Stochastic Processes and their Applications 172 (2024) 104338

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

1-stable fluctuation of the derivative martingale of branching random walk[☆]

Haojie Hou [a], Yan-Xia Ren [b,*,1], Renming Song [c,2]

[a] School of Mathematical Sciences, Peking University, Beijing, 100871, PR China
[b] LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, PR China
[c] Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, IL 61801, USA

ARTICLE INFO

MSC:
60J80
60F05
60G42

Keywords:
Branching random walk
Derivative martingale
Spine decomposition

ABSTRACT

In this paper, we study the functional convergence in law of the fluctuations of the derivative martingale of branching random walk on the real line. Our main result strengthens the results of Buraczewski et al. (2021) and is the branching random walk counterpart of the main result of Maillard and Pain (2019) for branching Brownian motion.

1. Introduction

Consider the following branching Brownian motion on $\mathbb{R}$: initially there is a particle at 0, it moves according to a standard Brownian motion with drift 1. After an exponentially distributed time with parameter $\beta > 0$, it dies and splits into a random number of offspring with law $\{p_k : k \geq 0\}$. The offspring repeat the parent's behavior independently from where they were born. We will use $N(t)$ to denote the set of particles alive at time t and for $u \in N(t)$, we will use $X_u(t)$ to denote the position of u. Without loss of generality, assume that

$$\beta\left(\sum_{k=1}^{\infty} kp_k - 1\right) = \frac{1}{2},$$

which implies that for any $t \geq 0$ (for example, see [11, (1.2) and (1.3)]),

$$\mathbb{E}\left(\sum_{u\in N(t)} e^{-X_u(t)}\right) = 1, \quad \mathbb{E}\left(\sum_{u\in N(t)} X_u(t)e^{-X_u(t)}\right) = 0 \quad \text{and} \quad \mathbb{E}\left(\sum_{u\in N(t)} (X_u(t))^2 e^{-X_u(t)}\right) = t.$$

The derivative martingale of the branching Brownian motion is defined as

$$Z_t := \sum_{u\in N(t)} X_u(t)e^{-X_u(t)}.$$

☆ The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900).
* Corresponding author.
E-mail addresses: houhaojie@pku.edu.cn (H. Hou), yxren@math.pku.edu.cn (Y.-X. Ren), rsong@illinois.edu (R. Song).
1 The research of this author is supported by NSFC, China (Grant Nos. 12071011 and 12231002) and The Fundamental Research Funds for the Central Universities, China, Peking University LMEQF.
2 Research supported in part by a grant from the Simons Foundation, USA (#960480, Renming Song).

https://doi.org/10.1016/j.spa.2024.104338
Received 23 November 2023; Received in revised form 29 February 2024; Accepted 10 March 2024
Available online 13 March 2024
0304-4149/

It was proved in [9,14] that Z_t converges almost surely to a non-degenerate non-negative limit Z_∞ if and only if $\sum_{k=1}^{\infty} k(\log k)^2 p_k < \infty$. Maillard and Pain [11] studied the fluctuation of $Z_\infty - Z_t$. They showed that, under the assumption $\sum_{k=1}^{\infty} k(\log k)^3 p_k < \infty$,

$$\left(\left(\sqrt{t}\left(Z_\infty - Z_{at} + \frac{\log t}{\sqrt{2\pi a t}} Z_\infty\right)\right)_{a\geq 1}, \mathbb{P}(\cdot \middle| \mathcal{F}_t^B)\right) \xrightarrow{f.d.d.} \left((S_{a^{-1/2}Z_\infty})_{a\geq 1}, \mathbb{P}\left(\cdot \middle| Z_\infty\right)\right)$$

in probability, where S_t is a spectrally positive 1-stable Lévy process independent of Z_∞ and $\{\mathcal{F}_t^B\}_{t\geq 0}$ is the filtration of the branching Brownian motion. More precisely, they showed that, for all $m \geq 1, a_1, \ldots, a_m \in [1, \infty)$ and bounded continuous $f : \mathbb{R}^m \to \mathbb{R}$,

$$\mathbb{E}\left(f\left(\sqrt{t}\left(Z_\infty - Z_{a_k t} + \frac{\log t}{\sqrt{2\pi a_k t}} Z_\infty\right), 1 \leq k \leq m\right) \middle| \mathcal{F}_t^B\right)$$
$$\xrightarrow[t\to\infty]{} \mathbb{E}\left(f\left(S_{a_k^{-1/2}Z_\infty}, 1 \leq k \leq m\right) \middle| Z_\infty\right), \quad \text{in probability.} \tag{1.1}$$

Now we turn to branching random walks. A branching random walk on $\mathbb{R}$ is defined as follows. At generation 0, there is a particle at the origin. At generation $n = 1$, this particle dies and gives birth to a set of offspring. The law of the positions of the offspring relative to their parent is given by a point process L. The offspring evolve independently and obey the same rule as their parent. The procedure goes on. Note that we allow the total number of offspring to be infinite with positive probability as in [5]. We will use $\mathbb{T}$ to denote the genealogical tree of the branching random walk, $\mathcal{N}(n)$ to denote the collection of particles in the nth generation, $|x|$ to denote the generation of particle x and $\left\{V(x), x \in \mathcal{N}(n)\right\}$ to denote the positions of the particles in the nth generation. We will use $\mathbb{P}$ to denote the law of the branching random walk above and use $\mathbb{E}$ to denote the expectation with respect to $\mathbb{P}$. If the initial particle is located at $x \in \mathbb{R}$ instead of the origin, we will use $\mathbb{P}_x$ to denote the law of the corresponding branching random walk and use $\mathbb{E}_x$ to denote the expectation with respect to $\mathbb{P}_x$. For $n \geq 0$, we denote by $\mathcal{F}_n$ the σ-field generated by the branching random walk up to generation n (including generation n). We will always assume that

(A1)

$$\mathbb{E}\left(\sum_{x\in\mathcal{N}(1)} e^{-V(x)}\right) = 1, \quad \mathbb{E}\left(\sum_{x\in\mathcal{N}(1)} V(x)e^{-V(x)}\right) = 0$$

and

$$\sigma^2 := \mathbb{E}\left(\sum_{x\in\mathcal{N}(1)} (V(x))^2\, e^{-V(x)}\right) < \infty.$$

Under **(A1)**,

$$W_n := \sum_{x\in\mathcal{N}(n)} e^{-V(x)}, \quad D_n := \sum_{x\in\mathcal{N}(n)} V(x)e^{-V(x)}, \quad n \geq 0,$$

are martingales with respect to $\{\mathcal{F}_n : n \geq 0\}$. They are called the additive martingale and the derivative martingale of the branching random walk respectively. Suppose that

(A2)

$$\mathbb{E}\left(W_1\left(\log_+ W_1\right)^2\right) + \mathbb{E}\left(\widetilde{W}_1 \log_+ \widetilde{W}_1\right) < \infty,$$

where $\log_+ y := \max\{0, \log y\}$ and

$$\widetilde{W}_1 := \sum_{x\in\mathcal{N}(1)} (V(x))_+ e^{-V(x)}$$

with $(V(x))_+ := \max\{V(x), 0\}$. It was proved in Aïdékon [1], Biggins and Kyprianou [4] and Chen [6] that, under **(A1)**, D_n converges almost surely to a non-negative limit D_∞ with $\mathbb{P}(D_\infty > 0) > 0$ if and only if **(A2)** holds. Aïdékon and Shi [2] studied the relationship between W_n and D_n, and showed that, under the assumptions **(A1)** and **(A2)**,

$$\lim_{n\to\infty} \sqrt{n}W_n = \sqrt{\frac{2}{\pi\sigma^2}} D_\infty, \quad \text{in probability.} \tag{1.2}$$

Under **(A1)**, **(A2)** and the additional assumption

(A3) The branching random walk is non-arithmetic, i.e., for any $\delta > 0$,

$$\mathbb{P}\left(L\left(\mathbb{R} \setminus \delta\mathbb{Z}\right) > 0\right) > 0,$$

Buraczewski, Iksanov and Mallein [5] proved that

$$\lim_{y\to+\infty}\left(\mathbb{E}\left(D_\infty 1_{\{D_\infty\leq y\}}\right) - \log y\right) = c_0 \tag{1.3}$$

for some real number c_0 if and only if

$$\mathbb{E}\left(W_1^+\left(\log_+ W_1^+\right)^3\right) + \mathbb{E}\left(\widetilde{W}_1\left(\log_+ \widetilde{W}_1\right)^2\right) + \mathbb{E}\left(\sum_{x\in\mathcal{N}(1)} e^{-V(x)}\left(-V(x)\right)_+^3\right) < \infty$$

and

$$\mathbb{E}\left(W_1^-\left(\log_+ W_1^-\right)^3 1_{\left\{\widehat{W}_1>C_0\right\}}\right)<\infty \text{ for some } C_0>0.$$

Here, W_1^+, W_1^- and $\widehat{W}_1$ are defined respectively by

$$W_1^+ := \sum_{x\in\mathcal{N}(1)} e^{-V(x)} 1_{\{V(x)\geq 0\}}, \quad W_1^- := \sum_{x\in\mathcal{N}(1)} e^{-V(x)} 1_{\{V(x)<0\}},$$

$$\widehat{W}_1 := \sum_{x\in\mathcal{N}(1)}\left(1+V(x)-\min_{y\in\mathcal{N}(1)}V(y)\right)e^{\min_{y\in\mathcal{N}(1)}V(y)-V(x)}1_{\{V(x)<0\}}.$$

The following sufficient condition for (1.3) was given in [5, Remark 2.3(2)]:

(A4)

$$\mathbb{E}\left(W_1\left(\log_+ W_1\right)^3\right)+\mathbb{E}\left(\widetilde{W}_1\left(\log_+\widetilde{W}_1\right)^2\right)<\infty.$$

[5, Theorem 2.4] says that, under conditions **(A1)–(A4)**, for any bounded continuous function $f:\mathbb{R}\to\mathbb{R}$, it holds that

$$\mathbb{E}\left(f\left(\sqrt{n}\left(D_\infty-D_n+\frac{\log n}{2}W_n\right)\right)\Big|\mathcal{F}_n\right)\stackrel{\mathbb{P}}{\to}\mathbb{E}\left(f(D_\infty X_1)\mid D_\infty\right),\quad n\to\infty, \tag{1.4}$$

where X_1 is a spectrally positive 1-stable random variable independent of D_∞ with generating triplet $\left((c_0+1-\gamma)\sqrt{2/\left(\pi\sigma^2\right)},\sqrt{\pi/\left(2\sigma^2\right)},1\right)$, γ is the Euler–Mascheroni constant and c_0 is the constant in (1.3). More precisely, the characteristic function of X_1 is given by

$$\begin{aligned}\mathbb{E}\left(e^{\mathrm{i}\lambda X_1}\right)&=\exp\left\{\mathrm{i}(c_0+1-\gamma)\sqrt{2/\left(\pi\sigma^2\right)}\lambda-\sqrt{\pi/\left(2\sigma^2\right)}|\lambda|(1+\mathrm{i}\,\mathrm{sgn}(\lambda)(2/\pi)\log|\lambda|)\right\}\\&=\exp\left\{-\psi_{\sqrt{\pi/(2\sigma^2)},(c_0+1-\gamma)\sqrt{2/(\pi\sigma^2)}}\right\},\quad\lambda\in\mathbb{R},\end{aligned} \tag{1.5}$$

where $\psi_{\sigma,\mu}(\lambda):=\sigma|\lambda|(1+\mathrm{i}\,\mathrm{sgn}(\lambda)(2/\pi)\log|\lambda|)-\mathrm{i}\mu\lambda$. Combining (1.2) and (1.4), we can easily get the following fact: for any $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\left(|D_\infty-D_n|>n^{\varepsilon-1/2}\right)=0. \tag{1.6}$$

The goal of this paper is to prove the counterpart of (1.1) for branching random walks. We work under the additional assumption

(A5) There exists a constant $\alpha\in(0,1]$ such that

$$\mathbb{E}\left(\sum_{u\in\mathcal{N}(1)}e^{-(1+\alpha)V(u)}\right)<\infty.$$

Let $\{(S_n)_{n\geq 0},\mathbf{P}\}$ be the random walk defined in (3.1) below and let $\mathbf{E}$ denote the corresponding expectation. Then **(A5)** says that $\mathbf{E}(e^{-\alpha S_1})<\infty$. This assumption is only used in (3.25).

2. Main result

We will always assume that **(A1)–(A5)** hold. Define $\lceil y\rceil:=\min\{k\in\mathbb{Z}:k\geq y\}$.

Theorem 2.1. *Let $(X_t)_{t\geq 0}$ be a spectrally positive 1-stable Lévy process with characteristic function given in (1.5), independent of D_∞. Then the conditional law of*

$$\left(\sqrt{n}\left(D_\infty-D_{\lceil an\rceil}+\frac{\log n}{2}W_{\lceil an\rceil}\right)\right)_{a\geq 1}$$

given $\mathcal{F}_n$ converges weakly in probability (in the sense of finite-dimensional distributions) to the conditional law of $\left(X_{a^{-1/2}D_\infty}\right)_{a\geq 1}$ given D_∞. In other words, for all $m\geq 1, a_1,\ldots,a_m\in[1,\infty)$ and bounded continuous $f:\mathbb{R}^m\to\mathbb{R}$, we have

$$\begin{aligned}&\mathbb{E}\left(f\left(\sqrt{n}\left(D_\infty-D_{\lceil a_k n\rceil}+\frac{\log n}{2}W_{\lceil a_k n\rceil}\right),1\leq k\leq m\right)\Big|\mathcal{F}_n\right)\\&\underset{n\to\infty}{\longrightarrow}\mathbb{E}\left(f\left(X_{a_k^{-1/2}D_\infty},1\leq k\leq m\right)\Big|D_\infty\right),\quad \textit{in probability.}\end{aligned}$$

Recall that $\{(S_n)_{n\geq 0},\mathbf{P}\}$ is the random walk defined in (3.1) below. It follows from [2, (2.8)] that there exists $\theta^*>0$ such that

$$\lim_{n\to\infty}\sqrt{n}\mathbf{P}\left(\min_{j\leq n}S_j\geq 0\right)=\theta^*.$$

Set

$$\delta_n:=\left(\theta^*\right)^{-1}\sqrt{n}\mathbf{P}\left(\min_{j\leq n}S_j\geq 0\right). \tag{2.1}$$

Proposition 2.2. *There exists a $\delta_+ > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\left| \sqrt{n} W_n - \sqrt{\frac{2}{\pi\sigma^2}} \delta_n D_\infty \right| \geq n^{-\delta_+} \right) = 0.$$

Consequently, for all $m \geq 1, a_1, \dots, a_m \in [1, \infty)$ and bounded continuous $f : \mathbb{R}^m \to \mathbb{R}$,

$$\mathbb{E}\left(f\left(\sqrt{n}\left(D_\infty - D_{\lceil a_k n\rceil} + \frac{\log n}{\sqrt{2\pi\sigma^2 \lceil a_k n\rceil}} \delta_{\lceil a_k n\rceil} D_\infty \right), 1 \leq k \leq m \right) \Big| \mathcal{F}_n \right)$$
$$\xrightarrow[n\to\infty]{} \mathbb{E}\left(f\left(X_{a_k^{-1/2} D_\infty}, 1 \leq k \leq m \right) \mid D_\infty \right), \quad \textit{in probability.}$$

If we want to replace δ_n by 1, we will need a slightly stronger condition:

(A6) For some $\gamma_0 > 0$,

$$\mathbb{E}\left(\sum_{u\in\mathcal{N}(1)} |V(u)|^{2+\gamma_0} e^{-V(u)} \right) < \infty.$$

The assumption **(A6)** says that the random walk S_n has finite $(2+\gamma_0)$th moment, which implies that $\delta_n - 1 = o(n^{-\epsilon_0})$ with some $\epsilon_0 > 0$ according to [7]. We summarize the result of [7, Theorem 2.7] as follows:

Lemma 2.3. *If* **(A1)–(A6)** *hold, then there exists $\epsilon_0 > 0$ such that*

$$\lim_{n\to\infty} n^{\epsilon_0} \left| \delta_n - 1 \right| = 0.$$

Combining Theorem 2.1, Proposition 2.2 and Lemma 2.3, we immediately get the following theorem:

Theorem 2.4. *Assume that* **(A1)–(A6)** *hold. Then for all $m \geq 1, a_1, \dots, a_m \in [1, \infty)$ and bounded continuous $f : \mathbb{R}^m \to \mathbb{R}$, we have*

$$\mathbb{E}\left(f\left(\sqrt{n}\left(D_\infty - D_{\lceil a_k n\rceil} + \frac{\log n}{\sqrt{2\pi\sigma^2 \lceil a_k n\rceil}} D_\infty \right), 1 \leq k \leq m \right) \Big| \mathcal{F}_n \right)$$
$$\xrightarrow[n\to\infty]{} \mathbb{E}\left(f\left(X_{a_k^{-1/2} D_\infty}, 1 \leq k \leq m \right) \mid D_\infty \right), \quad \textit{in probability.}$$

The main idea of this paper is a modification of that of [11]. To get the fluctuation of $D_\infty - D_{\lceil an\rceil}$, we choose a level γ_n and define a quantity $D_m^{\lceil an\rceil,\gamma_n}$, for $m \geq \lceil an\rceil$, which roughly takes care of the contributions to D_m by the paths that stay above the level γ_n between generations $\lceil an\rceil$ and m. We first show that $D_m^{\lceil an\rceil,\gamma_n}$ converges to a limit $D_\infty^{\lceil an\rceil,\gamma_n}$ as $m \to \infty$ and get a rate of convergence for $D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}$ as $n \to \infty$, see Lemma 4.1. Then we analyze the contribution of $D_\infty^{\lceil an\rceil,\gamma_n}$ to the limit behavior of $D_{\lceil an\rceil}$ in Proposition 4.3 . For contributions to D_∞ by the collection $\mathcal{L}^{\lceil an\rceil,\gamma_n}$ of particles x with $|x| > \lceil an\rceil$, $V(x) < \gamma_n$ and $\min_{j\in[an,|x|-1]\cap\mathbb{Z}} V(x_j) \geq \gamma_n$, we separate $\mathcal{L}^{\lceil an\rceil,\gamma_n}$ into two sets $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$ and $\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}$ and look at their respective contributions to the limit behavior of D_∞, see (4.8) in which $F_{good}^{\lceil an\rceil,\gamma_n}$ represents the contributions by $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$ and $F_{bad}^{\lceil an\rceil,\gamma_n}$ represents the contributions by $\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}$. We show in Proposition 4.9 that $F_{bad}^{\lceil an\rceil,\gamma_n}$ is asymptotically negligible. For the contributions by $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$, we define a sequence of random variables $\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}$ (see (4.30)). By using the branching property and the tail behavior of D_∞, we show in Proposition 4.10 that $\sqrt{n}(F_{good}^{\lceil an\rceil,\gamma_n} - \widehat{N}_{good}^{\lceil an\rceil,\gamma_n})$ converges in distribution to $c^* X_{a^{-1/2} D_\infty}$ with c^* being the positive constant defined in (3.4) below, which leads to the main result.

Although the general approach of this paper is similar to that of [11], adapting it to the case of branching random walk is pretty challenging. In [11], the continuity of the sample paths of Brownian motion makes things a lot easier. For instance, the counterpart of $\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}$ in the branching Brownian motion case takes care the contributions by the particles that hit a certain level at some time after at due to the continuity of Brownian motion. The main difficulty in the case of branching random walks is that a branching random walk can jump across the level and one needs to take care of the landing positions of the particles after crossing the level. This leads to many complications and many subtle modifications are needed to actually carry out the program.

3. Preliminaries

We will use $f(x) \lesssim g(x),\ x \in E$, to denote that there exists a constant C independent of $x \in E$ such that

$$f(x) \leq C g(x), \quad x \in E.$$

We will use $f(x) \asymp g(x), x \in E$ to denote $f(x) \lesssim g(x), x \in E$ and $g(x) \lesssim f(x), x \in E$.

3.1. Spine decomposition

Define a random walk $\{(S_n)_{n\geq 0}, \mathbf{P}\}$ such that for any $n \in \mathbb{N}$ and measurable function $g : \mathbb{R}^n \to [0, \infty)$,

$$\mathbb{E}\left(\sum_{x\in\mathcal{N}(n)} g\left(V(x_1), \dots, V(x_n)\right) \right) = \mathbf{E}\left(e^{S_n} g\left(S_1, \dots, S_n\right) \right), \tag{3.1}$$

where $\mathbf{E}$ stands for expectation with respect to $\mathbf{P}$ and for $x \in \mathcal{N}(n)$ and $j \le n$, x_j denotes the ancestor of x in the jth generation. (3.1) is also known as the many-to-one formula. See [12, Theorem 1.1] for more information about the random walk $\{S_n, n \ge 0\}$. By taking $n = 1$, $g(x) = xe^{-x}$ and $g(x) = x^2e^{-x}$ respectively in (3.1), we get that **(A1)** and **(A2)** imply that $\mathbf{E}S_1 = 0, \sigma^2 = \mathbf{E}S_1^2 < \infty$. For any $y \in \mathbb{R}$, we use $\mathbf{P}_y$ to denote the law of $\{y + S_n, n \ge 0\}$ and $\mathbf{E}_y$ to denote the expectation with respect to $\mathbf{P}_y$. Note that, under $\mathbf{P}_y$, $\{S_n, n \ge 0\}$ is a random walk starting from y.

We define a probability $\mathbb{Q}$ such that for all $n \ge 0$,

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_n} := W_n.$$

Denote by $\widehat{L}$ the law of L under $\mathbb{Q}$. Lyons [10] gave the following description of the law of the branching random walk under $\mathbb{Q}$: there is a spine process denoted by $\{w_n\}_{n\ge0}$ with $w_0 = \emptyset$ and the initial position of the spine is $V(w_0) = 0$. At generation $n = 1$, w_0 dies and splits into a set of offspring with law $\widehat{L}$. Choose one offspring x from all the offspring of w_0 with probability proportional to $e^{-V(x)}$, and call it w_1. w_1 evolves independently as w_0 and the other unmarked offspring evolve independently as in the original branching random walk. By Lyons [10], for any $x \in \mathcal{N}(n)$, we have

$$\mathbb{Q}\left(w_n = x|\mathcal{F}_n\right) = \frac{e^{-V(x)}}{W_n}. \tag{3.2}$$

Moreover, the position process $\left\{V(w_n)\right\}_{n\ge0}$ along the spine under $\mathbb{Q}$ is equal in law to $\left\{S_n\right\}_{n\ge0}$ defined in (3.1). Also, for $y \in \mathbb{R}$, we will use $\mathbb{Q}_y$ to denote the counterpart of $\mathbb{Q}$ in the case of branching random walk with the initial particle located at y.

