1_{\{|X_u(s)-a_ts|\le R\sqrt{s}\}}\,\mathrm{d}\xi\Bigg\}\Bigg).$$

By (4.16), uniformly for $(s,X_u(s))\in\Omega^R_{t,h}$, we have $\mathrm{y}=[1+o(1)]\frac{1}{2\sqrt{2}}t^{1-\frac{h}{2}}$. Then by Lemma 2.3, uniformly for $(s,X_u(s))\in\Omega^R_{t,h}$, we have

$$\begin{aligned}&\Phi_{\sqrt{2}}\left(t-s,X_u(s)-a_ts-\mathrm{y}+\frac{h}{2\sqrt{2}}\log t\right)\\&=[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{t^{h/2}\mathrm{y}}{(t-s)^{3/2}}\exp\left\{\sqrt{2}(X_u(s)-a_ts)-\sqrt{2}\mathrm{y}-\frac{[X_u(s)-a_ts-\mathrm{y}+O(\log t)]^2}{2(t-s)}\right\}\\&=[1+o(1)]\frac{\gamma_{\sqrt{2}}(\varphi)}{t^{1/2}}\exp\left\{(\sqrt{2}+\frac{\mathrm{y}}{t-s})(X_u(s)-a_ts)-\sqrt{2}\mathrm{y}-\frac{\mathrm{y}^2}{2(t-s)}-\frac{(X_u(s)-a_ts)^2}{2(t-s)}\right\}.\end{aligned}$$

We now simplify the term inside exponential. By (4.16), $\delta:=\delta_{t,\xi,h}=\frac{(b_t-a_t)}{t-s}t^{\frac{1+h}{2}}=[1+o(1)]\frac{2\sqrt{2}}{\sqrt{t}}$. Now $\mathrm{y}=(b_t-\sqrt{2})(t-s)+(b_t-a_t)t^{\frac{1+h}{2}}\xi$ can be written as $\sqrt{2}+\frac{\mathrm{y}}{t-s}=b_t+\delta\xi$. Part (ii) of Lemma 4.1 yields that $-\sqrt{2}\mathrm{y}-\frac{\mathrm{y}^2}{2(t-s)}=-(\beta_t-\frac{a_t^2}{2\sigma_t^2})s-(\sqrt{2}+1)\xi^2+o(1)$. Besides, $(\beta_t-\frac{a_t^2}{2\sigma_t^2})s+a_t(b_t+\delta\xi)s=[\beta_t+\frac{\sigma_t^2}{2}(b_t+\delta\xi)^2]s-\frac{\sigma_t^2}{2}\delta^2s\xi^2=[\beta_t+\frac{\sigma_t^2}{2}(b_t+\delta\xi)^2]s-2\xi^2+o(1)$ and $-\frac{(X_u(s)-a_ts)^2}{2(t-s)}=-\frac{(X_u(s)-a_ts)^2}{2s}+o(1)$. Thus,

$$\begin{aligned}&\Phi_{\sqrt{2}}\left(t-s,X_u(s)-a_ts-\mathrm{y}+\frac{h}{2\sqrt{2}}\log t\right)\\&=[1+o(1)]\frac{\gamma_{\sqrt{2}}(\varphi)}{t^{1/2}}\exp\left\{(b_t+\delta\xi)X_u(s)-[\beta_t+\frac{\sigma_t^2}{2}(b_t+\delta\xi)^2]s-\frac{(X_u(s)-a_ts)^2}{2s}-(\sqrt{2}-1)\xi^2\right\}.\end{aligned}$$

As a consequence, we have

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right)=\mathbb{E}\Bigg(\exp\Bigg\{-[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{-R}^R t^{h/2}W_s^G(b_t+\delta\xi)e^{-(\sqrt{2}-1)\xi^2}\,\mathrm{d}\xi\Bigg\}\Bigg),\tag{4.22}$$

where $G(x)=e^{-\frac{x^2}{2}}1_{\{|x|\le R\}}$ and

$$W_s^G(b_t+\delta\xi):=\sum_{u\in N^1_s}e^{(b_t+\delta\xi)X_u(s)-[\beta_t+\frac{\sigma_t^2}{2}(b_t+\delta\xi)^2]s}G\left(\frac{a_ts-X_u(s)}{\sqrt{s}}\right).$$

By the Brownian scaling property, $\left(X_u(s) : u \in N_s^1\right) \overset{law}{=} \left(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s') : u \in \mathsf{N}_{s'}\right)$, where $s' = \beta_t s$. Let $\lambda_t := \frac{\sigma_t}{\sqrt{\beta_t}}(b_t+\delta\xi) = a_t+\sigma_t^2\delta\xi$ (noticing that $\frac{\sigma_t}{\sqrt{\beta_t}} = \sigma_t^2$). So $W_s^G(b_t+\delta\xi)$ has the same distribution as

$$\mathsf{W}_{s'}^G(\lambda_t) := \sum_{u\in\mathsf{N}_{s'}} e^{\lambda_t\mathsf{X}_u(s')-(\frac{\lambda_t^2}{2}+1)s'} G\left(\frac{\lambda_t s' - \mathsf{X}_u(s')}{\frac{1}{\sigma_t}\sqrt{s'}} - \sigma_t^2\delta\sqrt{s}\xi\right).$$

By (4.16), $\sqrt{2}-\lambda_t \sim \sqrt{2}-a_t \sim \frac{1}{\sqrt{2}t^{h/2}}$, and $\lim_{t\to\infty}\sigma_t^2\delta\sqrt{s} = 2$. Applying part (ii) of Lemma 2.4, we have

$$\lim_{t\to\infty} t^{h/2}\mathsf{W}_{s'}^G(\lambda_t) = \sqrt{2}\mathsf{Z}_\infty \int_{\mathbb{R}} G(z-2\xi)e^{-\frac{z^2}{2}}\frac{\mathrm{d}z}{\sqrt{2\pi}}.$$

Letting $t\to\infty$ in (4.22), we get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\Bigg\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\int_{-R}^{R} e^{-(\sqrt{2}-1)\xi^2}\,\mathrm{d}\xi\int_{\mathbb{R}} G_R(z-2\xi)e^{-\frac{z^2}{2}}\frac{\mathrm{d}z}{\sqrt{\pi}}\Bigg\}\Bigg).$$

Then letting $R\to\infty$, by Corollary 2.7, we have

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \mathsf{E}\left(\exp\left\{-\sqrt{2}\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\iint_{\mathbb{R}^2} e^{-\sqrt{2}\xi^2}e^{-(z-\xi)^2}\frac{\mathrm{d}\xi\,\mathrm{d}z}{\sqrt{2\pi}}\right\}\right) = \mathsf{E}\left(e^{-\sqrt{\frac{\pi}{\sqrt{2}}}\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty}\right),$$

which is the Laplace functional of DPPP $\left(\sqrt{\frac{\pi}{\sqrt{2}}}\sqrt{2}C_\star\mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$.

Case 2: $h\in[1,\infty]$**.** Now $\sqrt{2}(t-s) - m_{h,3}^{(1,1)}(t) = -\sqrt{2}s + \frac{1}{2\sqrt{2}}\log t$. Making a change of variable $s = \xi t$, we can rewite $\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)$ as

$$\mathbb{E}\Bigg(\exp\Bigg\{-t\int_{\frac{1}{R}}^{1-\frac{1}{R}}\sum_{u\in N_s^1}\Phi_{\sqrt{2}}\left(t-s, X_u(s)-\sqrt{2}s+\frac{1}{2\sqrt{2}}\log t\right)1_{\{\sqrt{2}s-X_u(s)\in[\frac{1}{R}\sqrt{s},R\sqrt{s}]\}}\,\mathrm{d}\xi\Bigg\}\Bigg).$$

By Lemma 2.3, uniformly for $(s, X_u(s))\in\Omega_{t,h}^R$, we have

$$\begin{aligned}&\Phi_{\sqrt{2}}\left(t-s, X_u(s)-\sqrt{2}s+\frac{1}{2\sqrt{2}}\log t\right)\\ &= [1+o(1)]\frac{\gamma_{\sqrt{2}}(\varphi)t^{1/2}s^{1/2}}{(t-s)^{3/2}}\frac{\sqrt{2}s-X_u(s)}{\sqrt{s}}\exp\left\{\sqrt{2}(X_u(s)-\sqrt{2}s) - \frac{1}{2(t-s)}(X_u(s)-\sqrt{2}s)^2\right\}.\end{aligned}$$

Then we have

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathbb{E}\Bigg(\exp\Bigg\{-[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}}^{1-\frac{1}{R}}\frac{1}{(1-\xi)^{3/2}}\sqrt{\xi t}\,W_{\xi t}^{G_\xi}\,\mathrm{d}\xi\Bigg\}\Bigg). \tag{4.23}$$

where $G_\xi(x) := xe^{-\frac{\xi}{2(1-\xi)}x^2}1_{\{x\in[R^{-1},R]\}}$ and

$$W_s^{G_\xi} := \sum_{u\in N_s^1} e^{\sqrt{2}(X_u(s)-\sqrt{2}s)}G_\xi\left(\frac{\sqrt{2}s-X_u(s)}{\sqrt{s}}\right).$$

Since $\left(X_u(s) : u\in N_s^1\right) \overset{law}{=} \left(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s') : u\in\mathsf{N}_{s'}\right)$(where $s' = \beta_t s$) and $\beta_t\sigma_t^2 = 1$, $W_s^{G_\xi}$ has the same distribution as

$$\mathsf{W}_{s'}^{G_\xi} := \sum_{u\in\mathsf{N}_s} e^{\sqrt{2}[\mathsf{X}_u(s')-\sqrt{2}s']}e^{\sqrt{2}(1-\sigma_t^2)[\sqrt{2}s'-\mathsf{X}_u(s')]}G_\xi\left(\sigma_t\frac{\sqrt{2}s'-\mathsf{X}_u(s')}{\sqrt{s'}}\right).$$

Applying Lemma 2.5, we have for $s' = \beta_t s = \beta_t \xi t$, as $1-\sigma_t^2 \sim \frac{1}{t^{h/2}}$,

$$\lim_{t\to\infty} \sqrt{s'} \mathsf{W}_{s'}^{G_\xi} = \mathsf{Z}_\infty \sqrt{\frac{2}{\pi}} \int_{\frac{1}{R}}^{R} e^{\sqrt{2\xi}\mathbb{1}_{\{h=1\}}z} e^{-\frac{\xi}{2(1-\xi)}z^2} z^2 e^{-\frac{z^2}{2}}\, \mathrm{d}z.$$

Letting $t\to\infty$ in (4.22), and then $R\to\infty$, by Corollary 2.7, we have

$$\lim_{t\to\infty} \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t \varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\Bigg\{ -\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty \int_0^1 \frac{1}{(1-\xi)^{3/2}}\,\mathrm{d}\xi \int_0^\infty e^{\sqrt{2\xi}\mathbb{1}_{\{h=1\}}z} e^{-\frac{\xi}{2(1-\xi)}z^2} \mu_{\mathrm{Bes}}(\mathrm{d}z)\Bigg\}\Bigg).$$

which is the Laplace functional of DPPP $\left(C_{h,3}\sqrt{2}C_\star \mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$, where

$$\begin{aligned} C_{h,3} &= \int_0^1 \frac{1}{(1-\xi)^{3/2}}\,\mathrm{d}\xi \sqrt{\frac{2}{\pi}} \int_0^\infty z^2 e^{\sqrt{2\xi}\mathbb{1}_{\{h=1\}}z} e^{-\frac{1}{2(1-\xi)}z^2}\,\mathrm{d}z \\ &= \int_0^1 \mathrm{d}\xi \sqrt{\frac{2}{\pi}} \int_0^\infty z^2 e^{\sqrt{2\xi(1-\xi)}\mathbb{1}_{\{h=1\}}z} e^{-\frac{1}{2}z^2}\,\mathrm{d}z = 1_{\{h>1\}} + \int_0^1 \mathrm{d}\xi \sqrt{\frac{2}{\pi}} \int_0^\infty z^2 e^{\sqrt{2\xi(1-\xi)}z} e^{-\frac{1}{2}z^2}\,\mathrm{d}z 1_{\{h=1\}}. \end{aligned}$$

