Electronic Journal of Probability

Electron. J. Probab. **29** (2024), article no. 23, 1–41.
ISSN: 1083-6489 https://doi.org/10.1214/24-EJP1084

The extremal process of super-Brownian motion: A probabilistic approach via skeletons[*†]

Yan-Xia Ren[‡] Ting Yang[§] Rui Zhang[¶]

Abstract

Recently Ren et al. [Stoch. Proc. Appl., 137 (2021)] have proved that the extremal process of the super-Brownian motion converges in distribution in the limit of large times. Their techniques rely heavily on the study of the convergence of solutions to the Kolmogorov-Petrovsky-Piscounov equation along the lines of [M. Bramson, Mem. Amer. Math. Soc., 44 (1983)]. In this paper we take a different approach. Our approach is based on the skeleton decomposition of super-Brownian motion. The skeleton may be interpreted as immortal particles that determine the large time behaviour of the process. We exploit this fact and carry asymptotic properties from the skeleton over to the super-Brownian motion. Some new results concerning the probabilistic representations of the limiting process are obtained, which cannot be directly obtained through the results of [Y.-X. Ren et al., Stoch. Proc. Appl., 137 (2021)]. Apart from the results, our approach offers insights into the driving force behind the limiting process for super-Brownian motions.

Keywords: extremal process; super-Brownian motion; branching Brownian motion; skeleton decomposition.
MSC2020 subject classifications: Primary 60J68, Secondary 60F15; 60F25.
Submitted to EJP on September 3, 2022, final version accepted on January 11, 2024.

[*]The research of this project is supported by the National Key R&D Program of China (No. 2020YFA0712900). The research of Yan-Xia Ren is supported in part by NSFC (Grant Nos. 11731009, 12071011 and 12231002) and The Fundamental Research Funds for Central Universities, Peking University LMEQF. The research of Ting Yang is supported by NSFC (Grant Nos. 12271374 and 12371143). The research of Rui Zhang is supported by NSFC (Grant Nos. 11601354 and 12271374), Beijing Municipal Natural Science Foundation (Grant No. 1202004), and Academy for Multidisciplinary Studies, Capital Normal University.

[†]Ting Yang is the corresponding author.

[‡]LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, P.R. China. E-mail: yxren@math.pku.edu.cn

[§]School of Mathematics and Statistics & Beijing Key Laboratory on MCAACI, Beijing Institute of Technology, Beijing, 100081, P.R. China. E-mail: yangt@bit.edu.cn

[¶]School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, Beijing, 100048, P.R. China. E-mail: zhangrui27@cnu.edu.cn

1 Introduction

1.1 Super-Brownian motion

Throughout this paper, we use ":=" as a way of definition. We use $\mathcal{B}_b(\mathbb{R})$ (respectively, $\mathcal{B}^+(\mathbb{R})$) to denote the space of bounded (respectively, nonnegative) Borel functions on $\mathbb{R}$. The space of continuous (and compactly supported) functions on $\mathbb{R}$ will be denoted as $C(\mathbb{R})$ (and $C_c(\mathbb{R})$ resp.). Let $\mathcal{M}(\mathbb{R})$ be the space of all Radon measures on $\mathbb{R}$ equipped with the vague topology. Suppose $\mathcal{M}_F(\mathbb{R})$ (resp. $\mathcal{M}_c(\mathbb{R})$) is the set of finite (resp. finite and compactly supported) measures on $\mathbb{R}$. We use the notation $\langle f,\mu\rangle := \int_{\mathbb{R}} f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle$. A random variable taking values in $\mathcal{M}(\mathbb{R})$ is called a random Radon measure on $\mathbb{R}$. A sequence of random Radon measures $\{\xi_n : n\geq 1\}$ is said to converge in distribution to ξ if and only if the random variables $\langle f,\xi_n\rangle$ converges in distribution to $\langle f,\xi\rangle$ for any $f\in C_c^+(\mathbb{R})$.

The main process of interest in this paper is an $\mathcal{M}_F(\mathbb{R})$-valued Markov process $X=\{X_t : t\geq 0\}$ with evolution depending on two quantities P_t and ψ. Here P_t is the semigroup of the standard Brownian motion $\{((B_t)_{t\geq 0},\Pi_x) : x\in\mathbb{R}\}$ and ψ is the so-called branching mechanism, which is specified by the Lévy-Khintchine formula

$$\psi(\lambda) = -\alpha\lambda+\beta\lambda^2+\int_{(0,\infty)}\left(e^{-\lambda y}-1+\lambda y\right)\pi(dy) \quad \text{for } \lambda\geq 0, \tag{1.1}$$

where $\alpha>0$, $\beta\geq 0$ and $\pi(dy)$ is a measure on $(0,\infty)$ such that $\int_{(0,+\infty)}\left(y\wedge y^2\right)\pi(dy) < +\infty$. The distribution of X is denoted by P_μ if it is started at $\mu\in\mathcal{M}_F(\mathbb{R})$ at $t=0$. With abuse of notation, we also use P_μ to denote the expectation with respect to P_μ. X is called a (supercritical) (P_t,ψ)-*superprocess* or *super-Brownian motion with branching mechanism* ψ if X is an $\mathcal{M}_F(\mathbb{R})$-valued process such that for any $\mu\in\mathcal{M}_F(\mathbb{R})$, $f\in\mathcal{B}_b^+(\mathbb{R})$ and $t\geq 0$,

$$\mathrm{P}_\mu\left[e^{-\langle f,X_t\rangle}\right] = e^{-\langle u_f(t,\cdot),\mu\rangle}, \tag{1.2}$$

where

$$u_f(t,x) := -\log \mathrm{P}_{\delta_x}\left(e^{-\langle f,X_t\rangle}\right) \tag{1.3}$$

is the unique nonnegative locally bounded solution to the following integral equation:

$$u_f(t,x) = P_t f(x) - \int_0^t P_s\left(\psi(u_f(t-s,\cdot))\right)(x)ds \quad \text{for any } x\in\mathbb{R} \text{ and } t\geq 0.$$

We note that $u_f(t,x)$ is also a solution to the partial differential equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) - \psi(u(t,x)) \tag{1.4}$$

with initial condition $u(0,x)=f(x)$. Moreover, if f is a nonnegative bounded continuous function on $\mathbb{R}$, $\lim_{t\to 0}u_f(t,x)=f(x)$ for $x\in\mathbb{R}$. We will refer to (1.4) as the Kolmogorov-Petrovsky-Piscounov (K-P-P) equation. The existence of such a process X is established by [12]. Moreover, the super-Brownian motion X has a Hunt realization in $\mathcal{M}_F(\mathbb{R})$ (see, for example, [20, Theorem 5.12]) such that $t\mapsto\langle f,X_t\rangle$ is almost surely right continuous for any bounded continuous functions f. We shall always work with this version.

Super-Brownian motion is a stochastic model describing the evolution of a random cloud of Brownian molecules in space. A closely related model is branching Brownian motion (BBM), in which particles move in space according to the law of a standard Brownian motion and reproduce at a constant branching rate. It is well-known that a super-Brownian motion can be constructed as the high density limit of a sequence of BBMs. Another link between super-Brownian motion and BBM is provided by the so-called skeleton decomposition, which is developed in [7, 11, 13, 19]. A detailed review of the skeleton decomposition is given in Section 2.1. There has been great interest in the asymptotic behavior of the extremal configurations of BBM. We refer the reader to [1, 2, 3, 4, 9, 21, 25] and the references therein. We assume that the splitting time of a particle is exponentially distributed with rate 1, the particle splits into two new particles at the splitting time, and each of these two particles independently mimics the parent's behavior from its place of birth. Let M_t be the maximal displacement of BBM at time t. It is proved that, if $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, then, as $t \to +\infty$, $M_t - m(t)$ converges in distribution. Recently, the full statistics of the extremal configurations has been studied in [1, 4]. We use $n(t)$ to denote the number of particles alive at time t and $x_j(t)$ to denote the spatial position of the jth particle. It is proved in [1, 4] that for a BBM, the extremal process, namely the random point measure

$$\sum_{j=1}^{n(t)} \delta_{x_j(t)-m(t)}$$

converges in distribution to a limiting process as $t \to +\infty$. Recently, Ren et al. [23] have obtained the corresponding results for super-Brownian motions by analytic methods. In this paper we shall re-establish some results of [23], with improvements, by appealing to the skeleton decomposition. This decomposition provides a path-wise description of the super-Brownian motion in terms of immigration along a BBM, the skeleton. We shall use it to make a connection between the spatial asymptotic behavior of a BBM and that of a super-Brownian motion.

In analogy to the case of BBM, We call the random measure $\mathcal{E}_t := X_t - m(t)$, with some centering term $m(t)$, the extremal process of the super-Brownian motion, which is simply the super-Brownain motion seen from the position $m(t)$. Our aim is to show that $\mathcal{E}_t$ converges in distribution as $t \to +\infty$ and give an explicit construction of the limiting process.

1.2 Preview of main results and ideas of proofs

One of our main purpose is to use skeleton decomposition to show that the extremal process $\mathcal{E}_t$ converges in distribution as $t \to +\infty$. This relies on the convergence of the log-Laplace functional of the extremal process. So we start with a theorem which establishes an integral representation of $u_\phi(t, x - m(t))$ in the large time limit (Theorem 2.8). This is a fundamental result and will be used extensively in our paper. Results of this type have been proved in Ren et al. [23] by analytic methods. Here we take a different approach. The core of our argument is the skeleton decomposition that represents the supercritical super-Brownian motion as the sum of a subcritical super-Brownian motion X^* and an immigration process I along a BBM Z, called the skeleton. Then our proof of Theorem 2.8 follows two main steps: The first step is to show that the Laplace functional of the extremal process of the skeleton BBM (or, in other words, the solution to the K-P-P equation corresponding to a certain class of $[0, 1]$-valued initial conditions) converges as $t \to \infty$ and give an integral representation for the limit. This is done by Proposition 2.5, for which we take an approach similar to [4]: first, establish the convergence for the Laplace functionals of the skeleton BBM which are truncated by a certain cutoff, and then show the convergence still holds when the cutoff is lifted. The second step is to establish the convergence of the log-Laplace functionals of the extremal process of the full process X. Our main idea is as follows: Using the tree structure of the skeleton, we can represent the log-Laplace functional $u_\phi(t, x)$ of X as follows (eq. (3.16)):

$$u_\phi(t, x) = u^*_\phi(t, x) + 1 - V_\phi(t, x), \quad \forall t \geq 0,\ x \in \mathbb{R},$$

where $u_\phi^*(t,x)$ is the log-Laplace functional of the subcritical super-Brownian motion X^* and $V_\phi(t,x)$ is the Laplace functional of the immigration process I. On the other hand, it follows from the Markov property and spatial homogeneity of super-Brownian motion that for $t,s>0$ and $x\in\mathbb{R}$,

$$u_\phi(t+s,x-m(t+s)) = u_{u_\phi(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t)),$$

where for each $s>0$, $O_s(t)\to 0$ as $t\to+\infty$. Then using the comparison lemma (Lemma 3.1) we get two-sided bounds for $u_\phi(t,x-m(t))$ given by

$$\begin{aligned} u_{1-V_\phi(s,\cdot-\sqrt{2}s)}&(t,x-m(t)+O_s(t)) \leq u_\phi(t+s,x-m(t+s)) \\ &\leq u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t)) + u_{u_\phi^*(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t)). \end{aligned}$$

An advantage of the above representation is that both $u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t))$ and $u_{u_\phi^*(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t))$ (for large s) have $[0,1]$-valued initial conditions satisfying the assumptions of Proposition 2.5. So the convergence results obtained there can be applied directly. We thus finish our proof of Theorem 2.8 by showing that $u_{u_\phi^*(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t))$ converges to 0 and $u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(t,x-m(t)+O_s(t))$ converges to the desired limit as $t\to+\infty$ and $s\to+\infty$.

Following Theorem 2.8, we obtain in Theorem 2.10 the joint convergence of the extremal processes of the super-Brownian motion and the skeleton. The joint limiting process $(\mathcal{E}_\infty,\mathcal{E}_\infty^Z)$ is distributed in such a way that given $\mathcal{E}_\infty$, $\mathcal{E}_\infty^Z$ has the law of Poisson random measure with random intensity determined by $\mathcal{E}_\infty$. This is essentially a consequence of the weak bi-continuity of the Cox process (cf. [16, Lemma 4.17]), given the fact that, on the skeleton space, the skeleton Z_t has the law of a Poisson random measure with intensity determined by the full process X_t for each $t>0$.

In this paper we are also concerned with the construction of the limiting extremal process. Results of [1, 4] show that the limiting extremal process of BBM is a (randomly shifted) Poisson cluster process, also called a decorated Poisson point process, where the positions of the cluster form a Poisson point process with an exponential intensity measure and the law of the individual clusters is characterized as a BBM conditioned to perform unusually large displacements. A similar result is obtained in [23] for super-Brownian motion with branching mechanism being bounded from below by a stable branching mechanism. More precisely, for a super-Brownian motion with branching mechanism satisfying Assumption 2.15, [23] shows that the limiting extremal process is a decorated Poisson random measure, where the Poisson random measure has an exponential intensity, each atom is decorated by an independent copy of an auxiliary random measure, and the law of this auxiliary measure is characterized as a super-Brownian motion conditioned on the supremum of support being unusually large.

It is natural to wonder if one can give an explicit construction of the limiting extremal process for super-Brownian motions with more general branching mechanisms. To answer this, Theorem 2.10 provides an integral representation for the Laplace functional of the limiting extremal process, and more importantly, its proof tells us that, even though the immigration process I is a subprocess of X, the convergence for the Laplace functional of the extremal process of the full process X follows when that of the subprocess converges to the claimed limit. In fact, this key observation is formulated by Lemma 3.10 in terms of the corresponding convergence of the function C of initial conditions (see, (2.22) below, for the definition of the function C). This allows us to construct $\mathcal{E}_\infty$ as a limit from the random point measure $\mathcal{E}_\infty^Z$ to which, at each atom, is attached an independent random measure with the same law as $I_s-\sqrt{2}s$ (Theorem 2.12). On the other hand, since $\mathcal{E}_\infty^Z$ itself is a Poisson cluster process, we can also construct $\mathcal{E}_\infty$

from the limit of a Poisson point process in which each atom is decorated by independent copy of an auxiliary random measure (Proposition 2.13). The decoration laws needed for our analysis are obtained by using again the tree structure of the skeleton. To be more specific, by splitting the immigration that occurs after time t according to different branches of the skeleton at time t, we can represent the conditional expectation at time t for the immigration after time t as functionals of the skeleton at time t, and thus can investigate the joint conditional laws of the super-Brownian motion, the immigration process and the skeleton. Based on the above results, we prove that $\mathcal{E}_\infty$ is an infinitely divisible random measure. We then give a cluster representation of $\mathcal{E}_\infty$ and characterize its canonical measures in Theorem 2.14. Finally we discuss the special case where the branching mechanism satisfies Assumption 2.15 and give the explicit construction of $(\mathcal{E}_\infty, \mathcal{E}_\infty^Z)$ in Theorem 2.16.

Parts of our results (Theorem 2.8, Theorem 2.10 and Theorem 2.16) concerning the existence and construction of the limiting extremal process of super-Brownian motion overlap with the results obtained in Ren et al. [23]. Compared with [23], the present work takes a different approach, focuses on the sample paths, provides a self-contained proof of the stated results, as well as obtains new results concerning the probabilistic representation of the limiting process (Theorem 2.12, Proposition 2.13 and Theorem 2.14). Apart from the result itself, the idea of our proof provides structural insights into the driving force behind the spatial asymptotic behavior for super-Brownian motions. We believe that our idea, based on the skeleton decomposition, is (under certain assumption) feasible also for super-Brownian motions with spatially-dependent branching mechanisms. In fact, in the recent work [24], the authors use this idea to study the limiting distributions for super-Brownian motions with compactly supported branching mechanisms.

The rest of this paper is organized as follows. Section 2 collects the definition of skeleton decomposition, basic properties of the skeleton process and main results of this paper: In Subsection 2.1 we give a detailed description of the skeleton decomposition. In Subsection 2.2 we start with an introduction of the derivative martingale of the skeleton BBM. Then we review some facts about the convergence of the solutions to the K-P-P equation with applications to the skeleton BBM. At the end of this subsection, we compare the limit of derivative martingale of the skeleton BBM with that of the super-Brownian motion. In Subsection 2.3 we present our main results. The subsequent sections are devoted to the proofs of the results presented in Section 2. Some minor statements needed along the way are proved in the Appendix.

2 Preliminaries and results

2.1 The skeleton decomposition of super-Brownian motion

The skeleton decomposition makes a connection between superprocesses and branching Markov processes. This decomposition provides a path-wise description of a superprocesses in terms of immigrations along a branching Markov process called the skeleton. Recall that (X, P_μ) is a (P_t, ψ)-superprocess. The following condition is fundamental for the skeleton construction.

Assumption 2.1. $\psi(+\infty) = +\infty$.

Assumption 2.1 implies that there exists some $\lambda^* \in (0, +\infty)$ such that $\psi(\lambda^*) = 0$. The key property of λ^* used in the skeleton construction is that

$$\mathrm{P}_\mu\left(\lim_{t\to+\infty} \|X_t\| = 0\right) = \mathrm{e}^{-\lambda^*\|\mu\|} \quad \forall \mu \in \mathcal{M}_F(\mathbb{R}),$$

and so λ^* gives rise to the multiplicative P_μ-martingale $(e^{-\lambda^*\|X_t\|})_{t\geq 0}$. A more detailed description of skeleton construction is given in Proposition 2.2.

Proposition 2.2 (Kyrianou et al. [19]). *Suppose Assumption 2.1 holds. For every $\mu \in \mathcal{M}_F(\mathbb{R})$ and every finite point measure ν on $\mathbb{R}$, there exists a probability space with probability measure $\mathbb{P}_{\mu,\nu}$, called the skeleton space, that carries three processes: $Z = (Z_t)_{t\geq 0}$, $I = (I_t)_{t\geq 0}$ and $X^* = (X^*_t)_{t\geq 0}$, where $(Z, \mathbb{P}_{\mu,\nu})$ is a BBM with branching rate $q > 0$ and offspring distribution $\{p_k : k \geq 2\}$ uniquely determined by*

$$q(F(s) - s) = \frac{1}{\lambda^*}\psi(\lambda^*(1-s)) \quad \forall s \in [0,1], \tag{2.1}$$

here $F(s) := \sum_{k=2}^{+\infty} p_k s^k$, and $\mathbb{P}_{\mu,\nu}(Z_0 = \nu) = 1$; $(X^*, \mathbb{P}_{\mu,\nu})$ is a subcritical super-Brownian motion with branching mechanism $\psi^*(\lambda) = \psi(\lambda + \lambda^*)$ and $\mathbb{P}_{\mu,\nu}(X^*_0 = \mu) = 1$; $(I, \mathbb{P}_{\mu,\nu})$ is an $\mathcal{M}_F(\mathbb{R})$-valued process with $\mathbb{P}_{\mu,\nu}(I_0 = 0) = 1$ such that the following holds.

(i) $\mathbb{P}_{\mu,\sum_i \delta_{x_i}}\left[\mathrm{e}^{-\langle f, I_t\rangle}\right] = \prod_i \mathbb{P}_{\mu,\delta_{x_i}}\left[\mathrm{e}^{-\langle f, I_t\rangle}\right]$ for all $\mu \in \mathcal{M}_F(\mathbb{R})$, $t \geq 0$, countable $x_i \in \mathbb{R}$ and $f \in \mathcal{B}^+(\mathbb{R})$. Moreover, the distribution of I under $\mathbb{P}_{\mu,\nu}$ does not dependent on μ, and under $\mathbb{P}_{\mu,\nu}$, both Z and I are independent of X^*.

(ii) $((X^* + I, Z), \mathbb{P}_{\mu,\nu})$ is a Markov process.

(iii) If $\mathbb{P}_\mu$ denotes the measure $\mathbb{P}_{\mu,\nu}$ with ν replaced by a Poisson random measure with intensity $\lambda^*\mu(dx)$, then $\left(\widehat{X} := X^* + I; \mathbb{P}_\mu\right)$ is Markovian and has the same distribution as $(X; \mathrm{P}_\mu)$.

(iv) under $\mathbb{P}_\mu$, given $\widehat{X}_t$, the measure Z_t is a Poisson random measure with intensity $\lambda^*\widehat{X}_t(dx)$.

Since $(\widehat{X}; \mathbb{P}_\mu)$ is equal in distribution to the (P_t, ψ)-superprocess $(X; \mathrm{P}_\mu)$, we may work on this skeleton space whenever it is convenient. For notational simplification, we will abuse the notation and denote $\widehat{X}$ by X. We will refer to Z and I, respectively, as the skeleton BBM (skeleton) and the immigration process (immigration) of X. Since the distribution of X^* (resp. I) under $\mathbb{P}_{\mu,\nu}$ does not depend on ν (resp. μ), we sometimes write $\mathbb{P}_{\mu,\cdot}$ (resp. $\mathbb{P}_{\cdot,\nu}$) for $\mathbb{P}_{\mu,\nu}$.

We observe that up to a space-time scaling transform, the branching mechanism ψ can be assumed to satisfy that

$$\psi'(0+) = -1 \text{ and } \lambda^* = 1. \tag{2.2}$$

In fact, if $u(t,x)$ is a solution to (1.4) then $u(\alpha^{-1}t, \alpha^{-1/2}x)/\lambda^*$ is a solution to (1.4) with ψ replaced by $\widetilde{\psi}$ where $\widetilde{\psi}(\lambda) = \psi(\lambda^*\lambda)/\alpha\lambda^*$ satisfies that $\widetilde{\psi}'(0+) = -1$ and $\tilde{\psi}(1) = 0$. This implies that for a (P_t, ψ)-superprocess $(X_t)_{t\geq 0}$, if we define the random measures $\tilde{X}_t$ by

$$\langle f, \widetilde{X}_t\rangle = \lambda^*\langle f(\alpha^{1/2}\cdot), X_{\alpha^{-1}t}\rangle \quad \forall t \geq 0,\ f \in \mathcal{B}_b^+(\mathbb{R}), \tag{2.3}$$

then $(\widetilde{X}_t)_{t\geq 0}$ is a $(P_t, \widetilde{\psi})$-superprocess. It thus suffices to study the long time behavior of $\widetilde{X}_t$. In the rest of this paper the branching mechanism ψ is assumed to satisfy (2.2), which will simplify computations and notations later on.

2.2 The extremal process of the skeleton and facts

2.2.1 Derivative martingales for the skeleton

In this and the next two subsections we assume Assumption 2.1 and (2.2) hold. Additional conditions used are stated explicitly. Recall that $u \in Z_t$ and $z_u(t)$ denote, respectively, a particle of the skeleton BBM which is alive at time t and its spatial location at t. Define for $t \geq 0$,

$$\partial M_t := \sum_{u\in Z_t}\left(\sqrt{2}t - z_u(t)\right)\mathrm{e}^{\sqrt{2}\left(z_u(t)-\sqrt{2}t\right)}.$$

It is known that $((\partial M_t)_{t\geq 0}, \mathbb{P}_{\cdot,\nu})$ is a signed martingale for every compactly supported finite point measure ν on $\mathbb{R}$, which is referred to as the derivative martingale of the skeleton BBM $(Z_t)_{t\geq 0}$. This martingale is deeply related to the travelling wave solutions to the K-P-P equation and plays an important role in the limit theory of the skeleton BBM. [17] proved that the martingale $((\partial M_t)_{t\geq 0}, \mathbb{P}_{\cdot,\delta_0})$ has an almost sure nonnegative limit, and later [27] established the sufficient and necessary condition for the limit to be non-degenerate. We give the statement below which reproduces the same results in the setting of skeleton space.

