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Asymptotic Expansions for Additive Measures of Branching Brownian Motions

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Abstract

Let $N(t)$ be the collection of particles alive at time t in a branching Brownian motion in $\mathbb{R}^d$, and for $u \in N(t)$, let $\mathbf{X}_u(t)$ be the position of particle u at time t. For $\theta \in \mathbb{R}^d$, we define the additive measures of the branching Brownian motion by

$$\mu_t^\theta(\mathrm{d}\mathbf{x}) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)} \delta_{(\mathbf{X}_u(t)+\theta t)}(\mathrm{d}\mathbf{x}),$$

here $\|\theta\|$ is the Euclidean norm of θ.

In this paper, under some conditions on the offspring distribution, we give asymptotic expansions of arbitrary order for $\mu_t^\theta((\mathbf{a}, \mathbf{b}])$ and $\mu_t^\theta((-\infty, \mathbf{a}])$ for $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, where $(\mathbf{a}, \mathbf{b}] := (a_1, b_1] \times \cdots \times (a_d, b_d]$ and $(-\infty, \mathbf{a}] := (-\infty, a_1] \times \cdots \times (-\infty, a_d]$ for $\mathbf{a} = (a_1, \cdots, a_d)$ and $\mathbf{b} = (b_1, \cdots, b_d)$. These expansions sharpen the asymptotic results of Asmussen and Kaplan (Stoch Process Appl 4(1):1–13, 1976) and Kang (J Korean Math Soc 36(1): 139–157, 1999) and are analogs of the expansions in Gao and Liu (Sci China Math 64(12):2759–2774, 2021) and Révész et al. (J Appl Probab 42(4):1081–1094, 2005) for branching Wiener processes (a particular class of branching random walks) corresponding to $\theta = \mathbf{0}$.

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1 Introduction and Main Results

1.1 Introduction

A branching random walk in $\mathbb{R}^d$ is a discrete-time Markov process which can be defined as follows: At time 0, there is a particle at $\mathbf{0} \in \mathbb{R}^d$; at time 1, this particle is replaced by a random number of particles distributed according to a point process $\mathcal{L}$; at time 2, each individual, of generation 1, if located at $\mathbf{x} \in \mathbb{R}^d$, is replaced by a point process $\mathbf{x} + \mathcal{L}_{\mathbf{x}}$, where $\mathcal{L}_{\mathbf{x}}$ is an independent copy of $\mathcal{L}$. This procedure goes on. We use Z_n to denote the point process formed by the positions of the particles of generation n. $(Z_n)_{\geq 0}$ is called a branching random walk.

For any $n \in \mathbb{N}$ and $\theta \in \mathbb{R}^d$, define

$$W_n(\theta) := \frac{1}{m(\theta)^n} \int e^{-\theta \cdot \mathbf{x}} Z_n(\mathrm{d}\mathbf{x}),$$

where $m(\theta) := \mathbb{E}\left(\int e^{-\theta \cdot \mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\right)$. It is well known that, for any $\theta \in \mathbb{R}^d$, $(W_n(\theta))_{n\geq 0}$ is a martingale. $(W_n(\theta))_{n\geq 0}$ is called the additive martingale of the branching random walk. The additive martingale is a fundamental tool for studying various asymptotic behaviors of branching random walks, see [23] for some of its applications. Biggins [6] used the L^p convergence of the additive martingale to study the asymptotic behavior of $Z_n(n\mathbf{c} + I)$ for fixed $\mathbf{c}$ and bounded interval I. To describe Biggins' result, we introduce the following additive measure $\mu_n^{Z,\theta}$ of the branching random walk, which is a shifted version of the measure introduced before Theorem 4 in [6]:

$$\mu_n^{Z,\theta}(A) := m(\theta)^{-n} \int e^{-\theta \cdot \mathbf{y}} 1_A(\mathbf{y} - \mathbf{c}_\theta n) Z_n(\mathrm{d}\mathbf{y}), \quad A \in \mathcal{B}(\mathbb{R}^d), \tag{1.1}$$

with $(\mathbf{c}_\theta)_i := m(\theta)^{-1}\mathbb{E}\left(\int x_i e^{-\theta \cdot \mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\right)$. In the case $\theta = 0$, the additive measure above reduces to the normalization of Z_n. The additive measures appear as Gibbs measures in the study of directed polymers on trees in random environment, see Derrida and Spohn [9]. Theorem 4 in [6] implies that, in the weak disorder regime (i.e., $-\log m(\theta) < -\theta \cdot \nabla m(\theta)/m(\theta)$), if there exists $\gamma > 1$ such that $\mathbb{E}(W_1(\theta)^\gamma) < \infty$, then for $\mathbf{x} \in \mathbb{R}^d$ and $h > 0$, as $n \to \infty$,

$$n^{d/2} \mu_n^{Z,\theta}(\mathbf{x} + I_h) \longrightarrow \frac{(2h)^d W_\infty(\theta)}{(2\pi \det(\Sigma_\theta))^{d/2}}, \quad \text{a.s.} \tag{1.2}$$

where $W_\infty(\theta) := \lim_{n\to\infty} W_n(\theta)$, $I_h = [-h, h]^d$ and

$$(\Sigma_\theta)_{i,j} = m(\theta)^{-1}\mathbb{E}\Big(\int (x_i - (\mathbf{c}_\theta)_i)(x_j - (\mathbf{c}_\theta)_j)e^{-\theta\cdot\mathbf{x}} Z_1(\mathrm{d}\mathbf{x})\Big), \quad i, j \in \{1, ..., d\}.$$

From (1.2), we see that the limit of $n^{d/2}\mu_n^{Z,\theta}(\mathbf{x} + I_h)$ is proportional to the volume of I_h and the proportion is a multiple of the limit $W_\infty(\theta)$ of the additive martingale with parameter θ. The limit result (1.2) also tells us that $\mu_n^{Z,\theta}(\mathbf{x} + I_h)$ decays to 0 at the rate $n^{-d/2}$. In the case $d = 1$, Pain proved that, see [19, (1.14)], in the weak disorder regime, if there exists $\gamma > 1$ such that $\mathbb{E}(W_1(\theta)^\gamma) < \infty$, then for any $b \in \mathbb{R}$, as $n \to \infty$,

$$\mu_n^{Z,\theta}\left((-\infty, b\Sigma_\theta\sqrt{n}]\right) \to W_\infty(\theta)\Phi(b) \quad \text{in probability,} \tag{1.3}$$

where $\Phi(b) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{b} e^{-z^2/2}\mathrm{d}z$. The limit result (1.2) is a local limit theorem for the additive measure $\mu_n^{Z,\theta}$, and (1.3) is a central limit type theorem for $\mu_n^{Z,\theta}$.

For the case $\theta = \mathbf{0}$, there are many further asymptotic results. In the case when $d = 1$ and the point process $\mathcal{L}$ is given by $\mathcal{L} = \sum_{i=1}^{B}\delta_{X_i}$, where X_i are iid with common distribution G and B is an independent $\mathbb{N}$-valued random variable with $\mathbb{P}(B = k) = p_k$ and $\mu := \sum_k kp_k > 1$, Asmussen and Kaplan [2, 3] proved in 1976 that if G has mean 0, variance 1 and $\sum_{k=2}^{\infty} k(\log k)^{1+\varepsilon}p_k < \infty$ for some $\varepsilon > 0$, then conditioned on survival, for any $b \in \mathbb{R}$,

$$\mu_n^{Z,0}((-\infty, b\sqrt{n}]) \xrightarrow{n\to\infty} W_\infty(0)\Phi(b), \quad \text{a.s.} \tag{1.4}$$

They also proved that if G has finite 3rd moment and $\sum_{k=2}^{\infty} k(\log k)^{3/2+\varepsilon}p_k < \infty$ for some $\varepsilon > 0$, then, for any $a < b \in \mathbb{R}$, conditioned on survival,

$$\sqrt{2\pi n}\,\mu_n^{Z,0}([a, b]) \xrightarrow{n\to\infty} (b - a)W_\infty(0), \quad \text{a.s.} \tag{1.5}$$

Of course, (1.4) and (1.5) are special cases of (1.3) and (1.2), respectively. A natural and important next step is to study the convergence rates in these two limits above. Gao and Liu [13] gave first and second-order expansions of $\mu_n^{Z,0}((-\infty, b\sqrt{n}])$. A third-order expansion was proved by Gao and Liu [12, 14], where branching random walks in (time) random environment were studied. They also conjectured the form of asymptotic expansion of arbitrary order for $\mu_n^{Z,0}((-\infty, b\sqrt{n}])$. For general branching random walks, results similar to (1.4) and (1.5) were proved in Biggins [5].

When the point process $\mathcal{L}$ is given by $\mathcal{L} = \sum_{i=1}^{B}\delta_{\mathbf{X}_i}$ where $\mathbf{X}_1, \mathbf{X}_2, ...$ are independent d-dimensional standard normal random variables and B is an independent $\mathbb{N}$-valued random variable with $\mathbb{P}(B = k) = p_k$ and $\mu := \sum_k kp_k > 1$, Z_n is called a supercritical branching Wiener process. Révész [21] first proved the analogs of (1.4) and (1.5) for branching Wiener processes, and then, Chen [8] studied the corresponding convergence rates. Gao and Liu [11] proved that, for each $m \in \mathbb{N}$, when $\sum_{k=1}^{\infty} k(\log k)^{1+\lambda}p_k < \infty$ for some $\lambda > 3\max\{(m + 1), dm\}$, there exist random variables $\{V_{\mathbf{a}}, |\mathbf{a}| \le m\}$ such that for each $\mathbf{t} \in \mathbb{R}^d$,

$$\frac{1}{\mu^n} Z_n((-\infty, \mathbf{t}\sqrt{n}]) = \Phi_d(\mathbf{t}) V_{\mathbf{0}} + \sum_{\ell=1}^{m} \frac{(-1)^\ell}{n^{\ell/2}} \sum_{|\mathbf{a}|=\ell} \frac{D^{\mathbf{a}} \Phi_d(\mathbf{t})}{\mathbf{a}!} V_{\mathbf{a}} + o(n^{-m/2}), \text{ a.s.,} \tag{1.6}$$

where for $\mathbf{a} = (a_1, ..., a_d)$, $|\mathbf{a}| = a_1 + ... + a_d$, $\mathbf{a}! = a_1! \cdots a_d!$, $\Phi_d(\mathbf{t})$ is the distribution function of a d-dimensional standard normal random vector and $D^{\mathbf{a}}\Phi_d(\mathbf{t}) := \partial_{t_1}^{a_1} ... \partial_{t_d}^{a_d} \Phi_d(\mathbf{t})$. For the local limit theorem (1.5), Révész, Rosen and Shi [22] proved that when $\sum_{k=1}^{\infty} k^2 p_k < \infty$, for any bounded Borel set $A \subset \mathbb{R}^d$,

$$(2\pi n)^{d/2} \frac{1}{\mu^n} Z_n(A) = \sum_{\ell=0}^{m} \frac{(-1)^\ell}{(2n)^\ell} \sum_{|\mathbf{a}|=\ell} \frac{1}{\mathbf{a}!} \sum_{\mathbf{b} \le 2\mathbf{a}} C_{2\mathbf{a}}^{\mathbf{b}} (-1)^{|\mathbf{b}|} M_{\mathbf{b}}(A) V_{2\mathbf{a}-\mathbf{b}} + o(n^{-m}), \\ \text{a.s.,} \tag{1.7}$$

where $\mathbf{b} \le 2\mathbf{a}$ means that $b_i \le 2a_i$ for all $1 \le i \le d$, $C_{2\mathbf{a}}^{\mathbf{b}} := C_{2a_1}^{b_1} \cdots C_{2a_d}^{b_d}$, $C_n^k := \frac{n!}{k!(n-k)!}$ and $M_{\mathbf{b}}(A) := \int_A x_1^{b_1} \cdots x_d^{b_d} \mathrm{d}x_1 ... \mathrm{d}x_d$. The two results above give the asymptotic expansions of $\frac{1}{\mu^n} Z_n((-\infty, \mathbf{t}\sqrt{n}])$ and $(2\pi n)^{d/2} \frac{1}{\mu^n} Z_n(A)$ (with A being a bounded Borel sets of $\mathbb{R}^d$) up to arbitrary order.

For the lattice case, analogs of (1.4) and (1.5) can be found in [10, 16], and an asymptotic expansion similar to (1.7) for $Z_n(\{k\})$ was given by Grübel and Kabluchko [16].

In this paper, we are concerned with branching Brownian motions in $\mathbb{R}^d$. A branching Brownian motion in $\mathbb{R}^d$ is a continuous-time Markov process defined as follows: Initially, there is a particle at $\mathbf{0} \in \mathbb{R}^d$, it moves according to a d-dimensional standard Brownian motion and its lifetime is an exponential random variable of parameter 1, independent of the spatial motion. At the end of its lifetime, it produces k offspring with probability p_k for $k \in \mathbb{N}$ and the offspring move independently according to a d-dimensional standard Brownian motion from the death location of their parent and repeat their parent's behavior independently. This procedure goes on. Let $N(t)$ be the set of particles alive at time t and for $u \in N(t)$, we use $\mathbf{X}_u(t)$ to denote the position of particle u at time t. Define

$$Z_t := \sum_{u \in N(t)} \delta_{\mathbf{X}_u(t)}.$$

$(Z_t)_{t \ge 0}$ is called a branching Brownian motion. We will use $\mathbb{P}$ to denote the law of branching Brownian motion and $\mathbb{E}$ to denote the corresponding expectation. We will use $(\mathbf{B}_t)_{t \ge 0}$ to denote a standard Brownian motion in $\mathbb{R}^d$. For $\mathbf{x} \in \mathbb{R}^d$, we will use $\mathbf{P}_{\mathbf{x}}$ to the law of Brownian motion starting from $\mathbf{x}$ and use $\mathbf{E}_{\mathbf{x}}$ to denote the corresponding expectation. For $x \in \mathbb{R}$, we will also use $\mathbf{P}_x$ and $\mathbf{E}_x$ to denote, respectively, the law and the corresponding expectation of a 1-dimensional standard Brownian motion started at x. For convenience, we write $\mathbf{P}$ for $\mathbf{P_0}$ or $\mathbf{P}_0$, and $\mathbf{E}$ for $\mathbf{E_0}$ or $\mathbf{E}_0$. Without loss of generality, we assume that

$$\sum_{k=0}^{\infty} kp_k = 2.$$

This assumption is not essential and is assumed for convenience. For the general case, see the discussion in Remark 1.6. For $\theta = (\theta_1, ..., \theta_d) \in \mathbb{R}^d$,

$$W_t(\theta) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u \in N(t)} e^{-\theta \cdot \mathbf{X}_u(t)}$$

is a nonnegative martingale and is called the additive martingale of the branching Brownian motion, here $\|\theta\|$ is the Euclidean norm of θ. When θ is the zero vector, $W_t(\theta)$ reduces to $e^{-t} Z_t(\mathbb{R}^d)$. The additive martingale is very useful in studying the asymptotic behaviors of branching Brownian motions. For instance, in the case $d = 1$, it has been used to study the maximal position of branching Brownian motions. It was also used to give a probabilistic representation for the traveling wave solution of the KPP equation, see [18]. The limit of the additive martingale is related to the limit behavior of the number of particles whose speed is larger than θ, see [15]. The additive martingale and its limit also appear in the study of extremal processes of some inhomogeneous branching Brownian motions and reducible branching Brownian motions, see, for instance, [4, 7] and the references therein. It is well known that (for $d = 1$, see Kyprianou [18]), for each $\theta \in \mathbb{R}^d$, $W_t(\theta)$ converges to a non-trivial limit $W_\infty(\theta)$ if and only if $\|\theta\| < \sqrt{2}$ and

$$\sum_{k=1}^{\infty} k(\log k) p_k < \infty. \tag{1.8}$$

From now on, we will only consider $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any set $A \subset \mathbb{R}$ and $a \in \mathbb{R}$, we use $|A|$ to denote the Lebesgue measure of A and $aA := \{ax : x \in A\}$. Asmussen and Kaplan [3, Part 5] proved that when $d = 1$, under the assumption $\sum_{k=1}^{\infty} k^2 p_k < \infty$, for any Borel set B with $|\partial B| = 0$, as $t \to \infty$,

$$e^{-t} Z_t\left(\sqrt{t}B\right) \longrightarrow \frac{W_\infty(0)}{\sqrt{2\pi}} \int_B e^{-z^2/2} \mathrm{d}z, \quad \mathbb{P}\text{-a.s.} \tag{1.9}$$

and that for any bounded Borel set B with $|\partial B| = 0$, as $t \to \infty$,

$$\sqrt{2\pi t} e^{-t} Z_t(B) \longrightarrow |B| W_\infty(0), \quad \mathbb{P}\text{-a.s.} \tag{1.10}$$

Kang [17, Theorem 1] weakened the moment condition and proved that (1.9) holds with $B = (-\infty, b]$ under condition (1.8). The results (1.9) and (1.10) are the counterparts of (1.4) and (1.5) for branching Brownian motions.

