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Double jump in the maximum of two-type reducible branching Brownian motion

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Abstract. Consider a two-type reducible branching Brownian motion in which particles' diffusion coefficients and branching rates are influenced by their types. Here reducible means that type 1 particles can produce particles of type 1 and type 2, but type 2 particles can only produce particles of type 2. The maximum of this process is determined by two parameters: the ratio of the diffusion coefficients and the ratio of the branching rates for particles of different types. Belloum and Mallein (Electron. J. Probab. **26** (2021) 61) identified three phases of the maximum and the extremal process, corresponding to three regions in the parameter space.

We investigate how the extremal process behaves asymptotically when the parameters lie on the boundaries between these regions. An interesting phenomenon is that a double jump occurs in the maximum when the parameters cross the boundary of the so called anomalous spreading region, while only a single jump occurs when the parameters cross the boundary between the remaining two regions.

Résumé. Considérons un mouvement brownien branchant à deux types, réductible, dans lequel les coefficients de diffusion et les taux de branchement des particules dépendent de leur type. Ici, "réductible" signifie que les particules de type 1 peuvent produire des particules de type 1 et de type 2, alors que les particules de type 2 ne peuvent produire que des particules de type 2. Le maximum de ce processus est déterminé par deux paramètres : le rapport des coefficients de diffusion et le rapport des taux de branchement pour les particules de différents types. Belloum et Mallein (Electron. J. Probab. **26** (2021) 61) ont identifié trois phases du maximum et du processus extrémal, correspondant à trois régions dans l'espace des paramètres.

Nous étudions le comportement asymptotique du processus extrémal lorsque les paramètres se situent sur les frontières entre ces régions. Un phénomène intéressant est qu'un double saut se produit dans le maximum lorsque les paramètres franchissent la frontière de la région dite de diffusion anormale, tandis qu'un simple saut se produit lorsque les paramètres franchissent la frontière entre les deux autres régions.

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1. Introduction

Over the last few years, many people have studied the extreme values of the so-called *log-correlated* fields, which form a large universality class for the distributions of extreme values of correlated stochastic processes. One of the simple models in this class is the branching Brownian motion (BBM), which can be described as follows. Initially we have a particle moving as a standard Brownian motion. At rate 1 it splits into two particles. These particles behave independently of each other, continue to move and split, subject to the same rule.

Due to the presence of a tree structure and Brownian trajectories, many precise results for the extreme values of BBM were obtained. Bramson [27,28] gave the correct order of the maximum for BBM and the convergence in law of the centered maximum. Lalley and Sellke [42] obtained a probabilistic representation of the limit distribution. A remarkable contribution for the extreme value statistics is the construction of the limiting *extremal process* for BBM, obtained in Arguin, Bovier and Kistler [9], as well as in Aïdékon, Berestycki, Brunet and Shi [3]. With motivation from disordered systems [25,26], several works studied the extreme value for variable speed BBMs see, for example, [21,22,36,48]. Many results on BBM were extended to branching random walks [2,47], and other log-correlated fields, such as 2-dimensional

discrete Gaussian free fields [18,19,29,30], log-correlated Gaussian fields on d-dimensional boxes [1,33,45], and high values of the Riemann zeta-function [6,7,10]. For recent reviews see, e.g. [5,11].

This article concerns the extreme values of a *multi-type* branching Brownian motion, in which particles of different types have different branching mechanisms and diffusion coefficients. Just like in the case of Markov chains, we say a multi-type branching Brownian motion is *reducible* if particles of some type i can not have a descendant of some other type j; and otherwise call it *irreducible*. For irreducible multi-type BBMs (with a common diffusion coefficient for all types), the spreading speed was given by Ren and Yang [52], and recently Hou, Ren and Song [39] obtained the precise order of the maximum and the limiting extremal process. For reducible multi-type branching random walks (BRWs), Biggins [16,17] studied the leading coefficient of the maximum and found that in some cases these processes exhibit the so-called *anomalous spreading* phenomenon. More precisely, the leading coefficient of the maximum for a multi-type BRW is larger than that of a BRW consisting only of particles of a single type. Holzer [37,38] extended results of Biggins to the BBM setting, by studying the associated system of F-KPP equations.

Belloum and Mallein [13] studied the extremal process of a two-type reducible BBM and obtained in particular the precise order of the maximum. One can construct this process by first running a BBM with branching rate β and diffusion coefficient σ^2 (type 1 BBM), and then adding standard BBMs (type 2 BBMs) along each (β,σ^2) BBM path according to a Poisson process. There are three distinct regimes: type 1/type 2 domination, and anomalous spreading, corresponding to the parameters (β,σ^2) belonging to $\mathscr{C}_I$, $\mathscr{C}_{II}$, and $\mathscr{C}_{III}$, respectively (see Figure 1). However when the parameters (β,σ^2) are on the boundaries between these three sets, the precise order of the maximum and the behavior of extremal process are not clear, except the common intersection of these boundaries which was studied by Belloum [12].

In this article, we study the asymptotic behavior of extremal particles in the two-type reducible BBM above when the parameters (β,σ^2) lie on the boundaries between $\mathscr{C}_I$, $\mathscr{C}_{II}$, $\mathscr{C}_{III}$. We show that the extremal process converges in law towards a decorated Poisson point process and give the precise order of the maximum. Combined with the main results in [12,13], the phase diagram of the two-type reducible BBM is now complete and clear. As an interesting by-product, a *double jump* occurs in the maximum of the two-type reducible BBM when the parameters (β,σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$, and only a single jump occurs when the parameters cross the boundary between $\mathscr{C}_I$, $\mathscr{C}_{II}$.

1.1. *Standard branching Brownian motion*

Let $\{(\mathsf{X}_u(t), u\in\mathsf{N}_t)_{t\geq 0}, \mathsf{P}\}$ be a standard BBM, where N_t represents the set of all particles alive at time t, $\mathsf{X}_u(t)$ denotes the position of individual $u\in\mathsf{N}_t$, and P is the corresponding law of the BBM. Let $\mathsf{M}_t=\max_{u\in\mathsf{N}_t}\mathsf{X}_u(t)$ be the maximal displacement among all the particles alive at time t. Bramson [27,28] obtained an explicit asymptotic formula of M_t: If let $m_t:=\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t$, then $(\mathsf{M}_t-m_t : t>0)$ converges weakly, and the cumulative distribution function of the limit distribution is the unique (up to transition) travelling wave solution of a certain F-KPP equation. Lalley and Sellke [42] improved this result and they proved that M_t-m_t converges weakly to a random shift of the Gumbel distribution. Specifically, they showed that for some constant $C_\star>0$,

$$\lim_{t\to\infty}\mathsf{P}(\mathsf{M}_t-m_t\leq x)=\mathsf{E}\big[\exp\big\{-C_\star\mathsf{Z}_\infty e^{-\sqrt{2}x}\big\}\big], \tag{1.1}$$

where $\mathsf{Z}_\infty>0$ is the a.s. limit of the *derivative martingale* $(\mathsf{Z}_t)_{t>0}$ defined by $\mathsf{Z}_t=\sum_{u\in\mathsf{N}_t}(\sqrt{2}t-\mathsf{X}_u(t))e^{\sqrt{2}\mathsf{X}_u(t)-2t}$. The name derivative martingale comes from the fact that $\mathsf{Z}_t=-\frac{\partial}{\partial\lambda}|_{\lambda=\sqrt{2}}\mathsf{W}_t(\lambda)$, where $\mathsf{W}_t(\lambda):=\sum_{u\in\mathsf{N}_t}e^{\lambda X_u(t)-(1+\frac{\lambda^2}{2})t}$ is called the *additive martingale* for BBM. Aïdékon [2] proved the convergence in law of the centered maximum of branching random walks by probabilistic method.

After the maximum of the process was known, many researches focused on the full extreme value statistics for BBM, which can be encoded by the following point process, called *extremal process*

$$\mathcal{E}_t:=\sum_{u\in\mathsf{N}_t}\delta_{\mathsf{X}_u(t)-m_t}.$$

It was proved independently by Arguin, Bovier, Kistler [9], and Aïdékon, Berestycki, Brunet, Shi [3] that $\mathcal{E}_t$ converges in law to a random shifted *decorated Poisson point process* (DPPP for short) defined in the following paragraph. Bovier and Hartung [23] extended the convergence of the extremal process by adding an extra dimension that encodes the localization of the particle in the underlying Galton–Watson tree. We refer to [32] for further results on the structure of the extremal process; and to [31] for some conjectures, many of them still have not been rigorously proven yet.

A DPPP $\mathcal{E}$ is determined by an intensity measure μ and a decoration process $\mathfrak{D}$, where μ is a (random) measure on $\mathbb{R}$ and $\mathfrak{D}$ is the law of a random point process on $\mathbb{R}$. Conditioned on μ, sampling a Poisson point process $\sum_i\delta_{x_i}$ with

intensity μ, and an independent family of i.i.d. point processes $(\sum_j \delta_{d_j^i} : i \geq 0)$ with law $\mathfrak{D}$, then the point measure $\mathcal{E} \sim \mathrm{DPPP}(\mu, \mathfrak{D})$ can be constructed as $\mathcal{E} = \sum_{i,j} \delta_{x_i + d_j^i}$.

Using this notation, the main result in [3] and [9] can be stated as:

$$\lim_{t\to\infty} \sum_{u\in\mathsf{N}_t} \delta_{\mathsf{X}_u(t)-m_t} = \mathrm{DPPP}\big(\sqrt{2}C_\star \mathsf{Z}_\infty \mathrm{e}^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\big) \quad \text{in law in the vague topology.}^{1} \tag{1.2}$$

The decoration law $\mathfrak{D}^{\sqrt{2}}$ belongs to the family $(\mathfrak{D}^\varrho, \varrho \geq \sqrt{2})$, defined as the limits of the "gap processes" conditioned on $\mathsf{M}_t \geq \varrho t$:

$$\mathfrak{D}^\varrho(\cdot) := \lim_{t\to\infty} \mathsf{P}\bigg(\sum_{u\in\mathsf{N}_t} \delta_{\mathsf{X}_u(t)-\mathsf{M}_t} \in \cdot \;\Big|\; \mathsf{M}_t \geq \varrho t \bigg). \tag{1.3}$$

The existence of this limit is established in [9, Theorem 3.4] for $\rho = \sqrt{2}$ and in [21, Proposition 7.5] for $\rho > \sqrt{2}$. See also [14] for an alternative proof using the spine decomposition techniques. In [14], these point processes $(\mathfrak{D}^\rho : \rho \geq \sqrt{2})$ were used as decorations in the extremal processes of 2-speed BBMs.

1.2. *Two-type reducible branching Brownian motion*

Now we give the definition of a two-type reducible branching Brownian motion, which is the model we are going to study in this paper. The difference between our two-type reducible BBM and the standard BBM is that in our two-type BBM, each particle now has a type and the branching and movement depend on the type. Specifically, type 1 particles move according to a Brownian motion with diffusion coefficient σ^2. They branch at rate β into two children of type 1 and give birth to particles of type 2 at rate α. Type 2 particles move according to a standard Brownian motion and branch at rate 1 into 2 children of type 2, but can not give birth to offspring of type 1. We use N_t to represent all particles alive at time t, as well as N_t^1 and N_t^2 for particles of type 1 and type 2 alive at time t respectively. For $u \in N_t$ and $s \leq t$, let $X_u(s)$ be the position of the ancestor at time s of particle u. So we write $\{(X_u(t), u \in N_t)_{t\geq 0}, \mathbb{P}\}$ for a two-type reducible BBM and $M_t := \max_{u\in N_t} X_u(t)$ for its maximum.

As mentioned at the beginning of the paper, Biggins [16,17] found that anomalous spreading may occur for multi-type reducible BRWs. Belloum and Mallein [13] studied more details on the precise order of the maximum and the extremal process for this two-type BBM. Especially in the case when anomalous spreading occurs, they showed that the extremal process, formed by type 2 particles at time t, converges towards a DPPP weakly.