Let $\tau^+ := \inf\{k \ge 1 : S_k \ge 0\}$. Define the renewal function $R(y)$ by

$$R(y) := \mathbf{E}\Big(\sum_{j=1}^{\tau^+-1} 1_{\{S_j\ge -y\}}\Big), \quad y \in \mathbb{R}.$$

Using the facts that $\mathbf{E}S_1 = 0$ and $\mathbf{E}S_1^2 < \infty$, one can easily get that (see, for example, [2, Section 2.2]) $R(0) = 1$, $R(y) = 0$ for $y < 0$ and

$$R(y) \asymp (1 + y), \quad y \ge 0, \tag{3.3}$$

and the limit

$$c^* := \lim_{y\to+\infty} \frac{R(y)}{y} \tag{3.4}$$

exists in $(0, +\infty)$. According to [2, (2.4)], we also have

$$R(y) = \sum_{k=0}^{\infty} \mathbf{P}(|H_k| \le y), \tag{3.5}$$

where $H_k := S_{\sigma_k^-}$ with $\sigma_0^- := 0$ and $\sigma_k^- := \inf\left\{i > \sigma_{k-1}^- : S_i < \min_{0\le j\le\sigma_{k-1}^-} S_j\right\}$. For $y \ge 0$, define $\tau_{-y}^{-,H} := \inf\{k \ge 1 : H_k < -y\}$ and $\tau_{-y}^- := \inf\{n \ge 1 : S_n < -y\}$. Then we can rewrite (3.5) as

$$R(y) = \sum_{k=0}^{\infty} \mathbf{P}\left(H_k \ge -y\right) = \sum_{k=0}^{\infty} \mathbf{P}\left(\tau_{-y}^{-,H} > k\right) = \mathbf{E}\left(\tau_{-y}^{-,H}\right).$$

Note that $\mathbf{E}(H_1) \in (0, \infty)$ (see [5, Lemma A.4.(a)]) and that $H_k - k\mathbf{E}(H_1)$ is a martingale. Thus, combining the optional sampling theorem and the fact that $H_{\tau_{-y}^{-,H}} = S_{\tau_{-y}^-}$, we obtain that

$$(-\mathbf{E}H_1)\mathbf{E}\left(\tau_{-y}^{-,H}\right) = \mathbf{E}\left(-H_{\tau_{-y}^{-,H}}\right) \iff R(y)\mathbf{E}(|H_1|) = -\mathbf{E}\left(S_{\tau_{-y}^-}\right) = y - \mathbf{E}_y\left(S_{\tau_0^-}\right). \tag{3.6}$$

By [2, the first paragraph in the proof of Lemma 2.1], we have $c^* = \mathbf{E}(|H_1|)^{-1}$. Note that, as a consequence of **(A4)**, we have $\mathbf{E}\left((-S_1)_+^3\right) < \infty$. Thus, by [5, Lemma A.4.(d)], under **(A3)** and **(A4)**, there exists an $\alpha^* \in (0, \infty)$ such that

$$\lim_{y\to+\infty} \left(R(y) - c^*y\right) = \alpha^*. \tag{3.7}$$

One can easily check that

$$R(y) = \mathbf{E}\left(R(S_1 + y)1_{\{S_1\ge -y\}}\right), \quad y \ge 0. \tag{3.8}$$

Hence the sequence of random variables

$$D_n^{-y} := \sum_{x\in\mathcal{N}(n)} R\left(V(x) + y\right)e^{-V(x)}1_{\{\min_{j\le n} V(x_j)\ge -y\}}$$

is a non-negative $\mathbb{P}$-martingale with respect to $\{\mathcal{F}_n\}_{n\ge 0}$ with $\mathbb{E}\left(D_n^{-y}\right) = R(y)$ for all $n \ge 0$. Define a new probability measure $\mathbb{Q}^{-y}$ such that for all $n \ge 0$,

$$\frac{\mathrm{d}\mathbb{Q}^{-y}}{\mathrm{d}\mathbb{P}}\bigg|_{\mathcal{F}_n} := \frac{D_n^{-y}}{R(y)}. \tag{3.9}$$

Similar to the spine decomposition under $\mathbb{Q}$, we can also describe the spine decomposition for the branching random walk under $\mathbb{Q}^{-y}$ with a spine denoted by $\{w_n\}_{n\ge 0}$ and with spatial displacement following the law of the random walk $\{S_n\}$ conditioned to stay in $[-y, +\infty)$: there is a spine process denoted by $\{w_n\}_{n\ge 0}$ with $w_0 = \emptyset$ and the initial position of the spine is $V(w_0) = 0$. At generation $n = 1$, w_0 dies and gives birth to a set of offspring according to the law of L under $\mathbb{Q}^{-y}$. Choose one offspring x from all the offspring of w_0 with probability proportional to $R(V(x) + y)e^{-V(x)}1_{\{V(x)\ge -y\}}$, and call it w_1. At generation $n = 2$, given $V(w_1)$, w_1 gives birth to a set of offspring according to a point process with the same law as L under the law $\mathbb{Q}^{-(V(w_1)+y)}$ and again choose one offspring x from all the offspring of w_1 with probability proportional to $R(V(x)+y)e^{-V(x)}1_{\{V(x)\ge -y\}}$ named w_2. The other unmarked offspring evolve independently as in the original branching random walk. The procedure goes on. According to [2, Fact 3.2] or [6, Section 2.2], for $x \in \mathcal{N}(n)$,

$$\mathbb{Q}^{-y}\left(w_n = x|\mathcal{F}_n\right) = \frac{R\left(V(x)+y\right)e^{-V(x)}1_{\{\min_{j\le n} V(x_j)\ge -y\}}}{D_n^{-y}} \tag{3.10}$$

and the position process $\left(V(w_n)\right)_{n\ge 1}$ along the spine is equal in law to $\{S_n\}_{n\ge 1}$ conditioned to stay in $[-y, +\infty)$.

3.2. Elementary properties for centered random walk

Lemma 3.1. *(i) For all $a \ge 0$ and $n \ge 1$, it holds that*

$$\mathbf{P}_a\Big(\min_{j\le n} S_j \ge 0\Big) \lesssim \frac{(1+a)}{\sqrt{n}}.$$

(ii) For all $a, u \ge 0$, $b > 0$ and $n \ge 1$, it holds that

$$\mathbf{P}_a\Big(\min_{j\le n} S_j \ge 0, u \le S_n \le b+u\Big) \lesssim \frac{(b+1)(b+u+1)(a+1)}{\sqrt{n^3}}.$$

(iii) For all $a, b \ge 0$, it holds that

$$\sum_{n=0}^{\infty} \mathbf{P}_a\Big(\min_{j\le n} S_j \ge 0, S_n \le b\Big) \lesssim (1+b)\left(1+(a\wedge b)\right).$$

Here $a \wedge b := \min\{a, b\}$.

(iv) For any $\lambda > 0$, there exists a constant $C_1(\lambda) > 0$ such that

$$\sum_{k=0}^{\infty} \mathbf{E}_a\Big(e^{-\lambda S_k} 1_{\{\min_{j\le k} S_j \ge 0\}}\Big) \le C_1(\lambda), \quad a \ge 0.$$

Proof. For (i), see [1, (2.7)]; for (ii), see [2, Lemma 2.2]; for (iii) and (iv), see [1, Lemma B.2 (i) and (iii)]. □

Lemma 3.2. *For all $a \ge 0$ and $n \ge 1$, it holds that*

$$\mathbf{E}\Big(S_n^2 1_{\{\min_{j\le n} S_j \ge -a\}}\Big) \lesssim (1+a)\sqrt{n}.$$

Proof. Note that under $\mathbf{P}$, $\left\{S_n^2 - \sigma^2 n : n \ge 1\right\}$ is a mean 0 martingale. Thus, by Lemma 3.1(i),

$$\begin{aligned}
&\mathbf{E}\Big(S_n^2 1_{\{\min_{j\le n} S_j \ge -a\}}\Big) = \sigma^2 n \mathbf{P}\Big(\min_{j\le n} S_j \ge -a\Big) + \mathbf{E}\Big(\left(S_n^2 - \sigma^2 n\right) 1_{\{\min_{j\le n} S_j \ge -a\}}\Big)\\
&\lesssim (1+a)\sqrt{n} - \mathbf{E}\Big(\left(S_n^2 - \sigma^2 n\right) 1_{\{\min_{j\le n} S_j < -a\}}\Big)\\
&= (1+a)\sqrt{n} + \sum_{\ell=1}^{n} \mathbf{E}\Big(\left(\sigma^2 n - S_n^2\right) 1_{\{\min_{j\le \ell-1} S_j \ge -a\}} 1_{\{S_\ell < -a\}}\Big)\\
&= (1+a)\sqrt{n} + \sum_{\ell=1}^{n} \mathbf{E}\Big(\left(\sigma^2 \ell - S_\ell^2\right) 1_{\{\min_{j\le \ell-1} S_j \ge -a\}} 1_{\{S_\ell < -a\}}\Big)\\
&\le (1+a)\sqrt{n} + \sigma^2 \sum_{\ell=1}^{n} \ell\, \mathbf{P}\Big(\min_{j\le \ell-1} S_j \ge -a, S_\ell < -a\Big).
\end{aligned}$$

Using Lemma 3.1(i) again, we get

$$\sum_{\ell=1}^{n} \ell\, \mathbf{P}\Big(\min_{j\le \ell-1} S_j \ge -a, S_\ell < -a\Big) = \sum_{\ell=1}^{n} \ell\, \mathbf{P}\Big(\min_{j\le \ell-1} S_j \ge -a\Big) - \sum_{\ell=1}^{n} \ell\, \mathbf{P}\Big(\min_{j\le \ell} S_j \ge -a\Big)$$

$$= 1-(n+1)\mathbf{P}\Big(\min_{j\le n} S_j \ge -a\Big) + \sum_{\ell=1}^{n} \mathbf{P}\Big(\min_{j\le \ell} S_j \ge -a\Big)$$
$$\lesssim 1+(1+a)\sum_{\ell=1}^{n}\frac{1}{\sqrt{\ell}} \le 1+(1+a)\int_0^n \frac{1}{\sqrt{x}}\mathrm{d}x = 1+2(1+a)\sqrt{n} \lesssim (1+a)\sqrt{n}.$$

Combining the two displays above, we get the desired conclusion. □

Lemma 3.3. *If X and $\widetilde{X}$ are non-negative random variables such that*

$$\mathbb{E}\Big(X\left(\log_+ X\right)^3\Big) + \mathbb{E}\Big(\widetilde{X}\left(\log_+ \widetilde{X}\right)^2\Big) < \infty,$$

then

$$\mathbb{E}\bigg(X\Big(\log_+\left(\widetilde{X}+X\right)\Big)^3\bigg) + \mathbb{E}\bigg(\widetilde{X}\Big(\log_+\left(\widetilde{X}+X\right)\Big)^2\bigg) < \infty.$$

Proof. By the trivial inequality $\log_+(x+y) \le \log_+(2x) + \log_+(2y)$, we only need to show that

$$\mathbb{E}\Big(X\Big(\log_+ \widetilde{X}\Big)^3\Big) + \mathbb{E}\Big(\widetilde{X}\Big(\log_+ X\Big)^2\Big) < \infty.$$

For this, it suffices to prove that for any $x, \widetilde{x} > 0$,

$$x\left(\log_+ \widetilde{x}\right)^3 \le 8x\left(\log_+ x\right)^3 + 2\widetilde{x}\left(\log_+ \widetilde{x}\right)^2, \tag{3.11}$$
$$\widetilde{x}\left(\log_+ x\right)^2 \le 4\widetilde{x}\left(\log_+ \widetilde{x}\right)^2 + 2x\left(\log_+ x\right). \tag{3.12}$$

We will only prove (3.11), the proof of (3.12) is similar. Assume that $\widetilde{x} \ge 1$. If $\widetilde{x} \le x^2$, then $x\left(\log \widetilde{x}\right)^3 \le x\left(\log\left(x^2\right)\right)^3 = 8x(\log x)^3$. If $\widetilde{x} \ge x^2$, then by trivial inequality

$$\log \widetilde{x} \le 2\sqrt{\widetilde{x}}, \quad \widetilde{x} \ge 1,$$

we have $x\left(\log \widetilde{x}\right)^3 \le \sqrt{\widetilde{x}}\left(\log \widetilde{x}\right)^3 \le 2\widetilde{x}\left(\log \widetilde{x}\right)^2$. The proof is complete. □

Lemma 3.4. *Suppose that X, Y are random variables and that $\mathcal{H}$ is a σ-field. For any $\varepsilon > 0$ and $q > 0$, it holds that*

$$\mathbb{P}\left(|X-\mathbb{E}(X|\mathcal{H})| > 3\varepsilon|\mathcal{H}\right) \le \frac{2}{\varepsilon}\mathbb{E}\left(|X-Y|\Big|\mathcal{H}\right) + \frac{1}{\varepsilon^q}\mathbb{E}\left(|Y-\mathbb{E}(Y|\mathcal{H})|^q\Big|\mathcal{H}\right).$$

Proof. By Markov's inequality,

$$\mathbb{P}\left(|X-\mathbb{E}(X|\mathcal{H})| > 3\varepsilon|\mathcal{H}\right) \le \mathbb{P}\left(|X-Y| > \varepsilon|\mathcal{H}\right) + \mathbb{P}\left(|Y-\mathbb{E}(Y|\mathcal{H})| > \varepsilon|\mathcal{H}\right)$$
$$+ 1_{\{|\mathbb{E}(X-Y|\mathcal{H})|>\varepsilon\}}$$
$$\le \frac{1}{\varepsilon}\mathbb{E}\left(|X-Y|\Big|\mathcal{H}\right) + \frac{1}{\varepsilon^q}\mathbb{E}\left(|Y-\mathbb{E}(Y|\mathcal{H})|^q\Big|\mathcal{H}\right) + \frac{1}{\varepsilon}\left|\mathbb{E}(X-Y|\mathcal{H})\right|.$$

Now the desired conclusion follows immediately from the inequality $|\mathbb{E}(X-Y|\mathcal{H})| \le \mathbb{E}(|X-Y||\mathcal{H})$. □

3.3. Moment estimates for the truncated martingales

For $u \in \mathcal{N}(n), \Omega(u) := \left\{v \in \mathcal{N}(n) : v \ne u : v > u_{n-1}\right\}$ denotes the set of siblings of u. For $\kappa, y > 0$ and $m \in \mathbb{N}$, define

$$\mathcal{A}_\kappa^y := \Bigg\{x \in \mathbb{T} : \forall 1 \le j \le |x|, \sum_{u\in\Omega(x_j)}\left(1+\left(V(u)-V(x_{j-1})\right)_+\right)e^{-\left(V(u)-V(x_{j-1})\right)} \le \kappa e^{\left(V(x_{j-1})+y\right)/2}\Bigg\}, \tag{3.13}$$
$$D_{m,\kappa}^{-y} := \sum_{x\in\mathcal{N}(m)} R\left(V(x)+y\right)e^{-V(x)} 1_{\{\min_{j\le m} V(x_j)\ge -y\}} 1_{\{x\in\mathcal{A}_\kappa^y\}}.$$

Lemma 3.5. *There exists a decreasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{z\to+\infty} h(z) = 0$ such that for all $y, \kappa > 0$ and $m \in \mathbb{N}$,*

$$0 \le \mathbb{E}\left(D_m^{-y} - D_{m,\kappa}^{-y}\right) \lesssim h(\kappa).$$

Proof. The first inequality is trivial, so we only prove the second. For $j \ge 1$, set

$$E_j(y,\kappa) := \Bigg\{\sum_{u\in\Omega(w_j)\cup\{w_j\}}\left(1+\left(V(u)-V(w_{j-1})\right)_+\right)e^{-\left(V(u)-V(w_{j-1})\right)} > \kappa e^{\left(V(w_{j-1})+y\right)/2}\Bigg\}.$$

It follows from (3.10) that

$$\begin{aligned}
\mathbb{E}\left(D_m^{-y}-D_{m,\kappa}^{-y}\right) &= \mathbb{E}\Big(\sum_{x\in\mathcal{N}(m)} R\left(V(x)+y\right)e^{-V(x)}1_{\{\min_{j\le m}V(x_j)\ge -y\}}1_{\{x\notin\mathcal{A}_\kappa^y\}}\Big)\\
&= \mathbb{E}_{\mathbb{Q}^{-y}}\Big(\frac{R(y)}{D_n^{-y}}\sum_{x\in\mathcal{N}(m)} R\left(V(x)+y\right)e^{-V(x)}1_{\{\min_{j\le m}V(x_j)\ge -y\}}1_{\{x\notin\mathcal{A}_\kappa^y\}}\Big)\\
&= R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\Big(\sum_{x\in\mathcal{N}(m)}\mathbb{Q}^{-y}\left(w_m=x|\mathcal{F}_m\right)1_{\{x\notin\mathcal{A}_\kappa^y\}}\Big)\\
&= R(y)\mathbb{Q}^{-y}\left(w_m\notin\mathcal{A}_\kappa^y\right)\le R(y)\sum_{j=1}^\infty\mathbb{Q}^{-y}\left(E_j(y,\kappa)\right). \qquad (3.14)
\end{aligned}$$

By the Markov property, for any $z\ge -y$,

$$\begin{aligned}
&\mathbb{Q}^{-y}\left(E_j(y,\kappa)\middle|V(w_{j-1})=z\right)=\mathbb{Q}^{-y-z}\Big(\sum_{u\in\mathcal{N}(1)}\left(1+(V(u))_+\right)e^{-V(u)}>\kappa e^{(z+y)/2}\Big)\\
&=\mathbb{E}\Big(\frac{\sum_{u\in\mathcal{N}(1)}R(V(u)+z+y)e^{-V(u)}1_{\{V(u)\ge -y-z\}}}{R(z+y)}1_{\left\{\sum_{u\in\mathcal{N}(1)}(1+(V(u))_+)e^{-V(u)}>\kappa e^{(z+y)/2}\right\}}\Big)\\
&=\mathbb{E}\Big(\frac{\sum_{u\in\mathcal{N}(1)}R(V(u)+z+y)e^{-V(u)}}{R(z+y)}1_{\left\{\sum_{u\in\mathcal{N}(1)}(1+(V(u))_+)e^{-V(u)}>\kappa e^{(z+y)/2}\right\}}\Big).
\end{aligned}$$

Using (3.3), we have

$$\frac{R(V(u)+z+y)}{R(z+y)}\lesssim\frac{(V(u))_++z+y+1}{z+y+1}=1+\frac{(V(u))_+}{z+y+1}.$$

Thus,

$$\begin{aligned}
&\mathbb{Q}^{-y}\left(E_j(y,\kappa)\middle|V(w_{j-1})=z\right)\\
&\lesssim\mathbb{E}\Big(\sum_{u\in\mathcal{N}(1)}\left(1+\frac{(V(u))_+}{z+y+1}\right)e^{-V(u)}1_{\left\{\sum_{u\in\mathcal{N}(1)}(1+(V(u))_+)e^{-V(u)}>\kappa e^{(z+y)/2}\right\}}\Big)\\
&=\mathbb{E}\Big(\Big(W_1+\frac{\widetilde{W}_1}{z+y+1}\Big)1_{\left\{W_1+\widetilde{W}_1>\kappa e^{(z+y)/2}\right\}}\Big). \qquad (3.15)
\end{aligned}$$

Since the law of $(W_1,\widetilde{W}_1)$ is independent of z, we deduce from (3.15) that

$$\begin{aligned}
\mathbb{Q}^{-y}\left(E_j(y,\kappa)\right)&\lesssim\left(\mathbb{E}_{\mathbb{Q}^{-y}}\otimes\mathbb{E}\right)\Big(\Big(W_1+\frac{\widetilde{W}_1}{V(w_{j-1})+y+1}\Big)1_{\left\{W_1+\widetilde{W}_1>\kappa e^{(V(w_{j-1})+y)/2}\right\}}\Big)\\
&=\mathbb{E}\Big(\mathbb{E}_{\mathbb{Q}^{-y}}\Big(\Big(z_1+\frac{\widetilde{z}_1}{V(w_{j-1})+y+1}\Big)1_{\left\{V(w_{j-1})+y<2\log\left(\frac{z_1+\widetilde{z}_1}{\kappa}\right)\right\}}\Big)\Big|_{z_1=W_1,\widetilde{z}_1=\widetilde{W}_1}\Big). \qquad (3.16)
\end{aligned}$$

Here under $\mathbb{Q}^{-y}\otimes\mathbb{P}$, $(W_1,\widetilde{W}_1)$ is independent of $V(w_{j-1})$. Next, note that the law of $V(w_j)$ under $\mathbb{Q}^{-y}$ is equal to the law of the random walk S_j conditioned to stay in $[-y,+\infty)$. Summing j from 1 to ∞, and using (3.8) and the fact that $R(y)\lesssim 1+y$, we get

$$\begin{aligned}
&R(y)\sum_{j=1}^\infty\mathbb{E}_{\mathbb{Q}^{-y}}\Big(\Big(z_1+\frac{\widetilde{z}_1}{V(w_{j-1})+y+1}\Big)1_{\left\{V(w_{j-1})+y<2\log\left(\frac{z_1+\widetilde{z}_1}{\kappa}\right)\right\}}\Big)\\
&=\sum_{j=0}^\infty\mathbf{E}\Big(R(S_j+y)1_{\{\min_{\ell\le j}S_\ell\ge -y\}}\Big(z_1+\frac{\widetilde{z}_1}{S_j+y+1}\Big)1_{\left\{S_j+y<2\log\left(\frac{z_1+\widetilde{z}_1}{\kappa}\right)\right\}}\Big)\\
&\lesssim\sum_{j=0}^\infty\mathbf{E}\Big(\left(z_1\left(S_j+y+1\right)+\widetilde{z}_1\right)1_{\{\min_{\ell\le j}S_\ell\ge -y\}}1_{\left\{S_j+y<2\log\left(\frac{z_1+\widetilde{z}_1}{\kappa}\right)\right\}}\Big)\\
&\le\Big(z_1\Big(1+2\log_+\Big(\frac{z_1+\widetilde{z}_1}{\kappa}\Big)\Big)+\widetilde{z}_1\Big)\sup_{y\in\mathbb{R}}\sum_{j=0}^\infty\mathbf{P}\Big(\min_{\ell\le j}S_\ell\ge -y,S_j+y<2\log_+\Big(\frac{z_1+\widetilde{z}_1}{\kappa}\Big)\Big)\\
&=\Big(z_1\Big(1+2\log_+\Big(\frac{z_1+\widetilde{z}_1}{\kappa}\Big)\Big)+\widetilde{z}_1\Big)F\Big(2\log_+\Big(\frac{z_1+\widetilde{z}_1}{\kappa}\Big)\Big), \qquad (3.17)
\end{aligned}$$

where $F(x):=\sup_{y\in\mathbb{R}}\sum_{j=0}^\infty\mathbf{P}_y(\min_{l\le j}S_l\ge 0,S_j<x)$. Taking $\mathbb{P}$-expectation in (3.17), and combining the result with (3.14) and (3.16), we get

$$\begin{aligned}
&\mathbb{E}\left(D_m^{-y}-D_{m,\kappa}^{-y}\right)\le R(y)\sum_{j=1}^\infty\mathbb{Q}^{-y}\left(E_j(y,\kappa)\right)\\
&\lesssim\mathbb{E}\Big(\Big(W_1\Big(1+2\log_+\Big(\frac{W_1+\widetilde{W}_1}{\kappa}\Big)\Big)+\widetilde{W}_1\Big)F\Big(2\log_+\Big(\frac{W_1+\widetilde{W}_1}{\kappa}\Big)\Big)\Big).
\end{aligned}$$

It follows from Lemma 3.1 (iii) that $F(x) \lesssim (1+x)^2$ for all $x \geq 0$. Since $F(0)=0$ and F is increasing, we have for $\kappa > 1$,

$$\mathbb{E}\left(D_m^{-y} - D_{m,\kappa}^{-y}\right) \lesssim \mathbb{E}\left(\left(W_1\left(1+2\log_+\left(W_1+\widetilde{W}_1\right)\right)+\widetilde{W}_1\right) F\left(2\log_+(W_1+\widetilde{W}_1)\right) 1_{\left\{W_1+\widetilde{W}_1>\kappa\right\}}\right)$$
$$\lesssim \mathbb{E}\left(\left(W_1\left(1+\log_+(W_1+\widetilde{W}_1)\right)+\widetilde{W}_1\right)\left(1+\log_+(W_1+\widetilde{W}_1)\right)^2 1_{\left\{W_1+\widetilde{W}_1>\kappa\right\}}\right) =: h(\kappa).$$

By Lemma 3.3, we know that $h(\kappa)$ is finite for all $\kappa > 0$ and h is a decreasing function with $\lim_{z\to+\infty} h(z) = 0$. The proof is complete. □

Lemma 3.6. *For all* $y \geq 0, \kappa \geq 1$ *and* $m \geq 0$,

$$\mathbb{E}\left(\left(D_{m,\kappa}^{-y}\right)^2\right) \lesssim \kappa e^y.$$

Proof. Using (3.10) and the fact that $D_{m,\kappa}^{-y} \leq D_m^{-y}$, we get

$$\mathbb{E}\left(\left(D_{m,\kappa}^{-y}\right)^2\right) = \mathbb{E}\left(D_{m,\kappa}^{-y} \sum_{x\in\mathcal{N}(m)} R(V(x)+y)\, e^{-V(x)} 1_{\{\min_{j\leq m} V(x_j)\geq -y\}} 1_{\{x\in\mathcal{A}_\kappa^y\}}\right)$$
$$= R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(D_{m,\kappa}^{-y}\sum_{x\in\mathcal{N}(m)} \mathbb{Q}^{-y}\left(w_m = x | \mathcal{F}_m\right) 1_{\{x\in\mathcal{A}_\kappa^y\}}\right)$$
$$\leq R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(D_m^{-y} 1_{\{w_m\in\mathcal{A}_\kappa^y\}}\right). \tag{3.18}$$