Proposition 2.3. *Suppose $\mu \in \mathcal{M}_c(\mathbb{R})$. The limit $\partial M_\infty = \lim_{t\to+\infty} \partial M_t$ exists and is nonnegative $\mathbb{P}_\mu$-a.s. Moreover, if the Lévy measure π of ψ satisfies*

$$\int_{(1,+\infty)} x(\log x)^2 \pi(dx) < +\infty, \tag{2.4}$$

then

$$\mathbb{P}_\mu\left(\partial M_\infty = 0\right) = \mathrm{e}^{-\|\mu\|}.$$

If (2.4) *fails, then $\partial M_\infty = 0$ $\mathbb{P}_\mu$-a.s.*

Proof. By decomposing ∂M_t into contributions derived from the population at time $s \in [0,t)$, one has

$$\partial M_t = \sum_{u\in Z_s} \sum_{v\in Z_t, u\prec v} \left(\sqrt{2}t - z_v(t)\right) \mathrm{e}^{\sqrt{2}\left(z_v(t)-\sqrt{2}t\right)}.$$

Here $u \prec v$ means that u is an ancestor of v. We use $z_v^{(u)}(t-s)$ to denote $z_v(t) - z_u(s)$. Then we have

$$\begin{aligned}
&\partial M_t\\
= &\sum_{u\in Z_s} \sum_{v\in Z_t, u\prec v} \left(\sqrt{2}s - z_u(s) + \sqrt{2}(t-s) - z_v^{(u)}(t-s)\right) \mathrm{e}^{\sqrt{2}\left(z_u(s)-\sqrt{2}s+z_v^{(u)}(t-s)-\sqrt{2}(t-s)\right)}\\
= &\sum_{u\in Z_s} \mathrm{e}^{\sqrt{2}(z_u(s)-\sqrt{2}s)} \left[\sum_{v\in Z_t, u\prec v} \left(\sqrt{2}(t-s) - z_v^{(u)}(t-s)\right) \mathrm{e}^{\sqrt{2}\left(z_v^{(u)}(t-s)-\sqrt{2}(t-s)\right)}\right]\\
&+ \sum_{u\in Z_s} \left(\sqrt{2}s - z_u(s)\right) \mathrm{e}^{\sqrt{2}(z_u(s)-\sqrt{2}s)} \left[\sum_{v\in Z_t, u\prec v} \mathrm{e}^{\sqrt{2}\left(z_v^{(u)}(t-s)-\sqrt{2}(t-s)\right)}\right]\\
=: &\sum_{u\in Z_s} \mathrm{e}^{\sqrt{2}(z_u(s)-\sqrt{2}s)} \partial M_{t-s}^{(u)} + \sum_{u\in Z_s} (\sqrt{2}s - z_u(s)) \mathrm{e}^{\sqrt{2}(z_u(s)-\sqrt{2}s)} M_{t-s}^{(u)}.
\end{aligned}$$

In particular by setting $s=0$, we have

$$\partial M_t = \sum_{u\in Z_0} \mathrm{e}^{\sqrt{2}z_u(0)} \partial M_t^{(u)} - \sum_{u\in Z_0} z_u(0) \mathrm{e}^{\sqrt{2}z_u(0)} M_t^{(u)}. \tag{2.5}$$

Define for $t \geq 0$,

$$M_t := \sum_{u\in Z_t} \mathrm{e}^{\sqrt{2}(z_u(t)-\sqrt{2}t)}.$$

It is easy to see that for $u \in Z_0$, $\partial M_t^{(u)}$ and $M_t^{(u)}$ are independent copies of $(\partial M_t, \mathbb{P}_{\cdot,\delta_0})$ and $(M_t, \mathbb{P}_{\cdot,\delta_0})$, respectively. By [17] one has $\mathbb{P}_{\cdot,\delta_0}\left(\lim_{t\to+\infty} \partial M_t \in [0,+\infty)\right) = 1$ and $\mathbb{P}_{\cdot,\delta_0}\left(\lim_{t\to+\infty} M_t = 0\right) = 1$. We also note that Z_0 is a Poisson random measure with compactly supported intensity μ. So each of the sums on the right hand side of (2.5) contains finite terms almost surely. These facts together with (2.5) imply that the limit $\partial M_\infty = \lim_{t\to+\infty}\partial M_t$ exists and is nonnegative $\mathbb{P}_\mu$-a.s. for every $\mu \in \mathcal{M}_c(\mathbb{R})$.

By letting $t \to +\infty$ in (2.5), we have

$$\partial M_\infty \overset{\mathrm{d}}{=} \sum_{u\in Z_0} \mathrm{e}^{\sqrt{2}z_u(0)} \partial M_\infty^{(u)}. \tag{2.6}$$

where for $u \in Z_s$, $\partial M_\infty^{(u)}$ are independent copies of $(\partial M_\infty, \mathbb{P}_{\cdot,\delta_0})$. Recall that $(Z_0, \mathbb{P}_\mu)$ is a Poisson random measure with intensity $\mu(dx)$. Using the Poisson computations, we get by (2.6) that

$$\mathbb{P}_\mu\left[\mathrm{e}^{-\lambda\partial M_\infty}\right] = \exp\left\{-\int_{\mathbb{R}}\left(1-\mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\lambda\mathrm{e}^{\sqrt{2}x}\partial M_\infty}\right]\right)\mu(dx)\right\}, \quad \forall\lambda>0. \tag{2.7}$$

By letting $\lambda \to +\infty$ we have

$$\mathbb{P}_\mu\left(\partial M_\infty = 0\right) = \mathrm{e}^{-\left(1-\mathbb{P}_{\cdot,\delta_0}(\partial M_\infty=0)\right)\|\mu\|}. \tag{2.8}$$

By [27] $\mathbb{P}_{\cdot,\delta_0}(\partial M_\infty = 0) = 1$ if $\sum_k k(\log k)^2 p_k = +\infty$, and otherwise $\mathbb{P}_{\cdot,\delta_0}(\partial M_\infty = 0) = 0$. On the other hand by Lemma A.1 $\sum_k k(\log k)^2 p_k$ is finite if and only if so is $\int_{(1,+\infty)} x(\log x)^2\pi(dx)$. Thus it follows by (2.8) that $\mathbb{P}_\mu\left(\partial M_\infty = 0\right) = 1$ if and only if $\int_{(1,+\infty)} x(\log x)^2\pi(dx) = +\infty$, and otherwise $\mathbb{P}_\mu\left(\partial M_\infty = 0\right) = \mathrm{e}^{-\|\mu\|}$. □

2.2.2 The extremal process of the skeleton

For $x \in \mathbb{R}$ and a function f on $\mathbb{R}$, we define the shift operator $\mathcal{T}_x$ by $\mathcal{T}_x f(y) := f(x+y)$ for all $y \in \mathbb{R}$. For $\mu \in \mathcal{M}(\mathbb{R})$, we use $\mu + x$ and sometimes $\mathcal{T}_x\mu$ to denote the measure induced by $\mathcal{T}_x$, that is, $\int_{\mathbb{R}} f(y)\mathcal{T}_x\mu(dy) = \int_{\mathbb{R}} f(y)(\mu+x)(dy) = \int_{\mathbb{R}} \mathcal{T}_x f(y)\mu(dy)$ for all $f \in \mathcal{B}^+(\mathbb{R})$.

Given (2.2), (2.1) can be written as

$$q(F(s)-s) = \psi(1-s), \quad \forall s \in [0,1], \tag{2.9}$$

where $F(s) = \sum_{k=2}^{+\infty} p_k s^k$. Let $f : \mathbb{R} \to [0,1]$ be a Borel function. It is known that the function $(t,x) \mapsto \mathbb{P}_{\cdot,\delta_x}\left[\prod_{u\in Z_t} f(z_u(t))\right]$ is a solution of the equation

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x) + q(F(u(t,x)) - u(t,x)).$$

with initial condition $u(0,x) = f(x)$. Then $(t,x) \mapsto 1 - \mathbb{P}_{\cdot,\delta_x}\left[\prod_{u\in Z_t}(1-f(z_u(t)))\right]$ is a solution to the equation (1.4) with initial condition $u(0,x) = f(x)$.

Recall the definition of $u_f(t,x)$ for $f \in \mathcal{B}_b^+(\mathbb{R})$ given in (1.3). We note that $u_f(t,x)$ is the unique nonnegative solution to (1.4) with initial condition f. In particular, if $\|f\|_\infty \le 1$,

$$u_f(t,x) = 1 - \mathbb{P}_{\cdot,\delta_x}\left[\prod_{u\in Z_t}(1-f(z_u(t)))\right], \tag{2.10}$$

where $(Z_t)_{t\ge0}$ is the skeleton BBM.

Bramson [10] studied the asymptotic behavior of the solution to the K-P-P equation (1.4) with initial condition $u(0,x)$ taking values in $[0,1]$. Actually in [10], the nonlinear function $-\psi$ can be any function on $[0,1]$ satisfying that

$$\begin{aligned}&\psi \in C^1[0,1], \quad \psi(0) = \psi(1) = 0, \quad -\psi(u) > 0 \text{ for } u \in (0,1),\\ &\psi'(0) = -1, \quad -\psi'(u) \le 1 \text{ for } 0 < u \le 1.\end{aligned}$$

Bramson [10] showed in particular that if ψ also satisfies that

$$1+\psi'(u)=O(u^{\rho}) \quad \text{as } u\to 0 \tag{2.11}$$

for some $\rho>0$, then when the initial condition $u(0,x)$ satisfies a certain integrability condition, it holds that

$$u(t,x-m(t))\to w(-x) \quad \text{uniformly in } x\in\mathbb{R},\ \text{as } t\to+\infty.$$

where

$$m(t):=\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t, \tag{2.12}$$

and $w(x)$ is a travelling wave solution with speed $\sqrt{2}$, that is, $w(x)$ is the unique (up to translations) solution to the ordinary differential equation

$$\frac{1}{2}w''(x)+\sqrt{2}w'(x)-\psi(w(x))=0, \tag{2.13}$$

with $1-w(x)$ being a distribution function on $\mathbb{R}$. The integral representation of $w(x)$ is established in Lalley and Sellke [21] (see also, [17, 27]): When ∂M_∞ is nondegenerate, one has

$$w(x)=1-\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C\partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right] \tag{2.14}$$

for some constant $C>0$. Moreover, it holds that

$$\lim_{x\to+\infty}\frac{w(x)}{x\mathrm{e}^{-\sqrt{2}x}}=C. \tag{2.15}$$

Later [4] recovered the above representation of the form (2.14) and provided an expression for the constant C as a function of the initial condition. As a result [4] established the convergence in distribution of the extremal process of BBM. To apply the aforementioned results directly to the skeleton BBM, we assume the following condition holds.

Assumption 2.4. *For the Lévy measure π of ψ, there exists $\beta\in(0,1)$ such that*

$$\int_{(1,+\infty)} y^{1+\beta}\pi(dy)<+\infty.$$

It is easy to see from Proposition 2.3 that Assumption 2.4 is sufficient for ∂M_∞ to be nondegenerate. Besides, Assumption 2.4 is also sufficient for (2.11). This is because, by Lemma A.2, Assumption 2.4 holds if and only if $\int_0^1(1+\psi'(s))s^{-(1+\beta)}ds<+\infty$, and the latter implies that $1+\psi'(s)=O(s^{\beta})$.

Define

$$\mathcal{H}_1:=\left\{\phi\in\mathcal{B}_b^+(\mathbb{R}):\ \|\phi\|_\infty\le 1 \text{ and } \int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y}\phi(-y)dy<+\infty\right\}.$$

The following proposition establishes a Lalley-Sellke type representation for the Laplace functional of the extremal process of the skeleton BBM in the limit of large times. The proof is deferred to Section 3.1.

Proposition 2.5. *Suppose Assumption 2.4 holds. For every $\phi\in\mathcal{H}_1$, the limit*

$$C(\phi):=\lim_{r\to+\infty}\sqrt{\frac{2}{\pi}}\int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y}u_\phi(r,-y-\sqrt{2}r)dy$$

exists and is finite. Moreover,

$u_\phi(t,x-m(t))\to 1-\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi)\partial M_\infty \mathrm{e}^{\sqrt{2}x}\}\right]$ *locally uniformly in $x\in\mathbb{R}$, as $t\to+\infty$.*

We remark here that Proposition 2.5 refines [4, Lemma 4.10], where the convergence is obtained for continuous and compactly supported functions in $\mathcal{H}_1$.

Taking $\phi = 1 - \mathrm{e}^{-g}$ for some $g \in C_c^+(\mathbb{R})$, one can rewrite $u_\phi(t,x)$ as $1 - \mathbb{P}_{\cdot,\delta_x}\left[\mathrm{e}^{-\langle g, Z_t\rangle}\right]$ by (2.10), where $(Z_t)_{t\geq 0}$ is the skeleton BBM. Then the above result yields that

$$\mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle g, Z_t - m(t)\rangle}\right] \to \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi)\partial M_\infty\}\right] \quad \text{as } t \to +\infty. \tag{2.16}$$

This implies that under $\mathbb{P}_{\cdot,\delta_0}$, the extremal process of Z, defined by

$$\mathcal{E}_t^Z := Z_t - m(t), \quad \forall t \geq 0,$$

converges in distribution as $t \to +\infty$. Furthermore, by applying the results of [1, 4], one can deduce that $(\mathcal{E}_t^Z, \mathbb{P}_{\cdot,\delta_0})$ converges in distribution, as $t \to \infty$, to a random point measure $\mathcal{E}_\infty^Z$, which is a decorated Poisson point process with intensity $c_* \partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}z}dz$ and decoration law $\triangle^Z$, where

$$c_* := C(1_{(0,+\infty)}) = \lim_{r\to+\infty} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} z\mathrm{e}^{\sqrt{2}z} u_{1_{(0,+\infty)}}(r, -z - \sqrt{2}r)dz \in (0, +\infty), \tag{2.17}$$

and $\triangle^Z$ is a random point measure supported on $(-\infty, 0]$, with an atom at 0, which satisfies that

$$\mathrm{E}\left[\mathrm{e}^{-\langle \phi, \triangle^Z\rangle}\right] = \lim_{t\to+\infty} \mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle \phi, Z_t - \max Z_t\rangle} | \max Z_t > \sqrt{2}t\right], \quad \forall \phi \in \mathcal{C}_c^+(\mathbb{R}). \tag{2.18}$$

Here $\max Z_t := \max\{z_u(t) : \ u \in Z_t\}$. We denote $\mathcal{E}_\infty^Z$ by DPPP($c_* \partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}z}dz$, $\triangle^Z$).

The limiting extremal process $\mathcal{E}_\infty^Z$ can be constructed as follows. Given ∂M_∞, let $\{e_i : i \geq 1\}$ be the atoms of a Poisson point process on $\mathbb{R}$ with intensity $c_* \partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}z}dz$ and $\{\triangle_i^Z : i \geq 1\}$ be a sequence of i.i.d. point measures with the same law as $\triangle^Z$, then

$$\mathcal{E}_\infty^Z \overset{\mathrm{d}}{=} \sum_{i\geq 1} \mathcal{T}_{e_i} \triangle_i^Z.$$

Here $\mathcal{T}_{e_i}\triangle_i^Z$ denotes the point measure obtained from $\triangle_i^Z$ shifted by e_i.

Recall that $\max Z_t$ is the maximal displacement of the skeleton BBM. Then

$$u_{1_{(0,+\infty)}}(t,x) = \mathbb{P}_{\cdot,\delta_x}\left(\max Z_t > 0\right) = \mathbb{P}_{\cdot,\delta_0}\left(\max Z_t + x > 0\right)$$

is a solution to (1.4) with initial condition $1_{(0,+\infty)}(x)$. Proposition 2.5 yields that for any $x \in \mathbb{R}$,

$$\mathbb{P}_{\cdot,\delta_0}\left(\max Z_t - m(t) \leq x\right) \to \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-c_* \partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right] \text{ as } t \to +\infty.$$

This implies that under the assumptions of Proposition 2.5, the maximal displacement of the skeleton BBM centered by $m(t)$ converges in distribution to a randomly shifted Gumbel distribution. In fact, Proposition 2.5 implies the joint convergence of $(\mathcal{E}_t^Z, \max \mathcal{E}_t^Z)$ in distribution, see, for example, [6, Lemma 4.4].

2.2.3 Relation between the limits of the derivative martingales of super-Brownian motion and its skeleton

Define for $t \geq 0$,

$$\partial W_t := \langle (\sqrt{2}t - \cdot)\mathrm{e}^{\sqrt{2}(\cdot - \sqrt{2}t)}, X_t\rangle.$$

By [18] for every $\mu \in \mathcal{M}_c(\mathbb{R})$, $((\partial W_t)_{t\geq 0}, \mathbb{P}_\mu)$ is a martingale which is usually called the derivative martingale of the super-Brownian motion. Obviously $(\partial W_t)_{t\geq 0}$ is the counterpart of $(\partial M_t)_{t\geq 0}$ in the setting of superprocess. Interest of this martingale is stimulated by its close connection with the travelling wave solutions to the K-P-P equation (see, for example, [18] and the references therein). Note that $((\partial W_t)_{t\geq 0}, \mathbb{P}_\mu)$ is a signed martingale which does not necessarily converge almost surely. To study its convergence, [18] imposed the following condition on ψ:

$$\int_z^{+\infty} \frac{1}{\sqrt{\int_1^y \psi(u)du}} dy < +\infty \quad \forall z > 1. \tag{2.19}$$

The following result is from [18, Theorem 2.4].

Lemma 2.6. *Assume* (2.19) *holds. Then for every* $\mu \in \mathcal{M}_c(\mathbb{R})$*, the limit* $\partial W_\infty = \lim_{t\to+\infty} \partial W_t$ *exists and is nonnegative* $\mathbb{P}_\mu$*-a.s. Moreover,* ∂W_∞ *is non-degenerate if and only if* (2.4) *holds.*

Proposition 2.7. *Assume* (2.19) *holds. Then one has* $(\partial W_\infty, \mathbb{P}_\mu) \overset{\mathrm{d}}{=} (\partial M_\infty, \mathbb{P}_\mu)$ *for any* $\mu \in \mathcal{M}_c(\mathbb{R})$*.*

Proof. Fix an arbitrary $\mu \in \mathcal{M}_c(\mathbb{R})$. If (2.4) fails, then by Proposition 2.3 and Lemma 2.6, one has $\partial W_\infty = \partial M_\infty = 0$ $\mathbb{P}_\mu$-a.s. Now we assume (2.4) holds. Let $w(x)$ be the travelling wave solution to the K-P-P equation with speed $\sqrt{2}$. It follows by [18, Theorem 2.4 and Theorem 2.6] that under (2.19) and (2.4), $w(x)$ is given by

$$w(x) = -\log \mathbb{P}_{\delta_0}\left[\exp\{-C\partial W_\infty \mathrm{e}^{-\sqrt{2}x}\}\right]$$

for some constant $C > 0$ satisfying that $\lim_{x\to+\infty} \frac{w(x)}{x\mathrm{e}^{-\sqrt{2}x}} = C$. Hence by (2.14) one has

$$-\log \mathbb{P}_{\delta_0}\left[\exp\{-C\partial W_\infty \mathrm{e}^{-\sqrt{2}x}\}\right] = 1 - \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C\partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right] \tag{2.20}$$

for all $C > 0$ and $x \in \mathbb{R}$. We observe that for every $x \in \mathbb{R}$, the process $((X_t)_{t\geq 0}, \mathbb{P}_{\delta_x})$ is equal in law to $((X_t + x)_{t\geq 0}, \mathbb{P}_{\delta_0})$. It follows that $(\partial W_t, \mathbb{P}_{\delta_x}) \overset{\mathrm{d}}{=} (\mathrm{e}^{\sqrt{2}x}\partial W_t - x\mathrm{e}^{\sqrt{2}x} W_t, \mathbb{P}_{\delta_0})$ for any $t \geq 0$ where $W_t := \langle \mathrm{e}^{\sqrt{2}(\cdot - \sqrt{2}t)}, X_t\rangle$. Since by [18, Theorem 2.4(i)] $\lim_{t\to+\infty} W_t = 0$ $\mathbb{P}_{\delta_0}$-a.s., one gets that

$$(\partial W_\infty, \mathbb{P}_{\delta_x}) \overset{\mathrm{d}}{=} (\mathrm{e}^{\sqrt{2}x}\partial W_\infty, \mathbb{P}_{\delta_0}).$$

Using this and the branching property of superprocesses, one has for any $\lambda > 0$,

$$\begin{aligned} \mathbb{P}_\mu\left[\mathrm{e}^{-\lambda\partial W_\infty}\right] &= \exp\left\{\int_{\mathbb{R}} \log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\lambda\partial W_\infty}\right]\mu(dx)\right\} \\ &= \exp\left\{\int_{\mathbb{R}} \log \mathbb{P}_{\delta_0}\left[\mathrm{e}^{-\lambda\partial W_\infty \mathrm{e}^{\sqrt{2}x}}\right]\mu(dx)\right\}. \end{aligned}$$

Hence by (2.20) and (2.7) one gets $\mathbb{P}_\mu\left[\mathrm{e}^{-\lambda\partial W_\infty}\right] = \mathbb{P}_\mu\left[\mathrm{e}^{-\lambda\partial M_\infty}\right]$ for all $\lambda > 0$ and so $(\partial W_\infty, \mathbb{P}_\mu) \overset{\mathrm{d}}{=} (\partial M_\infty, \mathbb{P}_\mu)$. □

2.3 Statement of main results

In what follows and for the remainder of this paper we assume Assumptions 2.1, 2.4 and (2.2) hold. Additional conditions used are stated explicitly.

Define

$$\mathcal{H} := \left\{\phi \in \mathcal{B}_b^+(\mathbb{R}) : \int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y}\phi(-y)dy < +\infty\right\}. \tag{2.21}$$

Theorem 2.8. *Suppose $\phi \in \mathcal{H}$. The limit*

$$C(\phi) := \lim_{r\to+\infty} \int_0^{+\infty} y e^{\sqrt{2}y} u_\phi(r, -y-\sqrt{2}r)dy \tag{2.22}$$

exists and is finite. Moreover, as $t \to +\infty$,

$$u_\phi(t, x-m(t)) \to -\log \mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial M_\infty\}\right] \text{ locally uniformly in } x \in \mathbb{R}. \tag{2.23}$$

Remark 2.9. (i) We want to point out that the integral representation on the right hand side of (2.23) depends on ∂M_∞, the limit of the derivative martingale of the skeleton BBM. Assuming in addition that (2.19) holds, then by Proposition 2.7, ∂M_∞ is equal in distribution to ∂W_∞, and one has

$$u_\phi(t, x-m(t)) \to -\log \mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial W_\infty\}\right] \text{ locally uniformly in } x \in \mathbb{R}, \tag{2.24}$$

as $t \to +\infty$. This result is obtained independently in [23, Proposition 1.3(1)]. In fact, the constant $C(\phi)$ given in (2.22) is the same as the one given in [23, Proposition 1.3]. This is because our $u_\phi(t,x)$ is the solution to (1.4) with initial condition $u(0,x) = \phi(x)$, while $U_\phi(t,x)$ defined in [23] is the solution to (1.4) with initial condition $u(0,x) = \phi(-x)$. It holds that $U_\phi(t,x) = u_\phi(t,-x)$. Comparing the definitions of $C(\phi)$ in (2.22) and the one in [23, Proposition 1.3], we see that they are the same. We also note that the "locally uniform convergence" is slightly stronger than [23, Proposition 1.3(1)], where the convergence of (2.24) is established for each fixed $x \in \mathbb{R}$.

(ii) Let $\max X_t$ denote the supremum of the support of X_t, i.e., $\max X_t := \inf\{x \in \mathbb{R} : X_t(x, +\infty) = 0\}$. Here we take the convention that $\inf \emptyset = +\infty$. Unlike for the skeleton BBM, Theorem 2.8 does not imply the growth order of $\max X_t$ is $m(t)$. In fact, the asymptotic behavior of $\max X_t$ depends heavily on the branching mechanism $\psi(\lambda)$ and it may grow much faster than $m(t)$. We give such examples in Remark 3.12.

Theorem 2.8 yields the joint convergence of the extremal processes of super-Brownian motion and skeleton.

