

## Asymptotic expansion for branching killed Brownian motion with drift\*

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### Abstract

Let  $Z_t^{(0,\infty)}$  be the point process formed by the positions of all particles alive at time  $t$  in a branching Brownian motion with drift  $-\theta$  and killed upon reaching 0. We assume  $\theta \in [0, \sqrt{2})$  and study the asymptotic expansions of  $Z_t^{(0,\infty)}(A)$  for intervals  $A \subset (0, \infty)$  under the assumption that  $\sum_{k=1}^{\infty} k(\log k)^{1+\lambda} p_k < \infty$  for some large  $\lambda$ . These results extend and sharpen the results of Louidor and Saglietti [J. Stat. Phys, 2020] and that of Kesten [Stochastic Process. Appl., 1978].

**Keywords:** branching Brownian motion with absorption; asymptotic expansion; martingale approximation; spine decomposition.

**MSC2020 subject classifications:** 60J80; 60F15; 60G44.

Submitted to EJP on April 4, 2024, final version accepted on January 31, 2025.

## 1 Introduction and main results

### 1.1 Introduction

A branching Brownian motion with drift is a continuous-time Markov process defined as follows: At time 0, there is a particle at site  $x$  and it moves according to a Brownian motion with drift  $-\theta$ , where  $\theta \in \mathbb{R}$ . After an exponential time of parameter 1 independent of the movement, this particle dies and splits into  $k$  offspring with probability  $p_k$ . Each

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\*The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900). Haojie Hou's research is partially supported by the China Postdoctoral Science Foundation (No. 2024M764112). Yan-Xia Ren's research is partially supported by NSFC (Grant Nos. 12071011 and 12231002) and The Fundamental Research Funds for Central Universities, Peking University LMEQF. Renming Song's research is partially supported by the Simons Foundation (#960480, Renming Song). Part of the research for this paper was done while the third-named author was visiting Jiangsu Normal University, where he was partially supported by a grant from the National Natural Science Foundation of China (11931004, Yingchao Xie) and by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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of the offspring independently repeats its parent’s behavior from the death-place of its parent. This procedure goes on. We use  $\mathbb{P}_x$  and  $\mathbb{E}_x$  to denote the law of this process and the corresponding expectation operator. Let  $N(t)$  be the set of particles alive at time  $t$  and for each  $u \in N(t)$ , we will use  $X_u(t)$  to denote the position of the particle. For  $s < t$  and  $u \in N(t)$ , we will also use  $X_u(s)$  to denote the position of the ancestor of  $u$  at time  $s$ .

Suppose now that  $x > 0$  and that, once a particle hits  $(-\infty, 0]$ , we remove it (along with all its possible descendants) from the system. The resulting branching system is called a branching killed Brownian motion with drift. Let  $Z_t^{(0,\infty)}$  denote the point process formed by the positions of all the particles alive at time  $t$  in the branching killed Brownian motion with drift, i.e.,

$$Z_t^{(0,\infty)} := \sum_{u \in N(t)} 1_{\{\min_{s \leq t} X_u(s) > 0\}} \delta_{X_u(t)}.$$

We will use  $(\mathcal{F}_t : t \geq 0)$  to denote the natural filtration of branching Brownian motion. Assume that

$$\sum_{k=0}^{\infty} k p_k = 2.$$

Kesten [18] proved that for any  $\theta \in \mathbb{R}$ , there exists a constant  $C = C(x, \theta) > 0$  such that

$$\mathbb{E}_x \left( Z_t^{(0,\infty)}((0, \infty)) \right) \sim \begin{cases} C t^{-3/2} e^{(1-\frac{\theta^2}{2})t}, & \theta > 0, \\ C t^{-1/2} e^t, & \theta = 0, \\ C e^t, & \theta < 0. \end{cases}$$

Consequently, when  $\theta \geq \sqrt{2}$ , the branching killed Brownian motion with drift will die out with probability 1. It was proved in [18] that, when  $\theta < \sqrt{2}$  and  $\sum_{k=1}^{\infty} k^2 p_k < \infty$ , the branching killed Brownian motion with drift will survive with positive probability. In [18], Kesten also stated, without proof, that under the assumption  $\sum_{k=0}^{\infty} k^2 p_k < \infty$ , there exists a random variable  $W(\theta)$  such that

$$\mathbb{P}_x \left( W(\theta) > 0 \mid Z_t^{(0,\infty)}((0, \infty)) > 0, \forall t > 0 \right) = 1 \tag{1.1}$$

and that

- (i) if  $\theta \in [0, \sqrt{2})$ , then  $\mathbb{P}_x$ -a.s., simultaneously for all intervals  $\Delta \subset (0, \infty)$  (finite or infinite), it holds that

$$\frac{Z_t^{(0,\infty)}(\Delta)}{\mathbb{E}_x \left( Z_t^{(0,\infty)}(\Delta) \right)} \xrightarrow{t \rightarrow \infty} W(\theta); \tag{1.2}$$

- (ii) if  $\theta < 0$ , then  $\mathbb{P}_x$ -a.s.,

$$e^{-t} Z_t^{(0,\infty)}((0, \infty)) \xrightarrow{t \rightarrow \infty} W(\theta).$$

In [22], Loudidor and Saglietti proved that (1.1) and (1.2) hold for the case  $\theta \in (0, \sqrt{2})$ .

The purpose of this paper is to extend and sharpen the main result of [22]: We weaken the moment condition from  $\sum_{k=1}^{\infty} k^2 p_k < \infty$  to  $\sum_{k=1}^{\infty} k \log^{1+\lambda} k p_k < \infty$  for some  $\lambda > 0$  and, for any  $a \geq 0$ , give asymptotic expansions of arbitrary order for

$$\frac{Z_t^{(0,\infty)}((a, \infty))}{t^{-3/2} e^{(1-\frac{\theta^2}{2})t}}$$

under this weaker assumption. We emphasize here that we do not use the results of [22] in this paper and that, as a consequence of Theorem 1.1 below, we give another proof of (1.2) under the weaker condition. We also include the case  $\theta = 0$ . It is natural to study similar problems for the case  $\theta < 0$ . We believe things might be different in this case and we plan to tackle this in a future work.

Branching killed Brownian motions with drift have been studied in quite a few papers. The case  $\theta \geq \sqrt{2}$  has been studied in [2, 3, 4, 5, 6, 13, 20, 21, 24] and the references therein. Since the process will die out in finite time in the case  $\theta \geq \sqrt{2}$ , the limiting behavior in this case is quite different from the case  $\theta < \sqrt{2}$ . In the case  $\theta < \sqrt{2}$ , Harris et. al. [14] proved the existence and the uniqueness of the traveling-wave solution for one-sided F-KPP equation. Harris et. al. [14] also gave a strong law of large numbers for the Lyapunov exponent of  $Z_t^{(0,\infty)}((\lambda t, \infty))$  and proved a strong law of large numbers and a large deviation principle for the right-most position  $R_t$ . The weak convergence for the extremal process was recently proved by Yang and Zhu [28].

There are also many related researches on asymptotic expansions for supercritical branching random walks and branching Wiener processes. Kaplan and Asmussen [16, 17] dealt with branching random walks, they proved that, under the assumption that the step of the random walk has mean 0 and variance 1, for any  $b \in \mathbb{R}$ , almost surely,

$$\lim_{n \rightarrow \infty} \frac{1}{\mathbb{E}(Z_n(\mathbb{R}))} Z_n((-\infty, \sqrt{nb}]) = W_\infty \int_{-\infty}^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \tag{1.3}$$

and that, under the additional assumption that the step of the random walk has finite 3rd moment, for any  $a < b$ ,

$$\lim_{n \rightarrow \infty} \frac{\sqrt{2\pi n}}{\mathbb{E}(Z_n(\mathbb{R}))} Z_n((a, b)) = (b - a)W_\infty, \tag{1.4}$$

where  $W_\infty$  is a martingale limit. Subsequently, quite a few papers studied asymptotic behaviors of the type (1.3) and (1.4) for general branching random walks, for example, see [7, 8, 9, 10, 26] and the references therein. To the best of our knowledge no one has been able to establish the asymptotic expansions of  $Z_n((-\infty, \sqrt{nb}])$  and  $Z_n((a, b))$  for general branching random walks yet. See [10, Conjecture 2.7]. Asymptotic expansions have been established in the special case of branching Wiener processes or branching lattice random walks in [11, 12]. In the continuous-time setting, Kang [15] proved a strong law of large numbers for branching Brownian motions.

### 1.2 Main results

We will assume that

$$\sum_{k=1}^{\infty} k(\log k)^{1+\lambda} p_k < \infty, \tag{1.5}$$

for some  $\lambda > 0$ . Let  $H_k$  be the  $k$ -th order Hermite polynomial:  $H_0(x) := 1$  and for  $k \geq 1$ ,

$$H_k(x) := \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{k!(-1)^j}{2^j j!(k-2j)!} x^{k-2j}.$$

It is well known that, if  $\{(B_t)_{t \geq 0}, \Pi_0\}$  is a standard Brownian motion (throughout this paper, we use  $\Pi_x$  to denote both the law of a standard Brownian motion starting from  $x$  and the corresponding expectation), then, for any  $k \geq 1$ ,  $\{t^{k/2} H_k(B_t/\sqrt{t}), \sigma(B_s : s \leq t), \Pi_0\}$  is a martingale. Throughout this paper  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Now for  $\theta \in [0, \sqrt{2})$  and

$k \in \mathbb{N}$ , we define

$$M_t^{(2k+1,\theta)} := e^{-(1-\frac{\theta^2}{2})t} \sum_{u \in N(t)} 1_{\{\min_{s \leq t} X_u(s) > 0\}} e^{\theta X_u(t)} t^{(2k+1)/2} H_{2k+1} \left( \frac{X_u(t)}{\sqrt{t}} \right), \quad t \geq 0. \quad (1.6)$$

We will prove later (see Proposition 3.1) that, for any  $x > 0$ ,  $k \in \mathbb{N}$  and  $\theta \in [0, \sqrt{2})$ ,  $M_t^{(2k+1,\theta)}$  is a martingale, and if (1.5) holds for some large  $\lambda$ , then  $M_t^{(2k+1,\theta)}$  converges to a limit  $M_\infty^{(2k+1,\theta)}$   $\mathbb{P}_x$ -almost surely and in  $L^1(\mathbb{P}_x)$ .

**Theorem 1.1.** Assume the drift  $\theta$  is in  $(0, \sqrt{2})$ . For any given  $m \in \mathbb{N}$ , if (1.5) holds for some  $\lambda > 6m + 6$ , then for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all intervals  $A \subset (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-3/2}e^{(1-\frac{\theta^2}{2})t}} = -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} e^{-\theta z} dz + o(t^{-m}).$$

Note that, from Theorem 1.1, one can immediately get the asymptotic expansion of

$$\frac{Z_t^{(0,\infty)}((a, b))}{t^{-3/2}e^{(1-\frac{\theta^2}{2})t}}$$

for any finite interval  $(a, b) \subset (0, \infty)$ . For the case  $\theta = 0$ , the result is a little bit different. For finite intervals, the normalization function is the same as in Theorem 1.1. For infinite intervals, the normalization function is different.

**Theorem 1.2.** Assume that the drift  $\theta$  is 0. For any given  $m \in \mathbb{N}$ , if (1.5) holds for some  $\lambda > 6m + 6$ , then

(i) for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all bounded intervals  $A \subset (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-3/2}e^t} = -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} dz + o(t^{-m});$$

(ii) for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all  $A = (a, \infty)$  and  $A = [a, \infty)$  with  $a \geq 0$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-1/2}e^t} = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!} a^{2\ell-2k} + o(t^{-m}).$$

**Remark 1.3.** Note that we only dealt with the case that the branching rate is 1 and the mean number of offspring is 2 in the two theorems above. In the general case when the branching rate is  $\beta > 0$  and the mean number of offspring is  $\mu > 1$ , one can use the same argument to prove the following counterpart of Theorem 1.1: Let  $\theta \in (0, \sqrt{2\beta(\mu-1)})$ . For any given  $m \in \mathbb{N}$ , if (1.5) holds for some  $\lambda > 6m + 6$ , then for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all intervals  $A \subset (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-3/2}e^{(\beta(\mu-1)-\frac{\theta^2}{2})t}} = -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} e^{-\theta z} dz + o(t^{-m}),$$

with  $M_\infty^{(2k+1,\theta)}$  given by

$$M_\infty^{(2k+1,\theta)} := \lim_{t \rightarrow \infty} e^{-(\beta(\mu-1) - \frac{\theta^2}{2})t} \sum_{u \in N(t)} 1_{\{\min_{s \leq t} X_u(s) > 0\}} e^{\theta X_u(t)} t^{(2k+1)/2} H_{2k+1} \left( \frac{X_u(t)}{\sqrt{t}} \right). \quad (1.7)$$

The counterpart of Theorem 1.2 in the general case is as follows: For any given  $m \in \mathbb{N}$ , if (1.5) holds for some  $\lambda > 6m + 6$ , then

(i) for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all bounded intervals  $A \subset (0, \infty)$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-3/2} e^{\beta(\mu-1)t}} = -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} dz + o(t^{-m});$$

(ii) for any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all  $A = (a, \infty)$  and  $A = [a, \infty)$  with  $a \geq 0$ , as  $t \rightarrow \infty$ ,

$$\frac{Z_t^{(0,\infty)}(A)}{t^{-1/2} e^{\beta(\mu-1)t}} = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell}(0)}{t^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!} a^{2\ell-2k} + o(t^{-m}),$$

with  $M_\infty^{(2k+1,0)}$  given in (1.7).

**Remark 1.4.** Note that if  $\sum_{k=1}^{\infty} k^{1+\epsilon} p_k < \infty$  for some  $\epsilon > 0$ , then the conclusions of Theorems 1.1 and 1.2 hold for all  $m \in \mathbb{N}$ . Also, combining (1.2), Theorems 1.1 and 1.2, we easily see that for any  $x > 0$  and  $\theta \in [0, \sqrt{2})$ ,  $\mathbb{P}_x$ -almost surely,

$$W(\theta) = \frac{M_\infty^{(1,\theta)}}{\mathbb{E}_x(M_\infty^{(1,\theta)})} = \frac{1}{x e^{\theta x}} M_\infty^{(1,\theta)}.$$

Our moment condition  $\lambda > 6m + 6$  is not optimal. It is an interesting question to find the optimal moment condition. We do not explore this in the present paper.

**Remark 1.5.**  $M_\infty^{(1,\theta)}$  is a non-negative martingale. It was proved in [22] that, when  $\theta \in (0, \sqrt{2})$ , the limit  $M_\infty^{(1,\theta)}$  is almost surely strictly positive on the survival event. When  $k \geq 1$ , the martingales  $M_\infty^{(2k+1,\theta)}$  are not non-negative. We did not try to determine whether the limit  $M_\infty^{(2k+1,\theta)}$  is almost surely non-zero on the survival event.

Our strategy for proving these Theorems 1.1 and 1.2 is as follows. We will choose appropriate  $\kappa > 1$  depending on the order  $m$  of the expansion and define

$$r_n := n^{\frac{1}{\kappa}}, \quad n \in \mathbb{N}.$$

We first study the asymptotic expansion along  $\{r_n : n \in \mathbb{N}\}$ , which is given by Proposition 3.4, and then control the behavior for  $t \in (r_n, r_{n+1})$ , see Lemma 3.6 below.

More precisely, in the proof of Lemma 3.2, for a particle  $u$  alive at time  $r_n$ , we let  $G_{n,u}$  be the event that, for all  $v < u$  with death time  $d_v \in (\sqrt{r_n}, r_n)$ , the number  $O_v$  of offspring of  $v$  is no bigger than  $e^{c_0 \sqrt{r_n}}$  for some small  $c_0$  and then use  $G_{n,u}$  to define the process  $Z_{r_n}^{(0,\infty),G}$  in (3.16). For any interval  $A \subset (0, \infty)$ , we use the first moment method to show that  $Z_{r_n}^{(0,\infty)}(A) \approx Z_{r_n}^{(0,\infty),G}(A)$  and then use the  $\ell$ -th moment method to prove that  $Z_{r_n}^{(0,\infty),G}(A) \approx \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A) | \mathcal{F}_{\sqrt{r_n}} \right)$ , here  $\ell \in (1, (2 \wedge (2/\theta^2)))$  is chosen to ensure the decay of the  $\ell$ -th moment. Lemma 3.3 is a generalization of Lemma 3.2. In Lemma 3.3 we

show that one can find a  $\widehat{\Omega}$  with  $\mathbb{P}_x(\widehat{\Omega}) = 1$  such that  $Z_{r_n}^{(0,\infty)}(A) \approx \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A) \mid \mathcal{F}_{\sqrt{r_n}} \right)$  for all  $\omega \in \widehat{\Omega}$  and all intervals  $A \subset (0, \infty)$ . The key idea in this part is that for each  $n$ , we compare the quantities with the corresponding quantities associated with partitions of mesh size  $1/\lceil r_n^{m+\epsilon} \rceil$  for some  $\epsilon > 0$ , and use these comparisons to prove a uniformly result. Once Lemma 3.3 is proved, Proposition 3.4 follows directly by standard computation.

Although the focus of [17] was (1.3) and (1.4) for branching random walks, the method and idea used in [17] were guidelines in the study of asymptotic expansion of branching random walks (for example, see [9, 11]) and branching random walks in random environment (for example, see [8, 10]). The technique of [17] was improved and generalized in the literatures (see, for instance, [8, 9, 10, 11] and the references therein). The main idea of the present paper is also inspired by [17], but we have to make some modifications. Take one of the work, [8], as a comparison. In [8, beginning of the proof of Lemma 5.1], Gao and Liu defined a quantity  $X_{n,u}$  containing both of the jumping size and the population size, and truncated it appropriately. In our case, we do not mix the population size and the spatial motion together, and we use a different truncation. In (3.16), our choice for the truncation is to control the the offspring sizes of those ancestors of particles alive at time  $r_n$  with death time in  $(\sqrt{r_n}, r_n)$ , which is an effective way of separating the genealogy and the spatial motion. This truncation method has been used to study the limit behaviors of the derivative martingale of branching Brownian motions in [23].

To control the behaviors for  $t \in (r_n, r_{n+1})$ , we first give a lower bound of  $Z_t^{(0,\infty)}(A)$  (see Lemma 3.5), then prove Lemma 3.6 by finding an upper bound of  $Z_t^{(0,\infty)}((0, \infty))$ . The idea comes from Kang [15], but we need to make some modifications. Since the number of the particles grows exponentially, we can not just choose the same  $r_n = n\delta$  for some  $\delta > 0$  as in [15] to get the convergence rate of order  $r_n^m$ . We select particles alive at time  $r_n$  that do not split during the small time  $(r_n, r_{n+1})$  and show that the fluctuations of these particles are relative small. Then we can show that simultaneously for all intervals  $J \subset (0, \infty)$ ,  $Z_t^{(0,\infty)}(J) \approx Z_{r_n}^{(0,\infty)}(J)$  for all  $t \in (r_n, r_{n+1})$ .

Once we have Proposition 3.4 and Lemma 3.6, the proofs of Theorems 1.1 and 1.2 are straight-forward.