To describe the limiting extremal process in the form of (1.2), we introduce the additive and derivative martingales of type 1 particles. Note that particles $\{X_u(t) : u \in N_t^1\}$ of type 1 alone have the same law as the BBM with branching rate β and diffusion coefficient σ^2, which is denoted by $(\mathsf{X}_u^{\beta,\sigma^2}(t) : u \in \mathsf{N}_t)_{t\geq 0}$ (here N_t is a slight abuse of notation as it depends also on β). We write X for the standard BBM $\mathsf{X}^{1,1}$. Using the scaling property of Brownian motion, we have

$$\big(\mathsf{X}_u^{\beta,\sigma^2}(t) : u \in \mathsf{N}_t\big) \overset{\text{law}}{=} \bigg(\frac{\sigma}{\sqrt{\beta}} \mathsf{X}_u(\beta t) : u \in \mathsf{N}_{\beta t} \bigg). \tag{1.4}$$

So, by the results for standard BBM, the corresponding derivative martingale of type 1 individuals and its a.s. limit are given by

$$Z_t^{(1)} := \sum_{u\in N_t^1} \big(\sqrt{2\beta\sigma^2}t - X_u(t)\big) e^{\sqrt{2\beta/\sigma^2}X_u(t) - 2\beta t} \quad \text{and} \quad Z_\infty^{(1)} = \lim_{t\to\infty} Z_t^{(1)} > 0 \quad \text{a.s.} \tag{1.5}$$

The corresponding additive martingales of type 1 individuals and their a.s. limits are given by

$$W_t^{(1)}(\lambda) := \sum_{u\in N_t^1} e^{\lambda X_u(t) - (\frac{\lambda^2\sigma^2}{2} + \beta)t} \quad \text{and} \quad W_\infty^{(1)}(\lambda) = \lim_{t\to\infty} W_t^{(1)}(\lambda) \quad \text{a.s. for } \lambda \in \mathbb{R}. \tag{1.6}$$

[1] That is to say, for every continuous test function ϕ with compact support, $\int \phi \,\mathrm{d}\mathcal{E}_t$ converges weakly to $\int \phi \,\mathrm{d}\mathcal{E}$. In this paper when dealing with the weak convergence of point process, we always assume that we are using this setting.

Divide the parameter space $(\beta,\sigma^2)\in\mathbb{R}_+^2$ into three regions (see Figure 1):

$$\mathscr{C}_I=\left\{(\beta,\sigma^2):\sigma^2>\frac{1}{\beta}1_{\{\beta\le1\}}+\frac{\beta}{2\beta-1}1_{\{\beta>1\}}\right\},$$
$$\mathscr{C}_{II}=\left\{(\beta,\sigma^2):\sigma^2<\frac{1}{\beta}1_{\{\beta\le1\}}+(2-\beta)1_{\{\beta>1\}}\right\},$$
$$\mathscr{C}_{III}=\left\{(\beta,\sigma^2):\sigma^2+\beta>2\text{ and }\sigma^2<\frac{\beta}{2\beta-1}\right\}.$$

The main results in [13] are as follows. Recall the constant $C_\star$ in (1.1), (1.2), and the decorations $(\mathfrak{D}^\varrho)_{\rho\ge\sqrt{2}}$ in (1.3). Let $v:=\sqrt{2\beta\sigma^2}$ and $\theta:=\sqrt{2\beta/\sigma^2}$.

- If $(\beta,\sigma^2)\in\mathscr{C}_I$, then $M_t=vt-\frac{3}{2\theta}\log t+O_{\mathbb{P}}(1)$. For some constant C and some decoration law $\mathfrak{D}_{(I)}$ obtained implicitly in [13], we have
$$\lim_{t\to\infty}\sum_{u\in N_t^2}\delta_{X_u(t)-vt+\frac{3}{2\theta}\log t}=\mathrm{DPPP}\big(\theta CZ_\infty^{(1)}e^{-\theta x}\,\mathrm{d}x,\mathfrak{D}_{(I)}\big)\quad\text{in law.}$$
- If $(\beta,\sigma^2)\in\mathscr{C}_{II}$, then $M_t=\sqrt{2}t-\frac{3}{2\sqrt{2}}\log t+O_{\mathbb{P}}(1)$. There is some random variable $\bar{Z}_\infty\in(0,\infty)$ (see Lemma 5.3) such that
$$\lim_{t\to\infty}\sum_{u\in N_t^2}\delta_{X_u(t)-\sqrt{2}t+\frac{3}{2\sqrt{2}}\log t}=\mathrm{DPPP}\big(\sqrt{2}C_\star\bar{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x,\mathfrak{D}^{\sqrt{2}}\big)\quad\text{in law.}$$
- If $(\beta,\sigma^2)\in\mathscr{C}_{III}$, then $M_t=v^*t+O_{\mathbb{P}}(1)$, where $v^*=\frac{\beta-\sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}}>\max(v,\sqrt{2})$. For $\theta^*=\sqrt{2\frac{\beta-1}{1-\sigma^2}}$ and $C=\frac{\alpha C(\theta^*)}{2(\beta-1)}$ (where $C(\theta^*)$ is defined in Lemma 2.4(ii)), we have
$$\lim_{t\to\infty}\sum_{u\in N_t^2}\delta_{X_u(t)-v^*t}=\mathrm{DPPP}\big(\theta^*CW_\infty^{(1)}(\theta^*)e^{-\theta^*x}\,\mathrm{d}x,\mathfrak{D}^{\theta^*}\big)\quad\text{in law.}$$

Moreover, Belloum [12] showed that

- If $(\beta,\sigma^2)=(1,1)$, then $M_t=\sqrt{2}t-\frac{1}{2\sqrt{2}}\log t+O_{\mathbb{P}}(1)$. The extremal process
$$\lim_{t\to\infty}\sum_{u\in N_t^2}\delta_{X_u(t)-\sqrt{2}t+\frac{1}{2\sqrt{2}}\log t}=\mathrm{DPPP}\big(\sqrt{2}C_\star Z_\infty^{(1)}e^{-\sqrt{2}x}\,\mathrm{d}x,\mathfrak{D}^{\sqrt{2}}\big)\quad\text{in law.}$$

The above results were explained by Belloum and Mallein [13] as follows: If $(\beta,\sigma^2)\in\mathscr{C}_{III}$, the leading coefficient v^* is larger than $\max(\sqrt{2},v)$, and the extremal process is given by a mixture of the long-time behavior of the processes of particles of type 1 and 2. In other words, an extremal particle has an ancestral line that has been of type 1 for a positive proportion of the time, and also of type 2 for a positive proportion of the time. If $(\beta,\sigma^2)\in\mathscr{C}_i$ for $i=I,II$, the order of M_t is the same as a single BBM of particles of type $i=1,2$, and the extremal process is dominated by the long-time behavior of particles of type $i=1,2$, respectively.

The aim of this article is to obtain the asymptotic behavior of the maximum and the extremal process of the two-type branching process when parameters (β,σ^2) are on the boundaries between $\mathscr{C}_I,\mathscr{C}_{II},\mathscr{C}_{III}$, except the point $(1,1)$. In these cases there were some conjectures in [13, Section 2,4]. Our main results confirm that the conjectures are true with some coefficients being corrected and we also provide results for the case where no conjecture was made.

1.3. *Main results*

In the statements of our theorems, we continue to use the notation introduced earlier: the constant $C_\star$ in (1.2), the decorations $(\mathfrak{D}^\varrho:\rho\ge\sqrt{2})$ in (1.3), the derivative martingale $Z_\infty^{(1)}$ and additive martingales $W_\infty^{(1)}(\lambda)$ in (1.5) and (1.6). We note that, since $\sqrt{2}<\theta$, we have $W_\infty^{(1)}(\sqrt{2})>0$ almost surely. Denote by

$$\mathscr{B}_{I,II}=\partial\mathscr{C}_I\cap\partial\mathscr{C}_{II}\backslash\big\{(1,1)\big\}=\big\{(\beta,\sigma^2):\beta\sigma^2=1,\beta<1\big\};$$

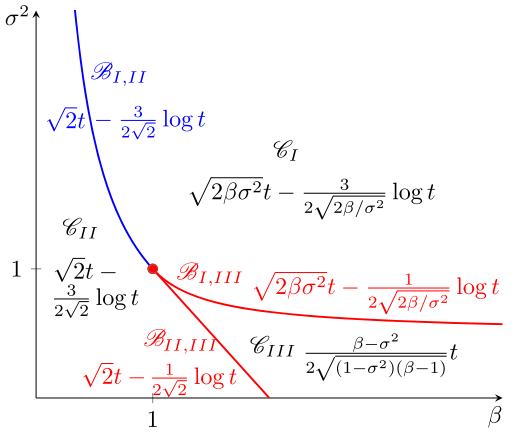


Fig. 1. Phase diagram for the maximum of two type reducible BBM.

$$\mathscr{B}_{II,III} = \partial\mathscr{C}_{II} \cap \partial\mathscr{C}_{III} \backslash \{(1,1)\} = \{(\beta,\sigma^2) : \beta + \sigma^2 = 2, \beta > 1\};$$
$$\mathscr{B}_{I,III} = \partial\mathscr{C}_{I} \cap \partial\mathscr{C}_{III} \backslash \{(1,1)\} = \left\{(\beta,\sigma^2) : \frac{1}{\beta} + \frac{1}{\sigma^2} = 2, \beta < 1\right\}.$$

See Figure 1 for an illustration.

Theorem 1.1 (Boundary between $\mathscr{C}_{II}$, $\mathscr{C}_{III}$). *Assume that $(\beta,\sigma^2) \in \mathscr{B}_{II,III}$. Let*

$$m_t^{2,3} := \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t.$$

Then for the constant $C = \frac{\alpha C_\star}{\sqrt{2}(1-\sigma^2)} > 0$, we have

$$\lim_{t\to\infty} \sum_{u\in N_t} \delta_{X_u(t)-m_t^{2,3}} = \lim_{t\to\infty} \sum_{u\in N_t^2} \delta_{X_u(t)-m_t^{2,3}} = \mathrm{DPPP}\big(\sqrt{2}C W_\infty^{(1)}(\sqrt{2}) e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\big) \quad \textit{in law}.$$

Theorem 1.1 confirms Conjecture 2.2 of Belloum and Mallein [13] with the constant before $\log t$ being corrected as $\frac{1}{2\sqrt{2}}$, and the random variable $\widetilde{Z}$ there taken to be $W_\infty^{(1)}(\sqrt{2})$.

Recall that throughout this paper, we set

$$v := \sqrt{2\beta\sigma^2} \quad \text{and} \quad \theta := \sqrt{2\beta/\sigma^2}.$$

Theorem 1.2 (Boundary between $\mathscr{C}_I$, $\mathscr{C}_{III}$). *Assume that $(\beta,\sigma^2) \in \mathscr{B}_{I,III}$. Let*

$$m_t^{1,3} := vt - \frac{1}{2\theta}\log t.$$

Then we have

$$\lim_{t\to\infty} \sum_{u\in N_t} \delta_{X_u(t)-m_t^{1,3}} = \lim_{t\to\infty} \sum_{u\in N_t^2} \delta_{X_u(t)-m_t^{1,3}} = \mathrm{DPPP}\big(\theta C Z_\infty^{(1)} e^{-\theta x}\,\mathrm{d}x, \mathfrak{D}^\theta\big) \quad \textit{in law},$$

with $C = \frac{2\alpha\sigma C(\theta)}{\sqrt{2\pi}\beta(1-\sigma^2)}$ (where $C(\theta)$ is defined in Lemma 2.4(ii)).

Theorem 1.2 confirms Conjecture 2.1 of Belloum and Mallein [13] with the coefficient before $\log t$ being corrected as $\frac{1}{2\theta}$ and the decoration law $\widetilde{\mathfrak{D}}$ there taken to be $\mathfrak{D}^\theta$.