Theorem 2.10. *For $t \geq 0$, set*

$$\mathcal{E}_t := X_t - m(t) \quad \text{and} \quad \mathcal{E}_t^Z := Z_t - m(t).$$

Then for every $x \in \mathbb{R}$, the process $((\mathcal{E}_t, \mathcal{E}_t^Z)_{t\geq 0}, \mathbb{P}_{\delta_x})$ converges in distribution to a limit $(\mathcal{E}_\infty, \mathcal{E}_\infty^Z)$ as $t \to +\infty$, where $\mathcal{E}_\infty$ is a random Radon measure and $\mathcal{E}_\infty^Z$ is a random point measure satisfying that

$$\mathrm{E}\left[e^{-\langle f, \mathcal{E}_\infty\rangle - \langle g, \mathcal{E}_\infty^Z\rangle}\right] = \mathbb{P}_{\delta_x}\left[\exp\{-C\left(f + 1 - e^{-g}\right)\partial M_\infty\}\right] \tag{2.25}$$

for all $f \in \mathcal{H}$ and $g \in \mathcal{B}^+(\mathbb{R})$ with $1 - e^{-g} \in \mathcal{H}$. Here $C(\cdot)$ is a function of the initial condition as defined in (2.22). Moreover, given $\mathcal{E}_\infty$, $\mathcal{E}_\infty^Z$ is a Poisson random measure on $\mathbb{R}$ with intensity $\mathcal{E}_\infty(dx)$.

In the above statement and for the remainder of this paper, when we talk about the distributional limit, we do not specify the probability space where the limit is defined, just use P to denote the probability measure, and E to denote the corresponding expectation. We remark that the distribution of $(\mathcal{E}_\infty, \mathcal{E}_\infty^Z)$ depends on x since it is the distributional limit of $((\mathcal{E}_t, \mathcal{E}_t^Z)_{t\geq 0}, \mathbb{P}_{\delta_x})$. We will not remark this dependence in similar situation later in this paper.

In the following proposition, we establish a dichotomy on the finiteness of the supremum of the support for the limiting process.

Proposition 2.11. *Suppose $x \in \mathbb{R}$ and $\mathcal{E}_\infty$ is the limit of $((\mathcal{E}_t)_{t\ge 0}, \mathbb{P}_{\delta_x})$ in distribution. Let $\max \mathcal{E}_\infty$ be the supremum of the support of $\mathcal{E}_\infty$. Then $\max \mathcal{E}_\infty$ is a.s. finite if and only if*

$$\sup_\lambda C\left(\lambda 1_{(0,+\infty)}\right) < +\infty. \tag{2.26}$$

Otherwise if (2.26) *fails, then* $\mathrm{P}\left(\max \mathcal{E}_\infty < +\infty\right) = \mathbb{P}_{\delta_x}\left(\partial M_\infty = 0\right) = \mathrm{e}^{-1}$.

We obtain some new results on the probabilistic representations for the limiting process. Recall that $I = (I_t)_{t\ge 0}$ is the immigration process on the skeleton space. For $u \in Z_t$, denote by $I_s^{(u)}$ the immigration at time $t+s$ that occurred along the subtree of the skeleton rooted at u with location $z_u(t)$. Lemma 4.4 below shows that under $\mathbb{P}_{\delta_x}$, for every $s > 0$, conditioned on $\{\max Z_t - \sqrt{2}t > 0\}$, $\left(\sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}s - \max Z_t, Z_t - \max Z_t\right)$ converges in distribution to a limit $\left(\triangle^{I,s}, \triangle^Z\right)$ as $t \to +\infty$, and that the law of the limit $\left(\triangle^{I,s}, \triangle^Z\right)$ does not depend on x.

Theorem 2.12. *Suppose $x \in \mathbb{R}$ and $(\mathcal{E}_\infty, \mathcal{E}_\infty^Z)$ is the limit of $((\mathcal{E}_t, \mathcal{E}_t^Z)_{t\ge 0}, \mathbb{P}_{\delta_x})$ in distribution. Then $\mathcal{E}_\infty^Z$ is a DPPP($c_*\partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}y}dy$, $\triangle^Z$), here $c_* = C(1_{(0,+\infty)})$. Let $\{d_i : i \ge 1\}$ be the atoms of $\mathcal{E}_\infty^Z$, and for every $s > 0$, let $\{\triangle_i^s : i \ge 1\}$ be an independent sequence of i.i.d. random measures with the same law as $(I_s - \sqrt{2}s, \mathbb{P}_{\cdot,\delta_0})$, then*

$$\mathcal{E}_\infty \overset{d}{=} \lim_{s\to+\infty} \sum_{i\ge 1} \mathcal{T}_{d_i} \triangle_i^s. \tag{2.27}$$

The limit in (2.27) can not be put into the summation. In fact, for each $i \ge 1$, $\triangle_i^s$ converges in distribution to the null measure as $s \to +\infty$, see Remark 4.7 below. The following result gives an alternative description of $\mathcal{E}_\infty$.

Proposition 2.13. *Suppose the assumptions of Theorem 2.12 hold. Given $(\partial M_\infty, \mathbb{P}_{\delta_x})$, let $\{e_i : i \ge 1\}$ be the atoms of a Poisson point process with intensity $c_*\partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}x}dx$, and for every $s > 0$, let $\{\triangle_i^{I,s} : i \ge 1\}$ be an independent sequence of i.i.d random measures with the same law as $\triangle^{I,s}$, then*

$$\mathcal{E}_\infty \overset{d}{=} \lim_{s\to+\infty} \sum_{i\ge 1} \mathcal{T}_{e_i} \triangle_i^{I,s}.$$

For every $s > 0$, $\sum_{i\ge 1} \mathcal{T}_{e_i} \triangle_i^{I,s}$ is a Poisson random measure with exponential intensity, in which each atom is decorated by an independent copy of an auxiliary measure. However, their distributional limit $\mathcal{E}_\infty$ may not inherit such a structure. This is revealed by the following theorem.

Theorem 2.14. *Suppose $x \in \mathbb{R}$ and $\mathcal{E}_\infty$ is the limit of $((\mathcal{E}_t)_{t\ge 0}, \mathbb{P}_{\delta_x})$ in distribution. There exist a constant $\iota \ge 0$ and a measure Λ on $\mathcal{M}(\mathbb{R}) \setminus \{0\}$ satisfying that*

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x} dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \left(1 \wedge \mathcal{T}_x\mu(A)\right) \Lambda(d\mu) < +\infty, \quad \forall \text{ bounded Borel set } A \subset \mathbb{R},$$

such that

$$\mathcal{E}_\infty \overset{d}{=} \iota\, \partial M_\infty \vartheta + \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \mu \eta(d\mu),$$

where $\vartheta(dx) := \mathrm{e}^{-\sqrt{2}x}dx$ is the (non-random) measure on $\mathbb{R}$ and given $(\partial M_\infty, \mathbb{P}_{\delta_x})$, η is a Poisson random measure on $\mathcal{M}(\mathbb{R}) \setminus \{0\}$ with intensity $c_*\partial M_\infty \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}x} \mathcal{T}_x\Lambda(d\mu)dx$. Moreover, ι and $\Lambda(d\mu)$ satisfy (4.27) *and* (4.28) *of Lemma 4.10, respectively.*

The constant ι may not be 0 in general. The argument of Remark 4.11 shows that $\iota = 0$ if the following condition holds:

Assumption 2.15. *There are constants $a, b > 0$ and $0 < \gamma \le 1$ such that*

$$\psi(\lambda) \ge -a\lambda + b\lambda^{1+\gamma}, \quad \forall \lambda > 0.$$

When Assumption 2.15 is satisfied, $\mathcal{E}_\infty$ is equal in law to a Poisson random measure on $\mathcal{M}(\mathbb{R})$ with intensity $c_* \partial M_\infty \int_{-\infty}^{+\infty} \sqrt{2} \mathrm{e}^{-\sqrt{2}x} \mathcal{T}_x \Lambda(d\mu) dx$. Furthermore, Theorem 2.16 below shows that $\Lambda(d\mu) = \frac{\tilde{c}_0}{c_*} \mathrm{P}\left(\widetilde{\triangle}^X \in d\mu\right)$, where $\tilde{c}_0$ is a constant given by (3.38) with $\phi = 0$, and $\widetilde{\triangle}^X$ is the limit of $X_t - \max X_t$ conditioned on $\{\max X_t - \sqrt{2}t > 0\}$. In fact, it is proved in Lemma 4.13 that under Assumption 2.15, conditioned on $\{\max X_t - \sqrt{2}t > 0\}$, the random measures $(X_t - \max X_t, Z_t - \max X_t)$ converges, as $t \to +\infty$, in distribution to a limit $(\widetilde{\triangle}^X, \widetilde{\triangle}^Z)$, and that given $\widetilde{\triangle}^X$, $\widetilde{\triangle}^Z$ is a Poisson random measure with intensity $\widetilde{\triangle}^X$.

Theorem 2.16. *Assume in addition that Assumption 2.15 holds. Suppose $x \in \mathbb{R}$ and $(\mathcal{E}_\infty, \mathcal{E}_\infty^Z)$ is the limit of $((\mathcal{E}_t, \mathcal{E}_t^Z)_{t\ge 0}, \mathbb{P}_{\delta_x})$ in distribution. Let $\tilde{c}_0$ be given by* (3.38) *with $\phi = 0$. Given $(\partial M_\infty, \mathbb{P}_{\delta_x})$, let $\{\tilde{e}_i : i \ge 1\}$ be the atoms of a Poisson point process with intensity $\tilde{c}_0 \partial M_\infty \sqrt{2} \mathrm{e}^{-\sqrt{2}y} dy$ and $\{(\widetilde{\triangle}_i^X, \widetilde{\triangle}_i^Z) : i \ge 1\}$ be an independent sequence of i.i.d. random measures with the same law as $(\widetilde{\triangle}^X, \widetilde{\triangle}^Z)$. Then we have*

$$(\mathcal{E}_\infty, \mathcal{E}_\infty^Z) \overset{d}{=} \left(\sum_{i\ge 1} \mathcal{T}_{\tilde{e}_i} \widetilde{\triangle}_i^X, \sum_{i \ge 1} \mathcal{T}_{\tilde{e}_i} \widetilde{\triangle}_i^Z \right).$$

Remark 2.17. It is easy to see that Assumption 2.15 implies both Assumption 2.1 and (2.19). So by Proposition 2.7 one can replace ∂M_∞ by ∂W_∞ in the statement of Theorem 2.16. This type of representation for $\mathcal{E}_\infty$ is due to [23, Theorem 1.6].

Remark 2.18. Assume that Assumptions 2.4, 2.15 and (2.2) hold. Theorem 2.16 implies that $\mathcal{E}_\infty^Z$ is a DPPP($\tilde{c}_0 \partial M_\infty \sqrt{2} \mathrm{e}^{-\sqrt{2}y} dy$, $\widetilde{\triangle}^Z$), while Theorem 2.12 says that $\mathcal{E}_\infty^Z$ is a DPPP($c_* \partial M_\infty \sqrt{2} \mathrm{e}^{-\sqrt{2}y} dy$, $\triangle^Z$). These two theorems give two interpretations of $\mathcal{E}_\infty^Z$ as a decorated Poisson point process. Though the two interpretations are equal in law, they have different intensities and then different decoration laws. To see this, we only need to show that $c_* < \tilde{c}_0$. Using $\mathbb{P}_{\delta_x}(\partial M_\infty > 0) > 0$ and Theorem 2.12, one has

$$\mathbb{P}_{\delta_x}\left(\mathcal{E}_\infty(0, +\infty) = 0\right) = \mathbb{P}_{\delta_x}\left(\max \mathcal{E}_\infty \le 0\right) \le \mathbb{P}_{\delta_x}\left(\max \mathcal{E}_\infty^Z \le 0\right) = \mathbb{P}_{\delta_x}\left[\exp\{-c_* \partial M_\infty\}\right] < 1.$$

Then we have, for $\lambda > 1$,

$$\begin{aligned} \mathbb{P}_{\delta_x}\left[e^{-C(\lambda 1_{(0,+\infty)})\partial M_\infty}\right] &= \mathrm{E}\left[e^{-\lambda \mathcal{E}_\infty(0,\infty)}\right] \\ &< \mathrm{E}\left[e^{-\mathcal{E}_\infty(0,\infty)}\right] = \mathbb{P}_{\delta_x}\left[e^{-C(1_{(0,+\infty)})\partial M_\infty}\right]. \end{aligned}$$

Using $\mathbb{P}_{\delta_x}(\partial M_\infty > 0) > 0$ again, we get $c_* = C(1_{(0,+\infty)}) < C(\lambda 1_{(0,+\infty)}) \le \tilde{c}_0$.

3 Convergence of the extremal process of super-Brownian motion

3.1 Proof of Proposition 2.5

Recall that $((B_t)_{t\ge 0}, \Pi_x)$ is a standard Brownian motion starting at x, and that $u_f(t, x)$ is the unique nonnegative solution to (1.4) with initial condition f. We start this section with a comparison lemma.

Lemma 3.1. *Suppose $f, f_1, f_2 \in \mathcal{B}_b^+(\mathbb{R})$, $s, t \ge 0$ and $x, y \in \mathbb{R}$. Then*

(1) $u_f(t, x) \le \mathrm{e}^t P_t f(x) \le \mathrm{e}^t \|f\|_\infty$.

(2) For any $M \geq 1$, $u_{Mf}(t,x) \leq M u_f(t,x)$.

(3) $u_{f_1}(t,x) \vee u_{f_2}(t,x) \leq u_{f_1+f_2}(t,x) \leq u_{f_1}(t,x) + u_{f_2}(t,x)$.

(4) $u_f(t+s, x+y) = u_{\mathcal{T}_y u_f(s,\cdot)}(t,x)$. *In particular,* $u_f(t,x+y) = u_{\mathcal{T}_y f}(t,x)$.

Proof. (1) and (2) follow from Jensen's inequality. In fact, one has

$$u_f(t,x) = -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] \leq -\log \mathrm{e}^{-\mathbb{P}_{\delta_x}[\langle f, X_t\rangle]} = \mathbb{P}_{\delta_x}\left[\langle f, X_t\rangle\right] = \mathrm{e}^t P_t f(x),$$

and

$$u_{Mf}(t,x) = -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-M\langle f, X_t\rangle}\right] \leq -\log\left(\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right]\right)^M = M u_f(t,x).$$

(3) follows directly from [23, Lemma 2.3(2)].

(4) Fix $s \geq 0$ and $y \in \mathbb{R}$. Let $v(t,x) := u_f(t+s, x+y)$ for all $t \geq 0$ and $x \in \mathbb{R}$. It is easy to verify that v is the unique nonnegative solution to (1.4) with initial condition $v(0,x) = u_f(s, x+y) = \mathcal{T}_y u_f(s,\cdot)(x)$. Hence we get $v(t,x) = u_{\mathcal{T}_y u_f(s,\cdot)}(t,x)$. In particular by setting $s = 0$ we get that $u_f(t,x+y) = u_{\mathcal{T}_y u_f(0,\cdot)}(t,x) = u_{\mathcal{T}_y f}(t,x)$. $\square$

We need the following lemmas, which are refinements of [4, Proposition 4.4 and Lemma 4.9]. In fact, Arguin et al. [4] proved the same results for $[0,1]$-valued functions with support bounded on the left. We extend their results to all functions of $\mathcal{H}_1$. Though the idea of our proofs is similar to [4], we give the details here for the reader's convenience.

Lemma 3.2. *Suppose* $\phi \in \mathcal{H}_1$. *Then for all* $r > 0$,

$$C_r(\phi) := \sqrt{\frac{2}{\pi}} \int_0^{+\infty} y \mathrm{e}^{\sqrt{2}y} u_\phi(r, -y - \sqrt{2}r)\, dy$$

exists and is finite. Moreover, the limit

$$C(\phi) := \lim_{r\to+\infty} C_r(\phi)$$

exists and is finite, and for every $x \in \mathbb{R}$,

$$\lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_\phi(t, x - \sqrt{2}t) = C(\phi).$$

Proof. By Lemma 3.1(1),

$$u_\phi(t,-x) \leq \mathrm{e}^t P_t \phi(-x) \quad \forall t \geq 0,\ x \in \mathbb{R}.$$

Then

$$C_r(\phi) \leq \sqrt{\frac{2}{\pi}} e^r \int_{-\infty}^{+\infty} |y| \mathrm{e}^{\sqrt{2}y} P_r \phi(-y - \sqrt{2}r)\, dy. \tag{3.1}$$

By Fubini's theorem and change of variables we have

$$\begin{aligned}
& \int_{-\infty}^{+\infty} |y| \mathrm{e}^{\sqrt{2}y} P_r \phi(-y - \sqrt{2}r)\, dy \\
= \; & \frac{1}{\sqrt{2\pi r}} \int_{-\infty}^{+\infty} |y| \mathrm{e}^{\sqrt{2}y} dy \int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(z-y-\sqrt{2}r)^2}{2r}} \phi(-z)\, dz \\
= \; & e^{-r} \frac{1}{\sqrt{2\pi r}} \int_{-\infty}^{+\infty} \phi(-z) e^{\sqrt{2}z} dz \int_{-\infty}^{+\infty} |y| \mathrm{e}^{-\frac{(z-y)^2}{2r}} dy \\
= \; & e^{-r} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(-z) e^{\sqrt{2}z} dz \int_{-\infty}^{+\infty} |\sqrt{r}x + z| \mathrm{e}^{-\frac{x^2}{2}} dx \\
\leq \; & e^{-r} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(-z) e^{\sqrt{2}z} (\sqrt{r}\Pi_0(|B_1|) + |z|)\, dz.
\end{aligned} \tag{3.2}$$

We get by (3.1) and (3.2) that

$$C_r(\phi) \le c_1 \int_{-\infty}^{+\infty} \phi(-z) \mathrm{e}^{\sqrt{2}z} (|z|+1)\, dz$$

for some constant $c_1 = c_1(r) > 0$. The fact that $\phi \in \mathcal{H}_1$ implies that the integral on the right hand side is finite. Thus we get that $C_r(\phi) < +\infty$.

Let $u(t,x) := u_\phi(t,-x)$ for $t \ge 0$ and $x \in \mathbb{R}$. Then u is a solution to (1.4) with initial condition $u(0,x) = \phi(-x)$ satisfying that

$$\int_0^{+\infty} y \mathrm{e}^{\sqrt{2}y} u(0,y) dy < +\infty.$$

It then follows by [4, Proposition 4.3] that for r large enough, $t \ge 8r$ and $x \ge 8r - \frac{3}{2\sqrt{2}} \log t$,

$$\gamma^{-1}(r) \Psi(r,t,x+\sqrt{2}t) \le u(t,x+\sqrt{2}t) \le \gamma(r) \Psi(r,t,x+\sqrt{2}t), \tag{3.3}$$

where $\gamma(r) \downarrow 1$ as $r \to +\infty$ and

$$\Psi(r,t,x+\sqrt{2}t) = \frac{\mathrm{e}^{-\sqrt{2}x}}{\sqrt{2\pi(t-r)}} \int_0^{+\infty} u(r,y+\sqrt{2}r) \mathrm{e}^{\sqrt{2}y} \mathrm{e}^{-\frac{(y-x)^2}{2(t-r)}} \left(1 - \mathrm{e}^{-2y \frac{\left(x + \frac{3}{2\sqrt{2}} \log t\right)}{t-r}}\right) dy.$$

We may rewrite $\Psi(r,t,x+\sqrt{2}t)$ as follows:

$$\begin{aligned} &\Psi(r,t,x+\sqrt{2}t) \\ = \quad & \mathrm{e}^{-\sqrt{2}x} \frac{x + \frac{3}{2\sqrt{2}} \log t}{(t-r)^{3/2}} \sqrt{\frac{2}{\pi}} \int_0^{+\infty} y \mathrm{e}^{\sqrt{2}y} u(r,y+\sqrt{2}r) \mathrm{e}^{-\frac{(y-x)^2}{2(t-r)}} G\left(\frac{2y\left(x + \frac{3}{2\sqrt{2}} \log t\right)}{t-r}\right) dy, \end{aligned}$$

where $G(z) := (1 - \mathrm{e}^{-z})/z$. Using the fact that $G(z) \in [0,1]$ for all $z > 0$ and $G(z) \sim 1$ as $z \to 0$, we get by the bounded convergence theorem that

$$\lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} \Psi(r,t,x+\sqrt{2}t) = C_r(\phi), \quad \forall x \in \mathbb{R}.$$

Consequently by letting $t \to +\infty$ in (3.3), we have

$$\begin{aligned} 0 \le \gamma^{-1}(r) C_r(\phi) \quad &\le \quad \liminf_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} u(t,x+\sqrt{2}t) \\ &\le \quad \limsup_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} u(t,x+\sqrt{2}t) \le \gamma(r) C_r(\phi) < +\infty. \end{aligned}$$

Then by letting $r \to +\infty$, we have

$$\begin{aligned} 0 \le \limsup_{r \to +\infty} C_r(\phi) \quad &\le \quad \liminf_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} u(t,x+\sqrt{2}t) \\ &\le \quad \limsup_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} u(t,x+\sqrt{2}t) \le \liminf_{r \to +\infty} C_r(\phi) < +\infty. \end{aligned}$$

This implies that the limit $\lim_{r \to +\infty} C_r(\phi)$ exists and is finite, and is equal to

$$\lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathrm{e}^{\sqrt{2}x} u(t,x+\sqrt{2}t).$$

Therefore we complete the proof. □

Corollary 3.3. *For all $\phi \in \mathcal{H}_1$ and $x \in \mathbb{R}$,*

$$C(\mathcal{T}_x\phi) = \mathrm{e}^{\sqrt{2}x} C(\phi).$$

Moreover, $\lim_{\delta \to +\infty} C(1_{[\delta,+\infty)}) = 0$.

Proof. It follows by Lemma 3.2 and Lemma 3.1(4) that

$$\begin{aligned} C(\mathcal{T}_x\phi) &= \lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}y} u_{\mathcal{T}_x\phi}(t, y - \sqrt{2}t) \\ &= \lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}y} u_{\phi}(t, x + y - \sqrt{2}t) \\ &= \mathrm{e}^{\sqrt{2}x} \lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}(x+y)} u_{\phi}(t, x + y - \sqrt{2}t) = \mathrm{e}^{\sqrt{2}x} C(\phi). \end{aligned}$$

We observe that $1_{[\delta,+\infty)}(x) = \mathcal{T}_{-\delta} 1_{[0,+\infty)}(x)$ for all $\delta, x \in \mathbb{R}$. Thus

$$C(1_{[\delta,+\infty)}) = C(\mathcal{T}_{-\delta}1_{[0,+\infty)}) = \mathrm{e}^{-\sqrt{2}\delta} C(1_{[0,+\infty)}) \to 0 \text{ as } \delta \to +\infty.$$

□

For $\phi \in \mathcal{H}_1$ and $\delta \in \mathbb{R}$, put

$$\phi^\delta(x) := \phi(x) 1_{(-\infty,\delta)}(x) + 1_{[\delta,+\infty)}(x). \tag{3.4}$$

Note that $\phi^\delta \in \mathcal{H}_1$ and

$$u_{\phi^\delta}(t,x) = 1 - \mathbb{P}_{\cdot,\delta_x}\left[\prod_{u\in Z_t} (1 - \phi(z_u(t)))\, 1_{\{z_u(t)<\delta\}}\right], \quad \forall t \ge 0,\ x \in \mathbb{R}.$$

The following lemma establishes an integral representation for $u_{\phi^\delta}(t, x - m(t))$ in the limit of large times.

Lemma 3.4. *Suppose $\phi \in \mathcal{H}_1$ and $\delta \in \mathbb{R}$. Then*

$$u_{\phi^\delta}(t, x - m(t)) \to 1 - \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi^\delta)\partial M_\infty \mathrm{e}^{\sqrt{2}x}\}\right] \text{ uniformly in } x \in \mathbb{R}, \text{ as } t \to +\infty,$$

where $m(t)$ is defined by (2.12).

