CONSTRUCTION OF SUPER-BROWNIAN MOTIONS

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ABSTRACT

Suppose A is a continuous additive functional of a Brownian motion. A d-dimensional super-Brownian motion with general branching rate functional A and general mechanism ψ is constructed under a condition on A, which is weaker than the conditions imposed by Dynkin (Ann. Prob. **1991**, *19*, 1157–1194; *An Introduction to Branching Measure-Valued Processes;* Amer. Math. Soc.: Providence, RI., 1994).

1. INTRODUCTION AND MAIN RESULT

For every Borel measurable space $(E, \mathcal{B}(E))$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}(E)$ endowed with the topology of weak convergence. The expression $\langle f, \mu \rangle$ stands for the integral of f with respect to μ and $\|\mu\|$ means $\langle 1, \mu \rangle$. We write $f \in \mathcal{B}(E)$ if f is a $\mathcal{B}(E)$-measurable function. Writing $f \in p\mathcal{B}(E)(b\mathcal{B}(E))$ means that, in addition, f is positive (bounded). We put $bp\mathcal{B}(E) = (b\mathcal{B}(E) \cap p\mathcal{B}(E))$. If $E = \mathbb{R}^d$ we simply write $\mathcal{B}$ instead of $\mathcal{B}(\mathbb{R}^d)$ and M instead

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of $M(\mathbb{R}^d)$. We put $f \in \mathcal{H}$ if $bp\mathcal{B}$ is supported by some interval $[0, T)$, $T > 0$. We will use the symbol $\overset{bp}{\to}$ to denote bounded pointwise convergence. (Recall that functions converge boundedly pointwise if they are uniformly bounded and converge pointwise).

Let $W := \{W, \Pi_{r,x}, r \geq 0, x \in \mathbb{R}^d\}$ denote the canonical Brownian motion in $\mathbb{R}^d$ with birth time α. $\Pi_{r,x}(\alpha = r, W_\alpha = x) = 1$. Set $\mathcal{F}^0_{\leq r} = \sigma(W_s, s \leq r)$; $\mathcal{F}^0_{>r} = \sigma(W_s, s > r)$ and $\mathcal{F}^0_\infty = \bigvee\{\mathcal{F}^0_{\leq r}, r \geq 0\}$.

Set $S = [0, \infty) \times \mathbb{R}^d$. To every set $Q \subset S$ there corresponds the first exit time $\tau = \inf\{t : t \geq \alpha, (t, W_t) \notin Q\}$. Put $(r, x) \in Q^0$ if $\Pi_{r,x}\{\tau > r\} = 1$. A set $Q \in \mathcal{B}(S)$ is called finely open if $Q^0 = Q$. We denote by $\mathcal{T}$ the set of all exit times from finely open sets $Q \in \mathcal{B}(S)$. Put $\tau \in \mathcal{T}$, Put $C \in \mathcal{F}^0_{\geq \tau}$ if $C \in \mathcal{F}^0_\infty$ and if, for each r, $\{C, \tau > r\} \in \mathcal{F}^0_{>r}$.

Let

$$\psi(s, x, z) = a(s, x)z + b(s, x)z^2 + \int_0^\infty (e^{-uz} - 1 + uz)n(s, x, du), s, z$$
$$\geq 0, x \in \mathbb{R}^d, \tag{1}$$

where a, b are positive measurable functions, n is a kernel from $\mathbb{R}^d$ to $(0, \infty)$ such that for every finite interval Δ, $a(s, x)$, $b(s, x)$ and $\int_0^\infty u \wedge u^2 n(s, x, du)$ are positive bounded Borel functions on $\Delta \times \mathbb{R}^d$.