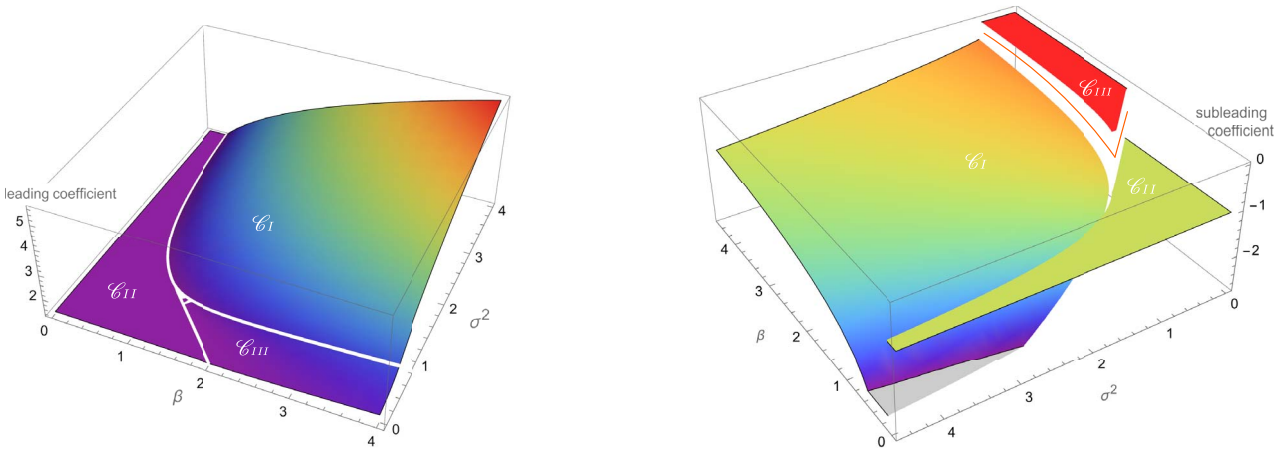


Fig. 2. Coefficients for term t and $\log t$ as function of (β, σ^2).

Theorem 1.3 (Boundary between $\mathscr{C}_I$, $\mathscr{C}_{II}$). *Assume that $(\beta, \sigma^2) \in \mathscr{B}_{I,II}$. Let*

$$m_t^{1,2} := m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t.$$

For some random variable $\bar{Z}_\infty$ (defined in Lemma 5.3), we have

$$\lim_{t\to\infty} \sum_{u\in N_t} \delta_{X_u(t)-m_t^{1,2}} = \lim_{t\to\infty} \sum_{u\in N_t^2} \delta_{X_u(t)-m_t^{1,2}} = \mathrm{DPPP}\big(\sqrt{2}C_\star \bar{Z}_\infty e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\big) \quad \textit{in law.}$$

Remark 1.4. Combining Theorems 1.1, 1.2, 1.3 and the results in [12,13], we can observe a double jump in the maximum $\max_{u\in N_t} X_u(t)$ when the parameters (β, σ^2) cross the boundary of the anomalous spreading region $\mathscr{C}_{III}$. The leading order varies continuously but the logarithmic correction changes from $-\frac{3}{2\sqrt{2}} \log t$ to $-\frac{1}{2\sqrt{2}} \log t$ then to 0; or from $-\frac{3}{2\theta} \log t$ to $-\frac{1}{2\theta} \log t$ then to 0. See Figure 2. Such a phase transition reminds us of the study of time inhomogeneous BRW [35], in which a constant multiplying the logarithmic correction changes from $-\frac{1}{2}$ to $-\frac{3}{2}$ then to $-\frac{6}{2}$. Also in a more general setting of [35], phase transitions become a little bit more complex and a double jump can occur as well (see [49]). A more interesting problem is to make the logarithmic correction smoothly interpolate from 1 to 6, which is done for variable speed BBM in [24] (see also [40]). For two-type reducible BBMs, we ask a similar question as follows.

Question 1. Can we let the parameters (β_t, σ_t^2) depend on the time horizon t, in order that the logarithmic correction for the order of the maximum at time t smoothly interpolate from $-\frac{3}{2\sqrt{2}} \log t$ to $-\frac{1}{2\sqrt{2}} \log t$ then to 0 or from $-\frac{3}{2\theta} \log t$ to $-\frac{1}{2\theta} \log t$ then to 0?

Remark 1.5. Question 1 above has now been addressed in our recent preprint [44].

Remark 1.6. When $(\beta, \sigma^2) \in \mathscr{B}_{I,II}$, the localization of paths of extremal particles is the same as in the case $(\beta, \sigma^2) \in \mathscr{C}_{II}$ (see Lemma 5.1). So when (β, σ^2) crosses the boundary between $\mathscr{C}_I$, $\mathscr{C}_{II}$, the maximum of the process only experience one jump: the subleading coefficient changes from $-\frac{3}{2\sqrt{2}}$ to $-\frac{3}{2\theta}$.

Remark 1.7. In the proof of Theorems 1.1, 1.2, 1.3, we only have to prove the convergence of the extremal process consisting of particles of type 2. Because by (1.1) and (1.4), the height of the highest type 1 particles is around $vt - \frac{3}{2\theta} \log t + O_{\mathbb{P}}(1)$, which is much below type 2 particles. Moreover, our results in Theorems 1.1, 1.2, 1.3 can be strengthened as the joint convergence of the extremal processes and its maximum $(\mathcal{E}_t, \max \mathcal{E}_t)$ to $(\mathcal{E}_\infty, \max \mathcal{E}_\infty)$, where $\mathcal{E}_\infty$ is a DPPP and $\max \mathcal{E}_\infty$ is its maximum. The reason is that, by [14, Lemma 4.4], to prove the convergence of $(\mathcal{E}_t, \max \mathcal{E}_t)$, it is sufficient to show the convergence of the Laplace functional $\mathbb{E}[e^{-\langle \phi, \mathcal{E}_t \rangle}]$ with certain test functions $\varphi \in \mathcal{T}$ introduced below.

Notation convention. Let $\mathcal{T}$ be the set of functions $\varphi \in C_b^+(\mathbb{R})$ such that $\inf \mathrm{supp}(\varphi) > -\infty$ and for some $a \in \mathbb{R}$, $\varphi(x) \equiv$ some positive constant for all $x > a$. $\mathcal{T}$ will serve as test functions in the Laplace functional (see [14, Lemma 4.4]). For

two quantities f and g, we write $f \sim g$ if $\lim f/g = 1$. We write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \le Cg$. We write $f \lesssim_\lambda g$ to stress that the constant C depends on parameter λ. We use the standard notation $\Theta(f)$ to denote a non-negative quantity such that there exists constant $c_1, c_2 > 0$ such that $c_1 f \le \Theta(f) \le c_2 f$.

1.4. *Heuristics*

We restate the optimization problem introduced in [13, Section 2.1] (see also Biggins [16]). First, we introduce the following definition:

Definition 1.8. If $u \in N_t^2$, we denote by T_u the time at which the oldest ancestor of type 2 of u was born. We say T_u is the *type transformation time* of u.

For $p \in [0,1]$, let $\mathcal{N}_{p,a,b}(t)$ be the expected number of type 2 particles at time t who have speed a before time $T_u \approx pt$ and speed b after time pt. Note that these particles are at level $[pa + (1-p)b]t$. First-moment computations yield that there are around $e^{(\beta - \frac{a^2}{2\sigma^2})pt + o(t)}$ type 1 particles at time pt at level apt, and a type 2 particle has probability $e^{(1-\frac{b^2}{2})(1-p)t + o(t)}$ of having a descendant at level $b(1-p)t$ at time $(1-p)t$. Hence we get $\mathcal{N}_{p,a,b}(t) = \exp\{[(\beta - \frac{a^2}{2\sigma^2})p + (1-\frac{b^2}{2})(1-p)]t + o(t)\}$. In order to get the maximum, we just maximize $pa + (1-p)b$ among all the parameter p, a, b such that $(\beta - \frac{a^2}{2\sigma^2})p \ge 0$ and $\mathcal{N}_{p,a,b}(t) \ge 1$. This turns to the following optimization problem:

$$v^* = \max\left\{ pa + (1-p)b : p \in [0,1], \left(\beta - \frac{a^2}{2\sigma^2}\right)p \ge 0, \left(\beta - \frac{a^2}{2\sigma^2}\right)p + \left(1 - \frac{b^2}{2}\right)(1-p) \ge 0 \right\}.$$

Write (p^*, a^*, b^*) for the maximizer of the problem above. If $(\beta, \sigma^2) \in \mathscr{C}_I$, then $p^* = 1$, $a^* = v$, and $v^* = v$; if $(\beta, \sigma^2) \in \mathscr{C}_{II}$, then $p^* = 0$, $b^* = \sqrt{2}$ and $v^* = \sqrt{2}$; if $(\beta, \sigma^2) \in \mathscr{C}_{III}$, then $p^* = \frac{\sigma^2 + \beta - 2}{2(1-\sigma^2)(\beta-1)}$, $b^* = \sqrt{2\frac{\beta-1}{1-\sigma^2}}$, $a^* = \sigma^2 b^*$, and $v^* = \frac{\beta - \sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}}$. Also if $(\beta, \sigma^2) = (1,1)$, then p^* can be arbitrary in $[0,1]$, $a^* = b^* = \sqrt{2}$ and $v^* = \sqrt{2}$.

When $(\beta, \sigma^2) \in \mathscr{B}_{II,III}$, the maximizer is $p^* = 0$, $v^* = b^* = \sqrt{2}$, and a^* can be arbitrary. Hence each individual $u \in N_t^2$ near the maximal position should satisfy $p^* = T_u/t \approx 0$. But now the order of T_u really matters, and if $T_u \gg 1$, a^* should be $\sqrt{2}\sigma^2$ as predicted by the formula for the case $\mathscr{C}_{III}$. When $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$, the maximizer $p^* = 1$, $v^* = a^* = v$, and b^* can be arbitrary. We can deduce that each individual $u \in N_t^2$ near the maximal position satisfies $p^* = T_u/t \approx 1$. The order of $t - T_u$ is also important, and if $t - T_u \gg 1$, b^* should be $\sqrt{2\beta/\sigma^2}$ predicted by the formula for the case $\mathscr{C}_{III}$. Similar problems occur when we consider the boundary $\mathscr{B}_{I,II}$. The following computations, based on a finer analysis, provide more insights for localization of paths of extremal particles.

The case $(\beta, \sigma^2) \in \mathscr{B}_{II,III}$. As in the computations above, the expected number of particles of type 1 at time $s = o(t)$ at level λs is roughly $e^{(\beta - \frac{\lambda^2}{2\sigma^2})s + O(\log t)}$. A typical particle of type 2 has probability $e^{-(2-\sqrt{2}\lambda)s - \frac{(\sqrt{2}-\lambda)^2}{2}\frac{s^2}{t-s} + O(\log t)}$ of having a descendant at level $\sqrt{2}t - \lambda s$ at time $t - s$. Hence there are around

$$\exp\left\{\left[\beta - \frac{\lambda^2}{2\sigma^2} - (2 - \sqrt{2}\lambda)\right]s - \frac{(\sqrt{2}-\lambda)^2}{2}\frac{s^2}{t-s} + O(\log t)\right\} \tag{1.7}$$

particles of type 2 at level $\sqrt{2}t$ at time t such that $T_u \approx s$ and $X_u(T_u) \approx \lambda s$. In order that the quantity in (1.7) is not zero as $t \to \infty$, first we have to ensure that $\beta - \frac{\lambda^2}{2\sigma^2} - (2 - \sqrt{2}\lambda) = -\frac{1}{2\sigma^2}(\lambda - \sqrt{2}\sigma^2)^2 \ge 0$ (here we used $\beta + \sigma^2 = 2$), which forces $\lambda = \sqrt{2}\sigma^2$. Secondly we have to ensure that $\frac{s^2}{t-s}$ is bounded, i.e., $s = O(\sqrt{t})$. In other words, the extremal particle $u \in N_t^2$ should satisfy $T_u = O(\sqrt{t})$ and $X_u(T_u) \approx \sqrt{2}\sigma^2 T_u$.