We now complete the proof. □

4.3 From $\mathscr{C}_{III}$ to $\mathscr{B}_{I,III}$

Assume that $(\beta,\sigma^2)\in\mathscr{B}_{I,III}$, and $(\beta_t,\sigma_t^2)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$. In this case

$$1-p_t \sim \frac{1}{2\left(\sigma^{-2}-1\right)^2 t^h},\ \theta_t - b_t \sim \frac{v}{2(1-\sigma^2)t^h},\ v_t - a_t \sim \frac{\sigma^2}{2(1-\sigma^2)}\frac{v}{t^h},\ v_t^* - v_t = \Theta(t^{-2h}). \tag{4.24}$$

In fact, $1-p_t = 1-\frac{\sigma_t^2+\beta_t-2}{2(\beta_t-1)(1-\sigma_t^2)} = \frac{\beta_t^{-1}+\sigma_t^{-2}-2}{2(1-\beta_t^{-1})(\sigma_t^{-2}-1)} \sim \frac{1}{2(\sigma^{-2}-1)^2t^h}$; $b_t = \theta_t(\frac{1-\beta_t^{-1}}{\sigma_t^{-2}-1})^{1/2} = \theta_t(1-\frac{1}{(\sigma_t^{-2}-1)t^h})^{1/2} = \theta_t - \frac{v}{2(1-\sigma^2)t^h} + o(\frac{1}{t^h})$; $v_t^* = v_t\frac{\sigma_t^{-2}-\beta_t^{-1}}{2\sqrt{(1-\beta_t^{-1})(\sigma_t^{-2}-1)}} = v_t[1+\frac{\theta_t}{2b_t}(\frac{b_t}{\theta_t}-1)^2]$.

Recall that $h' = \min\{h,1/2\}$, $m^{1,3}_{h,+}(t) := v_t^* t - \frac{h'}{\theta_t}\log t$. Define

$$\delta(x;s,t) := x - a_t s + (b_t-a_t)(p_t t - s), \tag{4.25}$$

and

$$\Omega^R_{t,h} = \begin{cases} \{(s,x) : |s-p_t t| \le R\sqrt{t},\ |\delta(x;s,t)| \le R\sqrt{t-s}\} \text{ for } h\in(0,\frac{1}{2}), \\ \{(s,x) : t-s \in [\frac{1}{R}\sqrt{t}, R\sqrt{t}], |\delta(x;s,t)| \le R\sqrt{t-s}\} \text{ for } h\in[\frac{1}{2},\infty]. \end{cases} \tag{4.26}$$

Lemma 4.4. *For all $A>0$, and $h\in(0,\infty]$*

$$\lim_{R\to\infty}\limsup_{t\to\infty} \mathbb{P}\left(\exists u\in N_t^2 : X_u(t) \ge m^{1,3}_{h,+}(t) - A, (T_u, X_u(T_u)) \notin \Omega_t^{R,h}\right) = 0.$$

Proof. Applying Corollary 2.7 with $m(t) = m^{1,3}_{h,+}(t)$ and $\Omega_t^R = \Omega^R_{t,h}$ defined in (4.26), it suffices to show that for each $A,K>0$, $I(t,R) = I(t,R;A,K)$ defined in (2.6) vanishes as first $t\to\infty$ and then $R\to\infty$. Conditioned on the Brownian motion B_s in (2.6) equals $\frac{a_t}{\sigma_t}s + x$, we have

$$\begin{aligned} I(t,R) &= \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathbf{P}\left(\sigma_t B_r \le v_t r + \sigma_t K, \forall r\le s \mid B_s = \frac{a_t}{\sigma_t}s + x\right) \\ &\quad \times e^{\beta_t s}\mathsf{F}_t\left(t-s, a_t s + \sigma_t x\right) 1_{\{(s,a_ts+\sigma_tx)\notin\Omega^R_{t,h}\}} \exp\left\{-\frac{a_t^2}{2\sigma_t^2}s - \frac{a_t}{\sigma_t}x - \frac{x^2}{2s}\right\}\frac{\mathrm{d}x}{\sqrt{2\pi s}} \\ &\lesssim \int_0^t \mathrm{d}s \int_{-\infty}^{\frac{(v_t-a_t)}{\sigma_t}s+K} \frac{(v_t-a_t)s + K - \sigma_t x}{\sigma_t s^{3/2}} \mathsf{F}_t\left(t-s, a_t s + \sigma_t x\right) 1_{\{(s,a_ts+\sigma_tx)\notin\Omega^R_{t,h}\}} \\ &\quad \times e^{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \frac{a_t}{\sigma_t}x - \frac{x^2}{2s}}\,\mathrm{d}x, \end{aligned}$$

where we used the fact that $\mathbf{P}\left(\sigma_t B_r \le v_t s + \sigma_t K, \forall r \le s | B_s = \frac{a_t}{\sigma_t} s + x\right) \lesssim_K \frac{(v_t - a_t)s + K - \sigma_t x}{\sigma_t s}$, which holds by Lemma 2.1.

- If $v_t^* t - a_t s - \sigma_t x - \mathrm{w} > 1$, where $\mathrm{w} := \frac{h'}{\theta_t} \log t + A$. By the Markov inequality and Gaussian tail inequality, we have

$$
\begin{aligned}
\mathsf{F}_t\left(t-s, a_t s + \sigma_t x\right) &= \mathsf{P}\left(\max_{u \in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) > v_t^* t - a_t s - \sigma_t x - \mathrm{w}\right) \\
&\lesssim_A \frac{\sqrt{t-s}}{v_t^* t - a_t s - \sigma_t x - \mathrm{w}} \exp\left\{(t-s) - \frac{1}{2(t-s)}[v_t^* t - a_t s - \sigma_t x - \mathrm{w}]^2\right\}.
\end{aligned}
\tag{4.27}
$$

- If $v_t^* t - a_t s - \sigma_t x - \mathrm{w} \le 1$, we simply upper bound $\mathsf{F}_t\left(t-s, v_t s - \sigma_t x\right)$ by 1. Note that, as $\sigma_t x + a_t s \le v_t s + \sigma_t K$ and $v_t^* > v_t$, we have $1 \ge v_t^* t - a_t s - \sigma_t x - \mathrm{w} \ge v_t(t-s) - \mathrm{w} - \sigma_t K$. So provided t is large, it must be $s \ge t - (\log t)^2$ and $-\sigma_t x \le -v_t^* t + a_t s + \mathrm{w} + 1$. Thanks to (4.3), we have

$$
\begin{aligned}
&\int_{t-(\log t)^2}^{t} \mathrm{d}s \int_{-\infty}^{O(\log t)} \frac{(v_t - a_t)s + K - \sigma_t x}{\sigma_t s^{3/2}} e^{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \frac{a_t}{\sigma_t}x - \frac{x^2}{2s}} 1_{\{v_t^* t - a_t s - \sigma_t x - \mathrm{w} \le 1\}} \,\mathrm{d}x \\
&\lesssim \int_{t-(\log t)^2}^{t} \mathrm{d}s \int_{-\infty}^{O(\log t)} \frac{O(\log t)}{t^{3/2}} e^{(\beta_t + \frac{\sigma_t^2 b_t^2}{2})s - b_t v_t^* t + b_t \mathrm{w}} \,\mathrm{d}x \lesssim O(\log t)^4 t^{\frac{b_t}{\theta_t} h' - 3/2} = o(1).
\end{aligned}
\tag{4.28}
$$

Let $J = J_{s,x,t} := \frac{|(v_t - a_t)s + K - \sigma_t x|}{s^{3/2}} \frac{\sqrt{t-s}}{|v_t^* t - a_t s - \sigma_t x - \mathrm{w}| + 1}$. Then we have

$$
I(t,R) \lesssim \int_0^t \mathrm{d}s \int_{-\infty}^{\frac{(v_t - a_t)}{\sigma_t}s + K} J e^{(\beta_t - \frac{a_t^2}{2\sigma_t^2})s + (t-s) - \frac{[v_t^* t - a_t s - \sigma_t x - \mathrm{w}]^2}{2(t-s)} - \frac{a_t}{\sigma_t}x} e^{-\frac{x^2}{2s}} 1_{\left\{(s, a_t s + \sigma_t x) \notin \Omega_{t,h}^R\right\}} \,\mathrm{d}x + o(1).
$$

Making a change of variable $s = p_t t + u$, we have

$$
\frac{(v_t^* t - a_t s - \sigma_t x - \mathrm{w})^2}{2(t-s)} = \frac{b_t^2}{2}(t-s) + b_t(b_t - a_t)u - \sigma_t b_t x - b_t \mathrm{w} + \frac{[(b_t - a_t)u - \sigma_t x - \mathrm{w}]^2}{2(t-s)},
$$

where we used $v_t^* t - a_t s = b_t(1-p_t)t - a_t u = b_t(t-s) + (b_t - a_t)u$. Applying (4.2) and the identity $a_t = \sigma_t^2 b_t$, we have

$$
\begin{aligned}
&(\beta_t - \frac{a_t^2}{2\sigma_t^2})s + (t-s) - \frac{[v_t^* t - a_t s - \sigma_t x - \mathrm{w}]^2}{2(t-s)} - \frac{a_t}{\sigma_t}x \\
&= (\beta_t - \frac{a_t^2}{2\sigma_t^2})s + (1 - \frac{b_t^2}{2})(t-s) - b_t^2(1-\sigma_t^2)u + b_t \mathrm{w} - \frac{[(b_t - a_t)u - \sigma_t x - \mathrm{w}]^2}{2(t-s)} \\
&= (\beta_t - \frac{a_t^2}{2\sigma_t^2})p_t t + (1 - \frac{b_t^2}{2})(1-p_t)t + \left(\beta_t - 1 + \frac{\sigma_t^2 - 1}{2} b_t^2\right) u + b_t \mathrm{w} - \frac{[(b_t - a_t)u - \sigma_t x - \mathrm{w}]^2}{2(t-s)} \\
&= b_t \mathrm{w} - \frac{[(b_t - a_t)u - \sigma_t x - \mathrm{w}]^2}{2(t-s)}.
\end{aligned}
$$

Therefore,

$$
I(t,R) \lesssim \int_{-p_t t}^{(1-p_t)t} \mathrm{d}u \int_{\mathbb{R}} e^{b_t \mathrm{w}} J e^{-\frac{[(b_t - a_t)u - \sigma_t x - \mathrm{w}]^2}{2(t-s)}} e^{-\frac{x^2}{2s}} 1_{\left\{(s, a_t s + \sigma_t x) \notin \Omega_{t,h}^R\right\}} \,\mathrm{d}x + o(1).
$$

By making a change of variable $x=\frac{(b_t-a_t)}{\sigma_t}u+z$, we get $(s,a_ts+\sigma_tx)\in\Omega^R_{t,h}$ if and only if $([1-p_t]\sqrt{t}1_{\{h\geq 1/2\}}-\frac{u}{\sqrt{t}},\frac{\sigma_t z}{\sqrt{t-s}})\in\Gamma^R_h\times[-R,R]$, where $\Gamma^R_h=[-R,R]$ if $h\in(0,1/2)$ and $\Gamma^R_h=[R^{-1},R]$ if $h\in[1/2,\infty]$. Hence,

$$I(t,R)\lesssim\int_{-p_tt}^{(1-p_t)t}\mathrm{d}u\int_{\mathbb{R}}Je^{b_t\mathrm{w}}e^{-\frac{(\sigma_tz+\mathrm{w})^2}{2(t-s)}-\frac{1}{2s}\left[\frac{(b_t-a_t)}{\sigma_t}u+z\right]^2}1_{\left\{\substack{(1-p_t)\sqrt{t}1_{\{h\geq 1/2\}}-\frac{u}{\sqrt{t}}\notin\Gamma^R_h\\ \text{or }|\frac{\sigma_tz}{\sqrt{t-s}}|>R}\right\}}\,\mathrm{d}z+o(1).$$