## 2 Preliminaries

### 2.1 Spine decomposition

Recall that  $(\mathcal{F}_t)$  is the natural filtration of our branching Brownian motion. Define

$$\frac{d\mathbf{P}_x}{d\mathbf{P}_x \Big|_{\mathcal{F}_t}} := \frac{\sum_{u \in N(t)} 1}{e^t}, \tag{2.1}$$

then, under  $\mathbf{P}_x$ , the branching Brownian motion has the following spine decomposition(see [19], or [25] for a more general case):

- (i) there is an initial marked particle at  $x \in \mathbb{R}$  which moves according to standard Brownian motion with drift  $-\theta$ ;
- (ii) the branching rate of this marked particle is 2;
- (iii) when the marked particle dies at site  $y$ , it gives birth to  $\widehat{L}$  children with  $\mathbf{P}_x(\widehat{L} = k) = kp_k/2$ ;
- (iv) one of these children is uniformly selected and marked, and the marked child evolves as its parent and the other children evolve with law  $\mathbf{P}_y$ , where  $\mathbf{P}_y$  denotes the law of a branching Brownian motion starting at  $y$ , and all the children evolve independently.

We use  $\xi_t$  and  $X_\xi(t)$  to denote the marked particle at time  $t$  and the position of this marked particle respectively. We also use  $d_i$  to denote the  $i$ -th splitting time of the spine and  $O_i$  to denote the number of children produced by the spine at time  $d_i$ . By [25, Theorem 2.11], we can get that for  $u \in N(t)$ ,

$$\mathbf{P}_x(\xi_t = u | \mathcal{F}_t) = \frac{1}{\sum_{u \in N(t)} 1}. \tag{2.2}$$

Using (2.2), we get the following many-to-one formula:

**Lemma 2.1.** *For any  $x \in \mathbb{R}$ ,  $t > 0$  and  $u \in N(t)$ , let  $\Gamma(u, t)$  be a non-negative  $\mathcal{F}_t$ -measurable random variable. Then*

$$\mathbf{E}_x\left(\sum_{u \in N(t)} \Gamma(u, t)\right) = e^t \mathbf{E}_x(\Gamma(\xi_t, t)).$$

*Proof.* Combining (2.1) and (2.2), we get

$$\begin{aligned} \mathbf{E}_x\left(\sum_{u \in N(t)} \Gamma(u, t)\right) &= e^t \mathbf{E}_x\left(\sum_{u \in N(t)} \frac{\Gamma(u, t)}{\sum_{v \in N(t)} 1}\right) = e^t \mathbf{E}_x\left(\sum_{u \in N(t)} \Gamma(u, t) \mathbf{P}_x(\xi_t = u | \mathcal{F}_t)\right) \\ &= e^t \mathbf{E}_x\left(\mathbf{E}_x\left(\sum_{u \in N(t)} 1_{\{\xi_t = u\}} \Gamma(u, t) | \mathcal{F}_t\right)\right) = e^t \mathbf{E}_x\left(\Gamma(\xi_t, t) \sum_{u \in N(t)} 1_{\{\xi_t = u\}}\right) = e^t \mathbf{E}_x(\Gamma(\xi_t, t)). \quad \square \end{aligned}$$

**2.2 Some useful facts**

We will use the notation  $f(x) \lesssim g(x), \forall x \in E$ , to denote that there exists some constant  $C$  independent of  $x$  such that  $f(x) \leq Cg(x)$  for all  $x \in E$ . For any subset  $A$  of  $[0, \infty)$ , we define  $\sup A := \sup\{y : y \in A\}$ .

**Lemma 2.2.** *(i) Let  $\ell \in [1, 2]$ . Then for any finite family of independent centered random variables  $\{X_i : i = 1, \dots, n\}$  with  $\mathbf{E}|X_i|^\ell < \infty$  for all  $i = 1, \dots, n$ , it holds that*

$$\mathbf{E}\left|\sum_{i=1}^n X_i\right|^\ell \leq 2 \sum_{i=1}^n \mathbf{E}|X_i|^\ell.$$

*(ii) For any  $\ell \in [1, 2]$  and any random variable  $X$  with  $\mathbf{E}|X|^\ell < \infty$ ,*

$$\mathbf{E}|X - \mathbf{E}X|^\ell \lesssim \mathbf{E}|X|^\ell \leq (\mathbf{E}X^2)^{\ell/2}.$$

*Proof.* For (i), see [27, Theorem 2]. (ii) follows easily from Jensen’s inequality. □

We will use  $\Pi_x^{-\theta}$  to denote both the law of a Brownian motion with drift  $-\theta$  starting from  $x$  and the corresponding expectation. Let  $\phi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$  and  $\Phi(x) := \int_{-\infty}^x \phi(y) dy$ .

**Lemma 2.3.** *(i) Let  $(B_t, \Pi_x^\uparrow)$  be a 3-dimensional Bessel process with transition density*

$$p_t^\uparrow(x, y) = 1_{\{y > 0\}} \frac{ye^{-(y-x)^2/(2t)}}{x\sqrt{2\pi t}} \left(1 - e^{-2xy/t}\right) = 1_{\{y > 0\}} \frac{y}{x\sqrt{t}} \left(\phi\left(\frac{y-x}{\sqrt{t}}\right) - \phi\left(\frac{y+x}{\sqrt{t}}\right)\right).$$

*Then for any  $\theta \in [0, \sqrt{2})$ , we have*

$$\left. \frac{d\Pi_x^\uparrow}{d\Pi_x^{-\theta}} \right|_{\sigma(B_s : s \leq t)} = \frac{B_t e^{\theta(B_t-x) + \frac{\theta^2}{2}t}}{x} 1_{\{\min_{s \leq t} B_s > 0\}}.$$

*(ii) For any  $t, x, y > 0$ ,*

$$p_t^\uparrow(x, y) \lesssim \frac{y^2}{t^{3/2}} e^{-(x-y)^2/(2t)} \leq \frac{y^2}{t^{3/2}}.$$

*Proof.* (ii) follows from the inequality  $1 - e^{-x} \leq x, x > 0$ . For (i), note that under  $\Pi_x$ , both

$$\frac{B_t}{x} 1_{\{\min_{s \leq t} B_s > 0\}} \quad \text{and} \quad e^{-\theta(B_t - x) - \frac{\theta^2}{2}t}$$

are mean 1 non-negative martingales and that we have the following change-of-measure:

$$\frac{d\Pi_x^\uparrow}{d\Pi_x} \Big|_{\sigma(B_s: s \leq t)} = \frac{B_t}{x} 1_{\{\min_{s \leq t} B_s > 0\}} \quad \text{and} \quad \frac{d\Pi_x^{-\theta}}{d\Pi_x} \Big|_{\sigma(B_s: s \leq t)} = e^{-\theta(B_t - x) - \frac{\theta^2}{2}t}.$$

Therefore, (i) follows from

$$\frac{d\Pi_x^\uparrow}{d\Pi_x^{-\theta}} \Big|_{\sigma(B_s: s \leq t)} = \frac{d\Pi_x^\uparrow}{d\Pi_x} \Big|_{\sigma(B_s: s \leq t)} \times \left( \frac{d\Pi_x^{-\theta}}{d\Pi_x} \Big|_{\sigma(B_s: s \leq t)} \right)^{-1} = \frac{B_t e^{\theta(B_t - x) + \frac{\theta^2}{2}t}}{x} 1_{\{\min_{s \leq t} B_s > 0\}}. \quad \square$$

**Lemma 2.4.** (i) Let  $\theta \in (0, \sqrt{2})$ . For any  $x, t > 0$  and Borel set  $A \subset (0, \infty)$ , it holds that

$$\Pi_x^{-\theta} \left( \min_{s \leq t} B_s > 0, B_t \in A \right) = \sqrt{\frac{2}{\pi}} x e^{\theta x} t^{-3/2} e^{-\frac{\theta^2}{2}t} \left( \int_A y e^{-\theta y} dy + \varepsilon_A(x, t) \right), \quad (2.3)$$

with  $\varepsilon_A(x, t)$  satisfying

$$|\varepsilon_A(x, t)| \leq C_\theta \left( 1 \wedge \frac{(x+1)^2}{t} \right)$$

for some constant  $C_\theta$  depending on  $\theta$  only. In particular, for any fixed  $\theta \in (0, \sqrt{2})$ ,

$$\Pi_x^{-\theta} \left( \min_{s \leq t} B_s > 0, B_t \in A \right) \lesssim x e^{\theta x} t^{-3/2} e^{-\frac{\theta^2}{2}t} (1_{\{\sup A = \infty\}} + (\sup A) |A| 1_{\{\sup A < \infty\}}), \quad (2.4)$$

where  $|A|$  represents the Lebesgue measure of  $A$ .

(ii) For any Borel set  $A \subset (0, \infty)$  and  $x, t > 0$ , it holds that

$$\Pi_x \left( \min_{s \leq t} B_s > 0, B_t \in A \right) \lesssim x t^{-1/2} 1_{\{\sup A = \infty\}} + x t^{-3/2} |A| (\sup A) 1_{\{\sup A < \infty\}}.$$

(iii) For any  $x, t > 0, \theta \in [0, \sqrt{2})$  and Borel set  $A \subset (0, \infty)$ , it holds that

$$\Pi_x^{-\theta} (B_t \in A) \leq e^{\theta x} e^{-\frac{\theta^2}{2}t}.$$

*Proof.* (i) For (2.3), see [22, Lemma 3.1]. Now we prove (2.4). According to Lemma 2.3, it holds that

$$\begin{aligned} \Pi_x^{-\theta} \left( \min_{s \leq t} B_s > 0, B_t \in A \right) &= x e^{\theta x} e^{-\frac{\theta^2}{2}t} \Pi_x^\uparrow \left( \frac{1}{B_t} e^{-\theta B_t} 1_{\{B_t \in A\}} \right) \\ &= x e^{\theta x} e^{-\frac{\theta^2}{2}t} \int_A \frac{1}{y} e^{-\theta y} p_t^\uparrow(x, y) dy \lesssim x e^{\theta x} e^{-\frac{\theta^2}{2}t} t^{-3/2} \int_A y e^{-\theta y} dy, \end{aligned}$$

as desired.

(ii) When  $\sup A = \infty$ , by the reflection principle for Brownian motion, we have

$$\Pi_x \left( \min_{s \leq t} B_s > 0 \right) = \Pi_0(|B_t| \leq x) = 2 \int_0^x \frac{1}{\sqrt{2\pi t}} e^{-y^2/(2t)} dy \lesssim \frac{x}{\sqrt{t}}.$$

When  $\sup A < \infty$ , by Lemma 2.3,

$$\Pi_x \left( \min_{s \leq t} B_s > 0, B_t \in A \right) = x \Pi_x^\uparrow \left( \frac{1_A(B_t)}{B_t} \right) = x \int_A \frac{1}{y} p_t^\uparrow(x, y) dy \lesssim \frac{x}{t^{3/2}} \int_A y dy \lesssim \frac{x |A| \sup A}{t^{3/2}}.$$

(iii) By Girsanov's theorem,

$$\Pi_x^{-\theta} (B_t \in A) \leq \Pi_x^{-\theta} (B_t > 0) = \Pi_x \left( e^{-\theta(B_t - x) - \frac{\theta^2}{2}t} 1_{\{B_t > 0\}} \right) \leq e^{\theta x} e^{-\frac{\theta^2}{2}t}. \quad \square$$

**Lemma 2.5.** (i) For any  $k \geq 1$  and  $x \in \mathbb{R}$ ,

$$|H_k(x)| \leq 2\sqrt{k!}e^{x^2/4}.$$

Consequently, it holds that

$$\sup_{y \in \mathbb{R}} |\phi(y)H_k(y)| \leq \sqrt{\frac{2}{\pi}}\sqrt{k!} \sup_{y \in \mathbb{R}} e^{-y^2/4} = \sqrt{\frac{2}{\pi}}\sqrt{k!}.$$

(ii) For any  $k \in \mathbb{N}$ , there exists a constant  $C(k)$  such that for all  $x \in \mathbb{R}$ ,

$$|H_{2k+1}(x)| \leq C(k)|x|(|x|^{2k} + 1).$$

*Proof.* For (i), see [11, (4.1)]; (ii) follows from the definition of  $H_{2k+1}(x)$ . □

**Lemma 2.6.** For  $(\rho, b, x) \in (-1, 1) \times \mathbb{R} \times \mathbb{R}$ , uniformly on compact subsets of  $(-1, 1) \times \mathbb{R} \times \mathbb{R}$ , it holds that

$$\Phi\left(\frac{b - \rho x}{\sqrt{1 - \rho^2}}\right) = \Phi(b) - \phi(b) \sum_{k=1}^{\infty} \frac{\rho^k}{k!} H_{k-1}(b)H_k(x).$$

*Proof.* See [11, Lemma 4.2.]. □

Recall that  $r_n = n^{1/\kappa}$ . Applying Lemma 2.6 with  $\rho = r_n^{-1/4}$ ,  $b = r_n^{-1/2}z$  and  $x = r_n^{-1/4}y$ , we get that for any  $z, y \in \mathbb{R}$ ,

$$\Phi\left(\frac{z - y}{\sqrt{r_n - \sqrt{r_n}}}\right) = \Phi\left(\frac{z}{\sqrt{r_n}}\right) - \phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{r_n^{k/2}} H_{k-1}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{k/4} H_k\left(\frac{y}{r_n^{1/4}}\right).$$

Noting that, for any  $k \in \mathbb{N}$ ,  $H_{2k}$  is an even function and  $H_{2k+1}$  is an odd function, we get that

$$\begin{aligned} & \Phi\left(\frac{z + y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \Phi\left(\frac{z - y}{\sqrt{r_n - \sqrt{r_n}}}\right) \\ &= 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^{\infty} \frac{1}{(2k + 1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right). \end{aligned} \tag{2.5}$$

**Lemma 2.7.** For any given  $m \in \mathbb{N}$  and  $\kappa > 1$ , let  $K > 0$  be a fixed constant and  $J$  be an integer such that  $J > 2m + \frac{K\kappa - 1}{2}$ . Then for any  $y, z \in \mathbb{R}$ , it holds that

$$\begin{aligned} & \Phi\left(\frac{z + y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \Phi\left(\frac{z - y}{\sqrt{r_n - \sqrt{r_n}}}\right) \\ &= 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^J \frac{1}{(2k + 1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right) + \varepsilon_{m,y,z,n,\kappa} \end{aligned} \tag{2.6}$$

with

$$r_n^{(2m+1)/2} \sup \left\{ |\varepsilon_{m,y,z,n,\kappa}| : z \in \mathbb{R}, |y| \leq \sqrt{K\sqrt{r_n} \log n} \right\} \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* By Lemma 2.5 (i), for all  $k \geq 1$ ,  $z \in \mathbb{R}$  and  $|y| \leq \sqrt{K\sqrt{r_n} \log n}$ ,

$$\begin{aligned} & \frac{r_n^{(2m+1)/2}}{(2k + 1)!} \frac{1}{r_n^{(2k+1)/2}} \left| \phi\left(\frac{z}{\sqrt{r_n}}\right) H_{2k}\left(\frac{z}{\sqrt{r_n}}\right) \right| \cdot \left| r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right) \right| \\ & \leq \frac{4r_n^{(2m+1)/2}}{r_n^{(2k+1)/4}} \frac{1}{\sqrt{2\pi}} e^{y^2/(4\sqrt{r_n})} \leq \frac{4}{\sqrt{2\pi}} \frac{1}{n^{(2k-1-4m)/(4\kappa)}} n^{K/4}. \end{aligned}$$

Combining this with (2.5), we get that (2.6) holds with

$$\begin{aligned} & r_n^{(2m+1)/2} \sup_{z \in \mathbb{R}, |y| \leq \sqrt{K\sqrt{r_n} \log n}} |\varepsilon_{m,y,z,n}| \\ & \leq \frac{8}{\sqrt{2\pi}} \sum_{k=J+1}^{\infty} \frac{1}{n^{(2k-1-4m)/(4\kappa)}} n^{K/4} \lesssim \frac{1}{n^{(2(J+1)-1-4m)/(4\kappa)}} n^{K/4}, \end{aligned}$$

which tends to 0 is since  $J > 2m + \frac{K\kappa-1}{2}$ . □

Taking derivative with respect to  $x$  in Lemma 2.6, and using the facts that

$$\frac{d^k}{dx^k} \Phi(x) = (-1)^{k-1} H_{k-1}(x) \phi(x), \quad H'_k(x) = k H_{k-1}(x), \quad k \geq 1, \quad (2.7)$$

we get that

$$\frac{1}{\sqrt{1-\rho^2}} \phi\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) = \phi(b) + \phi(b) \sum_{k=1}^{\infty} \frac{\rho^k}{k!} H_k(b) H_k(x). \quad (2.8)$$

Taking  $\rho = r_n^{-1/4}$ ,  $b = r_n^{-1/2} z$  and  $x = r_n^{-1/4} y$  in (2.8), we get that for any  $z, y \in \mathbb{R}$ ,

$$\begin{aligned} & \frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}} \phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) \\ & = \phi\left(\frac{z}{\sqrt{r_n}}\right) \left(1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{r_n^{k/2}} H_k\left(\frac{z}{\sqrt{r_n}}\right) r_n^{k/4} H_k\left(\frac{y}{r_n^{1/4}}\right)\right). \end{aligned} \quad (2.9)$$

Noting that, for any  $k \in \mathbb{N}$ ,  $H_{2k}$  is an even function and  $H_{2k+1}$  is an odd function, we deduce from (2.9) that

$$\begin{aligned} & \frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}} \left( \phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right) \right) \\ & = 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right). \end{aligned}$$

Using an argument similar to that leading to Lemma 2.7, we also have the following lemma. We omit the proof.