Similar to (1.1), we define the additive measure μ_t^θ of branching Brownian motion as

$$\mu_t^\theta(\mathrm{d}\mathbf{x}) := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} \delta_{(\mathbf{X}_u(t)+\theta t)}(\mathrm{d}\mathbf{x}).$$

The normalized random probability measure $\mu_t^\theta(\mathbb{R}^d)^{-1}\mu_t^\theta$ appears as the Gibbs measure of a directed polymer on trees in random environment, see Derrida and Spohn [9]. The goal of this paper is to sharpen the results in (1.9) and (1.10) and establish asymptotic expansions of the additive measure μ_t^θ in the subcritical case $\|\theta\| < \sqrt{2}$, see Theorems 1.1 and 1.2. These expansions sharpen the asymptotic results of [3, Part 5] and [17] mentioned above. The asymptotic expansions of [11, 22] are for the additive measure $\mu_n^{Z,0}$ of branching Wiener processes, while the asymptotic expansions of Theorems 1.1 and 1.2 are for the additive measure μ_t^θ of branching Brownian motions with θ not necessarily $\mathbf{0}$.

One might expect that the asymptotic expansions for branching Wiener processes, when considered along $\{t_n = n\delta, n \in \mathbb{N}\}$, can be used to get the expansions of this paper by letting $\delta \to 0$. However, it seems that this idea does not work due to two reasons. One of the reasons is that values along $\{n\delta, n \in \mathbb{N}\}$ are not good enough to control the behavior between the time intervals $[t_n, t_{n+1}]$. Another reason is that $\{Z_{n\delta} : n \in \mathbb{N}\}$ is not a branching Wiener process since in $Z_\delta = \sum_{u\in N(\delta)} \delta_{\mathbf{X}_u(\delta)}$, for $u, v \in N(\delta)$, $u \neq v$, $\mathbf{X}_u(\delta)$ and $\mathbf{X}_v(\delta)$ are not independent.

1.2 Notation

We list here some notation that will be used repeatedly below. Throughout this paper, $\mathbb{N} = \{0, 1, \dots\}$. Recall that $N(t)$ is the set of the particles alive at time t and that for $u \in N(t)$, $\mathbf{X}_u(t)$ is the position of u. For $u \in N(t)$, we use d_u and O_u to denote the death time and the offspring number of u, respectively. For v and u, we will use $v < u$ to denote that v is an ancestor of u. The notation $v \le u$ means that $v = u$ or $v < u$.

For $\mathbf{a} = (a_1, ..., a_d) \in \mathbb{R}^d$, define $(\mathbf{a})_j := a_j$ and $(-\infty, \mathbf{a}] := (-\infty, a_1] \times \cdots \times (-\infty, a_d]$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we use $\mathbf{a} < \mathbf{b}$ ($\mathbf{a} \le \mathbf{b}$) to denote that $(\mathbf{a})_j < (\mathbf{b})_j$ ($(\mathbf{a})_j < (\mathbf{b})_j$) for all $1 \le j \le d$. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, define $(\mathbf{a}, \mathbf{b}] := (a_1, b_1] \times \cdots \times (a_d, b_d]$. The definition of $[\mathbf{a}, \mathbf{b}]$ is similar. For $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{N}^d$, set $|\mathbf{k}| := k_1 + ... + k_d$ and $\mathbf{k}! := k_1! \cdots k_d!$. For a function f on $\mathbb{R}^d$, $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{k} \in \mathbb{N}^d$, let $D^{\mathbf{k}} f(\mathbf{x}) := \partial_{x_1}^{k_1} ... \partial_{x_d}^{k_d} f(\mathbf{x})$. We also use the notation $\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ and $\Phi_d(\mathbf{x}) := \prod_{j=1}^d \int_{-\infty}^{x_j} \phi(z)\mathrm{d}z$. Sometimes we write $\Phi(y)$ for $\Phi_1(y)$. For two functions f and g, we will use $f \lesssim g$ to denote that there exists a constant C such that $f(x) \le Cg(x)$ for all x in the common domain of definition of f and g.

1.3 Main Results

We will assume that

$$\sum_{k=1}^{\infty} k(\log k)^{1+\lambda} p_k < \infty \tag{1.11}$$

for appropriate $\lambda > 0$. Let H_k be the k-th-order Hermite polynomial: $H_0(x) := 1$ and for $k \geq 1$,

$$H_k(x) := \sum_{j=0}^{[k/2]} \frac{k!(-1)^j}{2^j j!(k-2j)!} x^{k-2j}.$$

It is well known that if $\{(B_t)_{t\geq 0}, \mathbf{P}\}$ is a standard Brownian motion in $\mathbb{R}$, then for any $k \geq 0$, $\{t^{k/2} H_k(B_t/\sqrt{t}), \sigma(B_s : s \leq t), \mathbf{P}\}$ is a martingale. Since the spine of the branching Brownian motion is a Brownian motion, we can use the martingales above and the many-to-one formula in Lemma 2.1 to construct martingales for branching Brownian motions. This motivates the definition of the following martingales:

Now for $\mathbf{k} \in \mathbb{N}^d$ and $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, we define

$$M_t^{(\mathbf{k},\theta)} := e^{-(1+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot \mathbf{X}_u(t)} t^{|\mathbf{k}|/2} \prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_u(t))_j + \theta_j t}{\sqrt{t}}\Big), \quad t \geq 0.$$

Note that $M_t^{(\mathbf{0},\theta)} = W_t(\theta)$. We will prove in Proposition 2.6 that $\{M_t^{(\mathbf{k},\theta)}, t \geq 0; \mathbb{P}\}$ is a martingale, and if (1.11) holds for λ large enough, $M_t^{(\mathbf{k},\theta)}$ will converge almost surely and in L^1 to a limit $M_\infty^{(\mathbf{k},\theta)}$. Here are the main results of this paper:

Theorem 1.1 *Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any given $m \in \mathbb{N}$, if (1.11) holds for some $\lambda > \max\{3m+8, d(3m+5)\}$, then for any $\mathbf{b} \in \mathbb{R}^d$, $\mathbb{P}$-almost surely, as $s \to \infty$,*

$$\begin{aligned}\mu_s^\theta\big((-\infty, \mathbf{b}\sqrt{s}]\big) &= \sum_{\mathbf{k}:|\mathbf{k}|\leq m} \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{1}{s^{|\mathbf{k}|/2}} D^{\mathbf{k}}\Phi_d(\mathbf{b}) M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}) \\ &= \sum_{\ell=0}^{m} \frac{(-1)^\ell}{s^{\ell/2}} \sum_{\mathbf{k}:|\mathbf{k}|=\ell} \frac{D^{\mathbf{k}}\Phi_d(\mathbf{b})}{\mathbf{k}!} M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}).\end{aligned} \tag{1.12}$$

Theorem 1.2 *Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$. For any given $m \in \mathbb{N}$, if (1.11) holds for some $\lambda > \max\{d(3m+5), 3m+3d+8\}$, then for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, $\mathbb{P}$-almost surely, as $s \to \infty$,*

$$s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}])$$

$$= \sum_{\ell=0}^{m} \frac{1}{s^{\ell/2}} \sum_{j=0}^{\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0})}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 \ldots \mathrm{d}z_d$$
$$+ o(s^{-m/2}), \tag{1.13}$$

where $\mathbf{1} := (1, ..., 1)$.

Remark 1.3 Theorems 1.1 and 1.2 give asymptotic expansions, up to arbitrary order, of $\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right)$ and $s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}])$, respectively. They give much more precise information than (1.9) and (1.10). When $d = 1$, $\theta = 0$ and $m = 0$, Theorem 1.1 is exactly (1.9), and Theorem 1.2 is exactly (1.10).

Remark 1.4 Taking $\theta = 0$ and $s = n\delta$, $n \in \mathbb{N}$, with δ being a positive constant, in (1.12), we see that it is consistent with (1.6). The asymptotic expansion (1.13) is also consistent with (1.7). In fact, combining the definition of H_k and (2.5), we see that $\frac{\mathrm{d}^{2\ell}}{\mathrm{d}x^{2\ell}}\Phi(x)\big|_{x=0} = 0$ for all $\ell \geq 1$. Let $m = 2m'$ with $m' \geq 1$. Then for each odd number $\ell = 2\ell' + 1$ with $0 \leq \ell' \leq m' - 1$ and any $(\mathbf{k}, \mathbf{i})$ such that $|\mathbf{k} + \mathbf{i}| = \ell$, there must be some $i_0 \in \mathbb{N}$ such that $(\mathbf{k} + \mathbf{i})_{i_0}$ is odd, which implies that

$$\frac{\mathrm{d}^{(\mathbf{k}+\mathbf{i})_{i_0}+1}}{\mathrm{d}x^{(\mathbf{k}+\mathbf{i})_{i_0}+1}}\Phi(x)\big|_{x=0} = 0 \implies D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0}) = 0.$$

Therefore, for $m = 2m'$, (1.13) can be written as

$$s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}])$$
$$= \sum_{\ell=0}^{m'} \frac{1}{s^{\ell}} \sum_{j=0}^{2\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=2\ell-j} \frac{D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0})}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 \ldots \mathrm{d}z_d$$
$$+ o(s^{-m'}).$$

Remark 1.5 We briefly explain here how the martingale limits $M_\infty^{(\mathbf{k},\theta)}$ appear in Theorem 1.1. The reason for their appearance in Theorem 1.2 is similar. Using the branching property, one can show that, under (1.11), for a certain sequence $(r_n)_{n\geq 1}$ of positive reals with $r_n \uparrow \infty$ and $s \in [r_n, r_{n+1})$, $\mathbb{E}\left(\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \big| \mathcal{F}_{\sqrt{r_n}}\right)$ is a good approximation of $\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right)$, see Lemma 3.2. The quantity $\mathbb{E}\left(\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \big| \mathcal{F}_{\sqrt{r_n}}\right)$ can be written as a sum involving the normal distribution function Φ, see (3.8). Combining this with the Taylor expansions (involving Hermite polynomials) of the normal distribution in Lemmas 2.3 and 2.4, the martingales $M_t^{(\mathbf{k},\theta)}$ appear naturally.

Remark 1.6 Note that we only dealt with the case when the branching rate is 1 and the mean number of offspring is 2 in the two theorems above. In the general case when the branching rate is $\beta > 0$ and the mean number of offspring is $\mu > 1$, one can use the same argument to prove the following counterpart of Theorem 1.1: Suppose

$\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2\beta(\mu-1)}$. For any given $m \in \mathbb{N}$, if (1.11) holds for some $\lambda > \max\{3m+8, d(3m+5)\}$, then for any $\mathbf{b} \in \mathbb{R}^d$, $\mathbb{P}$-almost surely, as $s \to \infty$,

$$\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) := e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} 1_{(-\infty,\mathbf{b}\sqrt{s}]}(\mathbf{X}_u(t)+\theta t)$$
$$= \sum_{\ell=0}^m \frac{(-1)^\ell}{s^{\ell/2}} \sum_{\mathbf{k}:|\mathbf{k}|=\ell} \frac{D^{\mathbf{k}}\Phi_d(\mathbf{b})}{\mathbf{k}!} M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}),$$

with $M_\infty^{(\mathbf{k},\theta)}$ given by

$$M_\infty^{(\mathbf{k},\theta)} := \lim_{t\to\infty} e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} t^{|\mathbf{k}|/2} \prod_{j=1}^d H_{k_j}\Big(\frac{(\mathbf{X}_u(t))_j+\theta_j t}{\sqrt{t}}\Big). \tag{1.14}$$

In the general case, the counterpart of Theorem 1.2 is as follows: Suppose $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2\beta(\mu-1)}$. For any given $m \in \mathbb{N}$, if (1.11) holds for some $\lambda > \max\{d(3m+5), 3m+3d+8\}$, then for any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$, $\mathbb{P}$-almost surely, as $s \to \infty$,

$$s^{d/2}\mu_s^\theta((\mathbf{a},\mathbf{b}]) = e^{-(\beta(\mu-1)+\frac{\|\theta\|^2}{2})t} \sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)} 1_{(\mathbf{a},\mathbf{b}]}(\mathbf{X}_u(t)+\theta t)$$
$$= \sum_{\ell=0}^m \frac{1}{s^{\ell/2}} \sum_{j=0}^\ell (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{D^{\mathbf{k}+\mathbf{i}+\mathbf{1}}\Phi_d(\mathbf{0})}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^d z_j^{i_j} \mathrm{d}z_1 \ldots \mathrm{d}z_d$$
$$+ o(s^{-m/2}),$$

with $M_\infty^{(\mathbf{k},\theta)}$ given in (1.14).

Remark 1.7 One could also consider asymptotic expansions for the additive measure $\mu_n^{Z,\theta}$ for branching random walks. Using the tools established in [12], it is possible to get fixed order expansions. However, getting asymptotic expansions of arbitrary order may be difficult.

We end this section with a few words about the strategy of the proofs and the organization of the paper. In Sect. 2, we introduce the spine decomposition and gather some useful facts. We also study the convergence rate of the martingales $M_t^{(\mathbf{k},\theta)}$ and moments of the additive martingale $W_t(\theta)$. In Sect. 3, we prove Theorems 1.1 and 1.2. To prove Theorem 1.1, we choose a sequence of discrete-time $r_n = n^{1/\kappa}$ for some $\kappa > 1$. To control the behavior of particles alive in (r_n, r_{n+1}), we need $r_{n+1} - r_n \to 0$. This is the reason we do not choose $r_n = n\delta$. We prove in Lemma 3.1 that $\mu_{r_n}^\theta\left((-\infty, \mathbf{b}\sqrt{r_n}]\right) \approx \mathbb{E}\left[\mu_{r_n}^\theta\left((-\infty, \mathbf{b}\sqrt{r_n}]\right) \middle| \mathcal{F}_{\sqrt{r_n}}\right]$, where $\mathcal{F}_t$ is the σ-field generated by the branching Brownian motion up to time t. To deal with $s \in (r_n, r_{n+1})$, we adapt some ideas from [3, Lemma 8] and [17, paragraph below (13)]. We prove in Lemma 3.2 that, for $s \in (r_n, r_{n+1})$, $\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \approx \mathbb{E}\left[\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \big| \mathcal{F}_{\sqrt{r_n}}\right]$. We complete the proof of Theorem 1.1 by using a series of identities proved in [11]. The proof of Theorem 1.2 is similar.

2 Preliminaries

2.1 Spine Decomposition

Define

$$\left.\frac{\mathrm{d}\mathbb{P}^{-\theta}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} := W_t(\theta). \tag{2.1}$$

Then under $\mathbb{P}^{-\theta}$, the evolution of our branching Brownian motion can be described as follows (spine decomposition) (see [18] for the case $d = 1$ or see [20] for a more general case):

(i) There is an initial marked particle at $\mathbf{0} \in \mathbb{R}^d$ which moves according to the law of a standard Brownian motion $\{\mathbf{B}_t - \theta t, \mathbf{P_0}\}$;
(ii) The branching rate of this marked particle is 2;
(iii) When the marked particle dies at site $\mathbf{y}$, it gives birth to $\widehat{L}$ children with $\mathbb{P}^{-\theta}(\widehat{L} = k) = kp_k/2$;
(iv) One of these children is uniformly selected and marked, and the marked child evolves as its parent independently and the other children evolve independently with law $\mathbb{P}_{\mathbf{y}}$, where $\mathbb{P}_{\mathbf{y}}$ denotes the law of a branching Brownian motion starting at $\mathbf{y}$.

Let d_i be the i-th splitting time of the spine and O_i be the number of children produced by the spine at time d_i. According to the spine decomposition, it is easy to see that $\{d_i : i \geq 1\}$ are the atoms for a Poisson point process with rate 2, $\{O_i : i \geq 1\}$ are iid with common law $\widehat{L}$ given by $\mathbb{P}^{-\theta}(\widehat{L} = k) = kp_k/2$, and that $\{d_i : i \geq 1\}$ and $\{O_i : i \geq 1\}$ and $\mathbf{X}_\xi$ are independent. This fact will be used repeatedly.

We use ξ_t and $\mathbf{X}_\xi(t)$ to denote the marked particle at time t and the position of this marked particle, respectively. By [20, Theorem 2.11], we have that, for $u \in N(t)$,

$$\mathbb{P}^{-\theta}\left(\xi_t = u \big| \mathcal{F}_t\right) = \frac{e^{-\theta\cdot\mathbf{X}_u(t)}}{\sum_{u\in N(t)} e^{-\theta\cdot\mathbf{X}_u(t)}} = \frac{e^{-(1+\frac{\|\theta\|^2}{2})t}e^{-\theta\cdot\mathbf{X}_u(t)}}{W_t(\theta)}. \tag{2.2}$$

Using (2.2), we can get the following many-to-one formula.