Let $\mathcal{G}$ be the σ-field consisting of all the information about the spine, including the set of children of the spine particles. By the spine decomposition, we have

$$\mathbb{E}_{\mathbb{Q}^{-y}}\left(D_m^{-y}\middle|\mathcal{G}\right) = R\left(V\left(w_m\right)+y\right) e^{-V(w_m)} + \sum_{\ell=1}^m \sum_{z\in\Omega(w_\ell)} R\left(V(z)+y\right) e^{-V(z)}. \tag{3.19}$$

Since

$$R(x+y) \leq R(x_+ + y) \lesssim (1+x_+ + y) \leq (1+x_+)(1+y), \quad y\geq 0,\ x\in\mathbb{R},$$

we get, for any $1 \leq \ell \leq m$,

$$R\left(V(z)+y\right) \lesssim \left(1+y+V\left(w_{\ell-1}\right)\right)\left(1+\left(V(z)-V\left(w_{\ell-1}\right)\right)_+\right), \quad z\in\Omega\left(w_\ell\right).$$

Thus, for $w_m \in B_{m,\kappa}$ and $1\leq \ell \leq m$,

$$\sum_{z\in\Omega(w_\ell)} R\left(V(z)+y\right) e^{-V(z)}$$
$$\lesssim \left(1+y+V\left(w_{\ell-1}\right)\right) e^{-V(w_{\ell-1})} \sum_{z\in\Omega(w_\ell)} \left(1+\left(V(z)-V\left(w_{\ell-1}\right)\right)_+\right) e^{-(V(z)-V(w_{\ell-1}))}$$
$$\leq \kappa e^{y/2}\left(1+y+V\left(w_{\ell-1}\right)\right) e^{-V(w_{\ell-1})/2}.$$

Combining this with (3.19) and the fact that $V(w_m)+y\geq 0$, we get

$$R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(D_m^{-y}1_{\{w_m\in\mathcal{A}_\kappa^y\}}\right) = R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(\mathbb{E}_{\mathbb{Q}^{-y}}\left(D_m^{-y}\middle|\mathcal{G}\right) 1_{\{w_m\in\mathcal{A}_\kappa^y\}}\right)$$
$$\lesssim R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(R\left(V\left(w_m\right)+y\right) e^{-V(w_m)} + \sum_{\ell=1}^m \kappa e^{y/2}\left(1+y+V\left(w_{\ell-1}\right)\right) e^{-V(w_{\ell-1})/2}\right)$$
$$\lesssim \kappa e^y R(y) \sum_{\ell=0}^m \mathbb{E}_{\mathbb{Q}^{-y}}\left(\left(1+y+V\left(w_\ell\right)\right) e^{-(y+V(w_\ell))/2}\right). \tag{3.20}$$

Since $\left((1+y)^2 e^{-y/4}\right) \lesssim 1$ on $[0,\infty)$, we have for all $\ell\geq 0$ and $y\geq 0$,

$$R(y)\mathbb{E}_{\mathbb{Q}^{-y}}\left(\left(1+y+V\left(w_\ell\right)\right) e^{-(y+V(w_\ell))/2}\right) = \mathbf{E}_y\left(R(S_\ell)\left(1+S_\ell\right) e^{-S_\ell/2} 1_{\{\min_{j\leq\ell} S_j\geq 0\}}\right)$$
$$\lesssim \mathbf{E}_y\left(\left(1+S_\ell\right)^2 e^{-S_\ell/2} 1_{\{\min_{j\leq\ell} S_j\geq 0\}}\right) \lesssim \mathbf{E}_y\left(e^{-S_\ell/4} 1_{\{\min_{j\leq\ell} S_j\geq 0\}}\right),$$

where in the equality we used (3.9). Applying Lemma 3.1 (iv) with $\lambda = \frac{1}{4}$, we get that for all $y\geq 0, m\geq 0$,

$$R(y)\sum_{\ell=0}^m \mathbb{E}_{\mathbb{Q}^{-y}}\left(\left(1+y+V\left(w_\ell\right)\right) e^{-(y+V(w_\ell))/2}\right)$$
$$\lesssim \sum_{\ell=0}^\infty \mathbf{E}_y\left(e^{-S_\ell/4} 1_{\{\min_{j\leq\ell} S_j\geq 0\}}\right) \leq C_1(1/4) \lesssim 1. \tag{3.21}$$

Combining (3.18), (3.20) and (3.21), we get the desired conclusion. □

3.4. Moment estimate for weighted number of particles hitting $-y$

Recall the definition of $\mathcal{A}_\kappa^y$ in (3.13). For $y \geq 0$, $\kappa > 0$ and $n, m \in \mathbb{N} := \{1, 2, \dots\}$ with $n \leq m$, define

$$N_{[n,m]}^y := \sum_{\ell=n}^{m} \sum_{x \in \mathcal{N}(\ell)} e^{-V(x)} 1_{\{V(x) < -y, \min_{j \leq \ell-1} V(x_j) \geq -y\}},$$

$$N_{[n,m],\kappa}^y := \sum_{\ell=n}^{m} \sum_{x \in \mathcal{N}(\ell)} e^{-V(x)} 1_{\{V(x) < -y, \min_{j \leq \ell-1} V(x_j) \geq -y\}} 1_{\{x \in \mathcal{A}_\kappa^y\}}.$$

We will use the notation $N_{[1,\infty)}^y := \lim_{m\to\infty} N_{[1,m]}^y$ and $N_{[1,\infty),\kappa}^y := \lim_{m\to\infty} N_{[1,m],\kappa}^y$.

Lemma 3.7. *(i) For any* $y \geq 0$,

$$\mathbb{E}\left(N_{[1,\infty)}^y\right) = 1.$$

(ii) There exists a decreasing function $g : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\lim_{z\to+\infty} g(z) = 0$ *such that*

$$\mathbb{E}\left(N_{[1,\infty)}^y - N_{[1,\infty),\kappa}^y\right) \lesssim \frac{g(\kappa)}{\log \kappa}, \quad y > 0, \kappa > 1.$$

Proof. (i) By the definition of $\mathbb{Q}$, we have

$$\begin{aligned}
\mathbb{E}\left(N_{[1,\infty)}^y\right) &= \sum_{k=1}^{\infty} \mathbb{E}_{\mathbb{Q}}\Big(\sum_{x \in \mathcal{N}(k)} \frac{e^{-V(x)}}{W_k} 1_{\{V(x) < -y, \min_{j \leq k-1} V(x_j) \geq -y\}}\Big) \\
&= \sum_{k=1}^{\infty} \mathbb{E}_{\mathbb{Q}}\Big(\sum_{x \in \mathcal{N}(k)} \mathbb{Q}\Big(w_k = x, V(w_k) < -y, \min_{j \leq k-1} V(w_j) \geq -y | \mathcal{F}_k\Big)\Big) \\
&= \sum_{k=1}^{\infty} \mathbb{Q}\Big(V(w_k) < -y, \min_{j \leq k-1} V(w_j) \geq -y\Big) = 1,
\end{aligned}$$

where in the second equality we used (3.2).

(ii) For any $m \in \mathbb{N}$, by the definition of $\mathbb{Q}$ and (3.2), we have

$$\begin{aligned}
\mathbb{E}\left(N_{[1,m]}^y - N_{[1,m],\kappa}^y\right) &= \sum_{\ell=1}^{m} \mathbb{E}\Big(\sum_{x \in \mathcal{N}(\ell)} e^{-V(x)} 1_{\{V(x) < -y, \min_{j \leq \ell-1} V(x_j) \geq -y\}} 1_{\{x \notin \mathcal{A}_\kappa^y\}}\Big) \\
&= \sum_{\ell=1}^{m} \mathbb{Q}\Big(V(w_\ell) < -y, \min_{j \leq \ell-1} V(w_j) \geq -y, w_\ell \notin \mathcal{A}_\kappa^y\Big) \\
&\leq \sum_{\ell=1}^{\infty} \mathbb{Q}\Big(V(w_\ell) < -y, \min_{j \leq \ell-1} V(w_j) \geq -y, w_\ell \notin \mathcal{A}_\kappa^y\Big).
\end{aligned}$$

Since

$$1_{\{w_\ell \notin \mathcal{A}_\kappa^y\}} \leq \sum_{q=1}^{\ell} 1_{\left\{\sum_{u \in \Omega(w_q)} \left(1 + (V(u) - V(w_{q-1}))_+\right) e^{-(V(u) - V(w_{q-1}))} > \kappa e^{(V(w_{q-1}) + y)/2}\right\}} =: \sum_{q=1}^{\ell} 1_{G_q},$$

we have

$$\begin{aligned}
&\sum_{\ell=1}^{\infty} \mathbb{Q}\Big(V(w_\ell) < -y, \min_{j \leq \ell-1} V(w_j) \geq -y, w_\ell \notin \mathcal{A}_\kappa^y\Big) \\
&\leq \sum_{\ell=1}^{\infty} \sum_{q=1}^{\ell} \mathbb{Q}\Big(V(w_\ell) < -y, \min_{j \leq \ell-1} V(w_j) \geq -y, G_q\Big) \\
&= \sum_{q=1}^{\infty} \sum_{\ell=q}^{\infty} \mathbb{Q}\Big(V(w_\ell) < -y, \min_{j \leq \ell-1} V(w_j) \geq -y, G_q\Big) = \sum_{q=1}^{\infty} \mathbb{Q}\Big(G_q, \min_{j \leq q-1} V(w_j) \geq -y\Big) \\
&\leq \sum_{q=1}^{\infty} \mathbb{E}_{\mathbb{Q}}\left(1_{\{\min_{j \leq q-1} V(w_j) \geq -y\}} \mathbb{Q}\Big(\sum_{u \in \mathcal{N}(1)} \left(1 + (V(u))_+\right) e^{-V(u)} > \kappa e^{(z+y)/2}\Big)\Big|_{z = V(w_{q-1})}\right).
\end{aligned}$$

Recalling the definition of F in (3.17) and using an argument similar to that of the proof of Lemma 3.5, we get

$$\begin{aligned}
&\sum_{q=1}^{\infty} \mathbb{E}_{\mathbb{Q}}\left(1_{\{\min_{j \leq q-1} V(w_j) \geq -y\}} \mathbb{Q}\Big(\sum_{u \in \mathcal{N}(1)} \left(1 + (V(u))_+\right) e^{-V(u)} > \kappa e^{(z+y)/2}\Big)\Big|_{z = V(w_{q-1})}\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(\sum_{q=1}^{\infty} \mathbb{Q}\Big(\min_{j \leq q-1} V(w_j) \geq -y, V(w_{q-1}) + y < 2\log_+\left(\frac{m_1}{\kappa}\right)\Big)\Big|_{m_1 = W_1 + \widetilde{W}_1}\right)
\end{aligned}$$

$$\leq \mathbb{E}_{\mathbb{Q}}\left(\sup_{y\in\mathbb{R}}\sum_{q=1}^{\infty}\mathbb{Q}\Big(\min_{j\leq q-1}V(w_j)\geq -y, V(w_{q-1})+y<2\log_+\Big(\frac{m_1}{\kappa}\Big)\Big)\Big|_{m_1=W_1+\widetilde{W}_1}\right)$$
$$=\mathbb{E}_{\mathbb{Q}}\left(F\left(2\log_+\left(\frac{W_1+\widetilde{W}_1}{\kappa}\right)\right)\right)=\mathbb{E}\left(W_1F\left(2\log_+\left(\frac{W_1+\widetilde{W}_1}{\kappa}\right)\right)\right).$$

It follows from Lemma 3.1(iii) that $F(x)\lesssim(1+x)^2$ for $x\geq 0$. Note that $F(0)=0$, we have

$$\mathbb{E}\left(N^y_{[1,\infty)}-N^y_{[1,\infty),\kappa}\right)\leq\mathbb{E}\left(W_1F\left(2\log_+\left(\frac{W_1+\widetilde{W}_1}{\kappa}\right)\right)1_{\{W_1+\widetilde{W}_1>\kappa\}}\right)$$
$$\lesssim\mathbb{E}\left(W_1\left(1+2\log_+\left(\frac{W_1+\widetilde{W}_1}{\kappa}\right)\right)^21_{\{W_1+\widetilde{W}_1>\kappa\}}\frac{\log_+\left(W_1+\widetilde{W}_1\right)}{\log\kappa}\right)=:\frac{g(\kappa)}{\log\kappa}.$$

The proof is complete. □

Lemma 3.8. *Let α be the constant in* **(A5)**. *Then*

$$\mathbb{E}\Big(\Big(N^y_{[1,\infty),\kappa}\Big)^{1+\alpha}\Big)\lesssim\kappa^\alpha e^{\alpha y},\quad y\geq 0,\kappa\geq 1.$$

Proof. By Lemma 3.7(i), we have

$$\mathbb{E}\big(N^y_{[1,\infty),\kappa}\big)\leq\mathbb{E}\big(N^y_{[1,\infty)}\big)=1. \tag{3.22}$$

Define $\tau^-_{-y}:=\inf\left\{k\geq 1: V(w_k)<-y\right\}$. For $m\geq 1$, it holds that

$$\mathbb{E}\Big(\Big(N^y_{[1,m],\kappa}\Big)^{1+\alpha}\Big)=\mathbb{E}\left(\sum_{k=1}^m\sum_{x\in\mathcal{N}(k)}e^{-V(x)}1_{\{V(x)<-y,\min_{j\leq k-1}V(x_j)\geq -y\}}1_{\{x\in\mathcal{A}^y_\kappa\}}\left(N^y_{[1,m],\kappa}\right)^\alpha\right)$$
$$=\mathbb{E}_{\mathbb{Q}}\left(1_{\{\tau^-_{-y}\leq m\}}1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\left(N^y_{[1,m],\kappa}\right)^\alpha\right).$$

Using the trivial inequality

$$(x+y)^\alpha\leq x^\alpha+y^\alpha,\quad x,y\geq 0,\alpha\in(0,1],$$

we get that

$$\mathbb{E}\left(\left(N^y_{[1,m],\kappa}\right)^{1+\alpha}\right)\leq\mathbb{E}_{\mathbb{Q}}\left(1_{\{\tau^-_{-y}\leq m\}}1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\left(N^y_{[1,m],\kappa}-e^{-V\left(w_{\tau^-_{-y}}\right)}\right)^\alpha\right)$$
$$+\mathbb{E}_{\mathbb{Q}}\left(1_{\left\{\tau^-_{-y}\leq m\right\}}1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}e^{-\alpha V\left(w_{\tau^-_{-y}}\right)}\right)=:I+II. \tag{3.23}$$

By the spine decomposition, we have

$$N^y_{[1,m],\kappa}=e^{-V\left(w_{\tau^-_{-y}\wedge m}\right)}+\sum_{k=1}^{\tau^-_{-y}\wedge m}\sum_{u\in\Omega(w_k)}e^{-V(u)}N^{y+V(u)}_{[1,m-k],\kappa}.$$

By the branching property, (3.22) and using Jensen's inequality $\mathbb{E}(|X|^\alpha)\leq\mathbb{E}(|X|)^\alpha$ (since $\alpha\in(0,1]$), we have

$$I^{1/\alpha}\leq\mathbb{E}_{\mathbb{Q}}\left(1_{\{\tau^-_{-y}\leq m\}}1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\left(N^y_{[1,m],\kappa}-e^{-V(w_{\tau^-_{-y}})}\right)\right)$$
$$=\mathbb{E}_{\mathbb{Q}}\left(1_{\{\tau^-_{-y}\leq m\}}1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\sum_{\ell=1}^{\tau^-_{-y}}\sum_{u\in\Omega(w_\ell)}e^{-V(u)}\mathbb{E}\left(N^z_{[1,m-\ell],\kappa}\right)_{z=V(u)+y}\right)$$
$$\leq\mathbb{E}_{\mathbb{Q}}\left(1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\sum_{\ell=1}^{\tau^-_{-y}}\sum_{u\in\Omega(w_\ell)}e^{-V(u)}\right)$$
$$\leq\mathbb{E}_{\mathbb{Q}}\left(1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\sum_{\ell=1}^{\tau^-_{-y}}e^{-V(w_{\ell-1})}\sum_{u\in\Omega(w_\ell)}\left(1+\left(V(u)-V(w_{\ell-1})\right)_+\right)e^{-(V(u)-V(w_{\ell-1}))}\right)$$
$$\leq\kappa\mathbb{E}_{\mathbb{Q}}\left(1_{\left\{w_{\tau^-_{-y}}\in\mathcal{A}^y_\kappa\right\}}\sum_{\ell=1}^{\tau^-_{-y}}e^{-(V(w_{\ell-1})-y)/2}\right)\leq\kappa e^y\mathbb{E}_{\mathbb{Q}}\left(\sum_{\ell=1}^{\tau^-_{-y}}e^{-(V(w_{\ell-1})+y)/2}\right).$$

Using Lemma 3.1 (iv) with $\lambda=1/2$, we get that

$$\mathbb{E}_{\mathbb{Q}}\left(\sum_{\ell=1}^{\tau^-_{-y}}e^{-(V(w_{\ell-1})+y)/2}\right)\leq 1+\mathbb{E}_{\mathbb{Q}}\left(\sum_{\ell=1}^{\tau^-_{-y}-1}e^{-(V(w_\ell)+y)/2}\right)$$

$$= 1 + \sum_{\ell=1}^{\infty} \mathbf{E}_y\left(e^{-S_\ell/2} 1_{\{\min_{j\le\ell} S_j \ge 0\}}\right) \le 1 + C_1(1/2) \lesssim 1.$$

Then we have

$$I \lesssim \kappa^\alpha e^{\alpha y}. \tag{3.24}$$

Finally, using Lemma 3.1 (iv) with $\lambda = 1$ and **(A5)**, we get

$$\begin{aligned} II &\le \mathbb{E}_{\mathbb{Q}}\left(e^{-\alpha V\left(w_{\tau^-_{-y}}\right)}\right) = \mathbf{E}\left(e^{-\alpha S_{\tau^-_{-y}}}\right) \\ &= e^{\alpha y} \sum_{\ell=1}^{\infty} \mathbf{E}_y\left(e^{-\alpha S_\ell} 1_{\{\min_{k\le\ell-1} S_k \ge 0\}} 1_{\{S_\ell < 0\}}\right) \\ &\le e^{\alpha y} \sum_{\ell=1}^{\infty} \mathbf{E}_y\left(e^{-\alpha S_{\ell-1}} 1_{\{\min_{k\le\ell-1} S_k \ge 0\}}\right) \mathbf{E}_y\left(e^{-\alpha(S_\ell - S_{\ell-1})}\right) \\ &\le C_1(1) e^{\alpha y} \mathbf{E}\left(e^{-\alpha S_1}\right) = C_1(1) e^{\alpha y} \mathbb{E}\left(\sum_{u\in\mathcal{N}(1)} e^{-(1+\alpha)V(u)}\right) \lesssim e^{\alpha y}. \end{aligned} \tag{3.25}$$

Hence, combining (3.23), (3.24) and (3.25), we get that

$$\mathbb{E}\left(\left(N^y_{[1,m],\kappa}\right)^{1+\alpha}\right) \lesssim \kappa^\alpha e^{\alpha y}.$$

This completes the proof of the Lemma. □

For a sequence $(\beta_n)_{n\ge1}$ of positive numbers, define

$$\widehat{N}^{y,n}_{[\ell,m]} := \sum_{q=\ell}^{m} \sum_{x\in\mathcal{N}(q)} \left(V(x) + \beta_n + y\right) e^{-V(x)} 1_{\{-y-\beta_n/2 \le V(x) < -y,\, \min_{j\le q-1} V(x_j) \ge -y\}},$$

$$\widehat{N}^{y,n}_{[\ell,m],\kappa} := \sum_{q=\ell}^{m} \sum_{x\in\mathcal{N}(q)} \left(V(x) + \beta_n + y\right) e^{-V(x)} 1_{\{-y-\beta_n/2 \le V(x) < -y,\, \min_{j\le q-1} V(x_j) \ge -y\}} 1_{\{x\in\mathcal{A}^y_\kappa\}},$$

and let $\widehat{N}^{y,n}_{[\ell,\infty)} := \lim_{m\to\infty} \widehat{N}^{y,n}_{[\ell,m]}$ and $\widehat{N}^{y,n}_{[\ell,\infty),\kappa} := \lim_{m\to\infty} \widehat{N}^{y,n}_{[\ell,m],\kappa}$. Then we have the following result similar to that in Lemma 3.7 and Lemma 3.8 for $\widehat{N}^{y,n}_{[1,\infty)}$ and $\widehat{N}^{y,n}_{[1,\infty),\kappa}$:

Lemma 3.9. *(i) Let g be the function in Lemma 3.7 (ii). Then*

$$\mathbb{E}\left(\widehat{N}^{y,n}_{[1,\infty)} - \widehat{N}^{y,n}_{[1,\infty),\kappa}\right) \lesssim \beta_n \frac{g(\kappa)}{\log \kappa}, \quad y > 0, \kappa > 1, n \ge 1.$$

(ii) Let α be the constant in **(A5)**. *Then*

$$\mathbb{E}\left(\left(\widehat{N}^{y,n}_{[1,\infty),\kappa}\right)^{1+\alpha}\right) \lesssim \beta_n^{1+\alpha} \kappa^\alpha e^{\alpha y}, \quad y \ge 0, \kappa \ge 1, n \ge 1.$$

Proof. (i) By direct calculation and Lemma 3.7 (ii), we have

$$\begin{aligned} &\mathbb{E}\left(\widehat{N}^{y,n}_{[1,\infty)} - \widehat{N}^{y,n}_{[1,\infty),\kappa}\right) \\ &\le \mathbb{E}\left(\sum_{q=1}^{\infty} \sum_{x\in\mathcal{N}(q)} \left(V(x) + \beta_n + y\right)_+ e^{-V(x)} 1_{\{V(x) < -y,\, \min_{j\le q-1} V(x_j) \ge -y\}} 1_{\{x\notin\mathcal{A}^y_\kappa\}}\right) \\ &\le \beta_n \mathbb{E}\left(\sum_{q=1}^{\infty} \sum_{x\in\mathcal{N}(q)} e^{-V(x)} 1_{\{V(x) < -y,\, \min_{j\le q-1} V(x_j) \ge -y\}} 1_{\{x\notin\mathcal{A}^y_\kappa\}}\right) \\ &= \beta_n \mathbb{E}\left(N^y_{[1,\infty)} - N^y_{[1,\infty),\kappa}\right) \lesssim \beta_n \frac{g(\kappa)}{\log \kappa}. \end{aligned}$$

(ii) Combining Lemma 3.8 and the inequality $0 \le \widehat{N}^{y,n}_{[1,\infty),\kappa} \le \beta_n N^y_{[1,\infty),\kappa}$, we immediately get the desired conclusion. □

4. Proof of Theorem 2.1

For $n \ge 1$, set

$$\gamma_n := \frac{1}{2} \log n + \beta_n, \tag{4.1}$$

where $(\beta_n)_{n\ge1}$ is a sequence of positive numbers with

$$\lim_{n\to\infty} \beta_n = +\infty, \quad \limsup_{n\to\infty} \frac{\log n}{\beta_n} < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\beta_n}{n^{1/16}} = 0. \tag{4.2}$$

An example of a sequence $(\beta_n)_{n\geq 1}$ satisfying the conditions above is $\beta_n = \log n$. We remark that the first condition in (4.2) is used in the proof of Lemma 4.1, the second condition in (4.2) is used in (4.38), and the third condition in (4.2) is needed at the end of the proof of Lemma 4.2.