The case $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$. The expected number of type 1 particles at time $s = t - o(t)$ at level $vs - a(t)$ (where $a(t)$ will be determined later) is roughly $e^{\theta a(t) - \frac{a(t)^2}{2\sigma^2 s} + O(\log t)}$. A typical particle of type 2 has probability $e^{-[(\frac{v^2}{2}-1)(t-s) + va(t) + \frac{a(t)^2}{2(t-s)}] + O(\log t)}$ of having a descendant at level $v(t-s) + a(t)$ at time $t - s$. Hence there are around

$$\exp\left\{-\left[\left(\frac{v^2}{2} - 1\right)(t-s) + \frac{a(t)^2}{2\sigma^2 s} + \frac{a(t)^2}{2(t-s)}\right] + (\theta - v)a(t) + O(\log t)\right\} \tag{1.8}$$

particles of type 2 at level vt at time t such that $T_u \approx s$ and $X_u(T_u) \approx vs - a(t)$. In order that the quantity in (1.8) is not zero as $t \to \infty$, using the prior knowledge $s \sim t$, first we have to ensure that $a(t)$ has the same order as $t - s$ or $\frac{a(t)^2}{t-s}$, and we get $a(t) = \Theta(t-s)$. We also need to ensure that $\frac{a(t)^2}{2s} = O(1)$, this implies $a(t) = \Theta(t-s) = O(\sqrt{t})$. So, letting $a(t) = a\sqrt{t}$ and $t - s = \lambda\sqrt{t}$, we can rewrite (1.8) as

$$\exp\left\{\left(\theta a - \frac{(a+\lambda v)^2}{2\lambda} + \lambda\right)\sqrt{t} + O(\log t)\right\}.$$

We now have to ensure $\theta a - \frac{(a+\lambda v)^2}{2\lambda} + \lambda = -\frac{1}{2\lambda}[a - (\theta - v)\lambda]^2 \geq 0$ (here we used $\frac{1}{\beta} + \frac{1}{\sigma^2} = 2$). This forces $a = (\theta - v)\lambda$ and hence $a(t) = a\sqrt{t} = (\theta - v)(t - s)$. In other words, the extremal particle $u \in N_t^2$ should satisfy $T_u = t - \Theta(\sqrt{t})$ and $X_u(T_u) \approx vT_u - (\theta - v)(t - T_u)$.

The case $(\beta, \sigma^2) \in \mathscr{B}_{I,II}$. We do the same computations as in the case $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$, and still get that the expected number of type 2 particles at level vt at time t satisfying $T_u \approx s$ and $X_u(T_u) \approx vs - a(t)$ is given by (1.8). However, now we are in the case $\mathscr{B}_{I,II}$ with $\beta\sigma^2 = 1$ and $\beta < 1$, and hence $v = \sqrt{2}$ and $\theta = \sqrt{2}\beta$. Therefore (1.8) becomes

$$\exp\left\{-\left[\sqrt{2}(1-\beta)a(t) + \frac{a(t)^2}{2\sigma^2 s} + \frac{a(t)^2}{2(t-s)}\right] + O(\log t)\right\}. \tag{1.9}$$

In order that (1.9) tends to a nonzero limit, we need $a(t) = O(\log t)$. This is very different from the case $(\beta, \sigma^2) = (1, 1)$ in [12], in which case we can get $a(t) = O(\sqrt{t})$ (and s is of order t). Now the simple first moment computations can not tell us more. However, we can still make a *guess*. Notice that the extremal particles $u \in N_t^2$ are also extremal (up to $O(\log t)$) at time T_u. This fact reminds us of the decreasing variances case in [35]: The maximum at time t is the highest value among the descendants of the maximal particle at time $t/2$. So we guess that $\max_{u\in N_t} X_u(t) \approx \max_{T_u\in[0,t]}\{\sqrt{2}T_u - \frac{3}{2\theta}\log T_u + \sqrt{2}(t - T_u) - \frac{3}{2\sqrt{2}}\log(t - T_u)\}$. As $\theta < \sqrt{2}$, we should choose T_u small. So we expect $T_u = O(1)$ and $\max_{u\in N_t} X_u(t)$ should be $\sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$.

2. Preliminary results

2.1. *Brownian motions estimates*

We always use $\{(B_t)_{t\geq 0}; \mathbf{P}\}$ to denote a standard Brownian motion (BM) starting from the origin. Here is a useful upper bound for the probability that a Brownian bridge is below a line, see [27, Lemma 2] for a proof.

Lemma 2.1. *Consider a line segment with endpoints* $(0, x_1)$, (t, x_2) *with* $x_1, x_2 \geq 0$. *We have*

$$\mathbf{P}\left(B_s \leq \frac{s}{t}x_2 + \frac{t-s}{t}x_1, \forall s \in [0,t] \,\middle|\, B_t = 0\right) = 1 - e^{-\frac{2x_1x_2}{t}} \leq \frac{2x_1x_2}{t}.$$

As a corollary, we also have an estimate for the probability that a BM stays below a line and ends up in a finite interval. For all $K \geq 1$ and $y \geq 0$ we have

$$\mathbf{P}\big(B_s \leq K, s \leq t, B_t - K \in [-y-1, -y]\big) \leq C\frac{K(y+1)}{(1+t)^{3/2}}. \tag{2.1}$$

In fact, the desired probability is less than the product of $\mathbf{P}(B_t - K + y \in [-1, 0]) \leq \frac{1}{\sqrt{2\pi t}}$ and $\max_{z\in[0,1]}\mathbf{P}(B_s \leq K, \forall s \in [0,t] \,|\, B_t = K - y - z) \leq 2K(y+1)/t$ by Lemma 2.1 and scaling invariance properties of Brownian bridge.

Later in the proof of Lemma 4.1, we will use a slight modification of Lemma 2.1 as follows. For completeness, we give its proof in the Appendix.

Lemma 2.2. *Let* $\widetilde{m}_t = vt - w_t$, *where* $w_t = \Theta(\log t)$. *Assume that* $\sigma^2 \leq 1$. *Fix* $K \geq 0$. *Then for sufficiently large* t, $s \in [t - \sqrt{t}\log t, t]$ *and* $x \in \mathbb{R}$,

$$\mathbf{P}(\sigma B_r \leq vr + K, \forall r \leq s \mid \sigma B_s + B_t - B_s = \widetilde{m}_t + x) \lesssim_{K,\beta,\sigma} \frac{t - s + w_t + |x|}{t}. \tag{2.2}$$

2.2. *Branching Brownian motions estimates*

We always use $\{(\mathsf{X}_u^{\beta,\sigma^2}(t) : u \in \mathsf{N}_t)_{t\geq 0}, \mathsf{P}\}$ to denote a BBM starting from one particle at the origin with branching rate β and diffusion coefficient σ^2. For the BBM, there is an upper envelope through which particles find it difficult to pass. In fact, letting $a_t = \frac{3}{2\theta}\log(t+1)$, for some constant $C > 0$ and for all $t > 0$, $K > 0$,

$$\mathsf{P}\big(\exists s \leq t, u \in \mathsf{N}_s : \mathsf{X}_u^{\beta,\sigma^2}(s) \geq vs - a_t + a_{t-s} + K\big) \leq C(K+1)e^{-\theta K}, \tag{2.3}$$

see [13, (6.1)] (or [50, Lemma 3.1]). In particular, we have

$$\mathsf{P}\big(\exists s \leq t, u \in \mathsf{N}_s : \mathsf{X}_u^{\beta,\sigma^2}(s) \geq vs + K\big) \leq C(K+1)e^{-\theta K}. \tag{2.4}$$

We collect several results for the standard BBM $(\mathsf{X}_u(t) : u \in \mathsf{N}_t)_{t\geq 0}$, (i.e., $\beta = \sigma^2 = 1$) that will be used frequently. Recall that $\mathsf{M}_t = \max_{u\in\mathsf{N}_t} \mathsf{X}_u(t)$. The following estimate of the upper tail of M_t was proved in [8, Corollary 10]

Lemma 2.3. *For $x > 1$ and $t \geq t_o$ (for t_o a numerical constant),*

$$\mathsf{P}\bigg(\mathsf{M}_t \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + x\bigg) \leq \rho \cdot x \cdot \exp\bigg(-\sqrt{2}x - \frac{x^2}{2t} + \frac{3}{2\sqrt{2}}x\frac{\log t}{t}\bigg)$$

for some constant $\rho > 0$.

As a consequence of Lemma 2.3, we have

$$\mathsf{P}\bigg(\mathsf{M}_t \geq \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t + x\bigg) \leq \rho \cdot x\exp(-\sqrt{2}x + 1), \tag{2.5}$$

for all $x > 1$ and $t > t_0$, since $-\frac{x^2}{2t} + \frac{3}{2\sqrt{2}}x\frac{\log t}{t} \leq 1$, for all $t > 100$ and $x > 1$.

The following estimates for the Laplace functionals of standard BBMs can be found in [12, Corollary 2.9] and [13, Lemma 3.7]. One can also obtain the same results from the large deviations probability $\mathsf{P}(\mathsf{M}_t > \rho t + x)$ with $\rho \geq \sqrt{2}$ and the conditioned convergence of the gap processes (1.3).

Lemma 2.4. *Let $\varphi \in \mathcal{T}$, $R > 0$ and $\rho > \sqrt{2}$. Then the following assertions hold.*

(i) *For $x \in [-R\sqrt{t}, -\frac{1}{R}\sqrt{t}]$ uniformly*

$$1 - \mathsf{E}\big(e^{-\sum_{u\in\mathsf{N}_t}\varphi(x+\mathsf{X}_u(t)-\sqrt{2}t)}\big) = \gamma_{\sqrt{2}}(\varphi)\frac{(-x)}{t^{3/2}}e^{\sqrt{2}x-\frac{x^2}{2t}}\big(1+o(1)\big),$$

as $t \to \infty$, where $\gamma_{\sqrt{2}}(\varphi) = \sqrt{2}C_\star\int e^{-\sqrt{2}z}(1 - \mathsf{E}(e^{-\langle\mathfrak{D}^{\sqrt{2}},\varphi(\cdot+z)\rangle}))\,\mathrm{d}z$.

(ii) *For $|x| \leq R\sqrt{t}$ uniformly*

$$1 - \mathsf{E}\big(e^{-\sum_{u\in\mathsf{N}_t}\varphi(x+\mathsf{X}_u(t)-\rho t)}\big) = \gamma_\rho(\varphi)\frac{e^{(1-\rho^2/2)t}}{\sqrt{t}}e^{\rho x-\frac{x^2}{2t}}\big(1+o(1)\big),$$

as $t \to \infty$, where $\gamma_\rho(\varphi) = \frac{C(\rho)}{\sqrt{2\pi}}\int e^{-\rho z}(1 - \mathsf{E}(e^{-\langle\mathfrak{D}^\rho,\varphi(\cdot+z)\rangle}))\,\mathrm{d}z$.

2.3. *Choosing an individual according to Gibbs measure*

For a standard BBM $\{(\mathsf{X}_u(t) : u \in \mathsf{N}_t), \mathsf{P}\}$, it is well known that the additive martingale

$$\mathsf{W}_t(\eta) = \sum_{u\in\mathsf{N}_t} e^{\eta\mathsf{X}_u(t)-(\frac{\eta^2}{2}+1)t}$$

converges almost surely and in L^1 to a non-degenerate random variable $\mathsf{W}_\infty(\eta)$ when $\eta \in (0, \sqrt{2})$. And, when $\eta = \sqrt{2}$, the non-negative martingale $\mathsf{W}_t(\eta)$ converges to zero with probability one (see e.g. [41]). Conditioned on BBM at time

t, we randomly choose a particle $u \in \mathsf{N}_t$ with probability $\frac{e^{\eta \mathsf{X}_u(t)}}{\sum_{u\in\mathsf{N}_t} e^{\eta \mathsf{X}_u(t)}}$, which is the so-called *Gibbs measure* at inverse temperature η. Hence the additive martingale can be viewed as the *normalized partition function* of the Gibbs measure.

Firstly we state a law of large number theorem for the particle chosen according to the Gibbs measure. This result is not new. Since we didn't find a reference, we offer a simple proof in Section 6 for completeness.

