Potential Analysis
https://doi.org/10.1007/s11118-025-10223-2

RESEARCH

Tails of Extinction Time and Maximal Displacement for Critical Branching Killed Lévy Process

Haojie Hou[1] · **Yan-Xia Ren**[2] · **Renming Song**[3]

Received: 15 May 2024 / Accepted: 4 May 2025

Abstract

In this paper, we study asymptotic behaviors of the tails of extinction time and maximal displacement of a critical branching killed Lévy process $(Z_t^{(0,\infty)})_{t\geq 0}$ in $\mathbb{R}$, in which all particles (and their descendants) are killed upon exiting $(0, \infty)$. Let $\zeta^{(0,\infty)}$ and $M_t^{(0,\infty)}$ be the extinction time and maximal position of all the particles alive at time t of this branching killed Lévy process and define $M^{(0,\infty)} := \sup_{t\geq 0} M_t^{(0,\infty)}$. Under the assumption that the offspring distribution belongs to the domain of attraction of an α-stable distribution, $\alpha \in (1, 2]$, and some moment conditions on the spatial motion, we give the decay rates of the survival probabilities

$$\mathbb{P}_y(\zeta^{(0,\infty)} > t), \quad \mathbb{P}_{\sqrt{t}y}(\zeta^{(0,\infty)} > t)$$

and the tail probabilities

$$\mathbb{P}_y(M^{(0,\infty)} \geq x), \quad \mathbb{P}_{xy}(M^{(0,\infty)} \geq x).$$

We also study the scaling limits of $M_t^{(0,\infty)}$ and the point process $Z_t^{(0,\infty)}$ under $\mathbb{P}_{\sqrt{t}y}(\cdot|\zeta^{(0,\infty)} > t)$ and $\mathbb{P}_y(\cdot|\zeta^{(0,\infty)} > t)$. The scaling limits under $\mathbb{P}_{\sqrt{t}y}(\cdot|\zeta^{(0,\infty)} > t)$ are represented in terms of super killed Brownian motion.

Keywords Branching killed Lévy process · Superprocess · Critical branching process · Extinction time · Maximal displacement · Yaglom limit · Feynman-Kac representation

Mathematics Subject Classification (2010) 60J80 · 60J68 · 60G51 · 60G57

✉ Yan-Xia Ren
yxren@math.pku.edu.cn

Haojie Hou
houhaojie@bit.edu.cn

Renming Song
rsong@.uiuc.edu

1 School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, P. R. China

2 LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, P.R. China

3 Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, U.S.A.

Published online: 21 May 2025

1 Introduction and Main Results

1.1 Background and Motivation

A branching Lévy process on $\mathbb{R}$ is defined as follows: initially there is a particle at position $x \in \mathbb{R}$ which moves according to a Lévy process $(\xi_t, \mathbf{P}_x)$ on $\mathbb{R}$. We will use $\mathbf{E}_x$ to denote expectation with respect to $\mathbf{P}_x$. The lifetime of this particle is an exponentially distributed random variable with parameter $\beta > 0$ and when it dies, this particle gives birth to a random number of offspring with law $\{p_k : k \geq 0\}$. The children of this particle independently repeat their parent's behavior from their birthplace. The procedure goes on. We use $N(t)$ to denote the set of particles alive at time t and for each $u \in N(t)$, we denote by $X_u(t)$ the position of u at time t. Also, for any $u \in N(t)$ and $s \leq t$, we use $X_u(s)$ to denote the position of u or its ancestor at time s. The point process $(Z_t)_{t\geq 0}$ defined by

$$Z_t := \sum_{u\in N(t)} \delta_{X_u(t)}$$

is called a branching Lévy process. We will use $\mathbb{P}_x$ to denote the law of this process and use $\mathbb{E}_x$ to denote the corresponding expectation. We will use the convention $\mathbb{P} := \mathbb{P}_0$ and $\mathbb{E} := \mathbb{E}_0$.

Suppose that $m := \sum_{k=0}^{\infty} kp_k \in (0, \infty)$. It is well known that the branching Lévy process $(Z_t)_{t\geq 0}$ will become extinct with probability one if and only if $m < 1$ (subcritical) or $m = 1$ and $p_1 \neq 1$(critical). In this paper, we will focus on the critical case, that is, we always assume that $m = 1$ and $p_1 \neq 1$.

For any t, let $M_t := \sup_{u\in N(t)} X_u(t)$ be the maximal position of all the particles alive at time t and we use the convention that $M_t = -\infty$ if $N(t) = \emptyset$. Now we define the maximal displacement and extinction time respectively by

$$M = \sup_{t\geq 0} M_t \quad \text{and} \quad \zeta := \inf\{t > 0 : N(t) = \emptyset\}. \tag{1.1}$$

Since we always assume $m = 1$ and $p_1 \neq 1$, we have $\mathbb{P}(M < \infty) = \mathbb{P}(\zeta < \infty) = 1$.

Due to the homogeneity of the branching rate β and offspring law $\{p_k : k \geq 0\}$, ζ is equal in law to the extinction time of a continuous-time Galton-Waston process with the same offspring distribution as $(Z_t)_{t\geq 0}$, so the decay rate of the survival probability $\mathbb{P}(\zeta > t)$ is clear. For example, suppose that

(H1) The offspring distribution $\{p_k : k \geq 0\}$ belongs to the domain of attraction of an α-stable, $\alpha \in (1, 2]$, distribution. More precisely, either there exist $\alpha \in (1, 2)$ and $\kappa(\alpha) \in (0, \infty)$ such that

$$\lim_{n\to\infty} n^{\alpha} \sum_{k=n}^{\infty} p_k = \kappa(\alpha),$$

or that (corresponding $\alpha = 2$)

$$\sum_{k=0}^{\infty} k^2 p_k < \infty.$$

Then it is known (see, for example, [14, 29, 31]) that, there exists a $C(\alpha) \in (0, \infty)$ such that

$$\lim_{t\to\infty} t^{\frac{1}{\alpha-1}} \mathbb{P}(\zeta > t) = C(\alpha). \tag{1.2}$$

The tail probability of the maximal displacement M has been intensively studied in the literature. Sawyer and Fleischman [28] proved that under the assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$ and that the spatial motion ξ is a standard Brownian motion, there exists a constant $\theta(2) > 0$ such that

$$\lim_{x\to\infty} x^2 \mathbb{P}(M \geq x) = \theta(2). \tag{1.3}$$

For corresponding results in the case of critical branching random walks with offspring distribution having finite third moment, see [18], and for these in the case of critical branching Lévy processes with offspring distribution having finite third moment, see [17, 25, 26]. In the case of critical branching Lévy processes with offspring distribution belonging to the domain of attraction of an α-stable distribution with $\alpha \in (1, 2]$, we assume

(H2)

$$\mathbf{E}_0(\xi_1) = 0, \quad \sigma^2 = \mathbf{E}_0(\xi_1^2) \in (0, \infty)$$

and

(H3) for α given in **(H1)**,

$$\mathbf{E}_0\left((\xi_1 \vee 0)^{r_0}\right) < \infty \quad \text{for some } r_0 > \frac{2\alpha}{\alpha - 1}.$$

Hou, Jiang, Ren and Song [9] proved that under **(H1)** (although [9] did not deal with the case $\alpha = 2$, the proof is actually the same as the case $\alpha \in (1, 2)$, without the additional assumption $\sum_{k=0}^{\infty} k^3 p_k < \infty$, see the argument beneath [9, Theorem 1.1]), **(H2)** and **(H3)**, there exists a constant $\theta(\alpha) > 0$ such that

$$\lim_{x\to\infty} x^{\frac{2}{\alpha-1}} \mathbb{P}(M \geq x) = \theta(\alpha). \tag{1.4}$$

The main concern of this paper is on critical branching Lévy processes. If, in the critical branching Lévy process, we kill all particles (and their potential descendants) once they exit $(0, \infty)$, we obtain a point process $(Z_t^{(0,\infty)})_{t\geq 0}$ with

$$Z_t^{(0,\infty)} := \sum_{u\in N(t)} 1_{\{\inf_{s\leq t} X_u(s) > 0\}} \delta_{X_u(t)}.$$

The process $(Z_t^{(0,\infty)})_{t\geq 0}$ is called a critical branching killed Lévy process. Let $(Z_t^0, \mathbb{P}_y)$ stand for a branching Markov process with spatial motion $\xi_{t\wedge\tau_0^-}$ where $\tau_0^- := \inf\{t > 0 : \xi_t \leq 0\}$, branching rate β and offspring distribution $\{p_k : k \geq 0\}$. Then it is easy to see that for any $t, y > 0$,

$$\left(Z_t^{(0,\infty)}, \mathbb{P}_y\right) \stackrel{\mathrm{d}}{=} \left(Z_t^0|_{(0,\infty)}, \mathbb{P}_y\right). \tag{1.5}$$

Define

$$M_t^{(0,\infty)} := \sup_{u\in N(t):\inf_{s\leq t} X_u(s)>0} X_u(t), \quad M^{(0,\infty)} := \sup_{t\geq 0} M_t^{(0,\infty)}$$

and

$$\zeta^{(0,\infty)} := \inf\left\{t > 0 : Z_t^{(0,\infty)}((0,\infty)) = 0\right\} \tag{1.6}$$

with the convention $M_t^{(0,\infty)} = -\infty$ when $Z_t^{(0,\infty)}((0,\infty)) = 0$. When the underlying motion ξ is a standard Brownian motion, Lalley and Zheng [19] proved that, if $\sum_{k=0}^{\infty} k^3 p_k < \infty$, then

$$\lim_{x\to\infty} x^3 \mathbb{P}_y(M^{(0,\infty)} \geq x) = \theta^{(0,\infty)}(2) y, \quad \text{for all } y > 0, \tag{1.7}$$

where $\theta^{(0,\infty)}(2) \in (0, \infty)$ is a constant independent of x and y. Comparing Eqs. 1.3 and 1.7, we see that the tail $\mathbb{P}_y(M^{(0,\infty)} \geq x)$ of critical branching killed Brownian motion decays to 0 in the order x^{-3}, while the tail $\mathbb{P}_y(M \geq x)$ of critical branching Brownian motion decays to 0 in the order x^{-2}. Lalley and Zheng [19] also showed that there exists a continuous function $(0, 1) \ni y \mapsto \theta_y^{(0,\infty)}(2)$ such that

$$\lim_{x\to\infty} x^2 \mathbb{P}_{xy}(M^{(0,\infty)} \geq x) = \theta_y^{(0,\infty)}(2), \quad y \in (0, 1). \tag{1.8}$$

The argument of [19] relies heavily on the construction of $\mathbb{P}_y(M^{(0,\infty)} \geq x)$ via Weierstrass' $\mathcal{P}$-functions in the special case $p_0 = p_2 = \frac{1}{2}$ and a comparison argument for general offspring distributions.

There are also some works on the survival probability and maximal displacement of branching killed Lévy processes when $m = \sum_{k=0}^{\infty} kp_k > 1$ and the spatial motion ξ is a Brownian motion with drift $-\mu$ where $\mu = \sqrt{2\beta(m-1)}$, see [2, 3, 13, 21, 22]. For these branching processes, $\sqrt{2\beta(m-1)}$ is the critical value of the drift in the sense that the process will die out with probability 1 if and only if $\mu \geq \sqrt{2\beta(m-1)}$. When $p_2 = 1$ and $\mu = \sqrt{2\beta}$ (critical drift case), Berestycki, Berestycki and Schweinsberg [2] studied, among other things, the asymptotic behavior of the position of the right-most particle as the position y of the initial particle tends to infinity. In the case $\sum_{k=0}^{\infty} kp_k = 2$ and $\mu = \sqrt{2\beta}$ (critical drift case), the survival probability was first studied by Kesten [13], and the result of [13] was later refined in [3, 22]. For the (all-time) maximal displacement $M^{(0,\infty)} = \max_{s\geq 0} M_s^{(0,\infty)}$ and the time when this all-time maximum is achieved $m^{(0,\infty)} := \arg\max_{s\geq 0} M_s^{(0,\infty)}$, Maillard and Schweinsberg [21] proved the weak convergence for the conditioned law of $(M^{(0,\infty)}, m^{(0,\infty)})$ on the event $\{\zeta^{(0,\infty)} > t\}$.

The purpose of the paper is to study the asymptotic behaviors of the tails of the survival probability and maximal displacement of critical branching killed Lévy processes. More precisely, our goals are as follows:

(i) generalize Eqs. 1.7 and 1.8 to critical branching killed Lévy processes with offspring distribution satisfying **(H1)** and spatial motion satisfying **(H2)–(H4)**, with **(H4)** given in Section 1.3 below;

(ii) find the exact decay rate of the survival probability $\mathbb{P}_y(\zeta^{(0,\infty)} > t)$;

(iii) give probabilisitic interpretations of the limit in the generalization of Eq. 1.8 and the limit of the survival probability when the initial position is at $\sqrt{t}y$ for fixed $y > 0$;

(iv) find scaling limits of $Z_t^{(0,\infty)}$ and $M_t^{(0,\infty)}$ under law $\mathbb{P}_y\big(\cdot\,|\zeta^{(0,\infty)} > t\big)$ and law $\mathbb{P}_{\sqrt{t}y}\big(\cdot\,|\zeta^{(0,\infty)} > t\big)$.

Our approach for proving the generalizations of Eqs. 1.7 and 1.8 is different from that of [19]. The probabilistic interpretations of the limits under $\mathbb{P}_{\sqrt{t}y}\big(\cdot\,|\zeta^{(0,\infty)} > t\big)$ are given in terms of a particular critical superprocess. So in the next subsection, we will give a description of this superprocess and some basic facts about it.

1.2 Critical Super Killed Brownian Motion

Set $\mathbb{R}_+ := [0, \infty)$. Let $\mathcal{M}_F(\mathbb{R}_+)$ and $\mathcal{M}_F((0, \infty))$ be the families of finite Borel measures on $\mathbb{R}_+$ and on $(0, \infty)$ respectively, endowed with the weak topology. We will use $\mathbf{0}$ to denote the null measure on $\mathbb{R}_+$ and on $(0, \infty)$. Let $B_b(\mathbb{R}_+)$ and $B_b^+(\mathbb{R}_+)$ be the spaces of bounded Borel functions and non-negative bounded Borel functions on $\mathbb{R}_+$ respectively. In this paper, whenever we are given a function f on $(0, \infty)$, we automatically extend it to $\mathbb{R}$ by setting

$f(x) = 0$ for $x \leq 0$. The meanings of $B_b((0,\infty))$ and $B_b^+((0,\infty))$ are similar. For any $f \in B_b(\mathbb{R}_+)$ and $\mu \in \mathcal{M}_F(\mathbb{R}_+)$, we use $\langle f, \mu\rangle$ to denote the integral of f with respect to μ. For any $\alpha \in (1, 2]$, the function

$$\varphi(\lambda) := \mathcal{C}(\alpha)\lambda^\alpha := \begin{cases} \frac{\beta\kappa(\alpha)\Gamma(2-\alpha)}{\alpha-1}\lambda^\alpha, & \text{when } \alpha \in (1,2), \\ \frac{\beta}{2}\left(\sum_{k=1}^\infty k(k-1)p_k\right)\lambda^2, & \alpha = 2, \end{cases} \tag{1.9}$$

where $\kappa(\alpha)$ is given in **(H1)** and $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ is the Gamma function, is a branching mechanism. Note that φ is the branching mechanism of a critical superprocess. For more information on general branching mechanisms, we refer the reader to [20, Sections 2.3 and 2.4]). For any $x \in \mathbb{R}_+$, let $(W_t, \mathbf{P}_x)$ be a Brownian motion starting from x, with variance $\sigma^2 t$, where σ^2 is given in **(H2)**. Let $W_t^0 := W_{t\wedge\tau_0^{W,-}}$ be the process W stopped at the first exit time $\tau_0^{W,-}$ of $(0,\infty)$. Note that, when starting from 0, W^0 stays at 0.

In this paper, for any $\mu \in \mathcal{M}_F(\mathbb{R}_+)$, we will use $X = \{(X_t)_{t\geq 0}; \mathbb{P}_\mu\}$ to denote a superprocess with spatial motion W^0 and branching mechanism φ, that is, an $\mathcal{M}_F(\mathbb{R}_+)$-valued Markov process such that for any $f \in B_b^+(\mathbb{R}_+)$,

$$-\log \mathbb{E}_\mu\left(\exp\{-\langle f, X_t\rangle\}\right) = \langle v_f^X(t,\cdot), \mu\rangle,$$

where $(t, x) \mapsto v_f^X(t, x)$ is the unique locally bounded non-negative solution to

$$v_f^X(t,x) = \mathbf{E}_x\left(f(W_t^0)\right) - \mathbf{E}_y\Big(\int_0^t \varphi\left(v_f^X(t-s, W_s^0)\right)\mathrm{d}s\Big). \tag{1.10}$$

Taking $f \equiv \theta 1_{\mathbb{R}_+}$ in Eq. 1.10, the uniqueness of the solution implies that

$$-\log \mathbb{E}_{\delta_y}\left(\exp\{-\langle\theta, X_r\rangle\}\right) = \left((\alpha-1)\mathcal{C}(\alpha)r^\alpha + \theta^{1-\alpha}\right)^{-\frac{1}{\alpha-1}}.$$

Therefore, letting $\theta \to +\infty$ in the above equation, we obtain that

$$\begin{aligned} -\log \mathbb{P}_{\delta_y}(X_r = \mathbf{0}) &= \lim_{\theta\to\infty}\left((\alpha-1)\mathcal{C}(\alpha)r^\alpha + \theta^{1-\alpha}\right)^{-\frac{1}{\alpha-1}} \\ &= \left((\alpha-1)\mathcal{C}(\alpha)r^\alpha\right)^{-\frac{1}{\alpha-1}}, \quad \text{for all } r > 0,\ y \in \mathbb{R}_+. \end{aligned} \tag{1.11}$$

Next, we introduce the $\mathbb{N}$-measures associated to the superprocess X. Without loss of generality, we assume that X is the coordinate process on

$$\mathbb{D} := \{w = (w_t)_{t\geq 0} : w \text{ is an } \mathcal{M}_F(\mathbb{R}_+)\text{-valued càdlàg function on } \mathbb{R}_+\}.$$

We assume that $(\mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0})$ is the natural filtration on $\mathbb{D}$, completed as usual with the $\mathcal{F}_\infty$-measurable and $\mathbb{P}_\mu$-negligible sets for every $\mu \in \mathcal{M}_F(\mathbb{R}_+)$. Let $\mathbb{W}_0^+$ be the family of $\mathcal{M}_F(\mathbb{R}_+)$-valued càdlàg functions on $(0,\infty)$ with $\mathbf{0}$ as a trap and with $\lim_{t\downarrow 0} w_t = \mathbf{0}$. Note that $\mathbb{W}_0^+$ can be regarded as a subset of $\mathbb{D}$.

By Eq. 1.11, $\mathbb{P}_{\delta_y}(X_t = \mathbf{0}) > 0$ for all $t > 0$ and $y \in \mathbb{R}_+$, which implies that there exists a unique family of σ-finite measures $\{\mathbb{N}_y; y \in \mathbb{R}_+\}$ on $\mathbb{W}_0^+$ such that for any $\mu \in \mathcal{M}_F(\mathbb{R}_+)$, if $\mathcal{N}(\mathrm{d}w)$ is a Poisson random measure on $\mathbb{W}_0^+$ with intensity measure

$$\mathbb{N}_\mu(\mathrm{d}w) := \int_{\mathbb{R}_+} \mathbb{N}_y(\mathrm{d}w)\mu(\mathrm{d}y),$$

then the process defined by

$$\widehat{X}_0 := \mu, \quad \widehat{X}_t := \int_{\mathbb{W}_0^+} w_t \mathcal{N}(\mathrm{d}w), \quad t > 0,$$

is a realization of the superprocess $X = \{(X_t)_{t\geq 0}; \mathbb{P}_\mu\}$. Furthermore, for any $t > 0$, $y \in \mathbb{R}_+$ and $f \in B_b^+(\mathbb{R}_+)$,

$$\mathbb{N}_y\left(1 - \exp\{-\langle f, w_t\rangle\}\right) = -\log \mathbb{E}_{\delta_y}\left(\exp\{-\langle f, X_t\rangle\}\right), \tag{1.12}$$

see [20, Theorems 8.27 and 8.28]. $\{\mathbb{N}_y; y \in \mathbb{R}_+\}$ are called the $\mathbb{N}$-measures associated to $\{\mathbb{P}_{\delta_y}; y \in \mathbb{R}_+\}$. One can also see [7] for the definition of $\{\mathbb{N}_y; y \in \mathbb{R}_+\}$. Note that for any $y > 0$, $\mathbb{P}_{\delta_y}(X_1((0,\infty)) = 0) > 0$. Thus by Eq. 1.12, we see that $\mathbb{N}_y(w_1((0,\infty)) \neq 0) < \infty$.

Now we define

$$X_t^{(0,\infty)} := X_t\big|_{(0,\infty)}, \quad t \geq 0. \tag{1.13}$$

$(X_t^{(0,\infty)})_{t\geq 0}$ is called a critical super killed Brownian motion. By the definition of $X^{(0,\infty)}$ we see that for any $t, y > 0$, under $\mathbb{P}_{\delta_y}$,

$$\langle f, X_t\rangle = \langle f, X_t^{(0,\infty)}\rangle, \quad \text{for any } f \in B_b^+((0,\infty)). \tag{1.14}$$

1.3 Main Results

The following condition is stronger than **(H3)** since it requires that $-\xi$ also satisfies **(H3)**:

(H4) For the $\alpha \in (1,2]$ in **(H1)**, it holds that

$$\mathbf{E}_0\left(|\xi_1|^{r_0}\right) < \infty \quad \text{for some} \quad r_0 > \frac{2\alpha}{\alpha - 1}.$$

The assumption **(H4)** will be used in the proofs of Lemmas 2.13, 3.2 and 3.8. In the proof of Lemma 3.2, we need to apply Eq. 1.4 to critical branching killed Lévy processes with spatial motion $-\xi$. The assumption **(H4)** is also essentially used in the proof of Lemma 3.8, see Eq. 4.25 below. In the case $\alpha = 2$, the assumption **(H4)** is the same as that in [18] and is weaker than that in [23, Remark 1.4].

For critical branching random walks, in the case $\alpha = 2$, [30, Theorem 1.3] says that the assumption **(H2)** is sufficient to get Eq. 1.4. We believe that one might be able get Eq. 1.4 under **(H1)** and **(H2)**, without **(H4)**. However, different arguments are needed since **(H4)** is used essentially in several places in our argument. We do not purse this here.

For any $x \in \mathbb{R}$, define

$$\tau_x^+ := \inf\{t > 0 : \xi_t \geq x\} \quad \text{and} \quad \tau_x^- := \inf\{t > 0 : \xi_t \leq x\}.$$

Since ξ oscillates under condition **(H2)**, we have $\mathbf{P}_x(\tau_0^- < \infty) = 1$. By Lemma 2.8 below, we have $\mathbf{E}_x|\xi_{\tau_0^-}| < \infty$. Define

$$R(x) := x - \mathbf{E}_x\left(\xi_{\tau_0^-}\right) = -\mathbf{E}_0\left(\xi_{\tau_{-x}^-}\right), \quad x \geq 0. \tag{1.15}$$

Note that $R(x) \geq x$ and that $R(x)$ is non-decreasing in x. In Lemma 2.8, we will show that R is harmonic in $(0,\infty)$ with respect to the process $\xi_{t\wedge\tau_0^-}$. When ξ is a Brownian motion, $R(x) \equiv x$.

Our main results are as follows.

Theorem 1.1 *Assume* **(H1)**, **(H2)** *and* **(H4)** *hold.*

(i) For any $y > 0$,

$$\lim_{t\to\infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y}(\zeta^{(0,\infty)} > t) = \mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right),$$

where $\mathbb{N}_y$ is the $\mathbb{N}$-measure of the super Brownian motion defined in Section 1.2.

(ii) There exists a constant $C^{(0,\infty)}(\alpha) \in (0, \infty)$ such that for any $y > 0$,

$$\lim_{t\to\infty} t^{\frac{1}{\alpha-1}+\frac{1}{2}} \mathbb{P}_y(\zeta^{(0,\infty)} > t) = C^{(0,\infty)}(\alpha) R(y).$$

Remark 1.2 For a critical branching Lévy process, the tail $\mathbb{P}(\zeta > t)$ of extinction time decays to zero like $t^{-1/(\alpha-1)}$, see Eq. 1.2. Theorem 1.1 tells that, for a critical branching killed Lévy process starting from a single particle at $y > 0$, the tail $\mathbb{P}_y(\zeta^{(0,\infty)} > t)$ decays to zero like $t^{-1/(\alpha-1)-1/2}$, while the tail $\mathbb{P}_{\sqrt{t}y}(\zeta^{(0,\infty)} > t)$ reverts back to $t^{-1/(\alpha-1)}$.

For any $t > 0$, we define the following scaled version of $Z_t^{(0,\infty)}$:

$$Z_1^{(0,\infty),t} := t^{-\frac{1}{\alpha-1}} \sum_{u\in N(t)} 1_{\{\inf_{s\le t} X_u(s)>0\}} \delta_{X_u(t)/\sqrt{t}}. \tag{1.16}$$

The next theorem is about the limits of $Z_1^{(0,\infty),t}$ under $\mathbb{P}_{\sqrt{t}y}(\cdot\big|\zeta^{(0,\infty)} > t)$ and $\mathbb{P}_y(\cdot\big|\zeta^{(0,\infty)} > t)$ as $t \to \infty$. It is similar in spirit to the result that the Dawson-Watanabe process is an appropriate scaling limit of branching Markov processes (see, for instance, [20, Proposition 4.6]). This result partially answers the question in [24, Question 1.8] when the domain is assumed to be the half line.

Theorem 1.3 *Assume* **(H1)**, **(H2)** *and* **(H4)** *hold.*

(i) For any $y > 0$,

$$\lim_{t\to\infty} \mathbb{P}_{\sqrt{t}y}\big(Z_1^{(0,\infty),t} \in \cdot \,\big|\zeta^{(0,\infty)} > t\big) = \mathbb{N}_y\left(w_1|_{(0,\infty)} \in \cdot\big| w_1((0,\infty)) \neq 0\right),$$

where $w_1|_{(0,\infty)}$ is the restriction of the random measure w_1 on $(0, \infty)$.

(ii) There exists a random measure $(\eta_1, \mathbb{P})$ on $(0, \infty)$ such that for any $y > 0$,

$$\lim_{t\to\infty} \mathbb{P}_y\big(Z_1^{(0,\infty),t} \in \cdot \,\big|\zeta^{(0,\infty)} > t\big) = \mathbb{P}(\eta_1 \in \cdot).$$

Remark 1.4 Powell [24] studied critical branching diffusion processes Z_t^D killed upon exiting a bounded domain $D \subset \mathbb{R}^d$. It was proved in [24, Theorem 1.6] that for any $y \in D$ and non-negative bounded continuous function f on D, $\frac{1}{t}\langle f, Z_t^D\rangle$ under $\mathbb{P}_y(\cdot|Z_t^{(D)}(D) > 0)$ converges weakly to an exponential random variable. In [24, Question 1.8], Powell asked what happens when D is unbounded. Our Theorem 1.3 answers this question in the case that D is the half-plane $(0, \infty)$.

The following theorem generalizes Eqs. 1.7 and 1.8 and also provides a probabilistic interpretation for the limit of the generalization of Eq. 1.8. When specialized to the case $\alpha = 2$, the next theorem also gives an alternative proof of [19, Theorem 6.1]. Define the maximal displacement of $X^{(0,\infty)}$ by

$$M^{(0,\infty),X} := \sup_{r>0} \inf\{y \in \mathbb{R} : X_r^{(0,\infty)}((y,\infty)) = 0\}.$$

Theorem 1.5 *Assume* **(H1)**, **(H2)** *and* **(H4)** *hold.*

(i) For any $y > 0$, it holds that

$$\lim_{x\to\infty} x^{\frac{2}{\alpha-1}} \mathbb{P}_{xy}(M^{(0,\infty)} \geq x) = -\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big). \tag{1.17}$$

(ii) Assume further that **(H4)** *holds with* $r_0 > 1 + \frac{2\alpha}{\alpha-1}$. *Then there exists a constant* $\theta^{(0,\infty)}(\alpha) \in (0, \infty)$ *such that for any* $y > 0$, *it holds that*

$$\lim_{x\to\infty} x^{\frac{2}{\alpha-1}+1}\mathbb{P}_y(M^{(0,\infty)} \geq x) = \theta^{(0,\infty)}(\alpha)R(y). \tag{1.18}$$

The higher moment condition in Theorem 1.5 (ii) is used in Eq. 3.26.

Remark 1.6 (1) When $\alpha = 2$, Theorem 1.5 (i) is consistent with Eq. 1.8, and Theorem 1.5 (ii) is consistent with Eq. 1.7. In Lalley and Zheng [19], the offspring distribution is assumed to have finite 3rd moment.

In [19], the constant $\theta^{(0,\infty)}(2)$ and the limit in Eq. 1.17 are given in terms of Weierstrass' $\mathcal{P}$-functions. Our limit in Eq. 1.17 is given in terms of superprocess and limit in Eq. 1.18 is given in terms of $R(y)$ defined in Eq. 1.15. The right hand side of Eq. 1.17 can also be given in terms of Weierstrass' $\mathcal{P}$-functions. In fact, by [19, p.12, line 1 from below], we see that $-\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big) = \frac{2\omega_1^2}{3}\mathcal{P}_{\mathcal{L}_1}(\frac{\omega_1}{3}(2+y))$, where $\mathcal{P}_{\mathcal{L}_1}$ is the Weierstrass' $\mathcal{P}$-functions defined in [19, (4.1)] with $\mathcal{L}_1$ and ω_1 given in [19, (5.4) and (5.5)] respectively. Our assumption **(H1)** on the offspring distribution is weaker and optimal in some sense. Since $\frac{1}{y}W_t 1_{\{\min_{s\leq t} W_s>0\}}$ is a martingale under $\mathbf{P}_y$, we define

$$\frac{\mathrm{d}\mathbf{P}_y^{\uparrow}}{\mathrm{d}\mathbf{P}_y}\bigg|_{\sigma(W_s,s\leq t)} := \frac{1}{y}W_t 1_{\{\min_{s\leq t} W_s>0\}}.$$

In the case $\sigma = 1$, it is well-known that $(W_t, \mathbf{P}_y^{\uparrow})$ is a Bessel-3 process. Combining Eq. 4.32 (with $z = \frac{1}{2}$) and Eq. 3.29, we can give the following probabilistic representation for $\theta^{(0,\infty)}(\alpha)$:

$$\begin{aligned}
\theta^{(0,\infty)}(\alpha) &= K^X\Big(\frac{1}{2}\Big)\lim_{y\to 0+}\frac{1}{y}\mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_{1/2}^{W,+}}\psi^X\Big(K^X(W_s)\Big)\,\mathrm{d}s\Big\};\tau_{1/2}^{W,+} < \tau_0^{W,-}\Big)\\
&= 2K^X\Big(\frac{1}{2}\Big)\lim_{y\to 0+}\mathbf{E}_y^{\uparrow}\Big(\exp\Big\{-\int_0^{\tau_{1/2}^{W,+}}\psi^X\Big(K^X(W_s)\Big)\,\mathrm{d}s\Big\}\Big)\\
&= 2K^X\Big(\frac{1}{2}\Big)\mathbf{E}_0^{\uparrow}\Big(\exp\Big\{-\int_0^{\tau_{1/2}^{W,+}}\psi^X\Big(K^X(W_s)\Big)\,\mathrm{d}s\Big\}\Big),
\end{aligned}$$

where $\psi^X(v) := \varphi(v)/v$ and $K^X(\cdot)$ is the unique solution of Eq. 3.24. It is interesting and natural to to ask whether $\theta^{(0,\infty)}(\alpha)$ is monotone or smooth in α. We have not pursued this.