We first give the main idea of the proof of Theorem 2.1. Recall that c^* and α^* are the constants in (3.4) and (3.7) respectively. Let $\ell(y) := c^* y + \alpha^*$. According to [3], under **(A1)**, the branching random walk is in the so-called boundary case, which implies that $\lim_{m\to\infty} W_m = 0$, $\mathbb{P}$-a.s. Therefore, for any $n \geq 1$, $D_\infty = \lim_{m\to\infty} \sum_{x\in\mathcal{N}(m)} V(x)e^{-V(x)} = \lim_{m\to\infty} \sum_{x\in\mathcal{N}(m)} (V(x)-\gamma_n)e^{-V(x)}$, from which we get that

$$c^* D_\infty = \lim_{m\to\infty} \sum_{x\in\mathcal{N}(m)} \ell\left(V(x)-\gamma_n\right) e^{-V(x)}. \tag{4.3}$$

For $x \in \mathbb{T}$ and $m \geq 0$, let $\mathbb{T}_x$ be the subtree of $\mathbb{T}$ with root at x and $\mathcal{N}(x,m)$ be the collections of particles in the mth generation of $\mathbb{T}_x$. Define

$$D_m(x) := \sum_{u\in\mathcal{N}(x,m)} (V(u)-V(x))e^{-(V(u)-V(x))},$$

and

$$D_\infty(x) := \lim_{m\to\infty} D_m(x).$$

For $n \geq 1$ and $m \geq \lceil an \rceil$, we define the quantity

$$D_m^{\lceil an\rceil,\gamma_n} := \sum_{x\in\mathcal{N}(m)} \ell\left(V(x)-\gamma_n\right) e^{-V(x)} 1_{\{\min_{j\in[an,m]\cap\mathbb{Z}} V(x_j)\geq\gamma_n\}}, \tag{4.4}$$

which roughly takes care of the contributions to D_m by the paths that stay above the level γ_n between generations $\lceil an\rceil$ and m. We show in Lemma 4.1(i) that $D_m^{\lceil an\rceil,\gamma_n}$ converges to a limit $D_\infty^{\lceil an\rceil,\gamma_n}$ as $m\to\infty$. Then by (4.3), the branching property, we get

$$c^* D_\infty = D_\infty^{\lceil an\rceil,\gamma_n} + \sum_{x\in\mathcal{L}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} D_\infty(x). \tag{4.5}$$

Here for any $n\in\mathbb{N}$ and $a \geq 1$, $\mathcal{L}^{\lceil an\rceil,\gamma_n} \subset \mathbb{T}$ is defined by

$$\mathcal{L}^{\lceil an\rceil,\gamma_n} := \left\{x\in\mathbb{T} : |x| \geq \lceil an\rceil, V(x) < \gamma_n \text{ and } \min_{j\in[an,|x|-1]\cap\mathbb{Z}} V(x_j) \geq \gamma_n\right\}. \tag{4.6}$$

Recall that, for $x \in \mathcal{L}^{\lceil an\rceil,\gamma_n}$, $D_\infty(x)$ is the limit of the derivative martingale of the branching random walk starting from a single particle at x. To consider contributions to D_∞ by particles in $\mathcal{L}^{\lceil an\rceil,\gamma_n}$, we separate $\mathcal{L}^{\lceil an\rceil,\gamma_n}$ into two sets $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$ and $\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}$, where

$$\begin{aligned}
\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n} &:= \left\{x\in\mathcal{L}^{\lceil an\rceil,\gamma_n} : \min_{n\leq j\leq\lceil an\rceil} V(x_j)\geq\gamma_n,\ V(x)\geq\gamma_n-\frac{\beta_n}{2}\right\},\\
\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n} &:= \left\{x\in\mathcal{L}^{\lceil an\rceil,\gamma_n} : \min_{n\leq j\leq\lceil an\rceil} V(x_j)<\gamma_n\right\}\\
&\quad \bigcup\left\{x\in\mathcal{L}^{\lceil an\rceil,\gamma_n} : |x|>\lceil an\rceil,\ \min_{n\leq j\leq|x|-1} V(x_j)\geq\gamma_n,\ V(x)<\gamma_n-\frac{\beta_n}{2}\right\}.
\end{aligned} \tag{4.7}$$

We define $F_{good}^{\lceil an\rceil,\gamma_n}$ and $F_{bad}^{\lceil an\rceil,\gamma_n}$ to be the contributions to D_∞ by particles in $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$ and $\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}$ respectively, that is,

$$F_{good}^{\lceil an\rceil,\gamma_n} := \sum_{x\in\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} D_\infty(x) \quad\text{and}\quad F_{bad}^{\lceil an\rceil,\gamma_n} := \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} D_\infty(x).$$

Then by (4.5), we get

$$\begin{aligned}
c^* D_\infty &= D_\infty^{\lceil an\rceil,\gamma_n} + \sum_{x\in\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} D_\infty(x) + \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} D_\infty(x)\\
&= D_\infty^{\lceil an\rceil,\gamma_n} + F_{good}^{\lceil an\rceil,\gamma_n} + F_{bad}^{\lceil an\rceil,\gamma_n}.
\end{aligned} \tag{4.8}$$

Using this, we get that

$$\begin{aligned}
D_\infty - D_{\lceil an\rceil} + \frac{\log n}{2} W_{\lceil an\rceil} &= \frac{1}{c^*}\left(D_\infty^{\lceil an\rceil,\gamma_n} - \left(c^* D_{\lceil an\rceil} - \left(c^*\gamma_n - \alpha^*\right) W_{\lceil an\rceil}\right)\right)\\
&\quad + \frac{1}{c^*}\left(F_{good}^{\lceil an\rceil,\gamma_n} - \left(c^*\beta_n - \alpha^*\right) W_{\lceil an\rceil}\right) + \frac{1}{c^*} F_{bad}^{\lceil an\rceil,\gamma_n}.
\end{aligned} \tag{4.9}$$

Now we give a heuristic description of the steps in the proof of Theorem 2.1. For two random sequences X_n and Y_n, we use $X_n \approx Y_n$ to mean $\sqrt{n}(X_n - Y_n) \to 0$ in probability. In Section 4.1, we prove that $D_\infty^{\lceil an\rceil,\gamma_n} \approx c^* D_{\lceil an\rceil} - \left(c^*\gamma_n - \alpha^*\right) W_{\lceil an\rceil}$, see Proposition 4.3 below. In Section 4.2, we analyze the weighted number of particles $\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}$ defined in (4.30) below, and prove that $\widehat{N}_{good}^{\lceil an\rceil,\gamma_n} \approx \left(c^*\beta_n - \alpha^*\right) W_{\lceil an\rceil}$, see Corollary 4.8 below. In Section 4.3, we prove that $F_{bad}^{\lceil an\rceil,\gamma_n}$ is negligible, see Proposition 4.9.

Then, by (4.9), we see that

$$c^*\left(D_\infty - D_{\lceil an\rceil} + \frac{\log n}{2} W_{\lceil an\rceil}\right) \approx F_{good}^{\lceil an\rceil,\gamma_n} - \widehat{N}_{good}^{\lceil an\rceil,\gamma_n}.$$

In Section 4.4, we prove the convergence of $\sqrt{n}(F_{good}^{\lceil an\rceil,\gamma_n} - \widehat{N}_{good}^{\lceil an\rceil,\gamma_n})$ to $c^* X_{a^{-1/2}D_\infty}$ in distribution, see Proposition 4.10 below. Using these, we can easily get the conclusion of Theorem 2.1.

4.1. Modifications of the martingales with level γ_n

For $a \geq 1$, $n \in \mathbb{N}$ and $m \geq \lceil an\rceil$, define

$$\widetilde{D}_m^{\lceil an\rceil,\gamma_n} := \sum_{x\in\mathcal{N}(m)} R\left(V(x)-\gamma_n\right) e^{-V(x)} 1_{\{\min_{j\in[an,m]\cap\mathbb{Z}} V(x_j)\geq\gamma_n\}}.$$

Recall the definition of $D_m^{\lceil an\rceil,\gamma_n}$ in (4.4). The quantities $\widetilde{D}_m^{\lceil an\rceil,\gamma_n}$ and $D_m^{\lceil an\rceil,\gamma_n}$ are related to contribution by particles that are not in $\mathcal{L}^{an,\gamma_n}$.

Lemma 4.1. *Let $a \geq 1$. Then*

(i) $D_\infty^{\lceil an\rceil,\gamma_n} := \lim_{m\to\infty} D_m^{\lceil an\rceil,\gamma_n}$ exists $\mathbb{P}$-almost surely.

(ii) Moreover, under $\mathbb{P}$,

$$\lim_{n\to\infty} \sqrt{n}\left|D_\infty^{\lceil an\rceil,\gamma_n} - D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\right| = 0, \quad \textit{in probability}.$$

Proof. (i) For $m \geq \lceil an\rceil$, by the branching property,

$$\begin{aligned}\mathbb{E}\left(\widetilde{D}_{m+1}^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_m\right) &= \sum_{x\in\mathcal{N}(m)} 1_{\{\min_{j\in[an,m]\cap\mathbb{Z}} V(x_j)\geq\gamma_n\}} e^{-V(x)} \\ &\quad\times \mathbb{E}\left(\sum_{x'\in\mathcal{N}(1)} R\left(V(x')+z-\gamma_n\right) e^{-V(x')} 1_{\{V(x')+z\geq\gamma_n\}}\right)\Big|_{z=V(x)} \\ &= \sum_{x\in\mathcal{N}(m)} 1_{\{\min_{j\in[an,m]\cap\mathbb{Z}} V(x_j)\geq\gamma_n\}} e^{-V(x)} R\left(V(x)-\gamma_n\right) = \widetilde{D}_m^{\lceil an\rceil,\gamma_n}.\end{aligned}$$

Thus $\left(\widetilde{D}_m^{\lceil an\rceil,\gamma_n}\right)_{m\geq\lceil an\rceil}$ is a non-negative martingale and hence $D_\infty^{\lceil an\rceil,\gamma_n} := \lim_{m\to\infty} \widetilde{D}_m^{\lceil an\rceil,\gamma_n}$ exists $\mathbb{P}$-almost surely. It follows from (3.7) that $\sup_{y\geq 0} |R(y) - \ell(y)| < \infty$. Therefore,

$$\left|D_m^{\lceil an\rceil,\gamma_n} - \widetilde{D}_m^{\lceil an\rceil,\gamma_n}\right| \leq \sup_{y\geq 0} |R(y)-\ell(y)| \sum_{x\in\mathcal{N}(m)} e^{-V(x)} = \sup_{y\geq 0} |R(y)-\ell(y)|\, W_m, \quad \mathbb{P}\text{-a.s.}$$

Since $\lim_{m\to\infty} W_m = 0$, we get the desired conclusion.

(ii) For $\kappa \geq 1$ and $m \geq \lceil an\rceil + 1$, define

$$\begin{aligned} B_{m,\kappa}^{\lceil an\rceil,\gamma_n} &:= \Bigg\{x\in\mathcal{N}(m):\ \forall \lceil an\rceil+1\leq j\leq m, \\ &\qquad \sum_{\Omega(x_j)} \left(1+\left(V(u)-V(x_{j-1})\right)_+\right) e^{-(V(u)-V(x_{j-1}))} \leq \kappa e^{(V(x_{j-1})-\gamma_n)/2}\Bigg\}, \\ \widetilde{D}_{m,\kappa}^{\lceil an\rceil,\gamma_n} &:= \sum_{x\in\mathcal{N}(m)} R\left(V(x)-\gamma_n\right) e^{-V(x)} 1_{\{\min_{j\in[an,m]\cap\mathbb{Z}} V(x_j)\geq\gamma_n\}} 1_{\left\{x\in B_{m,\kappa}^{\lceil an\rceil,\gamma_n}\right\}}.\end{aligned}$$

Then by the branching property and Lemma 3.5,

$$\begin{aligned}&\mathbb{E}\left(\widetilde{D}_m^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{m,\kappa}^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &= \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{V(u)\geq\gamma_n\}} \mathbb{E}\left(D_{m-\lceil an\rceil}^{-y} - D_{m-\lceil an\rceil,\kappa}^{-y}\right)\Big|_{y=V(u)-\gamma_n} \lesssim W_{\lceil an\rceil} h(\kappa),\end{aligned} \tag{4.10}$$

where h is the function in Lemma 3.5. By Lemma 3.6, we have

$$\begin{aligned}\mathrm{Var}\left(\widetilde{D}_{m,\kappa}^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) &\leq \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-2V(u)} \mathbb{E}\left(\left(D_{m-\lceil an\rceil,\kappa}^{-y}\right)^2\right)\Big|_{y=V(u)-\gamma_n} \\ &\lesssim \kappa \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-2V(u)} e^{V(u)-\gamma_n} = \kappa e^{-\gamma_n} W_{\lceil an\rceil}.\end{aligned} \tag{4.11}$$

Combining (4.10)–(4.11) with $\mathbb{E}\left(\widetilde{D}_m^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) = \widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}$ and Lemma 3.4 with $q = 2$, we get that, for any $\varepsilon > 0$,

$$\begin{aligned}&\mathbb{P}\left(\left|\widetilde{D}_m^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\right| \geq \frac{3\varepsilon}{\sqrt{\lceil an\rceil}}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &\leq 2\frac{\sqrt{\lceil an\rceil}}{\varepsilon}\mathbb{E}\left(\widetilde{D}_m^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{m,\kappa}^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) + \frac{\lceil an\rceil}{\varepsilon^2}\mathrm{Var}\left(\widetilde{D}_{m,\kappa}^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\end{aligned}$$

$$\lesssim \sqrt{\lceil an\rceil}W_{\lceil an\rceil}\left(\frac{2h(\kappa)}{\varepsilon}+\frac{\sqrt{\lceil an\rceil}}{\varepsilon^2}\kappa e^{-2^{-1}\log n-\beta_n}\right).$$

Letting $m\to\infty$, we get that for all $\kappa\ge 1$ and $n\ge 1$,

$$\mathbb{P}\Big(\Big|D_\infty^{\lceil an\rceil,\gamma_n}-\widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\Big|\ge\frac{3\varepsilon}{\sqrt{\lceil an\rceil}}\Big|\mathcal{F}_{\lceil an\rceil}\Big)\lesssim\sqrt{\lceil an\rceil}W_{\lceil an\rceil}\left(h(\kappa)+\kappa e^{-\beta_n}\right). \tag{4.12}$$

By (4.2), $\lim_{n\to\infty}\beta_n=\infty$. Using (1.2) and (4.12), first letting $n\to\infty$ and then $\kappa\to\infty$, we get

$$\lim_{n\to\infty}\sqrt{n}\Big|D_\infty^{\lceil an\rceil,\gamma_n}-\widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\Big|=0,\quad\text{in probability.} \tag{4.13}$$

On the other hand, by (3.7) we know that there exists $C>0$ such that $|R(y)-\ell(y)|\le C$ for all $y\ge 0$ and that, for any $\eta>0$, there exists $K=K(\eta)>0$ such that $|R(y)-\ell(y)|<\eta$ for $y>K$. Since $\min_{x\in\mathcal{N}(m)}V(x)\to+\infty$ as $m\to\infty$, for any $\delta>0$, there exists $L>0$ such that

$$\mathbb{P}\big(\min_{x\in\mathbb{T}}V(x)<-L\big)\le\delta. \tag{4.14}$$

Therefore,

$$\Big|\widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}-D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\Big|\le\eta W_{\lceil an\rceil}+C\sum_{x\in\mathcal{N}(\lceil an\rceil)}e^{-V(x)}1_{\{\gamma_n\le V(x)<\gamma_n+K\}}. \tag{4.15}$$

Combining (4.14), (4.15) and Markov's inequality, we get that, for any $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}-D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\Big|>\frac{\varepsilon}{\sqrt{n}}\Big)\le\delta+\mathbb{P}\left(\eta\sqrt{n}W_{\lceil an\rceil}>\varepsilon/2\right)\\
&\qquad+\mathbb{P}\Big(C\sqrt{n}\sum_{x\in\mathcal{N}(\lceil an\rceil)}e^{-V(x)}1_{\{\gamma_n\le V(x)<\gamma_n+K\}}1_{\{\min_{j\le\lceil an\rceil}V(x_j)\ge -L\}}>\varepsilon/2\Big)\\
&\le\delta+\mathbb{P}\left(\eta\sqrt{n}W_{\lceil an\rceil}>\varepsilon/2\right)\\
&\qquad+\frac{2C\sqrt{n}}{\varepsilon}\mathbb{E}\Big(\sum_{x\in\mathcal{N}(\lceil an\rceil)}e^{-V(x)}1_{\{\gamma_n\le V(x)<\gamma_n+K\}}1_{\{\min_{j\le\lceil an\rceil}V(x_j)\ge -L\}}\Big)\\
&=\delta+\mathbb{P}\left(\eta\sqrt{n}W_{\lceil an\rceil}>\varepsilon/2\right)+\frac{2C\sqrt{n}}{\varepsilon}\mathbf{P}\big(\gamma_n\le S_{\lceil an\rceil}<\gamma_n+K,\min_{j\le\lceil an\rceil}S_j\ge -L\big).
\end{aligned} \tag{4.16}$$

By the third condition in (4.2), we have $\gamma_n=o(n^{1/16})$. Thus by Lemma 3.1(ii) we have

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{2C\sqrt{n}}{\varepsilon}\mathbf{P}\big(\gamma_n\le S_{\lceil an\rceil}<\gamma_n+K,\min_{j\le\lceil an\rceil}S_j>-L\big)\\
&\lesssim\lim_{n\to\infty}\frac{2C\sqrt{n}}{\varepsilon}\frac{(L+1)(K+1)(\gamma_n+L+K+1)}{\sqrt{\lceil an\rceil^3}}=0.
\end{aligned}$$

Thus, using (1.2) and letting $n\to\infty$ in (4.16), we get

$$\limsup_{n\to\infty}\mathbb{P}\Big(\Big|\widetilde{D}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}-D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\Big|>\frac{\varepsilon}{\sqrt{n}}\Big)\le\delta+\mathbb{P}\Big(\eta\sqrt{\frac{2}{\pi\sigma^2}}a^{-1/2}D_\infty>\varepsilon/2\Big). \tag{4.17}$$

Letting $\delta,\eta\to 0$ in (4.17), and then combining the resulting fact with (4.13), we get the desired conclusion. □

For $m\ge n$, define

$$\widetilde{W}_m^{n,\gamma_n}:=\sum_{x\in\mathcal{N}(m)}e^{-V(x)}1_{\{\min_{j\in[n,m]\cap\mathbb{Z}}V(x_j)\ge\gamma_n\}}. \tag{4.18}$$

Lemma 4.2. *(i) For all $a\ge 1,b\in\mathbb{R}$, as $n\to\infty$, $(\gamma_n+b)\sqrt{n}\big(W_{\lceil an\rceil}-\widetilde{W}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\big)\to 0$ in probability.*

(ii) For all $a\ge 1,b\in\mathbb{R}$, as $n\to\infty$, $(\gamma_n+b)\sqrt{n}\big(W_{\lceil an\rceil}-\widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}\big)\to 0$ in probability. Moreover, if there exists a sequence of random variables $\{J_n\}$ such that for all $n\ge 1$,

$$\mathbb{E}\left(|J_n|\,\Big|\mathcal{F}_{\lceil an\rceil}\right)\lesssim(\gamma_n+b)\sqrt{n}\left(W_{\lceil an\rceil}-\widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}\right),$$

then $J_n\to 0$ in probability.

Proof. (i) Without loss of generality, we assume $b\ge 0$. Fix $\delta>0$ and let L be the constant in (4.14). By (4.18), $\widetilde{W}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}=\sum_{x\in\mathcal{N}(\lceil an\rceil)}e^{-V(x)}1_{\{V(x)\ge\gamma_n\}}$. Therefore, for any $\varepsilon>0$,

$$\mathbb{P}\Big(W_{\lceil an\rceil}-\widetilde{W}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\ge\frac{\varepsilon}{(\gamma_n+b)\sqrt{n}}\Big)$$

$$\le \delta + \mathbb{P}\Big(W_{\lceil an\rceil} - \widetilde{W}^{\lceil an\rceil,\gamma_n}_{\lceil an\rceil} \ge \frac{\varepsilon}{(\gamma_n+b)\sqrt{n}}, \min_{j\le\lceil an\rceil}\min_{x\in\mathcal{N}(\lceil an\rceil)} V(x_j) \ge -L\Big)$$
$$= \delta + \mathbb{P}\Big(\sum_{x\in\mathcal{N}(\lceil an\rceil)} e^{-V(x)} 1_{\{V(x)<\gamma_n\}} 1_{\{\min_{j\le\lceil an\rceil} V(x_j)\ge -L\}} \ge \frac{\varepsilon}{(\gamma_n+b)\sqrt{n}}, \min_{j\le\lceil an\rceil}\min_{x\in\mathcal{N}(\lceil an\rceil)} V(x_j) \ge -L\Big)$$
$$\le \delta + \mathbb{P}\Big(\sum_{x\in\mathcal{N}(\lceil an\rceil)} e^{-V(x)} 1_{\{V(x)<\gamma_n\}} 1_{\{\min_{j\le\lceil an\rceil} V(x_j)\ge -L\}} \ge \frac{\varepsilon}{(\gamma_n+b)\sqrt{n}}\Big).$$

Using Markov's inequality and Lemma 3.1 (ii), we obtain that

$$\mathbb{P}\Big(W_{\lceil an\rceil} - \widetilde{W}^{\lceil an\rceil,\gamma_n}_{\lceil an\rceil} \ge \frac{\varepsilon}{(\gamma_n+b)\sqrt{n}}\Big)$$
$$\le \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\mathbb{E}\Big(\sum_{x\in\mathcal{N}(\lceil an\rceil)} e^{-V(x)} 1_{\{V(x)<\gamma_n\}} 1_{\{\min_{j\le\lceil an\rceil} V(x_j)\ge -L\}}\Big)$$
$$= \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\mathbf{P}_L\Big(S_{\lceil an\rceil} < \gamma_n + L, \min_{j\le\lceil an\rceil} S_j \ge 0\Big) \lesssim \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\cdot\frac{(\gamma_n+1)^2(L+1)}{\sqrt{\lceil an\rceil^3}}.$$

Since $\gamma_n = o(n^{1/16})$, by taking $n\to\infty$ first and then $\delta\to 0$ in the display above, we get the conclusion of (i).

(ii) When $a=1$, the first assertion in (ii) is simply the assertion of (i). By Markov's inequality,

$$\mathbb{P}(|J_n|>\varepsilon\big|\mathcal{F}_n) \le \frac{1}{\varepsilon}\mathbb{E}\left(|J_n|\big|\mathcal{F}_n\right) \lesssim \frac{1}{\varepsilon}(\gamma_n+b)\sqrt{n}\left(W_n - \widetilde{W}^{n,\gamma_n}_n\right) \xrightarrow{\mathbb{P}} 0.$$

Since $\mathbb{P}(|J_n|>\varepsilon\big|\mathcal{F}_n)$ is bounded, the second assertion in (ii) now follows from the bounded convergence theorem. Now we assume $a>1$. Without loss of generality, we assume $b\ge 0$. It suffices to prove the second result since the first one holds by taking $J_n = (\gamma_n+b)\sqrt{n}\left(W_{\lceil an\rceil} - \widetilde{W}^{n,\gamma_n}_{\lceil an\rceil}\right)$. Using the same argument as in (i) gives

$$\mathbb{P}(|J_n|>\varepsilon) \le \delta + \frac{1}{\varepsilon}\mathbb{E}\left(\mathbb{E}\left(|J_n|\big|\mathcal{F}_{\lceil an\rceil}\right)1_{\left\{\min_{j\le\lceil an\rceil}\min_{x\in\mathcal{N}(\lceil an\rceil)} V(x_j)\ge -L\right\}}\right)$$
$$\le \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\mathbb{E}\left(\left(W_{\lceil an\rceil} - \widetilde{W}^{n,\gamma_n}_{\lceil an\rceil}\right)1_{\left\{\min_{j\le\lceil an\rceil}\min_{x\in\mathcal{N}(\lceil an\rceil)} V(x_j)\ge -L\right\}}\right)$$
$$\le \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\mathbb{E}\Big(\sum_{x\in\mathcal{N}(\lceil an\rceil)} e^{-V(x)} 1_{\{\min_{j\in[n,\lceil an\rceil]\cap\mathbb{Z}} V(x_j)<\gamma_n\}} 1_{\{\min_{j\le\lceil an\rceil} V(x_j)\ge -L\}}\Big)$$
$$= \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}\mathbf{P}_L\Big(\min_{n\le j\le\lceil an\rceil} S_j < \gamma_n + L, \min_{j\le\lceil an\rceil} S_j \ge 0\Big). \tag{4.19}$$

Let $f_k(S_k) := \mathbf{P}_{S_k}\left(\min_{j\le\lceil an\rceil-k} S_j \ge 0\right)$. By Lemma 3.1(i), we know that $f_k(S_k) \lesssim (1+S_k)(\lceil an\rceil - k)^{-1/2}$. By Lemma 3.1 (ii), we have

$$\mathbf{P}_L\Big(\min_{n\le j\le\lceil an\rceil} S_j < \gamma_n + L, \min_{j\le\lceil an\rceil} S_j \ge 0\Big) \le \sum_{k=n}^{\lceil an\rceil}\mathbf{P}_L\Big(S_k < \gamma_n + L, \min_{j\le\lceil an\rceil} S_j \ge 0\Big)$$
$$\le \sum_{k=n}^{\lceil an\rceil-[\sqrt{n}]}\mathbf{E}_L\Big(f_k(S_k)1_{\{S_k<\gamma_n+L\}}1_{\{\min_{j\le k} S_j\ge 0\}}\Big) + \sum_{k=\lceil an\rceil-[\sqrt{n}]+1}^{\lceil an\rceil}\mathbf{P}_L\Big(S_k<\gamma_n+L, \min_{j\le k} S_j\ge 0\Big)$$
$$\lesssim (1+\gamma_n+L)\sum_{k=n}^{\lceil an\rceil-[\sqrt{n}]}\frac{1}{\sqrt{\lceil an\rceil - k}}\mathbf{P}_L\Big(S_k<\gamma_n+L, \min_{j\le k} S_j\ge 0\Big) + \sqrt{n}\frac{(1+\gamma_n+L)^2(1+L)}{\sqrt{\left(\lceil an\rceil - [\sqrt{n}]+1\right)^3}}$$
$$\lesssim \frac{(1+\gamma_n+L)}{\sqrt{[\sqrt{n}]}}\cdot\lceil an\rceil\cdot\frac{(1+\gamma_n+L)^2(1+L)}{\sqrt{n^3}} + \sqrt{n}\frac{(1+\gamma_n+L)^2(1+L)}{\sqrt{\left(\lceil an\rceil - [\sqrt{n}]+1\right)^3}} =: b_n.$$

Using $\gamma_n\to\infty$ as $n\to\infty$, we can easily get $n^{3/4}b_n/\left(\gamma_n^3\right)\lesssim 1$. Combining this with (4.19), we get that as $n\to\infty$,

$$\mathbb{P}(|J_n|>\varepsilon) \lesssim \delta + \frac{(\gamma_n+b)\sqrt{n}}{\varepsilon}b_n \lesssim \delta + \frac{\gamma_n^4}{n^{1/4}}.$$

Note that by the third condition of (4.2), $\lim_{n\to\infty}\frac{\gamma_n^4}{n^{1/4}} = 0$. Letting $n\to\infty$ first and then $\delta\to 0$, we get the desired result. □

Recall that c^* and α^* are the constants in (3.4) and (3.7) and $\ell(y) = c^*y + \alpha^*$.