Proof. Let $v_\phi^\delta(t,x) := u_{\phi^\delta}(t,-x)$ for $t \ge 0$ and $x \in \mathbb{R}$. Then $v_\phi^\delta(t,x)$ is a solution to (1.4) with initial condition $v_\phi^\delta(0,x) = \phi(-x)1_{(-\delta,+\infty)}(x) + 1_{(-\infty,-\delta]}(x)$. Using the fact that $\phi \in \mathcal{H}_1$, one can easily verify that $v_\phi^\delta(0,x)$ satisfies conditions (8.1) and (1.17) of [10]. Hence by [10, Theorem 8.3], one has

$$v_\phi^\delta(t, x + m(t)) \to w(x) \text{ uniformly in } x \in \mathbb{R} \text{ as } t \to +\infty,$$

where w is the unique (up to translations) travelling wave solution with speed $\sqrt{2}$. It is established in [17] that

$$w(x) = 1 - \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C\partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right]$$

for some constant $C > 0$ which is determined by $C = \lim_{x\to+\infty} \frac{w(x)}{x\mathrm{e}^{-\sqrt{2}x}}$. Hence to prove this lemma, it suffices to show that $C = C(\phi^\delta)$, or equivalently,

$$\lim_{x\to+\infty} \lim_{t\to+\infty} \frac{v_\phi^\delta(t, x + m(t))}{x\mathrm{e}^{-\sqrt{2}x}} = C(\phi^\delta). \tag{3.5}$$

By (3.3), for r large enough and $t, x \geq 8r$, we have the bounds

$$\gamma(r)^{-1}\Psi(r,t,x+m(t)) \leq v_\phi^\delta(t,x+m(t)) \leq \gamma(r)\Psi(r,t,x+m(t)), \tag{3.6}$$

where $\gamma(r) \downarrow 1$ as $r \to +\infty$ and

$$\begin{aligned} &\Psi(r,t,x+m(t)) \\ = &\sqrt{\frac{2}{\pi}}\frac{t^{3/2}}{(t-r)^{3/2}}x\mathrm{e}^{-\sqrt{2}x}\int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y}v_\phi^\delta(r,y+\sqrt{2}r)\mathrm{e}^{-\frac{\left(y-x+\frac{3}{2\sqrt{2}}\log t\right)^2}{2(t-r)}}\frac{1-\mathrm{e}^{-\frac{2xy}{t-r}}}{2xy/(t-r)}dy. \end{aligned}$$

It follows by the bounded convergence theorem and Lemma 3.2 that

$$\lim_{t\to+\infty}\Psi(r,t,x+m(t)) = x\mathrm{e}^{-\sqrt{2}x}C_r(\phi^\delta).$$

This together with (3.6) yields that for r large enough and $x \geq 8r$,

$$\gamma(r)^{-1}C_r(\phi^\delta) \leq \lim_{t\to+\infty}\frac{v_\phi^\delta(t,x+m(t))}{x\mathrm{e}^{-\sqrt{2}x}} \leq \gamma(r)C_r(\phi^\delta). \tag{3.7}$$

Since $\gamma(r) \to 1$ and $C_r(\phi^\delta) \to C(\phi^\delta)$ as $r \to +\infty$, we get (3.5) by letting $r \to +\infty$ in (3.7). □

Corollary 3.5. *Suppose $\phi \in \mathcal{H}_1$. Then $C(\phi) = \lim_{\delta\to+\infty} C(\phi^\delta)$.*

Proof. We note that for every $\delta \in \mathbb{R}$,

$$\phi(x) \leq \phi^\delta(x) \leq \phi(x) + 1_{[\delta,+\infty)}(x), \quad \forall x \in \mathbb{R}.$$

Thus by Lemma 3.1(3), we have

$$u_\phi(t,x) \leq u_{\phi^\delta}(t,x) \leq u_\phi(t,x) + u_{1_{[\delta,+\infty)}}(t,x), \quad \forall t \geq 0,\ x \in \mathbb{R}, \tag{3.8}$$

which implies that $C(\phi) \leq C(\phi^\delta) \leq C(\phi) + C(1_{[\delta,+\infty)})$. Since $\lim_{\delta\to+\infty} C(1_{[\delta,+\infty)}) = 0$ by Corollary 3.3, it follows that $\lim_{\delta\to+\infty} C(\phi^\delta) = C(\phi)$. □

Proof of Proposition 2.5. The first part of this proposition follows from Lemma 3.2. We only need to show the second part. For $c > 0$ and $x \in \mathbb{R}$, let

$$w_c(x) := 1 - \mathbb{P}_{\cdot,\delta_0}\left[\exp\{-c\partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right].$$

By the uniqueness (up to translations) of the travelling wave solution, one has $w_c(x) = w_1(x - \ln c/\sqrt{2})$ for all $x \in \mathbb{R}$. We need to show that

$$u_\phi(t,x-m(t)) \to w_{C(\phi)}(-x) \quad \text{locally uniformly in } x \in \mathbb{R}, \text{ as } t \to +\infty. \tag{3.9}$$

For $\delta \geq 0$, by (3.8) we have that

$$\begin{aligned} &u_\phi(t,x-m(t)) - w_{C(\phi)}(-x) \\ \leq\ &\left(u_{\phi^\delta}(t,x-m(t)) - w_{C(\phi^\delta)}(-x)\right) + \left(w_{C(\phi^\delta)}(-x) - w_{C(\phi)}(-x)\right) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &u_\phi(t,x-m(t)) - w_{C(\phi)}(-x) \\ \geq\ &\left(u_{\phi^\delta}(t,x-m(t)) - w_{C(\phi^\delta)}(-x)\right) - \left(u_{1_{[\delta,+\infty)}}(t,x) - w_{C(1_{[\delta,+\infty)})}(-x)\right) \\ &+ \left(w_{C(\phi^\delta)}(-x) - w_{C(\phi)}(-x)\right) - w_{C(1_{[\delta,+\infty)})}(-x). \end{aligned} \tag{3.11}$$

We note that

$$w_{C(\phi^\delta)}(-x) - w_{C(\phi)}(-x) = w_1(-x - \frac{\ln C(\phi^\delta)}{\sqrt{2}}) - w_1(-x - \frac{\ln C(\phi)}{\sqrt{2}}), \tag{3.12}$$

and

$$w_{C(1_{[\delta,+\infty)})}(-x) = w_1(-x - \frac{\ln C(1_{[\delta,+\infty)})}{\sqrt{2}}). \tag{3.13}$$

By Corollary 3.5, $C(\phi^\delta) \to C(\phi)$ and $C(1_{[\delta,+\infty)}) \to 0$ as $\delta \to +\infty$. Then by the continuity of w_1 we get from (3.12) and (3.13) that

$$w_{C(\phi^\delta)}(-x) - w_{C(\phi)}(-x) \to 0, \text{ and } w_{C(1_{[\delta,+\infty)})}(-x) \to 0 \text{ locally uniformly in } x \in \mathbb{R},$$

as $\delta \to +\infty$. On the other hand, by Lemma 3.4 we have for $\delta \ge 0$,

$$u_{\phi^\delta}(t, x - m(t)) - w_{C(\phi^\delta)}(-x) \to 0, \text{ and } u_{1_{[\delta,+\infty)}}(t, x - m(t)) - w_{C(1_{[\delta,+\infty)})}(-x) \to 0$$

uniformly in $x \in \mathbb{R}$, as $t \to +\infty$. Hence we get (3.9) by letting first $t \to +\infty$ and then $\delta \to +\infty$ in both (3.10) and (3.11). □

3.2 Proof of Theorem 2.8

First of all, we introduce notation to refer to the different parts of the skeleton decomposition which will be used later in the computation. For $t \ge 0$, let $\mathcal{F}_t$ denote the σ-filed generated by Z, X^* and I up to time t. Denote by $I_s^{*,t}$ the immigration at time $t+s$ that occurred along the skeleton before time t. For $u \in Z_t$, denote by $I_s^{(u)}$ the immigration at time $t+s$ that occurred along the subtree of the skeleton rooted at u with location $z_u(t)$. We have

$$X_{s+t} = X^*_{s+t} + I_s^{*,t} + \sum_{u \in Z_t} I_s^{(u)} \quad \text{for all } s, t \ge 0. \tag{3.14}$$

It is known (see, e.g., [11]) that given $\mathcal{F}_t$, $(X^*_{s+t} + I_s^{*,t})_{s\ge 0}$ is equal in distribution to $((X_s^*)_{s\ge 0}; \mathbb{P}_{X_t})$ and $I^{(u)} := (I_s^{(u)})_{s \ge 0}$ is equal in distribution to $(I; \mathbb{P}_{\cdot,\delta_{z_u(t)}})$. Moreover, given $\mathcal{F}_t$, the processes $\{I^{(u)} : u \in Z_t\}$ are mutually independent and are independent of $(X^*_{s+t})_{s\ge 0}$. For $f \in \mathcal{B}_b^+(\mathbb{R})$, $t \ge 0$ and $x \in \mathbb{R}$, define

$$u_f^*(t,x) := -\log \mathbb{P}_{\delta_x,\cdot}\left[\mathrm{e}^{-\langle f, X_t^* \rangle}\right],$$

and

$$V_f(t,x) := \mathbb{P}_{\cdot,\delta_x}\left[\mathrm{e}^{-\langle f, I_t \rangle}\right]. \tag{3.15}$$

Since $X_t = X_t^* + \sum_{u \in Z_0} I_t^{(u)}$, we have

$$\begin{aligned} u_f(t,x) &= -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t\rangle}\right] = -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t^*\rangle}\right] - \log \mathbb{P}_{\delta_x}\left[\prod_{u\in Z_0} \mathrm{e}^{-\langle f, I_t^{(u)}\rangle}\right] \\ &= u_f^*(t,x) - \log \mathbb{P}_{\delta_x}\left[\prod_{u \in Z_0} V_f(t, z_u(0))\right] = u_f^*(t,x) - \log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{\langle \ln V_f(t,\cdot), Z_0\rangle}\right]. \end{aligned}$$

Using the fact that $(Z_0, \mathbb{P}_{\delta_x})$ is a Poisson random measure with intensity $\delta_x(dy)$, one has

$$u_f(t,x) = u_f^*(t,x) + 1 - V_f(t,x). \tag{3.16}$$

In this section we will make extensive use of (3.16), mostly when we deal with $u_f(t, x - \sqrt{2}t)$ for large t, in which case, $u_f^*(t, x - \sqrt{2}t)$ becomes relatively easy to handle.

Recall the definition of ϕ^δ given in (3.4). The following lemma gives an upperbound for the constant $C(\phi^0)$ which will be used later.

Lemma 3.6. *There exists a constant $c > 0$ such that for any $\phi \in \mathcal{H}_1$,*

$$C\left(\phi^0\right) \le c\left(\int_1^{+\infty} x\mathrm{e}^{\sqrt{2}x}\phi(-x)dx + 1\right). \tag{3.17}$$

Proof. Let $v(t,x) := u_{\phi^0}(t,-x)$ for $t \ge 0$ and $x \in \mathbb{R}$. It follows by (2.14), (2.15) and Proposition 2.5 that

$$C\left(\phi^0\right) = \lim_{x\to+\infty}\lim_{t\to+\infty}\frac{v(t,x+m(t))}{x\mathrm{e}^{-\sqrt{2}x}}. \tag{3.18}$$

Let $k(s,y) := -\psi(v(s,y))/v(s,y)$ for $s \ge 0$ and $y \in \mathbb{R}$. By the Feyman-Kac formula

$$\begin{aligned} v(t,x) &= \Pi_x\left[\mathrm{e}^{\int_0^t k(t-s,B_s)ds}v(0,B_t)\right] \\ &= \int_{-\infty}^{+\infty} v(0,y)\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(y-x)^2}{2t}}\,\mathrm{E}\left[\mathrm{e}^{\int_0^t k(t-s,\zeta^t_{x,y}(s))ds}\right]dy. \end{aligned} \tag{3.19}$$

Here $\{\zeta^{(t)}_{x,y}(s) : 0 \le s \le t\}$ denotes a Brownian bridge of length t starting at x and ending at y. Let $v^H(t,x)$ be the solution to (1.4) with heaviside initial condition $v^H(0,x) = 1_{\{x\le 0\}}$ and let $k^H(s,y) := -\psi(v^H(s,y))/v^H(s,y)$ for $s \ge 0$ and $y \in \mathbb{R}$. Define $m^H_{1/2}(t) := \sup\{x \in \mathbb{R} :\ v^H(t,x) \ge 1/2\}$ for $t \ge 0$. By [10, Proposition 8.1 and Proposition 8.2], there are constants C_1^H and C_2^H ($C_1^H < C_2^H$) such that for t large enough,

$$m(t) + C_1^H \le m^H_{1/2}(t) \le m(t) + C_2^H. \tag{3.20}$$

Moreover, it is established in [10, equation (8.21)] that there are constants $C_3^H > 0$ and $r >> 1$ such that for all $t \ge 3r$, $x \ge m^H_{1/2}(t) + 1$ and all $y \in \mathbb{R}$,

$$\mathrm{E}\left[\mathrm{e}^{\int_0^t k^H(t-s,\zeta^t_{x,y}(s))ds}\right] \le 2C_3^H r\mathrm{e}^t\left(1 - \mathrm{e}^{-\frac{2\hat{y}\bar{z}}{t}}\right)$$

where $\hat{y} := y \vee 1$ and $\bar{z} := x - (m(t) + C_1^H)$. Since $v(0,y) = 1$ for $y \in (-\infty, 0]$ and so $v(s,y) \ge v^H(s,y)$ for all $s \ge 0$ and $y \in \mathbb{R}$, it follows by the convexity of ψ that $k(s,y) \le k^H(s,y)$ for all $s \ge 0$ and $y \in \mathbb{R}$. Hence one has

$$\mathrm{E}\left[\mathrm{e}^{\int_0^t k(t-s,\zeta^t_{x,y}(s))ds}\right] \le C_4\mathrm{e}^t\left(1 - \mathrm{e}^{-\frac{2\hat{y}\bar{z}}{t}}\right)$$

for $C_4 = 2C_3^H r$. Putting this back in (3.19) one gets that for $t \ge 3r$ and $x \ge m^H_{1/2}(t) + 1$,

$$\begin{aligned} v(t,x) &\le C_4\mathrm{e}^t\int_{-\infty}^{+\infty} v(0,y)\frac{1}{\sqrt{2\pi t}}\mathrm{e}^{-\frac{(x-y)^2}{2t}}\left(1 - \mathrm{e}^{-\frac{2\hat{y}\bar{z}}{t}}\right)dy \\ &\le \frac{C_4\mathrm{e}^t}{\sqrt{2\pi t}}\left[\int_1^{+\infty}\phi(-y)\mathrm{e}^{-\frac{(x-y)^2}{2t}}\left(1 - \mathrm{e}^{-\frac{2y\bar{z}}{t}}\right)dy + \left(1 - \mathrm{e}^{-\frac{2\bar{z}}{t}}\right)\int_{-\infty}^1 \mathrm{e}^{-\frac{(x-y)^2}{2t}}dy\right] \\ &\le C_4\sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^t}{t^{3/2}}\bar{z}\left[\int_1^{+\infty}\phi(-y)y\mathrm{e}^{-\frac{(x-y)^2}{2t}}dy + \int_{-\infty}^1 \mathrm{e}^{-\frac{(x-y)^2}{2t}}dy\right]. \end{aligned}$$

The last inequality is from the fact that $1 - \mathrm{e}^{-x} \le x$ for all $x \ge 0$. This together with (3.20) yields that for $t \ge 3r$ and $x \ge C_2^H + 1$,

$$\begin{aligned} &v(t,x+m(t)) \\ \le\ & C_4\sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^t}{t^{3/2}}(x - C_1^H)\left[\int_1^{+\infty}\phi(-y)y\mathrm{e}^{-\frac{(x+m(t)-y)^2}{2t}}dy + \int_{-\infty}^1 \mathrm{e}^{-\frac{(x+m(t)-y)^2}{2t}}dy\right] \\ =\ & C_4\sqrt{\frac{2}{\pi}}(x - C_1^H)\mathrm{e}^{-\left(\sqrt{2}-\frac{3}{2\sqrt{2}}\frac{\log t}{t}\right)x - \frac{9}{16}\frac{\log^2 t}{t}}\Big[\int_1^{+\infty}\phi(-y)y\mathrm{e}^{-\frac{(x-y)^2}{2t}}\,\mathrm{e}^{\left(\sqrt{2}-\frac{3}{2\sqrt{2}}\frac{\log t}{t}\right)y}dy \\ &\quad + \int_{-\infty}^1 \mathrm{e}^{-\frac{(x-y)^2}{2t}}\,\mathrm{e}^{\left(\sqrt{2}-\frac{3}{2\sqrt{2}}\frac{\log t}{t}\right)y}dy\Big]. \end{aligned}$$

By letting $t \to +\infty$, we have

$$\lim_{t\to+\infty} v(t, x + m(t)) \le C_4\sqrt{\frac{2}{\pi}}(x - C_1^H)\mathrm{e}^{-\sqrt{2}x}\left[\int_1^{+\infty} y\mathrm{e}^{\sqrt{2}y}\phi(-y)dy + \frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{2}}\right]$$

for $x \ge C_2^H + 1$. Putting this back in (3.23), one gets that

$$C\left(\phi^0\right) \le C_4\sqrt{\frac{2}{\pi}}\left[\int_1^{+\infty} y\mathrm{e}^{\sqrt{2}y}\phi(-y)dy + \frac{1}{\sqrt{2}}\mathrm{e}^{\sqrt{2}}\right].$$

Hence we complete the proof. □

Lemma 3.7. *Suppose $\{\phi_s(x) : s \ge 0\}$ is a sequence of functions in $\mathcal{H}_1$. If $\phi_s(x) \to 0$ as $s \to +\infty$ for all $x \in \mathbb{R}$, and $\int_{-\infty}^{+\infty} |x|\mathrm{e}^{\sqrt{2}x}\phi_s(-x)dx \to 0$ as $s \to +\infty$, then $\lim_{s\to+\infty} C(\phi_s) = 0$.*

Proof. Suppose $\alpha(s) \ge 0$ for all $s > 0$. (The explicit value of $\alpha(s)$ will be given later.) By Lemma 3.1(3), one has

$$C(\phi_s) \le C\left(\phi_s 1_{(-\infty,\alpha(s))} + 1_{[\alpha(s),+\infty)}\right),\ \forall s \ge 0.$$

So it suffices to prove that

$$\lim_{s\to+\infty} C\left(\phi_s 1_{(-\infty,\alpha(s))} + 1_{[\alpha(s),+\infty)}\right) = 0. \tag{3.21}$$

We note that $\phi_s(x)1_{(-\infty,\alpha(s))}(x) + 1_{[\alpha(s),+\infty)}(x) = \mathcal{T}_{-\alpha(s)}\left(\mathcal{T}_{\alpha(s)}\phi_s \cdot 1_{(-\infty,0)} + 1_{[0,+\infty)}\right)(x)$ for all $x \in \mathbb{R}$. By Corollary 3.3 and Lemma 3.6, we have

$$\begin{aligned} C\left(\phi_s 1_{(-\infty,\alpha(s))} + 1_{[\alpha(s),+\infty)}\right) &= \mathrm{e}^{-\sqrt{2}\alpha(s)}C\left(\mathcal{T}_{\alpha_s}\phi_s \cdot 1_{(-\infty,0)} + 1_{[0,+\infty)}\right) \\ &\le \mathrm{e}^{-\sqrt{2}\alpha(s)}c\left[\int_1^{+\infty} x\mathrm{e}^{\sqrt{2}x}\mathcal{T}_{\alpha(s)}\phi_s(-x)dx + 1\right] \\ &= \mathrm{e}^{-\sqrt{2}\alpha(s)}c\left[\int_{1-\alpha(s)}^{+\infty}(y + \alpha(s))\mathrm{e}^{\sqrt{2}(y+\alpha(s))}\phi_s(-y)dy + 1\right] \\ &\le c\int_{-\infty}^{+\infty}(|y| + \alpha(s))\mathrm{e}^{\sqrt{2}y}\phi_s(-y)dy + c\mathrm{e}^{-\sqrt{2}\alpha(s)}. \end{aligned} \tag{3.22}$$

Let $\alpha(s) := s \wedge \left(\int_{-\infty}^{+\infty}\phi_s(-y)\mathrm{e}^{\sqrt{2}y}dy\right)^{-1/2}$. Then the right hand side of (3.22) is no larger than

$$c\left[\int_{-\infty}^{+\infty}|y|\mathrm{e}^{\sqrt{2}y}\phi_s(-y)dy + \alpha(s)^{-1} + \mathrm{e}^{-\sqrt{2}\alpha(s)}\right]. \tag{3.23}$$

Note that

$$\int_{-\infty}^{+\infty}\phi_s(-y)\mathrm{e}^{\sqrt{2}y}dy \le \int_{|y|\le 1}\phi_s(-y)\mathrm{e}^{\sqrt{2}y}dy + \int_{|y|>1}\phi_s(-y)|y|\mathrm{e}^{\sqrt{2}y}dy < +\infty.$$

Both integrals on the right hand side converge to 0 as $s \to +\infty$ given that $\phi_s(x) \to 0$ as $s \to \infty$ for all $x \in \mathbb{R}$ and $\int_{-\infty}^{+\infty}\phi_s(-y)|y|\mathrm{e}^{\sqrt{2}y}dy \to 0$ as $s \to \infty$. This implies $\alpha(s) \to +\infty$ as $s \to +\infty$. Thus (3.23) converges to 0 as $s \to +\infty$ and so we prove (3.21). □

We recall the definition of $\mathcal{H}$ given in (2.21).

Lemma 3.8. *For all $\phi \in \mathcal{H}$ and $s \ge 0$, the functions $u_\phi(s, \cdot - \sqrt{2}s)$, $u_\phi^*(s, \cdot - \sqrt{2}s)$ and $1 - V_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}$.*

Proof. Fix an arbitrary $\phi \in \mathcal{H}$ and $s \geq 0$. Since $0 \leq u^*_\phi(s, x-\sqrt{2}s),\ 1 - V_\phi(s, x-\sqrt{2}s) \leq u_\phi(s, x-\sqrt{2}s)$ for all $x \in \mathbb{R}$, it suffices to prove that

$$u_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}. \tag{3.24}$$

Let $M := \|\phi\|_\infty$. If $M \leq 1$, then $\phi \in \mathcal{H}_1$ and (3.24) follows directly from the first conclusion of Lemma 3.2. Now suppose $M > 1$. Let $\phi_1 = \phi/M$. Then by Lemma 3.1(2), one has $u_\phi(s, x-\sqrt{2}s) \leq M u_{\phi_1}(s, x-\sqrt{2}s)$ for all $x \in \mathbb{R}$, where $u_{\phi_1}(s, \cdot - \sqrt{2}s) \in \mathcal{H}$ by Lemma 3.2. This implies that $u_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}$. □

Lemma 3.9. *Suppose $\phi \in \mathcal{H}$. Then $u^*_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}_1$ for s large enough. Moreover,*

$$\lim_{s \to +\infty} C(u^*_\phi(s, \cdot - \sqrt{2}s)) = 0.$$

Proof. It follows by Jensen's inequality that

$$u^*_\phi(s,x) = -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle \phi, X^*_s\rangle}\right] \leq \mathbb{P}_{\delta_x}\left[\langle \phi, X^*_s\rangle\right] = \mathrm{e}^{\alpha^* s} P_s\phi(x) \leq \mathrm{e}^{\alpha^* s}\|\phi\|_\infty.$$

Since $\alpha^* = -\psi'(1) < 0$, we have $u^*_\phi(s,x) \to 0$ as $s \to +\infty$ for all $x \in \mathbb{R}$ and $\|u^*_\phi(s, \cdot - \sqrt{2}s)\|_\infty \leq 1$ for s large enough. Hence $u^*_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}_1$ by Lemma 3.8. We note that

$$u^*_\phi(s, -x-\sqrt{2}s) \leq \mathrm{e}^{\alpha^* s} P_s\phi(-x-\sqrt{2}s).$$

Thus by (3.2) we have

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{e}^{\sqrt{2}x}|x| u^*_\phi(s, -x-\sqrt{2}s)dx &\leq \mathrm{e}^{\alpha^* s}\int_{-\infty}^{+\infty} \mathrm{e}^{\sqrt{2}x}|x| P_s\phi(-x-\sqrt{2}s)dx \\ &\leq \mathrm{e}^{(\alpha^*-1)s}\int_{-\infty}^{+\infty} \phi(-y)\mathrm{e}^{\sqrt{2}y}\left(|y| + \sqrt{s}\Pi_0\left[|B_1|\right]\right)dy.\end{aligned}$$

The assumption that $\phi \in \mathcal{H}$ implies that $\int_{-\infty}^{+\infty}\phi(-y)\mathrm{e}^{\sqrt{2}y}|y|dy < \infty$. Since $\alpha^* < 0$, we get by the above inequality that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\sqrt{2}x}|x| u^*_\phi(s, -x-\sqrt{2}s)dx \to 0 \text{ as } s \to +\infty,$$

and thus by Lemma 3.7 $C(u^*_\phi(s, \cdot - \sqrt{2}s)) \to 0$ as $s \to +\infty$. □

The following lemma extends the result of Lemma 3.2 to all functions of $\mathcal{H}$.