Suppose A is a continuous additive functional of W. Fix a measurable space $(\Omega, \mathcal{F})$. Suppose that to every $\tau \in \mathcal{T}$ there corresponds a random measure X_τ on S, and to every $\mu \in M(S)$ there corresponds a probability measure P_μ on $(\Omega, \mathcal{F})$. According to Ref. (1) $X = (X_\tau, P_\mu; \mu \in M(S))$ is called a super-Brownian motion with parameters (A, ψ) if the following hold:

1. for every $f \in \mathcal{H}$, $\tau \in \mathcal{T}$ and $\mu \in M(S)$, we have

$$P_\mu \exp\langle -f, X_\tau \rangle = \exp\langle -u(r, \cdot), \mu \rangle, \tag{2}$$

where u is the unique bounded solution of the integral equation

$$u(r, x) + \Pi_{r,x} \int_r^\tau \psi(s, W_s, u(s, W_s))A(ds)$$
$$= \Pi_{r,x}(f(\tau, W_\tau)), x \in \mathbb{R}^d. \tag{3}$$

2. For $n \geq 2$, the joint probability distribution of $X_{\tau 1}, \ldots, X_{\tau n}$ is described as follows. Let

$$I = \{1, 2, \ldots, n\}, \quad \tau_I = \min\{\tau_1, \ldots, \tau_n\}, \quad \lambda = \{i : \tau_i = \tau_I\}$$

for every $f \in \mathcal{H}$, $i = 1, \ldots, n$,

$$P_\mu \exp\left\{ -\sum_{i=1}^n \langle f_i, X_{\tau_i} \rangle \right\} = \exp\langle -u_I(r, \cdot), \mu \rangle, \tag{4}$$

where the function u_I are determined recursively by the integral equations

$$u_I(r, x) + \Pi_{r,x} \int_r^{\tau_I} \psi(s, W_s, U_I(s, W_s))A(\mathrm{d}s) = \Pi_{r,x} G_I \qquad (5)$$

with $G_I = [f_\lambda + u_{I-\lambda}](\tau_\lambda, W_{\tau_\lambda})$.

Dynkin (Theorem 1.1 in Ref. (2)) showed that if A is a continuous additive functional of the d-dimensional Brownian motion W satisfying the following moment conditions

$$\Pi_{r,x} \exp[\lambda A(r, T)] < \infty, \quad \text{for every } \lambda > 0, r < T \text{ and } x \in \mathbb{R}^d; \qquad (6)$$

$$\sup\{\Pi_{r,x} A(r, T)] < \infty; r \in [L, T), x \in \mathbb{R}^d\} \quad \text{for every } L < T, \qquad (7)$$

then there exists a time-inhomogeneous super-Brownian motion $X = (X_\tau, P_\mu, \mu \in M(S))$ with parameters (A, ψ). But the above conditions (Eqs. (6) and (7)) are very strong and in many cases we need to consider super-Brownian motions with more general A. Dynkin (see Theorem 3.4.1 in Ref. (3)) proved the existence of a super-Brownian motion under conditions:

$$\Pi_{r,x} A(r, T) < \infty \quad \text{for every } r < T, x \in \mathbb{R}^d; \qquad (8)$$

$$\Pi_{r,x} A(r, T) \to 0 \quad \text{uniformly in } x \text{ as } r, T \to s \text{ for every } s. \qquad (9)$$

Comparing conditions (6) and (7) with conditions (8) and (9) we conjecture that conditions (8) is sufficient for the existence of a super-Brownian motion. The purpose of this paper is to prove that this conjecture is right. Now we state the main result of this paper.

Theorem 1.1. *Let ψ be given by Equation* (1). *Suppose A is a continuous additive functional of W satisfying Equation* (8). *Then there corresponds a super-Brownian motion with parameters* (A, ψ).

Note that our proof also holds if the Brownian motion W is replaced by a general Markov process ξ, i.e., if A is a continuous additive functional of a time-inhomogeneous Markov process ξ and satisfies condition (8), then there exists a superprocess X related to the integral Equation (3) with W replaced by ξ.

2. BASIC INTEGRAL EQUATION

The discussion about solutions of the basic integral Equation (3) plays a fundamental role in the construction of super-super-Brownian motions. so, we first investigate the existence, uniqueness and properties of the integral Equation (3).

For $c \in p\mathcal{B}$ put

$$H^c(r_1, r_2) = \exp\left(-\int_{r_1}^{r_2} c(s, W_s)A(\mathrm{d}s)\right), \quad 0 \le r_1 \le r_2. \tag{10}$$

The main result of this section about the integral Equation (3) is the following Theorem 2.1.