Proposition 4.3. *For any $a\ge 1$, under $\mathbb{P}$,*

$$\lim_{n\to\infty}\sqrt{n}\left|D^{\lceil an\rceil,\gamma_n}_\infty - \left(c^* D_{\lceil an\rceil} - (c^*\gamma_n - \alpha^*)W_{\lceil an\rceil}\right)\right| = 0 \qquad \textit{in probability.}$$

Proof. Define $\widetilde{D}_m^{\gamma_n} := \sum_{x\in\mathcal{N}(m)} \ell\left(V(x)-\gamma_n\right) e^{-V(x)} = c^* D_m - \left(c^*\gamma_n - \alpha^*\right) W_m$. Note that

$$\sqrt{n}\left|D_\infty^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{\lceil an\rceil}^{\gamma_n}\right| \le \sqrt{n}\left|D_\infty^{\lceil an\rceil,\gamma_n} - D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\right| + \sqrt{n}\left|D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{\lceil an\rceil}^{\gamma_n}\right|.$$

By Lemma 4.1 (ii), $\lim_{n\to\infty}\sqrt{n}\left|D_\infty^{\lceil an\rceil,\gamma_n} - D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\right| = 0$ in probability with respect to $\mathbb{P}$. On the set $\left\{\min_{|x|=\lceil an\rceil} V(x) > 0\right\}$, we have

$$\begin{aligned}\sqrt{n}\left|D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{\lceil an\rceil}^{\gamma_n}\right| &= \sqrt{n}\left|\sum_{x\in\mathcal{N}(\lceil an\rceil)} \ell\left(V(x)-\gamma_n\right) e^{-V(x)} 1_{\{V(x)<\gamma_n\}}\right| \\ &\le \left(c^*\gamma_n + \alpha^*\right)\sqrt{n}\left(W_{\lceil an\rceil} - \widetilde{W}_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n}\right).\end{aligned}$$

Combining Lemma 4.2(i) with the fact that $\lim_{m\to\infty}\mathbb{P}\left(\min_{x\in\mathcal{N}(m)} V(x) > 0\right) = 1$, we immediately get that $\lim_{n\to\infty}\sqrt{n}\left|D_{\lceil an\rceil}^{\lceil an\rceil,\gamma_n} - \widetilde{D}_{\lceil an\rceil}^{\gamma_n}\right| = 0$ in probability with respect to $\mathbb{P}$. Now the desired conclusion follows. □

4.2. Approximation of the martingales via weighted number of particles

Recall that γ_n is defined in (4.1). For any $n\in\mathbb{N}$ and $a\ge 1$, we define

$$N^{\lceil an\rceil,\gamma_n} := \sum_{k=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(k)} e^{-V(x)} 1_{\{\min_{n\le j\le k-1} V(x_j)\ge\gamma_n, V(x)<\gamma_n\}}.$$

Lemma 4.4. *For every $a\ge 1$, under $\mathbb{P}$,*

$$\lim_{n\to\infty}\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n} - \mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right| = 0, \quad \textit{in probability}.$$

Proof. Recall that $\mathcal{L}^{\lceil an\rceil,\gamma_n}$ is defined in (4.6). Define

$$\begin{aligned}\mathcal{A}_\kappa^{\lceil an\rceil,\gamma_n} &:= \Big\{x\in\mathcal{L}^{\lceil an\rceil,\gamma_n} : \forall\lceil an\rceil+1\le j\le|x|, \\ &\sum_{u\in\Omega(x_j)}\left(1+\left(V(u)-V(x_{j-1})\right)_+\right)e^{-(V(u)-V(x_{j-1}))}\le\kappa e^{(V(x_{j-1})-\gamma_n)/2}\Big\}, \\ N_\kappa^{\lceil an\rceil,\gamma_n} &:= \sum_{\ell=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell)} e^{-V(x)}1_{\{\min_{n\le j\le\ell-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n\}}1_{\{x\in\mathcal{A}_\kappa^{\lceil an\rceil,\gamma_n}\}}.\end{aligned} \tag{4.20}$$

Then by Lemma 3.4 with $q=1+\alpha$, for any $\varepsilon>0$, we have

$$\begin{aligned}&\mathbb{P}\left(\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n} - \mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right| > 3\varepsilon\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &\le \frac{2\sqrt{n}\beta_n}{\varepsilon}\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n} - N_\kappa^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &\quad + \left(\frac{\sqrt{n}\beta_n}{\varepsilon}\right)^{1+\alpha}\mathbb{E}\left(\left|N_\kappa^{\lceil an\rceil,\gamma_n} - \mathbb{E}\left(N_\kappa^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right).\end{aligned} \tag{4.21}$$

Using Lemma 3.7 (ii) and the Markov property, we get

$$\begin{aligned}&\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n} - N_\kappa^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &= \mathbb{E}\left(\sum_{\ell=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell)} e^{-V(x)}1_{\{\min_{n\le j\le\ell-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n\}}1_{\{x\notin\mathcal{A}_\kappa^{\lceil an\rceil,\gamma_n}\}}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &\le \mathbb{E}\left(\sum_{\ell=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell)} e^{-V(x)}1_{\{\min_{\lceil an\rceil\le j\le\ell-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n\}}1_{\{x\notin\mathcal{A}_\kappa^{\lceil an\rceil,\gamma_n}\}}\middle|\mathcal{F}_{\lceil an\rceil}\right) \\ &= \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)}1_{\{V(u)\ge\gamma_n\}}\mathbb{E}\left(N_{[1,\infty)}^z - N_{[1,\infty),\kappa}^z\right)\Big|_{z=V(u)-\gamma_n} \lesssim \frac{g(\kappa)}{\log\kappa}W_{\lceil an\rceil},\end{aligned} \tag{4.22}$$

where g is the function in Lemma 3.7 (ii). It follows from [13, Theorem 2] that, for any random variables $X_1,\dots,X_m$ with finite $(1+\alpha)$-moment satisfying $\mathbb{E}(X_{j+1}|X_1+\cdots X_j)=0$ for all $j=1,\dots,m-1$, we have $\mathbb{E}\left(\left|\sum_i X_i\right|^{1+\alpha}\right)\le 2\sum_i\mathbb{E}\left(|X_i|^{1+\alpha}\right)$. Note that, by the branching property, $N_\kappa^{\lceil an\rceil,\gamma_n} = \sum_{u\in\mathcal{N}(\lceil an\rceil)} H(u)$ with $\{H(u) : u\in\mathcal{N}(\lceil an\rceil)\}$ being independent random variables conditioned on $\mathcal{F}_{\lceil an\rceil}$. More precisely, for $u\in\mathcal{N}(\lceil an\rceil)$,

$$H(u) = \sum_{\ell=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell),x>u} e^{-V(x)}1_{\{\min_{n\le j\le\ell-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n\}}1_{\{x\in\mathcal{A}_\kappa^{\lceil an\rceil,\gamma_n}\}}.$$

Using these two observations, the branching property, the trivial inequality $\mathbb{E}\left(|X-\mathbb{E}X|^{1+\alpha}\right)\le 2^{1+\alpha}\mathbb{E}(|X|^{1+\alpha})$ and Lemma 3.8, we get

$$\begin{aligned}
&\mathbb{E}\left(\left|N_\kappa^{\lceil an\rceil,\gamma_n}-\mathbb{E}\left(N_\kappa^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&=\mathbb{E}\left(\left|\sum_{u\in\mathcal{N}(\lceil an\rceil)}\left(H(u)-\mathbb{E}\left(H(u)\middle|\mathcal{F}_{\lceil an\rceil}\right)\right)\right|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&\lesssim\sum_{u\in\mathcal{N}(\lceil an\rceil)}\mathbb{E}\left(\left|H(u)-\mathbb{E}\left(H(u)\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right)\lesssim\sum_{u\in\mathcal{N}(\lceil an\rceil)}\mathbb{E}\left(|H(u)|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&\lesssim\sum_{u\in\mathcal{N}(\lceil an\rceil)}e^{-(1+\alpha)V(u)}1_{\{V(u)\ge\gamma_n\}}\mathbb{E}\left(\left(N^z_{[1,\infty),\kappa}\right)^{1+\alpha}\right)\Big|_{z=V(u)-\gamma_n}\\
&\lesssim\kappa^\alpha\sum_{u\in\mathcal{N}(\lceil an\rceil)}e^{-(1+\alpha)V(u)}e^{\alpha(V(u)-\gamma_n)}=\kappa^\alpha e^{-\alpha\gamma_n}W_{\lceil an\rceil}.
\end{aligned}\tag{4.23}$$

Combining (4.21), (4.22) and (4.23), taking $\kappa=e^{\beta_n/2}$ and using the definition of γ_n, we get

$$\begin{aligned}
&\mathbb{P}\left(\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n}-\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|>3\varepsilon\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&\lesssim\sqrt{n}W_{\lceil an\rceil}\left(\frac{g\left(e^{\beta_n/2}\right)}{\varepsilon}+\frac{\beta_n^{1+\alpha}e^{-\alpha\beta_n/2}}{\varepsilon^{1+\alpha}}\right).
\end{aligned}\tag{4.24}$$

Letting $n\to\infty$, and combining (1.2), (4.2) and the fact that $\lim_{z\to\infty}g(z)=0$, we get the desired conclusion. □

Recall the definition of $\mathcal{L}_{good}^{\lceil an\rceil,\gamma_n}$ in (4.7). Define

$$\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}:=\sum_{k=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(k)}e^{-V(x)}1_{\{\min_{n\le j\le k-1}V(x_j)\ge\gamma_n,\gamma_n-\beta_n/2\le V(x)<\gamma_n\}}.\tag{4.25}$$

Proposition 4.5. *For any $a\ge 1$, under $\mathbb{P}$, it holds that*

$$\lim_{n\to\infty}\sqrt{n}\beta_n\left|\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}-W_{\lceil an\rceil}\right|=0,\quad\textit{in probability}.$$

Proof. By the branching property and Lemma 3.7(i), we have

$$\begin{aligned}
\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)&=\mathbb{E}\left(\sum_{k=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(k)}e^{-V(x)}1_{\{\min_{n\le j\le k-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n\}}\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&=\sum_{u\in\mathcal{N}(\lceil an\rceil)}e^{-V(u)}1_{\{\min_{n\le j\le\lceil an\rceil}V(u_j)\ge\gamma_n\}}\mathbb{E}\left(N^z_{[1,\infty)}\right)\Big|_{z=V(u)-\gamma_n}\\
&=\sum_{u\in\mathcal{N}(\lceil an\rceil)}e^{-V(u)}1_{\{\min_{n\le j\le\lceil an\rceil}V(u_j)\ge\gamma_n\}}=\widetilde{W}^{n,\gamma_n}_{\lceil an\rceil}.
\end{aligned}\tag{4.26}$$

Then we have

$$\begin{aligned}
\sqrt{n}\beta_n\left|\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}-W_{\lceil an\rceil}\right|\le{}&\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n}-\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}\right|+\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n}-\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|\\
&+\sqrt{n}\beta_n\left|\widetilde{W}^{n,\gamma_n}_{\lceil an\rceil}-W_{\lceil an\rceil}\right|.
\end{aligned}$$

It follows from Lemma 4.2 and Lemma 4.4 that the second and third terms on the right hand side of the above inequality converge to 0 in probability as $n\to\infty$. To prove the desired result, we only need to prove

$$\lim_{n\to\infty}\sqrt{n}\beta_n\left|N^{\lceil an\rceil,\gamma_n}-\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}\right|=0,\quad\text{in probability}.\tag{4.27}$$

Recall that $(S_n,\mathbf{P}_y)$ is the random walk defined in (3.1) with $S_0=y$. By the Markov property and (3.1), we have

$$\begin{aligned}
&\mathbb{E}\left(\sqrt{n}\beta_n\left(N^{\lceil an\rceil,\gamma_n}-\widetilde{N}_{good}^{\lceil an\rceil,\gamma_n}\right)\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&=\sqrt{n}\beta_n\mathbb{E}\left(\sum_{k=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(k)}e^{-V(x)}1_{\{\min_{n\le j\le k-1}V(x_j)\ge\gamma_n,V(x)<\gamma_n-\beta_n/2\}}\middle|\mathcal{F}_{\lceil an\rceil}\right)\\
&\le\sqrt{n}\beta_n\sum_{u\in\mathcal{N}(\lceil an\rceil)}e^{-V(u)}1_{\{V(u)\ge\gamma_n\}}\sum_{k=1}^{\infty}\mathbf{P}_y\left(\min_{j\le k-1}S_j\ge 0,S_k<-\beta_n/2\right)\Big|_{y=V(u)-\gamma_n}.
\end{aligned}\tag{4.28}$$

For $k\ge 2$, $U_k:=S_k-S_{k-1}$ is independent to $S_1,\dots,S_{k-1}$ and has the same law as S_1-S_0. Thus for all $y\ge 0$,

$$\sum_{k=1}^{\infty}\mathbf{P}_y\left(\min_{j\le k-1}S_j\ge 0,S_k<-\beta_n/2\right)=\mathbf{E}_y\left(\sum_{k=1}^{\infty}\mathbf{P}_y\left(\min_{j\le k-1}S_j\ge 0,S_{k-1}<-u_k-\beta_n/2\right)\Big|_{u_k=U_k}\right)$$

$$= \mathbf{E}_y\Big(\sum_{k=1}^{\infty} \mathbf{P}_y\Big(\min_{j\le k-1} S_j \ge 0, S_{k-1} < -u - \beta_n/2 \Big)\Big|_{u=S_1-y} \Big).$$

By Lemma 3.1 (iii), we see that

$$\sum_{k=1}^{\infty} \mathbf{P}_y\big(\min_{j\le k-1} S_j \ge 0, S_{k-1} < -u - \beta_n/2 \big) \lesssim (1-u-\beta_n/2)^2 1_{\{-u-\beta_n/2>0\}}.$$

Therefore, using the fact that $(S_1 - y, \mathbf{P}_y) \overset{\mathrm{d}}{=} (S_1, \mathbf{P})$, we obtain

$$\sum_{k=1}^{\infty} \mathbf{P}_y\big(\min_{j\le k-1} S_j \ge 0, S_k < -\beta_n/2 \big) \lesssim \mathbf{E}\big(\big(1 - S_1 - \beta_n/2\big)^2 1_{\{S_1<-\beta_n/2\}} \big).$$

Note that for $u < -\beta_n/2$ and $\beta_n/2 > 1$,

$$\big(1 - u - \beta_n/2\big)^2 \le 2\big(u^2 + (\beta_n/2 - 1)^2\big) \le 4u^2.$$

Hence, for all $y \ge 0$ and $n \ge 1$,

$$\sum_{k=1}^{\infty} \mathbf{P}_y\big(\min_{j\le k-1} S_j \ge 0, S_k < -\beta_n/2 \big) \lesssim 4\mathbf{E}\left(S_1^2 1_{\{S_1<-\beta_n/2\}} \right) \lesssim \frac{1}{\beta_n}\mathbf{E}\big((-S_1)^3 1_{\{S_1<-\beta_n/2\}} \big). \tag{4.29}$$

Combining (4.28) and (4.29), we obtain

$$\mathbb{E}\left(\sqrt{n}\beta_n \left(N^{\lceil an\rceil,\gamma_n} - \widetilde{N}^{\lceil an\rceil,\gamma_n}_{good} \right) \Big| \mathcal{F}_{\lceil an\rceil} \right) \lesssim \sqrt{n} W_{\lceil an\rceil} \mathbf{E}\left((-S_1)^3 1_{\{S_1<-\beta_n/2\}} \right),$$

which implies (4.27). The proof is complete. □

Corollary 4.6. *Let $m \ge 1, 1 \le a_1 < \cdots < a_m < a_{m+1} = \infty$ and $(z^n)_{n\in\mathbb{N}} \in (\mathbb{C}^m)^{\mathbb{N}}$, here $z^n = (z^n_1, z^n_2, \dots, z^n_m)$. Assume that for all $1 \le k \le m$ and $n \ge 0$, $\mathrm{Re}\left(z^n_k\right) \le 0$ and z^n converges to some $z = (z_1, \dots, z_m) \in \mathbb{C}^m$. Then under $\mathbb{P}$,*

$$\lim_{n\to\infty} \mathbb{E}\left(\exp\left\{ \sum_{k=1}^{m} z^n_k \sqrt{n} \left(\widetilde{N}^{\lceil a_k n\rceil,\gamma_n}_{good} - \widetilde{N}^{\lceil a_{k+1} n\rceil,\gamma_n}_{good} \right) \right\} \Big| \mathcal{F}_n \right)$$
$$= \exp\left\{ \sqrt{\frac{2}{\pi\sigma^2}} D_\infty \sum_{k=1}^{m} z_k \left(\frac{1}{\sqrt{a_k}} - \frac{1}{\sqrt{a_{k+1}}} \right) \right\}, \qquad \textit{in probability}.$$

Proof. Using (1.2) and Proposition 4.5, we have that, for all $1 \le a < b \le \infty$,

$$\lim_{n\to\infty} \left| \sqrt{n} \left(\widetilde{N}^{\lceil an\rceil,\gamma_n}_{good} - \widetilde{N}^{\lceil bn\rceil,\gamma_n}_{good} \right) - \sqrt{\frac{2}{\pi\sigma^2}} D_n \left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}} \right) \right| = 0, \quad \text{in probability.}$$

Noticing that $\widetilde{N}^{\lceil an\rceil,\gamma_n}_{good} \ge \widetilde{N}^{\lceil bn\rceil,\gamma_n}_{good}$ and $\mathrm{Re}\left(z^n_k\right) \le 0$, using [11, Remark A.3], it suffices to prove that

$$\lim_{n\to\infty} \mathbb{E}\left(\exp\left\{ \sum_{k=1}^{m} z^n_k \sqrt{\frac{2}{\pi\sigma^2}} D_n \left(\frac{1}{\sqrt{a_k}} - \frac{1}{\sqrt{a_{k+1}}} \right) \right\} \Big| \mathcal{F}_n \right)$$
$$= \exp\left\{ \sqrt{\frac{2}{\pi\sigma^2}} D_\infty \sum_{k=1}^{m} z_k \left(\frac{1}{\sqrt{a_k}} - \frac{1}{\sqrt{a_{k+1}}} \right) \right\}, \qquad \text{in probability.}$$

Since $D_n \in \mathcal{F}_n$ and $\lim_{n\to\infty} D_n = D_\infty$, the equality above is trivial. □

Define

$$\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} := \sum_{k=\lceil an\rceil+1}^{\infty} \sum_{x\in\mathcal{N}(k)} c^* e^{-V(x)} \left(V(x) - \frac{\log n}{2} \right) 1_{\{\min_{n\le j\le k-1} V(x_j) \ge \gamma_n\}} 1_{\{\gamma_n - \beta_n/2 \le V(x) < \gamma_n\}}. \tag{4.30}$$

Lemma 4.7. *For any $a \ge 1$, under $\mathbb{P}$, it holds that*

$$\lim_{n\to\infty} \sqrt{n} \left| \widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \left(c^*\beta_n - \alpha^*\right) \widetilde{N}^{\lceil an\rceil,\gamma_n}_{good} \right| = 0, \quad \textit{in probability}.$$

Proof. Note that

$$\sqrt{n} \left| \widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \left(c^*\beta_n - \alpha^*\right) \widetilde{N}^{\lceil an\rceil,\gamma_n}_{good} \right|$$
$$\le \sqrt{n} \left| \widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} \Big| \mathcal{F}_{\lceil an\rceil} \right) \right|$$
$$+ \sqrt{n} \left| \mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} \Big| \mathcal{F}_{\lceil an\rceil} \right) - \left(c^*\beta_n - \alpha^*\right) \mathbb{E}\left(N^{\lceil an\rceil,\gamma_n} \Big| \mathcal{F}_{\lceil an\rceil} \right) \right|$$

$$
\begin{aligned}
&+ \left(c^*\beta_n - \alpha^*\right)\sqrt{n}\left|\mathbb{E}\left(N^{\lceil an\rceil,\gamma_n}\middle|\mathcal{F}_{\lceil an\rceil}\right) - N^{\lceil an\rceil,\gamma_n}\right| \\
&+ \left(c^*\beta_n - \alpha^*\right)\sqrt{n}\left|N^{\lceil an\rceil,\gamma_n} - \widetilde{N}^{\lceil an\rceil,\gamma_n}_{good}\right| \\
&=: I_n + II_n + III_n + IV_n.
\end{aligned}
$$

It follows from Lemma 4.4 and (4.27) that III_n and IV_n tend to 0 in probability as $n \to \infty$.

We first show that $\lim_{n\to\infty} I_n = 0$ in probability. Define

$$
\begin{aligned}
\widehat{N}^{\lceil an\rceil,\gamma_n}_{good,\kappa} &:= \sum_{\ell=\lceil an\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell)} c^* e^{-V(x)}\Big(V(x) - \frac{\log n}{2}\Big) \\
&\quad \times 1_{\{\min_{n\le j\le \ell-1} V(x_j)\ge\gamma_n\}} 1_{\{\gamma_n-\beta_n/2\le V(x)<\gamma_n\}} 1_{\left\{x\in\mathcal{A}^{\lceil an\rceil,\gamma_n}_\kappa\right\}},
\end{aligned}
$$

where $\mathcal{A}^{\lceil an\rceil,\gamma_n}_\kappa$ is defined in (4.20). By Lemma 3.4, we get that, for any $\varepsilon > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(\sqrt{n}\left|\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right| > 3\varepsilon\middle|\mathcal{F}_{\lceil an\rceil}\right) \\
&\le \frac{2\sqrt{n}}{\varepsilon}\mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \widehat{N}^{\lceil an\rceil,\gamma_n}_{good,\kappa}\middle|\mathcal{F}_{\lceil an\rceil}\right) + \left(\frac{\sqrt{n}}{\varepsilon}\right)^{1+\alpha}\mathbb{E}\left(\left|\widehat{N}^{\lceil an\rceil,\gamma_n}_{good,\kappa} - \mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good,\kappa}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right|^{1+\alpha}\middle|\mathcal{F}_{\lceil an\rceil}\right).
\end{aligned}
$$

By an argument similar to that leading to (4.24), taking $\kappa = e^{\beta_n/2}$ and using Lemma 3.9, we can get

$$
\begin{aligned}
&\mathbb{P}\left(\sqrt{n}\left|\widehat{N}^{\lceil an\rceil,\gamma_n}_{good} - \mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good}\middle|\mathcal{F}_{\lceil an\rceil}\right)\right| > 3\varepsilon\middle|\mathcal{F}_{\lceil an\rceil}\right) \\
&\lesssim \sqrt{n}W_{\lceil an\rceil}\left(\frac{g\left(e^{\beta_n/2}\right)}{\varepsilon} + \frac{\beta_n^{1+\alpha}e^{-\alpha\beta_n/2}}{\varepsilon^{1+\alpha}}\right).
\end{aligned}
$$

Combining this with (1.2), (4.2) and the fact $\lim_{z\to\infty} g(z) = 0$, we immediately get $\lim_{n\to\infty} I_n = 0$ in probability.