**Lemma 2.8.** *For any given  $m \in \mathbb{N}$  and  $\kappa > 1$ , let  $K > 0$  be a fixed constant and  $J$  be an integer such that  $J > 2m + \frac{K\kappa+1}{2}$ . Then for any  $y, z \in \mathbb{R}$ , it holds that*

$$\begin{aligned} & \frac{\sqrt{r_n}}{\sqrt{r_n - \sqrt{r_n}}} \left( \phi\left(\frac{z-y}{\sqrt{r_n - \sqrt{r_n}}}\right) - \phi\left(\frac{z+y}{\sqrt{r_n - \sqrt{r_n}}}\right) \right) \\ & = 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{y}{r_n^{1/4}}\right) + \varepsilon_{m,y,z,n,\kappa} \end{aligned}$$

with

$$r_n^{m+1} \sup \left\{ |\varepsilon_{m,y,z,n,\kappa}| : z \in \mathbb{R}, |y| \leq \sqrt{K\sqrt{r_n} \log n} \right\} \xrightarrow{n \rightarrow \infty} 0.$$

### 3 Proofs of the main results

#### 3.1 Convergence rate for the martingales

**Proposition 3.1.** *Suppose  $x > 0$  and  $\theta \in [0, \sqrt{2})$ . (i) For any  $k \in \mathbb{N}$ ,  $\{M_t^{(2k+1, \theta)}, t \geq 0; \mathbb{P}_x\}$  is a martingale. (ii) If (1.5) holds for some  $\lambda > k$ , then  $M_t^{(2k+1, \theta)}$  converges to a limit  $M_\infty^{(2k+1, \theta)}$   $\mathbb{P}_x$ -a.s. and in  $L^1(\mathbb{P}_x)$ . Moreover, for any  $\eta \in (0, \lambda - k)$ , as  $t \rightarrow \infty$ ,*

$$M_t^{(2k+1, \theta)} - M_\infty^{(2k+1, \theta)} = o(t^{-(\lambda-k)+\eta}), \quad \mathbb{P}_x\text{-a.s.}$$

*Proof.* (i) We will use  $v < u$  to denote that  $v$  is an ancestor of  $u$  and  $v \leq u$  to denote  $v = u$  or  $v < u$ . By the Markov property and Lemma 2.1, for any  $t, s > 0$ ,

$$\begin{aligned} \mathbb{E}_x \left( M_{s+t}^{(2k+1, \theta)} | \mathcal{F}_t \right) &= e^{-(1-\frac{\theta^2}{2})(t+s)} \sum_{v \in N(t)} 1_{\{\min_{r \leq t} X_v(r) > 0\}} (t+s)^{(2k+1)/2} \\ &\quad \times \mathbb{E}_{X_v(t)} \left( \sum_{u \in N(t+s): v \leq u} 1_{\{\min_{r \leq s} X_v(t+r) > 0\}} e^{\theta X_v(t+s)} H_{2k+1} \left( \frac{X_u(t+s)}{\sqrt{t+s}} \right) | \mathcal{F}_t \right) \\ &= e^{-(1-\frac{\theta^2}{2})(t+s)} \sum_{v \in N(t)} 1_{\{\min_{r \leq t} X_v(r) > 0\}} \\ &\quad \times e^s \mathbb{E}_{X_v(t)} \left( 1_{\{\min_{r \leq s} X_\xi(r) > 0\}} e^{\theta X_\xi(s)} (t+s)^{(2k+1)/2} H_{2k+1} \left( \frac{X_\xi(s)}{\sqrt{t+s}} \right) \right) \\ &=: e^{-(1-\frac{\theta^2}{2})(t+s)} \sum_{v \in N(t)} 1_{\{\min_{r \leq t} X_v(r) > 0\}} F(s, t, X_v(t)). \end{aligned} \tag{3.1}$$

Note that  $X_\xi(s)$  under  $\mathbb{P}_x$  is a standard Brownian motion with drift  $-\theta$ . It follows from Lemma 2.3 (i) that

$$\begin{aligned} F(s, t, X_v(t)) &= e^{(1-\frac{\theta^2}{2})s} X_v(t) e^{\theta X_v(t)} \Pi_{X_v(t)}^\uparrow \left( \frac{(t+s)^{(2k+1)/2}}{B_s} H_{2k+1} \left( \frac{B_s}{\sqrt{t+s}} \right) \right) \\ &= e^{(1-\frac{\theta^2}{2})s} X_v(t) e^{\theta X_v(t)} \int_0^\infty \frac{(t+s)^{(2k+1)/2}}{y} H_{2k+1} \left( \frac{y}{\sqrt{t+s}} \right) \\ &\quad \times \frac{y}{X_v(t)\sqrt{s}} \left( \phi \left( \frac{y - X_v(t)}{\sqrt{s}} \right) - \phi \left( \frac{y + X_v(t)}{\sqrt{s}} \right) \right) dy \\ &= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_0^\infty (t+s)^{(2k+1)/2} H_{2k+1} \left( \frac{y}{\sqrt{t+s}} \right) \frac{\phi \left( \frac{y - X_v(t)}{\sqrt{s}} \right) - \phi \left( \frac{y + X_v(t)}{\sqrt{s}} \right)}{\sqrt{s}} dy. \end{aligned}$$

Using the fact that  $H_{2k+1}(\cdot)$  is an odd function and that  $\phi(\cdot)$  is an even function, we have

$$\begin{aligned} F(s, t, X_v(t)) &= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^\infty (t+s)^{(2k+1)/2} H_{2k+1} \left( \frac{y}{\sqrt{t+s}} \right) \frac{\phi \left( \frac{y - X_v(t)}{\sqrt{s}} \right)}{\sqrt{s}} dy \\ &= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^\infty (t+s)^{(2k+1)/2} H_{2k+1}(z) \frac{\sqrt{t+s}}{\sqrt{s}} \phi \left( \frac{\sqrt{t+s}z - X_v(t)}{\sqrt{s}} \right) dz, \end{aligned}$$

where for the last equality we used the change of variables  $z = y/\sqrt{t+s}$ . Taking  $\rho = \sqrt{t/(s+t)}, b = z, x = X_v(t)/\sqrt{t}$  in (2.8), we see that

$$\frac{\sqrt{t+s}}{\sqrt{s}} \phi \left( \frac{\sqrt{t+s}z - X_v(t)}{\sqrt{s}} \right) = \phi(z) + \phi(z) \sum_{\ell=1}^\infty \frac{t^{\ell/2}}{\ell!} \frac{1}{(t+s)^{\ell/2}} H_\ell(z) H_\ell \left( \frac{X_v(t)}{\sqrt{t}} \right).$$

Combining these with Lemma 2.5(i) and the fact  $\phi(z)e^{z^2/4} \lesssim e^{-z^2/4}$ , we can easily get that for any  $s, t > 0, y \in \mathbb{R}$ , the series

$$H_{2k+1}(z) \phi(z) \sum_{\ell=1}^{\infty} \frac{t^{\ell/2}}{\ell!} \frac{1}{(t+s)^{\ell/2}} H_{\ell}(z) H_{\ell}\left(\frac{y}{\sqrt{t}}\right)$$

is uniformly convergent in  $z \in \mathbb{R}$ . Now applying the property  $\int_{-\infty}^{\infty} H_m(z) H_n(z) \phi(z) dz = \delta_{m,n} n!$ , we get

$$\begin{aligned} &F(s, t, X_v(t)) \\ &= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} \int_{-\infty}^{\infty} (t+s)^{(2k+1)/2} \frac{t^{(2k+1)/2}}{(2k+1)!} \frac{H_{2k+1}(z) \phi(z)}{(t+s)^{(2k+1)/2}} H_{2k+1}(z) H_{2k+1}\left(\frac{X_v(t)}{\sqrt{t}}\right) dz \\ &= e^{(1-\frac{\theta^2}{2})s} e^{\theta X_v(t)} t^{(2k+1)/2} H_{2k+1}\left(\frac{X_v(t)}{\sqrt{t}}\right). \end{aligned}$$

Plugging this into (3.1), we get (i).

(ii) Suppose (1.5) holds for some  $\lambda > k$ . If the first assertion of (ii) holds along  $t = n \in \mathbb{N}$ , then it is valid along all  $t$  since for  $t \in (n, n+1)$ ,  $M_t^{(2k+1, \theta)} = \mathbb{E}_x \left( M_{n+1}^{(2k+1, \theta)} | \mathcal{F}_t \right) = \mathbb{E}_x \left( M_{\infty}^{(2k+1, \theta)} | \mathcal{F}_t \right)$ . In the following we use two steps to prove the assertion of (ii) holds for the case  $t = n \in \mathbb{N}$ .

**Step 1:** In this step, we will define a truncated process  $M_{n+1}^{(2k+1, \theta, B)}$  and give first moment estimate for  $M_{n+1}^{(2k+1, \theta)} - M_{n+1}^{(2k+1, \theta, B)}$ . Let  $d_v, O_v$  denote the death time and the number of offspring of  $v$  respectively. For  $u \in N(n+1)$ , let  $B_{n,u}$  be the event that, for all  $v < u$  with  $d_v \in (n, n+1)$ , it holds that  $O_v \leq e^{c_0 n}$ , where  $c_0 > 0$  is a small constant to be determined later. Define

$$\begin{aligned} M_{n+1}^{(2k+1, \theta, B)} &:= e^{-(1-\frac{\theta^2}{2})(n+1)} \\ &\times \sum_{u \in N(n+1)} \mathbf{1}_{\{\min_{s \leq n+1} X_u(s) > 0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right) \mathbf{1}_{B_{n,u}}. \end{aligned}$$

By the branching property, it holds that

$$\begin{aligned} &\left| M_{n+1}^{(2k+1, \theta)} - M_{n+1}^{(2k+1, \theta, B)} \right| \leq e^{-(1-\frac{\theta^2}{2})(n+1)} \\ &\quad \times \sum_{u \in N(n+1)} \mathbf{1}_{\{\min_{s \leq n+1} X_u(s) > 0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} \left| H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right) \right| \mathbf{1}_{B_{n,u}^c} \\ &= e^{-(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} \\ &\quad \times \sum_{u \in N(n+1): v \leq u} \mathbf{1}_{\{\min_{s \leq 1} X_u(n+s) > 0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} \left| H_{2k+1}\left(\frac{X_u(n+1)}{\sqrt{n+1}}\right) \right| \mathbf{1}_{B_{n,u}^c}. \end{aligned}$$

Using the Markov property first and then Lemma 2.5(ii), we get that

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \middle| \mathcal{F}_n \right) \\ & \leq e^{-(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} \\ & \quad \times \mathbb{E}_{X_v(n)} \left( \sum_{u \in N(1)} \mathbf{1}_{\{\min_{s \leq 1} X_u(s) > 0\}} e^{\theta X_u(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_u(1)}{\sqrt{n+1}} \right) \right| \mathbf{1}_{D_{n,u}^c} \right) \\ & \lesssim e^{-(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} \\ & \quad \times \mathbb{E}_{X_v(n)} \left( \sum_{u \in N(1)} \mathbf{1}_{\{\min_{s \leq 1} X_u(s) > 0\}} X_u(1) e^{\theta X_u(1)} \left( (X_u(1))^{2k} + (n+1)^k \right) \mathbf{1}_{D_{n,u}^c} \right), \end{aligned}$$

where for  $u \in N(1)$ ,  $D_{n,u}$  denotes the event that, for all  $w < u$  with  $d_w < 1$ , it holds that  $O_w \leq e^{c_0 n}$ . Recall that  $d_i$  is the  $i$ -th splitting time of the spine and  $O_i$  is the number of children produced by the spine at time  $d_i$ . Define  $D_{n,\xi_1}$  to be the event that, for all  $i$  with  $d_i < 1$ , it holds that  $O_i \leq e^{c_0 n}$ . By Lemma 2.1,

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \middle| \mathcal{F}_n \right) \\ & \lesssim e^{-(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} \\ & \quad \times e \mathbf{E}_{X_v(n)} \left( \mathbf{1}_{\{\min_{s \leq 1} X_\xi(s) > 0\}} X_\xi(1) e^{\theta X_\xi(1)} \left( X_\xi(1)^{2k} + (n+1)^k \right) \mathbf{1}_{D_{n,\xi_1}^c} \right) \\ & =: e^{1-(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} R(X_v(n)). \end{aligned} \tag{3.2}$$

Conditioned on the motion  $X_\xi$ ,  $\{d_i : i \geq 1\}$  are the atoms of a Poisson point process with rate 2 and  $\{O_i : i \in \mathbb{N}\}$  are iid copies of  $\widehat{L}$  with law  $\mathbf{P}_x(\widehat{L} = \ell) = \ell p_\ell / 2$  which are independent of  $\{d_i : i \in \mathbb{N}\}$ . Therefore,  $D_{n,\xi_1}$  is independent of  $X_\xi(t)$ . Together with Lemma 2.3 (i), we get that

$$\begin{aligned} R(X_v(n)) & = \mathbf{E}_{X_v(n)} \left( \mathbf{1}_{\{\min_{s \leq 1} X_\xi(s) > 0\}} X_\xi(1) e^{\theta X_\xi(1)} \left( X_\xi(1)^{2k} + (n+1)^k \right) \mathbf{E}_{X_v(n)} \left( \mathbf{1}_{D_{n,\xi_1}^c} \right) \right) \\ & \leq e^{-\frac{\theta^2}{2}} X_v(n) e^{\theta X_v(n)} \Pi_{X_v(n)}^\uparrow \left( B_1^{2k} + (n+1)^k \right) \mathbf{E}_{X_v(n)} \left( \sum_{i: d_i \leq 1} \mathbf{1}_{\{O_i > e^{c_0 n}\}} \right) \\ & = 2e^{-\frac{\theta^2}{2}} X_v(n) e^{\theta X_v(n)} \Pi_{X_v(n)}^\uparrow \left( B_1^{2k} + (n+1)^k \right) \mathbf{P}_x(\widehat{L} > e^{c_0 n}). \end{aligned} \tag{3.3}$$

Noticing that  $(B_t, \Pi_x^\uparrow)$  is a 3-dimensional Bessel process, we easily see that

$$\Pi_{X_v(n)}^\uparrow \left( B_1^{2k} + (n+1)^k \right) \leq \Pi_0^\uparrow \left( (B_1 + y)^{2k} + (n+1)^k \right) \Big|_{y=X_v(n)} \lesssim (X_v(n))^{2k} + n^k. \tag{3.4}$$

Noting that (1.5) implies  $\mathbf{E}_x(\log_+^{1+\lambda} \widehat{L}) < \infty$ , using (3.3) and (3.4), we obtain that

$$R(X_v(n)) \lesssim \frac{1}{n^{1+\lambda}} e^{-\frac{\theta^2}{2}} X_v(n) e^{\theta X_v(n)} \left( (X_v(n))^{2k} + n^k \right).$$

Plugging this inequality into (3.2), we conclude that

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \middle| \mathcal{F}_n \right) \\ & \lesssim \frac{e^{-(1-\frac{\theta^2}{2})n}}{n^{1+\lambda}} \sum_{v \in N(n)} \mathbf{1}_{\{\min_{s \leq n} X_v(s) > 0\}} X_v(n) e^{\theta X_v(n)} \left( (X_v(n))^{2k} + n^k \right). \end{aligned}$$

Taking expectation with respect to  $\mathbb{P}_x$ , applying Lemma 2.1 first and then Lemma 2.3 (i), we get that

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \right) \\ & \lesssim \frac{e^{-(1-\frac{\theta^2}{2})n}}{n^{1+\lambda}} \mathbb{E}_x \left( \sum_{v \in N(n)} 1_{\{\min_{s \leq n} X_v(s) > 0\}} X_v(n) e^{\theta X_v(n)} ((X_v(n))^{2k} + n^k) \right) \\ & = \frac{e^{\frac{\theta^2}{2}n}}{n^{1+\lambda}} \mathbb{E}_x \left( 1_{\{\min_{s \leq n} X_\xi(s) > 0\}} e^{\theta X_\xi(n)} X_\xi(n) ((X_\xi(n))^{2k} + n^k) \right) \\ & = \frac{x e^{\theta x}}{n^{1+\lambda}} \Pi_x^\uparrow ((B_n)^{2k} + n^k) \leq \frac{x e^{\theta x}}{n^{1+\lambda}} \Pi_0^\uparrow ((B_n + x)^{2k} + n^k) \\ & \lesssim \frac{1}{n^{1+\lambda}} \Pi_0^\uparrow ((B_n)^{2k} + n^k) = \frac{n^k}{n^{1+\lambda}} \Pi_0^\uparrow ((B_1)^{2k} + 1), \end{aligned} \tag{3.5}$$

where in the last equality, we used the fact that  $(B_t, \Pi_0^\uparrow) \stackrel{d}{=} (\sqrt{t}B_1, \Pi_0^\uparrow)$ .

**Step 2:** In this step, we will give an upper bound for the  $\ell$ -th moment of  $M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left( M_{n+1}^{(2k+1,\theta,B)} | \mathcal{F}_n \right)$  for appropriate  $\ell \in (1, 2)$ . Combining this with Step 1 will yield the first result of (ii). For  $v \in N(n)$ , set

$$J_{n,v} := \sum_{u \in N(n+1): v \leq u} 1_{\{\min_{s \leq 1} X_u(n+s) > 0\}} e^{\theta X_u(n+1)} (n+1)^{(2k+1)/2} H_{2k+1} \left( \frac{X_u(n+1)}{\sqrt{n+1}} \right) 1_{B_{n,u}}.$$

By the branching property,  $\{J_{n,v} : v \in N(n)\}$  are independent conditioned on  $\mathcal{F}_n$ . Thus, for any fixed  $1 < \ell < \min\{2/\theta^2, 2\}$  with  $(\ell - 1)^2 \theta^2 / 2 < (\ell - 1)(1 - \frac{\theta^2}{2})$ , by Lemma 2.2,

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left( M_{n+1}^{(2k+1,\theta,B)} | \mathcal{F}_n \right) \right|^\ell | \mathcal{F}_n \right) \\ & = e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \mathbb{E}_x \left( \left| \sum_{v \in N(n)} 1_{\{\min_{s \leq n} X_v(s) > 0\}} (J_{n,v} - \mathbb{E}_x(J_{n,v} | \mathcal{F}_n)) \right|^\ell | \mathcal{F}_n \right) \\ & \leq 2e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} 1_{\{X_v(n) > 0\}} \mathbb{E}_x \left( |J_{n,v} - \mathbb{E}_x(J_{n,v} | \mathcal{F}_n)|^\ell | \mathcal{F}_n \right) \\ & \lesssim e^{-\ell(1-\frac{\theta^2}{2})(n+1)} \sum_{v \in N(n)} 1_{\{X_v(n) > 0\}} (\mathbb{E}_x((J_{n,v})^2 | \mathcal{F}_n))^{\ell/2}. \end{aligned} \tag{3.6}$$

Define

$$J_n^* := \sum_{u \in N(1)} 1_{\{\min_{s \leq 1} X_u(s) > 0\}} e^{\theta X_u(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_u(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,u}}.$$

By the Markov property and Lemma 2.1,

$$\begin{aligned} & \mathbb{E}_x \left( (J_{n,v})^2 | \mathcal{F}_n \right) \leq \mathbb{E}_{X_v(n)} \left( (J_n^*)^2 \right) \\ & = \mathbb{E}_{X_v(n)} \left( \sum_{u \in N(1)} 1_{\{\min_{s \leq 1} X_u(s) > 0\}} e^{\theta X_u(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_u(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,u}} J_n^* \right) \\ & = e \mathbb{E}_{X_v(n)} \left( 1_{\{\min_{s \leq 1} X_\xi(s) > 0\}} e^{\theta X_\xi(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_\xi(1)}{\sqrt{n+1}} \right) \right| 1_{D_{n,\xi_1}} J_n^* \right). \end{aligned} \tag{3.7}$$

Conditioned on  $\{X_\xi, d_i, O_i : i \geq 1\}$ , by the Markov property, on the event  $D_{n,\xi_1}$ , we have

$$\begin{aligned}
 & \mathbf{E}_{X_v(n)} \left( J_n^* | X_\xi, d_i, O_i : i \geq 1 \right) \\
 &= \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} (O_i - 1) \mathbf{E}_{X_\xi(d_i)} \left( \sum_{u \in N(z)} \mathbf{1}_{\{\min_{s \leq z} X_u(s) > 0\}} \right. \\
 &\quad \left. \times e^{\theta X_u(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_u(z)}{\sqrt{n+1}} \right) \right| \mathbf{1}_{D_{n,u}} \right) \Big|_{z=1-d_i} \\
 &\leq \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} (O_i - 1) \\
 &\quad \times \mathbf{E}_{X_\xi(d_i)} \left( \sum_{u \in N(z)} \mathbf{1}_{\{\min_{s \leq z} X_u(s) > 0\}} e^{\theta X_u(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_u(z)}{\sqrt{n+1}} \right) \right| \right) \Big|_{z=1-d_i} \\
 &\leq e^{c_0 n} \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} e^{1-d_i} \\
 &\quad \times \mathbf{E}_{X_\xi(d_i)} \left( \mathbf{1}_{\{\min_{s \leq z} X_\xi(s) > 0\}} e^{\theta X_\xi(z)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_\xi(z)}{\sqrt{n+1}} \right) \right| \right) \Big|_{z=1-d_i} \\
 &=: e^{c_0 n} \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} e^{1-d_i} F_n(X_\xi(d_i), 1-d_i), \tag{3.8}
 \end{aligned}$$

where in the last inequality we used the fact that  $O_i - 1 \leq e^{c_0 n}$  on  $D_{n,\xi_1}$  and Lemma 2.1. Using Lemma 2.5(ii) in the first inequality, Lemma 2.3 (i) in the first equality and an argument similar to that leading to (3.4) in the second inequality, we get

$$\begin{aligned}
 & F_n(X_\xi(d_i), 1-d_i) \lesssim \mathbf{E}_{X_\xi(d_i)} \left( \mathbf{1}_{\{\min_{s \leq z} X_\xi(s) > 0\}} e^{\theta X_\xi(z)} X_\xi(z) \left( (X_\xi(z))^{2k} + (n+1)^k \right) \right) \Big|_{z=1-d_i} \\
 &= X_\xi(d_i) e^{\theta X_\xi(d_i)} e^{-\frac{\theta^2}{2}(1-d_i)} \Pi_{X_\xi(d_i)}^\uparrow \left( (B_z)^{2k} + (n+1)^k \right) \Big|_{z=1-d_i} \\
 &\lesssim X_\xi(d_i) e^{\theta X_\xi(d_i)} \left( (X_\xi(d_i))^{2k} + n^k \right) \sup_{z \in (0,1)} \Pi_0^\uparrow \left( (B_z)^{2k} + 1 \right) \\
 &\lesssim X_\xi(d_i) e^{\theta X_\xi(d_i)} \left( (X_\xi(d_i))^{2k} + n^k \right). \tag{3.9}
 \end{aligned}$$