Lemma 3.10. *Suppose $\phi \in \mathcal{H}$. Then for any $r > 0$,*

$$C_r(\phi) := \sqrt{\frac{2}{\pi}}\int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y} u_\phi(r, -y-\sqrt{2}r)dy$$

exists and is finite. The limit

$$C(\phi) := \lim_{r \to +\infty} C_r(\phi)$$

exists and is finite. Moreover, for every $x \in \mathbb{R}$,

$$C(\phi) = \lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t}\mathrm{e}^{-\sqrt{2}x} u_\phi(t, x-\sqrt{2}t) = \lim_{s \to +\infty} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right). \tag{3.25}$$

Proof. Without loss of generality we assume $\phi \in \mathcal{H} \setminus \mathcal{H}_1$. Let $M := \|\phi\|_\infty$ and $\phi_1 = \phi/M$. Then $\phi_1 \in \mathcal{H}_1$. The finiteness of $C_r(\phi)$ is immediate since by Lemma 3.1(2) $u_\phi(r, -y - \sqrt{2}r) \leq M u_{\phi_1}(r, -y - \sqrt{2}r)$ for all $r \geq 0$ and $y \in \mathbb{R}$ and so $C_r(\phi) \leq M C_r(\phi_1) < +\infty$. Since

$$u_\phi(t,x) = u_\phi^*(t,x) + 1 - V_\phi(t,x), \quad \forall t \geq 0,\ x \in \mathbb{R},$$

we get by Lemma 3.1(3)(4) that

$$\begin{aligned} & u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(r, x - \sqrt{2}r) \\ \leq\ & u_{u_\phi(s,\cdot-\sqrt{2}s)}(r, x-\sqrt{2}r) \\ =\ & u_\phi(s+r, x - \sqrt{2}(s+r)) \\ \leq\ & u_{u_\phi^*(s,\cdot-\sqrt{2}s)}(r, x-\sqrt{2}r) + u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(r, x-\sqrt{2}r). \end{aligned} \tag{3.26}$$

This implies that

$$\begin{aligned} C_r\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) &\leq C_{r+s}(\phi) \\ &\leq C_r\left(u_\phi^*(s, \cdot - \sqrt{2}s)\right) + C_r\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) \end{aligned} \tag{3.27}$$

for all $r > 0$. Since $1 - V_\phi(s, \cdot - \sqrt{2}s) \in \mathcal{H}_1$ and $u_\phi^*(s, \cdot - \sqrt{2}s) \in \mathcal{H}_1$ for s large enough, we get by (3.27) and Proposition 2.5 that

$$\begin{aligned} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) &\leq \liminf_{r \to +\infty} C_r(\phi) \\ &\leq \limsup_{r \to +\infty} C_r(\phi) \leq C\left(u_\phi^*(s, \cdot - \sqrt{2}s)\right) + C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right). \end{aligned}$$

Since $\lim_{s\to+\infty} C(u_\phi^*(s, \cdot - \sqrt{2}s)) = 0$, we have

$$\begin{aligned} \limsup_{s \to +\infty} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) &\leq \liminf_{r \to +\infty} C_r(\phi) \\ &\leq \limsup_{r \to +\infty} C_r(\phi) \leq \liminf_{s \to +\infty} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right). \end{aligned}$$

Since the $C_r(\phi) \leq M C_r(\phi_1)$ for all $r > 0$ and the latter is bounded in r, the above inequalities imply that the limit $C(\phi) = \lim_{r \to +\infty} C_r(\phi)$ exists and is finite, and satisfies that

$$C(\phi) = \lim_{s \to +\infty} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right). \tag{3.28}$$

On the other hand, it follows from Lemma 3.2 and (3.26) that

$$\begin{aligned} C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) &= \lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(t, x - \sqrt{2}t) \\ &\leq \liminf_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_\phi(t, x - \sqrt{2}t) \\ &\leq \limsup_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_\phi(t, x - \sqrt{2}t) \\ &\leq \lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_{u_\phi^*(s,\cdot-\sqrt{2}s)}(t, x - \sqrt{2}t) \\ &\qquad + \lim_{t \to +\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x} u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(t, x - \sqrt{2}t) \\ &= C\left(u_\phi^*(s, \cdot - \sqrt{2}s)\right) + C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right). \end{aligned}$$

Letting $s\to+\infty$, we get by (3.28) and Lemma 3.9 that

$$C(\phi)=\lim_{t\to+\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t}\mathrm{e}^{-\sqrt{2}x}u_\phi(t,x-\sqrt{2}t).$$

□

Corollary 3.11. *For $f,f_1,f_2\in\mathcal{H}$ and $M\geq 1$,*

$$\begin{aligned}&C(f)=C(u_f(s,\cdot-\sqrt{2}s)),\ \forall s\geq 0,\\&C(Mf)\leq MC(f),\\&C(f_1)\vee C(f_2)\leq C(f_1+f_2)\leq C(f_1)+C(f_2).\end{aligned}$$

Proof. This result follows directly from Lemma 3.10 and Lemma 3.1. □

Proof of Theorem 2.8. We first suppose that $\phi\in\mathcal{H}_1$. Then by Proposition 2.5 we have

$$u_\phi(t,x-m(t))\to 1-\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi)\mathrm{e}^{\sqrt{2}x}\partial M_\infty\}\right]\text{ locally uniformly in }x\in\mathbb{R},\tag{3.29}$$

as $t\to+\infty$. By (2.7) we have

$$\mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial M_\infty\}\right]=\exp\left\{-\left(1-\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi)\mathrm{e}^{\sqrt{2}x}\partial M_\infty\}\right]\right)\right\}.\tag{3.30}$$

Putting this back in (3.29), we get that

$$u_\phi(t,x-m(t))\to-\log\mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial M_\infty\}\right]\text{ locally uniformly in }x\in\mathbb{R},\text{ as }t\to+\infty.$$

Now we suppose $\phi\in\mathcal{H}\setminus\mathcal{H}_1$. For any $r,s>0$ and $x\in\mathbb{R}$, one can rewrite $u_\phi(r+s,x-m(r+s))$ as

$$u_{u_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r)),$$

where $O_s(r):=\frac{3}{2\sqrt{2}}\log\frac{r+s}{r}$. Note that for $s>0$, $O_s(r)\to 0$ as $r\to+\infty$. In view of this and (3.16) we get by Lemma 3.1(4) that

$$\begin{aligned}&u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r))\\\leq\ &u_\phi(r+s,x-m(r+s))\\\leq\ &u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r))+u_{u^*_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r)).\end{aligned}\tag{3.31}$$

It follows that

$$\begin{aligned}&\left|u_\phi(r+s,x-m(r+s))-\big(-\log\mathbb{P}_{\delta_x}\big[\exp\{-C(\phi)\partial M_\infty\}\big]\big)\right|\\\leq\ &\left|u_{1-V_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r))\right.\\&\quad\left.-\big(-\log\mathbb{P}_{\delta_x}\big[\exp\{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\big]\big)\right|\\&+\left|u_{u^*_\phi(s,\cdot-\sqrt{2}s)}(r,x-m(r)+O_s(r))-\big(-\log\mathbb{P}_{\delta_x}\big[\exp\{-C(u^*_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\big]\big)\right|\\&+\left|\log\mathbb{P}_{\delta_x}\big[\exp\{-C(\phi)\partial M_\infty\}\big]-\log\mathbb{P}_{\delta_x}\big[\exp\{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\big]\right|\\&+\left|\log\mathbb{P}_{\delta_x}\big[\exp\{-C(u^*_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\big]\right|.\end{aligned}\tag{3.32}$$

We have proved in the first part that the first two terms of (3.32) converge to 0 locally uniformly in $x\in\mathbb{R}$ as $r\to+\infty$. On the other hand we have by (3.30)

$$\begin{aligned}&\left|\log\mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial M_\infty\}\right]-\log\mathbb{P}_{\delta_x}\left[\exp\{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\right]\right|\\=\ &\left|\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(\phi)\mathrm{e}^{\sqrt{2}x}\partial M_\infty\}\right]-\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\mathrm{e}^{\sqrt{2}x}\partial M_\infty\}\right]\right|\\\leq\ &\left|C(\phi)-C(1-V_\phi(s,\cdot-\sqrt{2}s))\right|\mathrm{e}^{\sqrt{2}x}\mathbb{P}_{\cdot,\delta_0}\left[\partial M_\infty\right],\end{aligned}\tag{3.33}$$

where in the lase inequality we use the fact that $|e^{-x_1}-e^{-x_2}|\le|x_1-x_2|$, for all $x_1,x_2\ge 0$. Since $C(\phi)-C(1-V_\phi(s,\cdot-\sqrt{2}s))\to 0$ as $s\to+\infty$, we get by (3.33) that

$$\log\mathbb{P}_{\delta_x}\left[\exp\{-C(\phi)\partial M_\infty\}\right]-\log\mathbb{P}_{\delta_x}\left[\exp\{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\right]\to 0$$

locally uniformly in $x\in\mathbb{R}$, as $s\to+\infty$. Similarly one can prove that

$$\log\mathbb{P}_{\delta_x}\left[\exp\{-C(u^*_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty\}\right]\to 0 \text{ locally uniformly in } x\in\mathbb{R}, \text{ as } s\to+\infty.$$

Therefore we complete the proof. □

3.3 Proofs of Theorem 2.10 and Proposition 2.11

Proof of Theorem 2.10. Fix an arbitrary $x\in\mathbb{R}$ and functions f,g satisfying our assumptions. We have

$$\begin{aligned}\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f,\mathcal{E}_t\rangle-\langle g,\mathcal{E}_t^Z\rangle}\right] &= \mathbb{P}_{\delta_x}\left[\mathrm{e}^{\langle\mathcal{T}_{-m(t)}f,X_t\rangle}\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle\mathcal{T}_{-m(t)}g,Z_t\rangle}|X_t\right]\right]\\ &= \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle\mathcal{T}_{-m(t)}f+1-\mathrm{e}^{-\mathcal{T}_{-m(t)}g},X_t\rangle}\right]\\ &= \mathrm{e}^{-u_{\mathcal{T}_{-m(t)}(f+1-\mathrm{e}^{-g})}(t,x)}=\mathrm{e}^{-u_{f+1-\mathrm{e}^{-g}}(t,x-m(t))}.\end{aligned}$$

The second equality follows from the fact that given X_t, Z_t is a Poisson random measure with intensity $X_t(dx)$. Note that $f+1-\mathrm{e}^{-g}\in\mathcal{H}$ by our assumptions. We get by Theorem 2.8 that

$$\lim_{t\to+\infty}\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f,\mathcal{E}_t\rangle-\langle g,\mathcal{E}_t^Z\rangle}\right]=\mathbb{P}_{\delta_x}\left[\exp\{-C(f+1-\mathrm{e}^{-g})\partial M_\infty\}\right].$$

The above identity holds in particular for all $f,g\in C_c^+(\mathbb{R})$. On the other hand, it is easy to verify that the assumptions of Lemma 3.7 are satisfied by $\phi_s(x):=\phi(x)/s$ for every $\phi\in C_c^+(\mathbb{R})$. This implies that $\lim_{\lambda\to 0}C(\lambda\phi)=0$ for all $\phi\in C_c^+(\mathbb{R})$. Since $C(\lambda_1 f+1-\mathrm{e}^{-\lambda_2 g})\le C(\lambda_1 f)+C(\lambda_2 g)$ for $\lambda_1,\lambda_2>0$, we have $\lim_{\lambda_1,\lambda_2\to 0+}C(\lambda_1 f+1-\mathrm{e}^{-\lambda_2 g})=0$. Hence by [16, Chapter 4] we get the weak convergence of $(\mathcal{E}_t,\mathcal{E}_t^Z)$, and consequently (2.25) holds for all $f,g\in C_c^+(\mathbb{R})$. Using the monotone convergence theorem and Lemma 3.7, one can show by standard approximation that (2.25) holds for all $f\in\mathcal{H}$ and $g\in\mathcal{B}^+(\mathbb{R})$ with $1-\mathrm{e}^{-g}\in\mathcal{H}$. Also, (2.25) yields that

$$\begin{aligned}\mathrm{E}\left[\mathrm{e}^{-\langle f,\mathcal{E}_\infty\rangle}\mathrm{E}\left[\mathrm{e}^{-\langle g,\mathcal{E}_\infty^Z\rangle}\,|\,\mathcal{E}_\infty\right]\right] &= \mathbb{P}_{\delta_x}\left[\exp\{-C\left(f+1-\mathrm{e}^{-g}\right)\partial M_\infty\}\right]\\ &= \mathrm{E}\left[\mathrm{e}^{-\langle f,\mathcal{E}_\infty\rangle}\mathrm{e}^{-\langle 1-\mathrm{e}^{-g},\mathcal{E}_\infty\rangle}\right].\end{aligned}$$

Thus we get $\mathrm{E}\left[\mathrm{e}^{-\langle g,\mathcal{E}_\infty^Z\rangle}\,|\,\mathcal{E}_\infty\right]=\mathrm{e}^{-\langle 1-\mathrm{e}^{-g},\mathcal{E}_\infty\rangle}$ a.s., and the second conclusion follows immediately. □

Proof of Proposition 2.11. Suppose $M\in\mathbb{R}$. We have

$$\begin{aligned}\mathrm{P}\left(\max\mathcal{E}_\infty\le M\right) &= \mathrm{P}\left(\langle 1_{(M,+\infty)},\mathcal{E}_\infty\rangle=0\right)=\lim_{\lambda\to+\infty}\mathrm{E}\left[\mathrm{e}^{-\lambda\langle 1_{(M,+\infty)},\mathcal{E}_\infty\rangle}\right]\\ &= \lim_{\lambda\to+\infty}\mathbb{P}_{\delta_x}\left[\exp\{-\partial M_\infty C\left(\lambda 1_{(M,+\infty)}\right)\}\right]\\ &= \mathbb{P}_{\delta_x}\left[\exp\{-\partial M_\infty\lim_{\lambda\to+\infty}C\left(\lambda 1_{(M,+\infty)}\right)\}\right].\end{aligned}$$

The third equality follows from Theorem 2.10. Thus we have

$$\begin{aligned}\mathrm{P}\left(\max\mathcal{E}_\infty<+\infty\right) &= \lim_{M\to+\infty}\mathrm{P}\left(\max\mathcal{E}_\infty\le M\right)\\ &= \mathbb{P}_{\delta_x}\left[\exp\{-\partial M_\infty\lim_{M\to+\infty}\lim_{\lambda\to+\infty}C\left(\lambda 1_{(M,+\infty)}\right)\}\right].\end{aligned}\quad(3.34)$$

Note that

$$C\left(\lambda 1_{(M,+\infty)}\right) = C\left(\mathcal{T}_{-M}\lambda 1_{(0,+\infty)}\right) = \mathrm{e}^{-\sqrt{2}M} C\left(\lambda 1_{(0,+\infty)}\right).$$

It follows that

$$\lim_{M\to+\infty}\lim_{\lambda\to+\infty} C\left(\lambda 1_{(M,+\infty)}\right) = \lim_{M\to+\infty} \mathrm{e}^{-\sqrt{2}M}\left(\lim_{\lambda\to+\infty} C\left(\lambda 1_{(0,+\infty)}\right)\right) = 0 \text{ or } +\infty$$

corresponding to $\sup_\lambda C\left(\lambda 1_{(0,+\infty)}\right) < +\infty$ or $+\infty$. Hence by (3.34), $\mathrm{P}\left(\max \mathcal{E}_\infty < +\infty\right) = 1$ if and only if $\sup_\lambda C\left(\lambda 1_{(0,+\infty)}\right) < +\infty$. $\square$

Remark 3.12. Recall that $\max X_t$ denotes the supremum of the support of X_t. Unlike for the skeleton BBM, Theorem 2.8 does not imply the growth order of $\max X_t$ is $m(t)$. We make a short discussion here.

For $\phi \in \mathcal{B}_b^+(\mathbb{R})$, define

$$\tilde{u}_\phi(t,x) := -\log \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle \phi, X_t\rangle}; \max X_t \le 0\right], \quad \forall t\ge 0,\ x\in\mathbb{R}. \tag{3.35}$$

Then one has

$$\tilde{u}_\phi(t,x) = \lim_{\lambda\to+\infty} u_{\phi 1_{(-\infty,0]}+\lambda 1_{(0,+\infty)}}(t,x), \quad \forall t\ge 0,\ x\in\mathbb{R}. \tag{3.36}$$

In fact,

$$\begin{aligned}
\mathrm{e}^{-\tilde{u}_\phi(t,x)} &= \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle \phi 1_{(-\infty,0]}, X_t\rangle}; \langle 1_{(0,+\infty)}, X_t\rangle = 0\right] \\
&= \lim_{\lambda\to+\infty} \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle \phi 1_{(-\infty,0]}+\lambda 1_{(0,+\infty)}, X_t\rangle}\right] = \mathrm{e}^{-\lim_{\lambda\to+\infty} u_{\phi 1_{(-\infty,0]}+\lambda 1_{(0,+\infty)}}(t,x)}.
\end{aligned}$$

By (3.36) and the monotone convergence theorem, one has for $t, r\ge 0$ and $x\in\mathbb{R}$,

$$\begin{aligned}
\tilde{u}_\phi(t+r,x) &= \lim_{\lambda\to+\infty} u_{\phi 1_{(-\infty,0]}+\lambda 1_{(0,+\infty)}}(t+r,x) \\
&= \lim_{\lambda\to+\infty} u_{u_{\phi 1_{(-\infty,0]}+\lambda 1_{(0,+\infty)}}(r,\cdot)}(t,x) = u_{\tilde{u}_\phi(r,\cdot)}(t,x).
\end{aligned} \tag{3.37}$$

If we assume in addition that Assumption 2.15 holds, then by [23, Corollary 3.2] for all $\phi\in\mathcal{H}$ and $r>0$, $\tilde{u}_\phi(r,\cdot)\in\mathcal{B}_b^+(\mathbb{R})$. So by (3.37), for every $\phi\in\mathcal{H}$, $\tilde{u}_\phi(t,x)$ can be viewed as a solution to (1.4) with initial condition $u(0,x) = \phi(x)1_{(-\infty,0]}(x) + \infty 1_{(0,+\infty)}(x)$. Moreover, by [23, Lemma 2.1 and Corollary 3.2], for all $\phi\in\mathcal{H}$ and $r>0$, $\tilde{u}_\phi(r,\cdot-\sqrt{2}r)\in\mathcal{H}$. Hence applying Theorem 2.8 to the function $\tilde{u}_\phi(t+r, x-\sqrt{2}r) = u_{\tilde{u}_\phi(r,\cdot-\sqrt{2}r)}(t,x)$, one gets that for all $\phi\in\mathcal{H}$, the limit

$$\widetilde{C}(\phi) := \lim_{t\to+\infty}\sqrt{\frac{2}{\pi}}\int_0^{+\infty} y\mathrm{e}^{\sqrt{2}y}\tilde{u}_\phi(t,-y-\sqrt{2}t)dy \tag{3.38}$$

exists and is finite, and for all $x\in\mathbb{R}$,

$$\mathbb{P}_{\delta_0}\left[\mathrm{e}^{-\langle \phi, X_t - m(t) - x\rangle}; \max X_t - m(t) \le x\right] \to \mathbb{P}_{\delta_0}\left[\mathrm{e}^{-\widetilde{C}(\phi)\partial M_\infty \mathrm{e}^{-\sqrt{2}x}}\right] \text{ as } t\to+\infty.$$

Taking $\phi = 0$, one gets that

$$\mathbb{P}_{\delta_0}\left(\max X_t - m(t) \le x\right) \to \mathbb{P}_{\delta_0}\left[\exp\{-\tilde{c}_0 \partial M_\infty \mathrm{e}^{-\sqrt{2}x}\}\right] \text{ as } t\to+\infty, \tag{3.39}$$

where $\tilde{c}_0$ is the constant $\widetilde{C}(\phi)$ with $\phi = 0$.

On the other hand, if Assumption 2.15 fails, (3.39) may not be true as it is. For instance, we take the triplet $(\alpha,\beta,\pi(dy))$ of $\psi(\lambda)$ to be $(1,0,\delta_{y_0}(dy))$ where $y_0>1$ satisfies that $\mathrm{e}^{-y_0}=2-y_0$. In this case

$$\psi(\lambda)=(y_0-1)\lambda-\left(1-\mathrm{e}^{-\lambda y_0}\right),\quad \forall\lambda>0. \tag{3.40}$$

Let $k_\lambda(s,y):=-\psi(u_{\lambda 1_{(0,+\infty)}}(s,y))/u_{\lambda 1_{(0,+\infty)}}(s,y)$ for $s\ge 0$ and $y\in\mathbb{R}$. Then by Feynman-Kac formula, one has

$$\begin{aligned}
u_{\lambda 1_{(0,+\infty)}}(t,x) &= \Pi_x\left[\mathrm{e}^{\int_0^t k_\lambda(t-s,B_s)ds}u_{\lambda 1_{(0,+\infty)}}(0,B_t)\right]\\
&= \lambda\mathrm{e}^{-(y_0-1)t}\Pi_x\left[\exp\left\{\int_0^t\frac{1-\mathrm{e}^{-y_0u_{\lambda 1_{(0,+\infty)}}(t-s,B_s)}}{u_{\lambda 1_{(0,+\infty)}}(t-s,B_s)}ds\right\};\ B_t>0\right]\\
&\ge \lambda\mathrm{e}^{-(y_0-1)t}\Pi_x\left(B_t>0\right).
\end{aligned}$$

Since $\Pi_x(B_t>0)>0$ for all $t>0$ and $x\in\mathbb{R}$, one has $\tilde u_0(t,x)=\lim_{\lambda\to+\infty}u_{\lambda 1_{(0,+\infty)}}(t,x)=+\infty$. This implies that $\mathbb{P}_{\delta_0}(\max X_t\le x)=\mathbb{P}_{\delta_{-x}}(\max X_t\le 0)=\mathrm{e}^{-\tilde u_0(t,-x)}=0$ for all $t>0$ and $x\in\mathbb{R}$, and so by letting $x\to+\infty$, $\mathbb{P}_{\delta_0}(\max X_t=+\infty)=1-\mathbb{P}_{\delta_0}(\max X_t<+\infty)=1$ for all $t>0$. The branching mechanism given by (3.40) satisfies in particular that

$$\int_z^{+\infty}\frac{1}{\psi(y)}dy=+\infty\quad\forall z>1 \text{ and } \int_0^1 y\pi(dy)<+\infty. \tag{3.41}$$

In fact, [26, Theorem 4.4] shows that for a super-Brownian motion with branching mechanism ψ satisfying (3.41), it holds that $\mathbb{P}_\mu(\mathrm{supp}X_t=\mathbb{R})=1$ for all $t>0$.

4 Probabilistic representation of the limiting process

4.1 Laws of decorations

For the proofs of Theorem 2.12 and Proposition 2.13 we need to show the existence of the limit for $(Z_t-\max Z_t,\sum_{u\in Z_t}I_s^{(u)}-\sqrt{2}s-\max Z_t)$ conditioned on $\{\max Z_t-\sqrt{2}t>0\}$. This is completed by the following lemmas.