Making change of variables $u=-\xi\sqrt{t}$ and $z=\eta\sqrt{t-s}$ again, we have

$$\begin{aligned}I(t,R)\lesssim&\int_{-(1-p_t)\sqrt{t}}^{p_t\sqrt{t}}\mathrm{d}\xi\int_{\mathbb{R}}J\sqrt{t(t-s)}e^{b_t\mathrm{w}}e^{-\frac{t}{2s}\left[\frac{(b_t-a_t)}{\sigma_t}\xi-\eta\sqrt{\frac{t-s}{t}}\right]^2-\frac12(\sigma_t\eta+\frac{\mathrm{w}}{\sqrt{t-s}})^2}\\&\times 1_{\left\{\substack{(1-p_t)\sqrt{t}1_{\{h\geq 1/2\}}+\xi\notin\Gamma^R_h\\ \text{or }|\sigma_t\eta|>R}\right\}}\,\mathrm{d}\eta\,\mathrm{d}\xi+o(1).\end{aligned}$$

As $v^*_tt-a_ts-\sigma_tx=b_t(t-s)-\sigma_t\eta\sqrt{t-s}$ and $s=p_tt-\xi\sqrt{t}$, for each fixed ξ and η, by (4.24), we have

$$\begin{aligned}&\lim_{t\to\infty}J\sqrt{t(t-s)}e^{b_t\mathrm{w}}=\lim_{t\to\infty}\frac{t-s}{s^{3/2}}\,\frac{|(v_t-a_t)s+(b_t-a_t)\xi\sqrt{t}-\sigma_t\eta\sqrt{t-s}|+K}{|b_t(t-s)-\sigma_t\eta\sqrt{t-s}|+1}\,t^{\frac{b_t}{\theta}h'+1/2}\\&\leq\lim_{t\to\infty}\frac{1}{b_t}\,\frac{t^{h'}(v_t-a_t)t+t^{h'}(b_t-a_t)|\xi|\sqrt{t}}{t}=\frac{\sigma^4}{2(1-\sigma^2)}1_{\{h\leq\frac12\}}+(1-\frac{v}{\theta})|\xi|1_{\{h\geq 1/2\}}=:C(\xi;h).\end{aligned}$$

Put $\xi_0(h):=1_{\{h\geq 1/2\}}\lim_{t\to\infty}(1-p_t)\sqrt{t}=\frac{1}{2(\sigma^{-2}-1)^2}1_{\{h=1/2\}}$. Applying the dominated convergence theorem, we get

$$\limsup_{t\to\infty}I(t,R)\lesssim\int_{-\lim_t(1-p_t)\sqrt{t}}^{\infty}\mathrm{d}\xi\int C(\xi;h)e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2-\frac{\sigma^2\eta^2}{2}}1_{\left\{\substack{\xi\notin\Gamma^R_h-\xi_0(h),\text{or}\\ |\sigma\eta|>R}\right\}}\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Letting $R\to\infty$, we get $\lim\limits_{R\to\infty}\lim\limits_{t\to\infty}I(t,R)=0$ as desired. □

Proof of Theorem 1.1 for $(\beta_t,\sigma^2_t)_{t>0}\in\mathscr{A}^{h,+}_{(\beta,\sigma^2)}$. Take $\varphi\in\mathcal{T}$. Applying Corollary 2.7 with $m(t)=m^{1,3}_{h,+}(t)$, $\rho=b_t$ and $\Omega^R_t=\Omega^R_{t,h}$ defined in (4.26), it suffices to study the asymptotic of $\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}^R_t,\varphi\rangle}\right)$, which is equal to

$$\mathbb{E}\Bigg(\exp\Bigg\{-\int_0^t\sum_{u\in N^1_s}\Phi_{b_t}\big(t-s,\delta(X_u(s);s,t)+\frac{h'}{\theta_t}\log t\big)1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

where we used the fact that $X_u(s)+b_t(t-s)-m^{1,3}_{h,+}(t)=\delta(X_u(s);s,t)+\frac{h'}{\theta_t}\log t$, which holds by (4.25). By (4.24), $b_t\to\theta>\sqrt2$. Applying Lemma 2.3, we have, uniformly for $(s,X_u(s))\in\Omega^R_{t,h}$,

$$\begin{aligned}&\Phi_{b_t}\big(t-s,\delta(X_u(s);s,t)+\frac{h'}{\theta_t}\log t\big)\\&=(1+o(1))\frac{\gamma_\theta(\varphi)}{\sqrt{t-s}}e^{-(\frac{b_t^2}{2}-1)(t-s)}e^{b_t\delta(X_u(s);s,t)+b_t\frac{h'}{\theta_t}\log t}e^{-\frac{1}{2(t-s)}\delta(X_u(s);s,t)^2}\\&=(1+o(1))\gamma_\theta(\varphi)\frac{t^{h'}}{\sqrt{t-s}}e^{b_tX_u(s)-(\frac{\sigma_t^2b_t^2}{2}+\beta_t)s}e^{-\frac{1}{2(t-s)}\delta(X_u(s);s,t)^2},\end{aligned}$$

where we used the computation that $b_t\delta(X_u(s);s,t)-(\frac{b_t^2}{2}-1)(t-s)=b_tX_u(s)+[b_t(b_t-a_t)p_t-\frac{b_t^2}{2}+1]t-(\frac{b_t^2}{2}+1)s=b_tX_u(s)-(\frac{b_t^2}{2}+1)s=b_tX_u(s)-\left(\frac{\sigma_t^2b_t^2}{2}+\beta_t\right)s$. Thus $\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)$ equals

$$\mathbb{E}\Bigg(\exp\Bigg\{-(1+o(1))\gamma_\theta(\varphi)\int_0^t\sum_{u\in N_s^1}\frac{t^{h'}}{\sqrt{t-s}}e^{b_tX_u(s)-(\frac{\sigma_t^2b_t^2}{2}+\beta_t)s}e^{-\frac{\delta(X_u(s);s,t)^2}{2(t-s)}}1_{\{(s,X_u(s))\in\Omega_{t,h}^R\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

Case 1: $h\in(0,\frac{1}{2})$. Making a change of variable $s=p_tt-\xi\sqrt{t}$ and letting $r=r_{\xi,t}=\xi\sqrt{t}$, we have

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)=\mathbb{E}\Bigg(\exp\Big\{-(1+o(1))\gamma_\theta(\varphi)\int_{-R}^R t^h\frac{\sqrt{t}}{\sqrt{t-s}}W^G(p_tt-\xi\sqrt{t},\xi\sqrt{t};t)\,\mathrm{d}\xi\Big\}\Bigg), \tag{4.29}$$

where $G(x)=G_R(x):=e^{-\frac{x^2}{2}}1_{\{|x|\le R\}}$ and

$$W^G(s,r;t):=\sum_{u\in N_s^1}e^{b_tX_u(s)-(\frac{\sigma_t^2b_t^2}{2}+\beta_t)s}G\left(\frac{X_u(s)-a_ts+(b_t-a_t)r}{\sqrt{t-s}}\right).$$

By the Brownian scaling property, $(X_u(s):u\in N_s^1)\overset{law}{=}(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s'):u\in\mathsf{N}_{s'})$, where $s'=\beta_ts$. Then, letting $\lambda_t=b_t\frac{\sigma_t}{\sqrt{\beta_t}}$, $W^G(s,r;t)$ has the same distribution as

$$\mathsf{W}^G(s',r;t):=\sum_{u\in\mathsf{N}_{s'}}e^{\lambda_t\mathsf{X}_u(s')-(\frac{\lambda_t^2}{2}+1)s'}G\left(\frac{\frac{\lambda_ts'-\mathsf{X}_u(s')}{\sqrt{s'}}-\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\sqrt{s'}}}{\frac{1}{\sigma_t}\sqrt{\frac{\beta_tt-s'}{s'}}}\right).$$

Note that $\sqrt{2}-\lambda_t\sim\frac{1}{\sqrt{2}(\sigma^{-2}-1)t^h}$, $\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\sqrt{s'}}\sim\frac{\theta-v}{\sigma}\xi$ and $\frac{1}{\sigma_t}\sqrt{\frac{\beta_tt-s'}{s'}}\sim\frac{1}{\sigma}\sqrt{\frac{t-s}{s}}$. Applying part (ii) of Lemma 2.4, we have

$$\begin{aligned}\mathsf{W}^G(p_tt-\xi\sqrt{t},\xi\sqrt{t};t)&=(1+o(1))2\mathsf{Z}_\infty(\sqrt{2}-\lambda_t)\int_{\mathbb{R}}G\left(\frac{z-\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\sqrt{s'}}}{\frac{1}{\sigma_t}\sqrt{\frac{\beta_tt-s'}{s'}}}\right)e^{-\frac{z^2}{2}}\frac{\mathrm{d}z}{\sqrt{2\pi}}\\&=(1+o(1))2\mathsf{Z}_\infty(\sqrt{2}-\lambda_t)\left(\int_{\mathbb{R}}G(y)\frac{\mathrm{d}y}{\sqrt{2\pi}}\right)e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\frac{1}{\sigma}\sqrt{\frac{t-s}{s}}.\end{aligned}$$

As a consequence, using $\mathsf{Z}_\infty\overset{d}{=}\frac{\sqrt{\beta}}{\sigma}\mathsf{Z}_\infty^{\beta,\sigma^2}$, we have

$$\lim_{t\to\infty}t^h\frac{\sqrt{t}}{\sqrt{t-s}}W^G(p_tt-\xi\sqrt{t},\xi\sqrt{t};t)=\mathsf{Z}_\infty^{\beta,\sigma^2}\frac{\sqrt{2\beta}}{1-\sigma^2}\left(\int_{\mathbb{R}}G(y)\frac{\mathrm{d}y}{\sqrt{2\pi}}\right)e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\quad\text{in law.}$$

Letting $t\to\infty$ in (4.29), by the dominated convergence theorem we get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)=\mathsf{E}\Bigg(\exp\Bigg\{-\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\frac{\sqrt{2\beta}}{1-\sigma^2}\int_{-R}^Re^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\,\mathrm{d}\xi\int_{-R}^Re^{-\frac{y^2}{2}}\frac{\mathrm{d}y}{\sqrt{2\pi}}\Bigg\}\Bigg).$$

Applying Corollary 2.7, we finally get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right)=\lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)=\mathsf{E}\Big(\exp\big\{-C_{h,+}\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\big\}\Big),$$

which is the Laplace functional of DPPP $\left(C_{h,+}\frac{C(\theta)}{\theta\sqrt{2\pi}}\mathsf{Z}_\infty^{\beta,\sigma^2}\theta e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}^\theta\right)$, and where

$$C_{h,+} := \lim_{R\to\infty}\frac{\sqrt{2\beta}}{1-\sigma^2}\int_{-R}^{R} e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{2\pi}}{1-\sigma^2}\frac{v}{\theta-v}.$$

Case 2: $h\in[\frac{1}{2},\infty)$**.** Make a change of variable $s=t-\xi\sqrt{t}$ and let $r=r_{\xi,t}=\xi\sqrt{t}$. As before we have

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathbb{E}\Bigg(\exp\Big\{-(1+o(1))\gamma_\theta(\varphi)\int_{R^{-1}}^{R}\frac{t}{\sqrt{t-s}}W^G(t-\xi\sqrt{t},\xi\sqrt{t};t)\,\mathrm{d}\xi\Big\}\Bigg). \tag{4.30}$$