Proposition 2.5. *Let f be a bounded continuous function on $\mathbb{R}$. Suppose $\eta \in (0, \sqrt{2})$. Define*

$$\mathsf{W}_t^f(\eta) := \sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t)}{t}\right) e^{\eta \mathsf{X}_u(t) - (\frac{\eta^2}{2}+1)t}.$$

Then

$$\lim_{t\to\infty} \mathsf{W}_t^f(\eta) = f(\eta)\mathsf{W}_\infty(\eta) \quad \text{in } L^1(\mathsf{P}).$$

This law of large number holds since the Gibbs measure is supported on the particles at position around ηt for $\eta \in (0, \sqrt{2})$. One may further ask the fluctuations of $X_u(t) - \eta t$. Indeed a central limit theorem (CLT) holds (see [51, (1.14)]): for $\eta \in (0, \sqrt{2})$ and for each bounded continuous function f,

$$\lim_{t\to\infty} \sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t) - \eta t}{\sqrt{t}}\right) e^{\eta \mathsf{X}_u(t) - t(\frac{\eta^2}{2}+1)} = W_\infty(\eta) \int_{\mathbb{R}} f(z) e^{-\frac{z^2}{2}} \frac{\mathrm{d}z}{\sqrt{2\pi}} \quad \mathsf{P}\text{-a.s.}$$

We refer to [15, Corollary 4] for the corresponding local limit theorem. In the critical case $\eta = \sqrt{2}$, the limiting distribution in the CLT is no longer Gaussian, but the Rayleigh distribution $\mu(\mathrm{d}z) = z e^{-\frac{z^2}{2}} 1_{\{z>0\}}\, \mathrm{d}z$. Madaule [46, Theorem 1.2] showed that for every bounded continuous function F,

$$\lim_{t\to\infty} \sqrt{t} \sum_{u\in\mathsf{N}_t} F\left(\frac{\sqrt{2}t - \mathsf{X}_u(t)}{\sqrt{t}}\right) e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))} = \sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty \langle F, \mu\rangle \quad \text{in probability,} \tag{2.6}$$

where $\langle F, \mu\rangle := \int_0^\infty F(z)\mu(\mathrm{d}z)$. In fact, Madaule's result is a Donsker-type theorem for BRWs. A simple proof of (2.6) can be found in [48, Theorem B.1].

The following proposition gives a natural generalization of (2.6). We didn't find such a result in the literature and to our knowledge, it is new.

Proposition 2.6. *Let G be a non-negative bounded measurable function with compact support. Suppose $F_t(z) = G(\frac{z-r_t}{h_t})$, where r_t and h_t satisfy the condition that for large t, $\log^3(t)/\sqrt{t} \ll r_t \le \bar{r} < \infty$, $r_t + y h_t = \Theta(r_t)$ uniformly for $y \in \mathrm{supp}(G)$ and h_t decreases at most polynomially fast. Define*

$$\mathsf{W}_t^{F_t}(\sqrt{2}) := \sum_{u\in\mathsf{N}_t} F_t\left(\frac{\sqrt{2}t - \mathsf{X}_u(t)}{\sqrt{t}}\right) e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))}.$$

Then we have

$$\lim_{t\to\infty} \frac{\sqrt{t}}{\langle F_t, \mu\rangle} \mathsf{W}_t^{F_t}(\sqrt{2}) = \sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty \quad \textit{in probability.}$$

Taking $r_t = \lambda$ and $h_t = t^{-1/4}$, and denoting $F_{t,\lambda}(z) := G((z-\lambda)t^{1/4})$, for any finite interval $I \subset (0,\infty)$ with strictly positive endpoints we have

$$\lim_{t\to\infty} t^{3/4} \int_I \mathsf{W}_{t-\lambda\sqrt{t}}^{F_{t,\lambda}}(\sqrt{2})\, \mathrm{d}\lambda = \sqrt{\frac{2}{\pi}}\mu(I) \int_{\mathbb{R}} G(y)\, \mathrm{d}y \mathsf{Z}_\infty \quad \textit{in probability.} \tag{2.7}$$

Remark 2.7. To make Proposition 2.6 easier to understand, we choose the function G to be the indicator function $1_{[a,b]}$. Letting $h_t = 1$ and $r_t = \lambda > 0$, Proposition 2.6 gives the CLT (2.6):

$$\sqrt{t} \sum_{u\in\mathsf{N}_t} e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))} 1_{\{\sqrt{2}t - X_u(t) \in [(\lambda+a)\sqrt{t}, (\lambda+b)\sqrt{t}]\}} = \big[1 + o_{\mathsf{P}}(1)\big]\mu\big([\lambda+a, \lambda+b]\big)\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty.$$

Letting $h_t = t^{-1/2}$ and $r_t = \lambda > 0$, Proposition 2.6 yields that

$$\sqrt{t}\sum_{u\in\mathsf{N}_t} e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}1_{\{\sqrt{2}t-X_u(t)\in[\lambda\sqrt{t}+a,\lambda\sqrt{t}+b]\}} = \big[1+o_{\mathsf{P}}(1)\big]\frac{(b-a)}{\sqrt{t}}\lambda e^{-\frac{\lambda^2}{2}}\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty.$$

This can be thought of a *local limit theorem* (LLT) result for the position of a particle sampled according to the Gibbs measure with parameter $\eta = \sqrt{2}$. (See also [15, Theorem 4] for a LLT result in the case $\eta < \sqrt{2}$.) Letting $h_t = t^{-1/4}$ and $r_t = \lambda > 0$, we get

$$\sqrt{t}\sum_{u\in\mathsf{N}_t} e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}1_{\{\sqrt{2}t-X_u(t)\in[\lambda\sqrt{t}+at^{1/4},\lambda\sqrt{t}+bt^{1/4}]\}} = \big[1+o_{\mathsf{P}}(1)\big]\frac{(b-a)}{t^{1/4}}\lambda e^{-\frac{\lambda^2}{2}}\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty,$$

which can be regarded as an intermediate result between CLT and LLT.

Remark 2.8. Formally, applying Proposition 2.6 to the function $zF_t(z)$ and letting $\hat{\mu}(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}z^2e^{-z^2/2}1_{\{z>0\}}\,\mathrm{d}z$, we have

$$\frac{1}{\langle F_t,\hat{\mu}\rangle}\sum_{u\in\mathsf{N}_t} F_t\bigg(\frac{\sqrt{2}t-\mathsf{X}_u(t)}{\sqrt{t}}\bigg)\big(\sqrt{2}t-X_u(t)\big)e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))} \to \mathsf{Z}_\infty \quad \text{in probability.}$$

A rigorous proof can be achieved by slightly modifying the proof of Proposition 2.6.

Remark 2.9. In the proof of Proposition 2.6, we use a powerful method developed in [4]. The main step is to prove that a nonnegative random variable $U_t := U_t^{(K)}$ (see (6.7) for the definition of $U_t^{(K)}$) converges in L^1 under some probability $\mathsf{Q} := \mathsf{Q}^{(K)}$ (see (6.2) for the definition of $\mathsf{Q}^{(K)}$). In the CLT case (i.e., $r_t = \Theta(1) = h_t$), the 2nd moment of U_t is finite and one can find good events G_t so that $\limsup_{t\to\infty}\frac{\mathsf{E}_\mathsf{Q}[U_t^2 1_{G_t}]}{(\mathsf{E}_\mathsf{Q}[(U_t)])^2} \le 1$ and $\mathsf{E}_\mathsf{Q}[U_t^2 1_{G_t^c}]$ is negligible. However, when $h_t = o(1)$ or $r_t = o(1)$, the 2nd moment of U_t appears to explode, and $\mathsf{E}_\mathsf{Q}[U_t^2 I_{G_t^c}]$ is not negligible.

In the first version of this paper [43], we overcame this difficulty by using a bootstrap argument to show that the integral of U_t^p on G_t^c is negligible and $\limsup_{t\to\infty}\frac{\mathsf{E}_\mathsf{Q}[U_t^p I_{G_t}]}{(\mathsf{E}_\mathsf{Q}[U_t])^p} \le 1$ for $p = 1+\epsilon$ with small ϵ. But we need to further assume that $\sqrt{t}r_t \gg t^\epsilon$. We thank one of our referees for suggesting the current proof which is simpler than our first version. This type of argument is used in [51, Section 5] also based on the ideas in [4]. Indeed we do not have to prove the stronger result on the convergence of p-th moment. In order to show the L^1 convergence of U_t, it suffices to show that the expectation of U_t on G_t^c is negligible and the expectation of $(U_t)^2$ on G_t is less than the square of the expectation of U_t. Also, the conditions on r_t in [43] can be weakened to $\sqrt{t}r_t \gg \log^3 t$ and even weakened to $\sqrt{t}r_t \ge \log^{1+\epsilon} t$, by properly choosing the barrier in the good event G_t. However, as pointed out by the same referee, dealing with $r_t = O((\log t)/\sqrt{t})$ requires to work with a more precise barrier. In this regime what matters is not the distance between $X_u(t)$ and $\sqrt{2}t$, but rather the distance between $X_u(t)$ and m_t. This type of regime is covered in [51]. Moreover, results in [51, Corollary 1.3] are somewhat similar to our proposition: the scaling in the test function F is always $1/\sqrt{t}$, but the fact of having a weight $e^{\beta_t X_u(t)}$ instead of $e^{\sqrt{2}X_u(t)}$ implies that the considered sum is supported in some smaller windows.

2.4. *Many-to-one lemmas for two-type BBMs*

Let $\{(X_u(t), u\in N_t), \mathbb{P}\}$ be a two-type reducible BBM. Recall that α is the rate at which a type 1 individual gives birth to one type 2 children. Recall that for a type 2 particle u, T_u is the time at which the oldest ancestor of type 2 of u was born. Let b_u be the time when u was born. We write

$$\mathcal{B} = \bigg\{u \in \bigcup_{t\ge 0} N_t^2, T_u = b_u\bigg\}$$

for the set of particles of type 2 that are born from a particle of type 1. The following useful many-to-one lemmas were proved in [13, Proposition 4.1 and Corollary 4.3].

Lemma 2.10. *Let f be a non-negative measurable function. Then*

(i) $\mathbb{E}(\sum_{u\in N_t^2} f((X_u(r), r\le t), T_u)) = \alpha\int_0^t e^{\beta s+(t-s)}\mathbf{E}(f((\sigma B_{r\wedge s} + (B_r - B_{r\wedge s}), r\le t), s))\,\mathrm{d}s.$

(ii) $\mathbb{E}(\sum_{u\in\mathcal{B}} f(X_u(s), s\le T_u)) = \alpha\int_0^\infty e^{\beta t}\mathbf{E}(f(\sigma B_s, s\le t))\,\mathrm{d}t$.
(iii) $\mathbb{E}[\exp(-\sum_{u\in\mathcal{B}} f(X_u(s), s\le T_u))] = \mathbb{E}[\exp(-\alpha\int_0^\infty \sum_{u\in N_t^1}(1-e^{-f(X_u(s),s\le t)})\,\mathrm{d}t)]$.

3. Boundary between $\mathscr{C}_{II}$ and $\mathscr{C}_{III}$

In this section we always assume that $(\beta,\sigma^2)\in\mathscr{B}_{II,III}$, i.e., $\beta>1$, $\sigma^2=2-\beta<1$ and recall that $m_t^{2,3} := \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t$.

Lemma 3.1. *For any $A>0$, we have*

$$\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{P}\left(\exists u\in N_t^2 : T_u\notin\left[\frac{1}{R}\sqrt{t}, R\sqrt{t}\right], X_u(t)\ge m_t^{2,3}-A\right)=0.$$

We are going to show Theorem 1.1, postponing the proof of Lemma 3.1 in the end of this section.