Therefore, it remains to prove that $\lim_{n\to\infty} II_n = 0$ in probability. By the branching property, we have

$$
\begin{aligned}
&\mathbb{E}\left(\widehat{N}^{\lceil an\rceil,\gamma_n}_{good}\middle|\mathcal{F}_{\lceil an\rceil}\right) = c^*\sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)}1_{\{\min_{n\le j\le\lceil an\rceil} V(u_j)\ge\gamma_n\}} \\
&\quad\times\sum_{k=1}^{\infty}\mathbb{E}\Big(\sum_{x\in\mathcal{N}(k)} e^{-V(x)}\left(V(x)+\beta_n+y\right)1_{\{\min_{j\le k-1}V(x_j)\ge -y\}}1_{\{-y-\beta_n/2\le V(x)<-y\}}\Big)\Big|_{y=V(u)-\gamma_n} \\
&= c^*\sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)}1_{\{\min_{n\le j\le\lceil an\rceil} V(u_j)\ge\gamma_n\}} \\
&\quad\times\sum_{k=1}^{\infty}\mathbf{E}_{V(u)-\gamma_n}\left(\left(S_k+\beta_n\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{-\beta_n/2\le S_k<0\}}\right).
\end{aligned} \tag{4.31}
$$

For $y \ge 0, n \ge 1$, define

$$
\Lambda_n(y) := \sum_{k=1}^{\infty}\mathbf{E}_y\left(\left(S_k+\beta_n\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{S_k<-\beta_n/2\}}\right).
$$

By (4.29), we have

$$
\Lambda_n(y) \le \beta_n\sum_{k=1}^{\infty}\mathbf{P}_y\left(\min_{j\le k-1}S_j\ge0, S_k<-\beta_n/2\right) \lesssim \mathbf{E}\left(\left(-S_1\right)^3 1_{\{S_1<-\beta_n/2\}}\right).
$$

On the other hand, we have

$$
\begin{aligned}
-\Lambda_n(y) &\le \sum_{k=1}^{\infty}\mathbf{E}_y\left(\left(-S_k\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{S_k<-\beta_n/2\}}\right) \\
&= \sum_{k=1}^{\infty}\mathbf{E}_y\left(\left(-S_k\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{S_{k-1}<-U_k-\beta_n/2\}}\right) \\
&\le \sum_{k=1}^{\infty}\mathbf{E}_y\left(\left(-U_k\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{S_{k-1}<-U_k-\beta_n/2\}}\right),
\end{aligned}
$$

where $U_k = S_k - S_{k-1}$, and in the last inequality we used the fact that $-S_k \le -U_k$ on the set $\{S_{k-1} \ge 0\}$. By Lemma 3.1 (iii), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathbf{E}_y\big(\left(-U_k\right)1_{\{\min_{j\le k-1}S_j\ge0\}}1_{\{S_{k-1}<-U_k-\beta_n/2\}}\big) \\
&= \mathbf{E}\Big(\left(-U_1\right)1_{\{-U_1>\beta_n/2\}}\sum_{k=1}^{\infty}\mathbf{E}_y\Big(\min_{j\le k-1}S_j\ge0, S_{k-1}<-u_1-\beta_n/2\Big)\Big|_{u_1=U_1}\Big)
\end{aligned}
$$

$$\lesssim \mathbf{E}\left((-U_1)\left(1-U_1-\beta_n/2\right)^2 1_{\{-U_1>\beta_n/2\}}\right)$$
$$\lesssim \mathbf{E}\left((-U_1)^3 1_{\{-U_1>\beta_n/2\}}\right) = \mathbf{E}\left((-S_1)^3 1_{\{S_1<-\beta_n/2\}}\right).$$

Therefore, for all $y \geq 0, n \geq 1$,

$$|\Lambda_n(y)| \lesssim \mathbf{E}\left((-S_1)^3 1_{\{S_1<-\beta_n/2\}}\right).$$

Combining this inequality and the definition of Λ_n, we get

$$\begin{aligned}
&c^* \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} \left| \sum_{k=1}^{\infty} \mathbf{E}_{V(u)-\gamma_n}\left(\left(S_k+\beta_n\right) 1_{\{\min_{j\leq k-1} S_j\geq 0\}} 1_{\{S_k<-\beta_n/2\}}\right)\right| \\
&= c^* \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} \left|\Lambda_n(V(u)-\gamma_n)\right| \\
&\lesssim \mathbf{E}\left((-S_1)^3 1_{\{S_1<-\beta_n/2\}}\right) W_{\lceil an\rceil}.
\end{aligned} \tag{4.32}$$

By (3.6), we see that $R(y) = c^* y - c^* \mathbf{E}_y\left(S_{\tau_0^-}\right)$ for $y \geq 0$. Thus

$$\begin{aligned}
&c^* \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} \sum_{k=1}^{\infty} \mathbf{E}_{V(u)-\gamma_n}\left(\left(S_k+\beta_n\right) 1_{\{\min_{j\leq k-1} S_j\geq 0\}} 1_{\{S_k<0\}}\right) \\
&= \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} c^* \mathbf{E}_{V(u)-\gamma_n}\left(S_{\tau_0^-}+\beta_n\right) \\
&= \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} \left(c^*\beta_n + c^*\left(V(u)-\gamma_n\right) - R\left(V(u)-\gamma_n\right)\right).
\end{aligned} \tag{4.33}$$

For any $\varepsilon > 0$, let K be large enough such that $|R(y) - (c^* y + \alpha^*)| < \varepsilon$ for $y > K$. Therefore, when $V(u) - \gamma_n > K$, we have

$$\left|\left(c^*\beta_n + c^*\left(V(u)-\gamma_n\right) - R\left(V(u)-\gamma_n\right)\right) - \left(c^*\beta_n - \alpha^*\right)\right| < \varepsilon.$$

Recall that by (4.26),

$$\left(c^*\beta_n - \alpha^*\right) \mathbb{E}\left(N^{\lceil an\rceil,\gamma_n} \middle| \mathcal{F}_{\lceil an\rceil}\right) = \left(c^*\beta_n - \alpha^*\right) \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}}.$$

Combining this with (4.31), (4.32) and (4.33), we get

$$\begin{aligned}
II_n &= \sqrt{n}\left|\mathbb{E}\left(\widehat{N}_{good}^{\lceil an\rceil,\gamma_n} \middle| \mathcal{F}_{\lceil an\rceil}\right) - \left(c^*\beta_n - \alpha^*\right) \mathbb{E}\left(N^{\lceil an\rceil,\gamma_n} \middle| \mathcal{F}_{\lceil an\rceil}\right)\right| \\
&\lesssim \mathbf{E}\left((-S_1)^3 1_{\{S_1<-\beta_n/2\}}\right) \sqrt{n} W_{\lceil an\rceil} + \varepsilon\sqrt{n} \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} 1_{\{V(u)>\gamma_n+K\}} \\
&\quad + \sup_{y\in[0,K]} \left(\left|R(y) - (c^* y + \alpha^*)\right|\right) \sqrt{n} \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} 1_{\{V(u)\leq\gamma_n+K\}}.
\end{aligned} \tag{4.34}$$

Let L and δ be the constants in (4.14). Then for any $\theta > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sqrt{n} \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(u_j)\geq\gamma_n\}} 1_{\{V(u)\leq\gamma_n+K\}} > \theta\right) \\
&\leq \delta + \mathbb{P}\left(\sqrt{n} \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{j\leq\lceil an\rceil} V(u_j)\geq -L\}} 1_{\{V(u)\leq\gamma_n+K\}} > \theta\right) \\
&\leq \delta + \frac{\sqrt{n}}{\theta} \mathbb{E}\left(\sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{\min_{j\leq\lceil an\rceil} V(u_j)\geq -L\}} 1_{\{V(u)\leq\gamma_n+K\}}\right) \\
&= \delta + \frac{\sqrt{n}}{\theta} \mathbf{P}\left(\min_{j\leq\lceil an\rceil} S_j \geq -L, S_{\lceil an\rceil} \leq \gamma_n + K\right) \\
&\lesssim \delta + \frac{\sqrt{n}}{\theta} \frac{(\gamma_n + L + K)^2 (L+1)}{\sqrt{\lceil an\rceil^3}},
\end{aligned} \tag{4.35}$$

where in the last line we used Lemma 3.1 (ii). Letting $n \to \infty$ first and then $\delta \to 0$ in (4.35), we get that the third term on the right-side hand of (4.34) converges to 0 in probability as $n \to \infty$. Note that the second term on the right-side hand is bounded by $\varepsilon\sqrt{n} W_{\lceil an\rceil}$, so letting $n \to \infty$ first and then $\varepsilon \to 0$ in (4.34), by (1.2), we get that $\lim_{n\to\infty} II_n = 0$ in probability. The proof is now complete. □

Corollary 4.8. *For any $a \geq 1$, under $\mathbb{P}$,*

$$\lim_{n\to\infty} \sqrt{n}\left|\widehat{N}_{good}^{\lceil an\rceil,\gamma_n} - \left(c^*\beta_n - \alpha^*\right) W_{\lceil an\rceil}\right| = 0, \quad \textit{in probability}.$$

Proof. This is a direct consequence of Proposition 4.5 and Lemma 4.7. □

4.3. Limit behavior of $F_{bad}^{\lceil an\rceil,\gamma_n}$

Proposition 4.9. *For any $a \geq 1$, under $\mathbb{P}$,*

$$\lim_{n\to\infty} \sqrt{n} F_{bad}^{\lceil an\rceil,\gamma_n} = 0, \quad \textit{in probability}.$$

Proof. Set $Y_n := \sqrt{n}\left(F_{bad}^{\lceil an\rceil,\gamma_n} \wedge 1\right) > 0$. We only need to prove that $\lim_{n\to\infty} Y_n = 0$ in probability. We claim that (1.3) implies that

$$\mathbb{E}\left(D_\infty \wedge y\right) \lesssim \left(1+\log_+ y\right), \quad y \geq 0. \tag{4.36}$$

Indeed, (1.3) implies that $\mathbb{E}\left(D_\infty 1_{\{D_\infty \leq y\}}\right) - \log_+ y \lesssim 1, y \geq 0$. (1.3) also implies (see [5, Theorem 2.2]) that $\lim_{y\to+\infty} y\mathbb{P}\left(D_\infty > y\right) = 1$, which means $y\mathbb{P}\left(D_\infty > y\right) \lesssim 1, y \geq 0$. Therefore, for all $y \geq 0$,

$$\mathbb{E}\left(D_\infty \wedge y\right) = \mathbb{E}\left(D_\infty 1_{\{D_\infty \leq y\}}\right) + y\mathbb{P}\left(D_\infty > y\right) \lesssim 1 + \log_+ y.$$

Since $V(x) < \gamma_n$ for all $x \in \mathcal{L}^{\lceil an\rceil,\gamma_n}$, we have

$$\begin{aligned} F_{bad}^{\lceil an\rceil,\gamma_n} \wedge 1 &\leq \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)}\left(D_\infty(x) \wedge \left(\left(c^*\right)^{-1} e^{V(x)}\right)\right) \\ &\leq \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)}\left(D_\infty(x) \wedge \left(\left(c^*\right)^{-1} e^{\gamma_n}\right)\right). \end{aligned}$$

It follows from (4.36) that

$$\begin{aligned} &\mathbb{E}\left(F_{bad}^{\lceil an\rceil,\gamma_n} \wedge 1 \middle| \mathcal{F}_{\mathcal{L}^{\lceil an\rceil,\gamma_n}}\right) \leq \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} c^* e^{-V(x)} \mathbb{E}\left(D_\infty(x) \wedge \left(\left(c^*\right)^{-1} e^{\gamma_n}\right) \middle| \mathcal{F}_{\mathcal{L}^{\lceil an\rceil,\gamma_n}}\right) \\ &\lesssim \gamma_n \sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} e^{-V(x)}. \end{aligned} \tag{4.37}$$

By the branching property and the definition of $\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}$, we have

$$\begin{aligned} &\mathbb{E}\Bigg(\sum_{x\in\mathcal{L}_{bad}^{\lceil an\rceil,\gamma_n}} e^{-V(x)} \Big| \mathcal{F}_{\lceil an\rceil}\Bigg) = \sum_{x\in\mathcal{N}(\lceil an\rceil)} e^{-V(x)} 1_{\{\min_{n\leq j\leq\lceil an\rceil} V(x_j) < \gamma_n\}} \\ &\quad + \mathbb{E}\Bigg(\sum_{k=\lceil an\rceil+1}^{\infty} \sum_{x\in\mathcal{N}(k)} e^{-V(x)} 1_{\{\min_{n\leq j\leq k-1} V(x_j)\geq\gamma_n\}} 1_{\{V(x)<\gamma_n-\beta_n/2\}} \Big| \mathcal{F}_{\lceil an\rceil}\Bigg) \\ &\leq \left(W_{\lceil an\rceil} - \widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}\right) + \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{V(u)\geq\gamma_n\}} \\ &\quad \times \mathbb{E}\Bigg(\sum_{k=1}^{\infty} \sum_{x\in\mathcal{N}(k)} e^{-V(x)} 1_{\{\min_{0\leq j\leq k-1} V(x_j)\geq y\}} 1_{\{V(x)<y-\beta_n/2\}}\Bigg)\Bigg|_{y=V(u)-\gamma_n} \\ &= \left(W_{\lceil an\rceil} - \widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}\right) + \sum_{u\in\mathcal{N}(\lceil an\rceil)} e^{-V(u)} 1_{\{V(u)\geq\gamma_n\}} \sum_{k=1}^{\infty} \mathbf{P}\Big(\min_{0\leq j\leq k-1} S_j \geq y, S_k < y-\beta_n/2\Big)\Big|_{y=V(u)-\gamma_n}, \end{aligned}$$

where $\widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}$ is defined in (4.18). By (4.29) and (4.37), we have

$$\mathbb{E}\left(Y_n \middle| \mathcal{F}_{\lceil an\rceil}\right) \lesssim \gamma_n \sqrt{n}\left(W_{\lceil an\rceil} - \widetilde{W}_{\lceil an\rceil}^{n,\gamma_n}\right) + \frac{\gamma_n}{\beta_n}\sqrt{n} W_{\lceil an\rceil} \mathbf{E}\left(\left(-S_1\right)^3 1_{\{S_1<-\beta_n/2\}}\right). \tag{4.38}$$

By Lemma 4.2, the first term on the right hand side above converges to 0 in probability. Combining the second condition in (1.2) and (4.2), we get that the second term on the right hand side above also converges to 0 in probability. Thus $\mathbb{E}\left(Y_n \middle| \mathcal{F}_{\lceil an\rceil}\right)$ converges to 0 in probability. Finally, applying Jensen's inequality, we have

$$\mathbb{E}\left(e^{-Y_n}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{-Y_n} \middle| \mathcal{F}_{\lceil an\rceil}\right)\right) \geq \mathbb{E}\left(\exp\left\{-\mathbb{E}\left(Y_n \middle| \mathcal{F}_{\lceil an\rceil}\right)\right\}\right).$$

By the bounded convergence theorem, we get that

$$1 \geq \limsup_{n\to\infty} \mathbb{E}\left(e^{-Y_n}\right) \geq \liminf_{n\to\infty} \mathbb{E}\left(e^{-Y_n}\right) \geq \lim_{n\to\infty} \mathbb{E}\left(\exp\left\{-\mathbb{E}\left(Y_n \middle| \mathcal{F}_{\lceil an\rceil}\right)\right\}\right) = 1.$$

Therefore, for any $\varepsilon > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(Y_n > \varepsilon\right) = \limsup_{n\to\infty} \mathbb{P}\left(1-e^{-Y_n} > 1-e^{-\varepsilon}\right) \leq \frac{1}{1-e^{-\varepsilon}} \lim_{n\to\infty} \mathbb{E}\left(1-e^{-Y_n}\right) = 0,$$

which implies that $\lim_{n\to\infty} Y_n = 0$ in probability. □

4.4. Convergence in distribution for $\sqrt{n}\left(F_{good}^{\lceil an\rceil,\gamma_n}-\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}\right)$

Recall that $F_{good}^{\lceil an\rceil,\gamma_n}$ and $\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}$ are defined in (4.8) and (4.25) respectively.

Proposition 4.10. *Let $(X_t)_{t\geq 0}$ be the 1-stable Lévy process with characteristic function given by (1.5). For any $m\geq 1, 1\leq a_1<\cdots<a_m$ and $\lambda\in\mathbb{R}^m$, under $\mathbb{P}$,*

$$\lim_{m\to\infty}\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^m\lambda_k\sqrt{n}\left(F_{good}^{\lceil a_kn\rceil,\gamma_n}-\widehat{N}_{good}^{\lceil a_kn\rceil,\gamma_n}\right)\right\}\middle|\mathcal{F}_n\right)$$
$$=\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^m c^*\lambda_kX_{a_k^{-1/2}D_\infty}\right\}\middle|D_\infty\right),\qquad \textit{in probability}.$$

Proof. Let $\lambda_k^*:=\lambda_1+\cdots+\lambda_k$ and $a_{m+1}=\infty$, by the definitions of $F_{good}^{\lceil a_kn\rceil,\gamma_n}$ and $\widehat{N}_{good}^{\lceil a_kn\rceil,\gamma_n}$ in (4.8) and (4.30), we have

$$\sum_{k=1}^m\lambda_k\sqrt{n}\Big(F_{good}^{\lceil a_kn\rceil,\gamma_n}-\widehat{N}_{good}^{\lceil a_kn\rceil,\gamma_n}\Big)=\sum_{k=1}^m\lambda_k\sqrt{n}\Big(\sum_{\ell=\lceil a_kn\rceil+1}^{\infty}\sum_{x\in\mathcal{N}(\ell)}e^{-V(x)}$$
$$\times 1_{\{\min_{n\leq j\leq \ell-1}V(x_j)\geq\gamma_n,\gamma_n-\beta_n/2\leq V(x)<\gamma_n\}}c^*\left(D_\infty(x)-V(x)+\frac{\log n}{2}\right)\Big)$$
$$=\sum_{k=1}^m\sum_{\ell=\lceil a_kn\rceil+1}^{\lceil a_{k+1}n\rceil}\sum_{x\in\mathcal{N}(\ell)}1_{\{\min_{n\leq j\leq \ell-1}V(x_j)\geq\gamma_n,\gamma_n-\beta_n/2\leq V(x)<\gamma_n\}}$$
$$\times c^*\lambda_k^*\sqrt{n}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big).$$

Define $\Psi_{D_\infty}(\lambda):=\mathbb{E}(e^{\mathrm{i}\lambda D_\infty})$. For each $K\in\mathbb{N}$, define $a_k':=a_k$ for $k=1,\dots,m$ and $a_{m+1}':=a_m+K$. Also, for $n,\ell\in\mathbb{N}$ with $n<\ell$, define $\Gamma(\ell):=\{x\in\mathcal{N}(\ell):\min_{n\leq j\leq\ell-1}V(x_j)\geq\gamma_n,\gamma_n-\beta_n/2\leq V(x)<\gamma_n\}$ for simplicity. We first calculate

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^m\sum_{\ell=\lceil a_k'n\rceil+1}^{\lceil a_{k+1}'n\rceil}\sum_{x\in\Gamma(\ell)}c^*\lambda_k^*\sqrt{n}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\middle|\mathcal{F}_n\right)$$
$$=:\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=A}^{B}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\middle|\mathcal{F}_n\right),$$

where $A=\lceil a_1'n\rceil+1, B=\lceil a_{m+1}'n\rceil$ and $\lambda_{n,\ell}=c^*\lambda_k^*\sqrt{n}$ for $\lceil a_k'n\rceil+1\leq\ell\leq\lceil a_{k+1}'n\rceil$. Note that for every $\ell\in[A,B]$, if there is an $x\in\Gamma(\ell)$, then $x_j\notin\Gamma(j)$ for all $j\in[A,\ell-1]$. Therefore, by the branching property, conditioned on $\mathcal{F}_B$, $\{D_\infty(x),x\in\Gamma(\ell),\ell\in[A,B]\}$ are mutually independent, which implies that

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=A}^{B}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\middle|\mathcal{F}_B\right)$$
$$=\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=A}^{B-1}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\middle|\mathcal{F}_B\right)$$
$$\times\prod_{x\in\Gamma(B)}e^{-\mathrm{i}\lambda_{n,B}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B}e^{-V(x)})$$
$$=\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=A}^{B-1}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\right.$$
$$\left.\times\prod_{x\in\Gamma(B)}e^{-\mathrm{i}\lambda_{n,B}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B}e^{-V(x)})\middle|\mathcal{F}_B\right).$$

Using the fact that $x\in\Gamma(B)$ implies $x_{B-1}\notin\Gamma(B-1)$, we have

$$\prod_{x\in\Gamma(B)}e^{-\mathrm{i}\lambda_{n,B}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B}e^{-V(x)})$$
$$=\prod_{x\in\mathcal{N}(B-1)\setminus\Gamma(B-1)}\prod_{u\in\mathcal{N}(B):u>x}e^{-\mathrm{i}\lambda_{n,B}e^{-V(u)}(V(u)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B}e^{-V(u)}).$$

Therefore, again by the Markov property and the branching property, conditioned on $\mathcal{F}_{B-1}$, $\{D_\infty(x):x\in\Gamma(\ell),\ell\in[A,B-1]\}$ are independent and also independent of the σ-field generated by $\{u\in\mathbb{T},u_B\in\mathcal{N}(B)\}$, which implies that

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{\ell=A}^{B}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\right\}\middle|\mathcal{F}_{B-1}\right)$$

$$
\begin{aligned}
&= \mathbb{E}\bigg(\exp\bigg\{\mathrm{i}\sum_{\ell=A}^{B-2}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\bigg\}\bigg|\mathcal{F}_{B-1}\bigg)\\
&\quad\times\prod_{x\in\Gamma(B-1)}e^{-\mathrm{i}\lambda_{n,B-1}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B-1}e^{-V(x)})\\
&\quad\times\mathbb{E}\bigg(\prod_{x\in\mathcal{N}(B-1)\setminus\Gamma(B-1)}\prod_{u\in\mathcal{N}(B):u>x}e^{-\mathrm{i}\lambda_{n,B}e^{-V(u)}(V(u)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,B}e^{-V(u)})\bigg|\mathcal{F}_{B-1}\bigg)\\
&= \mathbb{E}\bigg(\exp\bigg\{\mathrm{i}\sum_{\ell=A}^{B-2}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\bigg\}\bigg|\mathcal{F}_{B-1}\bigg)\\
&\quad\times\mathbb{E}\bigg(\prod_{\ell=B-1}^{B}\prod_{x\in\Gamma(\ell)}e^{-\mathrm{i}\lambda_{n,\ell}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,\ell}e^{-V(x)})\bigg|\mathcal{F}_{B-1}\bigg).
\end{aligned}
$$

Iterating the above equation and using the fact that $\mathcal{F}_n\subset\mathcal{F}_A$, we finally get that

$$
\begin{aligned}
&\mathbb{E}\bigg(\exp\bigg\{\mathrm{i}\sum_{\ell=A}^{B}\sum_{x\in\Gamma(\ell)}\lambda_{n,\ell}e^{-V(x)}\Big(D_\infty(x)-V(x)+\frac{\log n}{2}\Big)\bigg\}\bigg|\mathcal{F}_n\bigg)\\
&= \mathbb{E}\bigg(\prod_{\ell=A}^{B}\prod_{x\in\Gamma(\ell)}e^{-\mathrm{i}\lambda_{n,\ell}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(\lambda_{n,\ell}e^{-V(x)})\bigg|\mathcal{F}_n\bigg)\\
&= \mathbb{E}\bigg(\prod_{k=1}^{m}\prod_{\ell=\lceil a'_k n+1\rceil}^{\lceil a'_{k+1}n\rceil}\prod_{x\in\Gamma(\ell)}e^{-\mathrm{i}\lambda^*_k c^*\sqrt{n}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}(c^*\lambda^*_k\sqrt{n}e^{-V(x)})\bigg|\mathcal{F}_n\bigg).
\end{aligned}
\tag{4.39}
$$