Letting $\lambda_t = b_t\frac{\sigma_t}{\sqrt{\beta_t}} = \sqrt{2}-\frac{1}{\alpha_t}$, we have $\alpha_t\sim\sqrt{2}(\sigma^{-2}-1)t^h$ by (4.24). $W^G(s,r;t)$ has the same distribution as

$$\begin{aligned}
\mathsf{W}^G(s',r;t) &:= \sum_{u\in\mathsf{N}_{s'}} e^{\lambda_t\mathsf{X}_u(s')-(\frac{\lambda_t^2}{2}+1)s'} G\left(\frac{\lambda_t s'-\mathsf{X}_u(s')-\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)r}{\frac{1}{\sigma_t}\sqrt{\beta_t t-s'}}\right)\\
&= e^{-\frac{(s')^2}{2\alpha_t^2}}\sum_{u\in\mathsf{N}_{s'}} e^{\sqrt{2}\mathsf{X}_u(s')-2s'} e^{\frac{\sqrt{2}s'-\mathsf{X}_u(s')}{\alpha_t}} G\left(\frac{\sqrt{2}s'-\mathsf{X}_u(s')-\frac{s'}{\alpha_t}-\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)r}{\frac{1}{\sigma_t}\sqrt{\beta_t t-s'}}\right)\\
&= [1+o(1)]e^{\frac{(s')^2}{2\alpha_t^2}+\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\alpha_t}}\sum_{u\in\mathsf{N}_{s'}} e^{\sqrt{2}\mathsf{X}_u(s')-2s'} G\left(\frac{\sqrt{2}s'-\mathsf{X}_u(s')-\frac{s'}{\alpha_t}-\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)r}{\frac{1}{\sigma_t}\sqrt{\beta_t t-s'}}\right).
\end{aligned}$$

Note that $r_{s'} = \frac{\sqrt{s'}}{\alpha_t}+\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\sqrt{s'}} \to \frac{\sqrt{\beta}}{\sqrt{2}(\sigma^{-2}-1)}1_{\{h=1/2\}}+\frac{\theta-v}{\sigma}\xi$ and $h_{s'}=\frac{1}{\sigma_t}\sqrt{\frac{\beta_t t-s'}{s'}}\sim\frac{1}{\sigma}\sqrt{\frac{t-s}{t}}$. Applying Lemma 2.5 and using the fact $\mathsf{Z}^\infty \overset{d}{=} \frac{\sqrt{\beta}}{\sigma}\mathsf{Z}_\infty^{\beta,\sigma^2}$, we have

$$\begin{aligned}
\mathsf{W}^G(s',r;t) &= \frac{[1+o(1)]}{\sqrt{s'}}e^{\frac{(s')^2}{2\alpha_t^2}+\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\alpha_t}}\int_0^\infty G\left(\frac{z-r_{s'}}{h_{s'}}\right)ze^{-\frac{z^2}{2}}\,\mathrm{d}z\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty\\
&= \frac{[1+o(1)]}{\sqrt{\beta t}}e^{\frac{(s')^2}{2\alpha_t^2}+\frac{\sqrt{\beta_t}}{\sigma_t}(b_t-a_t)\frac{r}{\alpha_t}}\int_{\mathbb{R}} G(y)\,\mathrm{d}y\,(r_{s'}e^{-\frac{r_{s'}^2}{2}})h_{s'}\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty\\
&= \frac{[1+o(1)]}{\sqrt{\beta t}}h_{s'}r_{s'}e^{-\frac{\beta_t}{2\sigma_t^2}(b_t-a_t)^2\frac{r^2}{s'}}\int_{\mathbb{R}} G(y)\,\mathrm{d}y\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty\\
&= [1+o(1)]\frac{\sqrt{t-s}}{t}\left[\frac{\sqrt{2\beta}}{(1-\sigma^2)}1_{\{h=1/2\}}+\frac{2(\theta-v)}{\sigma^3}\xi\right]e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\int_{\mathbb{R}} G(y)\frac{\mathrm{d}y}{\sqrt{2\pi}}\mathsf{Z}_\infty^{\beta,\sigma^2}.
\end{aligned}$$

Letting $t\to\infty$ in (4.29), we get

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)\\
=&\mathsf{E}\left(\exp\left\{-\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\int_{\frac{1}{R}}^{R}\left[\frac{\sqrt{2\beta}}{(1-\sigma^2)}1_{\{h=1/2\}}+\frac{2(\theta-v)}{\sigma^3}\xi\right]e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\,\mathrm{d}\xi\int_{-R}^{R}e^{-\frac{y^2}{2}}\frac{\mathrm{d}y}{\sqrt{2\pi}}\right\}\right).
\end{aligned}$$

Applying Corollary 2.7, we finally get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\Big\{-C_{h,+}\gamma_\theta(\varphi)\mathsf{Z}_\infty^{\beta,\sigma^2}\Big\}\Bigg),$$

which is the Laplace functional of DPPP $\left(C_{h,+}\frac{C(\theta)}{\theta\sqrt{2\pi}}\mathsf{Z}_\infty^{\beta,\sigma^2}\theta e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}^\theta\right)$, and where

$$\begin{aligned}
C_{h,+} :=& \lim_{R\to\infty}\int_{\frac{1}{R}}^{R}\left[\frac{\sqrt{2\beta}}{(1-\sigma^2)}1_{\{h=1/2\}}+\frac{2(\theta-v)}{\sigma^3}\xi\right]e^{-\frac{(\theta-v)^2}{2\sigma^2}\xi^2}\,\mathrm{d}\xi\\
=&\sqrt{\frac{\pi}{2}}\frac{\sigma^2}{(1-\sigma^2)^2}1_{\{h=1/2\}}+\frac{2}{\sigma(\theta-v)}.
\end{aligned}$$

We now complete the proof. $\square$

5 Approximation From $\mathscr{C}_{II}$

5.1 From $\mathscr{C}_{II}$ to $\mathscr{B}_{II,III}$

Assume that $(\beta, \sigma^2) \in \mathscr{B}_{II,III}$ and $(\beta_t, \sigma_t^2) \in \mathscr{A}^{h,-}_{(\beta,\sigma^2)}$. Recall that $m^{2,3}_{h,-}(t) = \sqrt{2}t - \frac{3-4h'}{2\sqrt{2}}\log t$, $h' = \min\{h, 1/2\}$. Define

$$\Omega^R_{t,h} := \left\{(s,x) : s \in [\frac{1}{R}t^{h'}, Rt^{h'}],\ |x - \sqrt{2}\sigma_t^2 s| \le R\sqrt{s}\right\}. \tag{5.1}$$

Lemma 5.1. *For any $A > 0$,*

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(\exists u \in N_t^2 : X_u(t) \ge m^{2,3}_{h,-}(t) - A, (T_u, X_u(T_u)) \notin \Omega^R_{t,h}\right) = 0.$$

Proof. Applying part (i) of Corollary 2.7 with $m(t) = m^{2,3}_{h,-}(t)$ and $\Omega^R_t = \Omega^R_{t,h}$ defined in (5.1), it suffices to show that for each $A, K > 0$, $I(t,R) = I(t,R;A,K)$ defined in (2.6) vanishes as first $t \to \infty$ and then $R \to \infty$. Conditioned on the Brownian motion B_s in (2.6) equals $\sqrt{2}\sigma_t s + x$, we have

$$\begin{aligned}
I(t,R) = &\int_0^t \mathrm{d}s \int_{-\infty}^{\sqrt{2}(\sqrt{\beta_t}-\sigma_t)s+K} \mathsf{F}_t\left(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x\right) 1_{\{(s,\sqrt{2}\sigma_t^2 s+\sigma_t x)\notin \Omega^R_{t,h}\}} \\
&\qquad \mathbf{P}(B_r \le \sqrt{2\beta_t}r + K, \forall r \le s \mid B_s = \sqrt{2}\sigma_t s + x)\frac{1}{\sqrt{2\pi s}} e^{\beta_t s} e^{-\frac{(\sqrt{2}\sigma_t s+x)^2}{2s}}\,\mathrm{d}x \\
\lesssim_K &\int_0^t \mathrm{d}s \int_{-\infty}^{\sqrt{2}(\sqrt{\beta_t}-\sigma_t)s+K} \mathsf{F}_t\left(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x\right) 1_{\{(s,\sqrt{2}\sigma_t^2 s+\sigma_t x)\notin \Omega^R_{t,h}\}} \\
&\qquad \left[\sqrt{2}(\sqrt{\beta_t} - \sigma_t) + \frac{K-x}{s}\right] \frac{1}{\sqrt{s}} e^{(\beta_t - \sigma_t^2)s} e^{-\sqrt{2}\sigma_t x - \frac{x^2}{2s}}\,\mathrm{d}x,
\end{aligned} \tag{5.2}$$

where in the inequality we used $\mathbf{P}(B_r \le \sqrt{2\beta_t}r + K, \forall r \le s \mid B_s = \sqrt{2}\sigma_t s + x) \lesssim_K \sqrt{2}(\sqrt{\beta_t} - \sqrt{\sigma_t^2}) + \frac{K-x}{s}$, which holds by Lemma 2.1. By the definition of $\mathsf{F}_t(r,x)$ in Corollary 2.7 with $m(t) = m^{2,3}_{h,-}(t)$, we have

$$\begin{aligned}
\mathsf{F}_t(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x) &= \mathsf{P}\left(\max_{u\in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) \ge \sqrt{2}t - \frac{3-4h'}{2\sqrt{2}}\log t - A - \sqrt{2}\sigma_t^2 s - \sigma_t x\right) \\
&= \mathsf{P}\left(\max_{u\in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) \ge \sqrt{2}(t-s) - \frac{3}{2\sqrt{2}}\log(t-s+1) + L_{s,t} - \sigma_t x\right),
\end{aligned}$$

where $L_{s,t} := \sqrt{2}(1-\sigma_t^2)s - \frac{3}{2\sqrt{2}}\log(\frac{t}{t-s+1}) + \sqrt{2}h'\log t - A$. Applying Lemma 2.2, provided that $L_{s,t} - \sigma_t x > 1$ we have

$$\mathsf{F}_t(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x) \lesssim_A (L_{s,t} - \sigma_t x) e^{-2(1-\sigma_t^2)s} (\frac{t}{t-s+1})^{3/2} \frac{1}{t^{2h'}} e^{\sqrt{2}\sigma_t x} e^{-\frac{(L_{s,t}-\sigma_t x)^2}{3(t-s)}}. \tag{5.3}$$

In fact, for large t, $L_{s,t} - \sigma_t x > 1$ holds for all $s \in [0,t]$, $x \le \sqrt{2}(\sqrt{\beta_t} - \sigma_t)s + K$. To see this, note that $L_{s,t} - \sigma_t x \ge (\sqrt{2} - v_t)s - \frac{3}{2\sqrt{2}}\log(\frac{t}{t-s+1}) + \sqrt{2}h'\log t - A - \sigma_t K$. By our assumption $(\beta_t, \sigma_t^2) \to (\beta, \sigma^2) \in \mathscr{B}_{II,III}$, for large t we have $\sqrt{2} - v_t > \delta > 0$. Then for each $\delta s > 2\log t$, $L_{s,t} - \sigma_t x > \sqrt{2}h'\log t - A - \sigma_t K > 1$; for each $\delta s \le 2\log t$, $t - s + 1 \ge t/2$ hence $L_{s,t} - \sigma_t x \ge \sqrt{2}h'\log t - 4\log(2) - A - \sigma_t K > 1$.