Proof of Theorem 1.1. In this proof, we set $\widehat{\mathcal{E}}_t := \sum_{u\in N_t^2}\delta_{X_u(t)-m_t^{2,3}}$, and for fixed $R>0$, let

$$\widehat{\mathcal{E}}_t^R := \sum_{u\in N_t^2} 1_{\{T_u\in[\frac{1}{R}\sqrt{t},R\sqrt{t}]\}}\delta_{X_u(t)-m_t^{2,3}}.$$

Thanks to Lemma 3.1, $\widehat{\mathcal{E}}_t^R$ is very close to $\widehat{\mathcal{E}}_t$. More precisely, for all $\varphi\in\mathcal{T}$, we have,

$$\left|\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\big)-\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\big)\right|\le\mathbb{P}\left(\exists u\in N_t^2 : X_u(t)\ge m_t^{2,3}-A, T_u\notin\left[\frac{1}{R}\sqrt{t},R\sqrt{t}\right]\right),$$

where A is chosen such that $\mathrm{supp}(\varphi)\subset[-A,\infty)$. Thus applying Lemma 3.1 we have

$$\lim_{R\to\infty}\limsup_{t\to\infty}\left|\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\big)-\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\big)\right|=0. \tag{3.1}$$

Hence in order to study the asymptotic behavior of $\mathbb{E}(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle})$, we only need to study the convergence of $\widehat{\mathcal{E}}_t^R$ as t and then R goes to infinity.

Fix $R>0$ and $\varphi\in\mathcal{T}$. Observe that we can rewrite $\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle$ as

$$\sum_{\substack{u\in\mathcal{B}\\ T_u\in[\frac{1}{R}\sqrt{t},R\sqrt{t}]}}\sum_{\substack{u'\in N_t^2\\ u'\succcurlyeq u}}\varphi\left(X_{u'}(t)-X_u(T_u)-\sqrt{2}(t-T_u)+X_u(T_u)-\sqrt{2}T_u+\frac{1}{2\sqrt{2}}\log t\right),$$

where $u'\succcurlyeq u$ means that u' is a descendant of u. Using the branching property first and then Lemma 2.10(iii), we have

$$\begin{aligned}\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\big)&=\mathbb{E}\left(\prod_{\substack{u\in\mathcal{B}\\ T_u\in[\frac{1}{R}\sqrt{t},R\sqrt{t}]}} f\left(t-T_u, X_u(T_u)-\sqrt{2}T_u+\frac{1}{2\sqrt{2}}\log t\right)\right)\\&=\mathbb{E}\left(\exp\left\{-\alpha\int_{\frac{1}{R}\sqrt{t}}^{R\sqrt{t}}\sum_{u\in N_s^1}F\left(t-s, X_u(s)-\sqrt{2}s+\frac{1}{2\sqrt{2}}\log t\right)\mathrm{d}s\right\}\right),\end{aligned}$$

where $F(r,x)=1-f(r,x)=1-\mathsf{E}(\exp\{-\sum_{u\in\mathsf{N}_r}\varphi(\mathsf{X}_u(r)-\sqrt{2}r+x)\})$. Additionally, as the speed $v=\sqrt{2\beta\sigma^2}$ of the BBM of type 1 is less than $\sqrt{2}$, we have, for all $s\in[\frac{1}{R}\sqrt{t},R\sqrt{t}]$ and $u\in N_s^1$, $\sqrt{2}s-X_u(s)=\Theta(s)$ a.s. since $\limsup_{t\to\infty}\frac{1}{t}\max_{u\in N_t^1}|X_u(t)|=v$ almost surely. Then applying part (i) of Lemma 2.4 we have, uniformly for $s\in[\frac{1}{R}\sqrt{t},R\sqrt{t}]$,

$$\begin{aligned}&F\left(t-s, X_u(s)-\sqrt{2}s+\frac{1}{2\sqrt{2}}\log t\right)\\&\quad=\big(1+o(1)\big)\gamma_{\sqrt{2}}(\varphi)\frac{(\sqrt{2}s-X_u(s))}{t}e^{\sqrt{2}(X_u(s)-\sqrt{2}s)}e^{-\frac{(X_u(s)-\sqrt{2}s)^2}{2t}}\quad\text{a.s.},\end{aligned}$$

as $t \to \infty$, where the $o(1)$ term is deterministic. Thus $\mathbb{E}(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle})$

$$
\begin{aligned}
&= \mathbb{E}\bigg(\exp\bigg\{-\big(1+o(1)\big)\alpha\gamma_{\sqrt{2}}(\varphi) \times \int_{\frac{1}{R}\sqrt{t}}^{R\sqrt{t}} \sum_{u\in N_s^1} \frac{(\sqrt{2}s - X_u(s))}{t} e^{\sqrt{2}(X_u(s)-\sqrt{2}s)} e^{-\frac{(X_u(s)-\sqrt{2}s)^2}{2t}}\,\mathrm{d}s\bigg\}\bigg) \\
&= \mathbb{E}\bigg(\exp\bigg\{-\big(1+o(1)\big)\alpha\gamma_{\sqrt{2}}(\varphi) \int_{\frac{1}{R}}^{R} \mathcal{W}(\lambda, \lambda\sqrt{t})\,\mathrm{d}\lambda\bigg\}\bigg),
\end{aligned} \tag{3.2}
$$

where

$$
\mathcal{W}(\lambda, t) := \sum_{u\in N_t^1} \lambda\bigg(\sqrt{2} - \frac{X_u(t)}{t}\bigg) e^{-\frac{\lambda^2}{2}(\sqrt{2}-\frac{X_u(t)}{t})^2} e^{\sqrt{2}(X_u(t)-\sqrt{2}t)}.
$$

Recall that $(\{X_u(t)\}_{u\in N_t^1}, \mathbb{P})$ has the same distribution as $(\{\frac{\sigma}{\sqrt{\beta}}\mathsf{X}_u(\beta t)\}_{u\in \mathsf{N}_{\beta t}}, \mathsf{P})$. Applying Proposition 2.5, we have, for each $\lambda > 0$,

$$
\lim_{t\to\infty} \mathcal{W}(\lambda, t) = \sqrt{2}\lambda\big(1-\sigma^2\big) e^{-(1-\sigma^2)^2\lambda^2} W_\infty^{(1)}(\sqrt{2}) \quad \text{in } L^1(\mathbb{P}).
$$

Then by the dominated convergence theorem, we have, as $t \to \infty$,

$$
\begin{aligned}
&\mathbb{E}\bigg|\int_{1/R}^{R} \mathcal{W}(\lambda, \lambda\sqrt{t})\,\mathrm{d}\lambda - \sqrt{2}\int_{1/R}^{R} \lambda\big(1-\sigma^2\big) e^{-(1-\sigma^2)^2\lambda^2} W_\infty^{(1)}(\sqrt{2})\,\mathrm{d}\lambda\bigg| \\
&\quad \le \int_{1/R}^{R} \mathbb{E}\big|\mathcal{W}(\lambda, \lambda\sqrt{t}) - \sqrt{2}\lambda\big(1-\sigma^2\big) e^{-(1-\sigma^2)^2\lambda^2} W_\infty^{(1)}(\sqrt{2})\big|\,\mathrm{d}\lambda \to 0.
\end{aligned}
$$

Here dominating functions exist since $\mathbb{E}[\mathcal{W}(\lambda, \lambda\sqrt{t})]$, $\mathbb{E}[\sqrt{2}\lambda(1-\sigma^2)e^{-(1-\sigma^2)^2\lambda^2} W_\infty^{(1)}(\sqrt{2})]$ are both bounded by $\max_{x\ge 0} x e^{-x^2/2}$. Letting $t \to \infty$ in (3.2), we have

$$
\lim_{t\to\infty} \mathbb{E}\big(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\big) = \mathbb{E}\bigg(\exp\bigg\{-\alpha\gamma_{\sqrt{2}}(\varphi) W_\infty^{(1)}(\sqrt{2}) \int_{\frac{1}{R}}^{R} \sqrt{2}\lambda\big(1-\sigma^2\big) e^{-(1-\sigma^2)^2\lambda^2}\,\mathrm{d}\lambda\bigg\}\bigg).
$$

Then letting $R \to \infty$, combining (3.1), we obtain

$$
\lim_{t\to\infty} \mathbb{E}\big(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\big) = \lim_{R\to\infty} \lim_{t\to\infty} \mathbb{E}\big(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\big) = \mathbb{E}\bigg(\exp\bigg\{-\frac{\alpha W_\infty^{(1)}(\sqrt{2})}{\sqrt{2}(1-\sigma^2)}\gamma_{\sqrt{2}}(\varphi)\bigg\}\bigg),
$$

which is the Laplace functional of DPPP($\frac{\alpha C_\star}{\sqrt{2}(1-\sigma^2)}\sqrt{2}W_\infty^{(1)}(\sqrt{2})e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}$). Using [14, Lemma 4.4], we complete the proof of Theorem 1.1. □

Now it suffices to show Lemma 3.1. First we give prior estimates for the type transformation times T_u for particles $u \in N_t^2$ with positions higher than $m_t^{2.3}$.

Lemma 3.2. *For any $A > 0$ we have*

$$
\limsup_{t\to\infty} \mathbb{P}\big(\exists u \in N_t^2 : T_u > \sqrt{t}\log t, X_u(t) \ge m_t^{2,3} - A\big) = 0. \tag{3.3}
$$

Proof. Let $Y_t^A := \sum_{u\in N_t^2} 1_{\{T_u > \sqrt{t}\log t, X_u(t) \ge m_t^{2,3} - A\}}$. By Markov's inequality, the probability in (3.3) is bounded by $\mathbb{E}(Y_t^A)$. It suffices to show that $\mathbb{E}(Y_t^A) \to 0$ as $t \to \infty$.

Using Lemma 2.10(i) and then the tail probability of Gaussian random variable ($\int_x^\infty e^{-y^2/2}\,\mathrm{d}y \le \frac{1}{x}e^{-x^2/2}$ for $x>0$), we have

$$\begin{aligned}\mathbb{E}\big(Y_t^A\big) &= \alpha\int_{\sqrt{t}\log t}^{t} e^{\beta s+t-s}\mathbf{P}\big(\sigma B_s+(B_t-B_s)\ge m_t^{2,3}-A\big)\,\mathrm{d}s\\ &\lesssim_\alpha \int_{\sqrt{t}\log t}^{t} e^{\beta s+(t-s)}\frac{\sqrt{\sigma^2 s+t-s}}{m_t^{2,3}-A}e^{-\frac{(m_t^{2,3}-A)^2}{2(\sigma^2 s+t-s)}}\,\mathrm{d}s\\ &\lesssim_{A,\alpha,\beta,\sigma^2} t^{-1/2}\int_{\sqrt{t}\log t}^{t} e^{\beta s+(t-s)-\frac{t^2}{(\sigma^2 s+t-s)}}e^{\frac{t\log t}{2(\sigma^2 s+t-s)}}\,\mathrm{d}s.\end{aligned}$$

Set $\varphi: u\mapsto \beta u+(1-u)-\frac{1}{(\sigma^2 u+1-u)}$. Making a change of variable $s=ut$, we have

$$\mathbb{E}\big(Y_t^A\big)\lesssim t^{1/2}\int_{\frac{\log t}{\sqrt{t}}}^{1}\exp\left\{t\varphi(u)+\frac{\log t}{2(\sigma^2 u+1-u)}\right\}\mathrm{d}u \le t^{\frac12+\frac{1}{2\sigma^2}}\int_{\frac{\log t}{\sqrt{t}}}^{1} e^{t\varphi(u)}\,\mathrm{d}u.$$

Since $\varphi'(u)=\beta-1-\frac{1-\sigma^2}{(\sigma^2u+1-u)^2}<0$ and $\varphi''(u)=-\frac{2(1-\sigma^2)^2}{(\sigma^2u+1-u)^3}<0$, φ is concave, and takes maximum $\varphi(0)=0$ at point $u=0$. By Taylor's expansion, there exists a constant $\delta>0$ (depending only on σ^2) such that $\varphi(u)\le -\delta u^2$ for all $u\in[0,1]$. Thus

$$\mathbb{E}\big(Y_t^A\big)\lesssim t^{\frac12+\frac{1}{2\sigma^2}}\int_{\frac{\log t}{\sqrt{t}}}^{1} e^{-\delta u^2 t}\,\mathrm{d}u \le t^{\frac{1}{2\sigma^2}}\int_{\log t}^{\infty} e^{-\delta z^2}\,\mathrm{d}z \overset{t\to\infty}{\longrightarrow} 0,$$

which gives the desired result. □

Proof of Lemma 3.1. Fix $A, K, t\ge 0$ and $R>1$. Set $I_t^R := [0,\frac{1}{R}\sqrt{t})\cup(R\sqrt{t},\sqrt{t}\log t]$, and

$$Y_t(A,R,K)=\sum_{u\in\mathcal{B}} 1_{\{T_u\in I_t^R\}}1_{\{|X_u(T_u)|\le vT_u+K\}}1_{\{M_t^u\ge m_t^{2,3}-A\}},$$

where M_t^u is the maximal position of the descendants at time t of the individual u. In other words, $Y_t(A,R,K)$ is the number of type 2 particles that are born from a type 1 particle during the time interval I_t^R and have a descendant at time t above $m_t^{2,3}-A$. By Markov's inequality,

$$\begin{aligned}&\mathbb{P}\left(\exists u\in N_t^2: T_u\notin\left[\frac{1}{R}\sqrt{t},R\sqrt{t}\right], X_u(t)\ge m_t^{2,3}-A\right)\\ &\quad\le \mathbb{P}\big(\exists s\le t, u\in N_s^1:\big|X_u(s)\big|\ge vs+K\big)\\ &\qquad+\mathbb{P}\big(\exists u\in N_t^2: T_u\ge\sqrt{t}\log t, X_u(t)\ge m_t^{2,3}-A\big)+\mathbb{E}\big(Y_t(A,R,K)\big).\end{aligned}$$

By (2.4) and Lemma 3.2, it suffices to show that $\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{E}(Y_t(A,R,K))=0$.