Theorem 2.1. *Under the conditions of Theorem* 1.1, *the following results hold.*

1. (*Existence and Uniqueness*). *For every $f \in \mathcal{H}$, there is exactly one $U(A, f) \in \mathcal{H}$ which solves Equation* (3). *Moreover if f is supported by $[0, T)$, then $U(A, f)$ is also supported by $[0, T)$.*
2. (*Continuity*). *Put $\mathcal{B}_T = \mathcal{B}([0, T)) \times \mathcal{B}$. $U(A, f)$ as a map of $bp\mathcal{B}_T \to b\mathcal{B}_T$ is continuous relative to the uniform convergence in $b\mathcal{B}_T$.*
3. (*First Derivative with Respect to a Small Parameter*). *For every $f \in \mathcal{H}$,*

$$\lambda^{-1}U(A, \lambda f) \xrightarrow{bp} \Pi_{\cdot,\cdot}[H^a(r, \tau)f(\tau, W_\tau)] \quad \text{as } \lambda \to 0,$$

where H^a is defined by Equation (10).

The main technique used in this paper is that we translate the integral Equation (3) into the following equivalent equation:

$$u(r, x) = \Pi_{r,x}[f(\tau, W_\tau)H^{a+\lambda}(r, \tau)]$$
$$+ \Pi_{r,x}\left[\int_r^\tau H^{a+\lambda}(r, s)R(\lambda, \psi)(s, W_s, u(s, W_s))A(\mathrm{d}s)\right], \tag{11}$$

where $\lambda(s, x) \in p\mathcal{B}(S)$ is a suitably chosen function, $R(\lambda, \psi)(s, x, z)$ is defined as

$$R(\lambda, \psi)(s, x, z) = [a(s, x) + \lambda(s, x)]z - \psi(s, x, z), \quad s, z \ge 0, x \in \mathbb{R}^d.$$

We do not directly discuss solutions of the integral Equation (3), but discuss the equivalent Equation (11). We will see the benefits of discussing solutions of the integral Equation (11) in the proof of Theorem 2.1 below.

Let us first state two lemmas on the integral Equation (3).

Lemma 2.1. *Suppose $A(\mathrm{d}t)$ is a nonnegative continuous additive functional of the Brownian motion W in $\mathbb{R}^d$. Let $\tau \in \mathcal{T}$, and $c, g \in bp\mathcal{B}(S)$. Assume that $\omega \in \mathcal{B}(S)$ and $F \in \mathcal{F}_{\ge r}^0$ satisfy*

$$\Pi_{r,x}\int_r^\tau |\omega(s, W_s)|A(\mathrm{d}s) < \infty; \quad \Pi_{r,x}|F| < \infty, \quad r \ge 0, x \in \mathbb{R}^d.$$

Then

$$g(r, x) = \Pi_{r,x}\left[H^c(r, \tau)F + \int_r^\tau H^c(r, s)\omega(s, W_s)A(\mathrm{d}s) \right] \qquad (12)$$

iff

$$g(r, x) + \Pi_{r,x} \int_r^\tau (cg)(s, W_s)A(\mathrm{d}s) = \Pi_{r,x}\left[F + \int_r^\tau \omega(s, W_s)A(\mathrm{d}s) \right]. \quad (13)$$

Proof: This lemma is taken from Dynkin (1) with a slight modification. The proof is similar to that of Lemma 2.1 in Ren and Wang (4). We omit the details here. $\qquad\square$

The following lemma is a generalization of Gronwall's lemma from deterministic time to random time.

Lemma 2.2 (Generalized Gronwall's lemma). *Let c, f and λ belong to $p\mathcal{B}(s)$. If $h_n \in p\mathcal{B}(s)$ satisfy the following conditions*:

$$\Pi_{r,x} \int_r^\tau (\lambda h_0)(s, W_s)A(\mathrm{d}s) < \infty,$$

$$h_n(r, x) \le \Pi_{r,x}[H^{c+\lambda}(r, \tau)f(\tau, W_\tau)] + q\Pi_{r,x} \int_r^\tau H^{c+\lambda}(r, s)A(\mathrm{d}s)$$