Substituting the inequality (5.3) into (5.2), and by the hypothesis $\beta_t+\sigma_t^2=2-1/t^h$, we get

$$I(t,R)\lesssim\int_0^t\int_{\mathbb{R}}1_{\{(s,\sqrt{2}\sigma_t^2s+\sigma_tx)\notin\Omega_{t,h}^R\}}\frac{1}{\sqrt{s}}\frac{|L_{s,t}-\sigma_tx|}{t^{2h'}}(\frac{t}{t-s+1})^{3/2}e^{-\frac{s}{t^h}}e^{-\frac{x^2}{2s}}e^{-\frac{(L_{s,t}-\sigma_tx)^2}{3(t-s)}}\,\mathrm{d}s\,\mathrm{d}x.$$

We now make change of variables $s=\xi t^{h'}$ and $x=\eta\sqrt{s}$ in the integral above. Then $(s,\sqrt{2}\sigma_t^2s+\sigma_tx)$ belongs to $\Omega_{t,h}^R$ if and only if $\xi\in[R^{-1},R]$ and $|\sigma_t\eta|\le R$. Applying the dominated convergence theorem twice we get

$$I(t,R)\lesssim\int_0^{t^{1-h'}}\int_{\mathbb{R}}1_{\left\{\substack{\xi\notin[R^{-1},R]\\ \text{or }|\sigma_t\eta|>R}\right\}}\frac{|L(\xi t^{h'},t)-\sigma_tx|}{t^{h'}}(\frac{t}{t-\xi t^{h'}+1})^{3/2}e^{-\xi t^{h'-h}-\frac{\eta^2}{2}}e^{-\frac{\left(L(\xi t^{h'},t)-\sigma_t\eta\sqrt{\xi t^{h'}}\right)^2}{3(t-\xi t^{h'})}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$\overset{t\to\infty}{\longrightarrow}\int_0^\infty\int_{\mathbb{R}}1_{\left\{\substack{\xi\notin[R^{-1},R]\\ \text{or }|\sigma^2\eta|>R}\right\}}\sqrt{2}(1-\sigma^2)\xi e^{-\xi 1_{\{h\le1/2\}}}e^{-\frac{\eta^2}{2}}e^{-(1-\sigma^2)^2\xi^21_{\{h\ge1/2\}}}\,\mathrm{d}\xi\,\mathrm{d}\eta\overset{R\to\infty}{\longrightarrow}0.$$

This completes the proof. ☐

Next we will prove Theorem 1.3 for $(\beta_t,\sigma_t^2)\in\mathscr{A}_{(\beta,\sigma^2)}^{h,-}$.

Proof of Theorem 1.3 for $(\beta_t,\sigma_t^2)\in\mathscr{A}_{(\beta,\sigma^2)}^{h,-}$. Take $\varphi\in\mathcal{T}$. Applying part (ii) of Corollary 2.7 with $m(t)=m_{h,-}^{2,3}(t)$, $\rho=\sqrt{2}$, and $\Omega_t^R=\Omega_{t,h}^R$ defined in (5.1), it suffices to study the asymptotic of

$$\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)=\mathbb{E}\Bigg(\exp\Big\{-\int_{\frac{1}{R}t^{h'}}^{Rt^{h'}}\sum_{u\in N_s^1}\Phi_{\sqrt{2}}\big(t-s,X_u(s)-\sqrt{2}s+\frac{3-4h'}{2\sqrt{2}}\log t\big)1_{\{(s,X_u(s))\in\Omega_{t,h}^R\}}\,\mathrm{d}s\Big\}\Bigg).$$

Observe that, uniformly for $(s,X_u(s))\in\Omega_{t,h}^R$, we have $\sqrt{2}s-X_u(s)=\sqrt{2}(1-\sigma_t^2)s+\Theta(\sqrt{s})$. Lemma 2.3 yields that, uniformly for $(s,X_u(s))\in\Omega_{t,h}^R$, as $t\to\infty$,

$$\begin{aligned}&\Phi_{\sqrt{2}}\left(t-s,X_u(s)-\sqrt{2}s+\frac{3-4h'}{2\sqrt{2}}\log t\right)\\&=[(1+o(1))]\gamma_{\sqrt{2}}(\varphi)\frac{(\sqrt{2}s-X_u(s))}{t^{3/2}}e^{\sqrt{2}X_u(s)-2s}t^{\frac{3-4h'}{2}}e^{-\frac{(X_u(s)-\sqrt{2}s)^2}{2t}}\\&=[(1+o(1))]\gamma_{\sqrt{2}}(\varphi)\frac{\sqrt{2}(1-\sigma_t^2)s}{t^{2h'}}e^{\sqrt{2}X_u(s)-(\beta_t+\sigma_t^2)s-\frac{s}{t^h}}e^{-\frac{(1-\sigma_t^2)^2s^2}{t}},\end{aligned}\tag{5.4}$$

where in last equality we replaced 2 by $\beta_t+\sigma_t^2+1/t^h$. Let

$$W(s;t):=\sum_{u\in N_s^1}e^{\sqrt{2}X_u(s)-(\beta_t+\sigma_t^2)s}1_{\{|X_u(s)-\sqrt{2}\sigma_t^2s|\le R\sqrt{s}\}}.$$

By the asymptotic equality (5.4), we have

$$\begin{aligned}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right)&=\exp\left\{[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}t^{h'}}^{Rt^{h'}}\frac{\sqrt{2}(1-\sigma_t^2)s}{t^{2h'}}e^{-\frac{s}{t^h}-\frac{(1-\sigma_t^2)^2s^2}{t}}W(s,t)\,\mathrm{d}s\right\}\\&=\exp\left\{[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\sqrt{2}(1-\sigma_t^2)\int_{\frac{1}{R}}^{R}\xi e^{-\xi t^{h'-h}-(1-\sigma_t^2)^2\xi^2t^{2h'-1}}W(\xi t^{h'},t)\,\mathrm{d}\xi\right\},\end{aligned}$$

where in the last equality we made a change variable $s = \xi t^{h'}$. By the Brownian scaling property, $(X_u(s) : u \in N_s^1)$ has the same distribution as $(\frac{\sigma_t}{\sqrt{\beta_t}}\mathsf{X}_u(s') : u \in \mathsf{N}_{s'})$, where $s' = \beta_t s$. Put $\lambda_t = \sqrt{2}\sigma_t/\sqrt{\beta_t}$. We have

$$W(s;t) \overset{law}{=} \sum_{u\in\mathsf{N}_{s'}} e^{\lambda_t X_u(s')-(1+\lambda_t^2/2)s'} 1_{\{|\mathsf{X}_u(s')-\lambda_t s'|\le R\sqrt{s'}/\sigma_t\}}.$$

Since $\lambda_t \to \sqrt{2}\sigma/\sqrt{\beta} < \sqrt{2}$, by part(i) of Lemma 2.4, we have

$$\lim_{t\to\infty} W(\xi t^{h'}, t) = \mathsf{W}_\infty(\frac{\sqrt{2}\sigma}{\sqrt{\beta}}) \int_{[-\frac{R}{\sigma},\frac{R}{\sigma}]} e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} = \mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2}) \int_{[-\frac{R}{\sigma},\frac{R}{\sigma}]} e^{-\frac{x^2}{2}} \frac{\mathrm{d}x}{\sqrt{2\pi}} \quad \text{in law.}$$

Applying the dominated convergence theorem, we get

$$\lim_{t\to\infty} \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Big(\exp\big\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2})C_{R,h}\big\}\Big),$$

where

$$C_{R,h} := \sqrt{2}\int_{\frac{1}{R}}^{R} (1-\sigma^2)\xi e^{-\xi 1_{\{h\le 1/2\}}-(1-\sigma^2)^2\xi^2 1_{\{h\ge 1/2\}}}\,\mathrm{d}\xi \int_{[-\frac{R}{\sigma},\frac{R}{\sigma}]} e^{-\frac{x^2}{2}}\frac{\mathrm{d}x}{\sqrt{2\pi}}$$
$$\overset{R\to\infty}{\longrightarrow} C_{h,-} := \sqrt{2}(1-\sigma^2)1_{\{h<1/2\}} + \frac{1}{\sqrt{2}(1-\sigma^2)}1_{\{h>1/2\}} + \sqrt{2}\int_0^\infty (1-\sigma^2)\xi e^{-\xi-(1-\sigma^2)^2\xi^2}\,\mathrm{d}\xi 1_{\{h=1/2\}}.$$

Letting $R\to\infty$ and applying Corollary 2.7, we get

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\right) = \mathsf{E}\Big(\exp\big\{-C_{h,-}\mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2})\gamma_{\sqrt{2}}(\varphi)\big\}\Big),$$

which is the Laplace functional of DPPP $\left(C_{h,-}\sqrt{2}C_\star\mathsf{W}_\infty^{\beta,\sigma^2}(\sqrt{2})e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$. Using [9, Lemma 4.4], we complete the proof. $\square$

5.2 From $\mathcal{C}_{II}$ to $(1,1)$

Assume that now $(\beta_t,\sigma_t^2)_{t>0} \in \mathscr{A}_{(1,1)}^{h,2}$, i.e., $\beta_t = \sigma_t^2 = 1-\frac{1}{2t^h}$. Let $m_{h,2}^{(1,1)}(t) = \sqrt{2}t - \frac{3-2h'}{2\sqrt{2}}\log t$, where $h' = \min\{h,1\}$. Define

$$\Omega_{t,h}^R = \begin{cases} \{(s,x) : s\in[\frac{1}{R}t^h, Rt^h],\ \sqrt{2}\sigma_t^2 s - x \in [\frac{1}{R}\sqrt{s}, R\sqrt{s}]\} & \text{for } h\in(0,1); \\ \{(s,x) : s\in[\frac{1}{R}t, (1-\frac{1}{R})t],\ \sqrt{2}\sigma_t^2 s - x\in[\frac{1}{R}\sqrt{s}, R\sqrt{s}]\} & \text{for } h\in[1,\infty]. \end{cases} \tag{5.5}$$

Lemma 5.2. *For any $A>0$,*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\exists u\in N_t^2 : X_u(t)\ge m_{h,2}^{(1,1)}(t)-A, (T_u, X_u(T_u))\notin\Omega_{t,h}^R\right) = 0.$$

Proof. As in the proof of Lemma 5.1, applying Corollary 2.7 with $m(t) = m_{h,2}^{(1,1)}(t)$, and $\Omega_t^R = \Omega_{t,h}^R$ defined in (5.5), it suffices to show that for each $A,K>0$, $I(t,R) = I(t,R;A,K)$, defined in (2.6), vanishes as first $t\to\infty$ and then $R\to\infty$. As in (5.2), noting that $\beta_t = \sigma_t^2$ now, we have

$$I(t,R) \lesssim_K \int_0^t \mathrm{d}s \int_{-\infty}^K \mathsf{F}_t\left(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x\right) 1_{\{(s,\sqrt{2}\sigma_t^2 s+\sigma_t x)\notin\Omega_{t,h}^R\}} \frac{K-x}{s^{3/2}} e^{-\sqrt{2}\sigma_t x - \frac{x^2}{2s}}\,\mathrm{d}x \tag{5.6}$$

vanishes as first $t\to\infty$ and then $R\to\infty$. Let $L_{s,t} := \sqrt{2}(1-\sigma_t^2)s + \frac{h'}{\sqrt{2}}\log t - \frac{3}{2\sqrt{2}}\log(\frac{t}{t-s+1}) - A$.