(2) Equation 1.18 says that, unlike Eq. 1.4 for critical branching Lévy processes, the tail $\mathbb{P}_y(M^{(0,\infty)} \geq x)$ decays to zero like $x^{-2/(\alpha-1)-1}$.

The following result gives a Yaglom-type limit for $M_t^{(0,\infty)}$. Define

$$M_1^{(0,\infty),X} := \inf\{y \in \mathbb{R} : X_1^{(0,\infty)}((y,\infty)) = 0\}.$$

Theorem 1.7 *Assume* **(H1)**, **(H2)** *and* **(H4)** *hold.*

(i) For any $y > 0$, *it holds that*

$$\lim_{t\to\infty}\mathbb{P}_{\sqrt{t}y}\Big(\frac{M_t^{(0,\infty)}}{\sqrt{t}} \in \cdot\big|\zeta^{(0,\infty)} > t\Big) = \mathbb{N}_y\big(M_1^{(0,\infty),X} \in \cdot\big|w_1((0,\infty)) \neq 0\big).$$

(ii) Let η_1 be the random measure in Theorem 1.3(ii) and $M^{\eta_1} := \inf\{y \in \mathbb{R} : \eta_1((y,\infty)) = 0\}$. Then for any $y > 0$,

$$\lim_{t\to\infty} \mathbb{P}_y\Big(\frac{M_t^{(0,\infty)}}{\sqrt{t}} \in \cdot \Big| \zeta^{(0,\infty)} > t\Big) = \mathbb{P}(M^{\eta_1} \in \cdot).$$

We mention in passing here that the proof of Theorem 1.7 does not use the conclusions of Theorem 1.5. So, we only need **(H4)**, not the enhanced version of **(H4)** in Theorem 1.5(ii).

Theorem 1.7 (ii) is similar in spirit to Lalley and Shao [18, Theorem 3] for branching random walks. Let M_n be the maximal position of a critical branching random walk at time n. [18, Theorem 3] says that, conditioned on survival at time n, $M_n/\sqrt{n}$ converges in distribution to the maximum of the support of a random measure Y_1, where Y_1 is the conditional limit of a super-Brownian motion X such that for any nonnegative bounded continuous test function f,

$$\lim_{t\to\infty} \mathbb{P}_{\delta_0}\big(t^{-1}\langle f(\sqrt{t}\cdot), X_t\rangle \in \cdot | X_t \neq \mathbf{0}\big) = \mathbb{P}(\langle f, Y_1\rangle \in \cdot),$$

see [18, Proposition 21]. Theorem 1.7 (i) corresponds to Lalley and Shao [18, Theorem 3], and note that since there is killing at 0, to get the conditional limit as $t \to \infty$, the starting point needs to be at $\sqrt{t}y$.

1.4 Proof Strategies and Organization of the Paper

Now we sketch the main idea of the proof of Theorem 1.1. The main ideas for the proofs of Theorem 1.3 and Theorem 1.5 are similar, and Theorem 1.7 follows from Theorems 1.1 and 1.3. For $t > 0$, $s \geq 0$ and $y > 0$, let

$$v_\infty^{(t)}(s,y) := t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(Z_{ts}^{(0,\infty)}((0,\infty)) > 0\big) = t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\zeta^{(0,\infty)} > ts\big).$$

In Section 2.1, we derive an integral equation for $v_\infty^{(t)}(s,y)$. In Section 2.3, we use the Feynman-Kac formula to prove some analytical properties of $v_\infty^{(t)}(s,y)$ and show that, for any $s_0 \in (0,1)$, $\{v_\infty^{(t)}(s,y) : s \geq s_0, y > 0\}_{t\geq 1}$ is tight. Then we show that the limit $v_\infty^X(s,y)$ exists, is unique and can be represented via the superprocess X.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. The proofs of the main results are given in Section 3. The proofs of some auxiliary results used in Section 3 are given in Section 4.

In the remainder of this paper, the notation $f(x) \lesssim g(x)$ means that there exists some constant C independent of x such that $f(x) \leq Cg(x)$ holds for all x.

2 Preliminaries

Recall that $(\xi_t, \mathbf{P}_y)$ is a Lévy process starting from y, and for any function f on $(0,\infty)$, we automatically extend it to $\mathbb{R}$ by setting $f(x) = 0$ for all $x \leq 0$. For a random variable X and events A, B, we will use $\mathbb{E}(X; A)$ and $\mathbb{E}(X; A, B)$ to denote $\mathbb{E}(X1_A)$ and $\mathbb{E}(X1_A1_B)$ respectively.

2.1 Feynman-Kac Representation

Define

$$\phi(v) := \beta\Big(\sum_{k=0}^{\infty} p_k(1-v)^k - (1-v)\Big), \quad v \in [0,1].$$

Let L be a random variable with law $\{p_k\}$, then by our assumption, $\mathbb{E}L = m = 1$. By Jensen's inequality, we have $\phi(v) = \beta\left(\mathbb{E}\left((1-v)^L\right) - (1-v)\right) \geq \beta\left((1-v)^{\mathbb{E}L} - (1-v)\right) = 0$, which implies that ϕ is a non-negative function on $[0,1]$.

Lemma 2.1 *For any $f \in B_b^+((0,\infty))$,*

$$u_f(t,y) := \mathbb{E}_y\Big(\exp\Big\{-\int f(y) Z_t^{(0,\infty)}(\mathrm{d}y)\Big\}\Big), \quad t > 0, y \in \mathbb{R}_+,$$

solves the equation

$$u_f(t,y) = \mathbf{E}_y\Big(\exp\Big\{-f(\xi_{t\wedge\tau_0^-})\Big\}\Big) + \beta\mathbf{E}_y\Big(\int_0^t \Big(\sum_{k=0}^{\infty} p_k u_f(t-s,\xi_{s\wedge\tau_0^-})^k - u_f(t-s,\xi_{s\wedge\tau_0^-})\Big)\mathrm{d}s\Big).$$

Consequently, $v_f(t,y) := 1 - u_f(t,y)$ satisfies

$$v_f(t,y) = \mathbf{E}_y\Big(1-\exp\Big\{-f(\xi_{t\wedge\tau_0^-})\Big\}\Big) - \mathbf{E}_y\Big(\int_0^t \phi(v_f(t-s,\xi_{s\wedge\tau_0^-}))\mathrm{d}s\Big). \tag{2.1}$$

Proof It follows from Eq. 1.5 that, for any $f \in B_b^+((0,\infty))$,

$$u_f(t,y) = \mathbb{E}_y\Big(\exp\Big\{-\int f(y) Z_t^0(\mathrm{d}y)\Big\}\Big).$$

By considering the first splitting time of the branching Markov process Z_t^0, we get

$$u_f(t,y) = e^{-\beta t}\mathbf{E}_y\Big(\exp\Big\{-f(\xi_{t\wedge\tau_0^-})\Big\}\Big) + \beta\mathbf{E}_y\Big(\int_0^t e^{-\beta s}\sum_{k=0}^{\infty} p_k u_f(t-s,\xi_{s\wedge\tau_0^-})^k \mathrm{d}s\Big).$$

Now the first result follows from [6, Lemma 4.1]. Equation 2.1 follows from the first result and the definition of v_f. This completes the proof of the lemma. $\square$

For any $t > 0$, $s \geq 0$, $x, y \in \mathbb{R}$ and $v \in [0, t^{\frac{1}{\alpha-1}}]$, define $f_{(t)}(\cdot) := f\left(\frac{\cdot}{\sqrt{t}}\right)$ and

$$v_f^{(t)}(s,y) := t^{\frac{1}{\alpha-1}} v_{f_{(t)}}(ts, \sqrt{t}y), \quad \phi^{(t)}(v) := t^{\frac{\alpha}{\alpha-1}}\phi\left(vt^{-\frac{1}{\alpha-1}}\right), \quad \psi^{(t)}(v) := \frac{\phi^{(t)}(v)}{v} \tag{2.2}$$

and

$$\xi_s^{(t)} := \frac{\xi_{st}}{\sqrt{t}}, \quad \tau_x^{(t),+} := \inf\{s > 0 : \xi_s^{(t)} \geq x\} \quad \tau_x^{(t),-} := \inf\{s > 0 : \xi_s^{(t)} \leq x\}. \tag{2.3}$$

With a slight abuse of notation, we also use $\mathbf{P}_y$ to denote the law of $\{\xi_s^{(t)}, s \geq 0\}$ with $\xi_0^{(t)} = y$. Then $(\xi_{(tr)\wedge\tau_0^-}, \mathbf{P}_{\sqrt{t}y}) \overset{\mathrm{d}}{=} (\sqrt{t}\xi^{(t)}_{r\wedge\tau_0^{(t),-}}, \mathbf{P}_y)$.

Corollary 2.2 *For any $t > 0$, $y \in \mathbb{R}_+$ and $0 \le w \le r$, it holds that*

$$v_f^{(t)}(r, y) = \mathbf{E}_y\big(v_f^{(t)}(r - w, \xi^{(t)}_{w\wedge\tau_0^{(t),-}})\big) - \mathbf{E}_y\Big(\int_0^w \phi^{(t)}(v_f^{(t)}(r - s, \xi^{(t)}_{s\wedge\tau_0^{(t),-}}))\mathrm{d}s\Big). \quad (2.4)$$

Proof It follows from Eq. 2.1 and the Markov property that for any $y \in \mathbb{R}_+$ and $0 \le r \le t$,

$$v_f(t, y) = \mathbf{E}_y(v_f(r, \xi_{(t-r)\wedge\tau_0^-})) - \mathbf{E}_y\Big(\int_0^{t-r} \phi(v_f(t - s, \xi_{s\wedge\tau_0^-}))\mathrm{d}s\Big).$$

By the equality above with t replaced by tr, r replaced by $t(r - w)$ and f replaced by $f_{(t)}$, we get

$$\begin{aligned} v_f^{(t)}(r, y) &= t^{\frac{1}{\alpha-1}}\mathbf{E}_{\sqrt{t}y}\big(v_{f_{(t)}}(t(r - w), \xi_{(tw)\wedge\tau_0^-})\big) - t^{\frac{1}{\alpha-1}}\mathbf{E}_{\sqrt{t}y}\Big(\int_0^{tw} \phi(v_{f_{(t)}}(tr - s, \xi_{s\wedge\tau_0^-}))\mathrm{d}s\Big) \\ &= \mathbf{E}_y\big(v_f^{(t)}(r - w, \xi^{(t)}_{w\wedge\tau_0^{(t),-}})\big) - \mathbf{E}_y\Big(\int_0^w \phi^{(t)}(v_f^{(t)}(r - s, \xi^{(t)}_{s\wedge\tau_0^{(t),-}}))\mathrm{d}s\Big), \end{aligned}$$

where in the last equality we used the fact that $(\xi_{(tr)\wedge\tau_0^-}, \mathbf{P}_{\sqrt{t}y}) \overset{\mathrm{d}}{=} (\sqrt{t}\xi^{(t)}_{r\wedge\tau_0^{(t),-}}, \mathbf{P}_y)$. This completes the proof of the corollary. □

Taking $f = \theta 1_{(0,\infty)}(\cdot)$ in Eq. 2.4 and then letting $\theta \to +\infty$, Corollary 2.2 tells us that

$$v_\infty^{(t)}(r, y) := t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(Z_{tr}^{(0,\infty)}((0, \infty)) > 0\big) = t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\zeta^{(0,\infty)} > tr\big) \quad (2.5)$$

satisfies the following equation: for any $t > 0$, $y \in \mathbb{R}_+$ and $0 \le w \le r$,

$$v_\infty^{(t)}(r, y) = \mathbf{E}_y\big(v_\infty^{(t)}(r - w, \xi^{(t)}_{w\wedge\tau_0^{(t),-}})\big) - \mathbf{E}_y\Big(\int_0^w \phi^{(t)}(v_\infty^{(t)}(r - s, \xi^{(t)}_{s\wedge\tau_0^{(t),-}}))\mathrm{d}s\Big). \quad (2.6)$$

Proposition 2.3 *For any $t > 0$, $y \in \mathbb{R}_+$ and any $0 < w < r$, it holds that*

$$v_\infty^{(t)}(r, y) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^w \psi^{(t)}\big(v_\infty^{(t)}(r - s, \xi_s^{(t)})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r - w, \xi_s^{(t)}); \tau_0^{(t),-} > w\Big).$$

Also, for any $f \in B_b^+((0, \infty))$, it holds that

$$v_f^{(t)}(r, y) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^w \psi^{(t)}\big(v_f^{(t)}(r - s, \xi_s^{(t)})\big)\mathrm{d}s\Big\}v_f^{(t)}(r - w, \xi_w^{(t)}); \tau_0^{(t),-} > w\Big).$$

Proof For fixed $t, r, w > 0$, write $H(x) := v_\infty^{(t)}(r - w, x)$, $Q_H(s, y) := v_\infty^{(t)}(s + r - w, y)$, $\eta_w := \xi^{(t)}_{w\wedge\tau_0^{(t),-}}$ and $I(q) := \int_0^q \psi^{(t)}(Q_H(w - s, \eta_s))\mathrm{d}s$ for short. For any $n \in \mathbb{N}$, iterating Eq. 2.6 n times and applying the Markov property, we get that

$$\begin{aligned} &Q_H(w, y) \\ &= \mathbf{E}_y\left(H(\eta_w)\right) - \mathbf{E}_y\Big(\int_0^w I'(q)\Big(\mathbf{E}_{\eta_s}(H(\eta_{w-s})) - \mathbf{E}_{\eta_s}\Big(\int_0^{w-s} \phi^{(t)}(Q_H(w - s - q, \xi_q))\mathrm{d}q\Big)\Big)\mathrm{d}s\Big) \\ &= \mathbf{E}_x\left(\sum_{i=0}^{1}\frac{(-I(w))^i}{i!}H(\eta_w)\right) + \mathbf{E}_y\Big(\int_0^w I'(q)I(q)Q_H(w - q, \eta_q)\mathrm{d}q\Big) \\ &= \cdots = \mathbf{E}_x\left(\sum_{i=0}^{n}\frac{(-I(w))^i}{i!}H(\eta_w)\right) + (-1)^{n+1}\mathbf{E}_y\Big(\int_0^w \frac{I^n(q)}{n!}I'(q)Q_H(w - q, \eta_q)\mathrm{d}q\Big). \end{aligned}$$

Since $0 \le H(x), Q_H(q,x) \le t^{\frac{1}{\alpha-1}}$ for all $x \ge 0, q \le w$ and $\psi^{(t)} \ge 0$ is locally bounded, we see that $\sup_{q \le w} |I(q)| < \infty$. Therefore, letting $n \uparrow \infty$ in the above equality, we obtain

$$v_\infty^{(t)}(r,y) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^w \psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r-w,\xi^{(t)}_{w\wedge\tau_0^{(t),-}})\Big).$$

Since $v_\infty^{(t)}(r-w,y) = 0$ for all $y \le 0$ and $0 < w < r$, we get the first result. The case for $v_f^{(t)}$ is similar. □

Remark 2.4 By the Markov property of $\xi^{(t)}_{r\wedge\tau_0^{(t),-}}$, for $y > 0$ and $w \in [0,r]$, it holds that

$$\begin{aligned}\Upsilon_w &:= \exp\Big\{-\int_0^w \psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r-w,\xi^{(t)}_{w\wedge\tau_0^{(t),-}})\\ &= \mathbf{E}_y\Big(\exp\Big\{-\int_0^r \psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(0,\xi^{(t)}_{r\wedge\tau_0^{(t),-}})\Big|\xi^{(t)}_{s\wedge\tau_0^{(t),-}} : s \le w\Big).\end{aligned}$$

Hence, $\{\Upsilon_w : w \in [0,r]\}$ is a $\mathbf{P}_y$-martingale. Thus, for any stopping time T of the Lévy process $\xi_s^{(t)}$ and any $t > 0$, $0 < w < r$, we have

$$\begin{aligned}&v_\infty^{(t)}(r,y) = \mathbf{E}_y\left(\Upsilon_{w\wedge T}\right)\\ &= \mathbf{E}_y\Big(\exp\Big\{-\int_0^{w\wedge T} \psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r-w\wedge T,\xi^{(t)}_{w\wedge T});\tau_0^{(t),-} > w\wedge T\Big).\end{aligned} \tag{2.7}$$

For any $0 < y < x$, define

$$v(y;x) := \mathbb{P}_y(M^{(0,\infty)} \ge x).$$

Proposition 2.5 *For any $0 < y < x$, it holds that*

$$v(y;x) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_x^+} \psi(v(\xi_s;x))\mathrm{d}s\Big\};\tau_x^+ < \tau_0^-\Big) \tag{2.8}$$

where

$$\psi(v) := \frac{\phi(v)}{v} = \frac{\beta\left(\sum_{k=0}^\infty p_k(1-v)^k - (1-v)\right)}{v}, \quad v \in [0,1].$$

Consequently, for $0 < y < z < x$, by the strong Markov property, we have

$$v(y;x) = \mathbf{E}_y\Big(v(\xi_{\tau_z^+};x)\exp\Big\{-\int_0^{\tau_z^+} \psi(v(\xi_s;x))\mathrm{d}s\Big\};\tau_z^+ < \tau_0^-\Big), \tag{2.9}$$

Proof Assume $0 < y < x$. Comparing the first branching time $\mathbf{e}$ with τ_0^-, we get

$$\begin{aligned}v(y;x) =&\mathbb{P}_y(M^{(0,\infty)} \ge x, \mathbf{e} \ge \tau_0^-) + \mathbb{P}_y(M^{(0,\infty)} \ge x, \mathbf{e} < \tau_0^-)\\ =&\int_0^\infty \beta e^{-\beta s}\mathbf{P}_y\left(\tau_x^+ < \tau_0^-, \tau_x^+ \le s\right)\mathrm{d}s\\ &+\int_0^\infty \beta e^{-\beta s}\mathbf{E}_y\Big(\Big(1-\sum_{k=0}^\infty p_k\,(1-v(\xi_s;x))^k\Big);s < \tau_0^- \wedge \tau_x^+\Big)\mathrm{d}s\\ =&\mathbf{E}_y\left(e^{-\beta\tau_x^+};\tau_x^+ < \tau_0^-\right) + \int_0^\infty \beta e^{-\beta s}\mathbf{E}_y\Big(\Big(1-\sum_{k=0}^\infty p_k\,(1-v(\xi_s;x))^k\Big);s < \tau_0^- \wedge \tau_x^+\Big)\mathrm{d}s.\end{aligned}$$

By [6, Lemma 4.1], the above equation is equivalent to

$$\begin{aligned} &v(y;x) + \beta \int_0^\infty \mathbf{E}_y\left(v(\xi_s;x); s < \tau_0^- \wedge \tau_x^+\right) \mathrm{d}s \\ &= \mathbf{P}_y\left(\tau_x^+ < \tau_0^-\right) + \beta \int_0^\infty \mathbf{E}_y\Big(1 - \sum_{k=0}^\infty p_k\Big(1 - v(\xi_s;x)\Big)^k; s < \tau_x^+ \wedge \tau_0^-\Big)\mathrm{d}s, \end{aligned}$$

which is also equivalent to

$$\begin{aligned} v(y;x) &= \mathbf{P}_y\left(\tau_x^+ < \tau_0^-\right) - \beta \int_0^\infty \mathbf{E}_y\Big(\sum_{k=0}^\infty p_k\,(1 - v(\xi_s;x))^k - (1 - v(\xi_s;x)); s < \tau_x^+ \wedge \tau_0^-\Big)\mathrm{d}s \\ &= \mathbf{P}_y\left(\tau_x^+ < \tau_0^-\right) - \mathbf{E}_y\Big(\int_0^{\tau_x^+ \wedge \tau_0^-} \psi(v(\xi_s;x))v(\xi_s;x)\mathrm{d}s\Big). \end{aligned}$$

Since $\psi(v) \geq 0$ for all $v \in [0,1]$, by the Feynman-Kac formula, we have

$$\begin{aligned} v(y;x) &= \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_x^+ \wedge \tau_0} \psi(v(\xi_s;x))\mathrm{d}s\Big\}; \tau_x^+ < \tau_0^-\Big) \\ &= \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_x^+} \psi(v(\xi_s;x))\mathrm{d}s\Big\}; \tau_x^+ < \tau_0^-\Big), \end{aligned}$$

which gives Eq. 2.8. □

For any $x > 0$ and $y \in \mathbb{R}_+$, define

$$K^{(x)}(y) := x^{\frac{2}{\alpha-1}} v(xy;x) = x^{\frac{2}{\alpha-1}} \mathbb{P}_{xy}(M^{(0,\infty)} \geq x). \tag{2.10}$$

Then $K^{(x)}(0) = 0$ and that $K^{(x)}(y) = x^{\frac{2}{\alpha-1}}$ when $y \geq 1$.

Lemma 2.6 *For every $x > 0$ and $0 < y < z < 1$, it holds that*

$$K^{(x)}(y) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{(x^2),+}} \psi^{(x^2)}\big(K^{(x)}\big(\xi_s^{(x^2)}\big)\big)\mathrm{d}s\Big\} K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big); \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big).$$

Proof By the definition of $\tau_x^{(t),+}$ in Eq. 2.3, $\big(x^2\tau_z^{(x^2),+}, \mathbf{P}_y\big) \overset{\mathrm{d}}{=} \big(\tau_{xz}^+, \mathbf{P}_{xy}\big)$. In fact, for any $a > 0$,

$$\mathbf{P}_y\big(x^2\tau_z^{(x^2),+} > a\big) = \mathbf{P}_y\big(\sup_{x^2 w < a} \xi_w^{(x^2)} < z\big) = \mathbf{P}_{xy}\big(\sup_{w<a} \xi_w < zx\big) = \mathbf{P}_{xy}\big(\tau_{xz}^+ > a\big). \tag{2.11}$$

Therefore, combining Eq. 2.9 and the definition of $\xi^{(t)}$ in Eq. 2.3, we get that

$$\begin{aligned}
K^{(x)}(y) &= x^{\frac{2}{\alpha-1}}\mathbf{E}_{xy}\Big(\exp\Big\{-\int_0^{\tau_{xz}^+}\psi(v(\xi_s;x))\mathrm{d}s\Big\}v(\xi_{\tau_{xz}^+};x);\,\tau_{xz}^+<\tau_0^-\Big)\\
&= x^{\frac{2}{\alpha-1}}\mathbf{E}_y\Big(\exp\Big\{-\int_0^{x^2\tau_z^{(x^2),+}}\psi(v(x\xi^{(x^2)}_{sx^{-2}};x))\mathrm{d}s\Big\}v\big(x\xi^{(x^2)}_{\tau_z^{(x^2),+}};x\big);\,\tau_z^{(x^2),+}<\tau_0^{(x^2),-}\Big)\\
&= \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{(x^2),+}}x^2\psi(v(x\xi_s^{(x^2)};x))\mathrm{d}s\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\,\tau_z^{(x^2),+}<\tau_0^{(x^2),-}\Big)\\
&= \mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{(x^2),+}}\psi^{(x^2)}\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\,\tau_z^{(x^2),+}<\tau_0^{(x^2),-}\Big),
\end{aligned}$$

where in the last equality we used the fact that

$$x^2\psi(vx^{-\frac{2}{\alpha-1}}) = x^2\frac{\phi(vx^{-\frac{2}{\alpha-1}})}{vx^{-\frac{2}{\alpha-1}}} = x^{\frac{2\alpha}{\alpha-1}}\frac{\phi(vx^{-\frac{2}{\alpha-1}})}{v} = \psi^{(x^2)}(v) \tag{2.12}$$

and the definition of $\psi^{(t)}$ in Eq. 2.2. This completes the proof. □

2.2 Some Useful Properties of Lévy Processes

In this subsection, we always assume that the Lévy process fulfills **(H2)**.

Lemma 2.7 *If* $\mathbf{E}_0\left(((-\xi_1)\vee 0)^\lambda\right) < \infty$ *for some* $\lambda > 2$*, then*

$$\sup_{x>0}\mathbf{E}_x\left(\left|\xi_{\tau_0^-}\right|^{\lambda-2}\right) < \infty.$$

If $\mathbf{E}_0\left((\xi_1\vee 0)^\lambda\right) < \infty$ *for some* $\lambda > 2$*, then*

$$\sup_{x>0}\mathbf{E}_{-x}\left(\xi_{\tau_0^+}^{\lambda-2}\right) < \infty.$$

Proof For the first result, see [9, Lemma 2.1]. The second result follows by the first result with ξ replaced by $-\xi$. □

Lemma 2.8 *(i) For any* $x > 0$*, it holds that* $\mathbf{E}_x|\xi_{\tau_0^-}| < \infty$ *and*

$$\lim_{x\to\infty}\frac{R(x)}{x} = 1 - \lim_{x\to\infty}\frac{\mathbf{E}_x\left(\xi_{\tau_0^-}\right)}{x} = 1. \tag{2.13}$$

Furthermore, $R(x) \lesssim x + 1$*.*

(ii) R *is harmonic with respect to* $\xi_{t\wedge\tau_0^-}$*, that is,*

$$R(x) = \mathbf{E}_x\left(R(\xi_s);\,\tau_0^- > s\right),\quad s > 0, x > 0.$$

Proof (i) The first equality in Eq. 2.13 is an immediate consequence of the definition of $R(x)$, so we only prove the second equality of Eq. 2.13. We will use the decomposition introduced in [4, p.208]. Suppose that π^ξ is the Lévy measure of ξ. If $\pi^\xi(|x| > 1) = 0$, then $\mathbf{E}_0(|\xi_1|^3) < \infty$, which implies the boundness of $\mathbf{E}_x\left(|\xi_{\tau_0^-}|\right)$ according to Lemma 2.7. Now assume that $\pi^\xi(|x| > 1) > 0$. Let σ_n be the n-th time that ξ has a jump of magnitude larger than 1, and put $\sigma_0 = 0$, then $\{\sigma_n - \sigma_{n-1}, n \geq 1\}$ are iid exponential random variables with parameter $\pi^\xi(\{|x| > 1\})$. We can define a random walk $\hat{Z}_n$ given in [4, p. 208]:

$$\hat{Z}_n = \xi_{\sigma_n}, \quad n \geq 1, \quad \text{and} \quad \hat{Z}_0 = \xi_0.$$

Similar to [9, (2.4) and (2.5)], under **(H2)**, $\mathbf{E}_0(\hat{Z}_1) = 0$ and $\mathbf{E}_0\left(\hat{Z}_1^2\right) < \infty$. For $x \geq 0$, define

$$R^Z(x) := x - \mathbf{E}_x\big(\hat{Z}_{\tau_0^{Z,-}}\big),$$

where $\tau_0^{Z,-} := \inf\{n : \hat{Z}_n < 0\}$. It is well-known that, under the assumption **(H2)**, $\mathbf{E}_x|\hat{Z}_{\tau_0^{Z,-}}| < \infty$. Using a martingale argument for the ladder heights process of $\hat{Z}$, we know that (for example, see [10, (3.4) and (3.6)])

$$\lim_{x\to\infty} \frac{R^Z(x)}{x} = 1,$$

which is equivalent to

$$\lim_{x\to\infty} \frac{\mathbf{E}_x\big(\big|\hat{Z}_{\tau_0^{Z,-}}\big|\big)}{x} = 0. \tag{2.14}$$

By [4, p.209], for any $z > 1$ and any $x > 0$,

$$\mathbf{P}_x\big(|\xi_{\tau_0^-}| > z\big) \leq \mathbf{P}_x\big(\big|\hat{Z}_{\tau_0^{Z,-}}\big| > z\big). \tag{2.15}$$

Combining Eqs. 2.14 and 2.15, we conclude that

$$\begin{aligned} \frac{1}{x}\mathbf{E}_x\big(|\xi_{\tau_0^-}|\big) &\leq \frac{1}{x} + \frac{1}{x}\int_1^\infty \mathbf{P}_x\big(|\xi_{\tau_0^-}| > z\big)\mathrm{d}z \\ &\leq \frac{1}{x} + \frac{1}{x}\int_0^\infty \mathbf{P}_x\big(|\hat{Z}_{\tau_0^{Z,-}}| > z\big)\mathrm{d}z = \frac{1 + \mathbf{E}_x\big(\big|\hat{Z}_{\tau_0^{Z,-}}\big|\big)}{x} \xrightarrow{x\to\infty} 0. \end{aligned}$$

The last assertion of (i) follows from Eq. 2.13 and the monotonicity of R.

(ii) Note that

$$x = \mathbf{E}_x\big(\xi_{\tau_0^-\wedge t}\big) = \mathbf{E}_x\big(\xi_t; \tau_0^- > t\big) + \mathbf{E}_x\big(\xi_{\tau_0^-}; \tau_0^- < t\big).$$

Letting $t \to \infty$ in the above equation, using the definition of $R(x)$ and the Markov property, we have

$$\begin{aligned} R(x) &= \lim_{t\to\infty} \mathbf{E}_x\left(\xi_t; \tau_0^- > t\right) = \lim_{t\to\infty} \mathbf{E}_x\left(\xi_{t+s}; \tau_0^- > t+s\right) \\ &= \lim_{t\to\infty} \mathbf{E}_x\left(\mathbf{E}_{\xi_s}\left(\xi_t; \tau_0^- > t\right); \tau_0^- > s\right) = \mathbf{E}_x\left(R(\xi_s), \tau_0^- > s\right), \end{aligned}$$

where in the last equality, we used dominated convergence theorem and the fact

$$|\mathbf{E}_x\left(\xi_t; \tau_0^- > t\right)| = |x - \mathbf{E}_x(\xi_{\tau_0^-}; \tau_0^- \leq t)| \leq x + R(x) \lesssim x + 1, x > 0.$$

The proof is complete. $\square$

Remark 2.9 It follows from Lemma 2.8(i) that, under **(H2)**, $\mathbf{E}_y(|\xi_{\tau_0^-}|) \lesssim y+1$ for any $y > 0$. Similarly, replacing ξ by $-\xi$, we see that

$$\mathbf{E}_{-y}\big(\xi_{\tau_0^+}\big) \lesssim y + 1, \quad \text{for all} \quad y > 0, \quad \text{and} \quad \lim_{y\to\infty} \frac{\mathbf{E}_{-y}\big(\xi_{\tau_0^+}\big)}{y} = 0.$$

Recall the definitions $\xi^{(t)}$, $\tau^{(t),+}$ and $\tau^{(t),-}$ in Eq. 2.3.