Lemma 4.1. *For any $f,g\in\mathcal{B}_b^+(\mathbb{R})$, $x,z\in\mathbb{R}$ and $t,y\ge 0$, we have*

$$\begin{aligned}
&\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f,X_t-\sqrt{2}t-z\rangle-\langle g,Z_t-\sqrt{2}t-z\rangle}1_{\{\max Z_t-\sqrt{2}t-z>y\}}\mid \max Z_t-\sqrt{2}t-z>0\right]\\
=&\ \frac{\mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})}(t,x-\sqrt{2}t-z)}-\mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}}(t,x-\sqrt{2}t-z)}}{1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}}.
\end{aligned} \tag{4.1}$$

Proof. We have

$$\begin{aligned}
&\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f,X_t-\sqrt{2}t-z\rangle-\langle g,Z_t-\sqrt{2}t-z\rangle}1_{\{\max Z_t-\sqrt{2}t-z>y\}}\right]\\
=&\ \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f,X_t-\sqrt{2}t-z\rangle}\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle g,Z_t-\sqrt{2}t-z\rangle}1_{\{\max Z_t-\sqrt{2}t-z>y\}}|X_t\right]\right].
\end{aligned} \tag{4.2}$$

Recall that given X_t, Z_t is a Poisson random measure with intensity $X_t(dx)$. Using Poisson computations, we have

$$\begin{aligned}
&\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle g,Z_t-\sqrt{2}t-z\rangle}1_{\{\max Z_t-\sqrt{2}t-z>y\}}|X_t\right]\\
=&\ \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle g,Z_t-\sqrt{2}t-z\rangle}\left(1-1_{\{\max Z_t-\sqrt{2}t-z\le y\}}\right)|X_t\right]\\
=&\ \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle g,Z_t-\sqrt{2}t-z\rangle}|X_t\right]-\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle g,Z_t-\sqrt{2}t-z\rangle}1_{\{\langle 1_{(y,+\infty)},Z_t-\sqrt{2}t-z\rangle=0\}}|X_t\right]\\
=&\ \mathrm{e}^{-\langle 1-\mathrm{e}^{-g},X_t-\sqrt{2}t-z\rangle}-\mathrm{e}^{-\langle(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)},X_t-\sqrt{2}t-z\rangle}.
\end{aligned}$$

Putting this back in (4.2) we get that

$$\begin{aligned}
& \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle}1_{\{\max Z_t-\sqrt{2}t-z>y\}}\right] \\
= \ & \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f+(1-\mathrm{e}^{-g}), X_t-\sqrt{2}t-z\rangle}\right] \\
& -\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f+(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}, X_t-\sqrt{2}t-z\rangle}\right] \\
= \ & \mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})}(t,x-\sqrt{2}t-z)}-\mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}}(t,x-\sqrt{2}t-z)}.
\end{aligned} \tag{4.3}$$

In particular by setting $f=g=0$ and $y=0$ in the above formula, we get that

$$\mathbb{P}_{\delta_x}\left(\max Z_t-\sqrt{2}t-z>0\right)=1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}. \tag{4.4}$$

Hence (4.1) follows by (4.3)/ (4.4). □

Lemma 4.2. *For any $f,g\in\mathcal{B}_b^+(\mathbb{R})$, $x,z\in\mathbb{R}$, $t\geq 0$ and $\lambda>0$,*

$$\begin{aligned}
& \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle-\lambda\left(\max Z_t-\sqrt{2}t-z\right)}\ \Big|\ \max Z_t-\sqrt{2}t-z>0\right] \\
= \ & \left(1-\exp\left\{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)\right\}\right)^{-1}\cdot \\
& \Big[\int_0^{+\infty}\mathrm{e}^{-y}\exp\left\{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,\frac{y}{\lambda}]}+1_{(\frac{y}{\lambda},+\infty)}}(t,x-\sqrt{2}t-z)\right\}dy \\
& -\exp\left\{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,0]}+1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)\right\}\Big].
\end{aligned} \tag{4.5}$$

Proof. We rewrite the left hand side of (4.5) as $I-II$ where

$$I:=\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle}\mid\max Z_t-\sqrt{2}t-z>0\right],$$

and

$$\begin{aligned}
II \ := \ & \mathbb{P}_{\delta_x}\Big[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle} \\
& \cdot\left(1-\mathrm{e}^{-\lambda\left(\max Z_t-\sqrt{2}t-z\right)}\right)\mid\max Z_t-\sqrt{2}t-z>0\Big].
\end{aligned}$$

By Lemma 4.1,

$$I=\frac{\mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})}(t,x-\sqrt{2}t-z)}-\mathrm{e}^{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,0]}+1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}}{1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}}. \tag{4.6}$$

On the other hand, by Fubini's theorem and Lemma 4.1 we have

$$\begin{aligned}
II \ = \ & \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle}\int_0^{\lambda(\max Z_t-\sqrt{2}t-z)}\mathrm{e}^{-y}dy\,\Big|\,\max Z_t-\sqrt{2}t-z>0\right] \\
= \ & \int_0^{+\infty}\mathrm{e}^{-y}\mathbb{P}_{\delta_x}\Big[\mathrm{e}^{-\langle f, X_t-\sqrt{2}t-z\rangle-\langle g, Z_t-\sqrt{2}t-z\rangle} \\
& \cdot 1_{\{\max Z_t-\sqrt{2}t-z>\frac{y}{\lambda}\}}\,\Big|\,\max Z_t-\sqrt{2}t-z>0\Big]dy \\
= \ & \left(1-\exp\left\{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)\right\}\right)^{-1}\Big[\exp\left\{-u_{f+(1-\mathrm{e}^{-g})}(t,x-\sqrt{2}t-z)\right\} \\
& -\int_0^{+\infty}\mathrm{e}^{-y}\exp\left\{-u_{f+(1-\mathrm{e}^{-g})1_{(-\infty,\frac{y}{\lambda}]}+1_{(\frac{y}{\lambda},+\infty)}}(t,x-\sqrt{2}t-z)\right\}dy\Big].
\end{aligned} \tag{4.7}$$

Hence we get (4.5) by letting (4.6) – (4.7). □

Lemma 4.3. *Suppose $x, z \in \mathbb{R}$ and $s > 0$. Under $\mathbb{P}_{\delta_x}$, conditioned on $\{\max Z_t - \sqrt{2}t - z > 0\}$, $\left(X_t - \sqrt{2}t - z, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}(s+t) - z, Z_t - \sqrt{2}t - z, \max Z_t - \sqrt{2}t - z\right)$ converges, as $t \to +\infty$, in distribution to a limit $\left(\bar{\mathcal{E}}_\infty^X, \bar{\mathcal{E}}_\infty^{I,s}, \bar{\mathcal{E}}_\infty^Z, Y\right)$, where the limit is independent of x and z, and Y is an exponential random variable with mean $1/\sqrt{2}$. Moreover, we have for any $f, h \in \mathcal{H}$, $g \in \mathcal{B}^+(\mathbb{R})$ with $1 - \mathrm{e}^{-g} \in \mathcal{H}_1$ and $\lambda \geq 0$,*

$$
\begin{aligned}
& \mathrm{E}\left[\exp\{-\langle f, \bar{\mathcal{E}}_\infty^X\rangle - \langle h, \bar{\mathcal{E}}_\infty^{I,s}\rangle - \langle g, \bar{\mathcal{E}}_\infty^Z\rangle - \lambda Y\}\right] \\
= & \lim_{t\to+\infty} \mathbb{P}_{\delta_x}\big[\exp\{-\langle f, X_t - \sqrt{2}t - z\rangle - \langle h, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}(s+t) - z\rangle \\
& \qquad -\langle g, Z_t - \sqrt{2}t - z\rangle - \lambda(\max Z_t - \sqrt{2}t - z)\} \mid \max Z_t - \sqrt{2}t - z > 0\big] \\
= & \frac{1}{c_*}\Big[C\left(f + \left(1 - \mathrm{e}^{-g}V_h(s, \cdot - \sqrt{2}s)\right)1_{(-\infty,0]} + 1_{(0,+\infty)}\right) \\
& \quad - \int_0^{+\infty} \mathrm{e}^{-y} C\left(f + \left(1 - \mathrm{e}^{-g}V_h(s, \cdot - \sqrt{2}s)\right)1_{(-\infty,\frac{y}{\lambda}]} + 1_{(\frac{y}{\lambda},+\infty)}\right)dy\Big],
\end{aligned}
\tag{4.8}
$$

where $c_* = C(1_{(0,+\infty)})$.

Proof. In view of (4.4) we have for any $x, z \in \mathbb{R}$ and $y \geq 0$,

$$
\begin{aligned}
& \lim_{t\to+\infty} \mathbb{P}_{\delta_x}\left(\max Z_t - \sqrt{2}t - z > y \mid \max Z_t - \sqrt{2}t - z > 0\right) \\
= & \lim_{t\to+\infty} \frac{\mathbb{P}_{\delta_x}\left(\max Z_t - \sqrt{2}t - z > y\right)}{\mathbb{P}_{\delta_x}\left(\max Z_t - \sqrt{2}t - z > 0\right)} \\
= & \lim_{t\to+\infty} \frac{1 - \exp\{-u_{1_{(0,+\infty)}}(t, x - \sqrt{2}t - z - y)\}}{1 - \exp\{-u_{1_{(0,+\infty)}}(t, x - \sqrt{2}t - z)\}} \\
= & \lim_{t\to+\infty} \frac{u_{1_{(0,+\infty)}}(t, x - \sqrt{2}t - z - y)}{u_{1_{(0,+\infty)}}(t, x - \sqrt{2}t - z)} = \mathrm{e}^{-\sqrt{2}y}.
\end{aligned}
$$

The final equality follows from Lemma 3.2. This implies that, conditioned on $\{\max Z_t - \sqrt{2}t - z > 0\}$, $\max Z_t - \sqrt{2}t - z$ converges in distribution to an exponentially distributed random variable with mean $1/\sqrt{2}$. Suppose f, h, g are functions satisfying our assumptions. Recall that $\mathcal{F}_t$ is the σ-filed generated by Z, X^* and I up to time t, and given $\mathcal{F}_t$, $I_s^{(u)} \stackrel{\mathrm{d}}{=} (I_s, \mathbb{P}_{\cdot,\delta_{z_u(t)}})$ for $u \in Z_t$. We have

$$
\begin{aligned}
\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle h, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}(t+s) - z\rangle} \,\Big|\, \mathcal{F}_t\right] &= \prod_{u\in Z_t} \mathbb{P}_{\cdot,\delta_{z_u(t)}}\left[\mathrm{e}^{-\langle h, I_s - \sqrt{2}(s+t) - z\rangle}\right] \\
&= \prod_{u\in Z_t} V_h(s, z_u(t) - \sqrt{2}(s+t) - z) \\
&= \mathrm{e}^{\langle \ln V_h(s, \cdot - \sqrt{2}s), Z_t - \sqrt{2}t - z\rangle},
\end{aligned}
\tag{4.9}
$$

where V_h is defined by (3.15) with f replaced by h. Thus we have

$$
\begin{aligned}
& \mathbb{P}_{\delta_x}\big[\exp\{-\langle f, X_t - \sqrt{2}t - z\rangle - \langle h, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}(s+t) - z\rangle \\
& \qquad -\langle g, Z_t - \sqrt{2}t - z\rangle - \lambda\big(\max Z_t - \sqrt{2}t - z\big)\} \mid \max Z_t - \sqrt{2}t - z > 0\big] \\
= & \mathbb{P}_{\delta_x}\Big[\exp\{-\langle f, X_t - \sqrt{2}t - z\rangle - \langle g, Z_t - \sqrt{2}t - z\rangle - \lambda\big(\max Z_t - \sqrt{2}t - z\big)\} \\
& \qquad \cdot \mathbb{P}_{\delta_x}\big[\exp\{-\langle h, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}(t+s) - z\rangle\} \mid \mathcal{F}_t\Big] \mid \max Z_t - \sqrt{2}t - z > 0\big] \\
= & \mathbb{P}_{\delta_x}\big[\exp\{-\langle f, X_t - \sqrt{2}t - z\rangle - \langle g - \ln V_h(s, \cdot - \sqrt{2}s), Z_t - \sqrt{2}t - z\rangle \\
& \qquad -\lambda\big(\max Z_t - \sqrt{2}t - z\big)\} \mid \max Z_t - \sqrt{2}t - z > 0\big].
\end{aligned}
\tag{4.10}
$$

By Lemma 4.2 the right hand side of (4.10) equals

$$
\begin{aligned}
& \big(1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}\big)^{-1} \\
& \quad \cdot\Big[\int_0^{+\infty}\mathrm{e}^{-y}\exp\big\{-u_{f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,\frac{y}{\lambda}]}+1_{(\frac{y}{\lambda},+\infty)}}(t,x-\sqrt{2}t-z)\big\}dy \\
& -\exp\big\{-u_{f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,0]}+1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)\big\}\Big] \\
= & \big(1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}\big)^{-1} \\
& \quad \cdot\Big[-\int_0^{+\infty}\mathrm{e}^{-y}\big(1-\mathrm{e}^{-u_{f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,\frac{y}{\lambda}]}+1_{(\frac{y}{\lambda},+\infty)}}(t,x-\sqrt{2}t-z)}\big)dy \\
& \quad +\big(1-\mathrm{e}^{-u_{f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,0]}+1_{(0,+\infty)}}(t,x-\sqrt{2}t-z)}\big)\Big]. \qquad (4.11)
\end{aligned}
$$

We observe that $1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)=\mathrm{e}^{-g}\left(1-V_h(s,\cdot-\sqrt{2}s)\right)+(1-\mathrm{e}^{-g})\in\mathcal{H}_1$ since $1-V_h(s,\cdot-\sqrt{2}s), 1-\mathrm{e}^{-g}\in\mathcal{H}_1$ by the assumptions. Using the facts that $1-\mathrm{e}^{-x}\sim x$ as $x\to 0$ and that

$$
\lim_{t\to+\infty}\frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t}\mathrm{e}^{-\sqrt{2}x}u_\phi(t,x-\sqrt{2}t)=C(\phi)\quad\forall\phi\in\mathcal{H},
$$

one can show by the bounded convergence theorem that the right hand side of (4.11) converges to

$$
\begin{aligned}
& \frac{1}{c_*}\Big[C\left(f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,0]}+1_{(0,+\infty)}\right) \\
& \quad -\int_0^{+\infty}\mathrm{e}^{-y}C\left(f+\left(1-\mathrm{e}^{-g}V_h(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,\frac{y}{\lambda}]}+1_{(\frac{y}{\lambda},+\infty)}\right)dy\Big]
\end{aligned}
$$

as $t\to+\infty$. In particular, for any $\lambda_i>0$, $i=1,2,3,4$ and $f,h,g\in C_c^+(\mathbb{R})$, one has

$$
\begin{aligned}
& \lim_{t\to+\infty}\mathbb{P}_{\delta_x}\Big[\exp\{-\lambda_1\langle f,X_t-\sqrt{2}t-z\rangle-\lambda_2\langle h,\sum_{u\in Z_t}I_s^{(u)}-\sqrt{2}(s+t)-z\rangle \\
& -\lambda_3\langle g,Z_t-\sqrt{2}t-z\rangle-\lambda_4\left(\max Z_t-\sqrt{2}t-z\right)\}\;\big|\;\max Z_t-\sqrt{2}t-z>0\Big] \\
= & \frac{1}{c_*}\Big[C\left(\lambda_1 f+\left(1-\mathrm{e}^{-\lambda_3 g}V_{\lambda_2 h}(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,0]}+1_{(0,+\infty)}\right) \\
& -\int_0^{+\infty}\mathrm{e}^{-y}C\left(\lambda_1 f+\left(1-\mathrm{e}^{-\lambda_3 g}V_{\lambda_2 h}(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,\frac{y}{\lambda_4}]}+1_{(\frac{y}{\lambda_4},+\infty)}\right)dy\Big]. \qquad (4.12)
\end{aligned}
$$

To show the convergence in distribution, it suffices to show the right hand side of (4.12) converges to 1 as $\lambda_i\to 0$, $i=1,2,3,4$. By Corollary 3.11 and Lemma 3.7 we have

$$
C(1-V_{\lambda_2 h}(s,\cdot-\sqrt{2}s))\le C(u_{\lambda_2 h}(s,\cdot-\sqrt{2}s))=C(\lambda_2 h)\to 0\text{ as }\lambda_2\to 0,
$$

and

$$
C(1-\mathrm{e}^{-\lambda_3 g})\le C(\lambda_3 g)\to 0,\quad\text{ as }\lambda_3\to 0.
$$

Thus one has

$$
\begin{aligned}
C(1-\mathrm{e}^{-\lambda_3 g}V_{\lambda_2 h}(s,\cdot-\sqrt{2}s)) & \le C\left(\mathrm{e}^{-\lambda_3 g}\left(1-V_{\lambda_2 h}(s,\cdot-\sqrt{2}s)\right)\right)+C(1-\mathrm{e}^{-\lambda_3 g}) \\
& \le C\left(1-V_{\lambda_2 h}(s,\cdot-\sqrt{2}s)\right)+C(1-\mathrm{e}^{-\lambda_3 g}) \\
& \to 0,\quad\text{ as }\lambda_2,\lambda_3\to 0. \qquad (4.13)
\end{aligned}
$$

On the other hand by Corollary 3.11 we have for any $\delta \in \mathbb{R}$

$$\begin{aligned} C(1_{(\delta,+\infty)}) &\leq C(\lambda_1 f + \left(1 - \mathrm{e}^{-\lambda_3 g} V_{\lambda_2 h}(s, \cdot - \sqrt{2}s)\right) 1_{(-\infty,\delta]} + 1_{(\delta,+\infty)}) \\ &\leq C(\lambda_1 f) + C\left(1 - \mathrm{e}^{-\lambda_3 g} V_{\lambda_2 h}(s, \cdot - \sqrt{2}s)\right) + C(1_{(\delta,+\infty)}), \end{aligned} \tag{4.14}$$

Using (4.13) and the fact that $\lim_{\lambda_1 \to 0} C(\lambda_1 f) = 0$ we get by (4.14) that

$$\lim_{\lambda_1,\lambda_2,\lambda_3 \to 0} C(\lambda_1 f + \left(1 - \mathrm{e}^{-\lambda_3 g} V_{\lambda_2 h}(s, \cdot - \sqrt{2}s)\right) 1_{(-\infty,\delta]} + 1_{(\delta,+\infty)}) = C(1_{(\delta,+\infty)}).$$

This implies that

$$\lim_{\lambda_1,\lambda_2,\lambda_3 \to 0} C(\lambda_1 f + \left(1 - \mathrm{e}^{-\lambda_3 g} V_{\lambda_2 h}(s, \cdot - \sqrt{2}s)\right) 1_{(-\infty,0]} + 1_{(0,+\infty)}) = C(1_{(0,+\infty)}), \tag{4.15}$$

and that for every $y \in \mathbb{R}$,

$$\begin{aligned} &\lim_{\lambda_1,\lambda_2,\lambda_3,\lambda_4 \to 0} C(\lambda_1 f + \left(1 - \mathrm{e}^{-\lambda_3 g} V_{\lambda_2 h}(s, \cdot - \sqrt{2}s)\right) 1_{(-\infty,\frac{y}{\lambda_4}]} + 1_{(\frac{y}{\lambda_4},+\infty)}) \\ = &\lim_{\lambda_4 \to 0} C(1_{(\frac{y}{\lambda_4},+\infty)}) = 0. \end{aligned} \tag{4.16}$$

In view of (4.15) and (4.16), one can use the bounded convergence theorem to show that the right hand side of (4.12) converges to 1 as $\lambda_i \to 0$, $i = 1,2,3,4$. Hence we complete the proof. □

Lemma 4.4. *Suppose $x, z \in \mathbb{R}$ and $s > 0$. Under $\mathbb{P}_{\delta_x}$, conditioned on $\{\max Z_t - \sqrt{2}t - z > 0\}$, $\left(X_t - \max Z_t, \sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}s - \max Z_t, Z_t - \max Z_t, \max Z_t - \sqrt{2}t - z\right)$ converges, as $t \to +\infty$, in distribution to a limit $\left(\triangle^X, \triangle^{I,s}, \triangle^Z, Y\right) := \left(\bar{\mathcal{E}}_\infty^X - Y, \bar{\mathcal{E}}_\infty^{I,s} - Y, \bar{\mathcal{E}}_\infty^Z - Y, Y\right)$, which is independent of x and z. Moreover $\left(\triangle^X, \triangle^{I,s}, \triangle^Z\right)$ is independent of Y.*

Proof. The first conclusion is a direct result of Lemma 4.3 and [4, Lemma 4.13]. We only need to show the independence. Suppose $f, g, h \in C_c^+(\mathbb{R})$ and $y \geq 0$. We have

$$\begin{aligned} &\mathrm{E}\left[\mathrm{e}^{-\langle f,\triangle^X\rangle - \langle g,\triangle^{I,s}\rangle - \langle h,\triangle^Z\rangle} 1_{\{Y>y\}}\right] \\ = &\lim_{t\to+\infty} \mathbb{P}_{\delta_0}\big[\mathrm{e}^{-\langle f,X_t-\max Z_t\rangle - \langle g,\sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}s - \max Z_t\rangle - \langle h,Z_t-\max Z_t\rangle} \\ &\qquad \cdot 1_{\{\max Z_t - \sqrt{2}t > y\}} | \max Z_t - \sqrt{2}t > 0\big] \\ = &\lim_{t\to+\infty} \mathbb{P}_{\delta_0}\big[\mathrm{e}^{-\langle f,X_t-\max Z_t\rangle - \langle g,\sum_{u\in Z_t} I_s^{(u)} - \sqrt{2}s - \max Z_t\rangle - \langle h,Z_t-\max Z_t\rangle} | \max Z_t - \sqrt{2}t > y\big] \\ &\cdot \mathbb{P}_{\delta_0}\left(\max Z_t - \sqrt{2}t > y | \max Z_t - \sqrt{2}t > 0\right) \\ = &\mathrm{E}\left[\mathrm{e}^{-\langle f,\triangle^X\rangle - \langle g,\triangle^{I,s}\rangle - \langle h,\triangle^Z\rangle}\right] \cdot \mathrm{P}(Y > y). \end{aligned}$$

This yields the independence. □

Remark 4.5. We have by (2.10) that for all $g \in C_c^+(\mathbb{R})$ and $y \geq 0$,

$$\begin{aligned} &\mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle g, Z_t - \sqrt{2}t\rangle}; \max Z_t - \sqrt{2}t > y\right] \\ = &u_{(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}}(t, -\sqrt{2}t) - u_{1-\mathrm{e}^{-g}}(t, -\sqrt{2}t). \end{aligned}$$

Thus by Lemma 4.1 with $f=0$ we have

$$
\begin{aligned}
& \mathbb{P}_{\delta_0}\left[\mathrm{e}^{-\langle g, Z_t-\sqrt{2}t\rangle}; \max Z_t-\sqrt{2}t>y \,\Big|\, \max Z_t-\sqrt{2}t>0\right] \\
= \quad & \frac{\mathrm{e}^{-u_{1-\mathrm{e}^{-g}}(t,-\sqrt{2}t)}-\mathrm{e}^{-u_{(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}}(t,-\sqrt{2}t)}}{1-\mathrm{e}^{-u_{1_{(0,+\infty)}}(t,-\sqrt{2}t)}} \\
\sim \quad & \frac{u_{(1-\mathrm{e}^{-g})1_{(-\infty,y]}+1_{(y,+\infty)}}(t,-\sqrt{2}t)-u_{1-\mathrm{e}^{-g}}(t,-\sqrt{2}t)}{u_{1_{(0,+\infty)}}(t,-\sqrt{2}t)} \\
= \quad & \mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle g, Z_t-\sqrt{2}t\rangle}; \max Z_t-\sqrt{2}t>y \,\Big|\, \max Z_t-\sqrt{2}t>0\right], \quad \text{as } t\to+\infty.
\end{aligned}
$$

This implies that the limit of $(Z_t-\sqrt{2}t, \max Z_t-\sqrt{2}t)$ under $\mathbb{P}_{\delta_0}\left(\cdot|\max Z_t-\sqrt{2}t>0\right)$ and that under $\mathbb{P}_{\cdot,\delta_0}(\cdot|\max Z_t-\sqrt{2}t>0)$ are equal in distribution. Therefore the definition of $\triangle^Z$ given in Lemma 4.4 coincides with that given in (2.18).