Lemma 2.1 *For any $t > 0$ and $u \in N(t)$, let $\Gamma(u, t)$ be a nonnegative $\mathcal{F}_t$-measurable random variable. Then,*

$$\mathbb{E}\Big(\sum_{u\in N(t)} \Gamma(u, t)\Big) = e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(e^{\theta\cdot\mathbf{X}_\xi(t)}\Gamma(\xi_t, t)\right).$$

Proof Combining (2.1) and (2.2), we get

$$\mathbb{E}\Big(\sum_{u\in N(t)}\Gamma(u,t)\Big)=\mathbb{E}^{-\theta}\Big(\sum_{u\in N(t)}\frac{\Gamma(u,t)}{W_t(\theta)}\Big)$$
$$=\mathbb{E}^{-\theta}\Big(\sum_{u\in N(t)}\Gamma(u,t)e^{(1+\frac{\|\theta\|^2}{2})t}e^{\theta\cdot\mathbf{X}_u(t)}\mathbb{P}^{-\theta}\left(\xi_t=u\middle|\mathcal{F}_t\right)\Big)$$
$$=e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\Big(\mathbb{E}^{-\theta}\Big(\sum_{u\in N(t)}1_{\{\xi_t=u\}}\Gamma(u,t)e^{\theta\cdot\mathbf{X}_u(t)}\Big|\mathcal{F}_t\Big)\Big)$$
$$=e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\Big(\Gamma(\xi_t,t)e^{\theta\cdot\mathbf{X}_\xi(t)}\sum_{u\in N(t)}1_{\{\xi_t=u\}}\Big)=e^{(1+\frac{\|\theta\|^2}{2})t}\mathbb{E}^{-\theta}\left(e^{\theta\cdot\mathbf{X}_\xi(t)}\Gamma(\xi_t,t)\right).$$

□

2.2 Some Useful Facts

In this subsection, we gather some useful facts that will be used later.

Lemma 2.2 *(i) Let $\ell\in[1,2]$ be a fixed constant. Then for any finite family of independent centered random variables $\{X_i : i=1,\dots,n\}$ with $\mathrm{E}|X_i|^\ell<\infty$ for all $i=1,\dots,n$, it holds that*

$$\mathrm{E}\Big|\sum_{i=1}^n X_i\Big|^\ell\le 2\sum_{i=1}^n \mathrm{E}|X_i|^\ell.$$

(ii) For any $\ell\in[1,2]$ and any random variable X with $\mathrm{E}|X|^2<\infty$,

$$\mathrm{E}\,|X-\mathrm{E}X|^\ell\lesssim \mathrm{E}|X|^\ell\le(\mathrm{E}X^2)^{\ell/2}.$$

Proof For (i), see [24, Theorem 2]. (ii) follows easily from Jensen's inequality. □

Lemma 2.3 *For any $\rho\in(0,1)$, $b,x\in\mathbb{R}$, it holds that*

$$\Phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right)=\Phi(b)-\phi(b)\sum_{k=1}^\infty\frac{\rho^k}{k!}H_{k-1}(b)H_k(x).$$

Proof See [11, Lemma 4.2.]. □

To prove Theorem 1.1, we will define $r_n:=n^{\frac{1}{\kappa}}$ for some $\kappa>1$. For $s\in[r_n,r_{n+1})$, applying Lemma 2.3 with $\rho=\sqrt{\sqrt{r_n}/s}$ and $x=r_n^{-1/4}y$, we get that for any $b,y\in\mathbb{R}$,

$$\Phi\Big(\frac{b\sqrt{s}-y}{\sqrt{s-\sqrt{r_n}}}\Big)=\Phi(b)-\phi(b)\sum_{k=1}^\infty\frac{1}{k!}\frac{1}{s^{k/2}}H_{k-1}(b)r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big).\tag{2.3}$$

Recall that (see [11, (4.1)]) for any $k \geq 1$ and $x \in \mathbb{R}$,

$$|H_k(x)| \leq 2\sqrt{k!}e^{x^2/4}. \tag{2.4}$$

Lemma 2.4 *For a given $m \in \mathbb{N}$, let $\kappa = m + 3$ and $r_n = n^{1/\kappa}$. Let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + K\kappa$. For any $b, y \in \mathbb{R}$ and $s \in [r_n, r_{n+1})$, it holds that*

$$\Phi\Big(\frac{b\sqrt{s} - y}{\sqrt{s - \sqrt{r_n}}}\Big) = \Phi(b) - \phi(b) \sum_{k=1}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} H_{k-1}(b) r_n^{k/4} H_k\Big(\frac{y}{r_n^{1/4}}\Big) + \varepsilon_{m,y,b,s},$$

and that

$$\sup\Big\{s^{m/2}\left|\varepsilon_{m,y,b,s}\right| : \ s \in [r_n, r_{n+1}), \ |y| \leq \sqrt{K\sqrt{r_n}\log n}, b \in \mathbb{R}\Big\} \stackrel{n\to\infty}{\longrightarrow} 0.$$

Proof It follows from (2.4) that there exists a constant C such that for all $b \in \mathbb{R}, n \geq 2$, $s \in [r_n, r_{n+1})$, $|y| \leq \sqrt{K\sqrt{r_n}\log n}$ and $k \geq m$,

$$s^{m/2}\frac{1}{k!}\frac{1}{s^{k/2}}\left(\phi(b)\left|H_{k-1}(b)\right|\right)\left|r_n^{k/4} H_k\Big(\frac{y}{r_n^{1/4}}\Big)\right| \leq C\frac{n^{k/(4\kappa)}}{n^{(k-m)/(2\kappa)}} n^{K/4}.$$

Combining this with (2.3), we get that for $J > 2m + K\kappa$, $n \geq 2$, $s \in [r_n, r_{n+1})$, $b \in \mathbb{R}$ and $|y| \leq \sqrt{K\sqrt{r_n}\log n}$,

$$s^{m/2}\left|\varepsilon_{m,y,b,s}\right| \leq C\sum_{k=J+1}^{\infty} n^{-(k-2m-K\kappa)/(4\kappa)} \lesssim n^{-(J+1-2m-K\kappa)/(4\kappa)}.$$

Thus, the assertions of the lemma are valid. □

Now, we give a result of similar flavor which will be used to prove Theorem 1.2. Taking derivative with respect to b in Lemma 2.3, and using the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}b^k}\Phi(b) = (-1)^{k-1}H_{k-1}(b)\phi(b), \tag{2.5}$$

we get that

$$\frac{1}{\sqrt{1-\rho^2}}\phi\Big(\frac{b-\rho x}{\sqrt{1-\rho^2}}\Big) = \phi(b) + \phi(b)\sum_{k=1}^{\infty}\frac{\rho^k}{k!}H_k(b)H_k(x). \tag{2.6}$$

Now letting $\rho = \sqrt{\sqrt{r_n}/s}$, $b = z/\sqrt{s}$ and $x = r_n^{-1/4} y$ in (2.6), we get that for any $z, y \in \mathbb{R}$,

$$\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\phi\Big(\frac{z-y}{\sqrt{s-\sqrt{r_n}}}\Big) = \phi\Big(\frac{z}{\sqrt{s}}\Big)\Big(1+\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{s^{k/2}}H_k\Big(\frac{z}{\sqrt{s}}\Big)r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big).$$

The proof of the following result is similar to that of Lemma 2.4, and we omit the details.

Lemma 2.5 *For a given $m \in \mathbb{N}$, let $\kappa = m+3$ and $r_n = n^{1/\kappa}$. Let $K > 0$ be a fixed constant and J be an integer such that $J > 2m + K\kappa$. For any $a < b \in \mathbb{R}$, $y, z \in \mathbb{R}$ and $s \in [r_n, r_{n+1})$, it holds that*

$$\begin{aligned}&\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}\phi\Big(\frac{z-y}{\sqrt{s-\sqrt{r_n}}}\Big)\\ &= \phi\Big(\frac{z}{\sqrt{s}}\Big)\Big(1+\sum_{k=1}^{J}\frac{1}{k!}\frac{1}{s^{k/2}}H_k\Big(\frac{z}{\sqrt{s}}\Big)r_n^{k/4}H_k\Big(\frac{y}{r_n^{1/4}}\Big)\Big) + \varepsilon_{m,y,z,s},\end{aligned}$$

and that

$$\sup\Big\{s^{m/2}\left|\varepsilon_{m,y,z,s}\right| : \ s \in [r_n, r_{n+1}),\ z \in [a,b],\ |y| \le \sqrt{K\sqrt{r_n}\log n}\Big\} \overset{n\to\infty}{\longrightarrow} 0.$$

2.3 Convergence Rate for the Martingales

Proposition 2.6 *For any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$ and $\mathbf{k} \in \mathbb{N}^d$, $\{M_t^{(\mathbf{k},\theta)}, t \ge 0; \mathbb{P}\}$ is a martingale. If (1.11) holds for some $\lambda > |\mathbf{k}|/2$, then $M_t^{(\mathbf{k},\theta)}$ converges to a limit $M_\infty^{(\mathbf{k},\theta)}$ $\mathbb{P}$-a.s. and in L^1. Moreover, for any $\eta \in (0, \lambda - |\mathbf{k}|/2)$, as $t \to \infty$,*

$$M_t^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)} = o(t^{-(\lambda-|\mathbf{k}|/2)+\eta}), \quad \mathbb{P}\text{-a.s.}$$

Before presenting the proof, we first sketch the main idea of the proof. Both of the existence of the limit $M_\infty^{(\mathbf{k},\theta)}$ and the convergence rate of $M_t^{(\mathbf{k},\theta)} - M_\infty^{(\mathbf{k},\theta)}$ rely on the convergence of the series $\sum_{n=1}^{\infty} n^{-|\mathbf{k}|/2+\lambda-\eta}\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right)$. Thus, our main effort is to analyze the decay rate of $\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right)$ under (1.11). The assumption (1.11) does not guarantee the finiteness of ℓth moment of $|M_n^{(\mathbf{k},\theta)}|$ for any $\ell > 1$. We appropriately truncate the number of offspring of particles born between time n and time $n+1$. The sequence $M_n^{(\mathbf{k},\theta),B}$ will have ℓth absolute moment for some $\ell \in (1,2)$, and $M_n^{(\mathbf{k},\theta),B}$ is also a good approximation for the martingale $M_n^{(\mathbf{k},\theta)}$. We will get the desired result by combining the trivial inequalities (2.19) with the moment estimate for $|M_n^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta),B}|$ and $|M_n^{(\mathbf{k},\theta),B}|^\ell$.

Proof of Proposition 2.6 Combining Lemma 2.1, the Markov property and the branching property, for any $t, s > 0$,

$$\begin{aligned}
\mathbb{E}\left(M_{t+s}^{(\mathbf{k},\theta)}\big|\mathcal{F}_t\right) &= e^{-(1+\frac{\|\theta\|^2}{2})(t+s)}\sum_{u\in N(t)} \\
&\quad\times \mathbb{E}_{\mathbf{X}_u(t)}\left(\sum_{v\in N(s)} e^{-\theta\cdot\mathbf{X}_v(s)}(t+s)^{|\mathbf{k}|/2}\prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_v(s))_j+\theta_j(t+s)}{\sqrt{t+s}}\Big)\right) \\
&= e^{-(1+\frac{\|\theta\|^2}{2})t}\sum_{u\in N(t)}\mathbb{E}^{-\theta}_{\mathbf{X}_u(t)}\left((t+s)^{|\mathbf{k}|/2}\prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_\xi(s))_j+\theta_j(t+s)}{\sqrt{t+s}}\Big)\right) \\
&= e^{-(1+\frac{\|\theta\|^2}{2})t}\sum_{u\in N(t)}\prod_{j=1}^{d}\mathbf{E}_{(\mathbf{X}_u(t))_j+\theta_j t}\left((t+s)^{k_j/2}H_{k_j}\Big(\frac{B_s}{\sqrt{t+s}}\Big)\right), \qquad (2.7)
\end{aligned}$$

where recall that $(\mathbf{X}_\xi(s)+\theta s, \mathbb{P}^{-\theta})$ is a d-dimensional standard Brownian motion. Since

$$\begin{aligned}
&\mathbf{E}_{B_t}\left((t+s)^{k_j/2}H_{k_j}\Big(\frac{B_s}{\sqrt{t+s}}\Big)\right) = \mathbf{E}\left((t+s)^{k_j/2}H_{k_j}\Big(\frac{B_{t+s}}{\sqrt{t+s}}\Big)\big|\sigma(B_r, r\le t)\right) \\
&= t^{k_j/2}H_{k_j}\left(\frac{B_t}{\sqrt{t}}\right),
\end{aligned}$$

we see that $\mathbf{E}_x\left((t+s)^{k_j/2}H_{k_j}\Big(\frac{B_s}{\sqrt{t+s}}\Big)\right) = t^{k_j/2}H_{k_j}\left(\frac{x}{\sqrt{t}}\right)$ for all $x\in\mathbb{R}$, $t, s > 0$ and $k_j\in\mathbb{N}$. Plugging this fact back to (2.7), we obtain that

$$\mathbb{E}\left(M_{t+s}^{(\mathbf{k},\theta)}\big|\mathcal{F}_t\right) = e^{-(1+\frac{\|\theta\|^2}{2})t}\sum_{u\in N(t)}\prod_{j=1}^{d} t^{k_j/2}H_{k_j}\left(\frac{(\mathbf{X}_u(t))_j+\theta_j t}{\sqrt{t}}\right) = M_t^{(\mathbf{k},\theta)},$$

which implies that $M_t^{(\mathbf{k},\theta)}$ is a martingale.

Now, we fix $\mathbf{k}\in\mathbb{N}^d$ and assume (1.11) holds for some $\lambda > |\mathbf{k}|/2$. We first look at the case when $t\to\infty$ along integers. Let $t = n\in\mathbb{N}$. Recall that $N(n+1)$ is the set of particles alive at time $n+1$. For $u\in N(n+1)$, define $B_{n,u}$ to be the event that, for all $v < u$ with $d_v\in(n, n+1)$, it holds that $O_v\le e^{c_0 n}$, where $c_0 > 0$ is a small constant to be determined later. Set

$$\begin{aligned}
M_{n+1}^{(\mathbf{k},\theta),B} &:= e^{-(1+\frac{\|\theta\|^2}{2})(n+1)} \\
&\quad\times\sum_{u\in N(n+1)} e^{-\theta\cdot\mathbf{X}_u(n+1)}(n+1)^{|\mathbf{k}|/2}\prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j+\theta_j(n+1)}{\sqrt{n+1}}\Big)1_{B_{n,u}}.
\end{aligned}$$

Since $|H_k(x)| \lesssim |x|^k + 1$ for all $x \in \mathbb{R}$ and $(|x|+|y|)^k \lesssim |x|^k + |y|^k$ for all $x, y \in \mathbb{R}$, we have

$$(n+1)^{k/2}\Big|H_k\Big(\frac{x+z}{\sqrt{n+1}}\Big)\Big| \lesssim (|x|+|z|)^k + (n+1)^{k/2} \lesssim |x|^k + |z|^k + n^{k/2},$$

which implies that for all $j \in \{1, ..., d\}$,

$$\begin{aligned}&(n+1)^{k_j/2}\Big|H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j + \theta_j(n+1)}{\sqrt{n+1}}\Big)\Big| \\ &\lesssim |(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2} + |(\mathbf{X}_u(n+1))_j - (\mathbf{X}_u(n))_j + \theta_j|^{k_j}. \qquad (2.8)\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right| \leq e^{-(1+\frac{\|\theta\|^2}{2})(n+1)} \\ &\times \sum_{u\in N(n+1)} e^{-\theta\cdot\mathbf{X}_u(n+1)}\Big|(n+1)^{|\mathbf{k}|/2}\prod_{j=1}^d H_{k_j}\Big(\frac{(\mathbf{X}_u(n+1))_j + \theta_j(n+1)}{\sqrt{n+1}}\Big)\Big| 1_{(B_{n,u})^c} \\ &\lesssim e^{-(1+\frac{\|\theta\|^2}{2})(n+1)} \sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)} \sum_{v\in N(n+1):u\leq v} e^{-\theta\cdot(\mathbf{X}_v(n+1)-\mathbf{X}_u(n))} \\ &\times \prod_{j=1}^d \Big(|(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2} + |(\mathbf{X}_v(n+1))_j - (\mathbf{X}_u(n))_j + \theta_j|^{k_j}\Big) 1_{(B_{n,v})^c}.\end{aligned}$$

By the branching property and the Markov property, we get that

$$\begin{aligned}&\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right| \big|\mathcal{F}_n\right) \lesssim e^{-(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)} e^{-(1+\frac{\|\theta\|^2}{2})} \\ &\times \mathbb{E}\Big(\sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} \prod_{j=1}^d \Big(\big|(\mathbf{X}_v(1))_j + \theta_j\big|^{k_j} + y_j\Big) 1_{(D_{n,v})^c}\Big)\Big|_{y_j = |(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2}} \\ &=: e^{-(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)} F(\mathbf{y})\Big|_{y_j = |(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2}}, \qquad (2.9)\end{aligned}$$

where, for $v \in N(1)$, $D_{n,v}$ denotes the event that, for all $w < v$, it holds that $O_w \leq e^{c_0 n}$. Recall that d_i is the i-th splitting time of the spine and O_i is the number of children produced by the spine at time d_i. Note that D_{n,ξ_1} is the event that, for all i with $d_i < 1$, it holds that $O_i \leq e^{c_0 n}$. By Lemma 2.1,

$$F(\mathbf{y}) = \mathbb{E}^{-\theta}\Big(\prod_{j=1}^d \Big(\big|(\mathbf{X}_\xi(1))_j + \theta_j\big|^{k_j} + y_j\Big) 1_{(D_{n,\xi_1})^c}\Big).$$