$$+ \Pi_{r,x} \int_r^\tau H^{c+\lambda}(r, s)(\lambda h_{n-1})(s, W_s)A(\mathrm{d}s), \quad \text{for } r \ge 0, x \in \mathbb{R}^d, n \in \mathbf{N},$$

where q is a positive constant. Then

$$h_n(r, x) \le \Pi_{r,x}[H^c(r, \tau)f(\tau, W_\tau)] + q\Pi_{r,x} \int_r^\tau H^c(r, s)A(\mathrm{d}s)$$

$$+ \Pi_{r,x} \int_r^\tau A(\mathrm{d}s)H^{c+\lambda}(r, s)\frac{\int_r^s \lambda(t, W_t)A(\mathrm{d}t))^{n-1}}{(n - 1)!}(\lambda h_0)(s, W_s). \tag{14}$$

In particular, if $h_0 = 0$ if h_n does not depend on n, then, for $r \ge 0, x \in \mathbb{R}^d$,

$$h_n(r, x) \le \Pi_{r,x}[H^c(r, \tau)f(\tau, W_\tau)] + q\Pi_{r,x} \int_r^\tau H^c(r, s)A(\mathrm{d}s). \tag{15}$$

Proof: By induction in n, we get

$$h_n(r, x) \leq \Pi_{r,x}\left[H^{c+\lambda}(r, \tau) \sum_{i=0}^{n-1} \frac{\left(\int_r^\tau \lambda(s, W_s)A(\mathrm{d}s) \right)^i}{i!} f(\tau, W_\tau) \right]$$

$$+ q \, \Pi_{r,x} \int_r^\tau A(\mathrm{d}s) H^{c+\lambda}(r, s) \sum_{i=0}^{n-1} \frac{\left(\int_r^s \lambda(t, W_t)A(\mathrm{d}t) \right)^i}{i!}$$

$$+ \Pi_{r,x} \int_r^\tau A(\mathrm{d}s) H^{c+\lambda}(r, s) \frac{\left(\int_r^s \lambda(t, W_t)A(\mathrm{d}t) \right)^{n-1}}{(n-1)!} (\lambda h_0)(s, W_s).$$

$$\square$$

Clearly, this implies Equation (14). If $h_0 = 0$, Equation (14) is exactly Equation (15). If h_n does not depend on n, letting $n \to \infty$ in inequality (Eq. (14)) and using the dominated convergence theorem, we get Equation (15).

For $0 < \beta < 1$, let

$$\psi_\beta(s, x, z) = a(s, x)z + \int_{(\beta,\infty)} (e^{-uz} - 1 + uz)n(s, x, \mathrm{d}u)$$

$$+ 2b(s, x)\beta^{-2}[e^{-2} - 1 + \beta z]; \tag{16}$$

$$\lambda_\beta(s, x) = 2b(s, x)\beta^{-1} + \int_{(\beta,\infty)} un(s, x, \mathrm{d}u); \tag{17}$$

$$R(\lambda_\beta, \psi_\beta)(s, x, z) = [a(s, x) + \lambda_\beta(s, x)]z - \psi_\beta(s, x, z)$$

$$= \int_{(\beta,\infty)} (1 - e^{-uz})n(s, x, \mathrm{d}u) + 2b(s, x)\beta^{-2}(1 - e^{-\beta z}). \tag{18}$$

Then λ_β is bounded in $[0, T) \times \mathbb{R}^d$ for any $T \geq 0$, $R(\lambda_\beta, \psi_\beta)(s, x, z)$ is increasing in z, and

$$0 \leq R(\lambda_\beta, \psi_\beta)(s, x, z) \leq \lambda_\beta(s, x)z, \quad \text{for } x \in \mathbb{R}^d, s, z \in [0, \infty). \tag{19}$$

Proposition 2.1. *For every $f \in \mathcal{H}$, there existis a positive solution $u(\beta, f)$ of the Equation* (3) *with ψ replaced by ψ_β, i.e.,*

$$u(\beta, f)(r, x) + \Pi_{r,x} \int_r^\tau \psi_\beta(s, W_s, u(\beta, f)(s, W_s))A(\mathrm{d}s) = \Pi_{r,x}(\tau, W_\tau). \tag{20}$$

Moreover, if f is supported by an interval $[0, T)$, then $u(\beta, f)$ is also supported by the interval $[0, T)$.