4.2 Proof of Theorem 2.12

In the following lemma, we establish an integral representation of $C(\phi)$ for $\phi\in\mathcal{H}_1$. It characterises the limiting extremal process of the skeleton BBM as a decorated Poisson point process.

Lemma 4.6. *For $\phi\in\mathcal{H}_1$,*

$$
C(\phi)=c_*\int_{-\infty}^{+\infty}\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{\langle \ln(1-\phi),\triangle^Z+y\rangle}\right]dy. \tag{4.17}
$$

Here $\triangle^Z$ is defined in Lemma 4.4, and $c_*=C(1_{(0,+\infty)})$.

Proof. First we suppose $\phi\in\mathcal{H}_1$ is a compactly supported continuous function with $\|\phi\|_\infty<1$. Let $g(x):=-\ln(1-\phi(x))$ for $x\in\mathbb{R}$. The argument below Proposition 2.5 implies that

$$
\mathrm{E}\left[\mathrm{e}^{-\langle g,\mathcal{E}^Z_\infty\rangle}\right]=\lim_{t\to+\infty}\mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle g,\mathcal{E}^Z_t\rangle}\right]=\mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-C(\phi)\partial M_\infty}\right], \tag{4.18}
$$

where $\mathcal{E}^Z_\infty$ is the limit of $((\mathcal{E}^Z_t)_{t\ge 0},\mathbb{P}_{\cdot,\delta_0})$ in distribution. It is known that $\mathcal{E}^Z_\infty$ is a DPPP($c_*\partial M_\infty\sqrt{2}\mathrm{e}^{\sqrt{2}y}dy$, $\triangle^Z$) where $\triangle^Z$ is the distributional limit of $Z_t-\max Z_t$ under $\mathbb{P}_{\cdot,\delta_0}\left(\cdot|\max Z_t-\sqrt{2}t>0\right)$ (and hence equal in law to $\triangle^Z$ defined in Lemma 4.4). Using Poisson computations one has

$$
\mathrm{E}\left[\mathrm{e}^{-\langle g,\mathcal{E}^Z_\infty\rangle}\right]=\mathbb{P}_{\cdot,\delta_0}\left[\exp\{-c_*\partial M_\infty\int_{-\infty}^{+\infty}\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\langle g,\triangle^Z+y\rangle}\right]dy\}\right].
$$

We note that under our assumptions, $\mathbb{P}_{\cdot,\delta_0}\left[\partial M_\infty>0\right]>0$. Hence (4.17) follows, otherwise there would be a contradiction.

For a general $\phi\in\mathcal{H}_1$, one can find a nondecreasing sequence of functions $\{\phi_n:n\ge 1\}\subset\mathcal{H}_1\cap C^+_c(\mathbb{R})$, such that $\|\phi_n\|_\infty<1$ and $\phi_n(x)\uparrow\phi(x)$ for all $x\in\mathbb{R}$ as $n\to+\infty$. Then by Corollary 3.11,

$$
C(\phi_n)\le C(\phi)\le C(\phi_n)+C(\phi-\phi_n).
$$

Note that by the dominated convergence theorem $\int_{-\infty}^{+\infty}|x|\mathrm{e}^{\sqrt{2}x}\left(\phi(-x)-\phi_n(-x)\right)dx\to 0$ as $n\to+\infty$. Thus by Lemma 3.7 $C(\phi-\phi_n)\to 0$ as $n\to+\infty$. So we get that $C(\phi_n)\uparrow C(\phi)$ as $n\to+\infty$, and (4.17) follows immediately by the monotone convergence theorem. $\square$

Proof of Theorem 2.12. It follows from Theorem 2.10 and Lemma 4.6 that for all $g \in \mathcal{B}^+(\mathbb{R})$ with $1 - \mathrm{e}^{-g} \in \mathcal{H}$,

$$\begin{aligned} \mathrm{E}\left[\mathrm{e}^{-\langle g, \mathcal{E}_\infty^Z \rangle}\right] &= \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-C(1-\mathrm{e}^{-g})\partial M_\infty}\right] \\ &= \mathbb{P}_{\delta_x}\left[\exp\{-c_* \partial M_\infty \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}y} \mathrm{E}\left[1 - \mathrm{e}^{-\langle g, \triangle^Z + y\rangle}\right] dy\}\right]. \end{aligned} \tag{4.19}$$

The above equations hold in particular for all $g \in C_c^+(\mathbb{R})$. This implies that $\mathcal{E}_\infty^Z$ is a decorated Poisson point process with intensity $c_* \partial M_\infty \sqrt{2}\mathrm{e}^{-\sqrt{2}y} dy$ and decoration law $\triangle^Z$.

We have for all $\phi \in C_c^+(\mathbb{R})$,

$$\begin{aligned} \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \sum_{i\ge 1} \mathcal{T}_{d_i} \triangle_i^s \rangle}\right] &= \mathrm{E}\left[\prod_{i\ge 1} \mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle \mathcal{T}_{d_i}\phi, I_s - \sqrt{2}s\rangle}\right]\right] = \mathrm{E}\left[\prod_{i\ge 1} V_{\mathcal{T}_{d_i}\phi}(s, -\sqrt{2}s)\right] \\ &= \mathrm{E}\left[\mathrm{e}^{\langle \ln V_\phi(s,\cdot - \sqrt{2}s), \mathcal{E}_\infty^Z\rangle}\right] = \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-C(1-V_\phi(s,\cdot-\sqrt{2}s))\partial M_\infty}\right]. \end{aligned}$$

The final equality follows from (4.19). Since $\lim_{s\to+\infty} C(1 - V_\phi(s, \cdot - \sqrt{2}s)) = C(\phi)$, we get by the above equality that

$$\lim_{s\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \sum_{i\ge 1} \mathcal{T}_{d_i} \triangle_i^s \rangle}\right] = \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-C(\phi)\partial M_\infty}\right] = \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \mathcal{E}_\infty\rangle}\right]$$

for all $\phi \in C_c^+(\mathbb{R})$. Hence we prove (2.27). □

Remark 4.7. We claim that for for each $i \ge 1$, $\triangle_i^s$ converges in distribution to the null measure as $s \to +\infty$. This is because, by (3.16) for all $\phi \in C_c^+(\mathbb{R})$,

$$\mathrm{E}\left[\mathrm{e}^{-\langle \phi, \triangle_i^s\rangle}\right] = \mathbb{P}_{\cdot,\delta_0}\left[\mathrm{e}^{-\langle \phi, I_s - \sqrt{2}s\rangle}\right] = V_\phi(s, -\sqrt{2}s) = 1 - \left(u_\phi(s, -\sqrt{2}s) - u_\phi^*(s, -\sqrt{2}s)\right).$$

Noticing that $0 \le u_\phi^*(s, x - \sqrt{2}s) \le u_\phi(s, x - \sqrt{2}s)$, (3.25) implies that

$$\lim_{s\to+\infty} u_\phi(s, -\sqrt{2}s) = \lim_{s\to+\infty} u_\phi^*(s, -\sqrt{2}s) = 0.$$

Then we have

$$\mathrm{E}\left[\mathrm{e}^{-\langle \phi, \triangle_i^s\rangle}\right] \to 1, \quad \text{as } s \to +\infty.$$

4.3 Proof of Proposition 2.13

Lemma 4.8. *Suppose $\phi \in \mathcal{H}$. Then for all $s > 0$ and $y \in \mathbb{R}$,*

$$\mathrm{E}\left[\mathrm{e}^{\langle \ln V_\phi(s,\cdot-\sqrt{2}s), \triangle^Z + y\rangle}\right] = \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \triangle^{I,s} + y\rangle}\right]. \tag{4.20}$$

Proof. Recall the definition of $(\bar{\mathcal{E}}_\infty^Z, Y)$ in Lemma 4.3. We use $x_j \in \bar{\mathcal{E}}_\infty^Z$ to denote an atom of the random point measure $\bar{\mathcal{E}}_\infty^Z$. For any $s > 0$, define random measure Θ_s by

$$\Theta_s := \sum_{x_j \in \bar{\mathcal{E}}_\infty^Z} \mathcal{T}_{x_j}(I_s^j - \sqrt{2}s),$$

where I^j, $j \ge 1$ are i.i.d. copies of $(I, \mathbb{P}_{\cdot,\delta_0})$, and are independent of $(\bar{\mathcal{E}}_\infty^Z, Y)$. Recall the Laplace functional of I_s given in (3.15). It follows from Lemma 4.3 that, for all $f \in \mathcal{H}$,

$s > 0$ and $\lambda > 0$

$$\begin{aligned}
\mathrm{E}\left[e^{-\langle f,\Theta_s\rangle-\lambda Y}\right] &= \mathrm{E}\left[e^{\langle \log V_f(s,\cdot-\sqrt{2}s),\bar{\mathcal{E}}^Z_\infty\rangle-\lambda Y}\right] \\
&= \frac{1}{c_*}\left[C\left(\left(1-V_f(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,0]}+1_{(0,+\infty)}\right)\right. \\
&\quad \left.-\int_0^{+\infty} e^{-y}C\left(\left(1-V_f(s,\cdot-\sqrt{2}s)\right)1_{(-\infty,y/\lambda]}+1_{(y/\lambda,+\infty)}\right)dy\right] \\
&= \mathrm{E}\Big[e^{-\langle f,\bar{\mathcal{E}}^{I,s}_\infty\rangle-\lambda Y}\Big].
\end{aligned}$$

This implies that $(\Theta_s, Y) \overset{\mathrm{d}}{=} (\bar{\mathcal{E}}^{I,s}_\infty, Y)$. Therefore, we obtain that

$$\triangle^{I,s} = \bar{\mathcal{E}}^{I,s}_\infty - Y \overset{\mathrm{d}}{=} \Theta_s - Y = \sum_{x_j\in\triangle^Z} \mathcal{T}_{x_j}(I^j_s - \sqrt{2}s),$$

which implies that, for all $\phi \in \mathcal{H}$ and $y \in \mathbb{R}$,

$$\mathrm{E}\left[\mathrm{e}^{-\langle \phi,\triangle^{I,s}+y\rangle}\right] = \mathrm{E}\left[\mathrm{e}^{-\langle \phi,\sum_{x_j\in\triangle^Z}\mathcal{T}_{x_j}(I^j_s-\sqrt{2}s)+y\rangle}\right] = \mathrm{E}\left[\mathrm{e}^{\langle \ln V_\phi(s,\cdot-\sqrt{2}s),\triangle^Z+y\rangle}\right].$$

Now we finish the proof. □

Proof of Proposition 2.13. It is easy to get by Poisson computations that

$$\begin{aligned}
&\mathrm{E}\left[\mathrm{e}^{-\langle \phi,\sum_{i\geq 1}\mathcal{T}_{e_i}\triangle^{I,s}_i\rangle}\right] \\
=\ &\mathbb{P}_{\delta_x}\left[\exp\left\{-c_*\partial M_\infty\int_{-\infty}^{+\infty}\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\langle\phi,\triangle^{I,s}+y\rangle}\right]dy\right\}\right] \qquad (4.21)
\end{aligned}$$

for all $\phi \in C_c^+(\mathbb{R})$. It follows from Lemma 3.10, Lemma 4.6 and Lemma 4.8 that for all $\phi \in C_c^+(\mathbb{R})$,

$$\begin{aligned}
C(\phi) &= \lim_{s\to+\infty} C(1-V_\phi(s,\cdot-\sqrt{2}s)) \\
&= \lim_{s\to+\infty} c_*\int_{-\infty}^{+\infty}\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{\langle \ln V_\phi(s,\cdot-\sqrt{2}s),\triangle^Z+y\rangle}\right]dy \\
&= \lim_{s\to+\infty} c_*\int_{-\infty}^{+\infty}\sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\langle\phi,\triangle^{I,s}+y\rangle}\right]dy. \qquad (4.22)
\end{aligned}$$

This together with (4.21) and Theorem 2.10 yields that

$$\lim_{s\to+\infty}\mathrm{E}\left[\mathrm{e}^{-\langle\phi,\sum_{i\geq1}\mathcal{T}_{e_i}\triangle^{I,s}_i\rangle}\right] = \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-C(\phi)\partial M_\infty}\right] = \mathrm{E}\left[\mathrm{e}^{-\langle\phi,\mathcal{E}_\infty\rangle}\right], \quad \forall \phi\in C_c^+(\mathbb{R}).$$

Hence we complete the proof. □

4.4 Proof of Theorem 2.14

We prove Theorem 2.14 in this section. First we relate $C(\phi)$ to the Laplace functional of a certain random Radon measure. Then we observe that this random measure is infinitely divisible and thus get an expression for $C(\phi)$ which leads to the probabilistic interpretation presented in Theorem 2.14. Our observation on $C(\phi)$ is inspired by the work of [22].

Lemma 4.9. *Let $\{e_i : i \geq 1\}$ be the atoms of a Poisson point process with intensity $c_*\sqrt{2}\mathrm{e}^{-\sqrt{2}y}dy$ and for every $s > 0$, $\{\triangle_i^{I,s} : i \geq 1\}$ be an independent sequence of i.i.d. random measures with the same law as $\triangle^{I,s}$. Set*

$$\mathcal{D}_s := \sum_{i\geq 1} \mathcal{T}_{e_i}\triangle_i^{I,s}.$$

Then as $s \to +\infty$, the random measures $\mathcal{D}_s$ converges in distribution to a random Radon measure $\mathcal{D}_\infty$. Moreover, we have for any $\phi \in \mathcal{H}$,

$$\mathrm{E}\left[\mathrm{e}^{-\langle \phi, \mathcal{D}_\infty\rangle}\right] = \lim_{s\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \mathcal{D}_s\rangle}\right] = \mathrm{e}^{-C(\phi)}. \tag{4.23}$$

Proof. By the definition of $\mathcal{D}_s$, one can use simple Poisson computations to get that

$$\mathrm{E}\left[\mathrm{e}^{-\langle f, \mathcal{D}_s\rangle}\right] = \exp\left\{-c_* \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1 - \mathrm{e}^{-\langle f, \triangle^{I,s}+y\rangle}\right] dy\right\}, \quad \forall f \in \mathcal{B}^+(\mathbb{R}). \tag{4.24}$$

Combining (4.24) and (4.22) we get that

$$\lim_{s\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle \phi, \mathcal{D}_s\rangle}\right] = \mathrm{e}^{-C(\phi)}, \quad \forall \phi \in \mathcal{H}. \tag{4.25}$$

The above identity holds in particular for $\phi \in C_c^+(\mathbb{R})$. Moreover one has by Lemma 3.7 that $\lim_{\lambda\to 0+} C(\lambda\phi) = 0$ for $\phi \in C_c^+(\mathbb{R})$. This implies the existence of the limit $\mathcal{D}_\infty$, and (4.23) follows immediately from (4.25). $\square$

A random measure μ is said to be exp-$\sqrt{2}$-stable if for any a, b satisfying $\mathrm{e}^{\sqrt{2}a} + \mathrm{e}^{\sqrt{2}b} = 1$, it holds that $\mathcal{T}_a\mu + \mathcal{T}_b\hat{\mu} \overset{\mathrm{d}}{=} \mu$, where $\hat{\mu}$ is an independent copy of μ. It is easy to see that an exp-$\sqrt{2}$-stable random measure is infinitely divisible. By (4.23) and the fact that $C(\mathcal{T}_a\phi) = \mathrm{e}^{\sqrt{2}a}C(\phi)$ for all $a \in \mathbb{R}$ and $\phi \in \mathcal{H}$, one can easily show that $\mathcal{D}_\infty$ is an exp-$\sqrt{2}$-stable random measure on $\mathbb{R}$.

Lemma 4.10. *There exist some constant $\iota \geq 0$ and some measure Λ on $\mathcal{M}(\mathbb{R}) \setminus \{0\}$ satisfying that*

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x}dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} (1 \wedge \mathcal{T}_x\mu(A))\,\Lambda(d\mu) < +\infty, \quad \forall \text{ bounded Borel set } A \subset \mathbb{R},$$

such that

$$C(\phi) = \iota \int_{-\infty}^{+\infty} \phi(x)\mathrm{e}^{-\sqrt{2}x}dx + c_* \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}x}dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \left(1 - \mathrm{e}^{-\langle \phi, \mathcal{T}_x\mu\rangle}\right) \Lambda(d\mu), \tag{4.26}$$

for every $\phi \in C_c^+(\mathbb{R})$. Moreover, it holds that

$$\iota = \frac{2c_*}{1 - \mathrm{e}^{-\sqrt{2}}} \lim_{\epsilon\to 0+} \mathrm{limsi}_{s\to+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x}\mathrm{E}\left[\langle 1_{(0,1)}, \mathcal{T}_x\triangle^{I,s}\rangle; \langle 1_{(0,1)}, \mathcal{T}_x\triangle^{I,s}\rangle < \epsilon\right] dx, \tag{4.27}$$

where "limsi" is supposed to hold with both $\liminf$ and $\limsup$, and

$$\begin{aligned} &\int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x}dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} f\left(\mathcal{T}_x\mu(A_1), \cdots, \mathcal{T}_x\mu(A_n)\right) \Lambda(d\mu) \\ = &\lim_{s\to+\infty} \int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x}\mathrm{E}\left[f\left(\mathcal{T}_x\triangle^{I,s}(A_1), \cdots, \mathcal{T}_x\triangle^{I,s}(A_n)\right)\right] dx \end{aligned} \tag{4.28}$$

for any $n \geq 1$, $f \in C_c\left(\bar{\mathbb{R}}^n \setminus \{0\}\right)$ and any bounded open sets $A_1, \cdots, A_n \subset \mathbb{R}$.

Proof. Since $\mathcal{D}_\infty$ is an exp-$\sqrt{2}$-stable random measure, (4.26) is a direct result of [22, Theorem 3.1]. We note that $\mathcal{D}^s$ converges in distribution to $\mathcal{D}_\infty$ and that the exponent of the Laplace functional of $\mathcal{D}^s$ is given by

$$-\ln \mathrm{E}\left[\mathrm{e}^{-\langle\phi,\mathcal{D}^s\rangle}\right] = C\left(1 - V_\phi(s, \cdot - \sqrt{2}s)\right) = c_* \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}x}\mathrm{E}\left[1 - \mathrm{e}^{-\langle\phi,\mathcal{T}_x \triangle^{I,s}\rangle}\right] dx$$

for all $\phi \in C_c^+(\mathbb{R})$. (4.27) and (4.28) follow immediately from [15, Exercise 6.4]. □

Lemma 4.10 yields a short proof for Theorem 2.14.

Proof of Theorem 2.14. We have that

$$\mathrm{E}\left[\mathrm{e}^{-\langle\phi,\mathcal{E}_\infty\rangle}\right] = \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\partial M_\infty C(\phi)}\right], \quad \forall\phi \in C_c^+(\mathbb{R})$$

where $C(\phi)$ can be represented by (4.26). This yields the result of this theorem. □

Remark 4.11. It follows by (4.26) that for all $\lambda > 0$ and $\phi \in C_c^+(\mathbb{R})$,

$$\frac{C(\lambda\phi)}{\lambda} = \iota \int_{-\infty}^{+\infty} \phi(x)\mathrm{e}^{-\sqrt{2}x} dx + c_* \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}x} dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \frac{1 - \mathrm{e}^{-\lambda\langle\phi,\mathcal{T}_x\mu\rangle}}{\lambda}\Lambda(d\mu). \tag{4.29}$$

Note that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x} dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \frac{1 - \mathrm{e}^{-\lambda\langle\phi,\mathcal{T}_x\mu\rangle}}{\lambda}\Lambda(d\mu) \le \int_{-\infty}^{+\infty} \mathrm{e}^{-\sqrt{2}x} dx \int_{\mathcal{M}(\mathbb{R})\setminus\{0\}} \frac{1}{\lambda} \wedge \langle\phi, \mathcal{T}_x\mu\rangle\Lambda(d\mu).$$

Thus by the dominated convergence theorem, the second term of (4.29) converges to 0 as $\lambda \to +\infty$. Consequently we get that

$$\iota \int_{-\infty}^{+\infty} \phi(x)\mathrm{e}^{-\sqrt{2}x} dx = \lim_{\lambda\to+\infty} \frac{C(\lambda\phi)}{\lambda}, \quad \forall\phi \in C_c^+(\mathbb{R}). \tag{4.30}$$

So a sufficient condition for the constant ι to be 0 is that

$$\sup_\lambda C(\lambda\phi) < +\infty \quad \text{for some } \phi \in C_c^+(\mathbb{R}).$$

This is true if the branching mechanism ψ satisfies Assumption 2.15, where one has $\sup_\lambda C(\lambda 1_{(0,+\infty)}) \le \tilde{c}_0 < +\infty$. In this case, $\mathcal{E}_\infty$ is equal in law to a Poisson random measure on $\mathcal{M}(\mathbb{R})$ with intensity $c_*\partial M_\infty \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}x}\mathcal{T}_x\Lambda(d\mu)dx$. We shall discuss this special case in detail in the next section.

4.5 Special case where Assumption 2.15 is satisfied and proof of Theorem 2.16

In this section we assume in addition that Assumption 2.15 is satisfied. Recall the definition of $\tilde{u}_\phi(t,x)$ given in (3.35). It is shown in the argument of Remark 3.12 that when Assumption 2.15 holds, $\tilde{u}_\phi(r, \cdot - \sqrt{2}r) \in \mathcal{H}$ for all $\phi \in \mathcal{H}$ and $r > 0$. Applying Lemma 3.10 to the function $u_{\tilde{u}_\phi(r,\cdot-\sqrt{2}r)}(t,x) = \tilde{u}_\phi(t + r, x - \sqrt{2}r)$, one gets that for all $x \in \mathbb{R}$ and $\phi \in \mathcal{H}$,

$$\lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x}\tilde{u}_\phi(t, x - \sqrt{2}t) = \tilde{C}(\phi),$$

where $\widetilde{C}(\phi)$ is defined in (3.38). In particular by taking $\phi = 0$, one gets that

$$\lim_{t\to+\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} \mathrm{e}^{-\sqrt{2}x}\mathbb{P}_{\delta_x}\left(\max X_t - \sqrt{2}t > 0\right) = \tilde{c}_0, \tag{4.31}$$

where $\tilde{c}_0 = \widetilde{C}(0)$.