Using the independence of $\{d_i : i \geq 1\}$, $\{O_i : i \geq 1\}$ and $\mathbf{X}_\xi$, we have that D_{n,ξ_1} is independent of $\mathbf{X}_\xi$, which implies that for $y_j \geq 1$,

$$\begin{aligned} F(\mathbf{y}) &= \prod_{j=1}^{d} \left(\mathbf{E}(|B_1|^{k_j}) + y_j\right) \mathbb{P}^{-\theta}\left(D^c_{n,\xi_1}\right) \\ &\leq \prod_{j=1}^{d} \left(\mathbf{E}(|B_1|^{k_j}) + y_j\right) \mathbb{E}^{-\theta}\Big(\sum_{i:d_i \leq 1} 1_{\{O_i > e^{c_0 n}\}}\Big) \\ &\lesssim \Big(\prod_{j=1}^{d} y_j\Big)\mathbb{P}^{-\theta}(\widehat{L} > e^{c_0 n}) \lesssim \frac{\prod_{j=1}^{d} y_j}{n^{1+\lambda}}, \end{aligned} \tag{2.10}$$

where in the first equality, we also used the fact that $\{\mathbf{X}_\xi(t) + \theta t, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, and in the last inequality, we used $\mathbb{E}^{-\theta}(\log_+^{1+\lambda} \widehat{L}) < \infty$ (which follows from (1.11)) and Markov's inequality. By (2.9) and (2.10), we have

$$\begin{aligned} &\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right| \middle| \mathcal{F}_n\right) \\ &\lesssim e^{-(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\theta\cdot \mathbf{X}_u(n)} \frac{\prod_{j=1}^{d}\left(|(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2}\right)}{n^{1+\lambda}}. \end{aligned}$$

Taking expectation with respect to $\mathbb{P}$, by Lemma 2.1, we obtain that

$$\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right|\right) \lesssim \frac{1}{n^{1+\lambda}} \prod_{j=1}^{d} \mathbf{E}\left(|B_n|^{k_j} + n^{k_j/2}\right) \lesssim \frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}. \tag{2.11}$$

On the other hand, by the branching property,

$$M_{n+1}^{(\mathbf{k},\theta),B} = e^{-(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\theta\cdot\mathbf{X}_u(n)} J_{n,u},$$

where

$$\begin{aligned} J_{n,u} := e^{-(1+\frac{\|\theta\|^2}{2})} &\sum_{v\in N(n+1):u\leq v} e^{-\theta\cdot(\mathbf{X}_v(n+1)-\mathbf{X}_u(n))} \\ &\times (n+1)^{|\mathbf{k}|/2} \prod_{j=1}^{d} H_{k_j}\Big(\frac{(\mathbf{X}_v(n+1))_j + \theta_j(n+1)}{\sqrt{n+1}}\Big) 1_{B_{n,v}} \end{aligned}$$

are independent given $\mathcal{F}_n$. For any fixed $1 < \ell < \min\{2/\|\theta\|^2, 2\}$, applying Lemma 2.2 (i) to the finite family $\{J_{n,u} - \mathbb{E}(J_{n,u}|\mathcal{F}_n) : u \in N(n)\}$ and Lemma 2.2

(ii) to $J_{n,u}$, together with (2.8), we get that

$$\mathbb{E}\Big(\Big|M_{n+1}^{(\mathbf{k},\theta),B}-\mathbb{E}\Big(M_{n+1}^{(\mathbf{k},\theta),B}\big|\mathcal{F}_n\Big)\Big|^{\ell}\big|\mathcal{F}_n\Big)$$
$$\lesssim e^{-\ell(1+\frac{\|\theta\|^2}{2})n}\sum_{u\in N(n)}e^{-\ell\theta\cdot\mathbf{X}_u(n)}\left(M_{n,u}\right)^{\ell/2}, \tag{2.12}$$

where $M_{n,u}$ is given by

$$M_{n,u}:=\mathbb{E}\Big(e^{-2(1+\frac{\|\theta\|^2}{2})}\Big(\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)1_{D_{n,v}}\Big)^2\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}$$

with $S_v(\mathbf{y},r):=\prod_{j=1}^d\Big(\big|(\mathbf{X}_v(r))_j+\theta_j r\big|^{k_j}+y_j\Big)$. Set

$$T_{n,w}:=S_w(\mathbf{y},1)1_{D_{n,w}}e^{-(1+\frac{\|\theta\|^2}{2})}\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)1_{D_{n,v}}.$$

By Lemma 2.1, we have

$$\begin{aligned}
M_{n,u}&=e^{-(1+\frac{\|\theta\|^2}{2})}\mathbb{E}\Big(\sum_{w\in N(1)}e^{-\theta\cdot\mathbf{X}_w(1)}T_{n,w}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}\\
&=\mathbb{E}^{-\theta}\left(T_{n,\xi_1}\right)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}\\
&=\mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}}S_{\xi_1}(\mathbf{y},1)e^{-(1+\frac{\|\theta\|^2}{2})}\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)1_{D_{n,v}}\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}\\
&\le\mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}}S_{\xi_1}(\mathbf{y},1)e^{-(1+\frac{\|\theta\|^2}{2})}\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)\Big)\Big|_{y_j=|(\mathbf{X}_u(n))_j+\theta_j n|^{k_j}+n^{k_j/2}}.
\end{aligned}\tag{2.13}$$

For each i with $d_i<1$, we use $N_i(1-d_i)$ to denote the set of particles whose most recent common ancestor is $\xi_{d_{i-1}}$. By spine decomposition, we have that

$$\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)=e^{-\theta\cdot\mathbf{X}_\xi(1)}S_{\xi_1}(\mathbf{y},1)+\sum_{i:d_i<1}\sum_{v\in N_i(1-d_i)\setminus\{\xi_1\}}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1).$$

By definition, we have $O_i\le e^{c_0 n}$ on the event D_{n,ξ_1}. Note that $\mathbf{E}\Big(\prod_{j=1}^d\Big(\big|(\mathbf{B}_s)_j+z_j\big|^{k_j}+y_j\Big)\Big)\lesssim\prod_{j=1}^d(|z_j|^{k_j}+y_j)$ for all $s\in(0,1)$, $z_j\in\mathbb{R}$ and $y_j\ge 1$. Using these and the branching property, we get

$$e^{-(1+\frac{\|\theta\|^2}{2})}\mathbb{E}^{-\theta}\Big(\sum_{v\in N(1)}e^{-\theta\cdot\mathbf{X}_v(1)}S_v(\mathbf{y},1)\big|\mathcal{G}\Big)$$

$$
\begin{aligned}
&= e^{-(1+\frac{\|\theta\|^2}{2})} e^{-\theta\cdot\mathbf{X}_\xi(1)} S_{\xi_1}(\mathbf{y},1) + \sum_{i:d_i<1} (O_i-1) e^{-(1+\frac{\|\theta\|^2}{2})d_i} e^{-\theta\cdot\mathbf{X}_\xi(d_i)} \\
&\quad \times \mathbf{E}\Big(\prod_{j=1}^d \Big(\big|(\mathbf{B}_{1-d_i})_j + z_j\big|^{k_j} + y_j \Big) \Big)\Big|_{z_j=(\mathbf{X}_\xi(d_i))_j+\theta_j d_i} \\
&\lesssim e^{-\theta\cdot\mathbf{X}_\xi(1)} S_{\xi_1}(\mathbf{y},1) + e^{c_0 n} \sum_{i:d_i<1} e^{-\theta\cdot\mathbf{X}_\xi(d_i)} \prod_{j=1}^d \Big(y_j + \big|(\mathbf{X}_\xi(d_i))_j + \theta_j d_i\big|^{k_j} \Big) \\
&= e^{-\theta\cdot\mathbf{X}_\xi(1)} S_{\xi_1}(\mathbf{y},1) + e^{c_0 n} \sum_{i:d_i<1} e^{-\theta\cdot\mathbf{X}_\xi(d_i)} S_{\xi_{d_i}}(\mathbf{y},d_i), \qquad (2.14)
\end{aligned}
$$

where $\mathcal{G}$ is the σ-field generated by all information along the spine, and in the first equality we also used Lemma 2.1. Plugging (2.14) into (2.13), noting that $\mathbf{X}_\xi$ and d_i are independent, we get that

$$
\begin{aligned}
&\mathbb{E}^{-\theta}\Big(1_{D_{n,\xi_1}} S_{\xi_1}(\mathbf{y},1) e^{-(1+\frac{\|\theta\|^2}{2})} \sum_{v\in N(1)} e^{-\theta\cdot\mathbf{X}_v(1)} S_v(\mathbf{y},1) \Big| \mathbf{X}_\xi \Big) \\
&\lesssim e^{\theta\cdot\mathbf{X}_\xi(1)} S_{\xi_1}(\mathbf{y},1)^2 + S_{\xi_1}(\mathbf{y},1) e^{c_0 n} \mathbb{E}^{-\theta}\Big(\sum_{i:d_i<1} e^{-\theta\cdot\mathbf{X}_\xi(d_i)} S_{\xi_{d_i}}(\mathbf{y},d_i) \Big| \mathbf{X}_\xi \Big) \\
&= e^{-\theta\cdot\mathbf{X}_\xi(1)} S_{\xi_1}(\mathbf{y},1)^2 + 2 S_{\xi_1}(\mathbf{y},1) e^{c_0 n} \int_0^1 e^{-\theta\cdot\mathbf{X}_\xi(s)} S_{\xi_s}(\mathbf{y},s)\mathrm{d}s \\
&\lesssim e^{c_0 n} \prod_{j=1}^d \left\{ \Big(y_j + \sup_{s<1} \big|(\mathbf{X}_\xi(s))_j + \theta_j s\big|^{k_j} \Big)^2 e^{\|\theta\| \sup_{s<1} |(\mathbf{X}_\xi(s))_j + \theta_j s|} \right\}. \qquad (2.15)
\end{aligned}
$$

Since $\{\mathbf{X}_\xi(s) + \theta s, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, combining with (2.13) and (2.15), we conclude that

$$
\begin{aligned}
M_{n,u} &\lesssim e^{c_0 n} \prod_{j=1}^d \mathbf{E}\Big(\big(y_j + \sup_{s<1} |(\mathbf{B}_s)_j|^{k_j} \big)^2 e^{\|\theta\| \sup_{s<1} |(\mathbf{B}_s)_j|} \Big)\Big|_{y_j = |(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2}} \\
&\lesssim \prod_{j=1}^d \Big(|(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2} \Big)^2 e^{c_0 n}. \qquad (2.16)
\end{aligned}
$$

Plugging (2.16) into (2.12), we conclude that

$$
\begin{aligned}
&\mathbb{E}\Big(\Big| M_{n+1}^{(\mathbf{k},\theta),B} - \mathbb{E}\big(M_{n+1}^{(\mathbf{k},\theta),B} \big| \mathcal{F}_n \big) \Big|^\ell \Big| \mathcal{F}_n \Big) \\
&\lesssim e^{c_0 \ell n/2} e^{-\ell(1+\frac{\|\theta\|^2}{2})n} \sum_{u\in N(n)} e^{-\ell\theta\cdot\mathbf{X}_u(n)} \prod_{j=1}^d \Big(|(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2} \Big)^\ell \\
&= e^{-\big((\ell-1)(1-\|\theta\|^2\ell/2) - c_0\ell/2\big)n} e^{-(1+\ell^2\|\theta\|^2/2)n}
\end{aligned}
$$

$$\times \sum_{u\in N(n)} e^{-\ell\theta\cdot \mathbf{X}_u(n)} \prod_{j=1}^{d} \left(|(\mathbf{X}_u(n))_j + \theta_j n|^{k_j} + n^{k_j/2}\right)^{\ell}. \tag{2.17}$$

Choose $c_0 > 0$ small so that $c_0\ell/2 < (\ell-1)(1-\|\theta\|^2\ell/2)$ and set $c_1 := (\ell-1)(1-\|\theta\|^2\ell/2) - c_0\ell/2 > 0$. Taking expectation with respect to $\mathbb{P}$ in (2.17), by Lemma 2.1 with θ replaced to $\ell\theta$, we get that

$$\begin{aligned} &\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta),B} - \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta),B}\middle|\mathcal{F}_n\right)\right|^{\ell}\right) \\ &\lesssim e^{-c_1 n} \prod_{j=1}^{d} \mathbf{E}\left(|(\mathbf{B}_n)_j - (\ell-1)\theta_j n|^{k_j} + n^{k_j/2}\right)^{\ell} \lesssim \left(\frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}\right)^{\ell}. \end{aligned} \tag{2.18}$$

Note that if X is a random variable with finite expectation and Y is a random variable with finite ℓ-th moment, then

$$\begin{aligned} &\mathrm{E}\left(|X - \mathrm{E}(X|\mathcal{F})|\right) \le \mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y - \mathrm{E}(Y|\mathcal{F})|\right) + \mathrm{E}\left(\left|\mathrm{E}\left(X-Y\middle|\mathcal{F}\right)\right|\right) \\ &\le 2\mathrm{E}\left(|X-Y|\right) + \mathrm{E}\left(|Y-\mathrm{E}(Y|\mathcal{F})|^{\ell}\right)^{1/\ell}. \end{aligned} \tag{2.19}$$

Combining this with (2.11), (2.18) and the fact that $M_n^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta)}\middle|\mathcal{F}_n\right)$, we get that

$$\begin{aligned} \mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) &\le 2\mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_{n+1}^{(\mathbf{k},\theta),B}\right|\right) \\ &\quad + \mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta),B} - \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta),B}\middle|\mathcal{F}_n\right)\right|^{\ell}\right)^{1/\ell} \lesssim \frac{n^{|\mathbf{k}|/2}}{n^{1+\lambda}}. \end{aligned} \tag{2.20}$$

Since $\lambda > |\mathbf{k}|/2$, we have $\sum_{n=1}^{\infty} \mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) < \infty$, which implies that $M_n^{(\mathbf{k},\theta)}$ converges to a limit $M_\infty^{(\mathbf{k},\theta)}$ $\mathbb{P}$-almost surely and in L^1. Therefore, $M_n^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_\infty^{(\mathbf{k},\theta)}\middle|\mathcal{F}_n\right), n \ge 1$.

For $s \in (n, n+1)$, $M_s^{(\mathbf{k},\theta)} = \mathbb{E}\left(M_{n+1}^{(\mathbf{k},\theta)}\middle|\mathcal{F}_s\right) = \mathbb{E}\left(M_\infty^{(\mathbf{k},\theta)}\middle|\mathcal{F}_s\right)$, thus the second assertion of the proposition is valid.

Now, we prove the last assertion of the proposition. For any $\eta \in (0, \lambda - |\mathbf{k}|/2)$, by (2.20),

$$\sum_{n=1}^{\infty} n^{-|\mathbf{k}|/2+\lambda-\eta} \mathbb{E}\left(\left|M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right|\right) \lesssim \sum_{n=1}^{\infty} \frac{1}{n^{1+\eta}} < \infty,$$

which implies that

$$\sum_{n=1}^{\infty} n^{\lambda-|\mathbf{k}|/2-\eta}\left(M_{n+1}^{(\mathbf{k},\theta)} - M_n^{(\mathbf{k},\theta)}\right) \text{ converges a.s.}$$

Thus, $n^{\lambda-|\mathbf{k}|/2-\eta}\left(M_n^{(\mathbf{k},\theta)}-M_\infty^{(\mathbf{k},\theta)}\right)\stackrel{n\to\infty}{\longrightarrow}0$, $\mathbb{P}$-a.s. (see, for example, [1, Lemma 2]). For $s\in[n,n+1]$, by Doob's inequality, for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty}\mathbb{P}\Big(n^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)}-M_n^{(\mathbf{k},\theta)}\right|>\varepsilon\Big)$$
$$\le\frac{1}{\varepsilon}\sum_{n=1}^{\infty}n^{-|\mathbf{k}|/2+\lambda-\eta}\mathbb{E}\Big(\left|M_{n+1}^{(\mathbf{k},\theta)}-M_n^{(\mathbf{k},\theta)}\right|\Big)<\infty.$$

Therefore, $n^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)}-M_n^{(\mathbf{k},\theta)}\right|\stackrel{n\to\infty}{\longrightarrow}0$, $\mathbb{P}$-a.s. Hence, $\mathbb{P}$-almost surely,

$$\sup_{n\le s\le n+1}s^{-|\mathbf{k}|/2+\lambda-\eta}\left|M_s^{(\mathbf{k},\theta)}-M_\infty^{(\mathbf{k},\theta)}\right|$$
$$\le(n+1)^{-|\mathbf{k}|/2+\lambda-\eta}\sup_{n\le s\le n+1}\left|M_s^{(\mathbf{k},\theta)}-M_n^{(\mathbf{k},\theta)}\right|+(n+1)^{-|\mathbf{k}|/2+\lambda-\eta}\left|M_n^{(\mathbf{k},\theta)}-M_\infty^{(\mathbf{k},\theta)}\right|$$
$$\stackrel{n\to\infty}{\longrightarrow}0,$$

which completes the proof of the last assertion of the proposition.

2.4 Moment Estimate for the Additive Martingale

In this subsection, we give an upper bound for $W_t(\theta)$ which will be used later.