Proof: Suppose f is supported by $[0, T)$. Using Lemma 2.1 with $g(r, x) = u(r, x), c(s, x) = a(s, x) + \lambda_\beta(s, x)$, Equation (20) can be rewritten as

$$u(\beta, f)(r, x) = \Pi_{r,x}[(H^{a+\lambda_\beta}(r, \tau)f(\tau, W_\tau)]$$
$$+ \Pi_{r,x} \int_r^\tau H^{a+\lambda_\beta}(r, s)R(\lambda_\beta, \psi_\beta)$$
$$\times (s, W_s, u(\beta, f)(s, W_s,))A(\mathrm{d}s). \tag{21}$$

$\square$

Therefore, we only need to prove that there exists a positive bounded function $u(\beta, f)$, supported by $[0, T)$, satisfies Equation (21).

Define a sequence $u_n(\beta, f)$ by the recursive formula:

$$u_0(\beta, f)(r, x) = 0$$
$$u_n(\beta, f)(r, x) = \Pi_{r,x}[H^{a+\lambda\beta}(r, \tau)f(\tau, W_\tau)]$$
$$+ \Pi_{r,x} \int_r^\tau H^{a+\lambda\beta}(r, s)R(\lambda_\beta, \psi_\beta)(s, W_s, u_{n-1}(\beta, f)$$
$$\times (s, W_s,))A(\mathrm{d}s), \tag{22}$$

with $r \in [0, \infty), x \in \mathbb{R}^d$. By Equation (19),

$$u_n((\beta, f)(r, x) \leq \Pi_{r,x}[(H^{a+\lambda_\beta}(r, \tau)f(\tau, W_\tau)]$$
$$+ \Pi_{r,x} \int_0^\tau H^{a+\lambda_\beta}(r, s)(\lambda_\beta u_{n-1}(\beta, f)(s, W_s)A(\mathrm{d}s).$$

Using Lemma 2.2 with $q = 0, c = a, \lambda = \lambda_\beta$ and $h_n = u_n(\beta, f)$, we get

$$0 \leq u_n(\beta, f)(r, x) \leq \Pi_{r,x}[H^a(r, \tau)f(\tau, W_\tau)] \leq \|f\|_\infty, \tag{23}$$

where $\|f\|_\infty = \sup_{s \geq 0, x \in \mathbb{R}^d} |f(s, x)|$. Since $R(\lambda_\beta, \psi_\beta)(s, x, z)$ is increasing in z, there exists a function $u(\beta, f) \in bp\mathcal{B}(S)$ such that for all $r \in [0, \infty), x \in \mathbb{R}^d$,

$$u_n(\beta, f)(r, x) \uparrow u(\beta, f)(r, x). \tag{24}$$

Using the monotone convergence theorem, letting $n \to \infty$ in Equation (22), we get $u(\beta, f)$ is a positive bounded solution of Equation (21), and therefore, a solution of Equation (20). By Equations (23) and (24), $u(\beta, f)$ is supported by $[0, T)$.

Proof of Theorem 2.1:

1. Suppose f is supported by $[0, T)$. For $0 < \beta < 1$, let $\psi_\beta(s, x, z)$ be defined by Equation (16), and let $u(\beta, f)$ be the solution of Equation (20)

constructed in Proposition 2.1. Since

$$|\psi(s, x, z) - \psi_\beta(s, x, z)| \leq \int_{[0,\beta]} (e^{-uz} - 1 + uz)n(s, x, \mathrm{d}u)$$
$$+ 2b(s, x)\beta^{-2}\left|\frac{1}{2}z^2\beta^2 - e^{-\beta z} + 1 - \beta z\right|$$
$$\leq \int_{[0,\beta]} \frac{1}{2}u^2 z^3 n(s, x, \mathrm{d}u) + \frac{1}{3}b(s, x)z^3\beta,$$