- If $L_{s,t} - \sigma_t x > 1$, by Lemma 2.2,

$$\mathsf{F}_t(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x) = \mathsf{P}\left(\max_{u \in \mathsf{N}_{t-s}} \mathsf{X}_u(t-s) \geq \sqrt{2}(t-s) - \frac{3}{2\sqrt{2}}\log(t-s+1) + L_{s,t} - \sigma_t x\right)$$
$$\lesssim_A (L_{s,t} - \sigma_t x) e^{-2(1-\sigma_t^2)s} (\frac{t}{t-s+1})^{3/2} \frac{1}{t^{h'}} e^{\sqrt{2}\sigma_t x} e^{-\frac{(L_{s,t}-\sigma_t x)^2}{3(t-s)}}.$$

- If $L_{s,t} - \sigma_t x \leq 1$, we simply upper bound $\mathsf{F}_t(t-s, \sqrt{2}\sigma_t^2 s + \sigma_t x)$ by 1. Moreover, $L_{s,t} - \sigma_t x \leq 1$ holds only if $h \geq 1$, $s \geq t/2$ and $-\sigma_t x \leq 1 - L_{s,t} \leq -\frac{h'}{\sqrt{2}} \log t + \frac{3}{2\sqrt{2}} \log(\frac{t}{t-s+1}) + A + 1$. Thus

$$\int_{t/2}^t \mathrm{d}s \int_{-O(\log t)}^K \frac{K-x}{s^{3/2}} e^{-\sqrt{2}\sigma_t x - \frac{x^2}{2s}} 1_{\{L_{s,t} - \sigma_t x \leq 1\}} \,\mathrm{d}x \leq \frac{O(\log t)^2}{t^{h'}} \int_0^{t/2} \frac{1}{(u+1)^{3/2}} \,\mathrm{d}u = o(1).$$

Substituting these inequalities into (5.6), and using the assumption $\beta_t + \sigma_t^2 - 2 = -t^{-h}$, we get

$$I(t,R) \lesssim$$
$$\int_0^t \int_{-\infty}^K 1_{\{(s, \sqrt{2}\sigma_t^2 s + \sigma_t x) \notin \Omega_{t,h}^R\}} \frac{K-x}{s^{3/2}} e^{-\frac{s}{t^h}} e^{-\frac{x^2}{2s}} \frac{|L_{s,t} - \sigma_t x|}{t^{h'}} (\frac{t}{t-s+1})^{3/2} e^{-\frac{(L_{s,t}-\sigma_t x)^2}{3(t-s)}} \,\mathrm{d}s \,\mathrm{d}x + o(1).$$

In the case $h \in (0,1)$, make change of variables $s = \xi t^h$ and $-x = \eta\sqrt{s}$. Noting that $(s, \sqrt{2}\sigma_t^2 s + \sigma_t x) \in \Omega_{t,h}^R$ if and only if $\xi \in [R^{-1}, R]$ and $\sigma_t \eta \in [R^{-1}, R]$, applying the dominated convergence theorem twice we get

$$I_1(t,R,K) \lesssim$$
$$\int_0^{t^{1-h}} \mathrm{d}\xi \int_{-K/\sqrt{\xi t^h}}^\infty 1_{\left\{\substack{\xi \notin [R^{-1},R] \\ \text{or } \sigma_t \eta \in [R^{-1},R]}\right\}} \frac{K + \eta\sqrt{\xi t^h}}{\xi t^h} e^{-\xi} e^{-\frac{\eta^2}{2}} [L(\xi t^h, t) + \sigma_t \eta \sqrt{\xi t^h}] (\frac{t}{t - \xi t^h + 1})^{3/2} \,\mathrm{d}\eta$$
$$\xrightarrow{t\to\infty} \int_0^\infty \int_0^\infty 1_{\left\{\substack{\xi \notin [R^{-1},R] \\ \text{or } \eta \in [R^{-1},R]}\right\}} e^{-\xi} e^{-\frac{\eta^2}{2}} \eta^2 \,\mathrm{d}\xi \,\mathrm{d}\eta \xrightarrow{R\to\infty} 0.$$

If $h \geq 1$, make change of variables $s = \xi t$ and $-x = \eta\sqrt{s}$. Noting that $(s, \sqrt{2}\sigma_t^2 s + \sigma_t x) \in \Omega_{t,h}^R$ if and only if $\xi \in [R^{-1}, 1 - R^{-1}]$ and $\sigma_t \eta \in [R^{-1}, R]$, applying the dominated convergence theorem twice we get

$$I_1(t,R,K) \lesssim \int_0^1 \int_{-\frac{K}{\sqrt{\xi t}}}^\infty 1_{\left\{\substack{\xi \notin [R^{-1},1-R^{-1}] \\ \text{or } \sigma_t \eta \in [R^{-1},R]}\right\}} e^{-\frac{\eta^2}{2}} \frac{K + \eta\sqrt{\xi t}}{\xi t} [L(\xi t, t) + \sigma_t \eta \sqrt{\xi t}] (1-\xi)^{-3/2} e^{-\frac{(L_{\xi t, t} - \sigma_t x)^2}{3(t - \xi t)}} \,\mathrm{d}\xi \,\mathrm{d}\eta$$
$$\xrightarrow{t\to\infty} \int_0^1 \int_0^\infty 1_{\left\{\substack{\xi \notin [R^{-1},R] \\ \text{or } \eta \in [R^{-1},R]}\right\}} e^{-\frac{\eta^2}{2}} \eta^2 (\frac{1}{1-\xi})^{3/2} e^{-\frac{\eta^2}{2}\frac{\xi}{1-\xi}} \,\mathrm{d}\xi \,\mathrm{d}\eta \xrightarrow{R\to\infty} 0,$$

where we used the fact $\int_0^1 (\frac{1}{1-\xi})^{3/2} \,\mathrm{d}\xi \int_0^\infty \eta^2 e^{-\frac{\eta^2}{2}\frac{1}{1-\xi}} \,\mathrm{d}\eta = \int_0^1 \mathrm{d}\xi \int_0^\infty \lambda^2 e^{-\frac{\lambda^2}{2}} \,\mathrm{d}\lambda < \infty$. $\square$

Proof of Theorem 1.4 for $(\beta_t, \sigma_t^2)_{t>0} \in \mathscr{A}_{(1,1)}^{h,2}$. Take $\varphi \in \mathcal{T}$. Applying Corollary 2.7 with $m(t) = m_{h,2}^{(1,1)}(t)$, $\rho = \sqrt{2}$, and $\Omega_t^R = \Omega_{t,h}^R$ defined in (5.1), it suffices to study the asymptotic of $\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right)$, which equals

$$\mathbb{E}\Big(\exp\Big\{-\int_0^t \sum_{u \in N_s^1} \Phi_{\sqrt{2}}\big(t-s, X_u(s) - \sqrt{2}s + \frac{3-4h'}{2\sqrt{2}} \log t\big) 1_{\{(s, X_u(s)) \in \Omega_{t,h}^R\}} \,\mathrm{d}s\Big\}\Big).$$

Note that uniformly for $(s, X_u(s)) \in \Omega^R_{t,h}$, we have $\sqrt{2}s - X_u(s) = \Theta(\sqrt{s}) = \Theta(t^{\frac{h'}{2}})$. Then Lemma 2.3 yields that

$$\begin{aligned}
&\Phi_{\sqrt{2}}(t-s, X_u(s) - \sqrt{2}s + \frac{3-2h'}{2\sqrt{2}}\log t) \\
&= [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{(\sqrt{2}s - X_u(s))}{(t-s)^{3/2}} e^{\sqrt{2}X_u(s)-2s} t^{\frac{3-2h'}{2}} e^{-\frac{(X_u(s)-\sqrt{2}s)^2}{2(t-s)}}
\end{aligned}$$

as $t \to \infty$ uniformly for $(s, X_u(s)) \in \Omega^R_{t,h}$.

Case 1: $h \in (0,1)$. Since now $s = \Theta(t^h) \ll t$, in fact

$$\Phi_{\sqrt{2}}(t-s, X_u(s) - \sqrt{2}s + \frac{3-2h'}{2\sqrt{2}}\log t) = (1+o(1))\gamma_{\sqrt{2}}(\varphi)\frac{\sqrt{2}\beta_t s - X_u(s)}{t^h} e^{\sqrt{2}X_u(s)-2\beta_t s - \frac{s}{t^h}}$$

as $t \to \infty$ uniformly in $(s, X_u(s)) \in \Omega^R_{t,h}$, where we used that $\beta_t = 1 - \frac{1}{2t^h}$. Thus $\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}^R_t, \varphi\rangle}\right)$ equals

$$\mathbb{E}\Bigg(\exp\Bigg\{-[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}t^h}^{Rt^h}\sum_{u\in N^1_s}\frac{\sqrt{2}\beta_t s - X_u(s)}{t^h}e^{\sqrt{2}X_u(s)-2\beta_t s - \frac{s}{t^h}}1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\Bigg\}\Bigg).$$

Making a change of variable $s = \lambda t^h$, and noticing that $\beta_t = \sigma_t^2$, $\{X_u(s) : u \in N^1_s\}$ has the same law of $\{\mathsf{X}_u(\beta_t s) : u \in \mathsf{N}_{\beta_t s}\}$, we have

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}^R_t, \varphi\rangle}\right) = \mathsf{E}\Bigg(\exp\Bigg\{-[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}}^{R}\mathsf{Z}^R(\lambda\beta_t t^h)e^{-\lambda}\,\mathrm{d}\lambda\Bigg\}\Bigg),$$

where

$$\mathsf{Z}^R(t) := \sum_{u\in\mathsf{N}_t}[\sqrt{2}t - \mathsf{X}_u(t)]e^{\sqrt{2}\mathsf{X}_u(t)-2t}1_{\{\sqrt{2}t-\mathsf{X}_u(t)\in[\frac{1}{R}\sqrt{t}, R\sqrt{t}]\}}.$$

By Lemma 2.5, for each $\lambda > 0$, $\mathsf{Z}^R(t) \to \mathsf{Z}_\infty\sqrt{\frac{2}{\pi}}\int_{1/R}^R z^2 e^{-\frac{z^2}{2}}\,\mathrm{d}z$ in probability. Letting $t \to \infty$ then $R \to \infty$ and applying the dominated convergence theorem and Corollary 2.7, we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\right) = \lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}^R_t, \varphi\rangle}\right) \\
&= \lim_{R\to\infty}\mathsf{E}\Bigg(\exp\Bigg\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\int_{\frac{1}{R}}^R e^{-\lambda}\,\mathrm{d}\lambda\sqrt{\frac{2}{\pi}}\int_{\frac{1}{R}}^R z^2 e^{-\frac{z^2}{2}}\,\mathrm{d}z\Bigg\}\Bigg) = \mathsf{E}\Bigg(\exp\Big\{-\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty\Big\}\Bigg),
\end{aligned}$$

which is the Laplace functional of DPPP $\left(\sqrt{2}C_\star \mathsf{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$.