Combining (3.8) and (3.9), we obtain that

$$\begin{aligned}
 & \mathbf{E}_{X_v(n)} \left( J_n^* | X_\xi, d_i, O_i : i \geq 1 \right) \\
 &\lesssim e^{c_0 n} \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} e^{1-d_i} X_\xi(d_i) e^{\theta X_\xi(d_i)} \left( (X_\xi(d_i))^{2k} + n^k \right) \\
 &\lesssim e^{c_0 n} \sum_{i:d_i \leq 1} \mathbf{1}_{\{\min_{s \leq d_i} X_\xi(s) > 0\}} X_\xi(d_i) e^{\theta X_\xi(d_i)} \left( (X_\xi(d_i))^{2k} + n^k \right). \tag{3.10}
 \end{aligned}$$

Plugging (3.10) into (3.7),

$$\begin{aligned}
 & \mathbb{E}_x \left( (J_{n,v})^2 | \mathcal{F}_n \right) \lesssim e^{c_0 n} \mathbf{E}_{X_v(n)} \left( \mathbf{1}_{\{\min_{s \leq 1} X_\xi(s) > 0\}} e^{\theta X_\xi(1)} (n+1)^{(2k+1)/2} \left| H_{2k+1} \left( \frac{X_\xi(1)}{\sqrt{n+1}} \right) \right| \right. \\
 &\quad \left. \times \int_0^1 \mathbf{1}_{\{\min_{s \leq r} X_\xi(s) > 0\}} X_\xi(r) e^{\theta X_\xi(r)} \left( (X_\xi(r))^{2k} + n^k \right) dr \right) \\
 &\lesssim e^{c_0 n} \mathbf{E}_{X_v(n)} \left( \mathbf{1}_{\{\min_{s \leq 1} X_\xi(s) > 0\}} e^{\theta X_\xi(1)} X_\xi(1) \left( (X_\xi(1))^{2k} + (n+1)^k \right) \right. \\
 &\quad \left. \times \int_0^1 \mathbf{1}_{\{\min_{s \leq r} X_\xi(s) > 0\}} X_\xi(r) e^{\theta X_\xi(r)} \left( (X_\xi(r))^{2k} + n^k \right) dr \right), \tag{3.11}
 \end{aligned}$$

where the last inequality follows by Lemma 2.5(ii). Using Lemma 2.3, we can continue the estimate (3.11) and get

$$\begin{aligned} & \mathbb{E}_x \left( (J_{n,v})^2 \middle| \mathcal{F}_n \right) \\ & \lesssim X_v(n) e^{\theta X_v(n)} e^{c_0 n} \Pi_{X_v(n)}^\uparrow \left( ((B_1)^{2k} + (n+1)^k) \int_0^1 B_r e^{\theta B_r} ((B_r)^{2k} + n^k) dr \right) \\ & \leq X_v(n) e^{2\theta X_v(n)} e^{c_0 n} \Pi_0^\uparrow \left( ((B_1 + y)^{2k} + (n+1)^k) \right. \\ & \quad \left. \times \int_0^1 (B_r + y) e^{\theta B_r} ((B_r + y)^{2k} + n^k) dr \right) \Big|_{y=X_v(n)} \\ & \lesssim e^{c_0 n} (X_v(n) + 1)^2 e^{2\theta X_v(n)} ((X_v(n))^{2k} + n^k)^2. \end{aligned} \tag{3.12}$$

Combining (3.6) and (3.12), we have

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left( M_{n+1}^{(2k+1,\theta,B)} \middle| \mathcal{F}_n \right) \right|^\ell \middle| \mathcal{F}_n \right) \\ & \lesssim e^{-\ell(1-\frac{\theta^2}{2})(n+1)} e^{c_0 \ell n/2} \sum_{v \in N(n)} 1_{\{X_v(n) > 0\}} (X_v(n) + 1)^\ell e^{\ell \theta X_v(n)} ((X_v(n))^{2k} + n^k)^\ell \\ & \leq e^{-\ell(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} \sum_{v \in N(n)} (|X_v(n)| + 1)^\ell e^{\ell \theta X_v(n)} ((X_v(n))^{2k} + n^k)^\ell. \end{aligned}$$

Taking expectation with respect to  $\mathbb{P}_x$  and applying Lemma 2.1, we conclude that

$$\begin{aligned} & \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left( M_{n+1}^{(2k+1,\theta,B)} \middle| \mathcal{F}_n \right) \right|^\ell \right) \\ & \lesssim e^{-\ell(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^n \mathbb{E}_x \left( (|X_\xi(n)| + 1)^\ell e^{\ell \theta X_\xi(n)} ((X_\xi(n))^{2k} + n^k)^\ell \right) \\ & = e^{-\ell(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^n \Pi_x^{-\theta} \left( (|B_n| + 1)^\ell e^{\ell \theta B_n} ((B_n)^{2k} + n^k)^\ell \right) \\ & = e^{\theta x} e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} \Pi_x \left( (|B_n| + 1)^\ell e^{(\ell-1)\theta B_n} ((B_n)^{2k} + n^k)^\ell \right) \\ & = e^{\theta \ell x} e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^{\frac{(\ell-1)^2 \theta^2}{2} n} \Pi_x^{(\ell-1)\theta} \left( (|B_n| + 1)^\ell ((B_n)^{2k} + n^k)^\ell \right) \\ & \lesssim e^{-(\ell-1)(1-\frac{\theta^2}{2})n} e^{c_0 \ell n/2} e^{\frac{(\ell-1)^2 \theta^2}{2} n} n^{2k\ell + \ell} =: n^{2k\ell + \ell} e^{-c_1 \ell n}, \end{aligned} \tag{3.13}$$

where in the second and third equalities we used the change-of-measure  $\frac{d\Pi_x^\eta}{d\Pi_x} \Big|_{\sigma(B_s: s \leq t)} = e^{\eta(B_t - x) - \frac{\eta^2}{2} t}$  for  $\eta = -\theta$  and  $\eta = (\ell - 1)\theta$  respectively. Let  $c_0 > 0$  be sufficiently small so that  $c_0 \ell / 2 < (\ell - 1)(1 - \frac{\theta^2}{2}) - (\ell - 1)^2 \theta^2 / 2$  with  $\ell \theta^2 / 2 < 1$ , which implies that  $c_1 > 0$ . Thus, using the inequality:

$$\begin{aligned} \mathbb{E} (|X - \mathbb{E}(X|\mathcal{F})|) & \leq \mathbb{E} (|X - Y|) + \mathbb{E} (|Y - \mathbb{E}(Y|\mathcal{F})|) + \mathbb{E} (|\mathbb{E}(X - Y|\mathcal{F})|) \\ & \leq 2\mathbb{E} (|X - Y|) + \mathbb{E} (|Y - \mathbb{E}(Y|\mathcal{F})|)^\ell)^{1/\ell}, \end{aligned} \tag{3.14}$$

(3.5) and (3.13), we get

$$\begin{aligned} & \sum_{n=1}^\infty \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) \\ & \leq 2 \sum_{n=1}^\infty \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_{n+1}^{(2k+1,\theta,B)} \right| \right) + \sum_{n=1}^\infty \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta,B)} - \mathbb{E}_x \left( M_{n+1}^{(2k+1,\theta,B)} \middle| \mathcal{F}_n \right) \right|^\ell \right)^{1/\ell} \\ & \lesssim \sum_{n=1}^\infty \frac{n^k}{n^{1+\lambda}} + \sum_{n=1}^\infty n^{2k+1} e^{-c_1 n}, \end{aligned}$$

which is finite since  $\lambda > k$ . Therefore,  $M_n^{(2k+1,\theta)}$  converges to a limit  $M_\infty^{(2k+1,\theta)}$   $\mathbb{P}_x$ -a.s. and in  $L^1(\mathbb{P}_x)$ .

**Step 3:** In this step, we prove the second assertion of (ii). For any  $\eta \in (0, \lambda - k)$ , by (3.5) and (3.13),

$$\sum_{n=1}^{\infty} n^{\lambda-k-\eta} \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) \lesssim \sum_{n=1}^{\infty} n^{\lambda-k-\eta} \frac{n^k}{n^{1+\lambda}} + \sum_{n=1}^{\infty} n^{\lambda-k-\eta} n^{2k+1} e^{-c_1 n} < \infty.$$

Thus,  $n^{\lambda-k-\eta} \left( M_n^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right) \xrightarrow{n \rightarrow \infty} 0$ ,  $\mathbb{P}_x$ -a.s. (see for example [1, Lemma 2]). For  $s \in [n, n + 1]$ , by Doob's inequality, for any  $\varepsilon > 0$ ,

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{P}_x \left( n^{\lambda-k-\eta} \sup_{n \leq s \leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| > \varepsilon \right) \\ & \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} n^{\lambda-k-\eta} \mathbb{E}_x \left( \left| M_{n+1}^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \right) < \infty, \end{aligned}$$

which implies that  $n^{\lambda-k-\eta} \sup_{n \leq s \leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \xrightarrow{n \rightarrow \infty} 0$ ,  $\mathbb{P}_x$ -a.s. Therefore, we have  $\mathbb{P}_x$ -almost surely,

$$\begin{aligned} \sup_{n \leq s \leq n+1} s^{\lambda-k-\eta} \left| M_s^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right| & \leq (n+1)^{\lambda-k-\eta} \sup_{n \leq s \leq n+1} \left| M_s^{(2k+1,\theta)} - M_n^{(2k+1,\theta)} \right| \\ & + (n+1)^{\lambda-k-\eta} \left| M_n^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right| \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

which completes the proof of (ii). □

### 3.2 Asymptotic expansions along discrete time

**Lemma 3.2.** Assume  $x > 0$  and  $\theta \in [0, \sqrt{2}]$ . Let  $\{A_n\}$  be a family of Borel subsets of  $(0, \infty)$  such that

$$\text{either } \sup_n \sup A_n < \infty \text{ or } \inf_n \sup A_n = \infty.$$

For any given  $m \in \mathbb{N}$  and  $\kappa > 1$ , if (1.5) holds for some  $\lambda > 2m + 2\kappa + 1$ , then

$$\lim_{n \rightarrow \infty} r_n^m \frac{Z_{r_n}^{(0,\infty)}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-b_\theta} e^{(1-\frac{\theta^2}{2})r_n}} = 0, \quad \mathbb{P}_x\text{-a.s.} \tag{3.15}$$

where

$$b_\theta := \begin{cases} 3/2, & \theta = 0 \text{ and } \sup_n \sup A_n < \infty \text{ or } \theta \in (0, \sqrt{2}); \\ 1/2, & \theta = 0 \text{ and } \inf_n \sup A_n = \infty. \end{cases}$$

In particular, for any Borel set  $A \subset (0, \infty)$ , as  $n \rightarrow \infty$ ,

$$r_n^m \frac{Z_{r_n}^{(0,\infty)}(A) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-b_\theta} e^{(1-\frac{\theta^2}{2})r_n}} \rightarrow 0, \quad \mathbb{P}_x\text{-a.s.}$$

*Proof.* Suppose  $m \in \mathbb{N}$ ,  $\kappa > 1$  and that (1.5) holds for some  $\lambda > 2m + 2\kappa + 1$ . We divide the proof into three steps. In Step 1, we define a truncated process  $Z_{r_n}^{(0,\infty),G}(A_n)$  and give a first-moment estimate for  $Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n)$ , see (3.23) below. In Step 2, we bound the  $\ell$ -th moment of  $Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right)$  for appropriate

$\ell \in (1, 2)$ , see (3.29) below; In Step 3, we combine the results obtained in Step 1 and Step 2 to get the assertion of the proposition.

**Step 1:** Recall that  $v < u$  and  $v \leq u$  mean that  $v$  is an ancestor of  $u$  and that  $v = u$  or  $v < u$  respectively. For  $u \in N(r_n)$ , define  $G_{n,u}$  to be the event that, for all  $v < u$  with death time  $d_v \in (\sqrt{r_n}, r_n)$ , it holds that  $O_v \leq e^{c_0\sqrt{r_n}}$ , where  $c_0 > 0$  is a small constant to be determined later. Define

$$Z_{r_n}^{(0,\infty),G}(A_n) := \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{A_n}(X_u(r_n)) 1_{G_{n,u}}. \tag{3.16}$$

By the branching property, it holds that

$$\begin{aligned} & Z_{r_n}^{(0,\infty),G}(A_n) \\ &= \sum_{v \in N(\sqrt{r_n})} 1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} \sum_{u \in N(r_n): v \leq u} 1_{\{\min_{\sqrt{r_n} < s \leq r_n} X_u(s) > 0\}} 1_{A_n}(X_u(r_n)) 1_{G_{n,u}}. \end{aligned} \tag{3.17}$$

Therefore, by the Markov property,

$$\begin{aligned} & \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n) \mid \mathcal{F}_{\sqrt{r_n}} \right) \\ &= \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{v \in N(\sqrt{r_n})} 1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} \\ & \quad \times \mathbb{E}_{X_v(\sqrt{r_n})} \left( \sum_{u \in N(r_n - \sqrt{r_n})} 1_{\{\min_{s \leq r_n - \sqrt{r_n}} X_u(s) > 0\}} 1_{A_n}(X_u(r_n - \sqrt{r_n})) 1_{D_{n,u}^c} \right) \\ &=: \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{v \in N(\sqrt{r_n})} 1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} T_{A_n}(X_v(\sqrt{r_n}), r_n - \sqrt{r_n}), \end{aligned} \tag{3.18}$$

where for  $u \in N(r_n - \sqrt{r_n})$ ,  $D_{n,u}$  denotes the event that, for all  $w < u$  with  $d_w < r_n - \sqrt{r_n}$ , it holds that  $O_w \leq e^{c_0\sqrt{r_n}}$ . Recall that  $d_i$  is the  $i$ -th splitting time of the spine and  $O_i$  is the number of children produced by the spine at time  $d_i$ . Define  $D_{n,\xi_{r_n - \sqrt{r_n}}}$  to be the event that, for all  $i$  with  $d_i < r_n - \sqrt{r_n}$ , it holds that  $O_i \leq e^{c_0\sqrt{r_n}}$ . By Lemma 2.1, we get that

$$\begin{aligned} & T_{A_n}(X_v(\sqrt{r_n}), r_n - \sqrt{r_n}) \\ &= e^{r_n - \sqrt{r_n}} \mathbb{E}_{X_v(\sqrt{r_n})} \left( 1_{\{\min_{s \leq r_n - \sqrt{r_n}} X_\xi(s) > 0\}} 1_{A_n}(X_\xi(r_n - \sqrt{r_n})) 1_{D_{n,\xi_{r_n - \sqrt{r_n}}}^c} \right). \end{aligned} \tag{3.19}$$

Note that given  $X_\xi$ ,  $\{d_i : i \geq 1\}$  are the atoms for a Poisson point process with rate 2,  $\{O_i : i \geq 1\}$  are iid with common law  $\widehat{L}$  given by  $\mathbf{P}_x(\widehat{L} = k) = kp_k/2$ , and that  $\{d_i : i \geq 1\}$  and  $\{O_i : i \geq 1\}$  are independent. By (1.5), we conclude that

$$\begin{aligned} & \mathbb{E}_{X_v(\sqrt{r_n})} \left( 1_{D_{n,\xi_{r_n - \sqrt{r_n}}}^c} \mid X_\xi(s) : s \geq 0 \right) \leq \mathbb{E}_{X_v(\sqrt{r_n})} \left( \sum_{i: d_i < r_n - \sqrt{r_n}} 1_{\{O_i > e^{c_0\sqrt{r_n}}\}} \mid X_\xi(s) : s \geq 0 \right) \\ &= 2 \int_0^{r_n - \sqrt{r_n}} \mathbf{P}_{X_v(\sqrt{r_n})} \left( \widehat{L} > e^{c_0\sqrt{r_n}} \right) ds \leq 2r_n \frac{\mathbf{E}_x \left( \log_+^{1+\lambda} L \right)}{(c_0\sqrt{r_n})^{1+\lambda}} \lesssim \frac{1}{r_n^{(\lambda-1)/2}}. \end{aligned} \tag{3.20}$$

Plugging (3.20) into (3.19), we get

$$\begin{aligned} & T_{A_n}(X_v(\sqrt{r_n}), r_n - \sqrt{r_n}) \\ & \lesssim \frac{e^{r_n - \sqrt{r_n}}}{r_n^{(\lambda-1)/2}} \mathbf{P}_{X_v(\sqrt{r_n})} \left( \min_{s \leq r_n - \sqrt{r_n}} X_\xi(s) > 0, X_{r_n - \sqrt{r_n}} \in A_n \right). \end{aligned} \tag{3.21}$$

Since, under  $\mathbf{P}_{X_v(\sqrt{r_n})}$ ,  $X_\xi(t)$  is a standard Brownian motion with drift  $-\theta$ . By Lemma 2.4 (i) and (ii) with  $A = A_n$ , we get that

$$\begin{aligned} & \mathbf{P}_{X_v(\sqrt{r_n})} \left( \min_{s \leq r_n - \sqrt{r_n}} X_\xi(s) > 0, X_{r_n - \sqrt{r_n}} \in A_n \right) \\ & \lesssim \frac{X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})}}{(r_n - \sqrt{r_n})^{b_\theta} e^{\frac{\theta^2}{2}(r_n - \sqrt{r_n})}} \lesssim \frac{X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})}}{r_n^{b_\theta} e^{\frac{\theta^2}{2}(r_n - \sqrt{r_n})}}. \end{aligned} \tag{3.22}$$

Combining (3.18), (3.21) and (3.22), we get that

$$\begin{aligned} & \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \\ & \lesssim \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{v \in N(\sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} \frac{e^{r_n - \sqrt{r_n}} X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})}}{r_n^{(\lambda-1)/2} r_n^{b_\theta} e^{\frac{\theta^2}{2}(r_n - \sqrt{r_n})}} \\ & = \frac{1}{r_n^{(\lambda-1-2m)/2}} e^{-(1-\frac{\theta^2}{2})\sqrt{r_n}} \sum_{v \in N(\sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})} \\ & = \frac{1}{r_n^{(\lambda-1-2m)/2}} M_{\sqrt{r_n}}^{(1,\theta)}, \end{aligned}$$

with  $M_{\sqrt{r_n}}^{(1,\theta)}$  given in (1.6). Now taking expectation with respect to  $\mathbb{P}_x$ , we get that

$$\frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n) \right) \lesssim \frac{1}{r_n^{(\lambda-1-2m)/2}}. \tag{3.23}$$