We have, for every $C \in (0, \infty)$, there exist constants $\alpha(\beta, T, C) \to 0$ as $\beta \to 0$ such that

$$|\psi(s, x, z) - \psi_\beta(s, x, z)| \leq \alpha(\beta, T, C), \tag{25}$$

for all $\beta \in (0, 1)$, $x \in \mathbb{R}^d$, $0 \leq s \leq T$, and $0 \leq z \leq C$. Let $M \geq 1$ be a constant such that $\|f\|_\infty \leq M$, and let

$$\lambda(s, x) = \left[2b(s, x) + \int_0^\infty u \wedge u^2 n(s, x, \mathrm{d}u)\right]M; \tag{26}$$
$$R(\lambda, \psi)(s, x, z) = [a(s, x) + \lambda(s, x)]z - \psi(s, x, z); \tag{27}$$
$$R(\lambda, \psi_\beta)(s, x, z) = [a(s, x) + \lambda(s, x)]z - \psi_\beta(s, x, z). \tag{28}$$

Then,

$$[R(\lambda, \psi)(s, x, z)]'_z = \lambda(s, x) - 2b(s, x)z - \int_0^\infty u(1 - e^{-uz})$$
$$\times n(s, x, \mathrm{d}u) \leq \lambda(s, x).$$

by Equation (26),

$$[R(\lambda, \psi)(s, x, z)]'_z = 2b(s, x)(M - z)$$
$$+ \int_0^1 u(Mu - 1 + e^{-uz})n(s, x, \mathrm{d}u)$$
$$+ \int_1^\infty u(M - 1 + e^{-uz})n(s, x, \mathrm{d}u) \geq 0.$$

Therefore, for all $x \in \mathbb{R}^d$, $0 \leq z \leq M$,

$$0 \leq [R(\lambda, \psi)(s, x, z)]'_z \leq \lambda(s, x). \tag{29}$$

Consequently

$$|R(\lambda, \psi)(s, x, z_1) - R(\lambda, \psi)(s, x, z_2)| \leq \lambda(s, x)|z_1 - z_2|, \tag{30}$$

for all $x \in \mathbb{R}^d$, $s \geq 0$, $0 \leq z_1, z_2 \leq M$. Combining Equations (25) and (30), we get, for $x \in \mathbb{R}^d$, $0 \leq s \leq T$, $0 \leq z_1, z_2 \leq M$, and $\beta, \beta' \in (0, 1)$,

$$
\begin{aligned}
&|R(\lambda, \psi_\beta)(s, x, z_1) - R(\lambda, \psi_{\beta'}(s, x, z_2)| \\
&\quad \leq |R(\lambda, \psi_\beta)(s, x, z_1) - R(\lambda, \psi)(s, x, z_1)| \\
&\qquad + |R(\lambda, \psi)(s, x, z_1) - R(\lambda, \psi)(s, x, z_2)| \\
&\qquad + |R(\lambda, \psi_{\beta'})(s, x, z_2) - R(\lambda, \psi)(s, x, z_2)| \\
&\quad \leq |\psi(s, x, z_1) - \psi_\beta(s, x, z_1)| + |\psi(s, x, z_2) - \psi_{\beta'}(s, x, z_2)| \\
&\qquad + \lambda(s, x)|z_1 - z_2| \\
&\quad \leq \alpha(\beta, T, M) + \alpha(\beta', T, M) + \lambda(s, x)|z_1 - z_2|, \quad\quad (31)
\end{aligned}
$$

where $\alpha(\beta, T, M)$, $\beta \in (0, 1)$ are constants satisfying $\alpha(\beta, T, M) \to 0$, as $\beta \to 0$. Using Lemma 2.1 with $g = u(\beta, f)$, $c = a + \lambda$, $F = f(\tau, W_\tau)$ and $\omega = R(\lambda, \psi_\beta)$, Equation (20) can be rewritten as

$$
\begin{aligned}
u(\beta, f)(r, x) = {} & \Pi_{r,x}[(H^{a+\lambda}(r, \tau)f(\tau, W_r)] \\
& + \Pi_{r,x} \int_r^\tau H^{a+\lambda}(r, s)R(\lambda, \psi_\beta)(s, W_s, u(\beta, f) \\
& \times (s, W_s))A(\mathrm{d}s). \quad\quad (32)
\end{aligned}
$$