Let $F_t(r,x):=\mathsf{P}(x+\mathsf{M}_r\ge m_t^{2,3}-A)$, where $\mathsf{M}_r=\max_{u\in\mathsf{N}_r}\mathsf{X}_u(r)$. Applying the branching property and Lemma 2.10(i), we have

$$\begin{aligned}\mathbb{E}\big(Y_t(A,R,K)\big)&=\mathbb{E}\left(\sum_{u\in\mathcal{B}}1_{\{T_u\in I_t^R\}}1_{\{|X_u(T_u)|\le vT_u+K\}}F_t\big(t-T_u,X_u(T_u)\big)\right)\\ &=\alpha\int_{I_t^R}e^{\beta s}\mathbf{E}\big(F_t(t-s,\sigma B_s)1_{\{|\sigma B_s|\le vs+K\}}\big)\,\mathrm{d}s\\ &=\alpha\int_{I_t^R}\mathbf{E}\big(e^{-\theta\sigma B_s}F_t(t-s,\sigma B_s+vs)\big)1_{\{-2vs-K\le\sigma B_s\le+K\}}\,\mathrm{d}s,\end{aligned}\tag{3.4}$$

where in the last equality we replaced σB_s by $\sigma B_s + vs$ by Girsanov's theorem. Moreover, since $\log t - \log(t-s) = o(1)$ for $s \le \sqrt{t}\log t$, we have for large t,

$$\begin{aligned} F_t(t-s, vs+z) &= \mathsf{P}\Big(\mathsf{M}_{t-s} \ge \sqrt{2}t - \frac{1}{2\sqrt{2}}\log t - A - vs - z\Big) \\ &\le \mathsf{P}\Big(\mathsf{M}_{t-s} \ge \sqrt{2}(t-s) - \frac{3}{2\sqrt{2}}\log(t-s) + (\sqrt{2}-v)s + \frac{1}{\sqrt{2}}\log t - z - 2A\Big). \end{aligned}$$

For $z \in [-2vs - K, K]$ and $s \le \sqrt{t}\log t$, we have $1 < (\sqrt{2}-v)s + \frac{1}{\sqrt{2}}\log t - z - 2A \lesssim \sqrt{t}\log t$ for large t. Applying Lemma 2.3, we have

$$F_t(t-s, vs+z) \lesssim_{A,\beta,\sigma^2} \frac{s + \log t + |z|}{t} e^{-\sqrt{2}(\sqrt{2}-v)s + \sqrt{2}z} e^{-\frac{(\sqrt{2}-v)^2 s^2}{2t}}, \tag{3.5}$$

where we used the fact that $-\frac{1}{2(t-s)}[(\sqrt{2}-v)s + \frac{1}{\sqrt{2}}\log t - z - 2A]^2 \le -\frac{(\sqrt{2}-v)^2 s^2}{2t}$ for large t. Replacing z by σB_s in (3.5) and then substituting (3.5) into (3.4), we obtain

$$\begin{aligned} \mathbb{E}\big(Y_t(A,R,K)\big) &\lesssim_{\alpha,A,\beta,\sigma^2} \int_{I_t^R} \mathbf{E}\bigg(\frac{s + \log t + |\sigma B_s|}{t} e^{-(\theta-\sqrt{2})\sigma B_s - \sqrt{2}(\sqrt{2}-v)s}\bigg) e^{-\frac{(\sqrt{2}-v)^2 s^2}{2t}}\,\mathrm{d}s \\ &= \int_{I_t^R} \mathbf{E}\bigg(\frac{s + \log t + |\sigma B_s - (\theta-\sqrt{2})\sigma^2 s|}{t}\bigg) e^{-\frac{(\sqrt{2}-v)^2 s^2}{2t}}\,\mathrm{d}s \\ &\lesssim \int_{[0,\frac{1}{R})\cup(R,\log t]} \bigg(\lambda + \frac{\log t + \mathbf{E}|B_{\lambda\sqrt{t}}|}{\sqrt{t}}\bigg) e^{-\frac{(\sqrt{2}-v)^2\lambda^2}{2}}\,\mathrm{d}\lambda, \end{aligned}$$

where in the equality we used the Girsanov's theorem and the fact that $\frac{1}{2}(\theta-\sqrt{2})^2\sigma^2 = \sqrt{2}(\sqrt{2}-v)$ when $\beta + \sigma^2 = 2$, and in the last $\lesssim$ we made a change of variable $s = \sqrt{t}\lambda$. Then the desired results follows from a simple computation. □

4. Boundary between $\mathscr{C}_I$ and $\mathscr{C}_{III}$

In this section we always assume that $(\beta,\sigma^2) \in \mathscr{B}_{I,III}$, i.e., $\beta > 1$, $\frac{1}{\beta} + \frac{1}{\sigma^2} = 2$. Recall that $m_t^{1,3} := vt - \frac{1}{2\theta}\log t$, where $v = \sqrt{2\beta\sigma^2}$ and $\theta = \sqrt{2\beta/\sigma^2}$.

Firstly, we describe the paths of extremal particles. It turns out that for a type 2 particle $u \in N_t^2$ above level $m_t^{1,3}$ at time t, its type transformation time T_u should be $s = t - \Theta(\sqrt{t})$, and its position x at time s should belong to the set

$$\Gamma_{s,t}^R := \big\{x : \big|\delta(x;s,t)\big| \le R\sqrt{t-s}\big\}, \quad \text{where } \delta(x;s,t) := x - vs + (\theta - v)(t-s).$$

Lemma 4.1. *For any $A > 0$, we have*

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\bigg(\exists u \in N_t^2 : t - T_u \notin \Big[\frac{1}{R}\sqrt{t}, R\sqrt{t}\Big], X_u(t) \ge m_t^{1,3} - A\bigg) = 0, \tag{4.1}$$

and

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\big(\exists u \in N_t^2 : X_u(t) \ge m_t^{1,3} - A, X_u(T_u) \notin \Gamma_{T_u,t}^R\big) = 0. \tag{4.2}$$

We will postpone the proof of Lemma 4.1 to the end of this section and show Theorem 1.2 first.

Proof of Theorem 1.2. In this proof, for simplicity, for $t > 0$, we set $\widehat{\mathscr{E}}_t := \sum_{u\in N_t^2} \delta_{X_u(t) - m_t^{1,3}}$, and for $R > 0$, we set $\Omega_t^R := \{(s,x) : t - s \in [\frac{1}{R}\sqrt{t}, R\sqrt{t}], x \in \Gamma_{s,t}^R\}$ and

$$\widehat{\mathscr{E}}_t^R := \sum_{u\in N_t^2} 1_{\{(T_u, X_u(T_u)) \in \Omega_t^R\}} \delta_{X_u(t) - m_t^{1,3}}.$$

Take $A > 0$ satisfying $\operatorname{supp}(\varphi) \subset [-A, \infty)$. For all $\varphi \in \mathcal{T}$, we have

$$\left|\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right) - \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\right)\right| \le \mathbb{P}\left(\exists u \in N_t^2,\, X_u(t) \ge m_t^{1,3} - A,\, \left(T_u, X_u(T_u)\right) \notin \Omega_t^R\right).$$

Thus by Lemma 4.1 we have

$$\lim_{R\to\infty} \limsup_{t\to\infty} \left|\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right) - \mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\right)\right| = 0. \tag{4.3}$$

We now show the convergence of $\widehat{\mathcal{E}}_t^R$ first as t and then R goes to ∞. Recall that $u' \succcurlyeq u$ means that u' is a descendant of u. We rewrite $\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle$ as

$$\sum_{\substack{u\in\mathcal{B}\\ (T_u, X_u(T_u))\in\Omega_t^R}} \sum_{\substack{u'\in N_t^2\\ u'\succcurlyeq u}} \varphi\left(X_{u'}(t) - X_u(T_u) - \theta(t - T_u) + \delta\left(X_u(T_u); T_u, t\right) + \frac{1}{2\theta}\log t\right).$$

Let $f(r,x) := \mathsf{E}(\exp\{-\sum_{u\in\mathsf{N}_r} \varphi(\mathsf{X}_u(r) - \theta r + x)\})$ and $F := 1 - f$. Using the branching property first and then applying Lemma 2.10(iii), we have

$$\begin{aligned}\mathbb{E}\left(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\right) &= \mathbb{E}\left[\prod_{\substack{u\in\mathcal{B}\\ (T_u, X_u(T_u))\in\Omega_t^R}} f\left(t - T_u, \delta\left(X_u(T_u); T_u, t\right) + \frac{1}{2\theta}\log t\right)\right]\\ &= \mathbb{E}\left(\exp\left\{-\alpha\int_{t-R\sqrt{t}}^{t-\frac{1}{R}\sqrt{t}} \sum_{u\in N_s^1} F\left(t - s, \delta\left(X_u(s); s, t\right) + \frac{1}{2\theta}\log t\right) 1_{\{X_u(s)\in\Gamma_{s,t}^R\}}\,\mathrm{d}s\right\}\right).\end{aligned}$$

Additionally, by Lemma 2.4(ii), uniformly in $s \in t - [\frac{1}{R}\sqrt{t}, R\sqrt{t}]$ and in $X_u(s) \in \Gamma_{s,t}^R$, the function

$$\begin{aligned}&F\left(t - s, \delta\left(X_u(s); s, t\right) + \frac{1}{2\theta}\log t\right)\\ &\sim \gamma_\theta(\varphi)\frac{e^{-(\frac{\theta^2}{2}-1)(t-s)}}{\sqrt{t-s}} e^{\theta[X_u(s) - vs + (\theta - v)(t-s) + \frac{1}{2\theta}\log t]} e^{-\frac{[X_u(s) - vs + (\theta-v)(t-s) + \frac{1}{2\theta}\log t]^2}{2(t-s)}}\\ &\sim \gamma_\theta(\varphi)\sqrt{\frac{t}{t-s}} e^{\theta[X_u(s) - vs]} e^{-\frac{[X_u(s) - vs + (\theta-v)(t-s)]^2}{2(t-s)}}, \quad \text{as } t\to\infty,\end{aligned}$$

where in the last "$\sim$" we used the fact that $\theta^2 - \theta v - (\frac{\theta^2}{2} - 1) = \frac{\beta}{\sigma^2} - 2\beta + 1 = 0$. Therefore, by making a change of variable $r = t - s$, we have $\mathbb{E}(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle})$

$$\begin{aligned}&= \mathbb{E}\left(\exp\left\{-\left(1 + o(1)\right)\alpha\gamma_\theta(\varphi)\int_{\frac{1}{R}\sqrt{t}}^{R\sqrt{t}} \sqrt{\frac{t}{r}} \sum_{\substack{u\in N_s^1\\ X_u(t-r)\in\Gamma_{t-r,t}^R}} e^{\theta[X_u(t-r) - v(t-r)]} e^{-\frac{[X_u(t-r) - v(t-r) + (\theta-v)r]^2}{2r}}\,\mathrm{d}r\right\}\right)\\ &= \mathbb{E}\left(\exp\left\{-\left(1 + o(1)\right)\alpha\gamma_\theta(\varphi)\int_{\frac{1}{R}}^{R} \frac{t}{\sqrt{r_{\lambda,t}}}\mathcal{W}^G(t - r_{\lambda,t}, r_{\lambda,t})\,\mathrm{d}\lambda\right\}\right),\end{aligned} \tag{4.4}$$

where in (4.4) we substitute r by $r_{\lambda,t} = \lambda\sqrt{t}$, and

$$\mathcal{W}^G(t,r) := \sum_{u\in N_t^1} e^{-\theta[vt - X_u(t)]} G\left(\frac{vt - X_u(t) - (\theta - v)r}{\sqrt{r}}\right), \qquad G(x) = e^{-\frac{x^2}{2}} 1_{\{|x|\le R\}}.$$