Lemma 2.7 *Suppose $\theta\in\mathbb{R}^d$ with $\|\theta\|<\sqrt{2}$. If (1.11) holds for some $\lambda>0$, then there exists a constant $C_{\theta,\lambda}$ such that for all $t>0$,*

$$\mathbb{E}\left((W_t(\theta)+1)\log^{1+\lambda}(W_t(\theta)+1)\right)\le C_{\theta,\lambda}(t+1).$$

Proof Since $\mathbb{E}W_t(\theta)=1$, it suffices to prove that there exists a constant $C_{\theta,\lambda}$ such that for all $t>0$, $\mathbb{E}\left(W_t(\theta)\log_+^{1+\lambda}(W_t(\theta))\right)\le C_{\theta,\lambda}(t+1)$. By using a projection argument, we can easily reduce to the one dimensional case. Indeed, for each $t>0$ and $u\in N(t)$, define $Y_u(t):=\theta\cdot\mathbf{X}_u(t)/\|\theta\|$ when $\theta\neq\mathbf{0}$ and for $Y_u(t)=(\mathbf{X}_u(t))_1$ when $\theta=\mathbf{0}$, then $\{Y_u(t):u\in N(t)\}_{t\ge 0}$ is a 1-dimensional branching Brownian motion with branching rate $\beta=1$ and offspring distribution $\{p_k\}$. Moreover, for each $\theta\in\mathbb{R}^d$, $W_t(\theta)=e^{-(1+\frac{\|\theta\|^2}{2})t}\sum_{u\in N(t)}e^{-\|\theta\|Y_u(t)}$. So we will only deal with the case $d=1$. By (2.1), we have $\mathbb{E}\left((W_t(\theta))\log_+^{1+\lambda}(W_t(\theta))\right)=\mathbb{E}^{-\theta}\left(\log_+^{1+\lambda}(W_t(\theta))\right)$. Using the spine decomposition, we have

$$W_t(\theta)=e^{-(1+\frac{\theta^2}{2})t}e^{-\theta X_\xi(t)}+\sum_{i:d_i\le t}e^{-(1+\frac{\theta^2}{2})d_i}e^{-\theta X_\xi(d_i)}\sum_{j=1}^{O_i-1}W_{t-d_i}^{i,j},$$

where d_i, O_i are the i-th fission time and the number of offspring of the spine at time d_i, respectively. Given all the information $\mathcal{G}$ about the spine, $(W^{i,j}_{t-d_i})_{j\geq 1}$ are independent with the same law as $W_{t-d_i}(\theta)$ under $\mathbb{P}$.

Using elementary analysis, one can easily show that there exists $A = A_\lambda > 1$ such that for any $x, y > A$,

$$\log_+^{1+\lambda}(x+y) \leq \log_+^{1+\lambda}(x) + \log_+^{1+\lambda}(y). \tag{2.21}$$

We set

$$\begin{aligned}
K_1 &:= e^{-(1+\frac{\theta^2}{2})t} e^{-\theta X_\xi(t)},\\
K_2 &:= \sum_{i:d_i\leq t} e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)} \sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i} 1_{\left\{e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)} \sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i} \leq A\right\}}\\
&\leq A \sum_{i:d_i\leq t} 1,\\
K_3 &:= \sum_{i:d_i\leq t} e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)} \sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i} 1_{\left\{e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)} \sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i} > A\right\}}.
\end{aligned}$$

Note that $\log_+^{1+\lambda}(x+y+z) \leq \log_+^{1+\lambda}(3x) + \log_+^{1+\lambda}(3y) + \log_+^{1+\lambda}(3z)$, $\log_+^{1+\lambda}(xy) \leq (\log_+ x + \log_+ y)^{1+\lambda} \lesssim \log_+^{1+\lambda}(x) + \log_+^{1+\lambda}(y)$ and $\log_+^{1+\lambda}(x) \lesssim x$. By (2.21), we have

$$\begin{aligned}
&\log_+^{1+\lambda}(W_t(\theta)) = \log_+^{1+\lambda}(K_1+K_2+K_3)\\
&\leq \log_+^{1+\lambda}(3K_1) + \log_+^{1+\lambda}(3K_2) + \log_+^{1+\lambda}(3K_3)\\
&\lesssim 1 + \log_+^{1+\lambda}(K_1) + \Big(\sum_{i:d_i\leq t} 1\Big) + \sum_{i:d_i\leq t} \log_+^{1+\lambda}\Big(e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)} \sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i}\Big)\\
&\lesssim 1 + \log_+^{1+\lambda}(K_1) + \Big(\sum_{i:d_i\leq t} 1 + \log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)}\big)\Big)\\
&\quad + \sum_{i:d_i\leq t} \log_+^{1+\lambda}\Big(\sum_{j=1}^{O_i-1} W^{i,j}_{t-d_i}\Big).
\end{aligned} \tag{2.22}$$

Put $\gamma := 1-\theta^2/2 > 0$. Recalling that $\{X_\xi(t)+\theta t, \mathbb{P}^{-\theta}\}$ is a standard Brownian motion and $\{d_i : i\geq 1\}$ are the atoms of a Poisson point process with rate 2 independent of X_ξ, we have

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}(K_1) + \Big(\sum_{i:d_i\leq t} 1 + \log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})d_i} e^{-\theta X_\xi(d_i)}\big)\Big)\Big)\\
&= \mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}(K_1) + 2\int_0^t \big(1 + \log_+^{1+\lambda}\big(e^{-(1+\frac{\theta^2}{2})s} e^{-\theta X_\xi(s)}\big)\mathrm{d}s\big)\Big)
\end{aligned}$$

$$\lesssim \mathbf{E}\Big((-\theta B_t - \gamma t)_+^{1+\lambda} + \int_0^t \big(1 + (-\theta B_s - \gamma s)_+^{1+\lambda}\big)\mathrm{d}s\Big) \lesssim t + 1, \tag{2.23}$$

where the last inequality follows from the following estimate:

$$\begin{aligned}
&\mathbf{E}\big((-\theta B_s - \gamma s)_+^{1+\lambda}\big) = s^{(1+\lambda)/2}\mathbf{E}\Big(\big(|\theta| B_1 - \gamma\sqrt{s}\big)^{1+\lambda} 1_{\{|\theta|B_1 > \gamma\sqrt{s}\}}\Big) \\
&\le s^{(1+\lambda)/2} e^{-\gamma\sqrt{s}}\mathbf{E}\big((|\theta||B_1|)^{1+\lambda} e^{|\theta|B_1}\big) \lesssim 1.
\end{aligned}$$

For the last term on the right-hand side of (2.22), conditioned on $\{d_i, O_i : i \ge 1\}$, we get

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\le t} \log_+^{1+\lambda}\big(\sum_{j=1}^{O_i-1} W_{t-d_i}^{i,j}\big)\big| d_i, O_i, i \ge 1\Big) \\
&\lesssim \sum_{i:d_i\le t} \log_+^{1+\lambda}(O_i - 1) + \sum_{i:d_i\le t} \mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\le O_i-1} W_{t-d_i}^{i,j}\big)\big| d_i, O_i, i \ge 1\Big).
\end{aligned} \tag{2.24}$$

Note that

$$\begin{aligned}
&\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\le O_i-1} W_{t-d_i}^{i,j}\big)\big| d_i, O_i, i \ge 1\Big) \\
&= (1+\lambda)\int_0^\infty y^\lambda \mathbb{P}^{-\theta}\Big(\max_{j\le O_i-1} W_{t-d_i}^{i,j} > e^y \Big| d_i, O_i, i \ge 1\Big)\mathrm{d}y \\
&= (1+\lambda)\int_0^\infty y^\lambda \Big(1 - \prod_{j=1}^{O_i-1} \big(1 - \mathbb{P}^{-\theta}\big(W_{t-d_i}^{i,j} > e^y \big| d_i, O_i, i \ge 1\big)\big)\Big)\mathrm{d}y \\
&\lesssim \int_0^\infty y^\lambda \big(1 - \big(1 - e^{-y}\big)^{O_i-1}\big)\mathrm{d}y,
\end{aligned} \tag{2.25}$$

where in the inequality we used Markov's inequality. When $O_i - 1 \ge e^{y/2}$ (which is equivalent to $y \le 2\log(O_i - 1)$), we have $y^\lambda \Big(1 - \big(1 - e^{-y}\big)^{O_i-1}\Big) \lesssim \log^\lambda(O_i - 1)$; when $O_i - 1 < e^{y/2}$, by the inequality $(1-x)^n \ge 1 - nx$, we get

$$y^\lambda\big(1 - \big(1 - e^{-y}\big)^{O_i-1}\big) \le y^\lambda(O_i - 1)e^{-y} \le y^\lambda e^{-y/2}.$$

Thus, by (2.25),

$$\mathbb{E}^{-\theta}\Big(\log_+^{1+\lambda}\big(\max_{j\le O_i-1} W_{t-d_i}^{i,j}\big)\big| d_i, O_i, i \ge 1\Big) \lesssim \log_+^{1+\lambda}(O_i - 1) + 1.$$

Note that condition (1.11) implies that $\mathbb{E}^{-\theta}\log_+^{1+\lambda}(O_i - 1) = \frac{1}{2}\sum_{k=2}^\infty k\log^{1+\lambda}(k - 1)p_k < \infty$. Plugging this back to (2.24) and taking expectation with respect to $\mathbb{P}^{-\theta}$, we conclude that

$$\begin{aligned}&\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\le t}\log_+^{1+\lambda}\big(\sum_{j=1}^{O_i-1}W^{i,j}_{t-d_i}\big)\Big)\\&\lesssim\mathbb{E}^{-\theta}\Big(\sum_{i:d_i\le t}\big(\log_+^{1+\lambda}(O_i-1)+1\big)\Big)\lesssim t+1.\end{aligned}\tag{2.26}$$

Combining (2.22), (2.23) and (2.26), we get the desired result. □

3 Proof of the Main Results

Let $\kappa > 1$ be fixed. Define

$$r_n := n^{\frac{1}{\kappa}},\quad n\in\mathbb{N}.$$

Lemma 3.1 *For any given $\alpha,\beta\ge 0$ and $\delta\in(0,1]$, assume that (1.11) holds for λ with $\lambda\delta-\alpha>\kappa(1+\beta)$.*

(i) For each n, let $a_n\le n^\beta$ and $\{Y_{n,u}: u\in N(r_n^\delta)\}$ be a family of random variables such that $\mathbb{E}\left(Y_{n,u}\middle|\mathcal{F}_{r_n^\delta}\right)=0$, and conditioned on $\mathcal{F}_{r_n^\delta}$, $Y_{n,u}$, $u\in N(r_n^\delta)$, are independent. If $|Y_{n,u}|\le W_{a_n}(\theta;u)+1$ for all n and $u\in N(r_n^\delta)$, with $\left(W_{a_n}(\theta;u),\mathbb{P}(\cdot|\mathcal{F}_{r_n^\delta})\right)$ being a copy of $W_{a_n}(\theta)$, then

$$r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u}\overset{n\to\infty}{\longrightarrow}0,\quad a.s.$$

(ii) Consequently, if $\lambda\delta-\alpha>\kappa+1$, then for any sequence $\{A_n\}$ of Borel sets in $\mathbb{R}^d$,

$$r_n^\alpha\left|\mu^\theta_{r_n}(A_n)-\mathbb{E}\left[\mu^\theta_{r_n}(A_n)\middle|\mathcal{F}_{r_n^\delta}\right]\right|\overset{n\to\infty}{\longrightarrow}0,\quad a.s.$$

Proof (i) Define

$$\overline{Y}_{n,u}:=Y_{n,u}1_{\{|Y_{n,u}|\le e^{c_* r_n^\delta}\}},\quad Y'_{n,u}=\overline{Y}_{n,u}-\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^\delta}\right),$$

where $c_*>0$ is a constant to be chosen later. Then for any $\varepsilon>0$,

$$\begin{aligned}&\mathbb{P}\Big(\Big|r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u}\Big|>\varepsilon\Big|\mathcal{F}_{r_n^\delta}\Big)\\&\le\mathbb{P}\Big(\Big|r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)}e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\left(Y_{n,u}-\overline{Y}_{n,u}\right)\Big|>\frac{\varepsilon}{3}\Big|\mathcal{F}_{r_n^\delta}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}\Big(r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\Big|\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)} Y'_{n,u}\Big| > \frac{\varepsilon}{3}\Big|\mathcal{F}_{r_n^\delta}\Big)\\
&+1_{\left\{r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\left|\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^\delta}\right)\right|>\frac{\varepsilon}{3}\right\}} =: I + II + III. \qquad (3.1)
\end{aligned}
$$

Using the inequality

$$
|Y_{n,u}-\overline{Y}_{n,u}| = |Y_{n,u}|1_{\{|Y_{n,u}|>e^{c_* r_n^\delta}\}} \le \left(W_{a_n}(\theta;u)+1\right)1_{\left\{W_{a_n}(\theta;u)+1>e^{c_* r_n^\delta}\right\}}
$$

and Markov's inequality, we have

$$
\begin{aligned}
I &\le \frac{3}{\varepsilon} r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\left|Y_{n,u}-\overline{Y}_{n,u}\right|\middle|\mathcal{F}_{r_n^\delta}\right)\\
&\lesssim r_n^{\alpha} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\left(W_{a_n}(\theta)+1\right)1_{\{W_{a_n}(\theta)+1>e^{c_* r_n^\delta}\}}\right)\\
&\le \frac{r_n^\alpha}{(c_* r_n^\delta)^{\lambda+1}} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\left(W_{a_n}(\theta)+1\right)\log_+^{1+\lambda}\left(W_{a_n}(\theta)+1\right)\right)\\
&\lesssim \frac{r_n^\alpha}{(c_* r_n^\delta)^{\lambda+1}} e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)} n^\beta, \qquad (3.2)
\end{aligned}
$$

where in the last inequality we used Lemma 2.7. For $1<\ell<\min\{2/\|\theta\|^2, 2\}$, set $b := e^{(\ell-1)(1-\|\theta\|^2\ell/2)/2} \in (1, e)$ and $c_* := \ln b$. Using Markov's inequality, Lemma 2.2 (ii) with $X=\overline{Y}_{n,u}$, the conditional independence of $Y'_{n,u}$, and the fact that $|\overline{Y}_{n,u}|^\ell \le e^{c_*(\ell-1)r_n^\delta}|Y_{n,u}| \le e^{c_* r_n^\delta}\left(W_{a_n}(\theta;u)+1\right)$, we have

$$
\begin{aligned}
II &\le \frac{3^\ell}{\varepsilon^\ell} r_n^{\alpha\ell} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^\delta}\mathbb{E}\Big(\Big|\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot \mathbf{X}_u(r_n^\delta)} Y'_{n,u}\Big|^\ell\Big|\mathcal{F}_{r_n^\delta}\Big)\\
&\lesssim \frac{3^\ell}{\varepsilon^\ell} r_n^{\ell\alpha} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\ell\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\left|Y'_{n,u}\right|^\ell\middle|\mathcal{F}_{r_n^\delta}\right)\\
&\lesssim \frac{3^\ell}{\varepsilon^\ell} r_n^{\ell\alpha} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\ell\theta\cdot \mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\left|\overline{Y}_{n,u}\right|^\ell\middle|\mathcal{F}_{r_n^\delta}\right)\\
&\le \frac{3^\ell}{\varepsilon^\ell} r_n^{\ell\alpha} e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\ell\theta\cdot \mathbf{X}_u(r_n^\delta)} e^{c_* r_n^\delta}\mathbb{E}\left(W_{a_n}(\theta;u)+1\middle|\mathcal{F}_{r_n^\delta}\right)\\
&\lesssim r_n^{\ell\alpha} b^{-2r_n^\delta} e^{-(1+\frac{\ell^2\|\theta\|^2}{2})r_n^\delta} b^{r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\ell\theta\cdot \mathbf{X}_u(r_n^\delta)}, \qquad (3.3)
\end{aligned}
$$

where in the last inequality, we used the identities $\mathbb{E}\left(W_{a_n}(\theta;u)+1\middle|\mathcal{F}_{r_n^\delta}\right) = \mathbb{E}\left(W_{a_n}(\theta)+1\right) = 2$, $e^{-\ell(1+\frac{\|\theta\|^2}{2})r_n^\delta} = b^{-2r_n^\delta}e^{-(1+\frac{\ell^2\|\theta\|^2}{2})r_n^\delta}$, and $e^{c_*} = b$. Therefore, by (3.3), we get

$$II \lesssim r_n^{\ell\alpha} b^{-r_n^\delta} W_{r_n^\delta}(\ell\theta). \tag{3.4}$$

Now taking expectation with respect to $\mathbb{P}$ in (3.1), and using (3.2) and (3.4), we get that

$$\begin{aligned}&\mathbb{P}\Big(\Big|r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u}\Big| > \varepsilon\Big)\\&\lesssim \frac{r_n^\alpha n^\beta}{(r_n^\delta)^{\lambda+1}} + r_n^{\ell\alpha}b^{-r_n^\delta} + \mathbb{P}\left(r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\Big|\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^\delta}\right)\Big| > \frac{\varepsilon}{3}\right).\end{aligned} \tag{3.5}$$

By Markov's inequality and the fact that $\mathbb{E}\left(\overline{Y}_{n,u}\middle|\mathcal{F}_{r_n^\delta}\right) = -\mathbb{E}\left(Y_{n,u}1_{\{|Y_{n,u}|>e^{c_*r_n^\delta}\}}\middle|\mathcal{F}_{r_n^\delta}\right)$, the third term on the right-hand side of (3.5) is bounded from above by

$$\begin{aligned}&\frac{3}{\varepsilon}r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\mathbb{E}\Big(\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\mathbb{E}\Big(|Y_{n,u}|1_{\left\{|Y_{n,u}|>e^{c_*r_n^\delta}\right\}}\Big|\mathcal{F}_{r_n^\delta}\Big)\Big)\\&\le \frac{3}{\varepsilon}r_n^\alpha(c_*r_n^\delta)^{-\lambda-1}\mathbb{E}\left(\left(W_{a_n}(\theta)+1\right)\log_+^{\lambda+1}(1+W_{a_n}(\theta))\right) \lesssim r_n^\alpha(r_n^\delta)^{-\lambda-1}n^\beta,\end{aligned}$$

where in the last inequality we used Lemma 2.7. Plugging the upper bound above into (3.5) and recalling $r_n = n^{\frac{1}{\kappa}}$, we get

$$\begin{aligned}&\sum_{n=1}^\infty \mathbb{P}\Big(\Big|r_n^\alpha e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u}\Big| > \varepsilon\Big)\\&\lesssim \sum_{n=1}^\infty\Big(\frac{r_n^\alpha n^\beta}{(r_n^\delta)^{\lambda+1}} + r_n^{\ell\alpha}b^{-r_n^\delta} + r_n^{-((\lambda+1)\delta-\alpha-\kappa\beta)}\Big),\end{aligned}$$

which is summable since $\lambda\delta - \alpha > \kappa(1+\beta)$. This completes the proof of (i).