**Step 2:** By (3.17) and the branching property,

$$Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) =: \sum_{v \in N(\sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} K_{n,v},$$

where conditioned on  $\mathcal{F}_{\sqrt{r_n}}$ ,  $\{K_{n,v} : v \in N(\sqrt{r_n})\}$  are centered independent random variables defined by

$$\begin{aligned} K_{n,v} & = \sum_{u \in N(r_n): v \leq u} \mathbf{1}_{\{\min_{\sqrt{r_n} < s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{A_n}(X_u(r_n)) \mathbf{1}_{G_{n,u}} \\ & \quad - \mathbb{E}_x \left( \sum_{u \in N(r_n): v \leq u} \mathbf{1}_{\{\min_{\sqrt{r_n} < s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{A_n}(X_u(r_n)) \mathbf{1}_{G_{n,u}} \middle| \mathcal{F}_{\sqrt{r_n}} \right). \end{aligned}$$

Thus, by Lemma 2.2, for any fixed  $\ell$  with  $1 < \ell < \min\{2, 2/\theta^2\}$  and  $(\ell - 1)\frac{\theta^2}{2} < 1 - \frac{\theta^2}{2}$ ,

$$\begin{aligned} & \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \middle| \mathcal{F}_{\sqrt{r_n}} \right) \\ & \leq 2 \sum_{v \in N(\sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} \mathbb{E}_x \left( |K_{n,v}|^\ell \middle| \mathcal{F}_{\sqrt{r_n}} \right) \lesssim \sum_{v \in N(\sqrt{r_n})} M_{n,v}, \end{aligned} \tag{3.24}$$

where for each  $v \in N(\sqrt{r_n})$ ,

$$M_{n,v} := \mathbb{E}_{X_v(\sqrt{r_n})} \left( \left( \sum_{u \in N(r_n - \sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} X_u(s) > 0\}} \mathbf{1}_{A_n}(X_u(r_n - \sqrt{r_n})) \mathbf{1}_{D_{n,u}} \right)^\ell \right).$$

Set  $V_n := \sum_{u \in N(r_n - \sqrt{r_n})} 1_{\{\min_{s \leq r_n - \sqrt{r_n}} X_u(s) > 0\}} 1_{A_n}(X_u(r_n - \sqrt{r_n})) 1_{D_{n,u}}$ . By Lemma 2.1 and the fact that  $A_n \subset (0, \infty)$ ,  $V_n \leq \sum_{u \in N(r_n - \sqrt{r_n})} 1_{(0, \infty)}(X_u(r_n - \sqrt{r_n}))$ , we have

$$\begin{aligned} M_{n,v} &= e^{r_n - \sqrt{r_n}} \mathbf{E}_{X_v(\sqrt{r_n})} \left( 1_{\{\min_{s \leq r_n - \sqrt{r_n}} X_\xi(s) > 0\}} 1_{A_n}(X_\xi(r_n - \sqrt{r_n})) 1_{D_{n,\xi_{r_n - \sqrt{r_n}}}} V_n^{\ell-1} \right) \\ &\leq e^{r_n - \sqrt{r_n}} \mathbf{E}_{X_v(\sqrt{r_n})} \left( 1_{D_{n,\xi_{r_n - \sqrt{r_n}}}} 1_{\{X_\xi(r_n - \sqrt{r_n}) > 0\}} \left( \sum_{u \in N(r_n - \sqrt{r_n})} 1_{\{X_u(r_n - \sqrt{r_n}) > 0\}} \right)^{\ell-1} \right). \end{aligned} \tag{3.25}$$

Given  $X_\xi, d_i$  and  $O_i$ , by the Markov property and the inequality  $(\sum_{i=1}^n x_i)^p \leq \sum_{i=1}^n x_i^p$  for all  $x_i \geq 0$  and  $p \in (0, 1)$ , it holds that

$$\begin{aligned} &1_{\{D_{n,\xi_{r_n - \sqrt{r_n}}}\}} \mathbf{E}_{X_v(\sqrt{r_n})} \left( \left( \sum_{u \in N(r_n - \sqrt{r_n})} 1_{\{X_u(r_n - \sqrt{r_n}) > 0\}} \right)^{\ell-1} \mid X_\xi, d_i, O_i : i \geq 1 \right) \\ &\leq 1_{\{D_{n,\xi_{r_n - \sqrt{r_n}}}\}} \sum_{i: d_i < r_n - \sqrt{r_n}} (O_i - 1) \mathbf{E}_{X_\xi(d_i)} \left( \left( \sum_{u \in N(z)} 1_{\{X_u(z) > 0\}} \right)^{\ell-1} \mid_{z=r_n - \sqrt{r_n} - d_i} \right) \\ &\leq e^{c_0 \sqrt{r_n}} \sum_{i: d_i < r_n - \sqrt{r_n}} \left[ \mathbf{E}_{X_\xi(d_i)} \left( \sum_{u \in N(z)} 1_{\{X_u(z) > 0\}} \right)^{\ell-1} \mid_{z=r_n - \sqrt{r_n} - d_i} \right], \end{aligned} \tag{3.26}$$

where in the last inequality, we used the fact that, on the event  $D_{n,\xi_{r_n - \sqrt{r_n}}}$ ,  $O_i - 1 \leq e^{c_0 \sqrt{r_n}}$  and the fact that  $\mathbf{E}(|X|^p) \leq \mathbf{E}|X|^p$  for  $p \in (0, 1)$ . Note that by Lemma 2.1 and Lemma 2.4 (iii),

$$\mathbf{E}_x \left( \sum_{u \in N(z)} 1_{\{X_u(z) > 0\}} \right) \leq e^{(1 - \frac{\theta^2}{2})z} e^{\theta x}.$$

Using the fact that  $d_i$  are the atoms of a Poisson process with rate 2, taking expectation with respect to  $\mathbf{P}_{X_v(\sqrt{r_n})}(\cdot \mid X_\xi)$  in (3.26), we get that

$$\begin{aligned} &\mathbf{E}_{X_v(\sqrt{r_n})} \left( 1_{\{D_{n,\xi_{r_n - \sqrt{r_n}}}\}} \left( \sum_{u \in N(r_n - \sqrt{r_n})} 1_{\{X_u(r_n - \sqrt{r_n}) > 0\}} \right)^{\ell-1} \mid X_\xi \right) \\ &\leq 2e^{c_0 \sqrt{r_n}} \int_0^{r_n - \sqrt{r_n}} e^{\theta(\ell-1)X_\xi(s)} e^{(1 - \frac{\theta^2}{2})(\ell-1)(r_n - \sqrt{r_n} - s)} ds. \end{aligned} \tag{3.27}$$

Combining (3.25) and (3.27), noting that  $X_\xi(s)$  under  $\mathbf{P}_x$  is a standard Brownian motion with drift  $-\theta$ , and applying Lemma 2.4(iii), we conclude that

$$\begin{aligned} &M_{n,v} \\ &\lesssim e^{r_n - \sqrt{r_n}} \mathbf{E}_{X_v(\sqrt{r_n})} \left( e^{c_0 \sqrt{r_n}} 1_{\{X_\xi(r_n - \sqrt{r_n}) > 0\}} \int_0^{r_n - \sqrt{r_n}} e^{\theta(\ell-1)X_\xi(s)} e^{(1 - \frac{\theta^2}{2})(\ell-1)(r_n - \sqrt{r_n} - s)} ds \right) \\ &= e^{r_n - \sqrt{r_n}} e^{c_0 \sqrt{r_n}} \int_0^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( e^{\theta(\ell-1)B_s} \Pi_{B_s}^{-\theta} (B_{r_n - \sqrt{r_n} - s} > 0) \right) \\ &\quad \times e^{(1 - \frac{\theta^2}{2})(\ell-1)(r_n - \sqrt{r_n} - s)} ds \\ &\leq e^{r_n - \sqrt{r_n}} e^{c_0 \sqrt{r_n}} \int_0^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( e^{\theta B_s} \right) e^{-\frac{\theta^2}{2}(r_n - \sqrt{r_n} - s)} e^{(1 - \frac{\theta^2}{2})(\ell-1)(r_n - \sqrt{r_n} - s)} ds. \end{aligned}$$

Using elementary calculus in the last integral, we get

$$\begin{aligned} M_{n,v} &\lesssim e^{\ell\theta X_v(\sqrt{r_n})} e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})} e^{c_0\sqrt{r_n}} \int_0^{r_n-\sqrt{r_n}} e^{-s(\ell-1)(1-\frac{\theta^2}{2})-(\ell-1)\frac{\theta^2}{2}} ds \\ &\lesssim e^{\ell\theta X_v(\sqrt{r_n})} e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})} e^{c_0\sqrt{r_n}}. \end{aligned}$$

Plugging this upper bound into (3.24), we obtain that

$$\begin{aligned} &\frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \middle| \mathcal{F}_{\sqrt{r_n}} \right) \\ &\lesssim \frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \sum_{v \in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})} e^{\ell(1-\frac{\theta^2}{2})(r_n-\sqrt{r_n})} e^{c_0\sqrt{r_n}} \\ &= \frac{r_n^{\ell(m+b_\theta)} e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}} \sum_{v \in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})}. \end{aligned} \tag{3.28}$$

Taking expectation in (3.28) with respect to  $\mathbb{P}_x$ , and using Lemma 2.1, we get

$$\begin{aligned} &\frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \right) \\ &\lesssim \frac{r_n^{\ell(m+b_\theta)} e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}} \mathbb{E}_x \left( \sum_{v \in N(\sqrt{r_n})} e^{\theta\ell X_v(\sqrt{r_n})} \right) = \frac{r_n^{\ell(m+b_\theta)} e^{c_0\sqrt{r_n}}}{e^{\ell(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\sqrt{r_n}} \mathbb{E}_x \left( e^{\theta\ell X_\xi(\sqrt{r_n})} \right) \\ &= r_n^{\ell(m+b_\theta)} e^{\theta\ell x} e^{-((\ell-1)(1-\frac{\theta^2}{2})-c_0)\sqrt{r_n}}. \end{aligned} \tag{3.29}$$

**Step 3:** Fix  $c_0 \in (0, (\ell-1)(1-\frac{\theta^2}{2}))$  and set  $c_1 := ((\ell-1)(1-\frac{\theta^2}{2}) - c_0)/\ell > 0$ . Using (3.14) with  $X = Z_{r_n}^{(0,\infty)}(A_n)$  and  $Y = Z_{r_n}^{(0,\infty),G}(A_n)$ , we have

$$\begin{aligned} &\mathbb{E}_x \left( \left| \frac{r_n^m}{r_n^{-b_\theta} e^{(1-\frac{\theta^2}{2})r_n}} \left| Z_{r_n}^{(0,\infty)}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right| \right) \right) \\ &\leq 2 \frac{r_n^{m+b_\theta}}{e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty)}(A_n) - Z_{r_n}^{(0,\infty),G}(A_n) \right| \right) \\ &\quad + \left( \frac{r_n^{\ell(m+b_\theta)}}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty),G}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \right) \right)^{1/\ell} \\ &\lesssim \frac{1}{r_n^{(\lambda-1-2m)/2}} + r_n^{(m+b_\theta)} e^{-c_1\sqrt{r_n}} = \frac{1}{n^{(\lambda-1-2m)/(2\kappa)}} + n^{(m+b_\theta)/\kappa} e^{-c_1 n^{1/(2\kappa)}}, \end{aligned}$$

where in the last inequality, we used (3.23) and (3.29). Since  $\lambda > 2m + 2\kappa + 1$ , we conclude that for any  $\varepsilon > 0$ ,

$$\begin{aligned} &\sum_{n=1}^\infty \mathbb{P}_x \left( \left| \frac{r_n^m}{r_n^{-b_\theta} e^{(1-\frac{\theta^2}{2})r_n}} \left| Z_{r_n}^{(0,\infty)}(A_n) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A_n) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right| > \varepsilon \right) \tag{3.30} \\ &\lesssim \sum_{n=1}^\infty \left( \frac{1}{n^{(\lambda-1-2m)/(2\kappa)}} + n^{(m+b_\theta)/\kappa} e^{-c_1 n^{1/(2\kappa)}} \right) < \infty, \end{aligned}$$

which completes the proof of the Lemma. □

Applying Lemma 3.2 with  $A_n = A$  being a subinterval of  $(0, \infty)$ , we see that (3.15) holds for any fixed interval  $A$ . It is not clear that (3.15) holds simultaneously for all intervals  $A$ . In the next lemma we prove a slightly stronger result.

**Lemma 3.3.** Assume  $x > 0$  and  $\theta \in [0, \sqrt{2})$ . For any given  $m \in \mathbb{N}$  and  $\kappa > 1$ , suppose that (1.5) holds for some  $\lambda > 2m + 2\kappa + 1$ . Then there exists  $\hat{\Omega} = \hat{\Omega}(x, \theta, m, \kappa, \lambda)$  with  $\mathbb{P}_x(\hat{\Omega}) = 1$  such that for any  $w \in \hat{\Omega}$  and any interval  $A \subset (0, \infty)$ , it holds that

$$\lim_{n \rightarrow \infty} r_n^m \frac{Z_{r_n}^{(0, \infty)}(A) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(A) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-b_\theta^A} e^{(1 - \frac{\theta^2}{2})r_n}} = 0, \tag{3.31}$$

where

$$b_\theta^A := \begin{cases} 3/2, & \theta = 0 \text{ and } \sup A < \infty \text{ or } \theta \in (0, \sqrt{2}); \\ 1/2, & \theta = 0 \text{ and } \sup A = \infty. \end{cases} \tag{3.32}$$

*Proof. Step 1:* Fix  $m \in \mathbb{N}$ ,  $\kappa > 1$  and  $\epsilon > 0$ . For any  $k \in \mathbb{N}$  and  $n \geq 1$ , define  $a_i^{(n)} := k + i/[r_n^{m+\epsilon}]$ ,  $\mathcal{B}_i(n) := (a_i^{(n)}, \infty)$  and  $\bar{\mathcal{B}}_i(n) := [a_i^{(n)}, \infty)$  for  $0 \leq i \leq 2[r_n^{m+\epsilon}]$ . In this step, we are going to prove that, for any fixed  $k \in \mathbb{N}$ ,

$$\lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|}{(r_n^{-1/2} \mathbf{1}_{\theta=0} + r_n^{-3/2} \mathbf{1}_{\theta \in (0, \sqrt{2})}) e^{(1 - \frac{\theta^2}{2})r_n}} = 0, \text{ } \mathbb{P}_x\text{-a.s.}, \tag{3.33}$$

$$\lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|}{(r_n^{-1/2} \mathbf{1}_{\theta=0} + r_n^{-3/2} \mathbf{1}_{\theta \in (0, \sqrt{2})}) e^{(1 - \frac{\theta^2}{2})r_n}} = 0, \text{ } \mathbb{P}_x\text{-a.s.}, \tag{3.34}$$

$$\lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)^c) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)^c) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} = 0, \text{ } \mathbb{P}_x\text{-a.s.} \tag{3.35}$$

and

$$\lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)^c) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)^c) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} = 0, \text{ } \mathbb{P}_x\text{-a.s.} \tag{3.36}$$

We only prove (3.33). The proofs of (3.34), (3.35) and (3.36) are similar. Recall the definition of  $Z_{r_n}^{(0, \infty), G}$  in (3.16) and the fact that  $\mathcal{B}_i(n) \subset (k, \infty)$ , we see that

$$\begin{aligned} & \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left( Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)) - Z_{r_n}^{(0, \infty), G}(\mathcal{B}_i(n)) \right) \\ &= \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{\mathcal{B}_i(n)}(X_u(r_n)) \mathbf{1}_{G_{n,u}^c} \\ &\leq \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{[k, \infty)}(X_u(r_n)) \mathbf{1}_{G_{n,u}^c} \\ &= Z_{r_n}^{(0, \infty)}([k, \infty)) - Z_{r_n}^{(0, \infty), G}([k, \infty)). \end{aligned}$$

Therefore, by (3.23) with  $A_n = [k, \infty)$ , we have

$$\begin{aligned} & \frac{r_n^{m+1/2} \mathbf{1}_{\theta=0} + r_n^{m+3/2} \mathbf{1}_{\theta \in (0, \sqrt{2})}}{e^{(1 - \frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left( Z_{r_n}^{(0, \infty)}(\mathcal{B}_i(n)) - Z_{r_n}^{(0, \infty), G}(\mathcal{B}_i(n)) \right) \right) \\ &\lesssim \frac{1}{r_n^{(\lambda-1-2m)/2}}. \end{aligned}$$

On the other hand, for any fixed  $\ell$  with  $1 < \ell < \min\{2, 2/\theta^2\}$  and  $(\ell - 1)\frac{\theta^2}{2} < 1 - \frac{\theta^2}{2}$ , it naturally holds that

$$\begin{aligned} & \frac{1}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_x \left( \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0,\infty),G}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(\mathcal{B}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \right) \\ & \leq \frac{1}{e^{\ell(1-\frac{\theta^2}{2})r_n}} \sum_{i=0}^{2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( \left| Z_{r_n}^{(0,\infty),G}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty),G}(\mathcal{B}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right|^\ell \right) \\ & \lesssim e^{\theta\ell x} e^{-((\ell-1)(1-\frac{\theta^2}{2})-c_0)\sqrt{r_n}} \cdot (2[r_n^{m+\epsilon}] + 1), \end{aligned}$$

where the last inequality follows from (3.29). Using a similar argument leading to (3.30), we get (3.33).

**Step 2:** In this step, we prove that

$$\lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = 0, \quad \mathbb{P}_x\text{-a.s.} \tag{3.37}$$

Using the Markov property and Lemma 2.1, we see that

$$\begin{aligned} & \frac{\sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ & = \sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_{X_v(\sqrt{r_n})} \left( Z_{r_n - \sqrt{r_n}}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \right) \\ & = \sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ & \quad \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbf{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbf{1}_{[a_i^{(n)}, a_{i+1}^{(n)}]}(B_{r_n - \sqrt{r_n}}) \right). \end{aligned}$$

Combining the above inequality, Lemma 2.4 and the fact that  $a_{i+1}^{(n)} \leq k + 2$ , we see that

$$\begin{aligned} & r_n^m \frac{\sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ & \lesssim \frac{(k+2)r_n^m}{[r_n^{m+\epsilon}]^{\frac{3}{2}}} \frac{r_n^{3/2}}{(r_n - \sqrt{r_n})^{3/2}} e^{-(1-\frac{\theta^2}{2})\sqrt{r_n}} \sum_{v \in N(\sqrt{r_n})} \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}} X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})} \\ & \lesssim \frac{k+2}{r_n^\epsilon} M_{\sqrt{r_n}}^{(1,\theta)} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

which implies (3.37).