Lemma 4.12. *Suppose $x, z \in \mathbb{R}$. Under $\mathbb{P}_{\delta_x}$, conditioned on $\{\max X_t - \sqrt{2}t - z > 0\}$, the random elements $\left(X_t - \sqrt{2}t - z, Z_t - \sqrt{2}t - z, \max X_t - \sqrt{2}t - z\right)$ converges, as $t \to +\infty$, in distribution to a limit $\left(\widetilde{\mathcal{E}}^X_\infty, \widetilde{\mathcal{E}}^Z_\infty, Y\right)$, where the limit is independent of x and z, and Y is an exponential random variable with mean $1/\sqrt{2}$. Moreover, given $\left(\widetilde{\mathcal{E}}^X_\infty, Y\right)$, $\widetilde{\mathcal{E}}^Z_\infty$ is a Poisson random measure with intensity $\widetilde{\mathcal{E}}^X_\infty$.*

Proof. Fix $x, z \in \mathbb{R}$. It has been proved in [23, Proposition 3.4] that conditioned on $\{\max X_t - \sqrt{2}t - z > 0\}$, $\left(X_t - \sqrt{2}t - z, \max X_t - \sqrt{2}t - z\right)$ converges, as $t \to +\infty$, in distribution to a limit $\left(\widetilde{\mathcal{E}}^X_\infty, Y\right)$, where the limit is independent of x and z, and Y is an exponential random variable with mean $1/\sqrt{2}$. We note that for all $f, g \in C_c^+(\mathbb{R})$ and $\lambda_i \ge 0$, $i = 1, 2, 3$,

$$
\begin{aligned}
& \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\lambda_1\langle f, X_t - \sqrt{2}t - z\rangle - \lambda_2\langle g, Z_t - \sqrt{2}t - z\rangle - \lambda_3\left(\max X_t - \sqrt{2}t - z\right)} \mid \max X_t - \sqrt{2}t - z > 0\right] \\
= \ & \mathbb{P}_{\delta_x}\Big[\mathrm{e}^{-\lambda_1\langle f, X_t - \sqrt{2}t - z\rangle - \lambda_3\left(\max X_t - \sqrt{2}t - z\right)} \\
& \quad \cdot \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\lambda_2\langle g, Z_t - \sqrt{2}t - z\rangle} \mid X_t\right] \mid \max X_t - \sqrt{2}t - z > 0\Big] \\
= \ & \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\langle \lambda_1 f + 1 - \mathrm{e}^{-\lambda_2 g}, X_t - \sqrt{2}t - z\rangle - \lambda_3\left(\max X_t - \sqrt{2}t - z\right)} \mid \max X_t - \sqrt{2}t - z > 0\right] \\
\to \ & \mathrm{E}\left[\mathrm{e}^{-\langle \lambda_1 f + 1 - \mathrm{e}^{-\lambda_2 g}, \widetilde{\mathcal{E}}^X_\infty\rangle - \lambda_3 Y}\right], \quad \text{as } t \to +\infty.
\end{aligned} \tag{4.32}
$$

Obviously the right hand side of (4.32) converges to 1 as $\lambda_i \to 0$, $i = 1, 2, 3$. Hence conditioned on $\{\max X_t - \sqrt{2}t - z > 0\}$, $\left(X_t - \sqrt{2}t - z, Z_t - \sqrt{2}t - z, \max X_t - \sqrt{2}t - z\right)$ converges in distribution to a limit $\left(\widetilde{\mathcal{E}}^X_\infty, \widetilde{\mathcal{E}}^Z_\infty, Y\right)$ with Y being an exponential random variable with mean $1/\sqrt{2}$. Moreover, it holds that

$$
\mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\mathcal{E}}^X_\infty\rangle - \langle g, \widetilde{\mathcal{E}}^Z_\infty\rangle - \lambda Y}\right] = \mathrm{E}\left[\mathrm{e}^{-\langle f + 1 - \mathrm{e}^{-g}, \widetilde{\mathcal{E}}^X_\infty\rangle - \lambda Y}\right]
$$

for all $f, g \in C_c^+(\mathbb{R})$ and $\lambda \ge 0$. In particular one has

$$
\mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\mathcal{E}}^X_\infty\rangle - \lambda Y} \mathrm{E}\left[\mathrm{e}^{-\langle g, \widetilde{\mathcal{E}}^Z_\infty\rangle} \mid \widetilde{\mathcal{E}}^X_\infty, Y\right]\right] = \mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\mathcal{E}}^X_\infty\rangle - \lambda Y} \cdot \mathrm{e}^{-\langle 1 - \mathrm{e}^{-g}, \widetilde{\mathcal{E}}^X_\infty\rangle}\right].
$$

This implies that $\mathrm{E}\left[\mathrm{e}^{-\langle g, \widetilde{\mathcal{E}}^Z_\infty\rangle} \mid \widetilde{\mathcal{E}}^X_\infty, Y\right] = \mathrm{e}^{-\langle 1 - \mathrm{e}^{-g}, \widetilde{\mathcal{E}}^X_\infty\rangle}$ P-a.s. So we prove the second conclusion of this lemma. □

Lemma 4.13. *Suppose $x, z \in \mathbb{R}$. Under $\mathbb{P}_{\delta_x}$, conditioned on $\{\max X_t - \sqrt{2}t - z > 0\}$, the random elements $(X_t - \max X_t, Z_t - \max X_t, \max X_t - \sqrt{2}t - z)$ converges, as $t \to +\infty$, in distribution to a limit $(\widetilde{\triangle}^X, \widetilde{\triangle}^Z, Y) := (\widetilde{\mathcal{E}}^X_\infty - Y, \widetilde{\mathcal{E}}^Z_\infty - Y, Y)$, where the limit is independent of x and z, and $(\widetilde{\triangle}^X, \widetilde{\triangle}^Z)$ is independent of Y. Moreover, given $\widetilde{\triangle}^X$, $\widetilde{\triangle}^Z$ is a Poisson random measure with intensity $\widetilde{\triangle}^X$.*

Proof. The first conclusion follows from Lemma 4.12 (in place of Lemma 4.3) in the same way as Lemma 4.4. We only need to show the second conclusion.

Since $\left(\widetilde{\triangle}^X, \widetilde{\triangle}^Z\right)$ is independent of Y, we have for all $f \in C_c^+(\mathbb{R})$,

$$
\begin{aligned}
\mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\triangle}^Z\rangle} \mid \widetilde{\triangle}^X\right] &= \mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\triangle}^Z\rangle} \mid \widetilde{\triangle}^X, Y\right] = \mathrm{E}\left[\mathrm{e}^{-\langle f, \widetilde{\mathcal{E}}^Z_\infty - Y\rangle} \mid \widetilde{\mathcal{E}}^X_\infty, Y\right] \\
&= \mathrm{e}^{-\langle 1 - \mathrm{e}^{-f}, \widetilde{\mathcal{E}}^X_\infty - Y\rangle} = \mathrm{e}^{-\langle 1 - \mathrm{e}^{-f}, \widetilde{\triangle}^X\rangle}.
\end{aligned}
$$

The third equality follows from the second conclusion of Lemma 4.12. Hence we prove the second conclusion. □

Lemma 4.14. *For any $f,g \in C_c^+(\mathbb{R})$,*

$$C\left(f+1-\mathrm{e}^{-g}\right) = \tilde{c}_0 \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\langle f, \widetilde{\triangle}^X + y\rangle - \langle g, \widetilde{\triangle}^Z + y\rangle}\right] dy.$$

Proof. Fix arbitrary $f,g \in C_c^+(\mathbb{R})$. It follows by [23, (3.12)] that

$$C\left(f+1-\mathrm{e}^{-g}\right) = \tilde{c}_0 \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\langle f+1-e^{-g}, \widetilde{\triangle}^X + y\rangle}\right] dy.$$

By Lemma 4.13, we have that

$$\mathrm{E}\left[1-\mathrm{e}^{-\langle f, \widetilde{\triangle}^X + y\rangle - \langle g, \widetilde{\triangle}^Z + y\rangle}\right] = \mathrm{E}\left[1-\mathrm{e}^{-\langle f+1-e^{-g}, \widetilde{\triangle}^X + y\rangle}\right].$$

Now the desired result follows immediately. □

Proof of Theorem 2.16. Using computation on Poisson point process, we have for all $f,g \in C_c^+(\mathbb{R})$,

$$\begin{aligned}
& \mathrm{E}\left[\mathrm{e}^{-\left\langle f, \sum_{i\geq 1} \mathcal{T}_{\tilde{e}_i}\widetilde{\triangle}_i^X\right\rangle - \left\langle g, \sum_{i\geq 1} \mathcal{T}_{\tilde{e}_i}\widetilde{\triangle}_i^Z\right\rangle}\right] \\
= \ & \mathrm{E}\left[\mathrm{e}^{\left\langle \ln \mathrm{E}\left[\mathrm{e}^{-\left\langle f, \widetilde{\triangle}^X + \cdot\right\rangle - \left\langle g, \widetilde{\triangle}^Z + \cdot\right\rangle}\right], \sum_{i\geq 1}\delta_{\tilde{e}_i}\right\rangle}\right] \\
= \ & \mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\tilde{c}_0 \partial M_\infty \int_{-\infty}^{+\infty} \sqrt{2}\mathrm{e}^{-\sqrt{2}y}\mathrm{E}\left[1-\mathrm{e}^{-\left\langle f, \widetilde{\triangle}^X + y\right\rangle - \left\langle g, \widetilde{\triangle}^Z + y\right\rangle}\right] dy}\right].
\end{aligned}$$

This together with Theorem 2.10 and Lemma 4.14 yields that

$$\mathrm{E}\left[\mathrm{e}^{-\langle f, \mathcal{E}_\infty\rangle - \langle g, \mathcal{E}_\infty^Z\rangle}\right] = \mathrm{E}\left[\mathrm{e}^{-\left\langle f, \sum_{i\geq 1} \mathcal{T}_{\tilde{e}_i}\widetilde{\triangle}_i^X\right\rangle - \left\langle g, \sum_{i\geq 1} \mathcal{T}_{\tilde{e}_i}\widetilde{\triangle}_i^Z\right\rangle}\right], \quad \forall f,g \in C_c^+(\mathbb{R}).$$

Hence we complete the proof. □

A Appendix

Lemma A.1. *Let L be the integer-valued random variable with distribution $\{p_k : k \geq 2\}$ as defined in Proposition 2.2. Suppose $f : [0,+\infty) \to [0,+\infty)$ satisfies the following conditions: There exist some constants $c, \kappa \geq 0$ such that*

(1) f is bounded in $[0,c)$ and convex on $[c,+\infty)$.

(2) $f(xy) \leq \kappa f(x) f(y)$ for all $x,y \in [c,+\infty)$.

Then the following statements are equivalent.

(i) $\mathrm{E}\left[f(L)\right] < +\infty$.

(ii) $\mathbb{P}_{\cdot, k\delta_0}\left[f(\|Z_t\|)\right] < \infty$ for all $t > 0$ and $k \in \mathbb{N}$.

(iii) $\mathbb{P}_\mu\left[f(\|Z_t\|)\right] < \infty$ for all $t > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$.

(iv) $\mathbb{P}_\mu\left[f(\|X_t\|)\right] < \infty$ for all $t > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$.

(v) $\int_{(1,+\infty)} f(x)\pi(dx) < +\infty$.

Proof. Without loss of generality we may and do assume that the function $f : [0,+\infty) \mapsto [0,+\infty)$ satisfies that there is some $\kappa > 0$ such that

(1') f is convex on $[0,+\infty)$;

(2') $f(xy) \leq \kappa f(x) f(y)$ for all $x, y \in [0,+\infty)$;

(3') f is nondecreasing on $[0,+\infty)$ and $f(x) > 1$ for all $x \geq 0$.

In fact, [5, Chapter IV, Lemma 1] shows that for any function f which satisfies the original hypothesis, there is a function $\tilde{f}$ satisfying (1')-(3') such that for any probability measure μ on $[0,+\infty)$, $\int_{[0,+\infty)} f(x)\mu(dx)$ is finite if and only if $\int_{[0,+\infty)} \tilde{f}(x)\mu(dx)$ is finite.

That $(i) \Leftrightarrow (ii)$ is established in [5, Chapter III, Theorem 2]. We note that $(\|X_t\|)_{t\geq 0}$ is a continuous-state branching process. Thus $(iv) \Leftrightarrow (v)$ follows directly from [14, Theorem 2.1].

(ii) $\Leftrightarrow$ (iii): Since the branching rate and offspring distribution of $(Z_t)_{t\geq 0}$ is spatially-independent, $(\|Z_t\|)_{t\geq 0}$ is a continuous-time Galton-Watson branching process. Thus we have for any nontrivial $\mu \in \mathcal{M}_c(\mathbb{R})$,

$$\mathbb{P}_\mu\left[f(\|Z_t\|)\right] = \mathbb{P}_\mu\left[\mathbb{P}_\mu\left[f(\|Z_t\|)|Z_0\right]\right] = \mathbb{P}_\mu\left[\mathbb{P}_{\cdot,\|Z_0\|\delta_0}\left[f(\|Z_t\|)\right]\right].$$

Note that $(\|Z_0\|, \mathbb{P}_\mu)$ is a Poinsson random variable with parameter $\|\mu\|$. Thus we have

$$\mathbb{P}_\mu\left[f(\|Z_t\|)\right] = \sum_{k=0}^{+\infty} \frac{\|\mu\|^k}{k!} \mathrm{e}^{-\|\mu\|} \mathbb{P}_{\cdot,k\delta_0}\left[f(\|Z_t\|)\right]. \tag{A.1}$$

Hence $\mathbb{P}_\mu\left[f(\|Z_t\|)\right] < +\infty$ if and only if $\mathbb{P}_{\cdot,k\delta_0}\left[f(\|Z_t\|)\right] < +\infty$ for all $k \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv): Fix $t > 0$ and a nontrivial $\mu \in \mathcal{M}_c(\mathbb{R})$. Suppose that $\mathbb{P}_\mu\left[f(\|Z_t\|)\right] < +\infty$. Recall that given X_t, Z_t is a Poisson random measure with intensity $X_t(dx)$, and so $\|Z_t\|$ is a Poisson random variable with parameter $\|X_t\|$. Thus we get

$$\mathbb{P}_\mu\left[\|Z_t\||X_t\right] = \|X_t\| \quad \mathbb{P}_\mu\text{-a.s.}$$

Since f in convex on $[0,+\infty)$, it follows by Jensen's inequality that

$$\mathbb{P}_\mu\left[f(\|X_t\|)\right] = \mathbb{P}_\mu\left[f\left(\mathbb{P}_\mu\left[\|Z_t\||X_t\right]\right)\right] \leq \mathbb{P}_\mu\left[\mathbb{P}_\mu\left[f(\|Z_t\|)|X_t\right]\right] = \mathbb{P}_\mu\left[f(\|Z_t\|)\right] < +\infty.$$

(iv) $\Rightarrow$ (iii): Fix $t, s > 0$ and a nontrivial $\mu \in \mathcal{M}_c(\mathbb{R})$. Suppose $\mathbb{P}_\mu\left[f(\|X_{t+s}\|)\right] < +\infty$. It follows by (3.14) that

$$\|X_{t+s}\| = \|X^*_{t+s}\| + \|I^{*,t}_s\| + \sum_{u\in Z_t} \|I^{(u)}_s\|.$$

We have

$$\begin{aligned}
\mathbb{P}_\mu\left[f(\|X_{t+s}\|)\right] &\geq \mathbb{P}_\mu\left[f\left(\sum_{u\in Z_t}\|I^{(u)}_s\|\right)\right] = \mathbb{P}_\mu\left[\mathbb{P}_\mu\left[f\left(\sum_{u\in Z_t}\|I^{(u)}_s\|\right)|Z_t\right]\right] \\
&\geq \mathbb{P}_\mu\left[f\left(\mathbb{P}_\mu\left[\sum_{u\in Z_t}\|I^{(u)}_s\||Z_t\right]\right)\right] = \mathbb{P}_\mu\left[f\left(\sum_{u\in Z_t}\mathbb{P}_{\cdot,\delta_{z_u(t)}}\left[\|I_s\|\right]\right)\right].
\end{aligned}$$

The first inequality follows from condition (3'), the second inequality from Jensen's inequality. We observe that the distribution of $\|I_s\|$ under $\mathbb{P}_{\cdot,\delta_x}$ is independent of the starting location x. If we define $g(s) := \mathbb{P}_{\cdot,\delta_x}\left[\|I_s\|\right]$ for all $s > 0$, then we get from the above argument that

$$\mathbb{P}_\mu\left[f\left(g(s)\|Z_t\|\right)\right] = \mathbb{P}_\mu\left[f\left(\sum_{u\in Z_t} g(s)\right)\right] \leq \mathbb{P}_\mu\left[f(\|X_{t+s}\|)\right] < +\infty.$$

Thus by (2′) we have

$$\mathbb{P}_\mu\left[f(\|Z_t\|)\right] = \mathbb{P}_\mu\left[f(g(s)g(s)^{-1}\|Z_t\|)\right] \le \kappa f(g(s)^{-1})\mathbb{P}_\mu\left[f(g(s)\|Z_t\|)\right] < +\infty.$$

Therefore we complete the proof. □

Lemma A.2. *Assumption 2.4 holds for some $0 < \beta < 1$ if and only if*

$$\int_0^1 \left(1 + \psi'(s)\right) s^{-(1+\beta)} ds < +\infty. \tag{A.2}$$

Proof. Let L be the integer-valued random variable with distribution $\{p_k : k \ge 2\}$. It follows from Lemma A.1 that Assumption 2.4 holds if and only if

$$\mathrm{E}\left[L^{1+\beta}\right] < +\infty. \tag{A.3}$$

Let $\varphi(s)$ be the Laplace transform of L, that is, $\varphi(s) := \mathrm{E}\left[\mathrm{e}^{-sL}\right]$ for all $s \ge 0$. Then by [8, Theorem B], (A.3) is equivalent to that

$$\int_0^1 f_1(s) s^{-(2+\beta) ds} < +\infty, \tag{A.4}$$

where $f_1(s) := \varphi(s) - 1 - \varphi'(0)s$. We use $F(s)$ to denote the generating function of L, i.e., $F(s) = \mathrm{E}\left[s^L\right]$ for $s \in [0,1]$. Since $\varphi(s) = F(\mathrm{e}^{-s})$ for all $s \ge 0$, setting $u = 1 - \mathrm{e}^{-s}$ we have

$$\varphi''(s) = F''(1-u)(1-u)^2 + F'(1-u)(1-u).$$

Since $u \sim s$ as $s \to 0$, one has

$$f_1(s) \sim \frac{s^2}{2}\varphi''(s) \sim \frac{u^2}{2}\left[F''(1-u)(1-u)^2 + F'(1-u)(1-u)\right] \sim c_1 u^2 F''(1-u) \quad \text{as } s \to 0, \tag{A.5}$$

for some constant $c_1 > 0$. On the other hand, in view of (2.1), one has $qF''(1-u) = \psi''(u)$. This together with (A.5) implies that

$$f_1(s) \sim c_2 u^2 \psi''(u) \sim c_2 u \left(\psi'(u) - \psi'(0)\right) = c_2 u \left(\psi'(u) + 1\right) \quad \text{as } s \to 0,$$

for some constant $c_2 > 0$. Hence we have $f_1(s)s^{-(2+\beta)} \sim c_2 u^{-(1+\beta)}(\psi'(u) + 1)$ as $s \to 0$. So (A.4) holds if and only if (A.2) holds. □

References

[1] Aïdékon, E., Berestycki, J., Brunet, É. and Shi, Z.: Branching Brownian motion seen from its tip, *Probab. Theory Related Fields*, **157** (2013), 405–451. MR3101852

[2] Arguin, L.-P., Bovier, A. and Kistler, N.: Genealogy of extremal particles of branching Brownian motion, *Comm. Pure Appl. Math.*, **64** (2011), 1647–1676. MR2838339

[3] Arguin, L.-P., Bovier, A. and Kistler, N.: Poissonian statistics in the extremal process of branching Brownian motion, *Ann. Appl. Probab.*, **22** (2012), 1693–1711. MR2985174

[4] Arguin, L.-P., Bovier, A. and Kistler, N.: The extremal process of branching Brownian motion, *Probab. Theory Related Fields*, **157** (2013), 535–574. MR3129797

[5] Athreya, K. and Ney, P.: Branching Processes, Springer-Verlag, Berlin, 1972. MR0373040

[6] Berestycki, J., Brunet, E., Cortines, A. and Mallein, B.: A simple backward construction of branching Brownian motion with large displacement and applications, *Ann. Instit. Henri Poincaré Probab. Stat.*, **58**(4) (2022), 2094–2113. MR4492972

[7] Berestycki, J., Kyprianou, A. E. and Murillo-Salas, A.: The prolific backbone for supercritical superprocesses, *Stoch. Process. Appl.* **121** (2011), 1315–1331. MR2794978

[8] Bingham, N. H. and Doney, R. A.: Asymptotic properties of supercritical branching processes I: The Galton-Watson process, *Adv. Appl. Prob.*, **6** (1974), 711–731. MR0362525

[9] Bramson, M.: Maximal displacement of branching Brownian motion, *Comm. Pure Appl. Math.*, **31** (1978), 531–581. MR0494541

[10] Bramson, M.: Convergence of solutions of the Kolmogorov equation to travelling waves, Mem. Amer. Math. Soc. **44** (1983), iv+190. MR0705746

[11] Chen, Z.-Q., Ren, Y.-X. and Yang, T.: Skeleton decomposition and law of large numbers for supercritical superprocesses, *Acta Appl. Math.*, **159**(1) (2019), 225–285. MR3904489

[12] Dynkin, E.: Superprocesses and partial differential equations, *Ann. Probab.*, **21**(3) (1993), 1185–1262. MR1235414

[13] Eckhoff, M., Kyprianou, A. E. and Winkel, M.: Spines, skeletons and the strong law of large numbers for superdiffusions, *Ann. Probab.*, **43**(5) (2015), 2545–2610. MR3395469

[14] Ji, L. and Li, Z.: Moments of continuous-state branching processes with or without immigration, *Acta Math. Appl. Sin. Engl. Ser.*, **36**(2) (2020), 361–373. MR4083449

[15] Kallenberg, O.: Random Measures, 4th ed., Akademie-Verlag and Academic Press, Berlin and London, 1986. MR0854102

[16] Kallenberg, O.: Random Measures, Theory and Applications, Springer, Cham, 2017. MR3642325

[17] Kyprianou, A. E.: Traveling wave solutions to the K-P-P equation: alternatives to Simon Harris' probability analysis, *Ann. Inst. H. Poincaré Probab. Statist.*, **40** (2004), 53–72. MR2037473

[18] Kyprianou, A. E., Liu, R.-L., Murillo-Salas, A. and Ren, Y.-X.: Supercritical super-Brownian motion with a general branching mechanism and travelling waves, *Ann. Inst. Henri Poincaré Probab. Stat.*, **48**(3) (2012), 661–687. MR2976558

[19] Kyprianou, A. E., Pérez, J.-L. and Ren, Y.-X.: The backbone decomposition for a general spatially dependent supercritical superprocesses, In: Lecture Notes in Mathematics, Séminaire de Probabilités, vol. 2123, **46** (2014), 33–59. MR3330813

[20] Li, Z.-H.: Measure-valued Branching Markov Processes, Springer, Heidelberg, 2011. MR2760602

[21] Lalley, S. P. and Sellke, T.: A conditional limit theorem for the frontier of a branching Brownian motion, *Ann. Probab.*, **15** (1987), 1052–1061. MR0893913

[22] Maillard, M.: A note on stable point processes occurring in branching Brownian motion, *Electron. Commun. Probab.*, **18**(5) (2013), 1–9. MR3019668

[23] Ren, Y.-X., Song, R., and Zhang, R.: The extremal process of super-Brownian motion, *Stoch. Proc. Appl.*, **137** (2021), 1–34. MR4242982

[24] Ren, Y.-X. and Yang, T.: Limiting distributions for a class of super-Brownian motions with spatially dependent branching mechanisms, *J. Theor. Probab.* (2023), https://doi.org/10.1007/s10959-023-01304-2.

[25] Roberts, M. I.: A simple path to asymptotics for the frontier of a branching Brownian motion, *Ann. Probab.*, **41** (2013), 3518–3541. MR3127890

[26] Sheu, Y.-C.: Lifetime and compactness of range for super-Brownian motion with general branching mechanism, *Stoch. Proc. Appl.*, **70** (1997), 129–141. MR1472962

[27] Yang, T. and Ren, Y.-X.: Limit theorem for derivative martingale at criticality w.r.t branching Brownian motion, *Statist. Probab. Lett.*, **81**(2) (2011), 195–200. MR2748182

Acknowledgments. The authors sincerely thank the anonymous reviewer for the valuable comments and suggestions that have led to the present improved version of the original manuscript.

Electronic Journal of Probability
Electronic Communications in Probability

Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS[1])
- Easy interface (EJMS[2])

Economical model of EJP-ECP

- Non profit, sponsored by IMS[3], BS[4], ProjectEuclid[5]
- Purely electronic

Help keep the journal free and vigorous

- Donate to the IMS open access fund[6] (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

[1]LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
[2]EJMS: Electronic Journal Management System: https://vtex.lt/services/ejms-peer-review/
[3]IMS: Institute of Mathematical Statistics http://www.imstat.org/
[4]BS: Bernoulli Society http://www.bernoulli-society.org/
[5]Project Euclid: https://projecteuclid.org/
[6]IMS Open Access Fund: https://imstat.org/shop/donation/