Lemma 2.10 *(i) For any* $y, s, t > 0$,

$$\mathbf{P}_y\left(\tau_0^{(t),-} > s\right) \lesssim \frac{\sqrt{t}y+1}{\sqrt{st}} \quad \textit{and} \quad \mathbf{P}_0\left(\tau_y^{(t),+} > s\right) \lesssim \frac{\sqrt{t}y+1}{\sqrt{st}}. \tag{2.16}$$

(ii) For any $0 < y < z$ *and any* $t > 0$,

$$\mathbf{P}_y\left(\tau_0^{(t),-} \le \tau_z^{(t),+}\right) \lesssim \frac{\sqrt{t}(z-y)+1}{\sqrt{t}z}.$$

Proof (i) By definition of $\tau_0^{(t),-}$, we have

$$\mathbf{P}_y\big(\tau_0^{(t),-} > s\big) = \mathbf{P}_y\big(\inf_{\ell \le s} \xi_\ell^{(t)} > 0\big) = \mathbf{P}_{\sqrt{t}y}\big(\inf_{\ell \le st} \xi_\ell > 0\big).$$

Now set $S_n := \xi_n$ for $n \in \mathbb{N}$. We use the trivial upper bound 1 in the case $st \le 1$. Now we assume that $st > 1$, then

$$\mathbf{P}_y\big(\tau_0^{(t),-} > s\big) \le \mathbf{P}_{\sqrt{t}y}\big(\inf_{j \le [st]} S_j > 0\big) \lesssim \frac{\sqrt{t}y+1}{\sqrt{[st]}} \le 2\frac{\sqrt{t}y+1}{\sqrt{st}},$$

where we used [1, (2.7)] in the second inequality above. For the second inequality in Eq. 2.16, noticing that

$$\mathbf{P}_0\big(\tau_y^{(t),+} > s\big) = \mathbf{P}_0\big(\sup_{\ell \le st} \xi_\ell < \sqrt{t}y\big) = \mathbf{P}_{\sqrt{t}y}\big(\inf_{\ell \le st} (-\xi_\ell) > 0\big),$$

an argument similar to that used to prove the first inequality in Eq. 2.16 with ξ replaced by $-\xi$ leads to the desired assertion.

(ii) By the definitions of $\tau_0^{(t),-}$ and $\tau_z^{(t),+}$,

$$\mathbf{P}_y\big(\tau_0^{(t),-} \le \tau_z^{(t),+}\big) = \mathbf{P}_{\sqrt{t}y}\big(\tau_0^- \le \tau_{\sqrt{t}z}^+\big).$$

Since $(\xi_s, \mathbf{P}_{\sqrt{t}y})$ is a martingale with mean $\sqrt{t}y$, it holds that

$$\sqrt{t}y = \mathbf{E}_{\sqrt{t}y}\big(\xi_{s\wedge\tau_0^-\wedge\tau_{\sqrt{t}z}^+}\big) = \mathbf{E}_{\sqrt{t}y}\big(\xi_s; s < \tau_0^- \wedge \tau_{\sqrt{t}z}^+\big) + \mathbf{E}_{\sqrt{t}y}\big(\xi_{\tau_0^-\wedge\tau_{\sqrt{t}z}^+}; s \ge \tau_0^- \wedge \tau_{\sqrt{t}z}^+\big). \tag{2.17}$$

According to Remark 2.9, we have

$$\begin{aligned}\mathbf{E}_{\sqrt{t}y}\big(|\xi_{\tau_0^-\wedge\tau_{\sqrt{t}z}^+}|\big) &\le \mathbf{E}_{\sqrt{t}y}\big(|\xi_{\tau_0^-}|\big) + \mathbf{E}_{\sqrt{t}y}\big(\xi_{\tau_{\sqrt{t}z}^+}\big)\\ &= \mathbf{E}_{\sqrt{t}y}\big(|\xi_{\tau_0^-}|\big) + \sqrt{t}z + \mathbf{E}_{\sqrt{t}y-\sqrt{t}z}\big(\xi_{\tau_0^+}\big) < \infty.\end{aligned}$$

Noticing that $|\xi_s| \le \sqrt{t}z$ when $s < \tau_0^- \wedge \tau_{\sqrt{t}z}^+$, taking $s \to \infty$ in Eq. 2.17, we get

$$\sqrt{t}y = \mathbf{E}_{\sqrt{t}y}\big(\xi_{\tau_0^-\wedge\tau_{\sqrt{t}z}^+}\big),$$

which implies

$$\mathbf{E}_{\sqrt{t}y}\big(\xi_{\tau_{\sqrt{t}z}^+}\big) - \sqrt{t}y = \mathbf{E}_{\sqrt{t}y}\big((\xi_{\tau_{\sqrt{t}z}^+} - \xi_{\tau_0^-}); \tau_0^- \le \tau_{\sqrt{t}z}^+\big). \tag{2.18}$$

By Remark 2.9, we conclude that

$$\mathbf{P}_y\big(\tau_0^{(t),-} \le \tau_z^{(t),+}\big) = \mathbf{P}_{\sqrt{t}y}\big(\tau_0^- \le \tau_{\sqrt{t}z}^+\big) \le \frac{1}{\sqrt{t}z}\mathbf{E}_{\sqrt{t}y}\big((\xi_{\tau_{\sqrt{t}z}^+} - \xi_{\tau_0^-}); \tau_0^- \le \tau_{\sqrt{t}z}^+\big)$$
$$= \frac{\mathbf{E}_{\sqrt{t}y}\big(\xi_{\tau_{\sqrt{t}z}^+}\big) - \sqrt{t}y}{\sqrt{t}z} = \frac{\sqrt{t}(z-y) + \mathbf{E}_{-(\sqrt{t}(z-y))}\big(\xi_{\tau_0^+}\big)}{\sqrt{t}z} \lesssim \frac{\sqrt{t}(z-y)+1}{\sqrt{t}z},$$

which completes the proof of (ii). □

Let S_n be the random walk defined by $S_n = \xi_n, n \in \mathbb{N}$. Sakhanenko [27] proved that (see also [8, Lemma 3.6]) under the assumption

$$\mathbf{E}_0\big(|\xi_1|^{2+\delta}\big) < \infty \quad \text{for some} \quad \delta > 0, \tag{2.19}$$

we can find a Brownian motion W_t with variance $\sigma^2 t$, starting from the origin, such that for any $\gamma \in (0, \frac{\delta}{2(2+\delta)})$ and any $t > 1$, there exists a constant $N_*(\gamma) > 1$ such that

$$\mathbf{P}_0\Big(\sup_{0\le s\le 1}\big|S_{[ts]} - W_{ts}\big| > \frac{1}{2}t^{\frac{1}{2}-\gamma}\Big) \le \frac{N_*(\gamma)}{t^{\delta/2-(2+\delta)\gamma}} = \frac{N_*(\gamma)}{t^{(\frac{1}{2}-\gamma)(\delta+2)-1}}. \tag{2.20}$$

By comparing S_n with ξ_t for $t \in [n, n+1]$, we immediately have the following result.

Lemma 2.11 *Assume that the inequality 2.19 holds. Let W_s be the Brownian motion in the inequality 2.20, then for any $\gamma \in (0, \frac{\delta}{2(2+\delta)})$ and any $t > 1$,*

$$\mathbf{P}_0\Big(\sup_{0\le s\le 1}|\xi_{ts} - W_{ts}| > t^{\frac{1}{2}-\gamma}\Big) \lesssim \frac{N_*(\gamma)}{t^{(\frac{1}{2}-\gamma)(\delta+2)-1}}.$$

Proof By Doob's inequality,

$$\mathbf{P}_0\Big(\sup_{0\le s\le 1}\big|\xi_{ts} - \xi_{[ts]}\big| > \frac{1}{2}t^{\frac{1}{2}-\gamma}\Big) \le \lceil t\rceil \mathbf{P}_0\Big(\sup_{0\le s\le 1}|\xi_s| > \frac{1}{2}t^{\frac{1}{2}-\gamma}\Big)$$
$$\le \lceil t\rceil \frac{2^{2+\delta}}{t^{(\frac{1}{2}-\gamma)(2+\delta)}}\mathbf{E}_0\Big(|\xi_1|^{2+\delta}\Big) \lesssim \frac{1}{t^{(\frac{1}{2}-\gamma)(2+\delta)-1}}.$$

Combining this with Eq. 2.20, we immediately get the desired result. □

Recall that the function R is defined in Eq. 1.15. The following result for Lévy processes is analogous to the corresponding result for random walks proved in [8, Theorem 2.9]. We postpone its proof to Section 4.1.

Lemma 2.12 *Assume that the inequality 2.19 holds. For any $y > 0$ and any bounded continuous function f on $(0, \infty)$, it holds that*

$$\lim_{t\to\infty}\sqrt{t}\mathbf{E}_y\Big(f\Big(\frac{\xi_t}{\sigma\sqrt{t}}\Big)1_{\{\tau_0^- > t\}}\Big) = \frac{2R(y)}{\sqrt{2\pi\sigma^2}}\int_0^\infty z e^{-\frac{z^2}{2}} f(z)\mathrm{d}z.$$

Consequently, for any $r > 0$ and any bounded continuous function f on $(0, \infty)$,

$$\lim_{t\to\infty}\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\Big(f\big(\xi_r^{(t)}\big)1_{\big\{\tau_0^{(t),-} > r\big\}}\Big) = \lim_{t\to\infty}\sqrt{t}\mathbf{E}_y\Big(f\Big(\frac{\xi_{tr}}{\sqrt{t}}\Big)1_{\{\tau_0^- > tr\}}\Big)$$
$$= \frac{1}{\sqrt{r}}\frac{2R(y)}{\sqrt{2\pi\sigma^2}}\int_0^\infty z e^{-\frac{z^2}{2}} f(z\sigma\sqrt{r})\mathrm{d}z.$$

Since $\xi_t^2 - \sigma^2 t$ is a martingale, under the assumption $\mathbf{E}_0(|\xi_1|^4) < \infty$, by the optional stopping theorem and Lemma 2.7, we have

$$y^2 + \sigma^2 \mathbf{E}_y\left(\tau_0^- \wedge \tau_x^+\right) = \mathbf{E}_y\left(\left(\xi_{\tau_0^- \wedge \tau_x^+}\right)^2\right) \le \mathbf{E}_y\left(\left(\xi_{\tau_0^-}\right)^2\right) + \mathbf{E}_y\left(\left(\xi_{\tau_x^+}\right)^2\right) < \infty. \tag{2.21}$$

The exit time estimates in the next result will be used to prove Eqs. 3.26–3.28. The requirement for the $(4+\varepsilon)$th moment of ξ is to ensure that the $(2+\varepsilon)$th moment of the overshoots $\xi_{\tau_0^-}$ and $\xi_{\tau_x^+}$ are finite, see Eq. 2.26 below. Note that when **(H4)** holds, the condition of Lemma 2.13 is fulfilled because that for $\alpha \in (1,2]$, we have $r_0 > (2\alpha)/(\alpha-1) = 2+2/(\alpha-1) \ge 2+2 = 4$, which implies that we can take $\varepsilon_0 = r_0 - 4 > 0$ in the following lemma.

Lemma 2.13 *Assume that* $\mathbf{E}_0(|\xi_1|^{4+\varepsilon_0}) < \infty$ *for some* $\varepsilon_0 > 0$.

(i) For any $y, z > 0$,

$$\lim_{x\to\infty} x\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-\right) = \lim_{x\to\infty} x\mathbf{P}_{yx^{-1}}\left(\tau_z^{(x^2),+} < \tau_0^{(x^2),-}\right) = \frac{R(y)}{z}. \tag{2.22}$$

(ii) For any $y > 0$, *there exists* $C > 0$ *such that for any* $z > 0$ *and any* $x > \max\{1, \frac{y}{z}\}$,

$$x\mathbf{E}_{yx^{-1}}\left(\tau_z^{(x^2),+}; \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\right) = \frac{1}{x}\mathbf{E}_y\left(\tau_{xz}^+; \tau_{xz}^+ < \tau_0^-\right) \le Cxz^2\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-\right) + \frac{C}{x}. \tag{2.23}$$

Proof (i) The first equality in Eq. 2.22 follows immediately from the definition of $\xi_t^{(x^2)}$, so we only need prove that the first limit in Eq. 2.22 is equal to the right hand side of Eq. 2.22. According to Eq. 2.18 and the definition of $R(y)$, we have

$$\begin{aligned} R(y) &= y - \mathbf{E}_y\left(\xi_{\tau_0^-}\right) = \mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right); \tau_{xz}^+ < \tau_0^-\right) \\ &\ge xz\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-\right). \end{aligned} \tag{2.24}$$

On the other hand, for any $\delta > 0$,

$$\begin{aligned} R(y) &= \mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right); \tau_{xz}^+ < \tau_0^-\right) \\ &\le (z+2\delta)x\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-, \xi_{\tau_0^-} > -\delta x, \xi_{\tau_{xz}^+} < (z+\delta)x\right) \\ &\quad + \mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right)\left(1_{\{\xi_{\tau_0^-} \le -\delta x\}} + 1_{\{\xi_{\tau_{xz}^+} \ge (z+\delta)x\}}\right); \tau_{xz}^+ < \tau_0^-\right) \\ &\le (z+2\delta)x\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-\right) \\ &\quad + \sqrt{2}\sqrt{\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right)^2\right)}\sqrt{\mathbf{P}_y(\xi_{\tau_0^-} \le -\delta x) + \mathbf{P}_y\left(\xi_{\tau_{xz}^+} \ge (z+\delta)x\right)}. \end{aligned} \tag{2.25}$$

Note that, by Lemma 2.7, for any fixed $y, z > 0$,

$$\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right)^2\right) \le 2\mathbf{E}_y\left(\xi_{\tau_{xz}^+}^2 + \xi_{\tau_0^-}^2\right) = 2\left(\mathbf{E}_{y-xz}\left(\xi_{\tau_0^+} + xz\right)^2 + \mathbf{E}_y\xi_{\tau_0^-}^2\right) \lesssim x^2.$$

By Markov's inequality, and using Lemma 2.7 again, we have that

$$\begin{aligned} &\sqrt{\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^+} - \xi_{\tau_0^-}\right)^2\right)}\sqrt{\mathbf{P}_y(\xi_{\tau_0^-} \le -\delta x) + \mathbf{P}_y\left(\xi_{\tau_{xz}^+} \ge (z+\delta)x\right)} \\ &\lesssim \sqrt{x^2}\sqrt{\frac{1}{(\delta x)^{2+\varepsilon_0}}\mathbf{E}_y\left(|\xi_{\tau_0^-}|^{2+\varepsilon_0}\right) + \frac{1}{(\delta x)^{2+\varepsilon_0}}\mathbf{E}_{-xz+y}\left(\xi_{\tau_0^+}^{2+\varepsilon_0}\right)} \\ &\lesssim x^{-\varepsilon_0/2}. \end{aligned} \tag{2.26}$$

Putting Eq. 2.26 into Eq. 2.25 and applying Eq. 2.24, we obtain

$$\frac{R(y)}{z+2\delta} \leq \liminf_{x\to\infty} x\mathbf{P}_y\left(\tau_{xz}^{+} < \tau_0^{-}\right) \leq \limsup_{x\to\infty} x\mathbf{P}_y\left(\tau_{xz}^{+} < \tau_0^{-}\right) \leq \frac{R(y)}{z}.$$

Since δ is arbitrary, we arrive at the desired result.

(ii) The first equality in Eq. 2.23 follows immediately from the definition of $\xi_t^{(x^2)}$, so we only need prove the inequality in Eq. 2.23. According to Eq. 2.21,

$$\begin{aligned}\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^{+}\wedge\tau_0^{-}}\right)^2\right) &= \sigma^2\mathbf{E}_y\left(\tau_{xz}^{+};\tau_{xz}^{+} < \tau_0^{-}\right) + \sigma^2\mathbf{E}_y\left(\tau_0^{-};\tau_0^{-} < \tau_{xz}^{+}\right) + y^2\\ &\geq \sigma^2\mathbf{E}_y\left(\tau_{xz}^{+};\tau_{xz}^{+} < \tau_0^{-}\right).\end{aligned}\tag{2.27}$$

For the left-hand side of Eq. 2.27, by Lemma 2.7,

$$\begin{aligned}\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^{+}\wedge\tau_0^{-}}\right)^2\right) &= \mathbf{E}_y\left(\left(\xi_{\tau_{xz}^{+}}\right)^2;\tau_{xz}^{+} < \tau_0^{-}\right) + \mathbf{E}_y\left(\left(\xi_{\tau_0^{-}}\right)^2;\tau_0^{-} < \tau_{xz}^{+}\right)\\ &\leq 2\mathbf{E}_y\left(\left(\xi_{\tau_{xz}^{+}} - xz\right)^2;\tau_{xz}^{+} < \tau_0^{-}\right) + 2x^2z^2\mathbf{P}_y\left(\tau_{xz}^{+} < \tau_0^{-}\right) + \mathbf{E}_y\left(\left(\xi_{\tau_0^{-}}\right)^2\right)\\ &\lesssim x^2z^2\mathbf{P}_y\left(\tau_{xz}^{+} < \tau_0^{-}\right) + 1.\end{aligned}$$

Plugging this upper bound back to Eq. 2.27, we conclude that

$$\frac{1}{x}\mathbf{E}_y\left(\tau_{xz}^{+};\tau_{xz}^{+} < \tau_0^{-}\right) \lesssim xz^2\mathbf{P}_y\left(\tau_{xz}^{+} < \tau_0^{-}\right) + \frac{1}{x}.$$

This implies the result of the lemma. □

2.3 Preliminary Estimates for the Survival Probability

Recall that $\psi^{(t)}$ is defined in Eq. 2.2.

Lemma 2.14 *Assume that* **(H1)** *and* **(H2)** *hold.*
(i) For any $t, r, y > 0$, it holds that

$$v_\infty^{(t)}(r,y) \lesssim \frac{1}{r^{\frac{1}{\alpha-1}}} \wedge \frac{\sqrt{t}y+1}{r^{\frac{1}{\alpha-1}+\frac{1}{2}}\sqrt{t}}.$$

(ii) For any $r > 0$ and $y \in \mathbb{R}_+$, it holds that

$$\psi^{(t)}\left(v_\infty^{(t)}(r,y)\right) \lesssim \frac{1}{r},$$

and that for each $K > 0$, uniformly for $v \in [0, K]$,

$$\lim_{t\to\infty}\frac{\psi^{(t)}(v)}{v^{\alpha-1}} = \mathcal{C}(\alpha),$$

where $\mathcal{C}(\alpha)$ is given in Eq. 1.9.

Proof (i) Recall the definitions of ζ, $\zeta^{(0,\infty)}$ and $v_\infty^{(t)}(r,y)$ in Eqs. 1.1, 1.6 and 2.5 respectively. Since $\zeta^{(0,\infty)} \leq \zeta$, by Eq. 1.2,

$$v_\infty^{(t)}(r,y) \leq t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}(\zeta > tr) \lesssim t^{\frac{1}{\alpha-1}}\frac{1}{(tr)^{\frac{1}{\alpha-1}}} = \frac{1}{r^{\frac{1}{\alpha-1}}}.\tag{2.28}$$

On the other hand, taking $w = \frac{r}{2}$ in Proposition 2.3, combining Eq. 2.28 and Lemma 2.10(i), we get

$$v_\infty^{(t)}(r, y) \le \mathbf{E}_y\big(v_\infty^{(t)}\big(r/2, \xi_{r/2}^{(t)}\big); \tau_0^{(t),-} > r/2\big) \lesssim \frac{1}{r^{\frac{1}{\alpha-1}}}\mathbf{P}_y\big(\tau_0^{(t),-} > r/2\big)$$
$$\lesssim \frac{1}{r^{\frac{1}{\alpha-1}}}\frac{\sqrt{t}y+1}{\sqrt{rt}} = \frac{\sqrt{t}y+1}{r^{\frac{1}{\alpha-1}+\frac{1}{2}}\sqrt{t}}. \tag{2.29}$$

Now the first result follows easily from Eqs. 2.28 and 2.29.

(ii) For $\alpha \in (1, 2)$, by [9, Lemma 3.1], we have

$$\lim_{v\to 0+}\frac{\phi(v)}{v^\alpha} = \frac{\beta\kappa(\alpha)\Gamma(2-\alpha)}{\alpha-1},$$

which implies

$$\phi(v) \lesssim v^\alpha, \quad v \in [0, 1]. \tag{2.30}$$

When $\alpha = 2$, Eq. 2.30 also holds since we have

$$\lim_{v\to 0+}\frac{\phi(v)}{v^2} = \phi''(0+) = \mathcal{C}(2).$$

Therefore, by part (i),

$$\psi^{(t)}\big(v_\infty^{(t)}(r, y)\big) \lesssim \frac{1}{v_\infty^{(t)}(r, y)} t^{\frac{\alpha}{\alpha-1}}\Big(v_\infty^{(t)}(r, y)t^{-\frac{1}{\alpha-1}}\Big)^\alpha = \big(v_\infty^{(t)}(r, y)\big)^{\alpha-1} \lesssim \frac{1}{r}, \quad r > 0, y \in \mathbb{R}.$$

Since, for any $K > 0$, $vt^{-\frac{1}{\alpha-1}}$ converges uniformly to 0 for $v \in [0, K]$ as $t \to \infty$, we have, uniformly for $v \in [0, K]$,

$$\lim_{t\to\infty}\frac{\psi^{(t)}(v)}{v^{\alpha-1}} = \lim_{t\to\infty}\frac{\phi(vt^{-\frac{1}{\alpha-1}})}{(vt^{-\frac{1}{\alpha-1}})^\alpha} = \frac{\beta\kappa(\alpha)\Gamma(2-\alpha)}{\alpha-1} = \mathcal{C}(\alpha).$$

Therefore, the assertion of (ii) is valid. □

Lemma 2.15 *Assume that* **(H1)** *and* **(H2)** *hold.*
(i) For any fixed $r_0 > 0$, there exists a constant $N_1(r_0) > 0$ such that for any $t > 0$,

$$\Big|v_\infty^{(t)}(r, y+w) - v_\infty^{(t)}(r, y)\Big| \le N_1(r_0)\frac{1+\sqrt{tw}}{\sqrt{tw}}, \quad r > 2r_0, y \in \mathbb{R}_+, w \in (0, r_0)\,.$$

(ii) For any fixed $r_0 > 0$, there exists a constant $N_2(r_0) > 0$ such that for any $t > 0$,

$$\Big|v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r+q, y)\Big| \le N_2(r_0)\frac{1+\sqrt{t}q^{1/4}}{\sqrt{t}q^{1/8}}, \quad r > 2r_0, y \in \mathbb{R}_+, q \in \big(0, r_0^4\big)\,.$$

Proof (i) Since $v_\infty^{(t)}(r, 0) = 0$, the assertion for $y = 0$ follows from Lemma 2.14(i). So we assume $y > 0$. Note that for $0 < y < z$ and $t > 0$,

$$\mathbb{P}_y(\zeta^{(0,\infty)} > t) = \mathbb{P}_y\big(\exists u \in N(t) : \inf_{s\le t} X_u(t) > 0\big)$$
$$= \mathbb{P}_z\Big(\exists u \in N(t) : \inf_{s\le t} X_u(t) > z - y\Big) \le \mathbb{P}_z\Big(\exists u \in N(t) : \inf_{s\le t} X_u(t) > 0\Big) = \mathbb{P}_z(\zeta^{(0,\infty)} > t),$$

which implies that

$$\left|v_\infty^{(t)}(r,y+w)-v_\infty^{(t)}(r,y)\right| = v_\infty^{(t)}(r,y+w)-v_\infty^{(t)}(r,y), \quad y,w>0. \tag{2.31}$$

Now for $y>0$ and $w>0$, taking $T=\tau_{y+w}^{(t),+}$ in Eq. 2.7, we get

$$\begin{aligned}
&v_\infty^{(t)}(r,y)\\
&=\mathbf{E}_y\Big(\exp\Big\{-\int_0^{w\wedge\tau_{y+w}^{(t),+}}\psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r-w\wedge\tau_{y+w}^{(t),+},\xi^{(t)}_{w\wedge\tau_{y+w}^{(t),+}});\tau_0^{(t),-}>w\wedge\tau_{y+w}^{(t),+}\Big)\\
&\geq\mathbf{E}_y\Big(\exp\Big\{-\int_0^{w}\psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\Big\}v_\infty^{(t)}(r-\tau_{y+w}^{(t),+},\xi^{(t)}_{\tau_{y+w}^{(t),+}});\tau_0^{(t),-}>\tau_{y+w}^{(t),+},\tau_{y+w}^{(t),+}\leq w\Big)\\
&\geq v_\infty^{(t)}(r,y+w)\mathbf{E}_y\Big(\exp\Big\{-\int_0^{w}\psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\Big\};\tau_0^{(t),-}>\tau_{y+w}^{(t),+},\tau_{y+w}^{(t),+}\leq w\Big),
\end{aligned}\tag{2.32}$$

where in the last inequality we used the facts that $\mathbb{P}_y(\zeta^{(0,\infty)}>t)$ is decreasing in t and increasing in y, and $\xi^{(t)}_{\tau_{y+w}^{(t),+}}\geq y+w$. By Lemma 2.14(ii), there exists a constant $\Gamma>0$ such that for any $r_0\in(0,r/2)$ and $w\in(0,r_0)$,

$$\int_0^w\psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\leq\Gamma\int_0^w\frac{1}{r-s}\mathrm{d}s\leq\frac{\Gamma w}{r-w}\leq\frac{\Gamma}{r_0}w. \tag{2.33}$$

Combining Eqs. 2.31, 2.32 and 2.33, we see that for any $t>0$, $y>0$, $w\in(0,r_0)$ and $r>2r_0$,

$$\begin{aligned}
&\left|v_\infty^{(t)}(r,y+w)-v_\infty^{(t)}(r,y)\right|\\
&\leq v_\infty^{(t)}(r,y+w)\Big(1-\mathbf{E}_y\Big(\exp\Big\{-\int_0^w\psi^{(t)}\big(v_\infty^{(t)}(r-s,\xi_s^{(t)})\big)\mathrm{d}s\Big\};\tau_0^{(t),-}>\tau_{y+w}^{(t),+},\tau_{y+w}^{(t),+}\leq w\Big)\Big)\\
&\leq v_\infty^{(t)}(r,y+w)\big(1-e^{-\frac{\Gamma}{r_0}w}\mathbf{P}_y\big(\tau_0^{(t),-}>\tau_{y+w}^{(t),+},\tau_{y+w}^{(t),+}\leq w\big)\big)\\
&\leq v_\infty^{(t)}(r,y+w)\big(1-e^{-\frac{\Gamma}{r_0}w}+\mathbf{P}_y\big(\tau_0^{(t),-}\leq\tau_{y+w}^{(t),+}\big)+\mathbf{P}_y\big(\tau_{y+w}^{(t),+}>w\big)\big).
\end{aligned}$$

Combining Lemma 2.14(i), the inequality $1-e^{-x}\leq x$ for $x\geq 0$, and Lemma 2.10, we conclude from the above inequality that

$$\begin{aligned}
&\left|v_\infty^{(t)}(r,y+w)-v_\infty^{(t)}(r,y)\right|\\
&\lesssim\frac{1}{r^{\frac{1}{\alpha-1}}}\left(\frac{\Gamma}{r_0}w+\frac{\sqrt{t}w+1}{\sqrt{t}(y+w)}+\frac{\sqrt{t}w+1}{\sqrt{wt}}\right)\lesssim\frac{1}{r_0^{\frac{1}{\alpha-1}}}\left(w+\frac{\sqrt{t}w+1}{\sqrt{t}(y+w)}+\frac{\sqrt{t}w+1}{\sqrt{wt}}\right)\\
&\leq\frac{1}{r_0^{\frac{1}{\alpha-1}}}\left(\frac{\sqrt{t}w+1}{\sqrt{t}(y+w)}+2\frac{\sqrt{t}w+1}{\sqrt{wt}}\right),
\end{aligned}$$

where in the last inequality, we used $w\leq\sqrt{r_0}\sqrt{w}\lesssim\frac{\sqrt{t}w}{\sqrt{wt}}<\frac{\sqrt{t}w+1}{\sqrt{wt}}$. Therefore, when $y>\sqrt{w}$, we have

$$\left|v_\infty^{(t)}(r,y+w)-v_\infty^{(t)}(r,y)\right|\lesssim\frac{\sqrt{t}w+1}{\sqrt{t}(\sqrt{w}+w)}+\frac{\sqrt{t}w+1}{\sqrt{wt}}\leq 2\frac{\sqrt{t}w+1}{\sqrt{wt}}. \tag{2.34}$$

On the other hand, when $y \le \sqrt{w}$, using the monotonicity of $v_\infty^{(t)}(r, y)$ in y and Lemma 2.14(i),

$$\begin{aligned}
&\left|v_\infty^{(t)}(r, y+w) - v_\infty^{(t)}(r, y)\right| \le v_\infty^{(t)}(r, y+w) \le v_\infty^{(t)}(r, \sqrt{w}+w) \\
&\lesssim \frac{\sqrt{t}(\sqrt{w}+w)+1}{r^{\frac{1}{\alpha-1}+\frac{1}{2}}\sqrt{t}} \le \frac{2}{(2r_0)^{\frac{1}{\alpha-1}+\frac{1}{2}}}\frac{\sqrt{tw}+1}{\sqrt{t}} \le \frac{2}{(2r_0)^{\frac{1}{\alpha-1}+\frac{1}{2}}}\frac{\sqrt{t}w+1}{\sqrt{tw}}.
\end{aligned} \tag{2.35}$$

Together with Eqs. 2.34 and 2.35, we complete the proof of (i).