Let $h_{\beta,\beta'} = |u(\beta, f) - u(\beta', f)|$. $h_{\beta,\beta'}$ is supported by $[0, T)$. By Equations (31) and (32) we have

$$
\begin{aligned}
h_{\beta,\beta'} \leq {} & q\Pi_{r,x} \int_r^T H^{c+\lambda}(r, s)A(\mathrm{d}s) \\
& + \Pi_{r,x} \int_r^T H^{c+\lambda}(r, s)(\lambda h_{\beta,\beta'})(s, W_s)A(\mathrm{d}s),
\end{aligned}
$$

where $q = \alpha(\beta, T, M) + \alpha(\beta', T, M)$. Since the additive function A satisfies Equation (8) with $I = [0, T)$, we have

$$
\Pi_{r,x} \int_r^T (\lambda h_{\beta,\beta'})(s, W_s)A(\mathrm{d}s) < C\Pi_{r,x}A(r, T) < \infty,
$$

where C is a constant. By Lemma 2.2, $h_{\beta,\beta'}(r, x) \leq q\Pi_{r,x}A(r, T)$, $r \in [0, T)$, $x \in \mathbb{R}^d$. Since $\|u(\beta, f)\|_\infty \leq \|f\|_\infty$ and $\Pi_{r,x}A(r, T) < \infty$, there exists a function $U(A, f) \in bp\mathcal{B}(S)$ supported by $[0, T)$ such that

$$
u(\beta, f) \xrightarrow{bp} U(f) \quad \text{as } \beta \to 0.
$$

The dominated convergence theorem implies that $U(A, f)$ is a bounded positive solution of Equation (3). Since f is supported by $[0, T)$, $U(A, f)$ is also supported by $[0, T)$.

The uniqueness can be proved similarly as above. In fact, assume that $u_1, u_2 \in bp\mathcal{B}(S)$ are two solutions of Equation (3), and $M \geq 1$ is a constant such that $0 \leq u_1, u_2 \leq M$. Since $0 \leq u_i(r, x) \leq \Pi_{r,x} f(\tau, X_\tau)$, $i = 1, 2$. If f is supported by $[0, T)$, the u_1 and u_2 are also supported by $[0, T)$. Let λ and $R(\lambda, \psi)$ be defined as in Equations (26) and (27), respectively. Then we similarly get

$$|u_1 - u_2| \leq \Pi_{r,x} \int_r^T H^{a+\lambda}(r, s)(\lambda|u_1 - u_2|)(s, W_s)A(ds).$$

By Lemma 2.2, $u_1 \equiv u_2$.

2. Let λ, $R(\lambda, \psi)$ be given by Equations (26) and (27), respectively, with constant M satisfies $M \geq 1 \vee \|f_1\|_\infty \vee \|f_2\|_\infty$. Then

$$|U(A, f_1) - U(A, f_2)| \leq \Pi_{r,x}[H^{a+\lambda}(r, \tau)|f_1 - f_2|(\tau, W_\tau)]$$
$$+ \Pi_{r,x} \int_r^T H^{a+\lambda}(r, s)(\lambda|U(A, f_1)$$
$$- U(A, f_2)|)(s, W_s)A(ds).$$

By Lemma 2.2, $|U(A, f_1) - U(A, f_2)| \leq \Pi_{r,x}[H^{a,\lambda}(r, \tau)|f_1 - f_2| (\tau, W_\tau)] \leq \|f_1 - f_2\|_\infty$, which means the statement 2 is valid.

3. Let $\Phi(s, x, z) = \psi(s, x, z) - a(s, x)z$. Using Lemma 3.1 with $c = a$, $F = \lambda f(\tau, W_\tau)$ and $\omega(\cdot, \cdot) = -\Phi(\cdot, \cdot, U(A, \lambda f)(\cdot, \cdot))$, we get $U(A, \lambda f)$ satisfies

$$U(A, \lambda f) = \lambda \Pi_{r,x}[H^a(r, \tau)f(\tau, W_\tau)]$$
$$- \Pi_{r,x} \int_r^\tau H^a(r, s)\Phi(s, W_s, U(A, \lambda f)(s, W_s)A(ds).$$
$$(33)$$