(ii) By the Markov property and Lemma 2.1,

$$\begin{aligned}&\mathbb{E}\left[\mu_{r_n}^\theta(A_n)\middle|\mathcal{F}_{r_n^\delta}\right]\\&= e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\mathbb{P}^{-\theta}\left(\mathbf{X}_\xi(r_n-r_n^\delta)+\theta r_n+\mathbf{y}\in A_n\right)\big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}.\end{aligned}$$

Since $\{\mathbf{X}_\xi(t)+\theta t, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we have

$$\begin{aligned}&\mathbb{E}\left[\mu^\theta_{r_n}(A_n)\,\middle|\mathcal{F}_{r_n^\delta}\right]\\ &= e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}\mathbf{P}\left(\mathbf{B}_{r_n-r_n^\delta}+\mathbf{y}+\theta r_n^\delta\in A_n\right)\Big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}. \end{aligned}\tag{3.6}$$

Therefore,

$$\mu^\theta_{r_n}(A_n)-\mathbb{E}\left[\mu^\theta_{r_n}(A_n)\,\middle|\mathcal{F}_{r_n^\delta}\right] =: e^{-(1+\frac{\|\theta\|^2}{2})r_n^\delta}\sum_{u\in N(r_n^\delta)} e^{-\theta\cdot\mathbf{X}_u(r_n^\delta)}Y_{n,u},$$

where

$$\begin{aligned}Y_{n,u} &:= e^{-(1+\frac{\|\theta\|^2}{2})(r_n-r_n^\delta)}\sum_{v\in N(r_n):u\le v} e^{-\theta\cdot\left(\mathbf{X}_v(r_n)-\mathbf{X}_u(r_n^\delta)\right)}1_{\{\mathbf{X}_v(r_n)+\theta r_n\in A_n\}}\\ &\quad -\mathbf{P}\left(\mathbf{B}_{r_n-r_n^\delta}+\mathbf{y}+\theta r_n^\delta\in A_n\right)\Big|_{\mathbf{y}=\mathbf{X}_u(r_n^\delta)}.\end{aligned}$$

By the branching property, we see that, conditioned on $\mathcal{F}_{r_n^\delta}$, $\{Y_{n,u}: u\in N(r_n^\delta)\}$ is a family of centered independent random variables. Furthermore, it holds that

$$\begin{aligned}|Y_{n,u}| &\le e^{-(1+\frac{\|\theta\|^2}{2})(r_n-r_n^\delta)}\sum_{v\in N(r_n):u\le v} e^{-\theta\cdot\left(\mathbf{X}_v(r_n)-\mathbf{X}_u(r_n^\delta)\right)}+1\\ &= W_{r_n-r_n^\delta}(\theta;u)+1.\end{aligned}$$

Therefore, the second result is valid by (i) by taking $\beta=1/\kappa$ and $a_n=r_n-r_n^\delta$. □

Now, we treat the case $s\in[r_n,r_{n+1})$. We will take $\delta=1/2$, $\beta=1/\kappa$ and $\alpha=m/2$ for $m\in\mathbb{N}$. Then, the condition $\lambda\delta-\alpha>\kappa(1+\beta)$ is equivalent to $\lambda>m+2(\kappa+1)$.

Lemma 3.2 *For $\mathbf{b}\in\mathbb{R}^d$, let $\mathbf{b}_s:=\mathbf{b}\sqrt{s}$ or $\mathbf{b}_s:=\mathbf{b}$. For any given $m\in\mathbb{N}$, assume that $\kappa>m+2$ and that (1.11) holds for some $\lambda>m+2(\kappa+1)$. Define $k_s:=\sqrt{r_n}$ for $s\in[r_n,r_{n+1})$. Then for any $\mathbf{b}\in\mathbb{R}^d$,*

$$s^{m/2}\left|\mu^\theta_s((-\infty,\mathbf{b}_s])-\mathbb{E}\left[\mu^\theta_s((-\infty,\mathbf{b}_s])\,\middle|\mathcal{F}_{k_s}\right]\right|\overset{s\to\infty}{\longrightarrow}0,\quad \mathbb{P}\text{-a.s.}$$

Proof *Step 1:* In this step, we prove that almost surely,

$$\sup_{r_n\le s<r_{n+1}} s^{m/2}\left|\mathbb{E}\left[\mu^\theta_s((-\infty,\mathbf{b}_s])\,\middle|\mathcal{F}_{k_s}\right]-\mathbb{E}\left[\mu^\theta_{r_n}\left((-\infty,\mathbf{b}_{r_n}]\right)\,\middle|\mathcal{F}_{k_s}\right]\right|\overset{n\to\infty}{\longrightarrow}0.\tag{3.7}$$

By the Markov property and Lemma 2.1, a similar argument as (3.6) yields that

$$\mathbb{E}\left[\mu^\theta_s((-\infty,\mathbf{b}_s])\,\middle|\mathcal{F}_{k_s}\right]$$

$$= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\mathbf{P}\left(\mathbf{B}_{s-\sqrt{r_n}}+\mathbf{y}+\theta\sqrt{r_n}\le \mathbf{b}_s\right)\Big|_{\mathbf{y}=\mathbf{X}_u(\sqrt{r_n})}$$

$$= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}\Phi_d\Big(\frac{\mathbf{b}_s-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big). \tag{3.8}$$

Thus, for $s\in[r_n, r_{n+1})$, it holds that

$$\begin{aligned} &s^{m/2}\left|\mathbb{E}\left[\mu_s^\theta\left((-\infty,b_s]\right)\middle|\mathcal{F}_{k_s}\right]-\mathbb{E}\left[\mu_{r_n}^\theta\left((-\infty,b_{r_n}]\right)\middle|\mathcal{F}_{k_s}\right]\right| \\ &\le r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \\ &\qquad\times\left|\Phi_d\Big(\frac{\mathbf{b}_s-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)-\Phi_d\Big(\frac{\mathbf{b}_{r_n}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{r_n-\sqrt{r_n}}}\Big)\right| \\ &=: r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}R(u,s). \end{aligned}$$

Let $E(u,r_n,\theta) := \cup_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|>\sqrt{r_n}\}$. Using Lemma 2.1, the trivial upper-bound $\sup_{r_n\le s<r_{n+1}} R(u,s)\le 2$ and the fact that $\{\mathbf{X}_\xi(t)+\theta t, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we have

$$\begin{aligned} &\sum_{n=1}^\infty r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\mathbb{E}\Big(\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{E(u,r_n,\theta)}\sup_{r_n\le s<r_{n+1}}R(u,s)\Big) \\ &\le 2\sum_{n=1}^\infty r_{n+1}^{m/2}\mathbf{P}\left(\cup_{j=1}^d\left\{|(\mathbf{B}_1)_j|>r_n^{1/4}\right\}\right)\le 2d\mathbf{E}(e^{|(\mathbf{B}_1)_1|})\sum_{n=1}^\infty r_{n+1}^{m/2}e^{-r_n^{1/4}}<\infty, \end{aligned}$$

which implies that $\mathbb{P}$-almost surely,

$$r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{E(u,r_n,\theta)}R(u,s)\overset{s\to\infty}{\longrightarrow}0. \tag{3.9}$$

On the other hand, on the event $E(u,r_n,\theta)^c=\cap_{j=1}^d\{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|\le\sqrt{r_n}\}$, in the case $\mathbf{b}_s=\mathbf{b}$, using the trivial inequality

$$|\Phi_d(\mathbf{a})-\Phi_d(\mathbf{b})|\le\sum_{j=1}^d\left|\Phi(a_j)-\Phi(b_j)\right|\le\frac{1}{\sqrt{2\pi}}\sum_{j=1}^d|a_j-b_j|,$$

we get that, uniformly for $s\in[r_n,r_{n+1})$,

$$R(u,s)\le\sum_{j=1}^d\Big(\frac{|b_j|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|$$

$$
\begin{aligned}
&+\frac{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|\Bigg)\\
\lesssim\; &\frac{|s-r_n|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}\left(\sqrt{s-\sqrt{r_n}}+\sqrt{r_n-\sqrt{r_n}}\right)}+\sqrt{r_n}\frac{\left|\sqrt{s-\sqrt{r_n}}-\sqrt{r_n-\sqrt{r_n}}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}\\
\lesssim\; &\frac{1}{r_n^{3/2}}(r_{n+1}-r_n)+\frac{\sqrt{r_n}}{r_n^{3/2}}(r_{n+1}-r_n)\lesssim\frac{1}{r_n}(r_{n+1}-r_n).
\end{aligned}
$$

In the case $\mathbf{b}_s=\mathbf{b}\sqrt{s}$, uniformly for $s\in[r_n,r_{n+1})$,

$$
\begin{aligned}
R(u,s)\le\;&\sum_{j=1}^{d}\Bigg(\frac{|b_j|}{\sqrt{2\pi}}\left|\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}}-\frac{\sqrt{r_n}}{\sqrt{r_n-\sqrt{r_n}}}\right|\\
&+\frac{|(\mathbf{X}_u(\sqrt{r_n}))_j+\theta_j\sqrt{r_n}|}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{s-\sqrt{r_n}}}-\frac{1}{\sqrt{r_n-\sqrt{r_n}}}\right|\Bigg)\\
\lesssim\;&\frac{\left|\sqrt{s(r_n-\sqrt{r_n})}-\sqrt{r_n(s-\sqrt{r_n})}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}+\sqrt{r_n}\frac{\left|\sqrt{s-\sqrt{r_n}}-\sqrt{r_n-\sqrt{r_n}}\right|}{\sqrt{s-\sqrt{r_n}}\sqrt{r_n-\sqrt{r_n}}}\\
\lesssim\;&\frac{1}{r_n}\frac{\left|s(r_n-\sqrt{r_n})-r_n(s-\sqrt{r_n})\right|}{r_n}+\frac{\sqrt{r_n}}{r_n^{3/2}}(r_{n+1}-r_n)\lesssim\frac{1}{r_n}(r_{n+1}-r_n).
\end{aligned}
$$

Thus in both cases, we have that

$$
\begin{aligned}
&r_{n+1}^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}}\sum_{u\in N(\sqrt{r_n})}e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})}1_{E(u,r_n,\theta)^c}\sup_{r_n\le s<r_{n+1}}R(u,s)\\
&\lesssim r_{n+1}^{m/2}\frac{1}{r_n}(r_{n+1}-r_n)W_{\sqrt{r_n}}(\theta). \qquad (3.10)
\end{aligned}
$$

We claim that the right-hand side of (3.10) goes to 0 almost surely as $s\to\infty$. In fact,

$$
r_{n+1}^{m/2}\frac{1}{r_n}(r_{n+1}-r_n)\lesssim r_n^{(-2+m)/2}(r_{n+1}-r_n)=n^{(-2+m)/(2\kappa)}\left((n+1)^{1/\kappa}-n^{1/\kappa}\right).
$$

By the mean value theorem, the right-hand side above is equal to

$$
n^{(-2+m)/(2\kappa)}\nu^{-1+\frac{1}{\kappa}}\lesssim n^{(m-2\kappa)/(2\kappa)},\qquad\text{for some }\nu\in[n,n+1].
$$

Since $\kappa>m+2$ and $\lim_{n\to\infty}W_{\sqrt{r_n}}(\theta)$ exists (due to the fact that $W_t(\theta)$ is a non-negative martingale), the claim is valid. Combining this with (3.9) and (3.10), we get (3.7).

Step 2: In this step, we prove that

$$
\sup_{r_n\le s<r_{n+1}}s^{m/2}\left|\mu_s^\theta\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^\theta\left((-\infty,\mathbf{b}_{r_n}]\right)\right|\overset{n\to\infty}{\longrightarrow}0,\quad\mathbb{P}\text{-a.s.}\qquad(3.11)
$$

Once we get (3.11), we can combine (3.7) and Lemma 3.1 (ii) (with $A_n = (-\infty, \mathbf{b}_{r_n}]$ and $\delta = 1/2$) to get the assertion of the lemma.

To prove (3.11), we first prove that

$$\liminf_{n\to\infty} \inf_{r_n \le s < r_{n+1}} s^{m/2} \left(\mu_s^\theta\left((-\infty, \mathbf{b}_s]\right) - \mu_{r_n}^\theta\left((-\infty, \mathbf{b}_{r_n}]\right)\right) \ge 0, \quad \mathbb{P}\text{-a.s.} \tag{3.12}$$

Define $\varepsilon_n := \sqrt{r_{n+1} - r_n}$. For $u \in N(r_n)$, let G_u be the event that u does not split before r_{n+1} and that $\max_{s\in(r_n, r_{n+1})} \|\mathbf{X}_u(s) - \mathbf{X}_u(r_n)\| \le \sqrt{r_n}\varepsilon_n$. Then,

$$\begin{aligned}\mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right) &= e^{-(r_{n+1}-r_n)}\mathbf{P}\Big(\max_{r\le r_{n+1}-r_n} \|\mathbf{B}_r\| \le \sqrt{r_n}\varepsilon_n\Big) \\ &= e^{-(r_{n+1}-r_n)}\mathbf{P}\Big(\max_{r\le 1} \|\mathbf{B}_r\| \le \sqrt{r_n}\Big).\end{aligned}$$

Recalling that $\mathbf{1} := (1, ..., 1)$, it holds that

$$\begin{aligned}\mu_s^\theta\left((-\infty, \mathbf{b}_s]\right) &= e^{-(1+\frac{\|\theta\|^2}{2})s} \sum_{u\in N(r_n)} e^{-\theta\cdot \mathbf{X}_u(r_n)} \\ &\quad \times \sum_{v\in N(s): u\le v} e^{-\theta\cdot(\mathbf{X}_v(s) - \mathbf{X}_u(r_n))} 1_{\{\mathbf{X}_v(s)+\theta s\le \mathbf{b}_s\}} \\ &\ge e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1}} e^{-\|\theta\|\sqrt{r_n}\varepsilon_n} \\ &\quad \times \sum_{u\in N(r_n)} e^{-\theta\cdot \mathbf{X}_u(r_n)} 1_{\{\mathbf{X}_u(r_n)+\theta r_n\le \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} 1_{G_u} \\ &= e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1} - \|\theta\|\sqrt{r_n}\varepsilon_n} \sum_{u\in N(r_n)} e^{-\theta\cdot \mathbf{X}_u(r_n)} \\ &\quad \times 1_{\{\mathbf{X}_u(r_n)+\theta r_n\le \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \left(1_{G_u} - \mathbb{P}\left(G_u\middle|\mathcal{F}_{r_n}\right)\right) \\ &\quad + e^{-(1+\frac{\|\theta\|^2}{2})r_{n+1} - \|\theta\|\sqrt{r_n}\varepsilon_n} \sum_{u\in N(r_n)} e^{-\theta\cdot \mathbf{X}_u(r_n)} \\ &\quad \times 1_{\{\mathbf{X}_u(r_n)+\theta r_n\le \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \mathbb{P}\left(G_u\middle|\mathcal{F}_{r_n}\right) =: I + II.\end{aligned} \tag{3.13}$$

For I, we will apply Lemma 3.1 (i) with $\alpha = m/2$, $\delta = 1$, $a_n = 0$, $\beta = 0$ and

$$Y_{n,u} := 1_{\{\mathbf{X}_u(r_n)+\theta r_n\le \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1}-r_n)\mathbf{1}\}} \left(1_{G_u} - \mathbb{P}\left(G_u\middle|\mathcal{F}_{r_n}\right)\right).$$

It is easy to see that $|Y_{n,u}| \le 2$, $r_{n+1} - r_n \to 0$ and $\sqrt{r_n}\varepsilon_n \lesssim \sqrt{n^{(2-\kappa)/\kappa}} \to 0$. Since $\lambda > m + 2(\kappa + 1)$, we have

$$\sup_{r_n\le s<r_{n+1}} s^{m/2}|I| \overset{n\to\infty}{\longrightarrow} 0, \quad \mathbb{P}\text{-a.s.} \tag{3.14}$$