(ii) Since $v_\infty^{(t)}(r, 0) = 0$, the assertion for $y = 0$ is trivial. So we assume $y > 0$. By the monotonicity property of $v_\infty^{(t)}(r, y)$ in r,

$$\left|v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r+q, y)\right| = v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r+q, y). \tag{2.36}$$

Using Proposition 2.3 with r replaced by $r+q$ and w replaced by q, and an argument similar to that for Eq. 2.33, we have

$$\begin{aligned}
&v_\infty^{(t)}(r+q, y) = \mathbf{E}_y\Big(\exp\Big\{-\int_0^q \psi^{(t)}\big(v_\infty^{(t)}(r+q-s, \xi_s^{(t)})\big)\mathrm{d}s\Big\} v_\infty^{(t)}(r, \xi_q^{(t)});\ \tau_0^{(t),-} > q\Big) \\
&\ge e^{-\Gamma\int_0^q \frac{1}{r+q-s}\mathrm{d}s}\mathbf{E}_y\big(v_\infty^{(t)}(r, \xi_q^{(t)});\ \tau_0^{(t),-} > q\big) \ge e^{-\frac{\Gamma}{2r_0}q}\mathbf{E}_y\Big(v_\infty^{(t)}(r, \xi_q^{(t)});\ \tau_0^{(t),-} > q\Big) \\
&\ge e^{-\frac{\Gamma}{2r_0}q}\mathbf{E}_y\big(v_\infty^{(t)}(r, \xi_q^{(t)});\ \tau_0^{(t),-} > q, |\xi_q^{(t)} - y| < q^{1/4}\big).
\end{aligned}$$

Plugging this into Eq. 2.36, we obtain

$$\begin{aligned}
&\left|v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r+q, y)\right| \le v_\infty^{(t)}(r, y) - e^{-\frac{\Gamma}{2r_0}q}\mathbf{E}_y\big(v_\infty^{(t)}(r, \xi_q^{(t)});\ \tau_0^{(t),-} > q, |\xi_q^{(t)} - y| < q^{1/4}\big) \\
&\le v_\infty^{(t)}(r, y)\big(1 - e^{-\frac{\Gamma}{2r_0}q}\mathbf{P}_y\big(\tau_0^{(t),-} > q, |\xi_q^{(t)} - y| < q^{1/4}\big)\big) \\
&\quad + e^{-\frac{\Gamma}{2r_0}q}\mathbf{E}_y\big(\big|v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r, \xi_q^{(t)})\big|;\ \tau_0^{(t),-} > q, |\xi_q^{(t)} - y| < q^{1/4}\big).
\end{aligned} \tag{2.37}$$

By part (i), the last term of Eq. 2.37 is bounded above by $N_1(r_0)\frac{1+\sqrt{t}q^{1/4}}{\sqrt{t}q^{1/8}}$. Similarly, combining Lemma 2.14(i), Doob's maximal inequality and Markov's inequality, we have

$$\begin{aligned}
&v_\infty^{(t)}(r, y)\big(1 - e^{-\frac{\Gamma}{2r_0}q}\mathbf{P}_y\big(\tau_0^{(t),-} > q, |\xi_q^{(t)} - y| < q^{1/4}\big)\big) \\
&\lesssim \frac{1}{r^{\frac{1}{\alpha-1}}}\big(1 - e^{-\frac{\Gamma}{2r_0}q} + \mathbf{P}_y\big(\tau_0^{(t),-} \le q\big) + \mathbf{P}_y\big(|\xi_q^{(t)} - y| > q^{1/4}\big)\big) \\
&\le \frac{1}{(2r_0)^{\frac{1}{\alpha-1}}}\Big(\frac{\Gamma}{2r_0}q + \mathbf{P}_y\big(\inf_{s\le q}\xi_s^{(t)} < 0\big) + \mathbf{P}_0\big(|\xi_q^{(t)}| > q^{1/4}\big)\Big) \\
&\lesssim q + \mathbf{P}_0\big(\sup_{s\le q}(-\xi_s^{(t)})^+ > y\big) + \frac{1}{q^{1/2}}\mathbf{E}_0\big((\xi_q^{(t)})^2\big) \\
&\le q + \frac{1}{y}\sqrt{\mathbf{E}_0\big((\xi_q^{(t)})^2\big)} + \mathbf{E}_0(\xi_1^2)q^{1/2} \lesssim q + \frac{\sqrt{q}}{y}.
\end{aligned} \tag{2.38}$$

When $y > q^{1/4}$, by combining $q \le r_0^{\frac{7}{2}}q^{1/8} \lesssim q^{1/8}$ with Eqs. 2.37 and 2.38, we get

$$\left|v_\infty^{(t)}(r, y) - v_\infty^{(t)}(r+q, y)\right| \lesssim q^{1/8} + \frac{\sqrt{q}}{y} + \frac{1+\sqrt{t}q^{1/4}}{\sqrt{t}q^{1/8}} \lesssim \frac{1+\sqrt{t}q^{1/4}}{\sqrt{t}q^{1/8}}. \tag{2.39}$$

On the other hand, when $y \le q^{1/4}$, by Lemma 2.14(i) and the monotonicity of $V_\infty^{(t)}(r,y)$ in r,

$$\left|v_\infty^{(t)}(r,y) - v_\infty^{(t)}(r+q,y)\right| \le v_\infty^{(t)}(r,y) \lesssim \frac{\sqrt{t}y+1}{r^{\frac{1}{\alpha-1}+\frac{1}{2}}\sqrt{t}} \le \frac{\sqrt{t}q^{1/4}+1}{(2r_0)^{\frac{1}{\alpha-1}+\frac{1}{2}}\sqrt{t}}. \tag{2.40}$$

Combining Eqs. 2.39 and 2.40, we complete the proof of (ii). □

2.4 Preliminary Estimates for the Tail Probability of $M^{(0,\infty)}$

Recall that, for $x, y > 0$, $K^{(x)}(y)$ is defined by Eq. 2.10.

Lemma 2.16 *Assume that* **(H1)***,* **(H2)** *and* **(H3)** *hold.*
(i) For any $x > 0$ and $y \in (0,1)$, it holds that

$$K^{(x)}(y) \lesssim \frac{1}{(1-y)^{\frac{2}{\alpha-1}}}.$$

(ii) There exists a constant $C_* > 0$ such that for any $0 < y < z < 1$ and $x > 0$,

$$\int_0^{\tau_z^{(x^2),+}} \psi^{(x^2)}\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s \le C_* \frac{1}{(1-z)^2}\tau_z^{(x^2),+}, \quad \mathbf{P}_y\text{-}a.s.$$

Proof (i) Since $M^{(0,\infty)} \le M$, by Eq. 1.4, we have

$$K^{(x)}(y) \le x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}(M \ge x) = x^{\frac{2}{\alpha-1}}\mathbb{P}(M \ge x(1-y)) \lesssim x^{\frac{2}{\alpha-1}}\frac{1}{(x(1-y))^{\frac{2}{\alpha-1}}} = \frac{1}{(1-y)^{\frac{2}{\alpha-1}}}.$$

(ii) Combining Eqs. 2.12, 2.30 and part (i), we get that for any $0 < y < z < 1$ and $x > 0$,

$$\begin{aligned}\int_0^{\tau_z^{(x^2),+}} \psi^{(x^2)}\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s &\lesssim \int_0^{\tau_z^{(x^2),+}} x^{\frac{2\alpha}{\alpha-1}}\frac{1}{K^{(x)}(\xi_s^{(x^2)})}\big(K^{(x)}(\xi_s^{(x^2)})x^{-\frac{2}{\alpha-1}}\big)^\alpha \mathrm{d}s\\ = \int_0^{\tau_z^{(x^2),+}} \big(K^{(x)}(\xi_s^{(x^2)})\big)^{\alpha-1}\mathrm{d}s &\lesssim \int_0^{\tau_z^{(x^2),+}} \left(\frac{1}{(1-\xi_s^{(x^2)})^{\frac{2}{\alpha-1}}}\right)^{\alpha-1}\mathrm{d}s \le \frac{1}{(1-z)^2}\tau_z^{(x^2),+}.\end{aligned}$$

□

Lemma 2.17 *Assume that* **(H1)***,* **(H2)** *and* **(H3)** *hold. For any $r_0 \in (0,1/4)$, there exists a constant $N_3(r_0)$ such that for any $x > 0$, any $y \in (0, 1-2r_0)$ and any $w \in (0, r_0^2)$, it holds that*

$$\left|K^{(x)}(y+w) - K^{(x)}(y)\right| \le N_3(r_0)\left(\frac{1+xw}{(y+w)x} + 1 - \mathbf{E}_0\left(\exp\left\{-N_3(r_0)\tau_w^{(x^2),+}\right\}\right)\right).$$

Proof Let $r_0 \in (0,1/4)$ and $x > 0$. For $z > y$, we have

$$\begin{aligned}K^{(x)}(y) &= x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}\big(\exists\, t>0,\ u\in N(t):\ X_u(t)\ge x,\ \inf_{s\le t}X_u(s)>0\big)\\ &\le x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}\big(\exists\, t>0,\ u\in N(t):\ X_u(t)\ge x-x(z-y),\ \inf_{s\le t}X_u(s)>-x(z-y)\big)\\ &= x^{\frac{2}{\alpha-1}}\mathbb{P}_{xz}\big(\exists\, t>0,\ u\in N(t):\ X_u(t)\ge x,\ \inf_{s\le t}X_u(s)>0\big) = K^{(x)}(z).\end{aligned}$$

Therefore,

$$\left|K^{(x)}(y+w)-K^{(x)}(y)\right|=K^{(x)}(y+w)-K^{(x)}(y).$$

Note that for $y\in(0,1-2r_0)$ and $w\in(0,r_0^2)$, we have $y+w<1-2r_0+r_0^2<1-r_0$. Thus, combining Lemma 2.6 and Lemma 2.16 (ii), we obtain that for $y\in(0,1-2r_0)$ and $w\in(0,r_0^2)$,

$$\begin{aligned}K^{(x)}(y)&\geq \mathbf{E}_y\Big(\exp\Big\{-C_*\frac{1}{(1-y-w)^2}\tau_{y+w}^{(x^2),+}\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_{y+w}^{(x^2),+}}\big);\tau_{y+w}^{(x^2),+}<\tau_0^{(x^2),-}\Big)\\&\geq K^{(x)}(y+w)\mathbf{E}_y\Big(\exp\Big\{-C_*\frac{1}{r_0^2}\tau_{y+w}^{(x^2),+}\Big\};\tau_{y+w}^{(x^2),+}<\tau_0^{(x^2),-}\Big).\end{aligned}\tag{2.41}$$

Together with Lemma 2.16(i) and Eq. 2.41, for all $y\in(0,1-2r_0)$ and $w\in(0,r_0^2)$,

$$\begin{aligned}&K^{(x)}(y+w)-K^{(x)}(y)\leq K^{(x)}(y+w)\Big(1-\mathbf{E}_y\Big(\exp\big\{-C_*\frac{1}{r_0^2}\tau_{y+w}^{(x^2),+}\big\};\tau_{y+w}^{(x^2),+}<\tau_0^{(x^2),-}\Big)\Big)\\&\lesssim\frac{1}{(1-y-w)^{\frac{2}{\alpha-1}}}\Big(\mathbf{P}_y\big(\tau_{y+w}^{(x^2),+}\geq\tau_0^{(x^2),-}\big)+1-\mathbf{E}_y\big(\exp\big\{-\frac{C_*}{r_0^2}\tau_{w+y}^{(x^2),+}\big\}\big)\Big)\\&\lesssim\frac{1}{r_0^{\frac{2}{\alpha-1}}}\Big(\frac{xw+1}{x(y+w)}+1-\mathbf{E}_y\big(\exp\big\{-\frac{C_*}{r_0^2}\tau_{w+y}^{(x^2),+}\big\}\big)\Big),\end{aligned}$$

where in the last inequality we used Lemma 2.10(ii). Therefore, there exists N depending on r_0 such that

$$K^{(x)}(y+w)-K^{(x)}(y)\leq N\left(\frac{xw+1}{x(y+w)}+1-\mathbf{E}_y\left(\exp\left\{-\frac{C_*}{r_0^2}\tau_{w+y}^{(x^2),+}\right\}\right)\right),$$

this completes the proof of the lemma with $N_3(r_0)=\max\{N,\frac{C_*}{r_0^2}\}$. □

3 Proofs of the Main Results

Throughout this section we assume **(H1)**, **(H2)** and **(H4)** hold.

3.1 Proof of Theorem 1.1

By Lemma 2.14(i), for any $r,y>0$, we have

$$\sup_{t>0}v_\infty^{(t)}(r,y)<\infty.$$

By a standard diagonalization argument, for any sequence of positive reals increasing to ∞, we can find a subsequence $\{t_k:k\in\mathbb{N}\}$ such that $\lim_{k\to+\infty}t_k=\infty$ and that the following limit exists

$$\lim_{k\to\infty}v_\infty^{(t_k)}(r,y)=:v_\infty^X(r,y),\quad\text{for all } r,y\in(0,\infty)\cap\mathbb{Q}.\tag{3.1}$$

Since $v_\infty^{(t)}(r, y)$ is decreasing in r and increasing in y, so is the limit $v_\infty^X(r, y)$ for rational number r and y. Therefore, for any $r, y > 0$, we can define

$$v_\infty^X(r, y) := \lim_{(0,\infty)\cap\mathbb{Q}\ni(r_k,y_k)\to(r,y)} v_\infty^X(r_k, y_k) = \sup_{w\in[r,\infty)\cap\mathbb{Q}, z\in(0,y]\cap\mathbb{Q}} v_\infty^X(w, z). \tag{3.2}$$

We define $v_\infty^X(r, 0) = 0$ for all $r > 0$.

Lemma 3.1 *The relation Eq. 3.1 holds for all $r, y > 0$.*

Proof For any $r, y > 0$, let $\{(r_m, y_m) : m \in \mathbb{N}\}$ be a sequence in $((0,\infty)\cap\mathbb{Q}) \times ((0,\infty)\cap\mathbb{Q})$ with $(r_m, y_m) \to (r, y)$. Note that

$$\begin{aligned}\left|v_\infty^X(r, y) - v_\infty^{(t_k)}(r, y)\right| &\le \left|v_\infty^X(r, y) - v_\infty^X(r_m, y_m)\right| \\ &+ \left|v_\infty^X(r_m, y_m) - v_\infty^{(t_k)}(r_m, y_m)\right| + \left|v_\infty^{(t_k)}(r_m, y_m) - v_\infty^{(t_k)}(r, y)\right|.\end{aligned} \tag{3.3}$$

Fix $r_0 \in \left(0, \left(\frac{1}{2}\inf_m r_m\right) \wedge 1\right)$, then there exists $A > 0$ such that $|y_m - y| < r_0$ and $|r_m - r| < r_0^4$ for all $m > A$. By Lemma 2.15, we have that

$$\begin{aligned}&\left|v_\infty^{(t_k)}(r_m, y_m) - v_\infty^{(t_k)}(r, y)\right| \le \left|v_\infty^{(t_k)}(r_m, y_m) - v_\infty^{(t_k)}(r_m, y)\right| + \left|v_\infty^{(t_k)}(r_m, y) - v_\infty^{(t_k)}(r, y)\right| \\ &\le N_1(r_0)\frac{1+\sqrt{t_k}|y_m - y|}{\sqrt{t_k|y_m - y|}}1_{\{y_m\neq y\}} + N_2(r_0)\frac{1+\sqrt{t_k}|r_m - r|^{1/4}}{\sqrt{t_k}|r_m - r|^{1/8}}1_{\{r_m\neq r\}}.\end{aligned} \tag{3.4}$$

Combining Eqs. 3.1, 3.3 and 3.4,

$$\limsup_{k\to\infty}\left|v_\infty^X(r, y) - v_\infty^{(t_k)}(r, y)\right| \lesssim \left|v_\infty^X(r, y) - v_\infty^X(r_m, y_m)\right| + \sqrt{|y_m - y|} + |r_m - r|^{1/8}. \tag{3.5}$$

By Eq. 3.2, letting $m \to \infty$ in Eq. 3.5, we complete the proof of lemma. □

Combining Lemma 2.14(i) and the definition of v_∞^X above, we can easily see that for $r_0 > 0$,

$$\sup_{r\ge r_0, y>0} v_\infty^X(r, y) < \infty. \tag{3.6}$$

To prove Theorem 1.1, we need some results on the uniform convergence of $v_\infty^{(t)}(s, y)$ to $v_\infty^X(s, y)$ as $t \to \infty$. For each $0 < w < r$, taking $t = t_k$ in Lemma 2.15 and letting $k \to \infty$, we see that for any fixed $r_0 \in (0, (r - w)/2)$, $s \in [r - w, r]$, $y \in \mathbb{R}_+$, $\delta \in (0, r_0)$ and $q \in (0, r_0^4)$,

$$|v_\infty^X(s, y + \delta) - v_\infty^X(s, y)| \le N_1(r_0)\sqrt{\delta}, \quad |v_\infty^X(s, y) - v_\infty^X(s + q, y)| \le N_2(r_0)\, q^{1/8}, \tag{3.7}$$

which implies that $v_\infty^X(s, y)$ is jointly continuous for all $s \in [r - w, r]$ and $y > 0$. Since $v_\infty^{(t)}(s, y)$ is increasing in y and $\sup_{t>1, y\in\mathbb{R}_+} v_\infty^{(t)}(s, y) < \infty$, we see that $v_\infty^X(s, y)$ is also increasing in y and that $\sup_{y\in\mathbb{R}_+} v_\infty^X(s, y) < \infty$, which implies the existence of $v_\infty^X(s, \infty) := \lim_{y\to\infty} v_\infty^X(s, y)$. Letting $t \to \infty$ first and then $y \to \infty$ in Lemma 2.15 (ii), we see that $v_\infty^X(s, \infty)$ is continuous in $s \in [r - w, r]$. Therefore, for any $\varepsilon > 0$, there exist $J, L \in \mathbb{N}$ and $s_0 = r - w < s_1 < \ldots < s_J = r$, $y_0 = 0 < y_1 < \ldots < y_L < y_{L+1} = \infty$ such that

$$\max_{j\in[1,J], \ell\in[0,L+1]}\left(\left|v_\infty^X(s_j, y_\ell) - v_\infty^X(s_{j-1}, y_\ell)\right| \vee \left|v_\infty^X(s_j, y_\ell) - v_\infty^X(s_j, y_{\ell-1})\right|\right) < \varepsilon \tag{3.8}$$

and that there exists $T_2 > 0$ such that for any $t > T_2$,

$$\max_{j\in[0,J],\ell\in[0,L]} \left| v_\infty^{(t)}(s_j, y_\ell) - v_\infty^X(s_j, y_\ell) \right| < \varepsilon.$$

Therefore, for all $s \in [s_{j-1}, s_j]$ and $y \in [y_{\ell-1}, y_\ell)$ with $j \in [1, J]$, $\ell \in [1, L+1]$, by the monotonicity of $v_\infty^{(t)}$, we get

$$v_\infty^{(t)}(s, y) \geq v_\infty^{(t)}(s_j, y_{\ell-1}) \geq -\varepsilon + v_\infty^X(s_j, y_{\ell-1}) \geq -3\varepsilon + v_\infty^X(s_{j-1}, y_\ell) \geq -3\varepsilon + v_\infty^X(s, y). \tag{3.9}$$

Similarly, we also have

$$v_\infty^{(t)}(s, y) \leq 3\varepsilon + v_\infty^X(s, y). \tag{3.10}$$

The next lemma shows that any subsequential limit $v_\infty^X(r, y)$ is a solution to some initial-boundary problem. We postpone its proof to Section 4.2.

Lemma 3.2 *The limit $v_\infty^X(r, y)$ solves the following initial-boundary value problem*

$$\begin{cases} \frac{\partial}{\partial r} v_\infty^X(r, y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} v_\infty^X(r, y) - \varphi\left(v_\infty^X(r, y)\right), & \text{in } (0, \infty) \times (0, \infty), \\ \lim_{r\to 0+} v_\infty^X(r, y) = \infty, & y \in (0, \infty), \\ \lim_{y\to 0+} v_\infty^X(r, y) = 0, & r \in (0, \infty), \end{cases} \tag{3.11}$$

and for each $r \in (0, \infty)$, $\sup_{y>0} v_\infty^X(r, y) < \infty$.

The next proposition is on the uniqueness of the solution to the problem Eq. 3.11.

Proposition 3.3 *The solution to the problem 3.11 is unique and can be written as*

$$v_\infty^X(r, y) = -\log \mathbb{P}_{\delta_y}(X_r^{(0,\infty)} = 0),$$

where $X_r^{(0,\infty)}$ is the process defined in Eq. 1.13.

Proof Suppose that u solves problem 3.11. For any $\delta > 0$, $v(r, y) := u(\delta + r, y)$ solves the following problem:

$$\begin{cases} \frac{\partial}{\partial r} v(r, y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} v(r, y) - \varphi\left(v(r, y)\right), & \text{in } (0, \infty) \times (0, \infty), \\ \lim_{r\to 0+} v(r, y) = u(\delta, y), & y \in (0, \infty), \\ \lim_{y\to 0+} v(r, y) = 0, & r \in (0, \infty), \end{cases}$$

which is equivalent to the integral equation Eq. 1.10 with $f = u(\delta, \cdot)$. By the uniqueness of the solution to Eq. 1.10, we get

$$u(r+\delta, y) = v_{u(\delta,\cdot)}^X(r, y) = -\log \mathbb{E}_{\delta_y}\left(\exp\left\{-\langle u(\delta, \cdot), X_t^{(0,\infty)}\rangle\right\}\right), \quad r > 0, y > 0.$$

Now letting $\delta \to 0+$ in the above equation, by Lemma 3.2 and the continuity of $v_\infty^X(r, y)$ in r,

$$\begin{aligned} u(r, y) &= \lim_{\delta\to 0+} u(r+\delta, y) = \lim_{\delta\to 0+} -\log \mathbb{E}_{\delta_y}\left(\exp\left\{-\langle u(\delta, \cdot), X_t^{(0,\infty)}\rangle\right\}\right) \\ &= -\log \mathbb{P}_{\delta_y}(X_r^{(0,\infty)} = 0). \end{aligned}$$

□

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 (i) Combining Lemmas 3.1, 3.2 and Proposition 3.3, we get that, for any $y > 0$,

$$\lim_{t\to+\infty} v_\infty^{(t)}(1, y) = \lim_{t\to\infty} t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y}\left(\zeta^{(0,\infty)} > t\right) = -\log \mathbb{P}_{\delta_y}(X_1^{(0,\infty)} = 0). \tag{3.12}$$

Taking $f = \theta 1_{(0,\infty)}$ in Eq. 1.12 and letting $\theta \to +\infty$, we have that

$$-\log \mathbb{P}_{\delta_y}(X_1^{(0,\infty)} = 0) = \lim_{\theta\to+\infty} \mathbb{N}_y\left(1 - \exp\left\{-\langle \theta 1_{(0,\infty)}(\cdot), w_1\rangle\right\}\right) = \mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right). \tag{3.13}$$

Combining Eqs. 3.12 and 3.13, we arrive at assertion (i).

(ii) By Eq. 2.5, we have

$$v_\infty^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) = t^{\frac{1}{\alpha-1}} \mathbb{P}_y\big(\zeta^{(0,\infty)} > t\big), \quad y > 0.$$

It suffices to show that there exists $C^{(0,\infty)}(\alpha) \in (0,\infty)$ such that

$$\lim_{t\to\infty} \sqrt{t} v_\infty^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) = R(y) C^{(0,\infty)}(\alpha). \tag{3.14}$$

By Proposition 2.3 and Eq. 2.33, there exists $\Gamma > 0$ such that for any $r_0 \in (0, 1/2)$ and $w \in (0, r_0)$,

$$\begin{aligned}&\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big)\\ &\geq \sqrt{t} v_\infty^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) \geq e^{-\frac{\Gamma}{r_0}w} \sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big).\end{aligned} \tag{3.15}$$

By Eq. 3.10, for any $\varepsilon > 0$, when t is large enough,

$$\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big) \leq \sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big((v_\infty^X(1-w, \xi_w^{(t)}) + 3\varepsilon); \tau_0^{(t),-} > w\big),$$

which, by Lemma 2.12, tends to

$$\frac{1}{\sqrt{w}} \frac{2R(y)}{\sqrt{2\pi\sigma^2}} \int_0^\infty z e^{-\frac{z^2}{2}} (v_\infty^X(1-w, z\sigma\sqrt{w}) + 3\varepsilon) \mathrm{d}z$$

as $t \to \infty$. Similarly, using Eq. 3.9, we have

$$\begin{aligned}&\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big) \geq \sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big((v_\infty^X(1-w, \xi_w^{(t)}) - 3\varepsilon); \tau_0^{(t),-} > w\big)\\ &\xrightarrow{t\to\infty} \frac{1}{\sqrt{w}} \frac{2R(y)}{\sqrt{2\pi\sigma^2}} \int_0^\infty z e^{-\frac{z^2}{2}} (v_\infty^X(1-w, z\sigma\sqrt{w}) - 3\varepsilon) \mathrm{d}z.\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we conclude that

$$\begin{aligned}&\lim_{t\to\infty} \sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big)\\ &\quad = \frac{1}{\sqrt{w}} \frac{2R(y)}{\sqrt{2\pi\sigma^2}} \int_0^\infty z e^{-\frac{z^2}{2}} v_\infty^X(1-w, z\sigma\sqrt{w}) \mathrm{d}z =: R(y) G(1-w, w).\end{aligned} \tag{3.16}$$

Plugging this limit into Eq. 3.15, we get that

$$R(y)G(1-w,w) \geq \limsup_{t\to\infty} \sqrt{t} v_{\infty}^{(t)}\left(1, \frac{y}{\sqrt{t}}\right)$$
$$\geq \liminf_{t\to\infty} \sqrt{t} v_{\infty}^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) \geq e^{-\frac{\Gamma}{r_0}w} R(y)G(1-w,w). \tag{3.17}$$

Using Eq. 3.17, for any $w \in (0,1)$, we easily see that $G(1-w,w) \in (0,\infty)$, which implies that

$$\infty > \limsup_{t\to\infty} \sqrt{t} v_{\infty}^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) \geq \liminf_{t\to\infty} \sqrt{t} v_{\infty}^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) > 0.$$

Therefore, letting $w \to 0+$ in Eq. 3.17, we finally conclude that

$$\lim_{t\to\infty} \sqrt{t} v_{\infty}^{(t)}\left(1, \frac{y}{\sqrt{t}}\right) = R(y) \lim_{w\to 0+} G(1-w,w) =: R(y)C^{(0,\infty)}(\alpha),$$

which is Eq. 3.14. The proof is complete. □

3.2 Proof of Theorem 1.3

To prove Theorem 1.3, we need to show the convergence of $v_f^{(t)}(r,y)$ for every continuous function $f \in B_b^+((0,\infty))$. The next lemma shows that we can assume additionally that f is Lipschitz.

Lemma 3.4 *Let μ_n and μ be non-negative finite random measures on $\mathbb{R}$, then the following conditions are equivalent:*

(i) For any continuous $f \in B_b^+(\mathbb{R})$, $\int f(x)\mu_n(\mathrm{d}x) \overset{\mathrm{d}}{\Longrightarrow} \int f(x)\mu(\mathrm{d}x)$;

(ii) For any Lipschitz continuous $f \in B_b^+(\mathbb{R})$, $\int f(x)\mu_n(\mathrm{d}x) \overset{\mathrm{d}}{\Longrightarrow} \int f(x)\mu(\mathrm{d}x)$;

Proof We only need to prove (ii) ⇒ (i). First note that (ii) implies that $(\mu_n)_{n\geq 1}$ is relatively compact in distribution. In fact, by [12, Lemma 16.15], $(\mu_n)_{n\geq 1}$ is tight if and only if any relatively compact Borel set B, $\mu_n(B)$ is tight. Taking $f \equiv 1$ in (ii), we see that $\mu_n(\mathbb{R})$ is tight. For any relatively compact Borel set B, using the fact that $\mu_n(B) \leq \mu_n(\mathbb{R})$, we get $\mu_n(B)$ is tight. Now it remains to show that the distribution of a random measure μ is determined by $\int f(x)\mu(\mathrm{d}x)$ for all Lipschitz continuous $f \in B_b^+(\mathbb{R})$, which can be shown via a routine argument. We omit the details. □

The next two results will be needed in the proof of Theorem 1.3. We postpone their proofs to Section 4.3.

Proposition 3.5 *Suppose that f is a bounded Lipschitz function on $\mathbb{R}_+$ with $f(0)=0$ and that $T>0$.*

(i) For any $r \in [0,T]$ and any $w>0$, it holds that

$$\sup_{y>0} \left| v_f^{(t)}(r,y) - v_f^{(t)}(r,y+w) \right| \lesssim \left(\frac{1}{\log t} + w\right)(1+r^{-1/2}).$$

(ii) For any $r,q \geq 0$ with $r+q \leq T$, it holds that

$$\sup_{y>0} \left| v_f^{(t)}(r,y) - v_f^{(t)}(r+q,y) \right| \lesssim \left(\frac{1}{\log t} + q^{1/4}\right)\left(1+r^{-1/2}\right).$$

Proposition 3.6 *For any continuous function $f \in B_b^+((0,\infty))$ and any $r, y > 0$, it holds that*

$$\lim_{t\to\infty} v_f^{(t)}(r,y) = v_f^X(r,y),$$

where $v_f^X(r,y)$ is the solution of Eq. 1.10.

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3 (i) By the definition of $Z_1^{(0,\infty),t}$ in Eq. 1.16, for any continuous function $f \in B_b^+((0,\infty))$,

$$\begin{aligned}
&\mathbb{E}_{\sqrt{t}y}\Big(\exp\Big\{-\int_{(0,\infty)} f(y)\, Z_1^{(0,\infty),t}(\mathrm{d}y)\Big\}\Big|\zeta^{(0,\infty)} > t\Big)\\
=&\mathbb{E}_{\sqrt{t}y}\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big).
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E}_{\sqrt{t}y}\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big)\\
&= 1 - \frac{1}{\mathbb{P}_{\sqrt{t}y}(\zeta^{(0,\infty)} > t)}\mathbb{E}_{\sqrt{t}y}\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big),
\end{aligned} \tag{3.18}$$

where in the equality we used the fact that $Z_t^{(0,\infty)}((0,\infty)) = 0$ on the set $\{\zeta^{(0,\infty)} \le t\}$. Recall the definitions of $v_f^{(t)}$ and $v_\infty^{(t)}$ in Eqs. 2.2 and 2.5, Eq. 3.18 is equivalent to

$$\mathbb{E}_{\sqrt{t}y}\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big) = 1 - \frac{v_f^{(t)}(1,y)}{v_\infty^{(t)}(1,y)}. \tag{3.19}$$

Combining Proposition 3.6 and Eq. 1.12, we get

$$\lim_{t\to\infty} v_f^{(t)}(1,y) = v_f^X(1,y) = -\log \mathbb{P}_{\delta_y}\left(-\langle f, X_1\rangle\right) = \mathbb{N}_y\left(1-\exp\{-\langle f, w_1\rangle\}\right). \tag{3.20}$$

Plugging Eqs. 3.12, 3.13 and 3.20 into Eq. 3.19, we conclude that

$$\begin{aligned}
&\lim_{t\to\infty}\mathbb{E}_{\sqrt{t}y}\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big)\\
&= 1 - \frac{\mathbb{N}_y\left(1-\exp\{-\langle f, w_1\rangle\}\right)}{\mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right)} = 1 - \frac{\mathbb{N}_y\left(1-\exp\left\{-\langle f, w_1|_{(0,\infty)}\rangle\right\}\right)}{\mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right)}\\
&= 1 - \mathbb{N}_y\left(1-\exp\left\{-\langle f, w_1|_{(0,\infty)}\rangle\right\}\Big| w_1((0,\infty)) \neq 0\right)\\
&= \mathbb{N}_y\left(\exp\left\{-\langle f, w_1|_{(0,\infty)}\rangle\right\}\Big| w_1((0,\infty)) \neq 0\right).
\end{aligned}$$

This completes the proof of (i).