**Step 3:** In this step, we prove (3.31). For any  $k \in \mathbb{N}$ , there exists  $\tilde{\Omega}_k$  with  $\mathbb{P}_x(\tilde{\Omega}_k) = 1$  such that (3.33), (3.34), (3.35), (3.36) and (3.37) hold for all  $\omega \in \tilde{\Omega}_k$ . Now define

$$\widehat{\Omega} := \bigcap_{k \in \mathbb{N}} \tilde{\Omega}_k, \tag{3.38}$$

then  $\mathbb{P}_x(\widehat{\Omega}) = 1$ . We first deal with the case of all  $A = (a, \infty)$  with  $a \geq 0$ . Suppose that  $a \in (k, k + 1]$  for some  $k \in \mathbb{N}$ , then for each  $n \in \mathbb{N}$ , there exists  $0 \leq i < [r_n^{m+\epsilon}]$  such that

$a \in (a_i^{(n)}, a_{i+1}^{(n)})$ . Thus,

$$\begin{aligned} Z_{r_n}^{(0,\infty)}(A) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A) \middle| \mathcal{F}_{\sqrt{r_n}} \right) &\leq Z_{r_n}^{(0,\infty)}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(\mathcal{B}_{i+1}(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \\ &\leq \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0,\infty)}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(\mathcal{B}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right| \\ &\quad + \sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right). \end{aligned}$$

Similarly, we also have the lower bound:

$$\begin{aligned} Z_{r_n}^{(0,\infty)}(A) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A) \middle| \mathcal{F}_{\sqrt{r_n}} \right) &\geq - \sup_{0 \leq i \leq 2[r_n^{m+\epsilon}]} \left| Z_{r_n}^{(0,\infty)}(\mathcal{B}_i(n)) - \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(\mathcal{B}_i(n)) \middle| \mathcal{F}_{\sqrt{r_n}} \right) \right| \\ &\quad - \sup_{0 \leq i < 2[r_n^{m+\epsilon}]} \mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right). \end{aligned}$$

Therefore, combining (3.33), (3.37) and the definition of  $\widehat{\Omega}$ , we conclude that (3.31) holds for all  $\omega \in \widehat{\Omega}$  and  $A = (a, \infty)$  with  $a \geq 0$ . The proofs of the other cases are similar with the help of (3.34), (3.35) and (3.36) and we omit the details. The proof is now complete.  $\square$

**Proposition 3.4.** *Let  $x > 0$ . For any given  $m \in \mathbb{N}$  and  $\kappa > 2m + 1$ , if (1.5) holds for some  $\lambda > 2m + 2\kappa + 2$ , then (i) for any  $\theta \in (0, \sqrt{2})$ ,  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all  $A = (a, \infty)$  and  $A = [a, \infty)$  with  $a \geq 0$ , as  $n \rightarrow \infty$ ,*

$$\begin{aligned} &\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ &= -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} e^{-\theta z} dz + o(r_n^{-m}); \end{aligned}$$

(ii) for  $\theta = 0$  and any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all bounded intervals  $A \subset (0, \infty)$ , as  $n \rightarrow \infty$ ,

$$\begin{aligned} &\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{r_n}} \\ &= -\sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} dz + o(r_n^{-m}); \end{aligned}$$

(iii) for  $\theta = 0$  and any  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, simultaneously for all  $A = (a, \infty)$  and  $A = [a, \infty)$  with  $a \geq 0$ , as  $n \rightarrow \infty$ ,

$$\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-1/2} e^{r_n}} = \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell}(0)}{r_n^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,0)}}{(2k+1)!(2\ell-2k)!} a^{2\ell-2k} + o(r_n^{-m}).$$

*Proof.* First using Lemma 3.3 and then the Markov property, we get that for any  $\omega \in \widehat{\Omega}$  and any interval  $A \subset (0, \infty)$ ,

$$\frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m}) + \frac{\mathbb{E}_x \left( Z_{r_n}^{(0,\infty)}(A) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}}$$

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$$\begin{aligned}
 &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{E}_{X_v(\sqrt{r_n})} \left( Z_{r_n - \sqrt{r_n}}^{(0, \infty)}(A) \right) \\
 &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \\
 &\quad \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbb{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbb{1}_A(B_{r_n - \sqrt{r_n}}) \right),
 \end{aligned}$$

where  $b_\theta^A$  is defined in (3.32), and in the last equality we used Lemma 2.1. Let  $K := 2m/\kappa + 3$  and fix a sufficiently small  $\varepsilon > 0$  such that

$$K(1 - \varepsilon) > \frac{2m + 1}{\kappa} + 2. \tag{3.39}$$

**Step 1:** In this step, we prove that, for  $\theta \in [0, \sqrt{2})$  and  $x > 0$ ,  $\mathbb{P}_x$ -almost surely, for all intervals  $A \subset (0, \infty)$ ,

$$\begin{aligned}
 \frac{Z_{r_n}^{(0, \infty)}(A)}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\
 &\quad \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbb{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbb{1}_A(B_{r_n - \sqrt{r_n}}) \right).
 \end{aligned} \tag{3.40}$$

Using Lemma 2.4 (i) (ii) first, and then Lemma 2.1, we get

$$\begin{aligned}
 &\sum_{n=2}^{\infty} r_n^m \mathbb{E}_x \left( \sup_{A \subset (0, \infty)} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right. \\
 &\quad \times \left. e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbb{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbb{1}_A(B_{r_n - \sqrt{r_n}}) \right) \right) \\
 &\lesssim \sum_{n=2}^{\infty} r_n^m \mathbb{E}_x \left( \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right. \\
 &\quad \times \left. X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})} \frac{e^{(1-\frac{\theta^2}{2})(r_n - \sqrt{r_n})}}{(r_n - \sqrt{r_n}) b_\theta^A} \right) \\
 &= \sum_{n=2}^{\infty} \frac{r_n^m r_n^{b_\theta^A} e^{\frac{\theta^2}{2} \sqrt{r_n}}}{(r_n - \sqrt{r_n}) b_\theta^A} \\
 &\quad \times \mathbb{E}_x \left( \mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_\xi(s) > 0\}} \mathbb{1}_{\{X_\xi(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} X_\xi(\sqrt{r_n}) e^{\theta X_\xi(\sqrt{r_n})} \right).
 \end{aligned} \tag{3.41}$$

Recall that  $(X_\xi(t), \mathbb{P}_x)$  is equal in law to a standard Brownian motion with drift  $-\theta$ . By Lemma 2.3 (i), recalling the choice of  $\varepsilon$  in (3.39), the left-hand side of (3.41) is bounded from above by

$$\begin{aligned}
 &\sum_{n=2}^{\infty} \frac{r_n^m r_n^{b_\theta^A}}{(r_n - \sqrt{r_n}) b_\theta^A} x e^{\theta x} \Pi_x^\uparrow \left( B_{\sqrt{r_n}} > \sqrt{K\sqrt{r_n} \log n} \right) \\
 &\leq x e^{\theta x} \sum_{n=2}^{\infty} \frac{r_n^m r_n^{b_\theta^A}}{(r_n - \sqrt{r_n}) b_\theta^A} \frac{1}{n^{K(1-\varepsilon)/2}} \Pi_x^\uparrow \left( e^{(1-\varepsilon)(B_{\sqrt{r_n}})^2 / (2\sqrt{r_n})} \right),
 \end{aligned} \tag{3.42}$$

where in the last inequality we used the Markov inequality. Using Lemma 2.3 (ii), we get

$$\begin{aligned} \Pi_x^\uparrow \left( e^{(1-\varepsilon)(B_{\sqrt{r_n}})^2/(2\sqrt{r_n})} \right) &= \int_0^\infty e^{(1-\varepsilon)y^2/(2\sqrt{r_n})} p_{\sqrt{r_n}}^\uparrow(x, y) dy \\ &\lesssim \int_0^\infty e^{(1-\varepsilon)y^2/(2\sqrt{r_n})} \frac{y^2}{r_n^{3/4}} e^{-(x-y)^2/(2\sqrt{r_n})} dy \\ &= \int_0^\infty e^{(1-\varepsilon)y^2/2} y^2 e^{-(xr_n^{-1/4}-y)^2/2} dy \lesssim 1. \end{aligned} \tag{3.43}$$

Combining (3.42), (3.43) and the fact that  $r_n(r_n - \sqrt{r_n})^{-1} \lesssim 1$ , we obtain

$$\begin{aligned} &\sum_{n=2}^\infty r_n^m \mathbb{E}_x \left( \sup_{A \subset (0, \infty)} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right. \\ &\quad \left. \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbb{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbb{1}_A(B_{r_n - \sqrt{r_n}}) \right) \right) \\ &\lesssim \sum_{n=2}^\infty \frac{r_n^m}{n^{K(1-\varepsilon)/2}} = \sum_{n=2}^\infty \frac{n^{m/\kappa}}{n^{K(1-\varepsilon)/2}} < \infty, \end{aligned}$$

which implies that  $\mathbb{P}_x$ -almost surely,

$$\begin{aligned} &r_n^m \sup_{A \subset (0, \infty)} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-b_\theta^A} e^{(1-\frac{\theta^2}{2})r_n}} \mathbb{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})}^{-\theta} \left( \mathbb{1}_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} \mathbb{1}_A(B_{r_n - \sqrt{r_n}}) \right) \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

and thus (3.40) holds.

**Step 2:** In this step, we prove (i) and (ii). Define  $\mathcal{B}_\theta$  to be the collection of all subintervals of  $(0, \infty)$  if  $\theta > 0$  and the collection of all bounded subintervals of  $(0, \infty)$  if  $\theta = 0$ . Recall that  $b_\theta^A = 3/2$  is defined in (3.32). Let  $\widehat{\Omega}_1$  with  $\mathbb{P}_x(\widehat{\Omega}_1) = 1$  be such that  $\lim_{t \rightarrow \infty} M_t^{(1, \theta)}$  exists and (3.40) holds for all  $\omega \in \widehat{\Omega}_1$  and all intervals  $A$ . In this case, let  $J := 2m + 2\kappa + 2 > 2m + \frac{K\kappa+1}{2}$ . By using Lemma 2.3 (i) in the first two equalities below, Lemma 2.8 in the third, we get that for all  $\omega \in \widehat{\Omega}_1$  and all  $A \in \mathcal{B}_\theta$ ,

$$\begin{aligned} &\frac{Z_{r_n}^{(0, \infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} X_v(\sqrt{r_n}) e^{\theta X_v(\sqrt{r_n})} \mathbb{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \Pi_{X_v(\sqrt{r_n})}^\uparrow \left( \frac{\mathbb{1}_A(B_{r_n - \sqrt{r_n}})}{B_{r_n - \sqrt{r_n}} e^{\theta B_{r_n - \sqrt{r_n}}}} \right) \\ &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \mathbb{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \int_A \frac{r_n^{3/2}}{e^{\theta z} \sqrt{r_n - \sqrt{r_n}}} \left( \phi \left( \frac{z - X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) - \phi \left( \frac{z + X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) \right) dz \\ &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbb{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \mathbb{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \int_A r_n e^{-\theta z} \\ &\quad \times \left( 2\phi \left( \frac{z}{\sqrt{r_n}} \right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k+1} \left( \frac{z}{\sqrt{r_n}} \right) r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_v(\sqrt{r_n})}{r_n^{1/4}} \right) + \varepsilon_{m, v, z, n} \right) dz, \end{aligned}$$

where the error term  $\varepsilon_{m,v,z,n}$  satisfies that

$$r_n^{m+1} \sup_{z>0} \sup_{v \in N(\sqrt{r_n})} |\varepsilon_{m,v,z,n}| \mathbf{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \xrightarrow{n \rightarrow \infty} 0.$$

Noticing that  $e^{-(1-\frac{\theta^2}{2})t} \sum_{v \in N(t)} e^{\theta X_v(t)}$  is a non-negative martingale, and that  $\int_A e^{-\theta z} dz < \infty$  for any  $A \in \mathcal{B}_\theta$ , we see that for all  $\omega \in \widehat{\Omega}_1$  and  $A \in \mathcal{B}_\theta$ ,

$$\begin{aligned} & r_n^m \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \mathbf{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \int_A r_n e^{-\theta z} |\varepsilon_{m,v,z,n}| dz \\ & \leq \left( e^{-(1-\frac{\theta^2}{2})\sqrt{r_n}} \sum_{v \in N(\sqrt{r_n})} e^{\theta X_v(\sqrt{r_n})} \right) \left( \int_A e^{-\theta z} dz \right) \\ & \quad \times r_n^{m+1} \sup_{z>0} \sup_{v \in N(\sqrt{r_n})} |\varepsilon_{m,v,z,n}| \mathbf{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore,  $\mathbb{P}_x$ -almost surely, for all  $A \in \mathcal{B}_\theta$ ,

$$\begin{aligned} & \frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ & = o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \mathbf{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \int_A r_n e^{-\theta z} \\ & \quad \times 2\phi\left(\frac{z}{\sqrt{r_n}}\right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right) dz \\ & = o(r_n^{-m}) + 2 \sum_{k=0}^J \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right) \\ & \quad \times \mathbf{1}_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) dz. \quad (3.44) \end{aligned}$$

Now we show that we can drop the indicator function from the last line above. Define  $\mathcal{B}_{\theta,N}$  to be  $\mathcal{B}_\theta$  if  $\theta > 0$  and to be the collection of all intervals  $A \subset (0, \infty)$  with  $\sup A \leq N$  if  $\theta = 0$ . Note that for all  $0 \leq k \leq J$ , applying Lemma 2.5(ii) and the inequality

$$\sup_{A \in \mathcal{B}_{\theta,N}} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \left| H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) \right| dz \lesssim \sup_{A \in \mathcal{B}_{\theta,N}} \int_A e^{-\theta z} dz < \infty$$

first and then Lemma 2.1, we get that for any  $N > 0$ ,

$$\begin{aligned} & \sum_{n=2}^{\infty} r_n^m \mathbb{E}_x \left( \sup_{A \in \mathcal{B}_{\theta,N}} \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \left| r_n^{(2k+1)/4} H_{2k+1}\left(\frac{X_v(\sqrt{r_n})}{r_n^{1/4}}\right) \right| \right. \\ & \quad \left. \times \mathbf{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) \right| dz \right) \\ & \lesssim \sum_{n=2}^{\infty} r_n^m \mathbb{E}_x \left( \sum_{v \in N(\sqrt{r_n})} \frac{\mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} r_n^{(2k+1)/4} \left( \left| \frac{X_v(\sqrt{r_n})}{r_n^{1/4}} \right|^{2k+1} + \left| \frac{X_v(\sqrt{r_n})}{r_n^{1/4}} \right| \right) \right. \\ & \quad \left. \times \frac{1}{r_n^{(2k-1)/2}} \mathbf{1}_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right) \\ & = \sum_{n=2}^{\infty} \frac{r_n^m e^{\frac{\theta^2}{2}\sqrt{r_n}}}{r_n^{(2k-1)/2}} \mathbb{E}_x \left( \mathbf{1}_{\{\min_{s \leq \sqrt{r_n}} X_\xi(s) > 0\}} X_\xi(\sqrt{r_n}) e^{\theta X_\xi(\sqrt{r_n})} \right. \\ & \quad \left. \times \left( X_\xi(\sqrt{r_n})^{2k} + r_n^{k/2} \right) \mathbf{1}_{\{X_\xi(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right). \quad (3.45) \end{aligned}$$

Recall the choice of  $\varepsilon$  in (3.39). By Lemma 2.3, the right-hand side of (3.45) is equal to

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2}} x e^{\theta x} \Pi_x^\uparrow \left( \left( B_{\sqrt{r_n}}^{2k} + r_n^{k/2} \right) 1_{\{B_{\sqrt{r_n}} > \sqrt{K\sqrt{r_n} \log n}\}} \right) \\
 & \leq \sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} x e^{\theta x} \Pi_x^\uparrow \left( \left( B_{\sqrt{r_n}}^{2k} + r_n^{k/2} \right) e^{(1-\varepsilon)(B_{\sqrt{r_n}})^2/(2\sqrt{r_n})} \right) \\
 & \lesssim \sum_{n=2}^{\infty} \frac{r_n^m}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} \int_0^\infty \frac{y^2}{r_n^{3/4}} e^{-(x-y)^2/(2\sqrt{r_n})} \left( y^{2k} + r_n^{k/2} \right) e^{(1-\varepsilon)y^2/(2\sqrt{r_n})} dy \\
 & = \sum_{n=2}^{\infty} \frac{r_n^m r_n^{k/2}}{r_n^{(2k-1)/2} n^{(1-\varepsilon)K/2}} \int_0^\infty y^2 e^{-(x r_n^{-1/4} - y)^2/2} \left( y^{2k} + 1 \right) e^{(1-\varepsilon)y^2/2} dy \\
 & \leq \sum_{n=2}^{\infty} \frac{n^{(2m+1)/(2\kappa)}}{n^{(1-\varepsilon)K/2}} \int_0^\infty y^2 e^{-(x r_n^{-1/4} - y)^2/2} \left( y^{2k} + 1 \right) e^{(1-\varepsilon)y^2/2} dy, \tag{3.46}
 \end{aligned}$$

which is finite. Hence, combining (3.45) and (3.46), we conclude that for all  $0 \leq k \leq J$ , almost surely for all  $A \in \mathcal{B}_{\theta, N}$ ,

$$\begin{aligned}
 & r_n^m \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{(1-\frac{\theta^2}{2})\sqrt{r_n}}} e^{\theta X_v(\sqrt{r_n})} \left| r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_v(\sqrt{r_n})}{r_n^{1/4}} \right) \right| \\
 & \times 1_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \int_A e^{-\theta z} \phi \left( \frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1} \left( \frac{z}{\sqrt{r_n}} \right) \right| dz \xrightarrow{n \rightarrow \infty} 0.
 \end{aligned}$$

Plugging this back to (3.44) and recalling the definition of the martingales in (1.6), we obtain that almost surely for all  $A \in \mathcal{B}_{\theta, N}$ ,

$$\begin{aligned}
 & \frac{Z_{r_n}^{(0, \infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m}) \\
 & + 2 \sum_{k=0}^J M_{\sqrt{r_n}}^{(2k+1, \theta)} \int_A e^{-\theta z} \phi \left( \frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1} \left( \frac{z}{\sqrt{r_n}} \right) dz. \tag{3.47}
 \end{aligned}$$

Letting  $N \rightarrow \infty$ , we see that almost surely (3.47) holds for all  $A \in \mathcal{B}_\theta$ .