If we can prove that

$$\sup_{r_n \le s < r_{n+1}} s^{m/2} \left| II - \mu_{r_n}^{\theta}\left((-\infty, \mathbf{b}_{r_n}]\right)\right| \xrightarrow{n\to\infty} 0, \quad \mathbb{P}\text{-a.s.}, \tag{3.15}$$

then (3.12) will follow from (3.13), (3.14) and (3.15). Now, we prove (3.15). Since $\kappa > m+2$, we have $r_n^{m/2}(r_{n+1} - r_n) \lesssim n^{-1+(m+2)/(2\kappa)} \to 0$. Thus,

$$\begin{aligned} s^{m/2}\left|1 - \mathbb{P}\left(G_u \middle| \mathcal{F}_{r_n}\right)\right| &\le r_{n+1}^{m/2}(1 - e^{-(r_{n+1}-r_n)}) + r_{n+1}^{m/2}\mathbf{P}\Big(\max_{r\le 1}\|\mathbf{B}_r\| > \sqrt{r_n}\Big) \\ &\lesssim r_{n+1}^{m/2}(r_{n+1} - r_n) + r_{n+1}^{m/2} e^{-\sqrt{r_n}} \to 0. \end{aligned}$$

Hence, using the fact that $\lim_{n\to\infty} W_{r_n}(\theta) = W_\infty(\theta)$ exists, we get

$$\begin{aligned} &s^{m/2}\left| e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} e^{\|\theta\|\sqrt{r_n}\varepsilon_n} II - \mu_{r_n}^{\theta}\left((-\infty, \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}]\right)\right| \\ &\lesssim W_{r_n}(\theta)\left(r_{n+1}^{m/2}(r_{n+1} - r_n) + r_{n+1}^{m/2}e^{-\sqrt{r_n}}\right) \xrightarrow{s\to\infty} 0, \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{3.16}$$

Note that $0 \le e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} e^{\|\theta\|\sqrt{r_n}\varepsilon_n} II \le W_{r_n}(\theta)$, $e^{(1+\frac{\|\theta\|^2}{2})(r_{n+1}-r_n)} = 1 + O(r_{n+1} - r_n) = 1 + o(r_n^{-m/2})$, and that $e^{\|\theta\|\sqrt{r_n}\varepsilon_n} = 1 + O(\sqrt{r_n}\sqrt{r_{n+1} - r_n}) = 1 + O(n^{1/\kappa - 1/2}) = 1 + o(r_n^{-m/2})$ by the assumption that $\kappa > m+2$. Therefore, (3.16) implies that

$$s^{m/2}\left| II - \mu_{r_n}^{\theta}\left((-\infty, \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}]\right)\right| \xrightarrow{s\to\infty} 0, \quad \mathbb{P}\text{-a.s.} \tag{3.17}$$

Now, we put $A_n = (-\infty, \mathbf{b}_{r_n}] \setminus (-\infty, \mathbf{b}_{r_n} - \varepsilon_n\sqrt{r_n}\mathbf{1} - \|\theta\|(r_{n+1} - r_n)\mathbf{1}] \subset \cup_{j=1}^d C_{n,j}$ where $C_{n,j} := \left\{\mathbf{x} = (x_1, ..., x_d) : x_j \in ((\mathbf{b}_{r_n})_j - \varepsilon_n\sqrt{r_n} - \|\theta\|(r_{n+1} - r_n), (\mathbf{b}_{r_n})_j]\right\}$. Then by Lemma 2.1 and the inequality $\mathbf{P}\left(\mathbf{B}_t + \mathbf{y} \in C_{n,j}\right) \le \frac{\varepsilon_n\sqrt{r_n} + \|\theta\|(r_{n+1} - r_n)}{\sqrt{2\pi t}}$, we obtain that

$$\begin{aligned} &r_n^{m/2}\mathbb{E}\left[\mu_{r_n}^{\theta}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}}\right] \\ &= r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \\ &\quad \times \mathbb{P}^{-\theta}\left(\mathbf{X}_\xi(r_n - \sqrt{r_n}) + \mathbf{y} + \theta r_n \in A_n\right)|_{\mathbf{y}=\mathbf{X}_u(\sqrt{r_n})} \\ &\le \sum_{j=1}^d r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \\ &\quad \times \mathbf{P}\left(\mathbf{B}_{r_n - \sqrt{r_n}} + \mathbf{y} + \theta\sqrt{r_n} \in C_{n,j}\right)|_{\mathbf{y}=\mathbf{X}_u(\sqrt{r_n})} \\ &\le d W_{\sqrt{r_n}}(\theta) \frac{r_n^{m/2}\left(\varepsilon_n\sqrt{r_n} + \|\theta\|(r_{n+1} - r_n)\right)}{\sqrt{2\pi(r_n - \sqrt{r_n})}} \xrightarrow{s\to\infty} 0, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

Here, the last assertion about the limit being 0 follows from the following argument:

$$\frac{r_n^{m/2}\left(\varepsilon_n\sqrt{r_n}+\|\theta\|(r_{n+1}-r_n)\right)}{\sqrt{2\pi(r_n-\sqrt{r_n})}} \lesssim r_n^{m/2}\varepsilon_n = n^{m/(2\kappa)}\sqrt{(n+1)^{1/\kappa}-n^{1/\kappa}}$$
$$\lesssim n^{(m+1-\kappa)/(2\kappa)} \to 0.$$

Using Lemma 3.1 (ii), we immediately get that $r_n^{m/2}\mu_{r_n}^{\theta}(A_n)\to 0$, $\mathbb{P}$-almost surely. Then by (3.17), we conclude that (3.15) holds.

Applying similar arguments for the interval $(\mathbf{b}_s,+\infty)$, we can also get

$$\liminf_{n\to\infty}\inf_{r_n\le s<r_{n+1}} s^{m/2}\left(\mu_s^{\theta}\left((\mathbf{b}_s,+\infty)\right)-\mu_{r_n}^{\theta}\left((\mathbf{b}_{r_n},+\infty)\right)\right)\ge 0, \quad \mathbb{P}\text{-a.s.} \tag{3.18}$$

Using Proposition 2.6 with $\mathbf{k}=\mathbf{0}$ and $\eta=2(\kappa+1)$, and the assumption $\lambda > m+2(\kappa+1)$, we get

$$\lim_{n\to\infty}\sup_{r_n\le s<r_{n+1}} s^{m/2}\left|\mu_s^{\theta}\left(\mathbb{R}^d\right)-\mu_{r_n}^{\theta}\left(\mathbb{R}^d\right)\right| = \lim_{n\to\infty}\sup_{r_n\le s<r_{n+1}} s^{m/2}\left|W_s(\theta)-W_{r_n}(\theta)\right|$$
$$=0. \tag{3.19}$$

Now, we prove (3.11) follows from (3.12), (3.18) and (3.19). Indeed, for any $\varepsilon>0$, (3.12), (3.18) and (3.19) imply that one can find a random time N such that for all $n>N$ and $r_n\le s<r_{n+1}$,

$$s^{m/2}\left(\mu_s^{\theta}\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^{\theta}\left((-\infty,\mathbf{b}_{r_n}]\right)\right)>-\varepsilon,$$
$$s^{m/2}\left(\mu_s^{\theta}\left((\mathbf{b}_s,+\infty)\right)-\mu_{r_n}^{\theta}\left((\mathbf{b}_{r_n},+\infty)\right)\right)>-\varepsilon \quad\text{and}\quad s^{m/2}\left|\mu_s^{\theta}\left(\mathbb{R}^d\right)-\mu_{r_n}^{\theta}\left(\mathbb{R}^d\right)\right|$$
$$<\varepsilon.$$

Thus,

$$s^{m/2}\left(\mu_s^{\theta}\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^{\theta}\left((-\infty,\mathbf{b}_{r_n}]\right)\right)$$
$$=s^{m/2}\left(\mu_s^{\theta}\left(\mathbb{R}^d\right)-\mu_{r_n}^{\theta}\left(\mathbb{R}^d\right)\right)-s^{m/2}\left(\mu_s^{\theta}\left((\mathbf{b}_s,+\infty)\right)-\mu_{r_n}^{\theta}\left((\mathbf{b}_{r_n},+\infty)\right)\right)<2\varepsilon.$$

Hence, we have that when $n>N$ and $r_n\le s<r_{n+1}$,

$$s^{m/2}\left|\mu_s^{\theta}\left((-\infty,\mathbf{b}_s]\right)-\mu_{r_n}^{\theta}\left((-\infty,\mathbf{b}_{r_n}]\right)\right|<2\varepsilon,$$

which implies (3.11). □

For any given $m\in\mathbb{N}$, we will take $\kappa:=m+3$ in the remainder of this section. It follows from Lemma 3.2 and (3.8) that if (1.11) holds for some $\lambda>m+2(\kappa+1)=3m+8$, then $\mathbb{P}$-almost surely for all $s\in[r_n,r_{n+1})$ and $\mathbf{b}_s=\mathbf{b}\sqrt{s}$,

$$\mu_s^{\theta}\left((-\infty,\mathbf{b}\sqrt{s}]\right)=\mathbb{E}\left[\mu_s^{\theta}\left((-\infty,\mathbf{b}\sqrt{s}]\right)\middle|\mathcal{F}_{k_s}\right]+o(s^{-m/2})$$

$$= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}\Phi_d\Big(\frac{\mathbf{b}\sqrt{s}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)$$
$$+ o(s^{-m/2}). \tag{3.20}$$

Note that, for any $\mathbf{a} < \mathbf{b}$, $(\mathbf{a}, \mathbf{b}] = \prod_{j=1}^d (a_j, b_j]$ can be expressed in terms $\prod_{j=1}^d E_j$ where $E_j \in \{(-\infty, a_j], (-\infty, b_j]\}$ using a finite number of set theoretic operations. Thus, applying Lemma 3.2 to $\prod_{j=1}^d E_j$ and by (3.8), we get that if (1.11) holds for some $\lambda > 3(m + d) + 8 = 3m + 3d + 8$, then

$$\mu_s^\theta((\mathbf{a},\mathbf{b}]) = o(s^{-(m+d)/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}$$
$$\times \prod_{j=1}^d \frac{1}{\sqrt{s-\sqrt{r_n}}}\int_{a_j}^{b_j} \phi\Big(\frac{z_j-\theta_j\sqrt{r_n}-(\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-\sqrt{r_n}}}\Big)\mathrm{d}z_j. \tag{3.21}$$

Proof of Theorem 1.1 Let $m \in \mathbb{N}$ and assume (1.11) holds for some $\lambda > \max\{3m + 8, d(3m + 5)\}$. Recall that $r_n = n^{1/\kappa}$ and $\kappa = m + 3$. Put $K := m/\kappa + 3$ and $F(u, r_n, \theta) := \cup_{j=1}^d \big\{\big|(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}\big| > \sqrt{K\sqrt{r_n}\log n}\big\}$. Combining Lemma 2.1, $\sup_{\mathbf{z}\in\mathbb{R}^d} \Phi_d(\mathbf{z}) = 1$ and the fact that $\{(\mathbf{X}_\xi(t) + \theta t)_{t\ge 0}, \mathbb{P}^{-\theta}\}$ is a d-dimensional standard Brownian motion, we get that

$$\sum_{n=2}^\infty r_n^{m/2}\mathbb{E}\Big(e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}1_{F(u,r_n,\theta)}$$
$$\times \sup_{r_n\le s<r_{n+1}} \Phi_d\Big(\frac{\mathbf{b}\sqrt{s}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big)\Big)$$
$$\le \sum_{n=2}^\infty n^{m/(2\kappa)}\sum_{j=1}^d \mathbf{P}\left(|(\mathbf{B}_{\sqrt{r_n}})_j| > \sqrt{K\sqrt{r_n}\log n}\right)$$
$$= d\sum_{n=2}^\infty n^{m/(2\kappa)}\mathbf{P}\left(|(\mathbf{B}_1)_1| > \sqrt{K\log n}\right) \lesssim \sum_{n=1}^\infty n^{m/(2\kappa)}n^{-K/2} < \infty, \tag{3.22}$$

where in the last inequality we used the fact that $\mathbf{P}(|(\mathbf{B}_1)_1| > x) \lesssim e^{-x^2/2}$. Therefore, $\mathbb{P}$-almost surely,

$$r_n^{m/2}e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}1_{F(u,r_n,\theta)}$$
$$\times \sup_{r_n\le s<r_{n+1}} \Phi_d\Big(\frac{\mathbf{b}\sqrt{s}-\theta\sqrt{r_n}-\mathbf{X}_u(\sqrt{r_n})}{\sqrt{s-\sqrt{r_n}}}\Big) \overset{n\to\infty}{\longrightarrow} 0. \tag{3.23}$$

Since $\lambda > 3m + 8$, by (3.20) and (3.23), for any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, $\mathbf{b} \in \mathbb{R}^d$ and $s \in [r_n, r_{n+1})$,

$$\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) = o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)^c} \times \Phi_d\Big(\frac{\mathbf{b}\sqrt{s} - \theta\sqrt{r_n} - \mathbf{X}_u(\sqrt{r_n})}{\sqrt{s - \sqrt{r_n}}}\Big).$$

Put $J := 6m + 10$. Then, $J > 2m + K\kappa = 3m + 3\kappa = 6m + 9$. By Lemma 2.4, we get that for any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, $\mathbf{b} \in \mathbb{R}^d$ and $s \in [r_n, r_{n+1})$,

$$\begin{aligned}
&\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) \\
&= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)^c} \\
&\quad \prod_{j=1}^d \Big(\Phi(b_j) - \phi(b_j) \sum_{k=1}^J \frac{1}{k!} \frac{1}{s^{k/2}} \\
&\quad \times H_{k-1}(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big) + \varepsilon_{m,u,s,j}\Big) \\
&= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)^c} \\
&\quad \prod_{j=1}^d \Big(\Phi(b_j) - \phi(b_j) \sum_{k=1}^J \frac{1}{k!} \frac{1}{s^{k/2}} \\
&\qquad \times H_{k-1}(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big),
\end{aligned} \tag{3.24}$$

where $\varepsilon_{m,u,s,j} = \varepsilon_{m,y,b,s}|_{y=\theta_j\sqrt{r_n}+(\mathbf{X}_u(\sqrt{r_n}))_j,\ b=b_j}$. To justify the last equality, we first apply Lemma 2.4 to get that, for each $u \in N(\sqrt{r_n})$, as $s \to \infty$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&s^{m/2} e^{-(1+\frac{\theta^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta X_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)^c} \left|\varepsilon_{m,u,s,j}\right| \\
&\le W_{\sqrt{r_n}}(\theta) \times s^{m/2} \sup_{j \le d} \sup_{u \in N(\sqrt{r_n})} |\varepsilon_{m,u,s,j}| 1_{F(u,r_n,\theta)^c} \to 0.
\end{aligned}$$

Then, note that by (2.4) and $|H_k(x)| \lesssim |x|^k + 1$, on the event $F(u, r_n, \theta)^c = \cap_{j=1}^d \left\{\left|(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}\right| \le \sqrt{K\sqrt{r_n}\log n}\right\}$,

$$|Q_{u,j}| := \Big|\Phi(b_j) - \phi(b_j) \sum_{k=1}^J \frac{1}{k!} \frac{1}{s^{k/2}} H_{k-1}(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big|$$

$$\lesssim 1 + \sum_{k=1}^{J} \frac{1}{r_n^{k/2}} r_n^{k/4} \Big(1 + \Big|\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j \sqrt{r_n}}{r_n^{1/4}}\Big|^k\Big) \lesssim 1 + \frac{(\log n)^{J/2}}{r_n^{1/4}} \lesssim 1.$$

Combining the two displays above, we get that, on the event $F(u, r_n, \theta)^c$,

$$\begin{aligned}
&\Big|\Phi_d\Big(\frac{\mathbf{b}\sqrt{s} - \theta\sqrt{r_n} - \mathbf{X}_u(\sqrt{r_n})}{\sqrt{s - \sqrt{r_n}}}\Big) - \prod_{j=1}^{d} Q_{u,j}\Big| \\
&\leq \sum_{j=1}^{d} \prod_{\ell \neq j} \big|Q_{v,\ell}\big|\, |\varepsilon_{m,u,s,j}| \lesssim d \sup_{j \leq d} \sup_{u \in N(\sqrt{r_n})} |\varepsilon_{m,u,s,j}|,
\end{aligned}$$

which implies (3.24).