(ii) Let f be an arbitrary non-negative bounded Lipschitiz function on $(0,\infty)$. By Eq. 3.19, we see that

$$\mathbb{E}_y\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} f\left(\frac{x}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big) = 1 - \frac{v_f^{(t)}\left(1, yt^{-\frac{1}{2}}\right)}{v_\infty^{(t)}\left(1, yt^{-\frac{1}{2}}\right)}.$$

By using an argument similar to that leading to Eq. 3.15, we get that exists $\Gamma > 0$ such that for any $r_0 \in (0, 1/2)$ and $w \in (0, r_0)$,

$$\begin{aligned}&\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_f^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big)\\ &\geq \sqrt{t}v_f^{(t)}\big(1, \frac{y}{\sqrt{t}}\big) \geq e^{-\frac{\Gamma}{r_0}w}\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_f^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big).\end{aligned}$$

Proposition 3.5 implies that, for any $T > r_0$,

$$\left|v_f^{(t)}(r, y) - v_f^{(t)}(s, z)\right| \lesssim \frac{1}{\log t} + |y-z| + |r-s|^{1/4}, \quad \text{for all } r, s \in (r_0, T) \text{ and } y, z > 0.$$

Therefore, for any large $N > 0$ and any $\varepsilon > 0$, we can find $s_0 = r_0 < .. < s_J = T$ and $y_0 = 0 < ... < y_{L+1} = N$ such that Eq. 3.8 holds, which in turn implies that Eqs. 3.9 and 3.10 hold for all $s \in (r_0, T)$ and $y \in (0, N)$ when t is large enough. Therefore, using an argument similar to that leading to Eq. 3.16 and the following consequence of Lemma 4.1

$$\begin{aligned}&\lim_{N\to\infty}\limsup_{t\to\infty}\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_\infty^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w, \xi_w^{(t)} > N\big)\\ &\lesssim \lim_{N\to\infty}\limsup_{t\to\infty}\sqrt{t}\mathbf{P}_{y/\sqrt{t}}\big(\tau_0^{(t),-} > w, \xi_w^{(t)} > N\big) = \lim_{N\to\infty}\frac{2}{\sqrt{2\pi\sigma^2}}R(y)\int_{N/\sigma}^\infty ze^{-\frac{z^2}{2}}\mathrm{d}z = 0,\end{aligned}$$

we get the following result analogous to Eq. 3.16:

$$\begin{aligned}&\lim_{t\to\infty}\sqrt{t}\mathbf{E}_{y/\sqrt{t}}\big(v_f^{(t)}(1-w, \xi_w^{(t)}); \tau_0^{(t),-} > w\big)\\ &\quad= \frac{1}{\sqrt{w}}\frac{2R(y)}{\sqrt{2\pi\sigma^2}}\int_0^\infty ze^{-\frac{z^2}{2}}v_f^X(1-w, z\sigma\sqrt{w})\mathrm{d}z.\end{aligned}$$

Therefore,

$$\lim_{t\to\infty}\sqrt{t}v_f^{(t)}\big(1, \frac{y}{\sqrt{t}}\big) = \lim_{w\to 0}\frac{1}{\sqrt{w}}\frac{2R(y)}{\sqrt{2\pi\sigma^2}}\int_0^\infty ze^{-\frac{z^2}{2}}v_f^X(1-w, z\sigma\sqrt{w})\mathrm{d}z,$$

Together with Eq. 3.14, we conclude that

$$\begin{aligned}&\lim_{t\to\infty}\mathbb{E}_y\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)}f\left(\frac{x}{\sqrt{t}}\right)Z_t^{(0,\infty)}(\mathrm{d}x)\Big\}\Big|\zeta^{(0,\infty)} > t\Big)\\ &= 1 - \frac{1}{C^{(0,\infty)}(\alpha)}\frac{2}{\sqrt{2\pi\sigma^2}}\lim_{w\to 0}\frac{1}{\sqrt{w}}\int_0^\infty ze^{-\frac{z^2}{2}}v_f^X(1-w, z\sigma\sqrt{w})\mathrm{d}z. \qquad (3.21)\end{aligned}$$

If we could show that

$$\lim_{t\to\infty}\mathbb{E}_y\Big(\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)}\varepsilon f\left(\frac{y}{\sqrt{t}}\right)Z_t^{(0,\infty)}(\mathrm{d}y)\Big\}\Big|\zeta^{(0,\infty)} > t\Big) \xrightarrow{\varepsilon\to 0+} 1, \qquad (3.22)$$

we would get that there exists a random measure η_1 such that the right -hand side of Eq. 3.21 is equal to $\mathbb{E}\big(\exp\big\{-\langle f, \eta_1\rangle\big\}\big)$. Combining this with Lemma 3.4, we arrive at the assertion (ii). Now we prove Eq. 3.22. By Eq. 1.10,

$$v_{\varepsilon f}^X(1, y) \leq \varepsilon\mathbf{E}_y\left(f(W_1^0)\right) \leq \varepsilon\sup_{x>0}|f(x)|\mathbf{P}_y\big(\tau_0^{W,-} > 1\big) \lesssim \varepsilon y,$$

which implies that

$$\lim_{w\to 0}\frac{1}{\sqrt{w}}\int_0^\infty ze^{-\frac{z^2}{2}}v_{\varepsilon f}^X(1-w, z\sigma\sqrt{w})\mathrm{d}z \lesssim \varepsilon\int_0^\infty z^2e^{-\frac{z^2}{2}}\mathrm{d}z \xrightarrow{\varepsilon\to 0+} 0.$$

Thus Eq. 3.22 is valid. □

3.3 Proof of Theorem 1.5

By Lemma 2.16, for any $y \in (0, 1)$,

$$\sup_{x>0} K^{(x)}(y) < \infty.$$

Therefore, using a diagonalization argument, we can find, for any sequence of positive reals increasing to ∞, a subsubsequence $\{x_k : k \in \mathbb{N}\}$ such that the following limit exists

$$\lim_{k\to\infty} K^{(x_k)}(y) =: K^X(y), \quad \text{for all } y \in (0, 1) \cap \mathbb{Q}. \tag{3.23}$$

Since $K^{(x)}(y)$ is monotone in $y \in (0, 1)$, we can define

$$K^X(y) := \lim_{(0,1)\cap\mathbb{Q}\ni y_m \to y} K^X(y_m) = \inf_{z\in[y,1)\cap\mathbb{Q}} K^X(z), \quad y \in (0, 1).$$

Using an argument similar to that used in the proof of Lemma 3.1, with Lemma 2.15 replaced by Lemma 2.17 and the fact that $\tau_w^{(x^2),+}$ converges in distribution to the first exit time $\tau_w^{W,+}$ of W from $(-\infty, w)$, we can easily get the following lemma, whose proof is omitted:

Lemma 3.7 *The relation Eq. 3.23 holds for all $y \in (0, 1)$.*

The following Lemma 3.8 says that the limit $K^X(y)$ solves the boundary value problem 3.24 below. Proposition 3.9 is about the uniqueness and probabilistic representation to problem Eq. 3.24. Since the main idea of the proof of Lemma 3.8 is similar to that of Lemma 3.2, and since we need to introduce exit measures of superprocesses in the proof of Proposition 3.9, we postpone the proofs to Section 4.4.

Lemma 3.8 *The limit $K^X(y)$ solves the following problem*

$$\begin{cases} \frac{\sigma^2}{2}(K^X)''(y) = \varphi(K^X(y)), \quad y \in (0, 1), \\ \lim_{y\to 0+} K^X(y) = 0, \quad \lim_{y\to 1-} K^X(y) = \infty. \end{cases} \tag{3.24}$$

Proposition 3.9 *The problem in 3.24 has a unique solution and the unique solution admits the representation $K^X(y) = -\log \mathbb{P}_{\delta_y}\left(M^{(0,\infty),X} < 1\right)$.*

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5: Combining Lemma 3.7 and Proposition 3.9, we get that

$$\lim_{x\to\infty} K^{(x)}(y) = \lim_{x\to\infty} x^{\frac{2}{\alpha-1}} \mathbb{P}_{xy}(M^{(0,\infty)} \geq x) = -\log \mathbb{P}_{\delta_y}\left(M^{(0,\infty),X} < 1\right),$$

which proves (i). For (ii), by the definition of $K^{(x)}(y)$ given in Eq. 2.10 and Lemma 2.6, for any fixed small $z < \frac{1}{2}$, when $y < xz$, we have

$$\begin{aligned} &x^{\frac{2}{\alpha-1}+1}\mathbb{P}_y(M^{(0,\infty)} \geq x) = xK^{(x)}(yx^{-1}) \\ &= x\mathbf{E}_{yx^{-1}}\left(\exp\left\{-\int_0^{\tau_z^{(x^2),+}} \psi^{(x^2)}\left(K^{(x)}\left(\xi_s^{(x^2)}\right)\right)\mathrm{d}s\right\}K^{(x)}\left(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\right); \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\right). \end{aligned}$$

Therefore, by Lemma 2.16(ii) and the fact that $\xi^{(x^2)}_{\tau_z^{(x^2),+}} \geq z$, we have

$$\begin{aligned} & xK^{(x)}(z)\mathbf{E}_{yx^{-1}}\big(\exp\big\{-C_*\frac{1}{(1-z)^2}\tau_z^{(x^2),+}\big\};\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) \leq xK^{(x)}(yx^{-1}) \\ & \leq x\mathbf{E}_{yx^{-1}}\big(K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big). \end{aligned} \tag{3.25}$$

It follows from Lemma 2.7 that, for any $\delta > 0$,

$$\begin{aligned} & x\mathbf{E}_{yx^{-1}}\big(K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) \\ & \leq K^{(x)}(z+\delta)x\mathbf{P}_{yx^{-1}}\big(\tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) + x^{1+\frac{2}{\alpha-1}}\mathbf{P}_{yx^{-1}}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}} > z+\delta\big) \\ & = K^{(x)}(z+\delta)x\mathbf{P}_{yx^{-1}}\big(\tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) + x^{1+\frac{2}{\alpha-1}}\mathbf{P}_{-xz+y}\big(\xi_{\tau_0^+} > x\delta\big) \\ & \leq K^{(x)}(z+\delta)x\mathbf{P}_{yx^{-1}}\big(\tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) + \frac{1}{\delta^{r_0-2}}\frac{1}{x^{r_0-1-\frac{2\alpha}{\alpha-1}}}\sup_{y>0}\mathbf{E}_{-y}(|\xi_{\tau_0^+}|^{r_0-2}), \end{aligned} \tag{3.26}$$

where in the last equality we used the fact that $\mathbf{P}_{yx^{-1}}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}} > z+\delta\big) = \mathbf{P}_y\big(\xi_{\tau_{xz}^+} > xz+x\delta\big)$, which holds by Eq. 2.11. Therefore, combining Eqs. 3.25, 3.26 and Lemma 2.13(i), letting $x \to \infty$ first and then $\delta \to 0$, we get that

$$\limsup_{x\to\infty} xK^{(x)}(yx^{-1}) \leq R(y)\frac{K^X(z)}{z}. \tag{3.27}$$

On the other hand, by Lemma 2.13 (ii), there exists a constant C such that

$$C_*x\mathbf{E}_{yx^{-1}}\big(\tau_z^{(x^2),+};\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) \leq Cxz^2\mathbf{P}_y\left(\tau_{xz}^+ < \tau_0^-\right) + \frac{C}{x}.$$

Thus, by Eq. 3.25, using the inequality $e^{-x} \geq 1 - x$ and Lemma 2.13 (i), we have

$$\begin{aligned} & \liminf_{x\to\infty} xK^{(x)}(yx^{-1}) \geq K^X(z)\liminf_{x\to\infty} x\mathbf{E}_{yx^{-1}}\Big(\Big(1 - C_*\frac{1}{(1-z)^2}\tau_z^{(x^2),+}\Big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big) \\ & \geq K^X(z)\Big(1 - \frac{Cz^2}{(1-z)^2}\Big)\lim_{x\to\infty} x\mathbf{P}_{yx^{-1}}\big(\tau_z^{(x^2),+} < \tau_0^{(x^2),-}\big) \\ & = R(y)\frac{K^X(z)}{z}\Big(1 - \frac{Cz^2}{(1-z)^2}\Big). \end{aligned} \tag{3.28}$$

Letting $z \to 0$, we conclude from Eqs. 3.27 and 3.28 that $\lim_{z\to 0+}\frac{K^X(z)}{z}$ exists. Define

$$\theta^{(0,\infty)}(\alpha) := \lim_{z\to 0+}\frac{K^X(z)}{z}.$$

Then we have

$$\lim_{x\to\infty} xK^{(x)}(yx^{-1}) = \theta^{(0,\infty)}(\alpha)R(y). \tag{3.29}$$

Choose $z_0 \in (0,1)$ such that $Cz_0^2/(1-z_0)^2 < 1$. Then taking $z = z_0$ in Eqs. 3.27 and 3.28, we get

$$0 < \frac{K^X(z_0)}{z_0}\Big(1 - \frac{Cz_0^2}{(1-z_0)^2}\Big) \leq \theta^{(0,\infty)}(\alpha) \leq \frac{K^X(z_0)}{z_0} < \infty,$$

which implies that $\theta^{(0,\infty)}(\alpha) \in (0, 1)$. We complete the proof of the theorem. □

3.4 Proof of Theorem 1.7

For $t, r, z > 0$, define $Q_z^{(t)}(r, y) := t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y}\big(M_{tr}^{(0,\infty)} > \sqrt{t}z\big)$.

Lemma 3.10 *Let $z > 0$ and $\varepsilon \in (0, (z/2) \wedge 1)$. There exists a constant $L = L(\varepsilon) > 0$ such that for any $\delta > 0$, there exists $T = T(z, \varepsilon, \delta)$ such that when $t > T$,*

$$Q_z^{(t)}(\delta, z - 2\varepsilon) \le L(\varepsilon)\delta.$$

Proof Note that for $t, r, y, z > 0$,

$$\mathbb{P}_{\sqrt{t}y}\big(M_{tr}^{(0,\infty)} > \sqrt{t}z\big) = \lim_{\theta\to+\infty} \Big(1 - \mathbb{E}_{\sqrt{t}y}\big(\exp\big\{ - \theta Z_{tr}^{(0,\infty)} \Big((\sqrt{t}z, \infty)\Big) \big\}\big)\Big).$$

Taking $f = \theta 1_{(z,\infty)}$ in Proposition 2.3 first and then letting $\theta \to +\infty$, we see that for any $w \in (0, r]$, $Q_z^{(t)}(r, y)$ solves the equation

$$Q_z^{(t)}(r, y) = \mathbf{E}_y\Big(\exp\Big\{ - \int_0^w \psi^{(t)}\big(Q_z^{(t)}(r - s, \xi_s^{(t)})\big)\mathrm{d}s\Big\} Q_z^{(t)}(r - w, \xi_w^{(t)}); \tau_0^{(t),-} > w\Big).$$

Taking $r = \delta$, $y = z - 2\varepsilon$ and using the argument leading to Eq. 2.7 with $T = \tau_{z-\varepsilon}^{(t),+}$, we get

$$\begin{aligned} Q_z^{(t)}(\delta, z - 2\varepsilon) &\le \mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}(\delta - \delta \wedge \tau_{z-\varepsilon}^{(t),+}, \xi^{(t)}_{\delta\wedge\tau_{z-\varepsilon}^{(t),+}})\big) \\ &= \mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}\big(\delta - \tau_{z-\varepsilon}^{(t),+}, \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}}\big); \delta > \tau_{z-\varepsilon}^{(t),+}\big), \end{aligned} \tag{3.30}$$

where in the equality we used the fact that $Q_z^{(t)}(0, y) = 0$. Using $M_{tr}^{(0,\infty)} \le M$, we see that for any $0 < y < z$ and $r > 0$,

$$Q_z^{(t)}(r, y) \le t^{\frac{1}{\alpha-1}} \mathbb{P}_{\sqrt{t}y}\big(M > \sqrt{t}z\big) = t^{\frac{1}{\alpha-1}} \mathbb{P}_0\big(M > \sqrt{t}(z - y)\big) \lesssim \frac{1}{(z - y)^{\frac{2}{\alpha-1}}},$$

where in the last inequality we used Eq. 1.4. Combining the inequality above with the monotonicity of $Q_z^{(t)}(r, y)$ in y, we get that

$$\begin{aligned} &\mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}\big(\delta - \tau_{z-\varepsilon}^{(t),+}, \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}}\big); \delta > \tau_{z-\varepsilon}^{(t),+}\big) \\ &= \mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}\big(\delta - \tau_{z-\varepsilon}^{(t),+}, \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}}\big); \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}} > z - 2^{-1}\varepsilon, \delta > \tau_{z-\varepsilon}^{(t),+}\big) \\ &\quad + \mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}\big(\delta - \tau_{z-\varepsilon}^{(t),+}, \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}}\big); \xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}} \le z - 2^{-1}\varepsilon, \delta > \tau_{z-\varepsilon}^{(t),+}\big) \\ &\le t^{\frac{1}{\alpha-1}} \mathbf{P}_{z-2\varepsilon}\big(\xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}} > z - 2^{-1}\varepsilon\big) + \mathbf{E}_{z-2\varepsilon}\big(Q_z^{(t)}\big(\delta - \tau_{z-\varepsilon}^{(t),+}, z - 2^{-1}\varepsilon\big); \delta > \tau_{z-\varepsilon}^{(t),+}\big) \\ &\lesssim t^{\frac{1}{\alpha-1}} \mathbf{P}_{z-2\varepsilon}\big(\xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}} > z - 2^{-1}\varepsilon\big) + \frac{1}{\varepsilon^{\frac{2}{\alpha-1}}} \mathbf{P}_{z-2\varepsilon}\big(\delta > \tau_{z-\varepsilon}^{(t),+}\big). \end{aligned} \tag{3.31}$$

Since $(r_0 - 2)/2 > 1/(\alpha - 1)$, by Markov's inequality and Lemma 2.7, we have

$$\begin{aligned} t^{\frac{1}{\alpha-1}} \mathbf{P}_{z-2\varepsilon}\Big(\xi^{(t)}_{\tau_{z-\varepsilon}^{(t),+}} > z - 2^{-1}\varepsilon\Big) &= t^{\frac{1}{\alpha-1}} \mathbf{P}_{-\sqrt{t}\varepsilon}\Big(\xi_{\tau_0^+} > \frac{\varepsilon\sqrt{t}}{2}\Big) \\ &\le t^{\frac{1}{\alpha-1}} \left(\frac{2}{\varepsilon\sqrt{t}}\right)^{r_0-2} \mathbf{E}_{-\sqrt{t}\varepsilon}\left(\xi_{\tau_0^+}^{r_0-2}\right) \lesssim \frac{1}{\log t} \frac{1}{\varepsilon^{r_0-2}}, \end{aligned} \tag{3.32}$$

where in the last inequality we used the fact that $t^{\frac{r_0-2}{2}-\frac{1}{\alpha-1}} \gtrsim \log t$ (since $r_0 > \frac{2\alpha}{\alpha-1}$). By Doob's inequality,

$$\mathbf{P}_{z-2\varepsilon}\big(\delta > \tau^{(t),+}_{z-\varepsilon}\big) = \mathbf{P}_0\big(\sup_{s\le\delta} \xi^{(t)}_s \ge \varepsilon\big) \le \frac{\mathbf{E}_0\big(|\xi^{(t)}_\delta|^2\big)}{\varepsilon^2} \lesssim \frac{\delta}{\varepsilon^2}. \tag{3.33}$$

Plugging Eqs. 3.32 and 3.33 into Eq. 3.31, we see that for $t > e^{1/\delta}$, we have

$$\mathbf{E}_{z-2\varepsilon}\Big(Q^{(t)}_z\Big(\delta - \tau^{(t),+}_{z-\varepsilon}, \xi^{(t)}_{\tau^{(t),+}_{z-\varepsilon}}\Big); \delta > \tau^{(t),+}_{z-\varepsilon}\Big) \lesssim \frac{\delta}{\varepsilon^{r_0-2}} + \frac{\delta}{\varepsilon^{\frac{2\alpha}{\alpha-1}}} \lesssim \frac{\delta}{\varepsilon^{r_0}}. \tag{3.34}$$

Combining Eqs. 3.30 and 3.34, we get the assertion of the lemma. □

Proof of Theorem 1.7 It suffices to study the limits of the conditional probabilities of the $\{M^{(0,\infty)}_t > \sqrt{t}z\}$ for $z > 0$.

(i) We first prove the lower bound. For any fixed $z > 0$, define $g_1(x) := \min\{1, (x-z)^+\}$. Then combining Theorem 1.3(i) and the fact that for any $\theta > 0$,

$$1_{\{y\le\sqrt{t}z\}} \le \exp\left\{-\frac{\theta}{t^{\frac{1}{\alpha-1}}} g_1\left(\frac{y}{\sqrt{t}}\right)\right\}, \quad \forall y \in \mathbb{R},$$

we conclude that for any $\theta > 0$,

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbb{P}_{\sqrt{t}y}\big(M^{(0,\infty)}_t > \sqrt{t}z \big| \zeta^{(0,\infty)} > t\big)\\
&= \liminf_{t\to\infty}\Big(1 - \mathbb{P}_{\sqrt{t}y}\big(M^{(0,\infty)}_t \le \sqrt{t}z \big| \zeta^{(0,\infty)} > t\big)\Big)\\
&\ge \lim_{t\to\infty} \mathbb{E}_{\sqrt{t}y}\Big(1 - \exp\Big\{-\frac{\theta}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} g_1\big(\frac{a}{\sqrt{t}}\big) Z^{(0,\infty)}_t(\mathrm{d}a)\Big\}\Big| \zeta^{(0,\infty)} > t\Big)\\
&= \mathbb{N}_y\Big(1 - \exp\Big\{-\theta\int_{(0,\infty)} g_1(a) w_1(\mathrm{d}a)\Big\}\Big| w_1((0,\infty)) \ne 0\Big).
\end{aligned}$$

Letting $\theta \to +\infty$, we conclude that

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbb{P}_{\sqrt{t}y}\big(M^{(0,\infty)}_t > \sqrt{t}z \big| \zeta^{(0,\infty)} > t\big)\\
&\ge \mathbb{N}_y\big(M^{(0,\infty),X}_1 > z \big| w_1((0,\infty)) \ne 0\big).
\end{aligned} \tag{3.35}$$

For the upper bound, we fix an arbitrary $z > 0$. Let $\varepsilon \in (0, (z/2) \wedge 1)$ and $\delta \in (0, 1)$. We note that for any $r > \delta$,

$$\mathbb{P}_{\sqrt{t}y}\big(M^{(0,\infty)}_{tr} > \sqrt{t}z\big) = \mathbb{E}_{\sqrt{t}y}\Big(1 - \exp\Big\{\int_{(0,\infty)} \log \mathbb{P}_a\big(M^{(0,\infty)}_{t\delta} \le \sqrt{t}z\big) Z^{(0,\infty)}_{t(r-\delta)}(\mathrm{d}a)\Big\}\Big).$$

Note that for all $a > 0$,

$$\mathbb{P}_a\big(M^{(0,\infty)}_{t\delta} \le \sqrt{t}z\big) \ge \mathbb{P}_a\big(\zeta^{(0,\infty)} \le t\delta\big) \ge \mathbb{P}(\zeta \le t\delta) \overset{t\to\infty}{\longrightarrow} 1.$$

Using the fact that $\log x \sim x - 1$ as $x \to 1$, we get that there exists $t_0 = t_0(\delta) > 0$ such that for all $t \ge t_0$,

$$\mathbb{P}_{\sqrt{t}y}\big(M^{(0,\infty)}_{tr} > \sqrt{t}z\big) \le \mathbb{E}_{\sqrt{t}y}\Big(1 - \exp\Big\{-\frac{1}{2}\int_{(0,\infty)} \mathbb{P}_a\big(M^{(0,\infty)}_{t\delta} > \sqrt{t}z\big) Z^{(0,\infty)}_{t(r-\delta)}(\mathrm{d}a)\Big\}\Big). \tag{3.36}$$

When $a < \sqrt{t}(z-\varepsilon)$, by Lemma 3.10 with ε replaced by $\frac{\varepsilon}{2}$ and using the monotonicity of $Q_z^{(t)}$, we get that when $t \geq T(z, \frac{\varepsilon}{2}, \delta)$,

$$t^{\frac{1}{\alpha-1}}\mathbb{P}_a\big(M_{t\delta}^{(0,\infty)} > \sqrt{t}z\big) \leq t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}(z-\varepsilon)}\big(M_{t\delta}^{(0,\infty)} > \sqrt{t}z\big) = Q_z^{(t)}(\delta, z-\varepsilon) \leq L\left(\frac{\varepsilon}{2}\right)\delta. \tag{3.37}$$

When $a \geq \sqrt{t}(z-\varepsilon)$, by Eq. 1.2, there exists a constant L_1 such that

$$t^{\frac{1}{\alpha-1}}\mathbb{P}_a\big(M_{t\delta}^{(0,\infty)} > \sqrt{t}z\big) \leq t^{\frac{1}{\alpha-1}}\mathbb{P}\big(\zeta > \sqrt{t}\delta\big) \leq \frac{L_1}{\delta^{\frac{1}{\alpha-1}}}. \tag{3.38}$$

For any fixed $\varepsilon \in (0, (z/2) \wedge 1)$, let $\delta_* > 0$ be small enough so that

$$L\left(\frac{\varepsilon}{2}\right)\delta_* < \frac{L_1}{\delta_*^{\frac{1}{\alpha-1}}}.$$

Define another non-negative bounded continuous function

$$\begin{aligned} g_2(x) &:= \frac{1}{2}L\left(\frac{\varepsilon}{2}\right)\delta_* 1_{\{x \leq z-2\varepsilon\}} + \frac{L_1}{2\delta_*^{\frac{1}{\alpha-1}}}1_{\{x \geq z-\varepsilon\}} \\ &\quad + \frac{1}{2}\left(\left(\frac{L_1}{\delta_*^{\frac{1}{\alpha-1}}} - L\left(\frac{\varepsilon}{2}\right)\delta_*\right)\frac{x-(z-2\varepsilon)}{\varepsilon} + L\left(\frac{\varepsilon}{2}\right)\delta_*\right)1_{\{x \in (z-2\varepsilon, z-\varepsilon)\}}. \end{aligned}$$

Then Eqs. 3.37 and 3.38 imply that for all $a \in (0, \infty)$,

$$t^{\frac{1}{\alpha-1}}\mathbb{P}_a\big(M_{t\delta_*}^{(0,\infty)} > \sqrt{t}z\big) \leq g_2(a). \tag{3.39}$$

Plugging this upper bound into Eq. 3.36, we conclude that

$$\begin{aligned} &\mathbb{P}_{\sqrt{t}y}\big(M_{tr}^{(0,\infty)} > \sqrt{t}z\big) \leq \mathbb{E}_{\sqrt{t}y}\left(1 - \exp\left\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} g_2\Big(\frac{a}{\sqrt{t}}\Big)Z_{t(r-\delta_*)}^{(0,\infty)}(\mathrm{d}a)\right\}\right) \\ &= t^{-\frac{1}{\alpha-1}}v_{g_2}^{(t)}(r-\delta_*, y), \quad r > \delta_*. \end{aligned} \tag{3.40}$$

Combining Proposition 3.6 (applied to g_2) and Theorem 1.1, we get that

$$\begin{aligned} &\limsup_{t\to\infty}\mathbb{P}_{\sqrt{t}y}\big(M_t^{(0,\infty)} > \sqrt{t}z \big| \zeta^{(0,\infty)} > t\big) = \limsup_{t\to\infty}\frac{\mathbb{P}_{\sqrt{t}y}\big(M_t^{(0,\infty)} > \sqrt{t}z\big)}{\mathbb{P}_{\sqrt{t}y}\big(\zeta^{(0,\infty)} > t\big)} \\ &= \frac{\limsup_{t\to\infty}(t(1+\delta_*))^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}\sqrt{1+\delta_*}y}\big(M_{t(1+\delta_*)}^{(0,\infty)} > \sqrt{t(1+\delta_*)}z\big)}{\lim_{t\to\infty}t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\zeta^{(0,\infty)} > t\big)} \\ &\leq (1+\delta_*)^{\frac{1}{\alpha-1}}\frac{\limsup_{t\to\infty}t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}\sqrt{1+\delta_*}y}\big(M_{t(1+\delta_*)}^{(0,\infty)} > \sqrt{t}z\big)}{\mathbb{N}_y\big(w_1((0,\infty)) \neq 0\big)} \\ &\leq (1+\delta_*)^{\frac{1}{\alpha-1}}\frac{\lim_{t\to\infty}v_{g_2}^{(t)}(1, y\sqrt{1+\delta_*})}{\mathbb{N}_y\,(w_1((0,\infty)) \neq 0)} \\ &= (1+\delta_*)^{\frac{1}{\alpha-1}}\frac{-\log\mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\big(\exp\big\{-\int_{(0,\infty)}g_2(a)X_1(\mathrm{d}a)\big\}\big)}{\mathbb{N}_y\big(w_1((0,\infty)) \neq 0\big)}, \end{aligned} \tag{3.41}$$

where in the second inequality we used Eq. 3.40 with $r = 1+\delta_*$. By the inequality $e^{-x} \geq 1-x$ and the fact that $\{X_1((z,\infty)) = 0\} = \{X_1^{(0,\infty)}((z,\infty)) = 0\} = \{M_1^{(0,\infty)} \leq z\}$, we have

$$\begin{aligned}
&\mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\Big(\exp\Big\{-\int_{(0,\infty)} g_2(a)X_1(\mathrm{d}a)\Big\}\Big) \\
&\geq \mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\left(\exp\left\{-L\left(\frac{\varepsilon}{2}\right)\delta_* X_1((-\infty, z-2\varepsilon])\right\}; X_1((z-2\varepsilon,\infty)) = 0\right) \\
&\geq \mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\left(\left(1 - L\left(\frac{\varepsilon}{2}\right)\delta_* X_1((-\infty, z-2\varepsilon])\right); X_1((z-2\varepsilon,\infty)) = 0\right) \\
&\geq \mathbb{P}_{\delta_{\sqrt{1+\delta_*}y}}\big(M_1^{(0,\infty)} \leq z-2\varepsilon\big) - L\left(\frac{\varepsilon}{2}\right)\delta_*\mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}} X_1(\mathbb{R}).
\end{aligned}$$

Note that $X_1(\mathbb{R})$ under $\mathbb{P}_{\delta_{\sqrt{1+\delta_*}y}}$ is a critical continuous-state branching process, the last term in the above inequality is equal to $-L\left(\frac{\varepsilon}{2}\right)\delta_*$. Therefore, letting $\delta_* \to 0$, we conclude that

$$\begin{aligned}
&\limsup_{\delta_*\to 0} -\log \mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\Big(\exp\Big\{-\int_{(0,\infty)} g_2(a)X_1(\mathrm{d}a)\Big\}\Big) \\
&\leq -\log \mathbb{P}_{\delta_y}\big(M_1^{(0,\infty)} \leq z-2\varepsilon\big).
\end{aligned}$$

Now letting $\varepsilon \to 0$, we conclude that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\limsup_{\delta_*\to 0}(1+\delta_*)^{\frac{1}{\alpha-1}} \frac{-\log \mathbb{E}_{\delta_{\sqrt{1+\delta_*}y}}\Big(\exp\Big\{-\int_{(0,\infty)} g_2(a)X_1(\mathrm{d}a)\Big\}\Big)}{\mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right)} \\
&\leq \frac{-\log \mathbb{P}_{\delta_y}\big(M_1^{(0,\infty)} \leq z\big)}{\mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right)} = \frac{\mathbb{N}_y\big(M_1^{(0,\infty)} > z\big)}{\mathbb{N}_y\left(w_1((0,\infty)) \neq 0\right)} \\
&= \mathbb{N}_y\left(M_1^{(0,\infty)} > z \,\middle|\, w_1((0,\infty)) \neq 0\right).
\end{aligned} \tag{3.42}$$

Combining Eqs. 3.35, 3.41 and 3.42, we get the assertion of (i).