Case 2: $h \in [1, \infty]$. Notice that for $(s, X_u(s)) \in \Omega^R_{t,h}$, we have $s = \Theta(t)$ and $\sqrt{2}s - X_u(s) = \Theta(\sqrt{t})$. Now Lemma 2.3 yields that

$$\begin{aligned}
&\Phi_{\sqrt{2}}(t-s, X_u(s) - \sqrt{2}s + \frac{3-2h'}{2\sqrt{2}}\log t) \\
&= [1+o(1)]\gamma_{\sqrt{2}}(\varphi)\frac{t^{3/2}}{(t-s)^{3/2}t}[\sqrt{2}\beta_t s - X_u(s)]e^{\sqrt{2}X_u(s)-2\beta_t s-\frac{s}{t^h}}e^{-\frac{s}{2(t-s)}\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{\beta_t s}}
\end{aligned}$$

as $t \to \infty$ uniformly for $(s, X_u(s)) \in \Omega^R_{t,h}$, where we used the fact that $\beta_t = 1 - \frac{1}{2t^h}$. Then $\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}^R_t, \varphi\rangle}\right)$ equals

$$\begin{aligned}
&\mathbb{E}\Bigg(\exp\Bigg\{-[1+o(1)]\gamma_{\sqrt{2}}(\varphi)\int_{\frac{1}{R}t}^{(1-\frac{1}{R})t}\sum_{u\in N^1_s}\frac{t^{3/2}[\sqrt{2}\beta_t s - X_u(s)]}{t(t-s)^{3/2}}e^{\sqrt{2}X_u(s)-2\beta_t s-\frac{s}{t^h}}e^{-\frac{(X_u(s)-\sqrt{2}\beta_t s)^2}{2\beta_t(t-s)}} \\
&\qquad \times 1_{\{(s,X_u(s))\in\Omega^R_{t,h}\}}\,\mathrm{d}s\Bigg\}\Bigg)
\end{aligned}$$

Making a change of variable $s = \lambda t$, we have

$$\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right) = \mathsf{E}\left(\exp\left\{ -[1+o(1)]\gamma_{\sqrt{2}}(\varphi) \int_{\frac{1}{R}}^{1-\frac{1}{R}} \frac{e^{-\lambda 1_{\{h=1\}}}}{(1-\lambda)^{3/2}} \sqrt{\lambda\beta_t t} \mathsf{W}^{G_\lambda}(\lambda\beta_t t; \sqrt{2})\, \mathrm{d}\lambda \right\}\right),$$

where $G_\lambda(x) = xe^{-\frac{\lambda}{2(1-\lambda)}x^2} 1_{\{x\in[R^{-1},R]\}}$, and

$$\mathsf{W}^{G_\lambda}(t;\sqrt{2}) = \sum_{u\in \mathsf{N}_t} e^{\sqrt{2}\mathsf{X}_u(t)-2t} G_\lambda\left(\frac{\sqrt{2}t - \mathsf{X}_u(t)}{\sqrt{t}}\right).$$

By Lemma 2.5, for every $\lambda > 0$, $\sqrt{t}\mathsf{W}^{G_\lambda}(t;\sqrt{2}) \to \mathsf{Z}_\infty \sqrt{\frac{2}{\pi}} \int_{1/R}^R G_\lambda(z) z e^{-\frac{z^2}{2}}\, \mathrm{d}z = \mathsf{Z}_\infty \sqrt{\frac{2}{\pi}} \int_{1/R}^R z^2 e^{-\frac{z^2}{2(1-\lambda)}}\, \mathrm{d}z$. Letting $t\to\infty$ first and then $R\to\infty$, and applying the dominated convergence theorem and Corollary 2.7, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\right) &= \lim_{R\to\infty}\lim_{t\to\infty} \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right) \\
&= \lim_{R\to\infty} \mathsf{E}\left(\exp\left\{ -\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty \int_{\frac{1}{R}}^{1-\frac{1}{R}} \frac{e^{-\lambda 1_{\{h=1\}}}}{(1-\lambda)^{3/2}} \sqrt{\frac{2}{\pi}} \int_{\frac{1}{R}}^{R} z^2 e^{-\frac{z^2}{2(1-\lambda)}}\, \mathrm{d}z\, \mathrm{d}\lambda \right\}\right) \\
&= \mathsf{E}\left(\exp\left\{ -C_{h,2}\gamma_{\sqrt{2}}(\varphi)\mathsf{Z}_\infty \right\}\right),
\end{aligned}$$

which is the Laplace functional of DPPP $\left(C_{h,2}\sqrt{2}C_\star \mathsf{Z}_\infty e^{-\sqrt{2}x}\, \mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\right)$. Note that $C_{h,2} = 1$ if $h > 1$ and $C_{h,2} = (1-e^{-1})$ if $h = 1$, and we complete the proof. □

A Proof of Lemma 2.4

Proof of Lemma 2.4. To prove (ii), we only need to prove

$$\lim_{t\to\infty} \frac{\alpha_t}{\langle F_t, \mu_{\mathrm{Gau}}\rangle} \mathsf{W}_t^{F_t}(\lambda_t) = 2\sqrt{2}\mathsf{Z}_\infty \quad \text{in probability.}$$

The case $F_t \equiv 1$ was proved in [33, Theorem 1.1 (iv)]. That is, $\alpha_t \mathsf{W}_t(\lambda_t)$ converges to $2\sqrt{2}\mathsf{Z}_\infty$ in probability. So it suffices to show that

$$\zeta_t := \alpha_t \left(\frac{\mathsf{W}_t^{F_t}(\lambda_t)}{\langle F_t, \mu_{\mathrm{Gau}}\rangle} - \mathsf{W}_t(\lambda_t)\right) \to 0 \text{ in probability.} \tag{A.1}$$

Note that the above is also sufficient for (i). We are now left to prove (A.1).

Take $k_t, t > 0$, such that $k_t \le t/2$, $k_t/\alpha_t^2 \to \infty$ and $k_t/t \to 0$ as $t\to\infty$. Sometimes we also write $k(t)$. Such $(k_t)_{t>0}$ exists by the hypothesis $\alpha_t = o(\sqrt{t})$. For each $v \in \mathsf{N}_{k(t)}$, let $X_u^v(s) := X_u(k_t+s) - X_v(k_t)$, $u\in \mathsf{N}_s^v$, where N_s^v are descendants of v at time $k_t + s$. The branching property yields that conditioned on $\mathcal{F}_{k(t)}$, $\{X^v : v \in \mathsf{N}_{k(t)}\}$ are independent standard BBMs. We rewrite $\mathsf{W}_t^{F_t}(\lambda_t)$ as

$$\mathsf{W}_t^{F_t}(\lambda_t) = \sum_{v\in \mathsf{N}_{k(t)}} e^{\lambda_t \mathsf{X}_v(k_t) - (\frac{\lambda_t^2}{2}+1)k_t} \mathsf{W}_{t-k_t}^{v,F_t}(\lambda_t),$$

where

$$\mathsf{W}_{t-k_t}^{v,F_t}(\lambda_t) := \sum_{u\in \mathsf{N}_{t-k(t)}^v} e^{\lambda_t \mathsf{X}_u^v(t-k_t) - (\frac{\lambda_t^2}{2}+1)(t-k_t)} F_t\left(\frac{\lambda_t k_t - \mathsf{X}_v(k_t)}{\sqrt{t}} + \frac{\lambda_t(t-k_t) - \mathsf{X}_u^v(t-k_t)}{\sqrt{t}}\right).$$

By the many-to-one formula,

$$\begin{aligned}\mathsf{E}\left[\mathsf{W}^{v,F_t}_{t-k_t}(\lambda_t)|\mathcal{F}_{k(t)}\right] &= \mathsf{E}\left[\mathsf{W}^{F_t(y+\cdot)}_{t-k_t}(\lambda_t)\right]|_{y=\frac{\lambda_t k_t - \mathsf{X}_v(k_t)}{\sqrt{t}}} \\ &= \frac{\sqrt{t}}{\sqrt{2\pi t - k_t}}\int F_t(y+z)e^{-\frac{t}{2(t-k_t)}z^2}\,\mathrm{d}z|_{y=\frac{\lambda_t k_t - \mathsf{X}_v(k_t)}{\sqrt{t}}}.\end{aligned}$$

For $y\in\mathbb{R}$, define $\delta_t(y) := \frac{\sqrt{t}}{\sqrt{t-k_t}}(\int F_t(y+z)e^{-\frac{t}{2(t-k_t)}z^2}\,\mathrm{d}z)/(\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z)$. Then

$$\mathsf{E}\left[\frac{\mathsf{W}^{v,F_t}_{t-k_t}(\lambda_t)}{\langle F_t,\mu_{\mathrm{Gau}}\rangle}|\mathcal{F}_{k(t)}\right] = \delta_t\left(\frac{\lambda_t k_t - \mathsf{X}_v(k_t)}{\sqrt{t}}\right).$$

Step 1. Let

$$\widetilde{\mathsf{W}}_{k_t}(\lambda_t) := \sum_{v\in\mathsf{N}_{k(t)}} e^{\lambda_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)k_t}\,\delta_t\left(\frac{\lambda_t k_t - \mathsf{X}_v(k_t)}{\sqrt{t}}\right),$$

and $p_t = 1+\frac{1}{2\alpha_t}$. Then, using the von Bahr-Esseen inequality which claims that for any sequence $(X_i)_{i\in\mathbb{N}}$ of independent centered random variables and for any $\gamma\in[1,2]$, $\mathbb{E}\left[|\sum X_i|^\gamma\right]\le 2\sum\mathbb{E}\left[|X_i|^\gamma\right]$, we get

$$\begin{aligned}&\mathsf{E}\left[\left|\frac{\mathsf{W}^{F_t}_t(\lambda_t)}{\langle F_t,\mu_{\mathrm{G}}\rangle}-\widetilde{\mathsf{W}}_{k_t}(\lambda_t)\right|^{p_t}|\mathcal{F}_{k(t)}\right] \\ &\le 2\sum_{v\in\mathsf{N}_{k(t)}} e^{\lambda_t p_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)p_t k_t}\,\mathsf{E}\left|\frac{\mathsf{W}^{v,F_t}_{t-k_t}(\lambda_t)}{\langle F_t,\mu_{\mathrm{G}}\rangle}-\delta_t\left(\frac{\lambda_t k_t-\mathsf{X}_v(k_t)}{\sqrt{t}}\right)\right|^{p_t} \\ &\le 2^{p_t+1}\sum_{v\in\mathsf{N}_{k(t)}} e^{\lambda_t p_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)p_t k_t}\left[\frac{\mathsf{E}|\mathsf{W}^{v,F_t}_{t-k_t}(\lambda_t)|^{p_t}}{\langle F_t,\mu_{\mathrm{G}}\rangle^{p_t}}+\delta_t\left(\frac{\lambda_t k_t-\mathsf{X}_v(k_t)}{\sqrt{t}}\right)^{p_t}\right],\end{aligned}$$

where in the last inequality we used $|x+y|^p\le 2^p\left(|x|^p+|y|^p\right)$.

- Firstly, since F_t is bounded, $\mathsf{E}|\mathsf{W}^{v,F_t}_{t-k_t}(\lambda_t)|^{p_t}\le\|G\|_\infty^2\mathsf{E}|\mathsf{W}_{t-k_t}(\lambda_t)|^{p_t}\le\|G\|_\infty^2\mathsf{E}|\mathsf{W}_\infty(\lambda_t)|^{p_t}\le\|G\|_\infty^2 c_8$, where $c_8>0$ is the constant in [33, (4.1)].
- Secondly, suppose $\mathrm{supp}(G)\subset[-A,A]$. If $F_t(z)>0$, then $|z-r_t|\le Ah_t$, and hence $|z|\le\bar{r}+A\bar{h}$. Then for large t

$$\begin{aligned}\delta_t(y) &= \frac{\sqrt{t}}{\sqrt{t-k_t}}\frac{\int F_t(z)e^{-\frac{t}{2(t-k_t)}(z-y)^2}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z}\le 2\frac{\int F_t(z)e^{-\frac{(z-y)^2}{2}}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z} \\ &\le 2\frac{\int F_t(z)e^{-\frac{z^2}{2}+|z||y|}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z}e^{-\frac{y^2}{2}}\le 2e^{(\bar{r}+A\bar{h})|y|-\frac{y^2}{2}}\le\max_y 2e^{(\bar{r}+A\bar{h})|y|-\frac{y^2}{2}} := c_0<\infty.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\mathsf{E}\left(\alpha_t^{p_t}\left|\frac{\mathsf{W}^{F_t}_t(\lambda_t)}{\langle F_t,\mu_{\mathrm{G}}\rangle}-\widetilde{\mathsf{W}}_{k_t}(\lambda_t)\right|^{p_t}\right)\le\alpha_t^{p_t}2^{p_t+1}\left[\frac{\|G\|_\infty^2 c_8}{\langle F_t,\mu_{\mathrm{G}}\rangle^{p_t}}+(c_0)^{p_t}\right]\mathsf{E}\left[\sum_{v\in\mathsf{N}_{k(t)}} e^{\lambda_t p_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)p_t k_t}\right] \\ &=\alpha_t^{p_t}2^{p_t+1}\left[\frac{\|G\|_\infty^2 c_8}{\langle F_t,\mu_{\mathrm{G}}\rangle^{p_t}}+(c_0)^{p_t}\right]\exp\left\{(\frac{\lambda_t^2 p_t^2}{2}+1)k_t-(\frac{\lambda_t^2}{2}+1)p_t k_t\right\}\overset{t\to\infty}{\longrightarrow}0,\end{aligned}$$

where for the limit above we used the fact that $(\frac{\lambda_t^2 p_t^2}{2}+1)k_t-(\frac{\lambda_t^2}{2}+1)p_tk_t=(\frac{\lambda_t^2 p_t}{2}-1)(p_t-1)k_t=-\Theta(\frac{k_t}{\alpha_t^2})\to-\infty$. As a consequence,

$$\alpha_t\left|\frac{\mathsf{W}_t^{F_t}(\lambda_t)}{\langle F_t,\mu_{\mathrm{G}}\rangle}-\widetilde{\mathsf{W}}_{k_t}(\lambda_t)\right|\to 0\quad\text{ in probability.}$$