By [11, (1.12)] and [11, Lemma 2.3], there exist continuous functions $\eta,q:\mathbb{R}\to\mathbb{C}$ with $\eta(0)=q(0)=0$ such that

$$
\begin{aligned}
&e^{-\mathrm{i}\lambda^*_k c^*\sqrt{n}e^{-V(x)}(V(x)-\log n/2)}\Psi_{D_\infty}\left(c^*\lambda^*_k\sqrt{n}e^{-V(x)}\right)\\
&= e^{-\mathrm{i}\lambda^*_k c^*\sqrt{n}e^{-V(x)}(V(x)-\log n/2)}\Psi_{\pi/2,c_0+1-\gamma}\left(c^*\lambda^*_k\sqrt{n}e^{-V(x)}\right)e^{c^*\lambda^*_k\sqrt{n}e^{-V(x)}\eta\left(c^*\lambda^*_k\sqrt{n}e^{-V(x)}\right)}\\
&=: \exp\Big\{-\sqrt{n}e^{-V(x)}\psi_{\pi/2,c_0+1-\gamma}\left(c^*\lambda^*_k\right)+c^*\lambda^*_k\sqrt{n}e^{-V(x)}q\left(c^*\lambda^*_k\sqrt{n}e^{-V(x)}\right)\Big\},
\end{aligned}
$$

where c_0 is the constant in (1.3). Define

$$
R_n := -\sum_{k=1}^{m}\sqrt{n}\left(\widetilde{N}^{\lceil a_k n\rceil,\gamma_n}_{good}-\widetilde{N}^{\lceil a_{k+1}n\rceil,\gamma_n}_{good}\right)\psi_{\pi/2,c_0+1-\gamma}(c^*\lambda^*_k),
$$

$$
Z_n := \sum_{k=1}^{m}\sum_{\ell=\lceil a_k n\rceil+1}^{\lceil a_{k+1}n\rceil}\sum_{x\in\mathcal{N}(\ell)}1_{\{\min_{n\le j\le \ell-1}V(x_j)\ge\gamma_n,\gamma_n-\beta_n/2\le V(x)<\gamma_n\}}c^*\lambda^*_k\sqrt{n}e^{-V(x)}q\left(c^*\lambda^*_k\sqrt{n}e^{-V(x)}\right).
$$

Letting $K\to\infty$ in (4.39), we get

$$
\mathbb{E}\bigg(\exp\bigg\{\mathrm{i}\sum_{k=1}^{m}\lambda_k\sqrt{n}\left(F^{\lceil a_k n\rceil,\gamma_n}_{good}-\widehat{N}^{\lceil a_k n\rceil,\gamma_n}_{good}\right)\bigg\}\bigg|\mathcal{F}_n\bigg)=\mathbb{E}\left(\exp\left\{R_n+Z_n\right\}\Big|\mathcal{F}_n\right).
$$

We claim that

$$
\Big|\mathbb{E}\left(\exp\left\{R_n+Z_n\right\}\Big|\mathcal{F}_n\right)-\mathbb{E}\left(\exp\left\{R_n\right\}\Big|\mathcal{F}_n\right)\Big|\to 0\quad\text{in probability.}
\tag{4.40}
$$

Indeed, for any $\varepsilon\in(0,1)$ and any complex number $z=a+\mathrm{i}b$ with $|z|\le\varepsilon$, we have $|a|,|b|\le\varepsilon$, which implies that

$$
\left|e^z-1\right|\le\left|e^{a+\mathrm{i}b}-e^{\mathrm{i}b}\right|+\left|e^{\mathrm{i}b}-1\right|\lesssim\varepsilon.
$$

Thus, we get that, for any $\varepsilon>0$,

$$
\Big|\mathbb{E}\left(\exp\left\{R_n+Z_n\right\}\Big|\mathcal{F}_n\right)-\mathbb{E}\left(\exp\left\{R_n\right\}\Big|\mathcal{F}_n\right)\Big|\lesssim\mathbb{P}\left(|Z_n|>\varepsilon\Big|\mathcal{F}_n\right)+\varepsilon.
\tag{4.41}
$$

Recalling that $\widetilde{N}^{\lceil a_1 n\rceil,\gamma_n}_{good}$ is defined in (4.25) with $a=a_1$, we see that, for any fixed $\lambda_1,\dots,\lambda_m$, we have

$$
\begin{aligned}
|Z_n|&\lesssim\max_{1\le k_0\le m}\max_{y\in[\beta_n/2,\beta_n)}\left|q\left(c^*\lambda^*_{k_0}e^{-y}\right)\right|\sum_{k=1}^{m}\sum_{\ell=\lceil a_k n\rceil+1}^{\lceil a_{k+1}n\rceil}\sum_{x\in\mathcal{N}(\ell)}\\
&\qquad 1_{\{\min_{n\le j\le \ell-1}V(x_j)\ge\gamma_n,\gamma_n-\beta_n/2\le V(x)<\gamma_n\}}\sqrt{n}e^{-V(x)}\\
&\lesssim\max_{1\le k_0\le m}\max_{y\in[\beta_n/2,\beta_n)}\left|q\left(c^*\lambda^*_{k_0}e^{-y}\right)\right|\sqrt{n}\widetilde{N}^{\lceil a_1 n\rceil,\gamma_n}_{good}.
\end{aligned}
$$

Applying Corollary 4.6 with $m=1$ and $z^n=\mathrm{i}a$ with $a\in\mathbb{R}$, we see that $\left(\sqrt{n}\widetilde{N}^{\lceil a_1 n\rceil,\gamma_n}_{good},\mathbb{P}(\cdot|\mathcal{F}_n)\right)$ converges in distribution. Since $\lim_{z\to 0}q(z)=0$, we have $\lim_{n\to\infty}\mathbb{P}\left(|Z_n|>\varepsilon\Big|\mathcal{F}_n\right)=0$. Letting $n\to\infty$ first and then $\varepsilon\to 0$ in (4.41), we get (4.40).

Combining (4.40) and Corollary 4.6, we get

$$\lim_{n\to\infty}\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^m \lambda_k\sqrt{n}\left(F_{good}^{\lceil a_k n\rceil,\gamma_n}-\widehat{N}_{good}^{\lceil a_k n\rceil,\gamma_n}\right)\right\}\middle|\mathcal{F}_n\right)$$
$$=\exp\left\{\sqrt{\frac{2}{\pi\sigma^2}}D_\infty\sum_{k=1}^m\psi_{\pi/2,c_0+1-\gamma}(c^*\lambda_k^*)\left(\frac{1}{\sqrt{a_k}}-\frac{1}{\sqrt{a_{k+1}}}\right)\right\},\quad \text{in probability.}$$

A standard computation yields

$$\mathbb{E}\left(\exp\left\{\mathrm{i}\sum_{k=1}^m c^*\lambda_k X_{a_k^{-1/2}D_\infty}\right\}\middle|D_\infty\right)=\prod_{k=1}^m\mathbb{E}\left(\exp\left\{\mathrm{i}c^*\lambda_k^*\left(X_{a_{k+1}^{-1/2}D_\infty}-X_{a_k^{-1/2}D_\infty}\right)\right\}\middle|D_\infty\right)$$
$$=\exp\left\{\sum_{k=1}^m D_\infty\left(\frac{1}{\sqrt{a_k}}-\frac{1}{\sqrt{a_{k+1}}}\right)\psi_{\sqrt{\pi/2\sigma^2},(c_0+1-\gamma)\sqrt{2/\pi\sigma^2}}(c^*\lambda_k^*)\right\}.$$

This implies the desired result since

$$\psi_{\sqrt{\pi/2\sigma^2},(c_0+1-\gamma)\sqrt{2/\pi\sigma^2}}(\lambda)=\sqrt{\frac{2}{\pi\sigma^2}}\psi_{\pi/2,c_0+1-\gamma}(\lambda),\quad \lambda\in\mathbb{R}.\quad\square$$

Proof of Theorem 2.1. By (4.8), we get that

$$\sqrt{n}\left(D_\infty-D_{\lceil an\rceil}+\frac{\log n}{2}W_{\lceil an\rceil}\right)=\frac{\sqrt{n}}{c^*}\left(D_\infty^{\lceil an\rceil,\gamma_n}-\left(c^*D_{\lceil an\rceil}-\left(c^*\gamma_n-\alpha^*\right)W_{\lceil an\rceil}\right)\right)$$
$$+\frac{\sqrt{n}}{c^*}\left(\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}-\left(c^*\beta_n-\alpha^*\right)W_{\lceil a_k n\rceil}\right)+\frac{\sqrt{n}}{c^*}F_{bad}^{\lceil an\rceil,\gamma_n}$$
$$+\frac{\sqrt{n}}{c^*}\left(F_{good}^{\lceil an\rceil,\gamma_n}-\widehat{N}_{good}^{\lceil an\rceil,\gamma_n}\right).$$

It follows from Proposition 4.3, Corollary 4.8 and Proposition 4.9 that the first three terms on the right hand side of the display above converge to 0 in probability as $n\to\infty$. Now the conclusion of Theorem 2.1 follows from Proposition 4.10. $\square$

5. Proof of Proposition 2.2

Recall that $\alpha\in(0,1]$ is the constant in **(A5)**. Fix an $r\in(0,1)$ such that

$$\frac{1+\alpha}{2}<\frac{r}{2}(1-\alpha)+\alpha\left((1+r)\wedge\left(\frac{3}{2}\right)\right),$$

which is equivalent to

$$r>\max\left\{\frac{1-\alpha}{1+\alpha},\ \frac{1-2\alpha}{1-\alpha}\right\}.$$

By the definition (4.18) of $\widetilde{W}_m^{n,\gamma_n}$, we have

$$\widetilde{W}_n^{\lceil n^r\rceil,0}=\sum_{x\in\mathcal{N}(n)}e^{-V(x)}1_{\{\min_{j\in[n^r,n]\cap\mathbb{Z}}V(x_j)\geq 0\}}.$$

Note that $\min_{x\in\mathcal{N}(n)}V(x)\to+\infty$, $\mathbb{P}$-a.s., so

$$\lim_{n\to\infty}\mathbb{P}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\neq W_n\right)=0. \tag{5.1}$$

As preparation for the proof of Proposition 2.2, we prove a few lemmas first.

Lemma 5.1. *For any $\beta\in\left(0,\frac{1}{2}(r\wedge(1-r))\right)$, it holds that*

$$\lim_{n\to\infty}n^\beta\left|\sqrt{n}\mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\middle|\mathcal{F}_{\lceil n^r\rceil}\right)-\sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_{\lceil n^r\rceil}\right|=0,\quad \textit{in probability}.$$

Proof. By the branching property, we have

$$\sqrt{n}\mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\middle|\mathcal{F}_{\lceil n^r\rceil}\right)-\sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_{\lceil n^r\rceil}$$
$$=\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\left(\sqrt{n}\mathbf{P}_{V(x)}\left(\min_{j\leq n-\lceil n^r\rceil}S_j\geq 0\right)-\sqrt{\frac{2}{\pi\sigma^2}}\delta_n V(x)\right).$$

It follows from [8, Lemma 2.2] that, for any $y\geq 0$ and positive integer n with $n-\lceil n^r\rceil>1$,

$$\left|\mathbf{P}_y\left(\min_{j\leq n-\lceil n^r\rceil}S_j\geq 0\right)-R(y)\mathbf{P}\left(\min_{j\leq n-\lceil n^r\rceil}S_j\geq 0\right)\right|\lesssim\frac{1+y}{\sqrt{n}}R(y)\mathbf{P}\left(\min_{j\leq n-\lceil n^r\rceil}S_j\geq 0\right).$$

Combining Lemma 3.1(i) with the facts that $n-\lceil n^r\rceil \asymp n$ and $|R(y)-c^*y|\lesssim 1, y\geq 0$, we get that for all $y\geq 0$ and large n,

$$\Big|\sqrt{n}\mathbf{P}_y\Big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\Big)-c^*y\sqrt{n}\mathbf{P}\Big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\Big)\Big|$$
$$\lesssim \sqrt{n}\frac{(1+y)^2}{\sqrt{n-\lceil n^r\rceil}}\mathbf{P}\big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\big)+\sqrt{n}\mathbf{P}\big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\big)$$
$$\lesssim \frac{(1+y)^2}{\sqrt{n}}+1.$$

Recall that $\tau_0^-:=\inf\left\{\ell\geq 0: S_\ell<0\right\}$. Then by [8, (2.18)],

$$\Big|\mathbf{P}\Big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\Big)-\mathbf{P}\Big(\min_{j\leq n} S_j\geq 0\Big)\Big|=\mathbf{P}\Big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0,\min_{n-\lceil n^r\rceil+1\leq j\leq n} S_j<0\Big)$$
$$=\sum_{\ell=n-\lceil n^r\rceil+1}^{n}\mathbf{P}\left(\tau_0^-=\ell\right)\lesssim\sum_{\ell=n-\lceil n^r\rceil+1}^{n}\frac{1}{\ell^{3/2}}\leq\int_{n-\lceil n^r\rceil}^{n}\frac{1}{x^{3/2}}\mathrm{d}x$$
$$=\Big(\frac{1}{\sqrt{n-\lceil n^r\rceil}}-\frac{1}{\sqrt{n}}\Big)\lesssim\frac{n^r}{n^{3/2}}.$$

Combining the two displays above with (2.1), we get that for all $y\geq 0$ and large n,

$$\Big|\sqrt{n}\mathbf{P}_y\Big(\min_{j\leq n-\lceil n^r\rceil} S_j\geq 0\Big)-\sqrt{\frac{2}{\pi\sigma^2}}\delta_n y\Big|\lesssim\frac{(1+y)^2}{\sqrt{n}}+1+\frac{y}{n^{1-r}},$$

which implies that

$$\Big|\sqrt{n}\mathbb{E}\Big(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big)-\sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_{\lceil n^r\rceil}\Big|$$
$$\lesssim\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\Big(\frac{(1+V(x))^2}{\sqrt{n}}+1+\frac{V(x)}{n^{1-r}}\Big).$$

By (1.2), we have

$$\lim_{n\to\infty}\sqrt{\lceil n^r\rceil}\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}\to\sqrt{\frac{2}{\pi\sigma^2}}D_\infty\quad\text{in probability.}\tag{5.2}$$

Since $\lim_{n\to\infty}D_n=D_\infty$ almost surely and D_∞ is non-negative, we have

$$\lim_{n\to\infty}\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}V(x)e^{-V(x)}1_{\{V(x)\geq 0\}}=D_\infty,\ \text{almost surely.}\tag{5.3}$$

Since $\beta\in(0,\frac{1}{2}((1-r)\wedge r))$, using the two displays above, we get that for any $\varepsilon>0$,

$$\lim_{n\to\infty}\mathbb{P}\Big(n^\beta\Big|\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\Big(1+\frac{V(x)}{n^{1-r}}\Big)\Big|>\varepsilon\Big)=0.$$

For any $\delta>0$, let L satisfy (4.14). Then for any $\varepsilon>0$,

$$\mathbb{P}\Big(n^\beta\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\frac{(1+V(x))^2}{\sqrt{n}}>\varepsilon\Big)$$
$$\leq\delta+\mathbb{P}\Big(n^\beta\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\frac{(1+V(x))^2}{\sqrt{n}}1_{\{\min_{j\leq\lceil n^r\rceil}V(x_j)\geq -L\}}>\varepsilon\Big)$$
$$\leq\delta+\frac{1}{\varepsilon}n^{\beta-\frac{1}{2}}\mathbb{E}\Big(\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}(1+V(x))^2e^{-V(x)}1_{\{V(x)\geq 0\}}1_{\{\min_{j\leq\lceil n^r\rceil}V(x_j)\geq -L\}}\Big)$$
$$=\delta+\frac{1}{\varepsilon}n^{\beta-\frac{1}{2}}\mathbf{E}\Big((1+S_{\lceil n^r\rceil})^2 1_{\{S_{\lceil n^r\rceil}\geq 0\}}1_{\{\min_{j\leq\lceil n^r\rceil}S_j\geq -L\}}\Big)$$
$$\lesssim\delta+\frac{1}{\varepsilon}n^{\beta-\frac{1}{2}}\Big((1+L)^2+(1+L)\sqrt{\lceil n^r\rceil}\Big),$$

where in the last inequality we used Lemma 3.2. Letting $n\to\infty$ first, and then $\delta\to 0$, we get

$$\lim_{n\to\infty}\mathbb{P}\Big(n^\beta\sum_{x\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(x)}1_{\{V(x)\geq 0\}}\frac{(1+V(x))^2}{\sqrt{n}}>\varepsilon\Big)=0.$$

This completes the proof. □

For $m<n$, define

$$\mathcal{A}_{n,1}^{m,0}:=\Big\{x\in\mathcal{N}(n):\ \text{for all } j=m+1,\ldots,n,$$

$$\sum_{u\in\Omega(x_j)}\left(1+\left(V(u)-V(x_{j-1})\right)_+\right)e^{-(V(u)-V(x_{j-1}))}\le e^{V(x_{j-1})/2}\Big\},$$

$$\widetilde{W}_{n,1}^{\lceil n^r\rceil,0}=\sum_{x\in\mathcal{N}(n)}e^{-V(x)}1_{\{\min_{j\in[n^r,n]\cap\mathbb{Z}}V(x_j)\ge 0\}}1_{\left\{x\in\mathcal{A}_{n,1}^{\lceil n^r\rceil,0}\right\}}. \tag{5.4}$$

Lemma 5.2. *For large n,*

$$\mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}-\widetilde{W}_{n,1}^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\lesssim\sum_{u\in\mathcal{N}(\lceil n^r\rceil)}\left(\frac{1}{\sqrt{n}}+\frac{V(u)}{n}\right)e^{-V(u)}1_{\{V(u)\ge 0\}}.$$

Proof. By (3.2) and the branching property, we have

$$\begin{aligned}
&\mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}-\widetilde{W}_{n,1}^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\\
&=\sum_{u\in\mathcal{N}(\lceil n^r\rceil)}1_{\{V(u)\ge 0\}}\mathbb{E}_y\left(\sum_{x\in\mathcal{N}(n-\lceil n^r\rceil-1)}e^{-V(x)}1_{\{\min_{j\le n-\lceil n^r\rceil}V(x_j)\ge 0\}}1_{\left\{x\notin\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\right)_{y=V(u)}\\
&\le\sum_{u\in\mathcal{N}(\lceil n^r\rceil)}e^{-V(u)}1_{\{V(u)\ge 0\}}\sum_{\ell=1}^{n-\lceil n^r\rceil-1}\mathbb{Q}\Bigg(\min_{j\le n-\lceil n^r\rceil-1}V(w_j)\ge -y,\\
&\qquad\sum_{u\in\Omega(w_\ell)}\left(1+\left(V(u)-V(w_{\ell-1})\right)_+\right)e^{-(V(u)-V(w_{\ell-1}))}>e^{(V(w_{\ell-1})+y)/2}\Bigg)\Bigg|_{y=V(u)}.
\end{aligned}\tag{5.5}$$

Suppose n is large so that $n-\lceil n^r\rceil>1$. For any positive integer $\ell\le n-\lceil n^r\rceil-1$, conditioned on $\mathcal{F}_\ell$, by Lemma 3.1(i), we have

$$\begin{aligned}
&\mathbb{Q}\left(\min_{j\le n-\lceil n^r\rceil-1}V(w_j)\ge -y,\sum_{u\in\Omega(w_\ell)}\left(1+\left(V(u)-V(w_{\ell-1})\right)_+\right)e^{-(V(u)-V(w_{\ell-1}))}>e^{(V(w_{\ell-1})+y)/2}\right)\\
&\lesssim\left(1\wedge\frac{1}{\sqrt{n-\lceil n^r\rceil-1-\ell}}\right)\mathbb{E}_{\mathbb{Q}}\Bigg(\left(1+V(w_\ell)+y\right)1_{\{\min_{j\le\ell}V(w_j)\ge -y\}}\\
&\qquad\times 1_{\left\{\sum_{u\in\Omega(w_\ell)\cup\{w_\ell\}}\left(1+\left(V(u)-V(w_{\ell-1})\right)_+\right)e^{-(V(u)-V(w_{\ell-1}))}>e^{(V(w_{\ell-1})+y)/2}\right\}}\Bigg)\\
&=:\left(1\wedge\frac{1}{\sqrt{n-\lceil n^r\rceil-1-\ell}}\right)\mathbb{E}_{\mathbb{Q}}(J_\ell).
\end{aligned}\tag{5.6}$$

Conditioned on $\mathcal{F}_{\ell-1}$, we get that, given $V(w_{\ell-1})=z$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}}(J_\ell\big|V(w_{\ell-1})=z)\\
&=\mathbb{E}_{\mathbb{Q}}\left(\left(1+V(w_1)+z+y\right)1_{\{V(w_1)\ge -y-z\}}1_{\left\{\sum_{x\in\mathcal{N}(1)}(1+(V(u))_+)e^{-V(u)}>e^{(z+y)/2}\right\}}\right)\\
&=\mathbb{E}\left(\sum_{x\in\mathcal{N}(1)}(1+V(x)+z+y)\,e^{-V(x)}1_{\{V(x)\ge -y-z\}}1_{\left\{z+y<2\log_+\left(W_1+\widetilde{W}_1\right)\right\}}\right)\\
&\le\mathbb{E}\left(\left(W_1\left(1+2\log_+\left(W_1+\widetilde{W}_1\right)\right)+\widetilde{W}_1\right)1_{\left\{z+y<2\log_+\left(W_1+\widetilde{W}_1\right)\right\}}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(J_\ell)&\le\mathbb{E}\Bigg(\left(W_1\left(1+2\log_+\left(W_1+\widetilde{W}_1\right)\right)+\widetilde{W}_1\right)\\
&\qquad\times\mathbf{P}\left(\min_{j\le\ell-1}S_j\ge -y,S_{\ell-1}+y<2\log_+(m_1)\right)\Big|_{m_1=W_1+\widetilde{W}_1}\Bigg)\\
&=:\mathbb{E}\left(W\mathbf{P}\left(\min_{j\le\ell-1}S_j\ge -y,S_{\ell-1}+y<2\log_+(m_1)\right)\Big|_{m_1=W_1+\widetilde{W}_1}\right),
\end{aligned}$$

where $W:=W_1\left(1+2\log_+\left(W_1+\widetilde{W}_1\right)\right)+\widetilde{W}_1$. Thus, if k_n is the integer such that $2k_n\le n-\lceil n^r\rceil-1<2k_n+1$, then by Lemma 3.1 (ii)(iii), we have

$$\begin{aligned}
&\sum_{\ell=1}^{n-\lceil n^r\rceil-1}\left(1\wedge\frac{1}{\sqrt{n-\lceil n^r\rceil-1-\ell}}\right)\mathbb{E}_{\mathbb{Q}}(J_\ell)\\
&\le\frac{1}{\sqrt{n-\lceil n^r\rceil-1-k_n}}\mathbb{E}\left(W\sum_{\ell=1}^{k_n}\mathbf{P}\left(\min_{j\le\ell-1}S_j\ge -y,S_{\ell-1}+y<2\log_+(m_1)\right)\Big|_{m_1=W_1+\widetilde{W}_1}\right)\\
&\quad+\sum_{\ell=k_n+1}^{n-\lceil n^r\rceil-1}\left(1\wedge\frac{1}{\sqrt{n-\lceil n^r\rceil-1-\ell}}\right)\mathbb{E}\left(W\frac{(y+1)\left(1+2\log_+\left(W_1+\widetilde{W}_1\right)\right)^2}{(\ell-1)^{3/2}}\right)
\end{aligned}$$

$$\lesssim \left(\frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - k_n}} + (y+1) \sum_{\ell=k_n+1}^{n-\lceil n^r \rceil -1} \left(1 \wedge \frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - \ell}} \right) \frac{1}{(\ell-1)^{3/2}} \right) F(1), \tag{5.7}$$

with $F(1)$ given by

$$F(1) := \mathbb{E}\left(\left(W_1 \left(1 + 2\log_+ (W_1 + \widetilde{W}_1)\right) + \widetilde{W}_1 \right) \left(1 + 2\log_+ (W_1 + \widetilde{W}_1)\right)^2 \right) < \infty.$$

If n is large enough, we have

$$\sum_{\ell=k_n+1}^{n-\lceil n^r \rceil -1} \left(1 \wedge \frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - \ell}} \right) \frac{1}{(\ell-1)^{3/2}}$$
$$\leq \frac{1}{(n - \lceil n^r \rceil - 1)^{3/2}} + \sum_{\ell=k_n+1}^{n-\lceil n^r \rceil -2} \frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - \ell}} \frac{1}{(\ell-1)^{3/2}}$$
$$\lesssim \frac{1}{n} + \sum_{\ell=k_n+1}^{n-\lceil n^r \rceil -2} \int_{\ell}^{\ell+1} \frac{1}{\sqrt{(n - \lceil n^r \rceil - 1 - x)}(x-2)^{3/2}} \mathrm{d}x$$
$$= \frac{1}{n} + \frac{1}{n} \int_{n^{-1}(k_n+1)}^{n^{-1}(n-\lceil n^r \rceil -2)} \frac{1}{\sqrt{(n^{-1}(n - \lceil n^r \rceil - 1) - x)}(x - 2n^{-1})^{3/2}} \mathrm{d}x \lesssim \frac{1}{n}.$$

This implies that for n large enough,

$$\frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - k_n}} + (y+1) \sum_{\ell=k_n+1}^{n-\lceil n^r \rceil -1} \left(1 \wedge \frac{1}{\sqrt{n - \lceil n^r \rceil - 1 - \ell}} \right) \frac{1}{(\ell-1)^{3/2}} \lesssim \frac{1}{\sqrt{n}} + \frac{y}{n}.$$

Combining this with (5.5), (5.6) and (5.7), we get the desired result. □

Recall that $\widetilde{W}_{n,1}^{\lceil n^r \rceil,0}$ is defined in (5.4).