(ii) The proof of (ii) is similar. In Eq. 3.35, by replacing $\sqrt{t}y$ by y and applying Theorem 1.3(ii), we get that

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbb{P}_y\big(M_{\sqrt{t}}^{(0,\infty)} > \sqrt{t}z \big| \zeta^{(0,\infty)} > t\big) \\
&\geq \lim_{\theta\to\infty} \mathbb{E}\Big(1 - \exp\Big\{-\theta\int_{(0,\infty)} g_1(a)\eta_1(\mathrm{d}a)\Big\}\Big) = \mathbb{P}(M^{\eta_1} > z).
\end{aligned}$$

For the upper bound, the argument in Eq. 3.39 still holds in this case. Therefore, by Eq. 3.40 with $r = 1+\delta_*$, we get that

$$\begin{aligned}
&t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{P}_y\big(M_{t(1+\delta_*)}^{(0,\infty)} > \sqrt{t}z\big) \\
&\leq t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{E}_y\Big(1 - \exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}\int_{(0,\infty)} g_2\left(\frac{a}{\sqrt{t}}\right) Z_t^{(0,\infty)}(\mathrm{d}a)\Big\}\Big).
\end{aligned} \tag{3.43}$$

Combining Eq. 3.43, Theorem 1.1(ii) and Theorem 1.3(ii), we see that

$$\begin{aligned}
&\limsup_{t\to\infty} t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{P}_y\big(M_t^{(0,\infty)} > \sqrt{t}z\big)\\
&= (1+\delta_*)^{\frac{1}{\alpha-1}+\frac{1}{2}}\limsup_{t\to\infty} t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{P}_y\big(M_{t(1+\delta_*)}^{(0,\infty)} > \sqrt{t(1+\delta_*)}z\big)\\
&\le (1+\delta_*)^{\frac{1}{\alpha-1}+\frac{1}{2}}\limsup_{t\to\infty} t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{P}_y\left(M_{t(1+\delta_*)}^{(0,\infty)} > \sqrt{t}z\right)\\
&\le (1+\delta_*)^{\frac{1}{\alpha-1}+\frac{1}{2}}C^{(0,\infty)}(\alpha)R(y)\mathbb{E}\Big(1-\exp\Big\{-\int_{(0,\infty)} g_2(a)\eta_1(\mathrm{d}a)\Big\}\Big).
\end{aligned}$$

Now letting $\delta_* \to 0$ first and then $\varepsilon \to 0$, we see that

$$\begin{aligned}
&\limsup_{t\to\infty}\mathbb{P}_y\big(M_t^{(0,\infty)} > \sqrt{t}z\big|\zeta^{(0,\infty)} > t\big)\\
&\le \frac{C^{(0,\infty)}(\alpha)R(y)\mathbb{P}(M^{\eta_1} > z)}{\lim_{t\to\infty} t^{\frac{1}{\alpha-1}+\frac{1}{2}}\mathbb{P}_y\left(\zeta^{(0,\infty)} > t\right)} = \mathbb{P}(M^{\eta_1} > z),
\end{aligned}$$

which completes the proof of (ii). □

4 Proofs of the Auxiliary Results

4.1 Proof of Lemma 2.12

In this subsection, we assume that the Lévy process ξ satisfies **(H2)** and Eq. 2.19. We use

$$\Phi^+(t) := \int_0^t z e^{-\frac{z^2}{2}}\mathrm{d}z, \quad t \ge 0$$

to denote the Rayleigh distribution function.

Lemma 4.1 *For any $y > 0$ and $a \in (0,\infty]$, it holds that*

$$\lim_{t\to\infty}\sqrt{t}\mathbf{P}_y\big(\xi_t \le a\sqrt{t}, \tau_0^- > t\big) = \frac{2}{\sqrt{2\pi\sigma^2}}R(y)\Phi^+\left(\frac{a}{\sigma}\right).$$

Proof Recall that W_t is the Brownian motion with variance $\sigma^2 t$ introduced in Section 1.2. For any $r > 0$ and $\varepsilon \in (0, \delta/(2(5+2\delta)))$, where δ is the constant in Eq. 2.19, we define

$$A_r := \Big\{\sup_{0\le s\le 1}|\xi_{sr} - W_{sr}| \le r^{\frac{1}{2}-2\varepsilon}\Big\}.$$

Recall that the random walk S_n is given by $S_n = \xi_n$. For any $b \in \mathbb{R}$, define

$$\tau_b^{S,+} := \inf\left\{j \in \mathbb{N} : S_j > b\right\}.$$

Then we have the following decomposition:

$$\sqrt{t}\mathbf{P}_y\left(\xi_t \le a\sqrt{t}, \tau_0^- > t\right) = \sum_{k=1}^4 I_k,$$

where I_k are defined by

$$I_1 := \sqrt{t}\mathbf{P}_y\Big(\xi_t \le a\sqrt{t}, \tau_0^- > t, \tau_{t^{1/2-\varepsilon}}^{S,+} > [t^{1-\varepsilon}]\Big),$$

$$I_2 := \sqrt{t}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(\mathbf{P}_{\xi_k}\big(\xi_{t-k} \le a\sqrt{t}, \tau_0^- > t-k, A_{t-k}^c\big); \tau_0^- > k, \tau_{t^{1/2-\varepsilon}}^{S,+} = k\big),$$

$$I_3 := \sqrt{t}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(\mathbf{P}_{\xi_k}\big(\xi_{t-k} \le a\sqrt{t}, \tau_0^- > t-k, A_{t-k}\big); \tau_0^- > k, \xi_k > t^{(1-\varepsilon)/2}, \tau_{t^{1/2-\varepsilon}}^{S,+} = k\big),$$

$$I_4 := \sqrt{t}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(\mathbf{P}_{\xi_k}\big(\xi_{t-k} \le a\sqrt{t}, \tau_0^- > t-k, A_{t-k}\big); \tau_0^- > k, \xi_k \le t^{(1-\varepsilon)/2}, \tau_{t^{1/2-\varepsilon}}^{S,+} = k\big).$$

(i) In this part, we show that $\lim_{t\to\infty} I_1 = 0$. Since $I_1 \le \sqrt{t}\mathbf{P}_y\big(\tau_{t^{1/2-\varepsilon}}^{S,+} > [t^{1-\varepsilon}]\big)$, it suffices to prove that

$$\lim_{t\to\infty} \sqrt{t}\mathbf{P}_y\big(\tau_{t^{1/2-\varepsilon}}^{S,+} > [t^{1-\varepsilon}]\big) = 0. \tag{4.1}$$

Since $[t^{1-\varepsilon}] \ge [t^{\varepsilon} - 1][t^{1-2\varepsilon}] =: L_1 \cdot L_2$, we have

$$\mathbf{P}_y\Big(\tau_{t^{1/2-\varepsilon}}^{S,+} > [t^{1-\varepsilon}]\Big) = \mathbf{P}_y\Big(\max_{j\le[t^{1-\varepsilon}]} |S_j| \le t^{1/2-\varepsilon}\Big) \le \mathbf{P}_y\Big(\max_{j\le L_1} |S_{L_2 j}| \le t^{1/2-\varepsilon}\Big). \tag{4.2}$$

Applying the Markov property repeatedly, we get that

$$\begin{aligned}\mathbf{P}_y\Big(\max_{j\le L_1} |S_{L_2 j}| \le t^{1/2-\varepsilon}\Big) &\le \sup_{x\in\mathbb{R}} \mathbf{P}_x\Big(|S_{L_2}| \le^{1/2-\varepsilon}\Big)\mathbf{P}_y\Big(\max_{j\le L_1-1} |S_{L_2 j}| \le t^{1/2-\varepsilon}\Big)\\ &\le \cdots \le \left(\sup_{x\in\mathbb{R}} \mathbf{P}_x\Big(|S_{L_2}| \le t^{1/2-\varepsilon}\Big)\right)^{L_1}.\end{aligned} \tag{4.3}$$

The classical central limit theorem implies that when $x > 2\sqrt{L_2}$

$$\begin{aligned}&\mathbf{P}_x\big(|S_{L_2}| \le t^{1/2-\varepsilon}\big) \le \mathbf{P}_x\big(|S_{L_2}| \le 2\sqrt{L_2}\big)\\ &= \mathbf{P}_0\big(-x - 2\sqrt{L_2} \le S_{L_2} \le 2\sqrt{L_2} - x\big) \le \mathbf{P}_0\big(S_{L_2} \le 0\big) \overset{t\to\infty}{\longrightarrow} \frac{1}{2}.\end{aligned} \tag{4.4}$$

Similarly, when $x < -2\sqrt{L_2}$, we have

$$\mathbf{P}_x\big(|S_{L_2}| \le t^{1/2-\varepsilon}\big) \le \mathbf{P}_0\big(S_{L_2} \ge 0\big) \overset{t\to\infty}{\longrightarrow} \frac{1}{2} \tag{4.5}$$

and that for $|x| \le 2\sqrt{L_2}$,

$$\mathbf{P}_x\big(|S_{L_2}| \le t^{1/2-\varepsilon}\big) \le \mathbf{P}_0\big(-4\sqrt{L_2} \le S_{L_2} \le 4\sqrt{L_2}\big) \overset{t\to\infty}{\longrightarrow} \int_{-4}^{4} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}}\,\mathrm{d}z. \tag{4.6}$$

Combining Eqs. 4.4, 4.5 and 4.6, we see that there exist $c \in (0,1)$ and $t_0 > 0$ such that

$$\mathbf{P}_x\big(|S_{L_2}| \le t^{1/2-\varepsilon}\big) < c, \quad x \in \mathbb{R}, t \ge t_0.$$

Plugging this into Eq. 4.3 and combining the conclusion with Eq. 4.2, we get Eq. 4.1.

(ii) In this part, we show that $\lim_{t\to\infty} I_2 = 0$. By Lemma 2.11 and the definition of A_r, we have

$$\begin{aligned}
I_2 &\le \sqrt{t}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\Big(\mathbf{P}_{\xi_k}\left(A^c_{t-k}\right); \tau_0^- > k, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\Big),\\
&\lesssim \sqrt{t}\sum_{k=1}^{[t^{1-\varepsilon}]} \frac{N_*(2\varepsilon)}{(t-k)^{(2+\delta)(\frac{1}{2}-2\varepsilon)-1}}\mathbf{P}_y\big(\tau_0^- > k, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big)\\
&\lesssim \frac{\sqrt{t}N_*(2\varepsilon)}{t^{(2+\delta)(\frac{1}{2}-2\varepsilon)-1}}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{P}_y\big(\tau_0^{S,-} > k, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big).
\end{aligned} \tag{4.7}$$

Since S_k is also a martingale under $\mathbf{P}_y$, using the fact that $S_k \ge t^{1/2-\varepsilon}$ on the event $\{\tau^{S,+}_{t^{1/2-\varepsilon}} = k\}$, by Eq. 4.7,

$$\begin{aligned}
I_2 &\lesssim \frac{\sqrt{t}N_*(2\varepsilon)}{t^{(2+\delta)(\frac{1}{2}-2\varepsilon)-1}t^{1/2-\varepsilon}}\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{P}_y\big(S_k;\, \tau_0^{S,-} > k, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big)\\
&\le \frac{\sqrt{t}N_*(2\varepsilon)}{t^{(2+\delta)(\frac{1}{2}-2\varepsilon)-1}t^{1/2-\varepsilon}}\mathbf{E}_y\big(S_{[t^{1-\varepsilon}]\wedge\tau_0^{S,-}\wedge\tau^{S,+}_{t^{1/2}-\varepsilon}}\big) = \frac{N_*(2\varepsilon)y}{t^{\delta/2-(5+2\delta)\varepsilon}},
\end{aligned}$$

we see that when ε is sufficiently small so that $\varepsilon < \delta/(2(5+2\delta))$, we have $\lim_{t\to\infty} I_2 = 0$.

(iii) In this part, we show that $\lim_{t\to\infty} I_3 = 0$. Set $x' = \xi_k > t^{(1-\varepsilon)/2}$, by Lemma 2.10(i) with $t = 1$, we have that

$$\mathbf{P}_{x'}\big(\xi_{t-k} \le a\sqrt{t}, \tau_0^- > t-k, A_{t-k}\big) \le \mathbf{P}_{x'}\left(\tau_0^- > t-k\right) \lesssim \frac{x'+1}{\sqrt{t-k}} \lesssim \frac{x'}{\sqrt{t}},$$

where in the last inequality we used the fact that $t \lesssim t-k$ for all $k \le [t^{1-\varepsilon}]$. Therefore, we have

$$\begin{aligned}
I_3 &\lesssim \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(\xi_k;\, \tau_0^- > k, \xi_k > t^{(1-\varepsilon)/2}, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big)\\
&\le \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(S_k;\, \tau_0^{S,-} > k, S_k > t^{(1-\varepsilon)/2}, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big).
\end{aligned} \tag{4.8}$$

Now we deal with the random walk S_k. Set $\Delta_k := S_k - S_{k-1}$. We have that

$$\begin{aligned}
&\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(S_k;\, \tau_0^{S,-} > k, S_k > t^{(1-\varepsilon)/2}, \tau^{S,+}_{t^{1/2-\varepsilon}} = k\big)\\
&\le \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big((S_{k-1}+\Delta_k);\, \tau_0^{S,-} > k-1, S_k > t^{(1-\varepsilon)/2}, \Delta_k > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big)
\end{aligned}$$

$$\leq \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(S_{k-1};\, \tau_0^{S,-} > k-1\big)\mathbf{P}_y\big(\Delta_k > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big)$$
$$+ \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{P}_y\big(\tau_0^{S,-} > k-1\big)\mathbf{E}_y\big(\Delta_k;\, \Delta_k > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big).$$

Noting that, for any fixed y, $\mathbf{E}_y\big(S_{k-1};\, \tau_0^{S,-} > k-1\big) \lesssim 1$, $\mathbf{P}_y\big(\tau_0^{S,-} > k-1\big) \lesssim \frac{1}{\sqrt{k}}$ and that $(\Delta_k, \mathbf{E}_y) \overset{\mathrm{d}}{=} (\xi_1, \mathbf{E}_0)$, we can continue the estimates in the display above to get

$$\begin{aligned}
&\sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(S_k;\, \tau_0^{S,-} > k,\, S_k > t^{(1-\varepsilon)/2},\, \tau_{t^{1/2-\varepsilon}}^{S,+} = k\big)\\
&\lesssim \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{P}_0\big(\xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big) + \sum_{k=1}^{[t^{1-\varepsilon}]} \frac{1}{\sqrt{k}}\mathbf{E}_0\big(\xi_1;\, \xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big)\\
&\leq [t^{1-\varepsilon}]\frac{\mathbf{E}_0\left(\xi_1^2;\, \xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\right)}{\left(t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\right)^2} + \int_0^{[t^{1-\varepsilon}]} \frac{1}{\sqrt{x}}\mathrm{d}x \frac{\mathbf{E}_0\left(\xi_1^2;\, \xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\right)}{t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}}\\
&\lesssim \mathbf{E}_0\left(\xi_1^2;\, \xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\right). \qquad (4.9)
\end{aligned}$$

Combining Eqs. 4.8 and 4.9, we get that

$$I_3 \lesssim \mathbf{E}_0\big(\xi_1^2;\, \xi_1 > t^{(1-\varepsilon)/2} - t^{1/2-\varepsilon}\big) \overset{t\to\infty}{\longrightarrow} 0.$$

(iv) In this part, we deal with I_4. We allow a to be ∞. For $k \leq [t^{1-\varepsilon}]$ and $x' > 0$, define

$$K(k, x') := \mathbf{P}_{x'}\left(\xi_{t-k} \leq a\sqrt{t},\, \tau_0^- > t-k,\, A_{t-k}\right).$$

By the definition of A_r, we see that

$$K(k, x') \leq \mathbf{P}_{x'}\left(W_{t-k} \leq a\sqrt{t} + (t-k)^{\frac{1}{2}-2\varepsilon},\, \min_{s\leq t-k} W_s > -(t-k)^{\frac{1}{2}-2\varepsilon}\right). \qquad (4.10)$$

Since $B_t := W_t/\sigma$ is a standard Brownian motion, we see that $(\sigma B_t, \mathbf{P}_y) \overset{\mathrm{d}}{=} (W_t, \mathbf{P}_{\sigma y})$. For any $z > 0$, define

$$\left.\frac{\mathrm{d}\mathbf{P}_z^{\uparrow}}{\mathrm{d}\mathbf{P}_z}\right|_{\sigma(B_s, s\leq t)} := \frac{B_t}{z} 1_{\{\min_{s\leq t} B_s > 0\}}. \qquad (4.11)$$

It is well-known (for example, see [11, (3.1)]) that under $\mathbf{P}_z^{\uparrow}$, B_s is a Bessel-3 process with transition density

$$p_t^{\uparrow}(x, y) = \frac{y}{x\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}} \left(1 - e^{-\frac{2xy}{t}}\right) 1_{\{y>0\}}. \qquad (4.12)$$

Set

$$x^* := \frac{x' + (t-k)^{\frac{1}{2}-2\varepsilon}}{\sigma} \quad \text{and} \quad a^* := \frac{a\sqrt{t}}{\sigma\sqrt{t-k}} + \frac{2}{\sigma(t-k)^{2\varepsilon}}.$$

Combining Eqs. 4.10, 4.11, 4.12 and the inequality $1-e^{-x} \le x$, we obtain that

$$
\begin{aligned}
K(k,x') &\le \mathbf{P}_{x^*}\big(B_{t-k} \le a^*\sqrt{t-k},\ \min_{s\le t-k} B_s > 0\big) = x^* \mathbf{E}^{\uparrow}_{x^*}\Big(\frac{1}{B_{t-k}};\ B_{t-k} \le a^*\sqrt{t-k}\Big) \\
&= \frac{1}{\sqrt{2\pi(t-k)}} \int_0^{a^*\sqrt{t-k}} e^{-\frac{(y-x^*)^2}{2(t-k)}}\big(1-e^{-\frac{2x^* y}{t-k}}\big)\mathrm{d}y \\
&\le \frac{2x^*}{\sqrt{2\pi(t-k)^3}} \int_0^{a^*\sqrt{t-k}} y e^{-\frac{(y-x^*)^2}{2(t-k)}}\,\mathrm{d}y = \frac{2x^*}{\sqrt{2\pi(t-k)}} \int_0^{a^*} y e^{-\frac{y^2}{2}+\frac{yx^*}{\sqrt{t-k}}-\frac{(x^*)^2}{2(t-k)}}\mathrm{d}y \\
&\le \frac{2}{\sqrt{2\pi(t-k)}}\Big(\frac{x'}{\sigma}+\frac{t^{\frac12-2\varepsilon}}{\sigma}\Big)\int_0^{a^*} y e^{-\frac{y^2}{2}+\frac{yx^*}{\sqrt{t-k}}}\mathrm{d}y \\
&= \frac{2}{\sqrt{2\pi(t-k)}}\Big(\frac{x'}{\sigma}+\frac{t^{\frac12-2\varepsilon}}{\sigma}\Big)\Big(\int_0^{a/\sigma} y e^{-\frac{y^2}{2}+\frac{yx^*}{\sqrt{t-k}}}\mathrm{d}y + \int_{a/\sigma}^{a^*} y e^{-\frac{y^2}{2}+\frac{yx^*}{\sqrt{t-k}}}\mathrm{d}y\Big), \qquad (4.13)
\end{aligned}
$$

where the last term on the right-hand side of the inequality above is 0 when $a=\infty$. Note that for all $x' \le t^{(1-\varepsilon)/2}$ and all $k \le [t^{1-\varepsilon}]$,

$$
\frac{x^*}{\sqrt{t-k}} \lesssim \frac{t^{(1-\varepsilon)/2}+t^{\frac12-2\varepsilon}}{\sqrt{t}} \lesssim t^{-\varepsilon/2} \to 0 \quad \text{as } t\to\infty.
$$

Note also that, for $a\in(0,\infty)$, $a^*-\sigma^{-1}a \to 0$ as $t\to\infty$. Therefore, we conclude from Eq. 4.13 that for any $a\in(0,\infty]$ and $\delta_0>0$, there exists $T>0$ such that when $t>T$, for all $x' \le t^{(1-\varepsilon)/2}$,

$$
K(k,x') \le \frac{2(1+\delta_0)}{\sqrt{2\pi t}}\Big(\frac{x'}{\sigma}+\frac{t^{\frac12-2\varepsilon}}{\sigma}\Big)\Phi^+\Big(\frac{a}{\sigma}\Big).
$$

Plugging this into the definition of I_4, we see that when t is large enough,

$$
I_4 \le \frac{2(1+\delta_0)}{\sqrt{2\pi}}\Phi^+\Big(\frac{a}{\sigma}\Big)\sum_{k=1}^{[t^{1-\varepsilon}]}\mathbf{E}_y\Big(\Big(\frac{\xi_k}{\sigma}+\frac{t^{\frac12-2\varepsilon}}{\sigma}\Big);\ \tau_0^->k,\ \xi_k\le t^{(1-\varepsilon)/2},\ \tau^{S,+}_{t^{1/2-\varepsilon}}=k\Big).
$$

Note that on $\{\tau^{S,+}_{t^{1/2-\varepsilon}}=k\}$, we have $\xi_k=S_k\ge t^{1/2-\varepsilon}$, which implies that $t^{\frac12-2\varepsilon}\le\delta_0\xi_k$ for t large enough. Hence, for t large enough, by the inequality $R(x)\ge x$, we have

$$
\begin{aligned}
I_4 &\le \frac{2(1+\delta_0)^2}{\sqrt{2\pi\sigma^2}}\Phi^+\Big(\frac{a}{\sigma}\Big)\sum_{k=1}^{[t^{1-\varepsilon}]}\mathbf{E}_y\big(\xi_k;\ \tau_0^->k,\ \xi_k\le t^{(1-\varepsilon)/2},\ \tau^{S,+}_{t^{1/2-\varepsilon}}=k\big) \\
&\le \frac{2(1+\delta_0)^2}{\sqrt{2\pi\sigma^2}}\Phi^+\Big(\frac{a}{\sigma}\Big)\mathbf{E}_y\Big(R\big(\xi_{\tau^{S,+}_{t^{1/2-\varepsilon}}}\big);\ \tau_0^->\tau^{S,+}_{t^{1/2-\varepsilon}},\ \tau^{S,+}_{t^{1/2-\varepsilon}}\le[t^{1-\varepsilon}]\Big) \\
&\le \frac{2(1+\delta_0)^2}{\sqrt{2\pi\sigma^2}}\Phi^+\Big(\frac{a}{\sigma}\Big)\mathbf{E}_y\Big(R\big(\xi_{\tau^{S,+}_{t^{1/2-\varepsilon}}\wedge[t^{1-\varepsilon}]}\big);\ \tau_0^->\tau^{S,+}_{t^{1/2-\varepsilon}}\wedge[t^{1-\varepsilon}]\Big) \\
&= \frac{2(1+\delta_0)^2}{\sqrt{2\pi\sigma^2}}R(y)\Phi^+\Big(\frac{a}{\sigma}\Big), \qquad (4.14)
\end{aligned}
$$

where in the last equality we used Lemma 2.8(ii).

For the lower bound, we have, similarly, for $t^{1/2-\varepsilon}\le x'\le t^{(1-\varepsilon)/2}$,

$$
K(k,x') \ge \mathbf{P}_{x'}\Big(W_{t-k}\le a\sqrt{t}-(t-k)^{\frac12-2\varepsilon},\ \min_{s\le t-k} W_s > (t-k)^{\frac12-2\varepsilon}\Big) - \mathbf{P}_{x'}\Big(A^c_{t-k}\Big).
$$

In this case, we define

$$x_* := \frac{x' - (t-k)^{\frac{1}{2}-2\varepsilon}}{\sigma} \quad \text{and} \quad a_* := \frac{a\sqrt{t}}{\sigma\sqrt{t-k}} - \frac{2}{\sigma(t-k)^{2\varepsilon}}.$$

Then combining the inequalities $(y-x)^2 \le y^2 + x^2$, $1 - e^{-x} \ge x(1-x)$ for all $x, y > 0$ and an argument similar to that used in Eq. 4.13, by Lemma 2.11, there exists some constant $C_\varepsilon > 0$ such that

$$\begin{aligned}
K(k, x') &\ge \mathbf{P}_{x_*}\Big(B_{t-k} \le a_*\sqrt{t-k}, \min_{s \le t-k} B_s > 0\Big) - \frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \\
&= \frac{1}{\sqrt{2\pi(t-k)}} \int_0^{a_*\sqrt{t-k}} e^{-\frac{(y-x_*)^2}{2(t-k)}} \big(1 - e^{-\frac{2x_* y}{t-k}}\big) \mathrm{d}y - \frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \\
&\ge \frac{2x_*}{\sqrt{2\pi(t-k)^3}} \int_0^{a_*\sqrt{t-k}} y\Big(1 - \frac{2x_* y}{t-k}\Big) e^{-\frac{y^2+x_*^2}{2(t-k)}} \mathrm{d}y - \frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \\
&= \frac{2x_* e^{-\frac{x_*^2}{2(t-k)}}}{\sqrt{2\pi(t-k)}} \int_0^{a_*} y\Big(1 - \frac{2x_* y}{\sqrt{t-k}}\Big) e^{-\frac{y^2}{2}} \mathrm{d}y - \frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \\
&\ge \frac{2e^{-\frac{x_*^2}{2(t-k)}}}{\sqrt{2\pi t}} \Big(\frac{x'}{\sigma} - \frac{t^{\frac{1}{2}-2\varepsilon}}{\sigma}\Big)\Big(\int_0^{a_*} y e^{-\frac{y^2}{2}} \mathrm{d}y - \frac{2x_*}{\sqrt{t-k}} \int_0^\infty y^2 e^{-\frac{y^2}{2}} \mathrm{d}y\Big) - \frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}}.
\end{aligned}$$

Noting that for all $t^{1/2-\varepsilon} \le x' \le t^{(1-\varepsilon)/2}$ and $k \le [t^{1-\varepsilon}]$,

$$\frac{x_*}{\sqrt{t-k}} \lesssim \frac{t^{(1-\varepsilon)/2} + t^{\frac{1}{2}-2\varepsilon}}{\sqrt{t}} \lesssim t^{-\varepsilon/2}, \quad \frac{t^{\frac{1}{2}-2\varepsilon}}{x'} \le \frac{t^{\frac{1}{2}-2\varepsilon}}{t^{1/2-\varepsilon}} = t^{-\varepsilon},$$

$$\frac{C_\varepsilon}{(t-k)^{(\frac{1}{2}-2\varepsilon)(\delta+2)-1}} \lesssim \frac{x'}{\sqrt{t}} t^{-(\frac{\delta}{2}-\varepsilon(2\delta+5))},$$

and for any $a \in (0, \infty)$, $|\sigma^{-1}a - a_*| \lesssim t^{-\varepsilon}$. Therefore, for any $\delta_0 \in (0, 1)$, when t is large enough, we have for all $k \le [t^{1-\varepsilon}]$ and $t^{1/2-\varepsilon} \le x' \le t^{(1-\varepsilon)/2}$,

$$K(k, x') \ge \frac{2(1-\delta_0)}{\sqrt{2\pi t\sigma^2}} x' \Phi^+\left(\frac{a}{\sigma}\right).$$

Therefore, for t large enough,

$$I_4 \ge \frac{2(1-\delta_0)}{\sqrt{2\pi\sigma^2}} \Phi^+\left(\frac{a}{\sigma}\right) \sum_{k=1}^{[t^{1-\varepsilon}]} \mathbf{E}_y\big(\xi_k; \tau_0^- > k, \xi_k \le t^{(1-\varepsilon)/2}, \tau_{t^{1/2-\varepsilon}}^{S,+} = k\big).$$