For all  $m+1 \leq k \leq J = 2m+2\kappa+2$ , we have  $\lambda > J \geq k$ . Therefore, for all  $m+1 \leq k \leq J$ , by Lemma 2.5 (ii), Proposition 3.1(ii) and the fact that  $\phi(y)|H_{2k+1}(y)| \lesssim \phi(y)|y|(|y|^{2k}+1) \lesssim |y|$ ,

$$\begin{aligned}
 & \left| M_{\sqrt{r_n}}^{(2k+1, \theta)} \right| \int_A e^{-\theta z} \phi \left( \frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1} \left( \frac{z}{\sqrt{r_n}} \right) \right| dz \\
 & \lesssim \left| M_{\sqrt{r_n}}^{(2k+1, \theta)} \right| \int_A e^{-\theta z} \frac{1}{r_n^{(2k-1)/2}} \frac{z}{\sqrt{r_n}} dz = \frac{\left| M_{\sqrt{r_n}}^{(2k+1, \theta)} \right|}{r_n^k} \int_A z e^{-\theta z} dz = o(r_n^{-m}). \tag{3.48}
 \end{aligned}$$

Combining (3.47) and (3.48), we get that

$$\begin{aligned}
 & \frac{Z_{r_n}^{(0, \infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m}) \\
 & + 2 \sum_{k=0}^m M_{\sqrt{r_n}}^{(2k+1, \theta)} \int_A e^{-\theta z} \phi \left( \frac{z}{\sqrt{r_n}} \right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1} \left( \frac{z}{\sqrt{r_n}} \right) dz.
 \end{aligned}$$

Noticing that  $\lambda > 2m$ , let  $\eta := (\lambda - 2m)/2 < \lambda - 2m$ . By Proposition 3.1(ii), similar to

(3.48), for all  $0 \leq k \leq m$ ,

$$\begin{aligned} & \left| M_{\sqrt{r_n}}^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right| \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} \left| H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) \right| dz \\ & \lesssim \frac{\left| M_{\sqrt{r_n}}^{(2k+1,\theta)} - M_\infty^{(2k+1,\theta)} \right|}{r_n^k} \int_A z e^{-\theta z} dz = o(\sqrt{r_n}^{-(\lambda-k)+\eta} r_n^{-k}) = o(r_n^{-m} r_n^{-\frac{k}{2}}) \\ & = o(r_n^{-m}). \end{aligned} \tag{3.49}$$

Since  $\lambda > 2m + 2\kappa + 2$ , we have

$$\begin{aligned} & \frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m}) \\ & + 2 \sum_{k=0}^m M_\infty^{(2k+1,\theta)} \int_A e^{-\theta z} \phi\left(\frac{z}{\sqrt{r_n}}\right) \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} H_{2k+1}\left(\frac{z}{\sqrt{r_n}}\right) dz. \end{aligned}$$

By (2.7) and Lemma 2.5(i), for any  $0 \leq k \leq m$  and  $x > 0$ , there exists  $\xi \in (0, x)$  such that

$$\begin{aligned} \phi(x) H_{2k+1}(x) &= \sum_{j=0}^{2m+1} \frac{d^j}{dx^j} (\phi H_{2k+1})(0) \frac{x^j}{j!} + \frac{x^{2m+2}}{(2m+2)!} \frac{d^{2m+2}}{dx^{2m+2}} (\phi H_{2k+1})(\xi) \\ &= (-1)^{2k+1} \sum_{j=0}^m \frac{d^{2k+2+2j} \phi(0)}{dx^{2k+2+2j}} \frac{x^{2j+1}}{(2j+1)!} + O(x^{2m+2}) \\ &= - \sum_{j=0}^m \frac{H_{2k+2j+2}(0)}{\sqrt{2\pi}} \frac{x^{2j+1}}{(2j+1)!} + O(x^{2m+2}), \end{aligned}$$

where in the second equality we used the property that  $H_{2\ell+1}(0) = 0$ . Therefore, taking  $x = z/\sqrt{r_n}$  in the above equation,

$$\begin{aligned} & \frac{Z_{r_n}^{(0,\infty)}(A)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} = o(r_n^{-m}) - \sum_{k=0}^m \frac{2}{(2k+1)!} \frac{1}{r_n^{(2k-1)/2}} M_\infty^{(2k+1,\theta)} \\ & \quad \times \int_A e^{-\theta z} \left( \sum_{j=0}^m \frac{H_{2k+2j+2}(0)}{(2j+1)! \sqrt{2\pi}} \frac{z^{2j+1}}{r_n^{(2j+1)/2}} + O\left(\frac{z^{2m+2}}{r_n^{m+1}}\right) \right) dz \\ &= o(r_n^{-m}) - \sqrt{\frac{2}{\pi}} \sum_{k=0}^m \sum_{j=0}^m \frac{1}{(2k+1)!(2j+1)!} \frac{H_{2k+2j+2}(0)}{r_n^{k+j}} M_\infty^{(2k+1,\theta)} \int_A z^{2j+1} e^{-\theta z} dz \\ &= o(r_n^{-m}) - \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^m \frac{H_{2\ell+2}(0)}{r_n^\ell} \sum_{k=0}^{\ell} \frac{M_\infty^{(2k+1,\theta)}}{(2k+1)!(2\ell-2k+1)!} \int_A z^{2\ell-2k+1} e^{-\theta z} dz, \end{aligned}$$

which completes the proof of (i) and (ii).

**Step 3:** In this step, we prove (iii). We only prove the case  $A = (a, \infty)$ . The proof of the case  $A = [a, \infty)$  is similar. Recall that when  $\theta = 0$  and  $A = (a, \infty)$ ,  $b_\theta^A = 1/2$ . By

(3.40) and Lemma 2.3 (i), almost surely for all  $a \geq 0$ ,

$$\begin{aligned} \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-1/2} e^{r_n}} &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-1/2} e^{r_n}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times e^{r_n - \sqrt{r_n}} \Pi_{X_v(\sqrt{r_n})} \left( 1_{\{\min_{s \leq r_n - \sqrt{r_n}} B_s > 0\}} 1_{(a, \infty)}(B_{r_n - \sqrt{r_n}}) \right) \\ &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-1/2} e^{r_n}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \frac{e^{r_n - \sqrt{r_n}}}{\sqrt{r_n - \sqrt{r_n}}} \int_a^\infty \left( \phi \left( \frac{y - X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) - \phi \left( \frac{y + X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) \right) dy \\ &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-1/2} e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \left( \Phi \left( \frac{a + X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) - \Phi \left( \frac{a - X_v(\sqrt{r_n})}{\sqrt{r_n - \sqrt{r_n}}} \right) \right). \end{aligned}$$

Put  $J := 2m + 2k + 1 > 2m + \frac{K\kappa - 1}{2}$ . By Lemma 2.7, it holds that

$$\begin{aligned} \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-1/2} e^{r_n}} &= o(r_n^{-m}) + \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-1/2} e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \left( 2\phi \left( \frac{a}{\sqrt{r_n}} \right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^{(2k+1)/2}} H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_u(\sqrt{r_n})}{r_n^{1/4}} \right) \right. \\ &\quad \left. + \varepsilon_{m,v,a,n} \right). \end{aligned}$$

where the error term  $\varepsilon_{m,v,a,n}$  satisfies that

$$r_n^{(2m+1)/2} \sup_{a > 0} \sup_{v \in N(\sqrt{r_n})} |\varepsilon_{m,v,a,n}| 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \xrightarrow{n \rightarrow \infty} 0.$$

Using the fact that  $e^{-t} \sum_{v \in N(t)} 1$  is a non-negative martingale, we have

$$\begin{aligned} &r_n^m \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{r_n^{-1/2} e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} |\varepsilon_{m,v,a,n}| \\ &\leq \left( e^{-\sqrt{r_n}} \sum_{v \in N(\sqrt{r_n})} 1 \right) r_n^{(2m+1)/2} \sup_{a > 0} \sup_{v \in N(\sqrt{r_n})} |\varepsilon_{m,v,a,n}| 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Therefore, almost surely for all  $a \geq 0$ ,

$$\begin{aligned} \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-1/2} e^{r_n}} &= o(r_n^{-m}) + 2 \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) \leq \sqrt{K\sqrt{r_n} \log n}\}} \\ &\quad \times \phi \left( \frac{a}{\sqrt{r_n}} \right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_u(\sqrt{r_n})}{r_n^{1/4}} \right). \end{aligned} \tag{3.50}$$

Similar to the argument leading to (3.45), for each  $0 \leq k \leq J$ , applying Lemma 2.5 first,

then Lemma 2.1 and Lemma 2.3(i) at last, we also have that

$$\begin{aligned} & \sum_{n=1}^{\infty} r_n^m \mathbb{E}_x \left( \sup_{a \geq 0} \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right. \\ & \quad \times \left. \phi \left( \frac{a}{\sqrt{r_n}} \right) \frac{1}{r_n^k} \left| H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) \right| \times \left| r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_u(\sqrt{r_n})}{r_n^{1/4}} \right) \right| \right) \\ & \lesssim \sum_{n=1}^{\infty} r_n^m \mathbb{E}_x \left( \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \right. \\ & \quad \times \left. \frac{1}{r_n^k} X_u(\sqrt{r_n}) \left( (X_u(\sqrt{r_n}))^{2k} + r_n^{k/2} \right) \right) \\ & = \sum_{n=1}^{\infty} r_n^{m-k} \mathbb{E}_x \left( 1_{\{\min_{s \leq \sqrt{r_n}} X_{\xi}(s) > 0\}} 1_{\{X_{\xi}(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} X_{\xi}(\sqrt{r_n}) \left( (X_{\xi}(\sqrt{r_n}))^{2k} + r_n^{k/2} \right) \right) \\ & = x \sum_{n=1}^{\infty} r_n^{m-k} \Pi_x^{\uparrow} \left( 1_{\{B_{\sqrt{r_n}} > \sqrt{K\sqrt{r_n} \log n}\}} \left( (B_{\sqrt{r_n}})^{2k} + r_n^{k/2} \right) \right) < \infty, \end{aligned}$$

where the last inequality follows from (3.46). Therefore, for  $0 \leq k \leq J$ ,  $\mathbb{P}_x$ -a.s.,

$$\begin{aligned} & \sup_{a \geq 0} r_n^m \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{\sqrt{r_n}}} 1_{\{X_v(\sqrt{r_n}) > \sqrt{K\sqrt{r_n} \log n}\}} \\ & \quad \times \phi \left( \frac{a}{\sqrt{r_n}} \right) \frac{1}{r_n^k} \left| H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) \right| \times \left| r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_u(\sqrt{r_n})}{r_n^{1/4}} \right) \right| \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Plugging this back to (3.50), we finally get that

$$\begin{aligned} & \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-1/2} e^{r_n}} = o(r_n^{-m}) + 2 \sum_{v \in N(\sqrt{r_n})} \frac{1_{\{\min_{s \leq \sqrt{r_n}} X_v(s) > 0\}}}{e^{\sqrt{r_n}}} \\ & \quad \times \phi \left( \frac{a}{\sqrt{r_n}} \right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) r_n^{(2k+1)/4} H_{2k+1} \left( \frac{X_u(\sqrt{r_n})}{r_n^{1/4}} \right) \\ & = o(r_n^{-m}) + 2\phi \left( \frac{a}{\sqrt{r_n}} \right) \sum_{k=0}^J \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) M_{\sqrt{r_n}}^{(2k+1,0)}, \end{aligned} \tag{3.51}$$

where  $M_t^{(2k+1,0)}$  is given in (1.6). Using the same argument as (3.48) and (3.49), also noting that  $\lambda > J = 2m + 2\kappa + 1$ , Proposition 3.1 (ii) and (3.51) imply that

$$\frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-1/2} e^{r_n}} = o(r_n^{-m}) + 2\phi \left( \frac{a}{\sqrt{r_n}} \right) \sum_{k=0}^m \frac{1}{(2k+1)!} \frac{1}{r_n^k} H_{2k} \left( \frac{a}{\sqrt{r_n}} \right) M_{\infty}^{(2k+1,0)}.$$

The remaining part is similar to the end of Step 2 we omit the details here. The proof of (iii) is complete.  $\square$

### 3.3 From discrete time to continuous time

**Lemma 3.5.** *Let  $x > 0$  and  $\theta \in [0, \sqrt{2})$ . For any given  $m \in \mathbb{N}$ , if  $\kappa > 2m + 2$  and (1.5) holds for  $\lambda > 2m + 2\kappa + 2$ , then,  $\mathbb{P}_x$ -almost surely, simultaneously for all intervals  $J \subset (0, \infty)$ , it holds that*

$$\liminf_{n \rightarrow \infty} r_n^m \inf_{t \in (r_n, r_{n+1})} \frac{Z_t^{(0,\infty)}(J) - Z_{r_n}^{(0,\infty)}(J)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \geq 0.$$

*Proof. Step 1:* Define  $\varepsilon_n := \sqrt{r_{n+1} - r_n}$ , by the mean value theorem,

$$\sqrt{r_n} \varepsilon_n = \sqrt{n^{1/\kappa} \left( (n+1)^{1/\kappa} - n^{1/\kappa} \right)} \stackrel{\exists \xi \in [n, n+1]}{=} \sqrt{\frac{n^{1/\kappa}}{\kappa} \xi^{(-\kappa+1)/\kappa}} \lesssim n^{(-\kappa+2)/(2\kappa)} \rightarrow 0. \tag{3.52}$$

For any  $\eta < |J|/2$ , define

$$J_\eta := \{y \in J : \text{dist}(y, J^c) \geq \eta\}.$$

For any  $u \in N(r_n)$ , let  $G_u$  be the event that  $u$  does not split before  $r_{n+1}$  and that  $\max_{s \in (r_n, r_{n+1})} |X_u(s) - X_u(r_n)| \leq \sqrt{r_n} \varepsilon_n$ . When  $n$  is large enough so that  $\eta_n := \sqrt{r_n} \varepsilon_n < |J|/2$ , for  $u \in N(r_n)$ , on the event  $G_u$ , for  $t \in (r_n, r_{n+1})$ , it must hold that

$$\{X_u(r_n) \in J_{\eta_n}\} \subset \{X_u(s) \in J, \forall s \in (r_n, r_{n+1})\} \subset \{X_u(t) \in J\} \cap \{X_u(s) > 0, \forall s \in (r_n, t)\}.$$

Therefore, for  $t \in (r_n, r_{n+1})$ , by the branching property,

$$\begin{aligned} & r_n^m \frac{Z_t^{(0, \infty)}(J)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ &= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \sum_{v \in N(t): u \leq v} \mathbf{1}_{\{\min_{s < t-r_n} X_v(r_n+s) > 0\}} \mathbf{1}_{\{X_v(t) \in J\}} \\ &\geq \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{\{X_u(r_n) \in J_{\eta_n}\}} \mathbf{1}_{G_u} =: I_n + II_n. \end{aligned} \tag{3.53}$$

Here  $I_n$  and  $II_n$  are given by

$$\begin{aligned} I_n &:= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{\{X_u(r_n) \in J_{\eta_n}\}} (1_{G_u} - \mathbb{P}_x(G_u | \mathcal{F}_{r_n})), \\ II_n &:= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{\{X_u(r_n) \in J_{\eta_n}\}} \mathbb{P}_x(G_u | \mathcal{F}_{r_n}). \end{aligned}$$

We claim the following two limits hold:

$$\sup_{J \subset (0, \infty)} \sup_{t \in (r_n, r_{n+1})} |I_n| = \sup_{J \subset (0, \infty)} |I_n| \xrightarrow{n \rightarrow \infty} 0, \quad \mathbb{P}_x\text{-a.s.} \tag{3.54}$$

and  $\mathbb{P}_x$ -almost surely for all interval  $J \subset (0, \infty)$ ,

$$\sup_{t \in (r_n, r_{n+1})} \left| II_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0, \infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}} \right| = \left| II_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0, \infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}} \right| \xrightarrow{n \rightarrow \infty} 0. \tag{3.55}$$

If (3.54) and (3.55) hold, then we complete the proof of the Lemma together with (3.53).

**Step 2:** In this step, we prove (3.55). Define

$$III_n := \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbf{1}_{\{X_u(r_n) \in J\}} \mathbb{P}_x(G_u | \mathcal{F}_{r_n}).$$

We only prove the case  $J = (a, \infty)$ . The other cases are similar. Suppose that  $a \in (k, k+1]$ . Set  $A_n := J \setminus J_{\eta_n} = (a, a + \eta_n]$ , combining (3.52) and the fact that  $\epsilon := \frac{k-2}{2} - m > 0$ , we see that there exists  $N \in \mathbb{N}$  such that  $\eta_n \leq N[r_n^{m+\epsilon}]^{-1}$ . Recall the notation  $a_i^{(n)}$  and

$\bar{\mathcal{B}}_i(n)$  in the proof of Lemma 3.3. For each  $n$ , suppose that  $a \in (a_i^{(n)}, a_i^{(n+1)})$  for some  $1 \leq i \leq \lceil r_n^{m+\epsilon} \rceil$ , then we see that  $A_n \subset (a_i^{(n)}, a_{i+N}^{(n)}) = \bar{\mathcal{B}}_{i+N}(n)^c \setminus \bar{\mathcal{B}}_i(n)^c$ , which implies that

$$\begin{aligned} & \sup_{a \in (k, k+1]} |II_n - III_n| \leq r_n^{m+\frac{3}{2}} \sup_{a \in (k, k+1]} \frac{Z_{r_n}^{(0, \infty)}(A_n)}{e^{(1-\frac{\theta^2}{2})r_n}} \\ & \leq \sup_{0 \leq i \leq \lceil r_n^{m+\epsilon} \rceil} r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_{i+N}(n)^c \setminus \bar{\mathcal{B}}_i(n)^c)}{e^{(1-\frac{\theta^2}{2})r_n}} \\ & \leq 2r_n^m \frac{\sup_{0 \leq i \leq 2\lceil r_n^{m+\epsilon} \rceil} |Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)^c) - \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}(\bar{\mathcal{B}}_i(n)^c) \middle| \mathcal{F}_{\sqrt{r_n}} \right)|}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ & \quad + N \lim_{n \rightarrow \infty} r_n^m \frac{\sup_{0 \leq i \leq 2\lceil r_n^{m+\epsilon} \rceil - 1} \mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}([a_i^{(n)}, a_{i+1}^{(n)}]) \middle| \mathcal{F}_{\sqrt{r_n}} \right)}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}}. \end{aligned}$$

Combining (3.36), (3.37) and the construction of  $\hat{\Omega}$  in (3.38),  $\mathbb{P}_x$ -almost surely, for any interval  $J = (a, \infty)$  with  $a \in (k, k + 1]$ ,

$$|II_n - III_n| \xrightarrow{n \rightarrow \infty} 0. \tag{3.56}$$

Taking the intersection over  $k \in \mathbb{N}$  of all the events such that (3.56) holds, we conclude that  $\mathbb{P}_x$ -almost surely, simultaneously for all intervals  $J = (a, \infty)$  with  $a \geq 0$ , (3.56) holds. By the Markov property and the definition of  $G_u$ ,

$$\mathbb{P}_x(G_u | \mathcal{F}_{r_n}) = e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| \leq \sqrt{r_n} \varepsilon_n \right).$$

Thus, we get that

$$\begin{aligned} & \sup_{J \subset (0, \infty)} \left| III_n - r_n^{m+\frac{3}{2}} \frac{Z_{r_n}^{(0, \infty)}(J)}{e^{(1-\frac{\theta^2}{2})r_n}} \right| \\ & \leq \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \cdot r_n^{m+1} \left( 1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| \leq \sqrt{r_n} \varepsilon_n \right) \right) \\ & \lesssim r_n^{m+1} \left( 1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| \leq \sqrt{r_n} \varepsilon_n \right) \right), \end{aligned} \tag{3.57}$$

where in the last inequality we used Proposition 3.4 (i) (iii) with  $m = 0$ . Note that under the assumption  $\kappa > 2m + 2$ ,

$$\begin{aligned} & r_n^{m+1} \left( 1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| \leq \sqrt{r_n} \varepsilon_n \right) \right) \\ & \leq r_n^{m+1} O(r_{n+1} - r_n) + r_n^{m+1} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| > \sqrt{r_n} \varepsilon_n \right) \\ & \leq r_n^{m+1} O(n^{-(\kappa+1)/\kappa}) + r_n^{m+1} \Pi_0 \left( \max_{t < r_{n+1}-r_n} |B_t| > \sqrt{r_n} \varepsilon_n - \theta(r_{n+1} - r_n) \right) \\ & = O(n^{-(\kappa-2-m)/\kappa}) + r_n^{m+1} \Pi_0 \left( \max_{t < 1} |B_t| > \sqrt{r_n} - \theta \sqrt{r_{n+1} - r_n} \right) = o(1). \end{aligned} \tag{3.58}$$

Hence, (3.55) follows according to (3.56), (3.57) and (3.58).