Suppose f is supported by $[0, T)$ then $U(A, \lambda f)$ is also supported by $[0, T)$, and therefore,

$$\Pi_{r,x} \int_r^\tau H^a(r, s)\Phi(s, W_s, U(A, \lambda f)(s, W_s))A(ds)$$
$$\leq \Pi_{r,x} \int_r^T H^a(r, s)\Phi(s, W_s, C\lambda)A(ds), \qquad (34)$$

where $C = \|f\|_\infty \vee 1$. If we can prove

$$\frac{1}{\lambda}\Pi_{r,x} \int_r^T H^a(r, s)\Phi(s, W_s, C\lambda)A(ds) \to 0, \text{ as } \lambda \to 0, \qquad (35)$$

then by Equations (34) and (35),

$$U(A, \lambda f)/\lambda \overset{bp}{\to} \Pi_{\cdot,\cdot}[H^a(r, \tau)f(\tau, W_\tau)], \text{ as } \lambda \to 0.$$

Now we are left to prove Equation (35). Not that for $\lambda \leq 1$,

$$\frac{1}{\lambda}(C\lambda u - 1 + e^{-C\lambda u}) \leq (Cu) \leq (C^2 u^2) \leq C^2(u \vee u^2), u \geq 0.$$

(36)

By the dominated convergence theorem and the assumption of $\int_0^\infty u \wedge u^2 n(s, x, du) \leq \infty$, we have, for fixed $s \geq 0$, $x \in \mathbb{R}^d$,

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \Phi(s, x, C\lambda)$$

$$= \lim_{\lambda \to 0} \int_0^\infty \frac{1}{\lambda}(C\lambda u - 1 + e^{-C\lambda u})n(s, x, du) = 0.$$

(37)

By Equation (36),

$$\sup_{s \in [0,T], x \in \mathbb{R}^d} \frac{1}{\lambda} \Phi(s, x, C\lambda)$$

$$\leq \sup_{s \in [0,T], x \in \mathbb{R}^d} \int_0^\infty u \wedge u^2 n(s, x, du) < \infty.$$

Then, using the dominated convergence theorem again and by noticing Equations (8) and (37), we get that Equation (35) holds.

A real-valued function u on the Abelian semigroup $G = bp\mathcal{B}(S)$ is called nagative definite if

$$\sum_{i,j=1}^n \lambda_i \lambda_j u(g_i + g_j) \leq 0,$$

for every $n \geq 2$, all $g_1, \ldots, g_n \in G$ and all $\lambda_1, \ldots, \lambda_n \in R$ such that $\sum_1^n \lambda_i = 0$. It is known that if u is negative definite, then $L(f) = e^{-u(f)}$ is positive definite. (See Berg et al. (5).) $\qquad\square$

Proof of Theorem 1.1: Fix $T > 0$ and restrict ourselves to functions f_i supported by the interval $[0, T)$. By Theorem 2.1, u_I satisfying Equation (5) exists and is unique. We consider G_I and u_I as functions of $(f_1, \ldots, f_n) \in (bp\mathcal{B}_T)^n$. By induction on n and the construction process of u_I given by Theorem 2.1, we can prove u_I is negative definite and vanishes if $f_1 = \cdots = f_n = 0$. (We omit the details. See, e.g., Dynkin (1).) Let

$$L_I(f_1, \ldots, f_n) = \exp\langle -u_I, \mu \rangle.$$

(38)

Then L_I is positive definite. By Theorem 2.1 L_I is continuous. It follows from Lemma 1.4 in Dynkin (1) that there exists a unique probability measure on $\mu(S_{\leq T})^n$ with Laplace transform given by Equation (38). By Lemma 1.3 and Section 1.6 in Dynkin (1), there exists a unique probability measure P_μ on $\mu(S)^n$ such that

Equations (4) and (5) hold. Obviously, $L_I(f_1, \ldots, f_n) = L_J(f_1, \ldots, f_{n-1})$ if $J = \{1, \ldots, n-1\}$ and $f_n = 0$, which means L_I satisfy consistency property. Therefore the existence of the stochastic process $(X_\tau, P_\mu; \mu \in M(S))$ subject to the statement of Theorem 1.1 follows from Kolmogorov's theorm. □

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