It follows from Lemma 2.8 that $x \ge (1-\delta_0)R(x)$ for $x \ge t^{1/2-\varepsilon}$ with t large enough. Thus, when t is large enough,

$$\begin{aligned}
I_4 &\ge \frac{2(1-\delta_0)^2}{\sqrt{2\pi\sigma^2}} \Phi^+\left(\frac{a}{\sigma}\right) \mathbf{E}_y\Big(R\big(\xi_{\tau_{t^{1/2-\varepsilon}}^{S,+}}\big); \tau_0^- > \tau_{t^{1/2-\varepsilon}}^{S,+}, \tau_{t^{1/2-\varepsilon}}^{S,+} \le [t^{1-\varepsilon}]\Big) \\
&= \frac{2(1-\delta_0)^2}{\sqrt{2\pi\sigma^2}} \Phi^+\left(\frac{a}{\sigma}\right) \Big(R(y) - \mathbf{E}_y\big(R\left(\xi_{[t^{1-\varepsilon}]}\right); \tau_0^- > [t^{1-\varepsilon}], \tau_{t^{1/2-\varepsilon}}^{S,+} > [t^{1-\varepsilon}]\big)\Big), \quad (4.15)
\end{aligned}$$

where in the last inequality we used the following fact:

$$
\begin{aligned}
&R(y)\\
=&\mathbf{E}_y\Big(R(\xi_{\tau^{S,+}_{t^{1/2-\varepsilon}}\wedge[t^{1-\varepsilon}]});\, \tau_0>\tau^{S,+}_{t^{1/2-\varepsilon}}\wedge[t^{1-\varepsilon}]\Big)\\
=&\mathbf{E}_y\Big(R(\xi_{\tau^{S,+}_{t^{1/2-\varepsilon}}});\, \tau_0^->\tau^{S,+}_{t^{1/2-\varepsilon}},\tau^{S,+}_{t^{1/2-\varepsilon}}\le[t^{1-\varepsilon}]\Big)+\mathbf{E}_y\big(R\left(\xi_{[t^{1-\varepsilon}]}\right);\tau_0^->[t^{1-\varepsilon}],\tau^{S,+}_{t^{1/2-\varepsilon}}>[t^{1-\varepsilon}]\big).
\end{aligned}
$$

Noting that $\xi_{[t^{1-\varepsilon}]}\le t^{1/2-\varepsilon}$ on $\{\tau^{S,+}_{t^{1/2-\varepsilon}}>[t^{1-\varepsilon}]\}$ and applying Lemma 2.10 with $t=1$, we get

$$
\begin{aligned}
&\mathbf{E}_y\Big(R\big(\xi_{[t^{1-\varepsilon}]}\big);\,\tau_0^->[t^{1-\varepsilon}],\tau^{S,+}_{t^{1/2-\varepsilon}}>[t^{1-\varepsilon}]\Big)\\
&\le R(t^{1/2-\varepsilon})\mathbf{P}_y\left(\tau_0^->[t^{1-\varepsilon}]\right)\lesssim t^{1/2-\varepsilon}\frac{y+1}{\sqrt{[t^{1-\varepsilon}]}}\lesssim(y+1)t^{-\varepsilon/2}.
\end{aligned}\tag{4.16}
$$

Combining Eqs. 4.14, 4.15 and 4.16, we conclude that

$$
\frac{2(1+\delta_0)^2}{\sqrt{2\pi\sigma^2}}R(y)\Phi^+\left(\frac{a}{\sigma}\right)\ge\limsup_{t\to\infty}I_4\ge\liminf_{t\to\infty}I_4\ge\frac{2(1-\delta_0)^2}{\sqrt{2\pi\sigma^2}}R(y)\Phi^+\left(\frac{a}{\sigma}\right).
$$

Letting $\delta_0\to0$, we arrive at the assertion of the lemma. □

Proof of Lemma 2.12 Define a sequence of measures

$$
\mu^{(t)}(D):=\sqrt{t}\mathbf{P}_y\Big(\frac{\xi_t}{\sigma\sqrt{t}}\in D,\tau_0^->t\Big)\quad\text{and}\quad\mu(D):=\frac{2R(y)}{\sqrt{2\pi\sigma^2}}\int_D ze^{-\frac{z^2}{2}}\mathrm{d}z,\quad D\in\mathcal{B}((0,\infty)).
$$

Lemma 4.1 implies that for any $y>0$ and any $a\in(0,\infty)$,

$$
\lim_{t\to\infty}\mu^{(t)}((0,a])=\mu((0,a])
$$

and that $\lim_{t\to\infty}\mu^{(t)}((0,\infty))=\mu((0,\infty))$. Therefore, $\mu^{(t)}$ weakly converge to μ and this completes the proof of the lemma. □

4.2 Proof of Lemma 3.2

Proof of Lemma 3.2 First, it follows from Lemma 2.14(i) that $\sup_{y>0}v_\infty^X(r,y)<\infty$ for any $r>0$. Next we prove that for any $y\in(0,\infty)$, $\lim_{r\to0+}v_\infty^X(r,y)=\infty$. By the definition of $v_\infty^{(t)}(r,y)$ in Eq. 2.5, we have

$$
\begin{aligned}
v_\infty^{(t)}(r,y)&\ge t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\zeta^{(0,\infty)}>tr,\inf_{s>0}\inf_{u\in N(s)}X_u(s)>0\big)\\
&=t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\zeta>tr,\inf_{s>0}\inf_{u\in N(s)}X_u(s)>0\big)\\
&\ge t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\left(\zeta>tr\right)-t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\big(\inf_{s>0}\inf_{u\in N(s)}X_u(s)\le0\big)\\
&=t^{\frac{1}{\alpha-1}}\mathbb{P}_{\sqrt{t}y}\left(\zeta>tr\right)-t^{\frac{1}{\alpha-1}}\mathbb{P}_0\big(\widetilde{M}\ge\sqrt{t}y\big),
\end{aligned}\tag{4.17}
$$

where in the last equality, $\widetilde{M}$ is the maximal displacement of the critical branching Lévy process with spatial motion $-\xi$, branching rate β and offspring distribution $\{p_k\}$. Combining Eqs. 1.2, 1.4 with M replaced by $\widetilde{M}$ and Eq. 4.17, we see that

$$v_\infty^X(r,y) \geq \frac{C(\alpha)}{r^{\frac{1}{\alpha-1}}} - \frac{\widetilde{\theta}(\alpha)}{y^{\frac{2}{\alpha-1}}} \stackrel{r\to 0+}{\longrightarrow} \infty,$$

which implies $\lim_{r\to 0+} v_\infty^X(r,y) = \infty$.

Letting $t \to \infty$ first and then $y \to 0+$ in Lemma 2.14 (i), we easily see that $\lim_{y\to 0+} v_\infty^X(r,y) = 0$ for any $r > 0$.

Now we prove that v_∞^X satisfies the partial differential differential equation in Eq. 3.11. For any $0 < w < r$ and $y > 0$, by Eq. 2.6,

$$v_\infty^{(t)}(r,y) = \mathbf{E}_y\big(v_\infty^{(t)}(r-w,\xi^{(t)}_{w\wedge\tau_0^{(t),-}})\big) - \mathbf{E}_y\Big(\int_0^w \phi^{(t)}(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}}))\mathrm{d}s\Big). \quad (4.18)$$

By Lemma 2.14(i), $K := \sup_{t>0,s\leq w,y\in\mathbb{R}} v_\infty^{(t)}(r-s,y) < \infty$. By Lemma 2.14(ii) and the definition of $\phi^{(t)}$ in Eq. 2.2, $\lim_{t\to\infty}\frac{\phi^{(t)}(v)}{v^\alpha} = C(\alpha)$ uniformly for $v \in [0,K]$. Note that $\varphi(\lambda) = C(\alpha)\lambda^\alpha$ by definition Eq. 1.9. Therefore, for any $\varepsilon > 0$, there exists $T_1 > 0$ such that when $t > T_1$,

$$\phi^{(t)}(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}})) \geq (1-\varepsilon)\varphi\left(v_\infty^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}})\right). \quad (4.19)$$

Plugging Eqs. 3.9, 3.10 and 4.19 into Eq. 4.18 we see that when $t > \max\{T_1,T_2\}$,

$$\begin{aligned} v_\infty^{(t)}(r,y) &\leq 3\varepsilon + \mathbf{E}_y\big(v_\infty^X(r-w,\xi^{(t)}_{w\wedge\tau_0^{(t),-}})\big) \\ &\quad - (1-\varepsilon)\mathbf{E}_y\Big(\int_0^{w\wedge\tau_0^{(t),-}} \varphi\big((v_\infty^X(r-s,\xi_s^{(t)})-3\varepsilon)^+\big)\mathrm{d}s\Big). \end{aligned} \quad (4.20)$$

Define $W_s^{(t)} = \frac{W_{st}}{\sqrt{t}}$, which has the same law as W_s. Let $\tau_x^{(t),W,-}$ be the exit time of $W_s^{(t)}$ from (x,∞), which has the same law as $\tau_x^{W,-}$. For any fixed $w \geq 0$, by Lemma 2.11 with t replaced by tw, we get that for any $0 \leq s \leq w$ and $a \geq 0$, for each $\gamma \in (0,\frac{\delta}{2(2+\delta)})$,

$$\begin{aligned} &\limsup_{t\to\infty}\mathbf{P}_y\left(\xi^{(t)}_{s\wedge\tau_0^{(t),-}} > a\right) \leq \limsup_{t\to\infty}\mathbf{P}_y\left(\xi^{(t)}_{s\wedge\tau_0^{(t),-}} > a,\ \sup_{0\leq q\leq w}\left|\xi_q^{(t)} - W_q^{(t)}\right| \leq t^{-\gamma}w^{\frac12-\gamma}\right) \\ &\leq \limsup_{t\to\infty}\mathbf{P}_y\left(W_s^{(t)} > a - t^{-\gamma}w^{\frac12-\gamma},\ \sup_{0\leq s\leq w}\left|\xi_s^{(t)} - W_s^{(t)}\right| \leq t^{-\gamma}w^{\frac12-\gamma},\ \tau^{(t),W,-}_{-t^{-\gamma}w^{\frac12-\gamma}} > s\right) \\ &\leq \limsup_{t\to\infty}\mathbf{P}_y\left(W_s > a - t^{-\gamma}w^{\frac12-\gamma},\ \tau^{W,-}_{-t^{-\gamma}w^{\frac12-\gamma}} > s\right) = \mathbf{P}_y\left(W_s^0 > a\right). \end{aligned}$$

Using a similar argument for the lower bound, we can get that $\lim_{t\to\infty}\mathbf{P}_y\left(\xi^{(t)}_{s\wedge\tau_0^{(t),-}} > a\right) = \mathbf{P}_y\left(W_s^0 > a\right)$ for all $a \geq 0$. Therefore, for any $s > 0$, $\left(\xi^{(t)}_{s\wedge\tau_0^{(t),-}}, \mathbf{P}_y\right)$ converges in distribution to $\left(W_s^0, \mathbf{P}_y\right)$. Taking $t = t_k$ in Eq. 4.20 and letting $k \to \infty$, we get that

$$v_\infty^X(r, y) \le 3\varepsilon + \mathbf{E}_y\left(v_\infty^X(r-w, W_w^0)\right) \\ - (1-\varepsilon)\mathbf{E}_y\Big(\int_0^w \varphi\big((v_\infty^X(r-s, W_s^0) - 3\varepsilon)^+\big)\mathrm{d}s\Big),$$

where W_s^0 is a Brownian motion with variance $\sigma^2 t$ stopped upon exiting $(0, \infty)$. Now letting $\varepsilon \to 0$, we finally conclude that

$$v_\infty^X(r, y) \le \mathbf{E}_y\left(v_\infty^X(r-w, W_w^0)\right) - \mathbf{E}_y\Big(\int_0^w \varphi\big(v_\infty^X(r-s, W_s^0)\big)\mathrm{d}s\Big).$$

A very similar argument for the lower bound implies that, for any $0 < w < r$ and $y > 0$, $v_\infty^X(r, y)$ solves the equation

$$v_\infty^X(r, y) = \mathbf{E}_y\left(v_\infty^X(r-w, W_w^0)\right) - \mathbf{E}_y\Big(\int_0^w \varphi(v_\infty^X(r-s, W_s^0))\mathrm{d}s\Big). \tag{4.21}$$

This implies the desired result. Indeed, we may rewrite Eq. 4.21 as

$$v_\infty^X(r+w, y) = \mathbf{E}_y\left(v_\infty^X(r, W_w^0)\right) - \mathbf{E}_y\left(\int_0^w \varphi(v_\infty^X(r+w-s, W_s^0))\mathrm{d}s\right), \quad y > 0, r, w \ge 0.$$

For each fixed $r > 0$, set $f(y) = v_\infty^X(r, y)$. Then the function $u(w, y) := v_\infty^X(r+w, y)$ is the solution of the integral equation

$$u(w, y) = \mathbf{E}_y\left(f(W_w^0)\right) - \mathbf{E}_y\Big(\int_0^w \varphi(u(w-s, W_s^0))\mathrm{d}s\Big), \quad y > 0, w \ge 0.$$

Recall that W_s^0 is the Brownian motion W (with diffusion coefficient σ^2) stopped at $\tau_0^{W,-}$ and the generator of W_s^0 in the domain $(0, \infty)$ is $\frac{\sigma^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$. Combining the display above with Eqs. 3.6 and 3.7, and repeating the argument in [5, Sections 8.1 and 8.2], we get that

$$\frac{\partial}{\partial w} v_\infty^X(r+w, y) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} v_\infty^X(r+w, y) - \varphi\left(v_\infty^X(r+w, y)\right), \quad r, w, y > 0.$$

Since $r > 0$ is arbitrary, we get

$$\frac{\partial}{\partial w} v_\infty^X(w, y) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2} v_\infty^X(w, y) - \varphi\left(v_\infty^X(w, y)\right), \quad \text{in } (0, \infty) \times (0, \infty).$$

The proof is now complete. □

4.3 Proofs of Propositions 3.5 and 3.6

In this subsection, we assume that **(H1) (H2)** and Eq. 2.19 hold. When ξ is a standard Brownian motion, $\left(\xi_r^{(t)}, \mathbf{P}_y\right) \overset{\mathrm{d}}{=} \left(\xi_r, \mathbf{P}_y\right)$, and Proposition 3.6 follows immediately from [20, Proposition 4.5].

Lemma 4.2 *Let f be a bounded Lipschitz function on $\mathbb{R}_+$ with $f(0) = 0$.*

(i) For any $r, y, w > 0$ and $t > 1$, it holds that

$$t^{\frac{1}{\alpha-1}}\Big|\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)-\mathbf{E}_{y+w}\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)\Big|$$
$$\lesssim\Big(\frac{1}{\log t}+w\Big)(1+r^{-1/2})=:G_1^{(t)}(r,w).$$

(ii) For any $r, y, q > 0$ and $t > 1$, it holds that

$$t^{\frac{1}{\alpha-1}}\Big|\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)-\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{(r+q)\wedge\tau_0^{(t),-}}\big)\Big\}\Big)\Big|$$
$$\lesssim\Big(\frac{1}{\log t}+q^{1/4}\Big)\big(1+r^{-1/2}\big)=:G_2^{(t)}(r,w).$$

Proof (i) By the inequality $x-(1-e^{-x})\lesssim x^2$ for all $x>0$, we have that

$$t^{\frac{1}{\alpha-1}}\Big|\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)-\mathbf{E}_{y+w}\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)\Big|$$
$$\lesssim\frac{(\sup_{x\in\mathbb{R}}|f(x)|)^2}{t^{\frac{1}{\alpha-1}}}+\Big|\mathbf{E}_y\Big(f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big)-\mathbf{E}_{y+w}\Big(f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big)\Big|$$
$$\lesssim\frac{1}{t^{\frac{1}{\alpha-1}}}+\Big|\mathbf{E}_y\Big(f\big(\xi^{(t)}_r\big);\,\tau_0^{(t),-}>r\Big)-\mathbf{E}_{y+w}\Big(f\big(\xi^{(t)}_r\big);\,\tau_0^{(t),-}>r\Big)\Big|.$$

Since f is a bounded Lipschitz function, we have

$$\begin{aligned}&\Big|\mathbf{E}_y\Big(f\big(\xi^{(t)}_r\big);\,\tau_0^{(t),-}>r\Big)-\mathbf{E}_{y+w}\Big(f\big(\xi^{(t)}_r\big);\,\tau_0^{(t),-}>r\Big)\Big|\\&=\Big|\mathbf{E}_y\Big(f\big(\xi^{(t)}_r\big);\,\inf_{s\le r}\xi^{(t)}_s>0\Big)-\mathbf{E}_y\Big(f\big(\xi^{(t)}_r+w\big);\,\inf_{s\le r}\xi^{(t)}_s>-w\Big)\Big|\\&\lesssim w+\mathbf{P}_0\Big(\inf_{s\le r}\xi^{(t)}_s\in(-w-y,-y]\Big).\end{aligned}\tag{4.22}$$

Recalling the coupling in Lemma 2.11 and setting $W^{(t)}_s=W_{st}/\sqrt{t}$, we see that, for any fixed $\gamma\in(0,\frac{\delta}{2(2+\delta)})$, $r>0$ and t large enough (so that $t^{-\gamma}r^{\frac12-\gamma}\le t^{-\gamma/2}$), it holds that

$$\begin{aligned}&\mathbf{P}_0\Big(\inf_{s\le r}\xi^{(t)}_s\in(-w-y,-y]\Big)\\&\lesssim\frac{1}{(tr)^{(\frac12-\gamma)(\delta+2)-1}}+\mathbf{P}_0\Big(\inf_{s\le r}\xi^{(t)}_s\in(-w-y,-y],\ \sup_{0\le s\le r}|\xi^{(t)}_s-W^{(t)}_s|\le t^{-\gamma/2}\Big)\\&\lesssim\frac{1}{t^{(\frac12-\gamma)(\delta+2)-1}}+\mathbf{P}_0\Big(\min_{s\le r}W^{(t)}_s\in(-w-y-t^{-\gamma/2},-y+t^{-\gamma/2}]\Big)\\&\lesssim\frac{1}{\log t}+\mathbf{P}_0\left(W_r\in[y-t^{-\gamma/2},y+w+t^{-\gamma/2})\right).\end{aligned}\tag{4.23}$$

Here the last inequality holds by the reflection principle. Therefore, by estimating the density of Brownian motion, we obtain that

$$\mathbf{P}_0\Big(\inf_{s\le r}\xi^{(t)}_s\in(-w-y,-y]\Big)\lesssim\frac{1}{\log t}+\frac{w+t^{-\gamma/2}}{\sqrt{r}}\lesssim\frac{1}{\log t}+\frac{w+(\log t)^{-1}}{\sqrt{r}},$$

which gives the assertion (i).

(ii) Similar to the beginning of the proof of (i), we also have

$$t^{\frac{1}{\alpha-1}}\Big|\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)-\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f\big(\xi^{(t)}_{(r+q)\wedge\tau_0^{(t),-}}\big)\Big\}\Big)\Big|$$
$$\lesssim\frac{1}{\log t}+\Big|\mathbf{E}_y\Big(f\big(\xi_r^{(t)}\big);\,\tau_0^{(t),-}>r\Big)-\mathbf{E}_y\Big(f\big(\xi_{r+q}^{(t)}\big);\,\tau_0^{(t),-}>r+q\Big)\Big|.$$

Again using the fact that f is bounded Lipschitz, we have

$$\begin{aligned}&\Big|\mathbf{E}_y\Big(f\big(\xi_r^{(t)}\big);\,\tau_0^{(t),-}>r\Big)-\mathbf{E}_y\Big(f\big(\xi_{r+q}^{(t)}\big);\,\tau_0^{(t),-}>r+q\Big)\Big|\\&\lesssim\sup_{x>0}|f(x)|\mathbf{P}_y\big(\tau_0^{(t),-}\in(r,r+q]\big)+\mathbf{E}_y\big(\big|\xi_r^{(t)}-\xi_{r+q}^{(t)}\big|\big)\\&\lesssim\mathbf{P}_0\Big(\inf_{s\le r}\xi_s^{(t)}\ge -y,\ \inf_{s\le r+q}\xi_s^{(t)}<-y\Big)+\sqrt{q}.\end{aligned}$$

Here in the last inequality we used the fact that $\mathbf{E}_y\big(\big|\xi_r^{(t)}-\xi_{r+q}^{(t)}\big|\big)\le\sqrt{\mathbf{E}_0(|\xi_1|^2)q}$. Therefore, using a coupling argument similar to that leading to Eq. 4.23, we get

$$\begin{aligned}&\mathbf{P}_0\Big(\inf_{s\le r}\xi_s^{(t)}\ge -y,\ \inf_{s\le r+q}\xi_s^{(t)}<-y\Big)\\&\lesssim\frac{1}{\log t}+\mathbf{P}_0\Big(\min_{s\le r}W_s>-y-t^{-\gamma},\ \min_{s\le r+q}W_s\le -y+t^{-\gamma}\Big).\end{aligned}$$

Using

$$\begin{aligned}&\mathbf{P}_0\Big(\min_{s\le r}W_s>-y-t^{-\gamma},\ \min_{s\le r+q}W_s\le -y+t^{-\gamma}\Big)\\&\le\mathbf{P}_0\big(W_r\in(-y-t^{-\gamma},-y+t^{-\gamma}+q^{1/4})\big)+\mathbf{P}_0\Big(W_r>-y+t^{-\gamma}+q^{1/4},\ \min_{s\le r+q}W_s\le -y+t^{-\gamma}\Big)\\&\lesssim\frac{t^{-\gamma}+q^{1/4}}{\sqrt{r}}+\mathbf{P}_0\big(\min_{s\le q}W_s<-q^{1/4}\big)\lesssim\frac{t^{-\gamma}+q^{1/4}}{\sqrt{r}}+q^{1/4}\lesssim\Big(\frac{1}{\log t}+q^{1/4}\Big)(1+r^{-1/2}),\end{aligned}$$

we easily get the assertion of (ii). □

The following lemma is a generalized Gronwall inequality. We omit the proof here since the proof is standard.

Lemma 4.3 *Suppose that F and G are two bounded non-negative measurable function on* $[0,T]$. *If for any $r\in[0,T]$,*

$$F(r)\le G(r)+C\int_0^r F(s)\mathrm{d}s,$$

then we have for all $r\in[0,T]$,

$$F(r)\le G(r)+C\int_0^r e^{C(r-s)}G(s)\mathrm{d}s$$

Proof of Proposition 3.5 (i) Combining Corollary 2.2 (with $w = r$) and Lemma 4.2(i), we see that

$$
\begin{aligned}
&\left|v_f^{(t)}(r, y) - v_f^{(t)}(r, y+w)\right| \\
&\le t^{\frac{1}{\alpha-1}} \left|\mathbf{E}_y\Big(1 - \exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}} f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big) - \mathbf{E}_{y+w}\Big(1 - \exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}} f\big(\xi^{(t)}_{r\wedge\tau_0^{(t),-}}\big)\Big\}\Big)\right| \\
&\quad + \left|\mathbf{E}_y\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s, \xi_s^{(t)}))1_{\{\tau_0^{(t),-} > s\}}\mathrm{d}s\Big) - \mathbf{E}_{y+w}\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s, \xi_s^{(t)}))1_{\{\tau_0^{(t),-} > s\}}\mathrm{d}s\Big)\right| \\
&\lesssim G_1^{(t)}(r, w) + \left|\mathbf{E}_y\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s, \xi_s^{(t)}))1_{\{\tau_0^{(t),-} > s\}}\mathrm{d}s\Big)\right. \\
&\qquad \left. - \mathbf{E}_{y+w}\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s, \xi_s^{(t)}))1_{\{\tau_0^{(t),-} > s\}}\mathrm{d}s\Big)\right|.
\end{aligned}
\tag{4.24}
$$

Using the fact that $\left|\phi^{(t)}(u) - \phi^{(t)}(v)\right| \lesssim |u - v|$ for all $u, v \in [0, K]$ and $t > K$, and an argument similar to that leading to Eq. 4.22, we get that, for t large enough so that $t > \sup_{x\in\mathbb{R}} |f(x)|$, the second term on the right-hand side of Eq. 4.24 is bounded above by a constant multiple of

$$
\begin{aligned}
&\int_0^r \sup_{y\in\mathbb{R}} \left|v_f^{(t)}(r-s, y) - v_f^{(t)}(r-s, y+w)\right|\mathrm{d}s + \int_0^r \mathbf{P}_0\big(\inf_{\ell\le s} \xi_\ell^{(t)} \in (-w-y, -y]\big)\mathrm{d}s \\
&\lesssim \int_0^r \sup_{y\in\mathbb{R}} \left|v_f^{(t)}(r-s, y) - v_f^{(t)}(r-s, y+w)\right|\mathrm{d}s + \int_0^r \Big(\frac{1}{\log t} + \frac{w + (\log t)^{-1}}{\sqrt{s}}\Big)\mathrm{d}s \\
&\lesssim \int_0^r \sup_{y\in\mathbb{R}} \left|v_f^{(t)}(r-s, y) - v_f^{(t)}(r-s, y+w)\right|\mathrm{d}s + \frac{1}{\log t} + w \\
&\le \int_0^r \sup_{y\in\mathbb{R}} \left|v_f^{(t)}(r-s, y) - v_f^{(t)}(r-s, y+w)\right|\mathrm{d}s + G_1^{(t)}(r, w).
\end{aligned}
$$

Plugging this into Eq. 4.24, we conclude that there exists a constant L independent of r and t such that for all $r \in [0, T]$ and $t > 1$,

$$
\begin{aligned}
\sup_{y>0} \left|v_f^{(t)}(r, y) - v_f^{(t)}(r, y+w)\right| &\le L G_1^{(t)}(r, w) \\
&\quad + L \int_0^r \sup_{y\in\mathbb{R}} \left|v_f^{(t)}(r-s, y) - v_f^{(t)}(r-s, y+w)\right|\mathrm{d}s.
\end{aligned}
$$

Applying Lemma 4.3, we obtain that for all $r \in [0, T]$, we have that

$$
\begin{aligned}
&\sup_{y>0} \left|v_f^{(t)}(r, y) - v_f^{(t)}(r, y+w)\right| \le L G_1^{(t)}(r, w) + L \int_0^r e^{C(r-s)} G_1^{(t)}(s, w)\mathrm{d}s \\
&\lesssim \Big(\frac{1}{\log t} + w\Big)(1 + r^{-1/2}) + \int_0^r \Big(\frac{1}{\log t} + w\Big)(1 + s^{-1/2})\mathrm{d}s \\
&\lesssim \Big(\frac{1}{\log t} + w\Big)(1 + r^{-1/2}).
\end{aligned}
$$

This completes the proof of (i).