Lemma 5.3. *For large n,*

$$\mathbb{E}\left(\left| \widetilde{W}_{n,1}^{\lceil n^r \rceil,0} - \mathbb{E}\left(\widetilde{W}_{n,1}^{\lceil n^r \rceil,0} \middle| \mathcal{F}_{\lceil n^r \rceil} \right) \right|^{1+\alpha} \middle| \mathcal{F}_{\lceil n^r \rceil} \right)$$
$$\lesssim \sum_{u \in \mathcal{N}(\lceil n^r \rceil)} e^{-V(u)} \left(\left(\frac{1}{n} + V(u) \frac{(\log n)^2}{n^{3/2}} \right) + \left(\frac{1}{n} + V(u) \frac{(\log n)^2}{n^{3/2}} \right)^{\alpha} \right) 1_{\{V(u) \geq 0\}}$$
$$\lesssim U_{\lceil n^r \rceil} + W_{\lceil n^r \rceil}^{1-\alpha} U_{\lceil n^r \rceil}^{\alpha}$$

with

$$U_{\lceil n^r \rceil} := \sum_{u \in \mathcal{N}(\lceil n^r \rceil)} e^{-V(u)} \left(\frac{1}{n} + V(u) \frac{(\log n)^2}{n^{3/2}} \right) 1_{\{V(u) \geq 0\}}.$$

Proof. By the branching property,

$$\mathbb{E}\left(\left| \widetilde{W}_{n,1}^{\lceil n^r \rceil,0} - \mathbb{E}\left(\widetilde{W}_{n,1}^{\lceil n^r \rceil,0} \middle| \mathcal{F}_{\lceil n^r \rceil} \right) \right|^{1+\alpha} \middle| \mathcal{F}_{\lceil n^r \rceil} \right)$$
$$\lesssim \sum_{u \in \mathcal{N}(\lceil n^r \rceil)} 1_{\{V(u) \geq 0\}} \mathbb{E}_y \left(\left(\sum_{x \in \mathcal{N}(n - \lceil n^r \rceil - 1)} e^{-V(x)} 1_{\{\min_{j \leq n - \lceil n^r \rceil -1} V(x_j) \geq 0\}} 1_{\left\{x \in \mathcal{A}^{0,0}_{n - \lceil n^r \rceil -1,1}\right\}} \right)^{1+\alpha} \right) \Bigg|_{y=V(u)}$$
$$=: \sum_{u \in \mathcal{N}(\lceil n^r \rceil)} 1_{\{V(u) \geq 0\}} \mathbb{E}_y \left(\Gamma^{1+\alpha} \right) \Big|_{y=V(u)}.$$

Recalling the definition of $\mathbb{Q}_y$ in beginning of Section 3.1, we similarly have

$$\mathbb{E}_y \left(\Gamma^{1+\alpha} \right) = e^{-y} \mathbb{E}_{\mathbb{Q}_y} \left(1_{\{\min_{j \leq n - \lceil n^r \rceil -1} V(w_j) \geq 0\}} 1_{\left\{w \in \mathcal{A}^{0,0}_{n - \lceil n^r \rceil -1,1}\right\}} \Gamma^{\alpha} \right).$$

For $u < x$, define $V(x; u) := V(x) - V(u)$. Recall that $\mathcal{N}(u, m) := \left\{ x \in \mathcal{N}(|u| + m) : x > u \right\}$. By the spine decomposition, we have

$$\Gamma = \sum_{x \in \mathcal{N}(n - \lceil n^r \rceil -1)} e^{-V(x)} 1_{\{\min_{j \leq n - \lceil n^r \rceil -1} V(x_j) \geq 0\}} 1_{\left\{x \in \mathcal{A}^{0,0}_{n - \lceil n^r \rceil -1}\right\}}$$
$$\leq e^{-V(w_{n-\lceil n^r \rceil -1})} + \sum_{k=1}^{n-\lceil n^r \rceil -1} \sum_{u \in \Omega(w_k)} e^{-V(u)} \sum_{x \in \mathcal{N}(u, n - \lceil n^r \rceil -1-k)} e^{-V(x;u)} 1_{\{\min_{j \leq n - \lceil n^r \rceil -1-k} V(x_j;u) \geq -V(u)\}}$$
$$=: e^{-V(w_{n-\lceil n^r \rceil -1})} + \sum_{k=1}^{n-\lceil n^r \rceil -1} \sum_{u \in \Omega(w_k)} e^{-V(u)} H(u),$$

where $H(u) := \sum_{x\in\mathcal{N}(u,n-\lceil n^r\rceil-1-k)} e^{-V(x;u)}1_{\{\min_{j\le n-\lceil n^r\rceil-1-k} V(x_j;k)\ge -V(u)\}}$. Therefore, using the inequality $\mathbb{E}(|X|^\alpha)\le(\mathbb{E}(|X|))^\alpha$, we get

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_y}\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\Gamma^\alpha\right)\\
&\le \mathbb{E}_{\mathbb{Q}_y}\left(e^{-\alpha V(w_{n-\lceil n^r\rceil-1})}1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(x_j)\ge 0\}}\right)\\
&\quad+\left(\mathbb{E}_{\mathbb{Q}_y}\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\sum_{k=1}^{n-\lceil n^r\rceil-1}\sum_{u\in\Omega(w_k)}e^{-V(u)}H(u)\right)\right)^\alpha
\end{aligned}$$

By the branching property and Lemma 3.1(i), we have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_y}\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\sum_{k=1}^{n-\lceil n^r\rceil-1}\sum_{u\in\Omega(w_k)}e^{-V(u)}H(u)\right)\\
&=\mathbb{E}_{\mathbb{Q}_y}\Bigg(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\\
&\qquad\times\sum_{k=1}^{n-\lceil n^r\rceil-1}\sum_{u\in\Omega(w_k)}e^{-V(u)}\mathbf{P}\left(\min_{j\le n-\lceil n^r\rceil-1-k}S_j\ge -z\right)\Big|_{z=V(u)}\Bigg)\\
&\lesssim\mathbb{E}_{\mathbb{Q}_y}\Bigg(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\\
&\qquad\times\sum_{k=1}^{n-\lceil n^r\rceil-1}\sum_{u\in\Omega(w_k)}e^{-V(u)}\left(1\wedge\frac{(1+V(u))}{\sqrt{n-\lceil n^r\rceil-1-k}}\right)\Bigg).
\end{aligned}$$

Since

$$\begin{aligned}
1\wedge\frac{(1+V(u))}{\sqrt{n-\lceil n^r\rceil-1-k}}&\le 1\wedge\frac{\left(1+\left(V(u)-V(w_{k-1})\right)_++V(w_{k-1})\right)}{\sqrt{n-\lceil n^r\rceil-1-k}}\\
&\le 1\wedge\frac{\left(1+\left(V(u)-V(w_{k-1})\right)_+\right)(1+V(w_{k-1}))}{\sqrt{n-\lceil n^r\rceil-1-k}}\\
&\le\left(1+\left(V(u)-V(w_{k-1})\right)_+\right)\left(1\wedge\frac{(1+V(w_{k-1}))}{\sqrt{n-\lceil n^r\rceil-1-k}}\right),
\end{aligned}$$

and, on the set $\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}$, it holds that

$$\sum_{u\in\Omega(w_k)}e^{-V(u)}\left(1+\left(V(u)-V(w_{k-1})\right)_+\right)\le e^{-V(w_{k-1})/2}.$$

We have

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}_y}\Bigg(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}1_{\left\{w\in\mathcal{A}^{0,0}_{n-\lceil n^r\rceil-1,1}\right\}}\\
&\qquad\times\sum_{k=1}^{n-\lceil n^r\rceil-1}\sum_{u\in\Omega(w_k)}e^{-V(u)}\left(1\wedge\frac{(1+V(u))}{\sqrt{n-\lceil n^r\rceil-1-k}}\right)\Bigg)\\
&\lesssim\mathbb{E}_{\mathbb{Q}_y}\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1} V(w_j)\ge 0\}}\sum_{k=1}^{n-\lceil n^r\rceil-1}e^{-V(w_{k-1})/2}\left(1\wedge\frac{(1+V(w_{k-1}))}{\sqrt{n-\lceil n^r\rceil-1-k}}\right)\right).
\end{aligned}$$

Therefore, by the arguments above, we conclude that

$$\begin{aligned}
&\mathbb{E}_y\left(\Gamma^{1+\alpha}\right)\lesssim e^{-y}\mathbf{E}_y\left(e^{-\alpha S_{n-\lceil n^r\rceil-1}}1_{\{\min_{j\le n-\lceil n^r\rceil-1} S_j\ge 0\}}\right)\\
&\quad+e^{-y}\left(\sum_{k=1}^{n-\lceil n^r\rceil-1}\mathbf{E}_y\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1} S_j\ge 0\}}e^{-S_{k-1}/2}\left(1\wedge\frac{(1+S_{k-1})}{\sqrt{n-\lceil n^r\rceil-1-k}}\right)\right)\right)^\alpha\\
&:=e^{-y}(I+II^\alpha).
\end{aligned}\tag{5.8}$$

By Lemma 3.1 (ii), for all $y\ge 0$ and large n, it holds that

$$\begin{aligned}
I&\le\frac{1}{n}+\mathbf{P}_y\left(\min_{j\le n-\lceil n^r\rceil-1}S_j\ge 0, S_{n-\lceil n^r\rceil-1}\le\alpha^{-1}\log n\right)\\
&\lesssim\frac{1}{n}+\frac{(1+y)(1+\alpha^{-1}\log n)^2}{(n-\lceil n^r\rceil-1)^{3/2}}\lesssim\frac{1}{n}+y\frac{(\log n)^2}{n^{3/2}}.
\end{aligned}\tag{5.9}$$

For II, we have

$$II\le\mathbf{E}_y\left(e^{-S_{n-\lceil n^r\rceil-1}/2}1_{\{\min_{j\le n-\lceil n^r\rceil-1} S_j\ge 0\}}\right)+\mathbf{E}_y\left(e^{-S_{n-\lceil n^r\rceil-2}/2}1_{\{\min_{j\le n-\lceil n^r\rceil-2} S_j\ge 0\}}\right)$$

$$
\begin{aligned}
&+\sum_{k=1}^{n-\lceil n^r\rceil-3}\mathbf{E}_y\left(1_{\{\min_{j\le n-\lceil n^r\rceil-1}S_j\ge 0\}}e^{-S_{k-1}/2}\frac{(1+S_{k-1})}{\sqrt{n-\lceil n^r\rceil-1-k}}\right)\\
&:= II_1+II_2+II_3. \qquad (5.10)
\end{aligned}
$$

Applying Lemma 3.1 (ii) with $k=n-\lceil n^r\rceil-1$ or $n-\lceil n^r\rceil-2$, we get

$$
\mathbf{E}_y\left(e^{-S_k/2}1_{\{\min_{j\le k}S_j\ge 0\}}\right)\lesssim \frac{1}{n}+\frac{(1+y)(1+2\log n)^2}{k^{3/2}}\lesssim \frac{1}{n}+y\frac{(\log n)^2}{n^{3/2}}.
$$

Thus

$$
II_1\vee II_2\lesssim \frac{1}{n}+y\frac{(\log n)^2}{n^{3/2}}. \qquad (5.11)
$$

For II_3, we note that for, $k=1,\dots,n-\lceil n^r\rceil-3$,

$$
\begin{aligned}
&\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})}{\sqrt{n-\lceil n^r\rceil-1-k}}1_{\{\min_{j\le n-\lceil n^r\rceil-1}S_j\ge 0\}}\right)\\
&=\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})}{\sqrt{n-\lceil n^r\rceil-1-k}}1_{\{\min_{j\le k-1}S_j\ge 0\}}\mathbf{P}_{S_{k-1}}\left(\min_{j\le n-\lceil n^r\rceil-k}S_j\ge 0\right)\right)\\
&\lesssim \mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})^2}{n-\lceil n^r\rceil-1-k}1_{\{\min_{j\le k-1}S_j\ge 0\}}\right).
\end{aligned}
$$

Recall that k_n is the integer such that $2k_n\le n-\lceil n^r\rceil-1<2k_n+1$. Combining Lemma 3.1 (iv) with $\lambda=\frac{1}{4}$ and the fact that $\sup_{x>0}\left(e^{-x/2}(1+x)^2\right)<\infty$,

$$
\begin{aligned}
&\sum_{k=1}^{k_n}\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})}{\sqrt{n-\lceil n^r\rceil-1-k}}1_{\{\min_{j\le n-\lceil n^r\rceil-1}S_j\ge 0\}}\right)\\
&\lesssim \frac{1}{n-\lceil n^r\rceil-k_n-1}\sum_{k=0}^{\infty}\mathbf{E}_y\left(e^{-S_k/4}1_{\{\min_{j\le k}S_j\ge 0\}}\right)\lesssim \frac{1}{n}. \qquad (5.12)
\end{aligned}
$$

On the other hand, for $k_n+1\le k\le n-\lceil n^r\rceil-3$, by Lemma 3.1 (ii), we have

$$
\begin{aligned}
&\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})^2}{n-\lceil n^r\rceil-1-k}1_{\{\min_{j\le k-1}S_j\ge 0\}}\right)\\
&=\sum_{\ell=0}^{\infty}\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})^2}{n-\lceil n^r\rceil-1-k}1_{\{\min_{j\le k-1}S_j\ge 0\}}1_{\{S_{k-1}\in[\ell,\ell+1)\}}\right)\\
&\le \frac{1}{n-\lceil n^r\rceil-1-k}\sum_{\ell=0}^{\infty}e^{-\ell/2}(2+\ell)^2\mathbf{P}_y\left(\min_{j\le k-1}S_j\ge 0,S_{k-1}\in[\ell,\ell+1)\right)\\
&\lesssim \frac{1}{n-\lceil n^r\rceil-1-k}\frac{(y+1)}{(k-1)^{3/2}}\sum_{\ell=0}^{\infty}e^{-\ell/2}(2+\ell)^2(\ell+2)\lesssim \frac{(1+y)}{(n-\lceil n^r\rceil-1-k)(k-1)^{3/2}}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\sum_{k=k_n+1}^{n-\lceil n^r\rceil-3}\mathbf{E}_y\left(e^{-S_{k-1}/2}\frac{(1+S_{k-1})^2}{n-\lceil n^r\rceil-1-k}1_{\{\min_{j\le k-1}S_j\ge 0\}}\right)\\
&\lesssim (1+y)\sum_{k=k_n+1}^{n-\lceil n^r\rceil-3}\frac{1}{(n-\lceil n^r\rceil-1-k)(k-1)^{3/2}}\\
&\le (1+y)\int_{k_n}^{n-\lceil n^r\rceil-3}\frac{\mathrm{d}x}{(n-\lceil n^r\rceil-2-x)(x-1)^{3/2}}\\
&=(1+y)\left(\frac{\log\left(n-\lceil n^r\rceil-2-k_n\right)}{(k_n-1)^{3/2}}-\frac{3}{2}\int_{k_n}^{n-\lceil n^r\rceil-2}\frac{\log(n-\lceil n^r\rceil-2-x)\,\mathrm{d}x}{(x-1)^{5/2}}\right)\\
&\le (1+y)\frac{\log\left(n-\lceil n^r\rceil-2-k_n\right)}{(k_n-1)^{3/2}}\lesssim (1+y)\frac{(\log n)^2}{n^{3/2}}.
\end{aligned}
$$

Combining this with (5.12), we get

$$
II_3\lesssim \frac{1}{n}+y\frac{(\log n)^2}{n^{3/2}}. \qquad (5.13)
$$

Combining (5.8), (5.9),(5.10), (5.11) and (5.13), we get the desired result. □

Lemma 5.4. *Let* $\beta_+ \in (0, \frac{1}{2}((1-r)\wedge r))$ *satisfy*

$$\left(\beta_+ + \frac{1}{2}\right)(1+\alpha) < \min\left\{(1+r)\wedge\left(\frac{3}{2}\right),\ \frac{r}{2}(1-\alpha)+\alpha(1+r)\wedge\left(\frac{3}{2}\right)\right\}. \tag{5.14}$$

For any $\beta \in (0, \beta^+)$*, it holds that*

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\widetilde{W}_n^{\lceil n^r\rceil,0} - \mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\right| > 3n^{-\beta-\frac{1}{2}}\right) = 0.$$

Proof. We only need to prove that

$$\mathbb{P}\Big(\left|\widetilde{W}_n^{\lceil n^r\rceil,0} - \mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\right| > 3n^{-\beta-\frac{1}{2}}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big) \to 0 \ \text{ in probability.} \tag{5.15}$$

Recall that $\alpha \in (0,1]$ is given in **(A5)** and that $\widetilde{W}_{n,1}^{\lceil n^r\rceil,0}$ is defined in (5.4). Then by Lemma 3.4,

$$\begin{aligned}
&\mathbb{P}\Big(\left|\widetilde{W}_n^{\lceil n^r\rceil,0} - \mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\right| > 3n^{-\beta-\frac{1}{2}}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big)\\
&\lesssim 2n^{\beta+\frac{1}{2}}\mathbb{E}\Big(\widetilde{W}_n^{\lceil n^r\rceil,0} - \widetilde{W}_{n,1}^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big)\\
&\quad + n^{(\beta+\frac{1}{2})(1+\alpha)}\mathbb{E}\Big(\left|\widetilde{W}_{n,1}^{\lceil n^r\rceil,0} - \mathbb{E}\left(\widetilde{W}_{n,1}^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\right|^{1+\alpha}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big).
\end{aligned} \tag{5.16}$$

Now combining (5.16), Lemma 5.2 and Lemma 5.3, we get

$$\begin{aligned}
&\mathbb{P}\Big(\left|\widetilde{W}_n^{\lceil n^r\rceil,0} - \mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right)\right| > 3n^{-\beta-\frac{1}{2}}\Big|\mathcal{F}_{\lceil n^r\rceil}\Big)\\
&\lesssim n^\beta W_{\lceil n^r\rceil} + n^{\beta-\frac{1}{2}}\sum_{u\in\mathcal{N}(\lceil n^r\rceil)} V(u)e^{-V(u)}1_{\{V(u)\geq 0\}} + n^{(\beta+\frac{1}{2})(1+\alpha)}\left(U_{\lceil n^r\rceil} + W_{\lceil n^r\rceil}^{1-\alpha}U_{\lceil n^r\rceil}^{\alpha}\right).
\end{aligned}$$

Using (5.2) and (5.3), we get that for any $\varepsilon \in (0,(1+r)\wedge(\frac{3}{2}))$, $n^\varepsilon U_{n^r} \to 0$ in probability. Therefore, we have (5.15) and the proof is complete. □

Proof of Proposition 2.2. Recall the definition (2.1) of δ_n. Let $\beta_+ \in (0, \frac{1}{2}((1-r)\wedge r))$ satisfy (5.14). If we can show that for any $\beta \in (0,\beta_+)$,

$$\lim_{n\to\infty}\mathbb{P}\left(n^\beta\left|\sqrt{n}W_n - \sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_\infty\right| \geq 5\right) = 0,$$

then the desired conclusion follows immediately. Note that

$$\begin{aligned}
&\mathbb{P}\left(n^\beta\left|\sqrt{n}W_n - \sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_\infty\right| \geq 5\right)\\
&\leq \mathbb{P}\Big(\widetilde{W}_n^{\lceil n^r\rceil,0} \neq W_n\Big) + \mathbb{P}\Big(n^{\beta+\frac{1}{2}}\left|\widetilde{W}_n^{\lceil n^r\rceil,0} - \mathbb{E}\big(\widetilde{W}_n^{\lceil n^r\rceil,0}\big|\mathcal{F}_{\lceil n^r\rceil}\big)\right| > 3\Big)\\
&\quad + \mathbb{P}\left(\left(\sup_k \delta_k\right)\sqrt{\frac{2}{\pi\sigma^2}}|D_\infty - D_{\lceil n^r\rceil}| > n^{-\beta}\right) + \mathbb{P}\left(n^\beta\left|\sqrt{n}\mathbb{E}\left(\widetilde{W}_n^{\lceil n^r\rceil,0}\Big|\mathcal{F}_{\lceil n^r\rceil}\right) - \sqrt{\frac{2}{\pi\sigma^2}}\delta_n D_{\lceil n^r\rceil}\right| > 1\right).
\end{aligned}$$

Using (1.6) and (5.1), we know that the first and third term on the right hand side above tend to 0 as $n \to \infty$. Lemma 5.1 says that the fourth term on the right hand side above tend to 0 as $n \to \infty$, and Lemma 5.4 says that the second term on the right hand side above tend to 0 as $n \to \infty$. This completes the proof of Proposition 2.2. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We thank the referee for very helpful comments on the first version of this paper. Part of the research for this paper was done while the third-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie).

References

[1] E. Aïdékon, Convergence in law of the minimum of a branching random walk, Ann. Probab. 41 (2013) 1362–1426.
[2] E. Aïdékon, Z. Shi, The Seneta-Heyde scaling for the branching random walk, Ann. Probab. 42 (2014) 959–993.
[3] J.D. Biggins, Martingale convergence in the branching random walk, J. Appl. Probab. 14 (1977) 25–37.
[4] J.D. Biggins, A.E. Kyprianou, Measure change in multitype branching, Adv. Appl. Probab. 36 (2004) 544–581.

[5] D. Buraczewski, A. Iksanov, B. Mallein, On the derivative martingale in a branching random walk, Ann. Probab. 49 (2021) 1164–1204.

[6] X. Chen, A necessary and sufficient condition for the nontrivial limit of the derivative martingale in a branching random walk, Adv. Appl. Probab. 47 (3) (2015) 741–760.

[7] I. Grama, H. Xiao, Conditioned local limit theorems for random walks on the real line, Ann. Inst. H. Poincaré Probab. Statist (2024) arXiv:2110.05123 in press.

[8] Y. Hu, The almost sure limits of the minimal position and the additive martingale in a branching random walk, J. Theoret. Probab. 28 (2) (2015) 467–487.

[9] A.E. Kyprianou, Travelling wave solutions to the K-P-P equation: alternatives to Simon Harris' probabilistic analysis, Ann. Inst. H. Poincaré Probab. Statist. 40 (2004) 53–72.

[10] R. Lyons, A simple path to Biggins' martingale convergence for branching random walk, in: Classical and Modern Branching Processes, in: IMA Math. Appl., vol. 84, Springer, New York, 1997, pp. 217–221.

[11] P. Maillard, M. Pain, 1-stable fluctuations in branching Brownian motion at critical temperature I: The derivative martingale, Ann. Probab. 47 (5) (2019) 2953–3002.

[12] Z. Shi, Branching random walks, in: Lecture Notes from the 42nd Probability Summer School Held in Saint Flour, 2012, in: Lecture Notes in Math., vol. 2151, Springer, Cham, 2015, p. x+133.

[13] B. von Bahr, C.-G. Esseen, Inequalities for the rth absolute moment of a sum of random variables, $1 \leq r \leq 2$, Ann. Math. Stat. 36 (1965) 299–303.

[14] T. Yang, Y.-X. Ren, Limit theorem for derivative martingale at criticality w.r.t. branching Brownian motion, Statist. Probab. Lett. 81 (2) (2011) 195–200.