**Step 3:** In this step, we prove (3.54). Noticing that

$$\begin{aligned} |I_n| &\leq \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{\{X_u(r_n) \in J_{\eta_n}\}} 1_{G_u^c} \\ &\quad + \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{\{X_u(r_n) \in J_{\eta_n}\}} \mathbb{P}_x(G_u^c | \mathcal{F}_{r_n}) \\ &\leq \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{\{X_u(r_n) > 0\}} (1_{G_u^c} - \mathbb{P}_x(G_u^c | \mathcal{F}_{r_n})) \\ &\quad + 2 \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbb{P}_x(G_u^c | \mathcal{F}_{r_n}) =: I_n^{(1)} + 2I_n^{(2)}. \end{aligned}$$

Therefore, we conclude that

$$\sup_{J \subset (0, \infty)} \sup_{t \in (r_n, r_{n+1})} |I_n| \leq I_n^{(1)} + 2I_n^{(2)}. \tag{3.59}$$

On one hand, combining Proposition 3.4 (with  $m = 0$ ), (3.58) and the fact that  $b_\theta^{(0, \infty)} \geq \frac{1}{2}$ , we see that

$$\begin{aligned} I_n^{(2)} &\leq \frac{r_n^{m+1+b_\theta^{(0, \infty)}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbb{P}_x(G_u^c | \mathcal{F}_{r_n}) \\ &= \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-b_\theta^{(0, \infty)}} e^{(1-\frac{\theta^2}{2})r_n}} \times r_n^{m+1} \left( 1 - e^{-(r_{n+1}-r_n)} \Pi_0^{-\theta} \left( \max_{t < r_{n+1}-r_n} |B_t| \leq \sqrt{r_n} \varepsilon_n \right) \right) \\ &\xrightarrow{n \rightarrow \infty} 0, \quad \mathbb{P}_x\text{-a.s.} \end{aligned} \tag{3.60}$$

On the other hand, since, given  $\mathcal{F}_{r_n}$ ,  $\{G_u : u \in N(r_n)\}$  are independent, we have

$$\begin{aligned} \mathbb{E}_x \left( |I_n^{(1)}|^2 | \mathcal{F}_{r_n} \right) &= \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \\ &\quad \times \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{\{X_u(r_n) \in (0, \infty)\}} \mathbb{E}_x \left( (1_{G_u} - \mathbb{P}_x(G_u | \mathcal{F}_{r_n}))^2 | \mathcal{F}_{r_n} \right) \\ &\leq 4 \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} 1_{\{X_u(r_n) \in (0, \infty)\}} \\ &= \frac{4r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \times \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}}. \end{aligned}$$

Now taking expectation with respect to  $\mathbb{P}_x$ , by Lemma 2.1 and Lemma 2.4(i)(ii), for any  $\varepsilon > 0$ ,

$$\begin{aligned} \sum_{n=2}^{\infty} \mathbb{P}_x \left( |I_n^{(1)}| > \varepsilon \right) &\leq \frac{1}{\varepsilon^2} \sum_{n=2}^{\infty} \mathbb{E}_x \left( |I_n^{(1)}|^2 \right) \lesssim \sum_{n=2}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \frac{\mathbb{E}_x \left( Z_{r_n}^{(0, \infty)}((0, \infty)) \right)}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ &= \sum_{n=2}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \frac{e^{r_n} \mathbb{P}_x \left( \min_{s \leq r_n} X_\xi(s) > 0 \right)}{r_n^{-1/2} e^{(1-\frac{\theta^2}{2})r_n}} \lesssim \sum_{n=1}^{\infty} \frac{r_n^{2m+\frac{5}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} < \infty, \end{aligned} \tag{3.61}$$

Thus, (3.54) follows directly by (3.59), (3.60) and (3.61). This completes the proof of the lemma.  $\square$

**Lemma 3.6.** *Let  $x > 0$  and  $\theta \in [0, \sqrt{2})$ . For any  $m \in \mathbb{N}$ , suppose that  $\kappa > 2m + 2$  and (1.5) holds with  $\lambda > 2m + 2\kappa + 2$ , then  $\mathbb{P}_x$ -almost surely, simultaneously for all intervals  $A \subset (0, \infty)$ ,*

$$r_n^m \sup_{t \in (r_n, r_{n+1})} \left| \frac{Z_t^{(0, \infty)}(A) - Z_{r_n}^{(0, \infty)}(A)}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} \right| \xrightarrow{n \rightarrow \infty} 0.$$

*Proof.* Suppose  $x > 0, m \in \mathbb{N}$ , and that  $\kappa > 2m + 2$  and (1.5) holds with  $\lambda > 2m + 2\kappa + 2$ . Fix  $\theta \in [0, \sqrt{2})$ . Note that if

$$\liminf_{n \rightarrow \infty} x_n \geq 0, \quad \liminf_{n \rightarrow \infty} y_n \geq 0, \quad \liminf_{n \rightarrow \infty} z_n \geq 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} (x_n + y_n + z_n) \leq 0,$$

then  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} z_n = 0$ . Applying Lemma 3.5 with  $J = (a, b), [b, \infty)$  and  $(0, a]$ , we see that to prove Lemma 3.6, only need to prove that

$$\limsup_{n \rightarrow \infty} r_n^m \sup_{t \in (r_n, r_{n+1})} \frac{Z_t^{(0, \infty)}((0, \infty)) - Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} \leq 0. \quad \text{a.s.}$$

For any  $t \in (r_n, r_{n+1})$ , by the branching property, we see that

$$\begin{aligned} & r_n^m \sup_{t \in (r_n, r_{n+1})} \frac{Z_t^{(0, \infty)}((0, \infty)) - Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} \\ & \leq \frac{r_n^{m + \frac{3}{2}}}{e^{(1 - \frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \sup_{t \in (r_n, r_{n+1})} \left( \sum_{v \in N(t): u \leq v} 1 - 1 \right). \end{aligned} \quad (3.62)$$

To drop the ‘sup’ above, we modify the branching particle system such that when a particle dies in  $(r_n, r_{n+1})$  and it splits into  $L$  offspring, we modify the number of the offspring with  $L + 1$ . For  $t \in (r_n, r_{n+1})$ , we use  $\tilde{N}(t)$  to denote the set of the particles alive at time  $t$  in the modified process. It is obvious that the mean of the number in the modified process is equal to  $\sum_{k=0}^{\infty} (k + 1)p_k = 3$  and that for each  $u \in N(r_n)$ ,

$$\sup_{t \in (r_n, r_{n+1})} \left( \sum_{v \in N(t): u \leq v} 1 - 1 \right) \leq \sup_{t \in (r_n, r_{n+1})} \left( \sum_{v \in \tilde{N}(t): u \leq v} 1 - 1 \right) = \left( \sum_{v \in \tilde{N}(r_{n+1}): u \leq v} 1 \right) - 1.$$

Define

$$\tilde{Z}_{n+1} := \frac{r_n^{m + \frac{3}{2}}}{e^{(1 - \frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \sum_{v \in \tilde{N}(r_{n+1}): u \leq v} 1.$$

We claim that  $\mathbb{P}_x$  almost surely,

$$\tilde{Z}_{n+1} - \mathbb{E}_x \left( \tilde{Z}_{n+1} | \mathcal{F}_{r_n} \right) \rightarrow 0. \quad (3.63)$$

If the claim is true, then

$$\tilde{Z}_{n+1} - \mathbb{E}_x \left( \tilde{Z}_{n+1} | \mathcal{F}_{r_n} \right) = \tilde{Z}_{n+1} - e^{2(r_{n+1} - r_n)} r_n^m \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} \rightarrow 0.$$

Using this and (3.62), we get

$$\begin{aligned} & r_n^m \sup_{t \in (r_n, r_{n+1})} \frac{Z_t^{(0, \infty)}((0, \infty)) - Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-3/2} e^{(1 - \frac{\theta^2}{2})r_n}} \leq \tilde{Z}_{n+1} - r_n^{m + \frac{3}{2}} \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{e^{(1 - \frac{\theta^2}{2})r_n}} \\ & = \tilde{Z}_{n+1} - \mathbb{E}_x \left( \tilde{Z}_{n+1} | \mathcal{F}_{r_n} \right) + r_n^{m+1} (e^{2(r_{n+1} - r_n)} - 1) \frac{Z_{r_n}^{(0, \infty)}((0, \infty))}{r_n^{-1/2} e^{(1 - \frac{\theta^2}{2})r_n}} \rightarrow 0, \end{aligned}$$

where for the last limit, we used Proposition 3.4 (i) (iii) with  $m = 0$  and the fact that  $r_n^{m+1}(e^{2(r_{n+1}-r_n)} - 1) = r_n^{m+1}O(r_{n+1} - r_n) = o(1)$  under the assumption  $\kappa > 2m + 2$ . Thus the assertion of the lemma is valid.

Now we prove the claim (3.63). We consider another branching Brownian motion with underlying motion according to a standard Brownian motion with drift  $-\theta$ , with branching rate equal to 1 and with offspring distribution according to  $\mathbb{P}_x(\tilde{L} = k + 1) = p_k$  for all  $k \in \mathbb{N}$ , then we may define another change-of-measure

$$\frac{d\tilde{\mathbb{P}}_x}{d\mathbb{P}_x} \Big|_{\tilde{\mathcal{F}}_t} := \frac{\sum_{u \in \tilde{N}(t)} 1}{e^{2t}},$$

then a similar formula as Lemma 2.1 can be established:

For any  $t > 0$  and  $u \in \tilde{N}(t)$ , let  $\Gamma(u, t)$  be a non-negative  $\tilde{\mathcal{F}}_t$ -measurable random variable. Then

$$\mathbb{E}_x \left( \sum_{u \in \tilde{N}(t)} \Gamma(u, t) \right) = e^{2t} \tilde{\mathbf{E}}_x (\Gamma(\xi_t, t)). \tag{3.64}$$

For  $w \in \tilde{N}(t)$  with  $t \in (r_n, r_{n+1})$ , let  $\tilde{d}_w, \tilde{O}_w$  denote the death time and the number of offspring of  $w$  respectively. For  $v \in \tilde{N}(r_{n+1})$ , define  $\tilde{B}_{n,v}$  to be the event that, for all  $w < v$  with  $\tilde{d}_w \in (r_n, r_{n+1})$ , it holds that  $\tilde{O}_w \leq e^{c_0 n}$ , where  $0 < c_0 < 1 - \frac{\theta^2}{2}$  is fixed. Define

$$\tilde{Z}_{n+1}^B := \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \sum_{v \in \tilde{N}(r_{n+1}): u \leq v} 1_{\tilde{B}_{n,v}}.$$

Now for  $v \in \tilde{N}(r_{n+1} - r_n)$ ,  $\tilde{D}_{n,v}$  denotes the event that, for all  $w < v$ , it holds that  $\tilde{O}_w \leq e^{c_0 n}$ . Let  $\tilde{d}_i$  be the  $i$ -th splitting time of the spine and  $\tilde{O}_i$  be the number of children produced by the spine at time  $\tilde{d}_i$ . Define  $\tilde{D}_{n,\xi_{r_{n+1}-r_n}}$  to be the event that, for all  $i$  with  $\tilde{d}_i < r_{n+1} - r_n$ , it holds that  $\tilde{O}_i \leq e^{c_0 n}$ . Then by the branching property, the Markov property and (3.64), we have

$$\begin{aligned} & \mathbb{E}_x \left( \left| \tilde{Z}_{n+1} - \tilde{Z}_{n+1}^B \right| \Big| \mathcal{F}_{r_n} \right) \\ &= \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbb{E}_{X_u(r_n)} \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} 1_{\tilde{D}_{n,v}^c} \right) \\ &= e^{2(r_{n+1}-r_n)} \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \tilde{\mathbf{E}}_{X_u(r_n)} \left( 1_{\tilde{D}_{n,\xi_{r_{n+1}-r_n}}^c} \right). \end{aligned}$$

Noticing that (1.5) implies  $\tilde{\mathbf{E}}_x(\log_+^{1+\lambda} \tilde{O}_1) = \tilde{\mathbf{E}}_0(\log_+^{1+\lambda} \tilde{O}_1) < \infty$ , we obtain

$$\begin{aligned} & \mathbb{E}_x \left( \left| \tilde{Z}_{n+1} - \tilde{Z}_{n+1}^B \right| \Big| \mathcal{F}_{r_n} \right) \\ & \lesssim \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \tilde{\mathbf{E}}_{X_u(r_n)} \left( \sum_{i: \tilde{d}_i < r_{n+1}-r_n} 1_{\{\tilde{O}_i > e^{c_0 n}\}} \right) \\ & \lesssim \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} 1_{\{\min_{s \leq r_n} X_u(s) > 0\}} \frac{r_{n+1} - r_n}{n^{1+\lambda}}. \end{aligned}$$

Now taking expectation with respect to  $\mathbb{P}_x$ , using Lemma 2.1 and Lemma 2.4(i) (ii), we

get that

$$\begin{aligned} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1} - \tilde{Z}_{n+1}^B \right| \right) &\lesssim \frac{r_{n+1} - r_n}{n^{1+\lambda}} \frac{r_n^{m+\frac{3}{2}}}{e^{(1-\frac{\theta^2}{2})r_n}} e^{r_n} \mathbf{P}_x \left( \min_{s \leq r_n} X_\xi(s) > 0 \right) \\ &\lesssim x e^{\theta x r_{n+1} - r_n} r_n^{m+1}. \end{aligned} \tag{3.65}$$

Similarly, by the branching property and the Markov property,

$$\begin{aligned} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1}^B - \mathbb{E}_x \left( \tilde{Z}_{n+1}^B | \mathcal{F}_{r_n} \right) \right|^2 | \mathcal{F}_{r_n} \right) &= \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \\ &\quad \times \mathbb{E}_{X_u(r_n)} \left( \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\tilde{D}_{n,v}} - \mathbb{E}_{X_u(r_n)} \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\tilde{D}_{n,v}} \right) \right)^2 \right) \\ &\leq \frac{r_n^{2m+3}}{e^{2(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}} \mathbb{E}_{X_u(r_n)} \left( \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\tilde{D}_{n,v}} \right)^2 \right). \end{aligned} \tag{3.66}$$

By (3.64), we see that

$$\begin{aligned} \mathbb{E}_{X_u(r_n)} \left( \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\tilde{D}_{n,v}} \right)^2 \right) &= e^{2(r_{n+1}-r_n)} \tilde{\mathbf{E}}_{X_u(r_n)} \left( \mathbf{1}_{\tilde{D}_{n,\xi_{r_{n+1}-r_n}}} \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\tilde{D}_{n,v}} \right) \\ &\lesssim \tilde{\mathbf{E}}_{X_u(r_n)} \left( \mathbf{1}_{\tilde{D}_{n,\xi_{r_{n+1}-r_n}}} \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1} \right). \end{aligned}$$

On the set  $\tilde{D}_{n,\xi_{r_{n+1}-r_n}}$ , we have

$$\begin{aligned} \tilde{\mathbf{E}}_{X_u(r_n)} \left( \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1}_{\{\tilde{d}_i, \tilde{O}_i : i \geq 1\}} \right) &= \sum_{i: \tilde{d}_i < r_{n+1}-r_n} (\tilde{O}_i - 1) e^{2(r_{n+1}-r_n-d_i)} \\ &\lesssim e^{c_0 n} \sum_{i: \tilde{d}_i < r_{n+1}-r_n} \mathbf{1}, \end{aligned}$$

which implies that

$$\tilde{\mathbf{E}}_{X_u(r_n)} \left( \mathbf{1}_{\tilde{D}_{n,\xi_{r_{n+1}-r_n}}} \sum_{v \in \tilde{N}(r_{n+1}-r_n)} \mathbf{1} \right) \lesssim e^{c_0 n} \tilde{\mathbf{E}}_{X_u(r_n)} \left( \sum_{i: \tilde{d}_i < r_{n+1}-r_n} \mathbf{1} \right) \lesssim e^{c_0 n} (r_{n+1} - r_n).$$

Therefore, plugging this upper bound back to (3.66), we have

$$\mathbb{E}_x \left( \left| \tilde{Z}_{n+1}^B - \mathbb{E}_x \left( \tilde{Z}_{n+1}^B | \mathcal{F}_{r_n} \right) \right|^2 | \mathcal{F}_{r_n} \right) \lesssim \frac{r_n^{2m+3} e^{c_0 n} (r_{n+1} - r_n)}{e^{2(1-\frac{\theta^2}{2})r_n}} \sum_{u \in N(r_n)} \mathbf{1}_{\{\min_{s \leq r_n} X_u(s) > 0\}}.$$

Taking expectation with respect to  $\mathbf{P}_x$ , using Lemma 2.1 and Lemma 2.4(i), we conclude that

$$\begin{aligned} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1}^B - \mathbb{E}_x \left( \tilde{Z}_{n+1}^B | \mathcal{F}_{r_n} \right) \right|^2 \right) &\lesssim \frac{r_n^{2m+3} e^{c_0 n} (r_{n+1} - r_n)}{e^{2(1-\frac{\theta^2}{2})r_n}} e^{r_n} \mathbf{E}_x \left( \mathbf{1}_{\{\min_{s \leq r_n} X_\xi(s) > 0\}} \right) \\ &\lesssim \frac{r_n^{2m+3} e^{c_0 n} (r_{n+1} - r_n)}{e^{(1-\frac{\theta^2}{2})r_n}}. \end{aligned} \tag{3.67}$$

First using (3.14) with  $X = \tilde{Z}_{n+1}, Y = \tilde{Z}_{n+1}^B$  and  $\ell = 2$ , and then (3.65) and (3.67), we

get that

$$\begin{aligned} & \sum_{n=1}^{\infty} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1} - \mathbb{E}_x \left( \tilde{Z}_{n+1} | \mathcal{F}_{r_n} \right) \right| \right) \\ & \leq 2 \sum_{n=1}^{\infty} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1} - \tilde{Z}_{n+1}^B \right| \right) + \sum_{n=1}^{\infty} \mathbb{E}_x \left( \left| \tilde{Z}_{n+1}^B - \mathbb{E}_x \left( \tilde{Z}_{n+1}^B | \mathcal{F}_{r_n} \right) \right|^2 \right)^{1/2} \\ & \lesssim \sum_{n=1}^{\infty} \left( \frac{r_{n+1} - r_n}{n^{1+\lambda}} r_n^{m+1} + \left( \frac{r_n^{2m+3} e^{c_0 n} (r_{n+1} - r_n)}{e^{(1-\frac{\theta^2}{2})r_n}} \right)^{1/2} \right), \end{aligned}$$

which is finite since for  $\lambda > 0$  and  $\kappa > 2m + 2$ ,  $(r_{n+1} - r_n)r_n^{m+1} = O(n^{(m-\kappa+2)/\kappa}) = o(1)$  and  $\sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda}} < \infty$ . Then we finished the proof of the claim (3.63).  $\square$

*Proof of Theorem 1.1.* For any  $\lambda > 6m + 6$ , we can find an appropriate  $\kappa$  satisfying the conditions of Proposition 3.4 and Lemma 3.6. For instance, we can take  $\kappa := 2m + 2 + (\lambda - 6m - 6)/4 > 2m + 2$ , then

$$\lambda = 4(\kappa - 2m - 2) + 6m + 6 > 6m + 6 + 2(\kappa - 2m - 2) = 2m + 2\kappa + 2.$$

For any  $\ell \in [0, m]$ , since  $\kappa > 2m + 2$ , we have

$$r_{n+1}^\ell - r_n^\ell = \frac{\ell}{\kappa} \int_n^{n+1} y^{-(\kappa-\ell)/\kappa} dy \leq \frac{m}{\kappa} n^{-(\kappa-m)/\kappa} = o(r_n^{-m}). \tag{3.68}$$

For  $t \in (r_n, r_{n+1})$ , by Lemma 3.6 and (3.68), we get that for any  $a \geq 0$  and  $t \in (r_n, r_{n+1})$ ,

$$\begin{aligned} \frac{Z_t^{(0,\infty)}((a, \infty))}{t^{-3/2} e^{(1-\frac{\theta^2}{2})t}} &= \frac{r_n^{-3/2}}{t^{-3/2}} \cdot \frac{e^{(1-\frac{\theta^2}{2})r_n}}{e^{(1-\frac{\theta^2}{2})t}} \cdot \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \\ &= (1 + o(r_n^{-m})) \cdot \left( o(r_n^{-m}) + \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}} \right) \\ &= o(r_n^{-m}) + \frac{Z_{r_n}^{(0,\infty)}((a, \infty))}{r_n^{-3/2} e^{(1-\frac{\theta^2}{2})r_n}}. \end{aligned}$$

By (3.68), we have

$$r_n^m \sup_{t \in (r_n, r_{n+1})} \left| \frac{1}{t^\ell} - \frac{1}{r_n^\ell} \right| = o(1).$$

Combining the above with Proposition 3.4(i), we get the assertion of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* The proof is similar to that of Theorem 1.1 and we omit the details.  $\square$

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**Acknowledgments.** We thank the the associate editor and the referee for helpful comments on the first version of this paper. We also thank Zhenyao Sun for helpful discussions.