Let $\varepsilon \in (0, 1)$ be small enough so that $K(1-\varepsilon) \geq m/\kappa + 2$. For any $\mathbf{k} \in \mathbb{N}^d$ with $1 \leq |\mathbf{k}| \leq J$, using the inequality $|H_k(x)| \lesssim 1 + |x|^k$ first and then Lemma 2.1, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} r_n^{m/2} \mathbb{E}\Big(e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)} \\
&\quad \times \sup_{r_n \leq s < r_{n+1}} \prod_{j=1}^{d} \frac{\big(\sqrt{r_n}\big)^{k_j/2}}{s^{k_j/2}} \Big|H_{k_j}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big|\Big) \\
&\lesssim \sum_{n=2}^{\infty} r_n^{m/2} \mathbf{E}\Big(\sum_{j=1}^{d} 1_{\{|(\mathbf{B}_{\sqrt{r_n}})_j| > \sqrt{K\sqrt{r_n}\log n}\}} \prod_{\ell=1}^{d} \frac{1}{r_n^{k_\ell/4}} \Big(1 + \Big|\frac{(\mathbf{B}_{\sqrt{r_n}})_\ell}{\sqrt{\sqrt{r_n}}}\Big|^{k_\ell}\Big)\Big) \\
&= \sum_{j=1}^{d} \sum_{n=2}^{\infty} n^{(2m-|\mathbf{k}|)/(4\kappa)} \mathbf{E}\Big(1_{\{|(\mathbf{B}_1)_j| > \sqrt{K\log n}\}} \prod_{\ell=1}^{d} \Big(1 + |(\mathbf{B}_1)_\ell|^{k_\ell}\Big)\Big) \\
&\lesssim \mathbf{E}\left(\Big(1 + |B_1|^J\Big) e^{(1-\varepsilon)|B_1|^2/2}\right) \sum_{n=2}^{\infty} n^{(2m-|\mathbf{k}|)/(4\kappa)} n^{-(1-\varepsilon)K/2} \\
&\lesssim \sum_{n=2}^{\infty} n^{-(4\kappa+1)/(4\kappa)} < \infty.
\end{aligned} \tag{3.25}$$

Thus, we have that $\mathbb{P}$-almost surely,

$$\begin{aligned}
&\lim_{n\to\infty} r_n^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)} \\
&\quad \times \sup_{r_n \leq s < r_{n+1}} \prod_{j=1}^{d} \frac{\big(\sqrt{r_n}\big)^{k_j/2}}{s^{k_j/2}} \Big|H_{k_j}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big| = 0.
\end{aligned}$$

Consequently, for $s \in [r_n, r_{n+1})$, as $n \to \infty$,

$$e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)}$$

$$\times \prod_{j=1}^{d} \left| \Phi(b_j) - \phi(b_j) \sum_{k=1}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} H_{k-1}(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\right|$$

$$= e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)}$$

$$\times \prod_{j=1}^{d} \left| \sum_{k=0}^{J} \frac{(-1)^k}{k!} \frac{1}{s^{k/2}} \frac{\mathrm{d}^k}{\mathrm{d}b_j^k} \Phi(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\right|$$

$$\le \sum_{\mathbf{k}\le J\mathbf{1}} \frac{\left|D^{\mathbf{k}}\Phi_d(\mathbf{b})\right|}{\mathbf{k}!} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)}$$

$$\times \prod_{j=1}^{d} \frac{\left(\sqrt{r_n}\right)^{k_j/2}}{s^{k_j/2}} \left| H_{k_j}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\right|$$

$$= o(s^{-m/2}).$$

Now combining the above limit and (3.24), we may drop the indicator function in (3.24) to get

$$\mu_s^{\theta}\left((-\infty, \mathbf{b}\sqrt{s}]\right) = o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}$$

$$\times \prod_{j=1}^{d} \Big(\Phi(b_j) + \sum_{k=1}^{J} \frac{(-1)^k}{k!} \frac{1}{s^{k/2}} \frac{\mathrm{d}^k}{\mathrm{d}b_j^k} \Phi(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)$$

$$= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}$$

$$\times \prod_{j=1}^{d} \Big(\sum_{k=0}^{J} \frac{(-1)^k}{k!} \frac{1}{s^{k/2}} \frac{\mathrm{d}^k}{\mathrm{d}b_j^k} \Phi(b_j) \left(\sqrt{r_n}\right)^{k/2} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big)$$

$$= o(s^{-m/2}) + \sum_{\mathbf{k}:\mathbf{k}\le J\mathbf{1}} \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{1}{s^{|\mathbf{k}|/2}} D^{\mathbf{k}}\Phi_d(\mathbf{b}) M^{(\mathbf{k},\theta)}_{\sqrt{r_n}}.$$

Since $\lambda > |J\mathbf{1}|/2 = d(3m+5)$, it follows from Proposition 2.6 that for any $\mathbf{k} \in \mathbb{N}^d$ with $m+1 \le |\mathbf{k}| \le |J\mathbf{1}|$, $s^{-|\mathbf{k}|/2} M^{(\mathbf{k},\theta)}_{\sqrt{r_n}} = o(s^{-m/2})$. Thus,

$$\mu_s^{\theta}\left((-\infty, \mathbf{b}\sqrt{s}]\right) = \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{1}{s^{|\mathbf{k}|/2}} D^{\mathbf{k}}\Phi_d(\mathbf{b}) M^{(\mathbf{k},\theta)}_{\sqrt{r_n}} + o(s^{-m/2}).$$

Take $\eta > 0$ sufficient small so that $\lambda > \frac{3m}{2} + \eta$. Then by Proposition 2.6, for any $\mathbf{k} \in \mathbb{N}^d$ with $0 \le |\mathbf{k}| \le m$,

$$\begin{aligned} M_{\sqrt{r_n}}^{(\mathbf{k},\theta)} - M_{\infty}^{(\mathbf{k},\theta)} &= o(r_n^{-(\lambda-|\mathbf{k}|/2)/2+\eta/2}) = o\left(r_n^{-m/2} r_n^{m/2-(\lambda-m/2)/2+\eta/2}\right) \\ &= o(r_n^{-m/2}), \end{aligned}$$

which implies that as $s \to \infty$,

$$\mu_s^\theta\left((-\infty, \mathbf{b}\sqrt{s}]\right) = \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|}}{\mathbf{k}!} \frac{1}{s^{|\mathbf{k}|/2}} D^{\mathbf{k}} \Phi_d(\mathbf{b}) M_\infty^{(\mathbf{k},\theta)} + o(s^{-m/2}), \quad \mathbb{P}\text{-a.s.}$$

Therefore, the assertion of the theorem is valid under the assumption $\lambda > \max\{3m+8, d(3m+5)\}$.

Proof of Theorem 1.2 Let $m \in \mathbb{N}$ and assume (1.11) holds for some $\lambda > \max\{d(3m+5), 3m+3d+8\}$. Recall that $r_n = n^{1/\kappa}$ and $\kappa = m+3$. Put $K := m/\kappa + 3$, $F(u, r_n, \theta) := \cup_{j=1}^d \left\{\left|(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}\right| > \sqrt{K\sqrt{r_n}\log n}\right\}$ and define

$$Y_{s,n,u} := e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \prod_{j=1}^d \frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}} \int_{a_j}^{b_j} \phi\Big(\frac{z_j - \theta_j\sqrt{r_n} - (\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-\sqrt{r_n}}}\Big) \mathrm{d}z_j.$$

Since $\lambda > 3m+3d+8$, by (3.21), for any $\theta \in \mathbb{R}^d$ with $\|\theta\| < \sqrt{2}$, any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $\mathbf{a} < \mathbf{b}$ and $s \in [r_n, r_{n+1})$, $\mathbb{P}$-almost surely, as $s \to \infty$,

$$s^{d/2}\mu_s^\theta\left((\mathbf{a}, \mathbf{b}]\right) = e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} Y_{s,n,u} + o(s^{-m/2}). \tag{3.26}$$

Noticing that

$$0 \le \sup_{r_n\le s<r_{n+1}} Y_{s,n,u} \le e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})} \prod_{j=1}^d \frac{(b_j-a_j)\sqrt{r_{n+1}}}{\sqrt{r_n-\sqrt{r_n}}} \lesssim e^{-\theta\cdot\mathbf{X}_u(\sqrt{r_n})},$$

and using (3.22), we get

$$\begin{aligned} &\sum_{n=2}^\infty r_n^{m/2}\mathbb{E}\Big(\sup_{r_n\le s<r_{n+1}} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u\in N(\sqrt{r_n})} Y_{s,n,u}1_{F(u,r_n,\theta)}\Big) \\ &\lesssim d\sum_{n=2}^\infty n^{m/(2\kappa)}\mathbf{P}\left(|(\mathbf{B}_{\sqrt{r_n}})_1| > \sqrt{K\sqrt{r_n}\log n}\right) < \infty. \end{aligned}$$

Therefore, $\mathbb{P}$-almost surely,

$$s^{m/2} e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} Y_{s,n,u} 1_{F(u,r_n,\theta)} \xrightarrow{s\to\infty} 0.$$

By (3.26), for $s \in [r_n, r_{n+1})$,

$$s^{d/2} \mu_s^\theta((\mathbf{a}, \mathbf{b}]) = e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} Y_{s,n,u} 1_{F(u,r_n,\theta)^c} + o(s^{-m/2}). \tag{3.27}$$

Using Lemma 2.5, on the event $F(u, r_n, \theta)^c$, for $J = 6m + 10$,

$$\begin{aligned}
&\frac{\sqrt{s}}{\sqrt{s-\sqrt{r_n}}} \int_{a_j}^{b_j} \phi\Big(\frac{z_j - \theta_j\sqrt{r_n} - (\mathbf{X}_u(\sqrt{r_n}))_j}{\sqrt{s-\sqrt{r_n}}}\Big) \mathrm{d}z_j \\
&= \int_{a_j}^{b_j} \phi\left(\frac{z_j}{\sqrt{s}}\right) \Big(\sum_{k=0}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} H_k\Big(\frac{z_j}{\sqrt{s}}\Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big) \mathrm{d}z_j \\
&\quad + \varepsilon_{m,u,s,j} \\
&= \sum_{k=0}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_k\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j\Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big) \\
&\quad + \varepsilon_{m,u,s,j},
\end{aligned}$$

where, as $n \to \infty$,

$$r_{n+1}^{m/2} \sup_{s \in [r_n, r_{n+1})} \sup_{j \le d} \sup_{u \in N(\sqrt{r_n})} |\varepsilon_{m,u,s,j}| 1_{F(u,r_n,\theta)^c} \to 0.$$

Therefore, using (3.27) and an argument similar to that leading to (3.24), we get

$$\begin{aligned}
s^{d/2} \mu_s^\theta((\mathbf{a}, \mathbf{b}]) &= o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)^c} \\
&\times \prod_{j=1}^{d} \Big\{ \sum_{k=0}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_k\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j\Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j\sqrt{r_n}}{r_n^{1/4}}\Big)\Big\}.
\end{aligned} \tag{3.28}$$

By Lemma 2.1 and (3.25) and the fact that $\left|\int_a^b \phi\left(\frac{z}{\sqrt{s}}\right) H_k\left(\frac{z}{\sqrt{s}}\right) \mathrm{d}z\right| \lesssim |b-a|$, we have that for any $\mathbf{k} \in \mathbb{N}^d$ with $0 \le |\mathbf{k}| \le J$,

$$\sum_{n=2}^{\infty} r_n^{m/2} \mathbb{E} \sup_{r_n \le s < r_{n+1}} \Big| e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u,r_n,\theta)}$$

$$\times \prod_{\ell=1}^{d} \Big(\int_{a_\ell}^{b_\ell} \phi\Big(\frac{z_\ell}{\sqrt{s}}\Big) H_{k_\ell}\Big(\frac{z_\ell}{\sqrt{s}}\Big) \mathrm{d}z_\ell \Big) \frac{\big(\sqrt{r_n}\big)^{k_\ell/2}}{s^{k_\ell/2}} H_{k_\ell}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_\ell + \theta_\ell \sqrt{r_n}}{r_n^{1/4}}\Big)\Big|$$
$$\lesssim \sum_{n=2}^{\infty} r_n^{m/2} r_n^{-|\mathbf{k}|/4} \mathbf{E}\Big(1_{\{\cup_{j=1}^{d}\{|(\mathbf{B}_{\sqrt{r_n}})_j| > \sqrt{K\sqrt{r_n}\log n}\}\}} \prod_{\ell=1}^{d} \Big| H_{k_\ell}\Big(\frac{(\mathbf{B}_{\sqrt{r_n}})_\ell}{r_n^{1/4}}\Big)\Big|\Big)$$
$$\lesssim \sum_{j=1}^{d} \sum_{n=2}^{\infty} n^{(2m-|\mathbf{k}|)/(4\kappa)} \mathbf{E}\Big(1_{\{|(\mathbf{B}_1)_j| > \sqrt{K\log n}\}} \prod_{\ell=1}^{d} \Big(1 + |(\mathbf{B}_1)_\ell|^J\Big)\Big) < \infty.$$

Thus, $\mathbb{P}$-almost surely, as $s \to \infty$,

$$\lim_{n\to\infty} r_n^{m/2} \sup_{r_n \le s < r_{n+1}} \Big| e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})} 1_{F(u, r_n, \theta)}$$
$$\times \prod_{\ell=1}^{d} \Big(\int_{a_\ell}^{b_\ell} \phi\Big(\frac{z_\ell}{\sqrt{s}}\Big) H_{k_\ell}\Big(\frac{z_\ell}{\sqrt{s}}\Big) \mathrm{d}z_\ell \Big) \frac{\big(\sqrt{r_n}\big)^{k_\ell/2}}{s^{k_\ell/2}} H_{k_\ell}\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_\ell + \theta_\ell \sqrt{r_n}}{r_n^{1/4}}\Big)\Big| = 0.$$

Therefore, by (3.28), since $\lambda > 3m + 3d + 8$,

$$s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}]) = o(s^{-m/2}) + e^{-(1+\frac{\|\theta\|^2}{2})\sqrt{r_n}} \sum_{u \in N(\sqrt{r_n})} e^{-\theta\cdot \mathbf{X}_u(\sqrt{r_n})}$$
$$\times \prod_{j=1}^{d} \Big\{ \sum_{k=0}^{J} \frac{1}{k!} \frac{1}{s^{k/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_k\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) r_n^{k/4} H_k\Big(\frac{(\mathbf{X}_u(\sqrt{r_n}))_j + \theta_j \sqrt{r_n}}{r_n^{1/4}}\Big)\Big\}$$
$$= o(s^{-m/2}) + \sum_{\mathbf{k}:\mathbf{k}\le J\mathbf{1}} \prod_{j=1}^{d} \frac{1}{k_j!} \frac{1}{s^{k_j/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) M_{\sqrt{r_n}}^{(\mathbf{k},\theta)}.$$

Since $\lambda > \max\{d(3m+5), 3m+3d+8\}$, using Proposition 2.6 and argument similar to that used in the proof of Theorem 1.1, we get that

$$s^{d/2}\mu_s^\theta((\mathbf{a}, \mathbf{b}])$$
$$= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \prod_{j=1}^{d} \frac{1}{k_j!} \frac{1}{s^{k_j/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) M_{\infty}^{(\mathbf{k},\theta)}. \quad (3.29)$$

By Taylor's expansion, as $x \to 0$,

$$\phi(x) = \sum_{j=0}^{m} \frac{\phi^{(j)}(0)}{j!} x^j + o(x^m). \quad (3.30)$$

Note that $\phi^{(k)}(x) = (-1)^k H_k(x)\phi(x)$ and that, for each $1 \le k \le m$,

$$\phi(x)H_k(x) = (-1)^k \sum_{j=0}^{m} \frac{\phi^{(k+j)}(0)}{j!} x^j + o(x^m). \tag{3.31}$$

Note that for all $\mathbf{k}$ with $|\mathbf{k}| \le m$ and all $\mathbf{i}$ with $\mathbf{i} \le m\mathbf{1}$, it holds that $s^{-|\mathbf{k}|/2} s^{-|\mathbf{i}|/2} = o(s^{-m/2})$ if $|\mathbf{i}| + |\mathbf{k}| = |\mathbf{i} + \mathbf{k}| > m$. Thus, combining (3.30) and (3.31),

$$\begin{aligned}
&\sum_{\mathbf{k}:|\mathbf{k}|\le m} \prod_{j=1}^{d} \frac{1}{k_j!} \frac{1}{s^{k_j/2}} \Big(\int_{a_j}^{b_j} \phi\Big(\frac{z_j}{\sqrt{s}}\Big) H_{k_j}\Big(\frac{z_j}{\sqrt{s}}\Big) \mathrm{d}z_j \Big) M_\infty^{(\mathbf{k},\theta)} \\
&= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:\mathbf{i}\le m\mathbf{1}} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 \ldots \mathrm{d}z_d \\
&= o(s^{-m/2}) + \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:|\mathbf{i}+\mathbf{k}|\le m} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 \ldots \mathrm{d}z_d.
\end{aligned}$$

Therefore, by (3.29), we conclude that

$$\begin{aligned}
&s^{d/2} \mu_s^\theta((\mathbf{a}, \mathbf{b}]) \\
&= \sum_{\mathbf{k}:|\mathbf{k}|\le m} \frac{(-1)^{|\mathbf{k}|} M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}! s^{|\mathbf{k}|/2}} \int_{[\mathbf{a},\mathbf{b}]} \sum_{\mathbf{i}:|\mathbf{i}+\mathbf{k}|\le m} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0) z_j^{i_j}}{\mathbf{i}! s^{|\mathbf{i}|/2}} \mathrm{d}z_1 \ldots \mathrm{d}z_d + o(s^{-m/2}) \\
&= \sum_{\ell=0}^{m} \frac{1}{s^{\ell/2}} \sum_{j=0}^{\ell} (-1)^j \sum_{\mathbf{k}:|\mathbf{k}|=j} \frac{M_\infty^{(\mathbf{k},\theta)}}{\mathbf{k}!} \sum_{\mathbf{i}:|\mathbf{i}|=\ell-j} \frac{\prod_{j=1}^{d} \phi^{(k_j+i_j)}(0)}{\mathbf{i}!} \int_{[\mathbf{a},\mathbf{b}]} \prod_{j=1}^{d} z_j^{i_j} \mathrm{d}z_1 \ldots \mathrm{d}z_d \\
&\quad + o(s^{-m/2}).
\end{aligned}$$

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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