Step 2. By [33, Theorem 1.1 (iv)] again, as $\sqrt{k_t}/\alpha_t\to\infty$, we have $\alpha_t\mathsf{W}_{k_t}(\lambda_t)\to 2\sqrt{2}\mathsf{Z}_\infty$ in probability. To prove (A.1), it suffices to show that

$$\alpha_t[\widetilde{\mathsf{W}}_{k_t}(\lambda_t)-\mathsf{W}_{k_t}(\lambda_t)]=\alpha_t\mathsf{W}_{k_t}(\lambda_t)\sum_{v\in\mathsf{N}_{k(t)}}\frac{e^{\lambda_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)k_t}}{\mathsf{W}_{k_t}(\lambda_t)}\left[\delta_t\left(\frac{\lambda_tk_t-\mathsf{X}_v(k_t)}{\sqrt{t}}\right)-1\right]\to 0$$

in probability as $t\to\infty$. By [33, Corollary 1.3], for any $\epsilon>0$, there exists $K>0$ such that for t large enough, with probability at least $1-\epsilon/2$ we have

$$\sum_{v\in\mathsf{N}_{k(t)}}\frac{e^{\lambda_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)k_t}}{\mathsf{W}_{k_t}(\lambda_t)}1_{\{|\lambda_tk_t-\mathsf{X}_v(k_t)|>K\sqrt{k_t}\}}<\epsilon.$$

Again, by $\lim_{t\to\infty}\alpha_t\mathsf{W}_{k_t}(\lambda_t)=2\sqrt{2}\mathsf{Z}_\infty$, there exists $K'>0$ such that with probability at least $1-\epsilon/2$, we have $|\alpha_t\mathsf{W}_{k_t}(\lambda_t)|\le K'$. Then with probability $1-\epsilon$,

$$\begin{aligned}&\alpha_t|\widetilde{\mathsf{W}}_{k_t}(\lambda_t)-\mathsf{W}_{k_t}(\lambda_t)|\\&\le K'\sum_{v\in\mathsf{N}_{k(t)}}\frac{e^{\lambda_t\mathsf{X}_v(k_t)-(\frac{\lambda_t^2}{2}+1)k_t}}{\mathsf{W}_{k_t}(\lambda_t)}\left|\delta_t\left(\frac{\lambda_tk_t-\mathsf{X}_v(k_t)}{\sqrt{t}}\right)-1\right|1_{\{|\lambda_tk_t-\mathsf{X}_v(k_t)|\le K\sqrt{k_t}\}}+(c_0+1)\epsilon\\&\le K'\left(\sup\{|\delta_t(y)-1|:|y|\le K\sqrt{k_t/t}\}+(c_0+1)\epsilon\right).\end{aligned}$$

Noticing that $\delta_t(y)\le 2e^{(\bar r+A\bar h)|y|-\frac{y^2}{2}}$ and

$$\begin{aligned}\delta_t(y)&=\frac{\sqrt{t}}{\sqrt{t-k_t}}\frac{\int F_t(z)e^{-\frac{t}{2(t-k_t)}(z-y)^2}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z}\ge\frac{\int F_t(z)e^{-\frac{(z-y)^2}{2}-\frac{k_t}{2(t-k_t)}(z-y)^2}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z}\\&\ge\frac{\int F_t(z)e^{-\frac{z^2}{2}-|z||y|-\frac{k_t}{t}(|z|+|y|)^2}\,\mathrm{d}z}{\int F_t(z)e^{-\frac{z^2}{2}}\,\mathrm{d}z}e^{-\frac{y^2}{2}}\ge e^{(\bar r+A\bar h)|y|-\frac{y^2}{2}}e^{-\frac{k_t}{t}(|y|+\bar r+A\bar h)^2},\end{aligned}$$

we have $\sup\{|\delta_t(y)-1|:|y|\le K\sqrt{k_t/t}\}\to 0$ as $t\to\infty$. Then $\alpha_t|\widetilde{\mathsf{W}}_{k_t}(\lambda_t)-\mathsf{W}_{k_t}(\lambda_t)|\to 0$ in probability as $t\to\infty$ by the arbitrarily of ϵ, and the desired result follows. $\square$

B Proof of Lemma 4.1

Proof of Lemma 4.1. By definition $v_t^* = a_t p_t + b_t(1-p_t)$, so we have $y = (b_t - \sqrt{2})(1-p_t)t + (\sqrt{2}-a_t)u$. Consider first

$$\begin{aligned}
L_1 &:= (\beta_t - \frac{a_t^2}{2\sigma_t^2})s - \sqrt{2}y - \frac{y^2}{2(1-p_t)t} \\
&= (\beta_t - \frac{a_t^2}{2\sigma_t^2})(p_t t + u) - \sqrt{2}(b_t - \sqrt{2})(1-p_t)t - \sqrt{2}(\sqrt{2}-a_t)u - \frac{[(b_t-\sqrt{2})(1-p_t)t + (\sqrt{2}-a_t)u]^2}{2(1-p_t)t} \\
&= \left[(\beta_t - \frac{a_t^2}{2\sigma_t^2})p_t - (\sqrt{2}b_t - 2)(1-p_t) - \frac{(b_t-\sqrt{2})^2(1-p_t)}{2}\right] t \\
&\qquad + \left[(\beta_t - \frac{a_t^2}{2\sigma_t^2}) - \sqrt{2}(\sqrt{2}-a_t) - (b_t - \sqrt{2})(\sqrt{2}-a_t)\right] u - \frac{(\sqrt{2}-a_t)^2}{2(1-p_t)}\frac{u^2}{t}.
\end{aligned}$$

We write $\Delta_t := \frac{b_t}{\sqrt{2}} - 1$. Recalling our definition (4.4) and (4.1), a little computation yields the following claims.

- The coefficient for term t equals $(\beta_t - \frac{a_t^2}{2\sigma_t^2})p_t + (1-\frac{b_t^2}{2})(1-p_t) = 0$ by (4.2).
- The coefficient for term u equals $\beta_t + \frac{\sigma_t^2 b_t^2}{2} - \sqrt{2}b_t = \beta_t + \sigma_t^2\frac{\beta_t - 1}{1-\sigma_t^2} - \sqrt{2}b_t = \frac{\beta_t - \sigma_t^2}{1-\sigma_t^2} - \sqrt{2}b_t = \frac{b_t^2}{2} + 1 - \sqrt{2}b_t = \Delta_t^2$.

Thus $L_1 = \Delta_t^2 u - \frac{(\sqrt{2}-a_t)^2}{2(1-p_t)}\frac{u^2}{t}$. Secondly, letting $A_t = \frac{(\sqrt{2}-a_t)}{(1-p_t)t}$, we have

$$\begin{aligned}
L_2 &= \frac{y^2}{2(1-p_t)t} - \frac{y^2}{2(1-p_t)t - 2u} = \frac{-u[(b_t-\sqrt{2})(1-p_t)t + (\sqrt{2}-a_t)u]^2}{2(1-p_t)t[(1-p_t)t - u]} \\
&= -u\left[(\frac{b_t}{\sqrt{2}} - 1) + \frac{(\sqrt{2}-a_t)u}{(1-p_t)t}\right]^2 (1 - \frac{u}{(1-p_t)t})^{-1} \\
&= -u\left(\Delta_t + A_t u\right)^2 (1 + \frac{u}{(1-p_t)t - u}) = -\Delta_t^2 u - r(u,t),
\end{aligned}$$

where $r(u,t) := (2\Delta_t + A_t u)A_t u^2 + \frac{(\Delta_t + A_t u)^2}{(1-p_t)t - u}u^2$. Therefore

$$L(u,t) = L_1(u,t) + L_2(u,t) = -\frac{(\sqrt{2}-a_t)^2}{2(1-p_t)}\frac{u^2}{t} - r(u,t).$$

Case (i). By (4.6) we have $p_t \sim \frac{1}{2(1-\sigma^2)^2 t^h}$, $\Delta_t \sim \frac{1}{2(1-\sigma^2)t^h}$, $\sqrt{2} - a_t \sim \sqrt{2}(1-\sigma^2)$ and $A_t \sim \frac{\sqrt{2}(1-\sigma^2)}{t}$. Then, for fixed ξ,

$$L(\xi\sqrt{t}, t) = -\frac{(\sqrt{2}-a_t)^2}{2(1-p_t)}\xi^2 - r(\xi\sqrt{t}, t) \to -(1-\sigma^2)^2\xi^2 \text{ as } t \to \infty.$$

Moreover, we claim that for large t, $r(u;t) \geq 0$ for all $u > -p_t t$. In fact, the claim is true if we have $2\Delta_t + A_t u \geq 2\Delta_t - A_t p_t t \geq 0$. Note that $2\Delta_t - A_t p_t t = 2\Delta_t - \frac{(\sqrt{2}-a_t)}{(1-p_t)}p_t = [2\Delta_t t^h - \frac{(\sqrt{2}-a_t)}{(1-p_t)}p_t t^h]t^{-h}$. By (4.6), we have $[2\Delta_t t^h - \frac{(\sqrt{2}-a_t)}{(1-p_t)}p_t t^h] \to \frac{1}{1-\sigma^2} - \frac{1}{\sqrt{2}(1-\sigma^2)} > 0$. This proves our claim, and thus $L(\xi\sqrt{t}, t) \leq -\frac{(\sqrt{2}-a_t)^2}{2(1-p_t)}\xi^2 \leq -c\xi^2$ for some constant c.

Case (ii). By (4.16), we have $p_t = 1/2$, $\Delta_t \sim \frac{1}{2t^{h/2}}$, $\sqrt{2} - a_t \sim \frac{1}{\sqrt{2}t^{h/2}}$ and $A_t \sim \frac{\sqrt{2}}{t^{1+\frac{h}{2}}}$. Then $\lim_{t\to\infty} r(\xi\sqrt{t}, t) = 2\Delta_t A_t t^{1+h}\xi^2 + 2\Delta_t^2 t^h \xi^2 = (\sqrt{2} + \frac{1}{2})\xi^2$, and

$$L(\xi t^{\frac{1+h}{2}}, t) = -(\sqrt{2} - a_t)^2 t^h \xi^2 - r(\xi\sqrt{t}, t) \to -(\sqrt{2} + 1)\xi^2.$$

Finally we prove that for large t, $r(u; t) \geq 0$ for all $u > -1/2t$ by showing that $2\Delta_t + A_t u \geq 2\Delta_t - \frac{1}{2}A_t t \geq 0$. This follows from the fact that $\lim_{t\to\infty} t^{h/2}[2\Delta_t - \frac{1}{2}A_t t] \to 1 - \frac{\sqrt{2}}{2} > 0$. Hence $L(\xi\sqrt{t}, t) \leq -(\sqrt{2} - a_t)^2 t^h \xi^2 \leq -c\xi^2$ for some constant c. $\square$

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