(ii) By Lemma 4.2 (ii), we see that

$$\begin{aligned}\left|v_f^{(t)}(r,y)-v_f^{(t)}(r+q,y)\right| &\lesssim G_2^{(t)}(r,w)+\left|\mathbf{E}_y\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s,\xi_s^{(t)}))1_{\{\tau_0^{(t),-}>s\}}\mathrm{d}s\Big)\right.\\ &\quad\left.-\mathbf{E}_y\Big(\int_0^{r+q}\phi^{(t)}(v_f^{(t)}(r+q-s,\xi_s^{(t)}))1_{\{\tau_0^{(t),-}>s\}}\mathrm{d}s\Big)\right|\\ &\lesssim G_2^{(t)}(r,w)+q+\mathbf{E}_y\Big(\int_0^r\left|\phi^{(t)}(v_f^{(t)}(r-s,\xi_s^{(t)}))-\phi^{(t)}(v_f^{(t)}(r+q-s,\xi_s^{(t)}))\right|1_{\{\tau_0^{(t),-}>s\}}\mathrm{d}s\Big).\end{aligned}$$

Again by the inequality $|\phi^{(t)}(u)-\phi^{(t)}(v)|\lesssim|u-v|$, we get that the last term on the right-hand side of the inequality above is bounded from above by a constant multiple of

$$\int_0^r\sup_{y>0}\left|v_f^{(t)}(r-s,y)-v_f^{(t)}(r+q-s,y)\right|\mathrm{d}s.$$

Therefore, there exists a constant L independent of t, q and r such that for all $r+q\le T$,

$$\begin{aligned}&\sup_{y>0}\left|v_f^{(t)}(r,y)-v_f^{(t)}(r+q,y)\right|\\ &\le LG_2^{(t)}(r,w)+Lq+L\int_0^r\sup_{y>0}\left|v_f^{(t)}(r-s,y)-v_f^{(t)}(r+q-s,y)\right|\mathrm{d}s.\end{aligned}$$

Applying Lemma 4.3 for any fixed q yields that

$$\begin{aligned}&\sup_{y>0}\left|v_f^{(t)}(r,y)-v_f^{(t)}(r+q,y)\right|\le LG_2^{(t)}(r,w)+Lq+L\int_0^r e^{L(r-s)}\big(LG_2^{(t)}(s,w)+Lq\big)\mathrm{d}s\\ &\lesssim G_2^{(t)}(r,w)+q+\int_0^r\Big(\frac{1}{\log t}+q^{1/4}\Big)\big(1+s^{-1/2}\big)\mathrm{d}s\\ &\lesssim\Big(\frac{1}{\log t}+q^{1/4}\Big)\big(1+r^{-1/2}\big),\end{aligned}$$

which completes the proof of (ii). □

Proof of Proposition 3.6 Fix a continuous function $f\in B_b^+((0,\infty))$ and $T>0$. By Lemma 3.4, without loss of generality, we assume that f is Lipschitz continuous. Since $v_f^{(t)}(r,y)$ is uniformly bounded for all $r\in[0,T]$, $y>0$ and $t>1$, we can find a sequence $\{t_k\}$ and a limit $v_f^X(r,y)$ such that

$$v_f^X(r,y)=\lim_{k\to\infty}v_f^{(t_k)}(r,y),\quad \text{for all } r\in[0,T]\cap\mathbb{Q},\ y\in(0,\infty)\cap\mathbb{Q}.$$

Proposition 3.5 implies that for any $r\in(0,T)$, $y>0$ and any $((0,T)\cap\mathbb{Q})\times((0,\infty)\times\mathbb{Q})\ni(r_m,y_m)\to(r,y)$, we have that $v_f^X(r_m,y_m)$ is a Cauchy sequence. Thus we define, for any $r\in(0,T)$ and $y>0$,

$$v_f^X(r,y):=\lim_{((0,T)\cap\mathbb{Q})\times((0,\infty)\times\mathbb{Q})\ni(r_m,y_m)\to(r,y)}v_f^X(r_m,y_m).$$

Using an argument similar to that leading to Lemma 3.1, we can get

$$v_f^X(r,y)=\lim_{k\to\infty}v_f^{(t_k)}(r,y),\quad \text{for all } r\in(0,T),\ y\in(0,\infty).$$

Combining Corollary 2.2 and the fact that $v_f^{(t)}(0,x) = t^{\frac{1}{\alpha-1}}\Big(1-\exp\big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f(x)\big\}\Big)$, we see that $v_f^{(t)}$ solves the equation

$$v_f^{(t)}(r,y) = t^{\frac{1}{\alpha-1}}\mathbf{E}_y\Big(1-\exp\Big\{-\frac{1}{t^{\frac{1}{\alpha-1}}}f(\xi^{(t)}_{r\wedge\tau_0^{(t),-}})\Big\}\Big) - \mathbf{E}_y\Big(\int_0^r \phi^{(t)}(v_f^{(t)}(r-s,\xi^{(t)}_{s\wedge\tau_0^{(t),-}}))\mathrm{d}s\Big).$$

Using the invariance principle and an argument similar to that leading to Eq. 4.21, we arrive at the desired result. □

4.4 Proof of Lemma 3.8 and Proposition 3.9

Proof of Lemma 3.8 We first show that $\lim_{y\to 0+}K^X(y)=0$. Taking $z=\frac{1}{2}$ in Lemma 2.6 and applying Lemma 2.16 (i), we get that for $y<\frac{1}{2}$,

$$\begin{aligned}
K^{(x)}(y) &\le \mathbf{E}_y\Big(K^{(x)}\Big(\xi^{(x^2)}_{\tau_{1/2}^{(x^2),+}}\Big);\,\tau_{1/2}^{(x^2),+}<\tau_0^{(x^2),-}\Big)\\
&\le K^{(x)}\big(\frac{2}{3}\big)\mathbf{P}_y\Big(\tau_{1/2}^{(x^2),+}<\tau_0^{(x^2),-}\Big)+x^{\frac{2}{\alpha-1}}\mathbf{P}_y\Big(\xi^{(x^2)}_{\tau_{1/2}^{(x^2),+}}>\frac{2}{3}\Big)\\
&= K^{(x)}\big(\frac{2}{3}\big)\mathbf{P}_y\Big(\tau_{1/2}^{(x^2),+}<\tau_0^{(x^2),-}\Big)+x^{\frac{2}{\alpha-1}}\mathbf{P}_{-\frac{1}{2}x+xy}\Big(\xi_{\tau_0^+}>\frac{1}{6}x\Big)\\
&\lesssim \mathbf{P}_y\Big(\tau_{1/2}^{(x^2),+}<\tau_0^{(x^2),-}\Big)+x^{\frac{2}{\alpha-1}}\frac{6^{r_0-2}}{x^{r_0-2}}\mathbf{E}_{-\frac{1}{2}x+xy}\Big(\Big|\xi_{\tau_0^+}\Big|^{r_0-2}\Big),
\end{aligned}\tag{4.25}$$

where in the last inequality we used Markov's inequality. It follows from Lemma 2.7 that

$$\mathbf{E}_{-\frac{1}{2}x+xy}\Big(\Big|\xi_{\tau_0^+}\Big|^{r_0-2}\Big)\le C$$

for some constant $C>0$. Thus, since $r_0-2>\frac{2}{\alpha-1}$, taking $x=x_k$ and letting $k\to\infty$ in Eq. 4.25, we get that

$$K^X(y)\lesssim \mathbf{P}_y\big(\tau_{1/2}^{W,+}<\tau_0^{W,-}\big)\overset{y\to 0+}{\longrightarrow}0.$$

Next we show that $\lim_{y\to 1-}K^X(y)=\infty$. Note that

$$\begin{aligned}
K^{(x)}(y) &\ge x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}\Big(M^{(0,\infty)}\ge x,\ \inf_{t>0}\inf_{u\in N(t)}X_u(t)>0\Big)\\
&= x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}\Big(M\ge x,\ \inf_{t>0}\inf_{u\in N(t)}X_u(t)>0\Big)\\
&\ge x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}(M\ge x)-x^{\frac{2}{\alpha-1}}\mathbb{P}_{xy}\Big(\inf_{t>0}\inf_{u\in N(t)}X_u(t)\le 0\Big)\\
&= x^{\frac{2}{\alpha-1}}\mathbb{P}(M\ge x(1-y))-x^{\frac{2}{\alpha-1}}\mathbb{P}(\widetilde{M}\ge xy),
\end{aligned}$$

where $\widetilde{M}$ is the maximal displacement of the critical branching Lévy process with branching rate β, offspring distribution $\{p_k\}$ and spatial motion $-\xi$. Applying Eq. 1.4 to $\widetilde{M}$ and M we see that under **(H4)**,

$$K^X(y)=\lim_{k\to\infty}K^{(x_k)}(y)\ge\frac{\theta(\alpha)}{(1-y)^{\frac{2}{\alpha-1}}}-\frac{\widetilde{\theta}(\alpha)}{y^{\frac{2}{\alpha-1}}}\overset{y\to 1-}{\longrightarrow}+\infty.$$

Finally, we show that $K^X(\cdot)$ satisfies the differential equation in Eq. 3.24. We fix an arbitrary $z \in (0,1)$ in the remainder of this proof. By Lemma 2.16,

$$\sup_{s\in(0,\tau_z^{(x_k^2),+})} K^{(x_k)}\big(\xi_s^{(x_k^2)}\big) \le K^{(x_k)}(z) \lesssim \frac{1}{(1-z)^{\frac{2}{\alpha-1}}}.$$

Therefore, by Lemma 2.14(ii), for any $\varepsilon > 0$, there exists $N > 0$ such that for any $k > N$ and $s \in (0, \tau_z^{(x_k^2),+})$,

$$\mathcal{C}(\alpha)(1-\varepsilon) \le \frac{\psi^{(x_k^2)}\big(K^{(x_k)}\big(\xi_s^{(x_k^2)}\big)\big)}{\big(K^{(x_k)}\big(\xi_s^{(x_k^2)}\big)\big)^{\alpha-1}} \le \mathcal{C}(\alpha)(1+\varepsilon).$$

Recall that $\varphi(\lambda) = \mathcal{C}(\alpha)\lambda^\alpha$ defined in Eq. 1.9. Set $\psi^X(v) := \varphi(v)/v$. For simplicity, we will use x_k as x in the remainder of this proof. Applying the display above to Lemma 2.6, we see that for $k > N$,

$$\begin{aligned}
&\mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big) \ge K^{(x)}(y)\\
&\ge \mathbf{E}_y\Big(\exp\Big\{-(1+\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big). \qquad (4.26)
\end{aligned}$$

Now we will let $k \to +\infty$ in Eq. 4.26. For the upper bound, we note that for any $\delta \in (0, 1-z)$, $K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big) \le K^{(x)}(z+\delta)$ on the event $\big\{\xi^{(x^2)}_{\tau_z^{(x^2),+}} \le z+\delta\big\}$ and $K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big) \le x^{\frac{2}{\alpha-1}}$ on the event $\big\{\xi^{(x^2)}_{\tau_z^{(x^2),+}} > z+\delta\big\}$. Thus,

$$\begin{aligned}
&\mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\}K^{(x)}\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}}\big);\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big)\\
&\le K^{(x)}(z+\delta)\mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\};\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big)\\
&\quad + x^{\frac{2}{\alpha-1}}\mathbf{P}_y\big(\xi^{(x^2)}_{\tau_z^{(x^2),+}} > z+\delta\big). \qquad (4.27)
\end{aligned}$$

The last term of the upper bound converges to 0 as $k \to \infty$. Indeed, since $r-2 > 2/(\alpha-1)$, by Lemma 2.7, we have

$$x^{\frac{2}{\alpha-1}}\mathbf{P}_y\Big(\xi^{(x^2)}_{\tau_z^{(x^2),+}} > z+\delta\Big) = x^{\frac{2}{\alpha-1}}\mathbf{P}_{-x(z-y)}\big(\xi_{\tau_0^+} > x\delta\big) \le \frac{x^{\frac{2}{\alpha-1}}}{(\delta x)^{r-2}}\sup_{w>0}\mathbf{E}_{-w}\big(\xi_{\tau_0^+}^{r-2}\big) \xrightarrow{x\to\infty} 0.$$

Therefore, combining Eqs. 4.26 and 4.27, letting $k \to \infty$, we get

$$\begin{aligned}
&K^X(y)\\
&\le K^X(z+\delta)\limsup_{k\to\infty}\mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s\Big\};\, \tau_z^{(x^2),+} < \tau_0^{(x^2),-}\Big).
\end{aligned} \qquad (4.28)$$

Using the continuity of $K^X(\cdot)$ and the fact that $\lim_{y\to 0+} K^X(y) = 0$, we get that, for any $\varepsilon > 0$, there exist $L \in \mathbb{N}$ and $0 = w_0 < w_1 < \ldots < w_L = z$ such that

$$\max_{j\in\{1,\ldots,L\}} \left| K^X(w_j) - K^X(w_{j-1}) \right| < \varepsilon.$$

Let $T = T(L, \varepsilon)$ be large enough so that for all $k \ge T$,

$$\max_{j\in\{0,\ldots,L\}} \left| K^{(x_k)}(w_j) - K^X(w_j) \right| < \varepsilon.$$

For $w \in [0, z)$, we have $w \in [w_{j-1}, w_j)$ for some $j \in \{1, \ldots, L\}$. Using the fact that both $K^{(x_k)}(w)$ and $K^X(w)$ are increasing in w, we get

$$K^{(x_k)}(w) \ge K^{(x_k)}(w_{j-1}) \ge K^X(w_{j-1}) - \varepsilon \ge K^X(w_j) - 2\varepsilon \ge K^X(w) - 2\varepsilon.$$

Therefore, when k is sufficiently large,

$$\begin{aligned} &\mathbf{E}_y\Big(\exp\Big\{ -(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(K^{(x)}(\xi_s^{(x^2)})\big)\mathrm{d}s \Big\};\ \tau_z^{(x^2),+} < \tau_0^{(x^2),-} \Big) \\ &\le \mathbf{E}_y\Big(\exp\Big\{ -(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\Big(\big(K^X(\xi_s^{(x^2)}) - 2\varepsilon\big)^+\Big)\mathrm{d}s \Big\};\ \tau_z^{(x^2),+} < \tau_0^{(x^2),-} \Big). \end{aligned} \tag{4.29}$$

Plugging Eq. 4.29 into Eq. 4.28, we obtain

$$\begin{aligned} &\frac{K^X(y)}{K^X(z+\delta)} \\ &\le \limsup_{k\to\infty} \mathbf{E}_y\Big(\exp\Big\{ -(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\Big(\big(K^X(\xi_s^{(x^2)}) - 2\varepsilon\big)^+\Big)\mathrm{d}s \Big\};\ \tau_z^{(x^2),+} < \tau_0^{(x^2),-} \Big). \end{aligned} \tag{4.30}$$

Fix a large real number A and an integer N, and set $t_i = \frac{A}{N} i$ for $i \in \{0, \ldots, N\}$. Then we have

$$\begin{aligned} &\int_0^{\tau_z^{(x^2),+}\wedge A} \psi^X\Big(\big(K^X(\xi_s^{(x^2)}) - 2\varepsilon\big)^+\Big)\mathrm{d}s = \sum_{i=1}^N \int_{t_{i-1}}^{t_i} 1_{\{s<\tau_z^{(x^2),+}\}} \psi^X\Big(\big(K^X(\xi_s^{(x^2)}) - 2\varepsilon\big)^+\Big)\mathrm{d}s \\ &\ge \sum_{i=1}^N \frac{A}{N} 1_{\{t_i<\tau_z^{(x^2),+}\}} \psi^X\Big(\Big(K^X\Big(\inf_{s\in[t_{i-1},t_i]} \xi_s^{(x^2)}\Big) - 2\varepsilon\Big)^+\Big). \end{aligned}$$

Using an argument similar to that in [9, Step 1 in Lemma 3.3], with [9, Lemma 2.4] there replaced by Lemma 2.11, we see that

$$\begin{aligned} &\limsup_{k\to\infty} \mathbf{E}_y\Big(\exp\Big\{ -(1-\varepsilon)\int_0^{\tau_z^{(x^2),+}} \psi^X\big(\big(K^X(\xi_s^{(x^2)}) - 2\varepsilon\big)^+\big)\mathrm{d}s \Big\};\ \tau_z^{(x^2),+} < \tau_0^{(x^2),-} \Big) \\ &\le \limsup_{k\to\infty} \mathbf{P}_y\big(\tau_0^{(x^2),-} > A\big) + \mathbf{E}_y\Big(\exp\Big\{ -(1-\varepsilon)\sum_{i=1}^N \frac{A}{N} 1_{\{t_i<\tau_z^{W,+}\}} \\ &\qquad \times \psi^X\big(\big(K^X\big(\inf_{s\in[t_{i-1},t_i]} W_s\big) - 2\varepsilon\big)^+\big) \Big\};\ \tau_z^{W,+} < \tau_0^{W,-} < A \Big), \end{aligned} \tag{4.31}$$

where $\tau_z^{W,+}$ is the exit time of the process W on $(-\infty, z)$. Combining Eqs. 4.30 and 4.31, taking $N \to \infty$ first and then $A \to +\infty$, we get

$$\begin{aligned}
&\frac{K^X(y)}{K^X(z+\delta)} \\
&\leq \limsup_{A\to\infty} \mathbf{P}_y\big(\tau_0^{W,-} > A\big) + \limsup_{A\to\infty}\limsup_{N\to\infty} \mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\sum_{i=1}^{N}\frac{A}{N}1_{\{t_i<\tau_z^{W,+}\}} \\
&\qquad \times \psi^X\big(\big(K^X\big(\inf_{s\in[t_{i-1},t_i]} W_s\big)-2\varepsilon\big)^+\big)\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}<A\Big) \\
&= \limsup_{A\to\infty} \mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{A\wedge\tau_z^{W,+}} \psi^X\big(\big(K^X(W_s)-2\varepsilon\big)^+\big)\mathrm{d}s\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}<A\Big) \\
&= \mathbf{E}_y\Big(\exp\Big\{-(1-\varepsilon)\int_0^{\tau_z^{W,+}} \psi^X\big(\big(K^X(W_s)-2\varepsilon\big)^+\big)\mathrm{d}s\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}\Big).
\end{aligned}$$

Since ε and δ are independent, letting $\varepsilon, \delta \to 0$ in the above inequality, we conclude that

$$K^X(y) \leq K^X(z)\mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{W,+}} \psi^X\big(K^X(W_s)\big)\mathrm{d}s\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}\Big).$$

Using a similar argument, we can prove that

$$K^X(y) \geq K^X(z)\mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{W,+}} \psi^X\big(K^X(W_s)\big)\mathrm{d}s\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}\Big).$$

Therefore,

$$K^X(z)\mathbf{E}_y\Big(\exp\Big\{-\int_0^{\tau_z^{W,+}} \psi^X\big(K^X(W_s)\big)\mathrm{d}s\Big\};\ \tau_z^{W,+}<\tau_0^{W,-}\Big) = K^X(y). \tag{4.32}$$

Note that z is fixed. The display above implies that $K^X(y)$ satisfies the differential equation in Eq. 3.24. The proof is now complete. $\square$

To prove Proposition 3.9, we first recall some basics on exit measures of superprocesses. Let $S := \mathbb{R}_+ \times \mathbb{R}_+$ and we consider the evolution of the superprocess in S. Let $\mathbb{O} \subset \mathcal{B}(\mathcal{S})$ be the class of open subsets of S. Roughly speaking, we obtain the exit measures $\{X_O;\ O \in \mathbb{O}\}$ by freezing "particles" once they exit O. For supercritical branching Brownian motion, similar ideas but with a different terminology "stopping line" are used in [15]. For applications of exit measures of supercritical super Brownian motion, one can see [16]. Now we formally introduce the exit measures. For any $r > 0$ and $x \geq 0$, we use $\mathbf{P}_{r,x}$ to denote the law $\mathbf{P}(\cdot\,|W_r = x)$. Let $\mathcal{B}(S)$ be the Borel σ-field on S, and $\mathcal{M}_F(S)$ the space of finite Borel measures on S. A measure $\mu \in \mathcal{M}_F(\mathbb{R}_+)$ is identified with the corresponding measure on S concentrated on $\{0\}\times\mathbb{R}_+$. According to Dynkin [6], there exists a family of random measures $\{(X_Q, \mathbb{P}_\mu);\ Q \in \mathbb{O}, \mu \in \mathcal{M}_F(S)\}$ such that for any $Q \in \mathbb{O}$, $\mu \in \mathcal{M}_F(S)$ with supp $\mu \subset Q$, and bounded non-negative Borel function $f(t, x)$ on S,

$$\mathbb{E}_\mu\big(\exp\big\{-\langle f, X_Q\rangle\big\}\big) = \exp\big\{-\langle v_f^{X,Q}, \mu\rangle\big\},$$

where $v_f^{X,Q}(s,x)$ is the unique positive solution of the equation

$$\begin{aligned}v_f^{X,Q}(s,x) &= \mathbf{E}_{s,x}\left(f(\tau, W_\tau^0)\right) - \mathbf{E}_{s,x}\int_s^\tau \varphi\big(v_f^{X,Q}(r, W_r^0)\big)\mathrm{d}r\\ &= \mathbf{E}_{s,x}\big(f(\tau, W_{\tau\wedge\tau_0^{W,-}})\big) - \mathbf{E}_{s,x}\int_s^{\tau\wedge\tau_0^{W,-}} \varphi\big(v_f^{X,Q}(r, W_r)\big)\mathrm{d}r, \end{aligned} \tag{4.33}$$

with $\tau := \inf\{r > 0 : (r, W_r) \notin Q\}$. For $Q = D_z := (0,\infty)\times(0,z)$, $\tau = \tau_0^{W,-}\wedge\tau_z^{W,+}$. Taking $f(t,x) = \theta 1_{\{x>0\}}$ in Eq. 4.33 and using the time-homogeneity of W, we get that $v_f^{X,D_z}(s,x) =: v_\theta^{X,D_z}(x)$ is independent of s and is the unique positive solution of the equation of

$$v_\theta^{X,D_z}(x) = \theta\mathbf{P}_x\big(\tau_z^{W,+} < \tau_0^{W,-}\big) - \mathbf{E}_x\int_0^{\tau_z^{W,+}\wedge\tau_0^{W,-}} \varphi\big(v_\theta^{X,D_z}(W_r)\big)\mathrm{d}r. \tag{4.34}$$

Moreover, by Eq. 1.14,

$$\begin{aligned}v_\theta^{X,D_z}(x) &= -\log\mathbb{E}_{\delta_x}\left(\exp\left\{-\theta X_{D_z}([0,\infty)\times\{z\})\right\}\right)\\ &= -\log\mathbb{E}_{\delta_x}\big(\exp\big\{-\theta X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\})\big\}\big).\end{aligned} \tag{4.35}$$

Letting $\theta\to+\infty$ in the display above, by the definition of X_{D_z}, we see that

$$\begin{aligned}v_\infty^{X,D_z}(x) &= -\log\mathbb{P}_{\delta_x}\big(X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\}) = 0\big)\\ &= -\log\mathbb{P}_{\delta_x}\big(M^{(0,\infty),X} < z\big).\end{aligned}$$

Proof of Proposition 3.9 Note that if K^X is a solution to the problem 3.24, then for any $0 < y < z < 1$,

$$K^X(y) + \mathbf{E}_y\Big(\int_0^{\tau_z^{W,+}\wedge\tau_0^{W,-}} \varphi(K^X((W_s))\mathrm{d}s\Big) = \mathbf{E}_y\big(K^X(W_{\tau_z^{W,+}\wedge\tau_0^{W,-}})\big).$$

Thus, for each fixed $z\in(0,1)$, $K^X(y)$ is a solution to the equation

$$\begin{aligned}&f(y) + \mathbf{E}_y\Big(\int_0^{\tau_z^{W,+}\wedge\tau_0^{W,-}} \varphi(f(W_s))\mathrm{d}s\Big)\\ &= \mathbf{E}_y\big(K^X(W_{\tau_z^{W,+}\wedge\tau_0^{W,-}})\big) = K^X(z)\mathbf{P}_y\big(\tau_z^{W,+} < \tau_0^{W,-}\big), \quad y\in(0,z),\end{aligned} \tag{4.36}$$

where the last inequality holds since $K^X(0) = 0$. By Eqs. 4.34 and 4.35, Eq. 4.36 has a unique solution given by

$$v_{K^X(z)}^{X,D_z}(y) = -\log\mathbb{E}_{\delta_y}\big(\exp\big\{-K^X(z)X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\})\big\}\big).$$

Since K^X is a solution to Eq. 4.36, we have

$$K^X(y) = -\log\mathbb{E}_{\delta_y}\big(\exp\big\{-K^X(z)X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\})\big\}\big), \quad y\in(0,z).$$

On one hand,

$$K^X(y) \le -\log\mathbb{P}_{\delta_y}\big(X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\}) = 0\big) = -\log\mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < z\big). \tag{4.37}$$

On the other hand, for any fixed $z_0 \in (y, 1)$, we choose $z \in (z_0, 1)$ so that $K^X(z) > K^X(z_0)$. Then

$$K^X(y) \geq -\log \mathbb{E}_{\delta_y}\big(\exp\big\{-K^X(z_0)X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\})\big\}\big) =: K_{z_0}^X(y;z). \tag{4.38}$$

Note that $K_{z_0}^X(\cdot;z)$ is the unique bounded solution to

$$K_{z_0}^X(y;z) + \mathbf{E}_y\Big(\int_0^{\tau_z^{W,+}\wedge\tau_0^{W,-}} \varphi(K_{z_0}^X(W_s;z))\mathrm{d}s\Big) = K^X(z_0)\mathbf{P}_y\big(\tau_z^{W,+} < \tau_0^{W,-}\big).$$

Define $\widehat{K}_{z_0}^X(y) := z^{\frac{2}{\alpha-1}} K_{z_0}^X(yz;z)$, then the above equation is equivalent to

$$\begin{aligned}
&\widehat{K}_{z_0}^X\Big(\frac{y}{z}\Big) + z^{\frac{2}{\alpha-1}}\mathbf{E}_y\Big(\int_0^{\tau_z^{W,+}\wedge\tau_0^{W,-}} \varphi\big(z^{-\frac{2}{\alpha-1}}\widehat{K}_{z_0}^X(z^{-1}W_s;z)\big)\mathrm{d}s\Big)\\
&= z^{\frac{2}{\alpha-1}}K^X(z_0)\mathbf{P}_y\big(\tau_z^{W,+} < \tau_0^{W,-}\big).
\end{aligned} \tag{4.39}$$

Using the scaling property of Brownian motion and the fact that $x^{\frac{2}{\alpha-1}}\varphi\big(x^{-\frac{2}{\alpha-1}}v\big)x^2 = \varphi(v)$, Eq. 4.39 is equivalent to

$$\begin{aligned}
&\widehat{K}_{z_0}^X\Big(\frac{y}{z}\Big) + \mathbf{E}_{y/z}\Big(\int_0^{\tau_1^{W,+}\wedge\tau_0^{W,-M}} \varphi\big(\widehat{K}_{z_0}^X(z^{-1}W_s;z)\big)\mathrm{d}s\Big)\\
&= z^{\frac{2}{\alpha-1}}K^X(z_0)\mathbf{P}_{y/z}\big(\tau_1^{W,+} < \tau_0^{W,-}\big).
\end{aligned}$$

Again using the uniqueness of the solution to Eq. 4.34, we conclude that

$$\begin{aligned}
&-\log \mathbb{E}_{\delta_y}\big(\exp\big\{-K^X(z_0)X_{D_z}^{(0,\infty)}([0,\infty)\times\{z\})\big\}\big) = K_{z_0}^X(y;z)\\
&= z^{-\frac{2}{\alpha-1}}\widehat{K}_{z_0}^X\Big(\frac{y}{z}\Big) = \Big(-\log \mathbb{E}_{\delta_{y/z}}\big(\exp\big\{-z^{\frac{2}{\alpha-1}}K^X(z_0)X_{D_1}^{(0,\infty)}([0,\infty)\times\{1\})\big\}\big)\Big)\cdot z^{-\frac{2}{\alpha-1}}\\
&\geq \Big(-\log \mathbb{E}_{\delta_{y/z}}\big(\exp\big\{-z_0^{\frac{2}{\alpha-1}}K^X(z_0)X_{D_1}^{(0,\infty)}([0,\infty)\times\{1\})\big\}\big)\Big)\cdot z^{-\frac{2}{\alpha-1}}.
\end{aligned} \tag{4.40}$$

Therefore, plugging Eq. 4.40 into Eq. 4.38 and then letting $z \to 1-$ in both Eqs. 4.37 and 4.38, we conclude that

$$-\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big) \geq K^X(y) \geq -\log \mathbb{E}_{\delta_y}\big(\exp\big\{-z_0^{\frac{2}{\alpha-1}}K^X(z_0)X_{D_1}^{(0,\infty)}([0,\infty)\times\{1\})\big\}\big).$$

Since $K^X(z_0) \to +\infty$ as $z_0 \to 1-$, letting $z_0 \to 1-$ in the above inequality yields that

$$\begin{aligned}
&-\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big) \geq K^X(y) \geq -\log \mathbb{P}_{\delta_y}\big(X_{D_1}^{(0,\infty)}([0,\infty)\times\{1\}) = 0\big)\\
&= -\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big),
\end{aligned}$$

which implies that $K^X(y) = -\log \mathbb{P}_{\delta_y}\big(M^{(0,\infty),X} < 1\big)$. This completes the proof of the proposition. □

Acknowledgements We thank the referee for very helpful comments on the first version of this paper. Part of the research for this paper was done while the third-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie).

Author Contributions All authors contribute equally.

Funding The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900). Haojie Hou was supported by the China Postdoctoral Science Foundation (No. 2024M764112). Yan-Xia Ren was supported by NNSFC (Grant Nos.12071011 and 12231002) and the Fundamental Research Funds for the Central Universities, Peking University LMEQF. Renming Song was supported in part by a grant from the Simons Foundation (#960480, Renming Song).

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

Ethical approval Not applicable.

References

1. Aïdékon, E.: Convergence in law of the minimum of a branching random walk. Ann. Probab. **41**, 1362–1426 (2013)
2. Berestycki, J., Berestycki, N., Schweinsberg, J.: Critical branching Brownian motion with absorption: particle configurations. Ann. Inst. H. Poincaré Probab. Statist. **51**, 1215–1250 (2015)
3. Berestycki, J., Berestycki, N., Schweinsberg, J.: Critical branching Brownian motion with absorption: survival probability. Probab. Theory Related Fields **160**, 489–520 (2014)
4. Doney, R.A., Maller, R.A.: Stability of the overshoot for Lévy processes. Ann. Probab. **30**, 188–212 (2002)
5. Durrett, R.: Browninan motion and martingales in Analysis. Wadsworth, Belmont, California (1984)
6. Dynkin, E.B.: Branching exit Markov systems and superprocesses. Ann. Probab. **29**, 1833–1858 (2001)
7. Dynkin, E.B., Kuznetsov, S.E.: $\mathbb{N}$-measures for branching exit Markov systems and their applications to differential equations. Probab. Theory Relat. Fields. **130**(1), 135–150 (2004)
8. Grama, I., Xiao, H.: Conditioned local limit theorems for random walks on the real line. Ann. Inst. H. Poincaré Probab. Statist. **61**, 403–456 (2025)
9. Hou, H., Jiang, Y., Ren, Y.-X., Song, R.: Tail probability of maximal displacement in critical branching Levy process with stable branching. Bernoulli **31**, 630–648 (2025)
10. Hou, H., Ren, Y.-X., Song, R.: 1-stable fluctuation of the derivative martingale of branching random walk. Stochastic Process. Appl. **7**, No. 104338 (2024)
11. Imhof, J.-P.: Density factorizations for Brownian motion, meander and the three dimensional Bessel process, and applications. J. Appl. Probab. **21**, 500–510 (1984)
12. Kallenberg, O.: Foundations of modern probability, 2nd edn. Springer-Verlag, New York (2002)
13. Kesten, H.: Branching Brownian motion with absorption. Stochastic Process. Appl. **7**(1), 9–47 (1978)
14. Kolmogorov, A.: Zur lösung einer biologischen aufgabe. Comm. Math. Mech. Chebyshev Univ. Tomsk. **2**(1), 1–12 (1938)
15. Kyprianou, A.E.: Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis. Ann. Inst. Henri Poincaré Probab. Statist. **40**, 53–72 (2004)
16. Kyprianou, A.E., Liu, R.-L., Murillo-Salas, A., Ren, Y.-X.: Supercritical super-Brownian motion with a general branching mechanism and travelling waves. Ann. Inst. Henri Poincaré Probab. Statist. **48**, 661–687 (2012)
17. Lalley, S.P., Shao, Y.: Maximal displacement of critical branching symmetric stable processes. Ann. Inst. Henri Poincaré Probab. Stat. **52**(3), 1161–1177 (2016)
18. Lalley, S.P., Shao, Y.: On the maximal displacement of critical branching random walk. Probab. Theory Relat. Fields. **162**(1–2), 71–96 (2015)
19. Lalley, S.P., Zheng, B.: Critical branching Brownian motion with killing. Electron. J. Probab. **20**(118), 1–29 (2015)
20. Li, Z.: Measure valued branching Markov processes. 2nd edition, Probab. Theory Stoch. Model. **103** Springer, Berlin, (2022)
21. Maillard, P. and Schweinsberg, J.:The all-time maximum for branching Brownian motion with absorption conditioned on long-time survival. To appear in *Ann. Inst. Henri Poincaré Probab. Statist.* (2024)
22. Maillard, P., Schweinsberg, J.: Yaglom-type limit theorems for branching Brownian motion with absorption. Ann. H. Lebesgue **5**, 921–985 (2022)

23. Neuman, E., Zheng, X.: On the maximal displacement of near-critical branching random walks. Probab. Theory Relat. Fields. **180**, 199–232 (2021)
24. Powell, E.: An invariance principle for branching diffusions in bounded domains. Probab. Theory Relat. Fields. **173**(3–4), 999–1062 (2019)
25. Profeta, C.: Extreme values of critical and subcritical branching stable processes with positive jumps. ALEA, Lat. Am. J. Probab. Math. Stat. **19**(2), 1421–1433 (2022)
26. Profeta, C.: Maximal displacement of spectrally negative branching Lévy processes. Bernoulli **30**, 961–982 (2024)
27. Sakhanenko, A.I.: Estimates in the invariance principle in terms of truncated power moments. Siberian Math. J. **47**(6), 1113–1127 (2006)
28. Sawyer, S., Fleischman, J.: Maximum geographic range of a mutantallele considered as a subtype of a brownian branching random field. Proceedings of the National Academy of Sciences. **76**(2), 872–875 (1979)
29. Slack, R.: A branching process with mean one and possibly infinite variance. Z. Wahrscheinlichkeitstheor. Verwandte Geb. **9**, 139–145 (1968)
30. Zhang, S.: On empty balls of critical 2-dimensional branching random walks. (2025) arXiv:2502.16225
31. Zolotarev, V.: More exact statements of several theorems in the theory of branching processes. Teor. Veroyatn. Primen. **2**, 256–266 (1957)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.