Recall that $(\{X_u(t)\}_{u\in N_t^1}, \mathbb{P})$ has the same distribution as $(\{\frac{\sigma}{\sqrt{\beta}}\mathsf{X}_u(\beta t)\}_{u\in\mathsf{N}_{\beta t}}, \mathsf{P})$. Applying Proposition 2.6, we have, for fixed $\lambda > 0$ and $r_{\lambda,t} = \lambda\sqrt{t}$,

$$\lim_{t\to\infty} \frac{t}{\sqrt{r_{\lambda,t}}}\mathcal{W}^G(t - r_{\lambda,t}, r_{\lambda,t}) = Z_\infty^{(1)}\sqrt{\frac{2}{\pi}}\frac{(\theta - v)}{\sigma^3}\lambda e^{-\frac{(\theta-v)^2}{2\sigma^2}\lambda^2}\int_{\mathbb{R}} e^{-\frac{x^2}{2}} 1_{\{|x|\le R\}}\,\mathrm{d}x$$

in probability, and by using (2.7) the integral of $\frac{t}{\sqrt{r_{\lambda,t}}}\mathcal{W}^G(t-r_{\lambda,t},r_{\lambda,t})$ with respect to λ over the interval $[1/R,R]$ converges in probability to the integral of right hand side. By (4.4) and the dominated convergence theorem, we have

$$\lim_{t\to\infty}\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\big)=\mathbb{E}\bigg(\exp\bigg\{-\alpha\gamma_\theta(\varphi)Z_\infty^{(1)}\sqrt{\frac{2}{\pi}}\int_{\frac{1}{R}}^R\frac{(\theta-v)\lambda}{\sigma^3}e^{-\frac{(\theta-v)^2}{2\sigma^2}\lambda^2}\,\mathrm{d}\lambda\int_{\mathbb{R}}e^{-\frac{x^2}{2}}1_{\{|x|\le R\}}\,\mathrm{d}x\bigg\}\bigg).$$

Letting $R\to\infty$, combining (4.3), we have

$$\lim_{t\to\infty}\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t,\varphi\rangle}\big)=\lim_{R\to\infty}\lim_{t\to\infty}\mathbb{E}\big(e^{-\langle\widehat{\mathcal{E}}_t^R,\varphi\rangle}\big)=\mathbb{E}\bigg(\exp\bigg\{-\gamma_\theta(\varphi)Z_\infty^{(I)}\frac{2\alpha}{\sigma(\theta-v)}\bigg\}\bigg),$$

which is the Laplace functional of DPPP($\frac{2\alpha\sigma C(\theta)}{\sqrt{2\pi}\beta(1-\sigma^2)}\theta Z_\infty^{(1)}e^{-\theta x}\,\mathrm{d}x,\mathfrak{D}^\theta$). Using [14, Lemma 4.4], we complete the proof of Theorem 1.2. □

Before proving Lemma 4.1, we show a weaker result first.

Lemma 4.2. *For any $A>0$ we have*

$$\limsup_{t\to\infty}\mathbb{P}\big(\exists u\in N_t^2:T_u\le t-\sqrt{t}\log t,X_u(t)\ge m_t^{1,3}-A\big)=0.$$

Proof. For $A>0$, we compute the mean of $Y_t^A:=\sum_{u\in N_t^2}1_{\{T_u\le t-\sqrt{t}\log t,X_u(t)\ge m_t^{1,3}-A\}}$. Applying Lemma 2.10(i) and Gaussian tail bounds, we have

$$\begin{aligned}\mathbb{E}\big(Y_t^A\big)&=\alpha\int_0^{t-\sqrt{t}\log t}e^{\beta s+t-s}\mathbf{P}\big(\sigma B_s+(B_t-B_s)\ge m_t^{1,3}-A\big)\,\mathrm{d}s\\&\lesssim\int_0^{t-\sqrt{t}\log t}e^{\beta s+(t-s)}\frac{\sqrt{\sigma^2s+t-s}}{m_t^{1,3}-A}e^{-\frac{(m_t^{1,3}-A)^2}{2(\sigma^2s+t-s)}}\,\mathrm{d}s\\&\lesssim_A t^{-1/2}\int_{\sqrt{t}\log t}^t e^{\beta(t-u)+u-\frac{v^2t^2}{2(\sigma^2(t-u)+u)}}e^{\frac{vt\log t}{\theta(\sigma^2(t-u)+u)}}\,\mathrm{d}u.\end{aligned}$$

Set $\varphi:\lambda\mapsto\beta(1-\lambda)+\lambda-\frac{\beta\sigma^2}{\sigma^2+(1-\sigma^2)\lambda}$. Making a change of variable $u=t\lambda$, we get

$$\mathbb{E}\big(Y_t^A\big)\lesssim_A t^{\frac{1}{2}+\frac{v}{\theta\sigma^2}}\int_{\frac{\log t}{\sqrt{t}}}^1 e^{t\varphi(\lambda)}\,\mathrm{d}\lambda.$$

As $\varphi'(\lambda)=1-\beta+\beta\sigma^2\frac{1-\sigma^2}{(\sigma^2+(1-\sigma^2)\lambda)^2}<0$ and $\varphi''(u)=-\beta\sigma^2\frac{2(1-\sigma^2)^2}{(\sigma^2+(1-\sigma^2)\lambda)^3}<0$, φ is concave, and takes maximum $\varphi(0)=0$ at point $\lambda=0$. By Taylor's expansion, there is a constant $\delta>0$ (depending only on σ^2) such that $\varphi(\lambda)\le-\delta\lambda^2$ for all $\lambda\in[0,1]$. Hence

$$\mathbb{E}\big(Y_t^A\big)\lesssim t^{\frac{1}{2}+\frac{v}{\theta\sigma^2}}\int_{\frac{\log t}{\sqrt{t}}}^1 e^{-\delta\lambda^2t}\,\mathrm{d}\lambda\le t^{\frac{v}{\theta\sigma^2}}\int_{\log t}^\infty e^{-\delta z^2}\,\mathrm{d}z\overset{t\to\infty}{\longrightarrow}0,$$

which gives the desired result. □

Proof of Lemma 4.1. *Step 1.* We first prove (4.1). As shown in the proof of Lemma 3.1, it suffices to bound the mean of

$$Y_t(A,K,R):=\sum_{u\in N_t^2}1_{\{t-T_u\in I_t^R\}}1_{\{X_u(r)\le vr+K,\forall r\le T_u\}}1_{\{X_u(t)\ge m_t^{1,3}-A\}},$$

where $A, R, K \geq 0$ and $I_t^R := [0, \frac{1}{R}\sqrt{t}) \cup (R\sqrt{t}, \sqrt{t}\log t]$. Applying Lemma 2.10(i), we have

$$\begin{aligned}\mathbb{E}\big[Y_t(A,K,R)\big] &= \alpha \int_{t-I_t^R} e^{\beta s+(t-s)} \mathbf{P}\begin{pmatrix}\sigma B_s + B_t - B_s \geq m_t^{1,3} - A,\\ \sigma B_r \leq vr + K, \forall r \leq s\end{pmatrix} \mathrm{d}s \\ &\lesssim \int_{t-I_t^R}\int_{-A}^{\infty} e^{\beta s+(t-s)} \mathbf{P}\begin{pmatrix}\sigma B_r \leq vr + K, \\ \forall r \leq s\end{pmatrix}\Bigg| \begin{matrix}\sigma B_s + B_t - B_s \\ = m_t^{1,3} + x\end{matrix}\Bigg) e^{-\frac{(m_t^{1,3}+x)^2}{2(\sigma^2 s+t-s)}} \frac{\mathrm{d}s\,\mathrm{d}x}{\sqrt{t}} \\ &\lesssim \int_{-A}^{\infty} \mathrm{d}x \int_{I_t^R} \frac{u+\log t+|x|}{t} e^{\beta(t-u)+u} e^{-\frac{(m_t^{1,3}+x)^2}{2(\sigma^2 t+(1-\sigma^2)u)}} \frac{\mathrm{d}u}{\sqrt{t}},\end{aligned} \tag{4.5}$$

where in the third line we used Lemma 2.2 and we substitute $t-s$ by u in the integral. Note that $\frac{1}{2\sigma^2 t}(m_t^{1,3}+x)^2 \geq \frac{1}{2\sigma^2 t}[v^2t^2 - 2vt(\frac{1}{2\theta}\log t - x)] = (\beta t - \frac{1}{2}\log t + \theta x)$. Then uniformly in $u \in [0, \sqrt{t}\log t]$, we have $1 - \frac{\sigma^2 t}{\sigma^2 t+(1-\sigma^2)u} = O(\frac{\log t}{\sqrt{t}})$, and hence

$$\begin{aligned}&\beta(t-u)+u - \frac{(m_t^{1,3}+x)^2}{2(\sigma^2 t+(1-\sigma^2)u)} \\ &= \beta(t-u)+u - \frac{(m_t^{1,3}+x)^2}{2\sigma^2 t}\frac{\sigma^2 t}{\sigma^2 t+(1-\sigma^2)u} \\ &\leq -(\beta-1)u + \frac{1}{2}\log t + \beta t\frac{(1-\sigma^2)u}{\sigma^2 t+(1-\sigma^2)u} - \big[\theta - o(1)\big]x + o(1).\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\begin{aligned}\mathbb{E}\big[Y_t(A,K,R)\big] &\lesssim \int_{-A}^{\infty} \mathrm{d}x e^{-\theta x/2} \int_{I_t^R} \frac{u+\log t+|x|}{t} e^{\frac{(1-\sigma^2)t}{\sigma^2 t+(1-\sigma^2)u}\beta u - (\beta-1)u}\,\mathrm{d}u \\ &= \int_{-A}^{\infty} \mathrm{d}x e^{-\theta x/2} \int_{I_t^R} \frac{u+\log t+|x|}{t} e^{-\frac{(\beta-1)(1-\sigma^2)u^2}{\sigma^2 t+(1-\sigma^2)u}}\,\mathrm{d}u,\end{aligned}$$

where in the equality we used the fact $\frac{(1-\sigma^2)t}{\sigma^2 t+(1-\sigma^2)u}\beta u - (\beta-1)u = \frac{-(\beta-1)(1-\sigma^2)u^2}{\sigma^2 t+(1-\sigma^2)u}$, which follows from the assumption $\frac{1}{\sigma^2} + \frac{1}{\beta} = 2$. Making a change of variable $u = \lambda\sqrt{t}$, we get

$$\mathbb{E}\big[Y_t(A,K,R)\big] \lesssim \int_{-A}^{\infty} \mathrm{d}x e^{-\theta x/2} \int_{[0,\frac{1}{R})\cup(R,\infty)} \frac{\lambda\sqrt{t}+\log t+|x|}{\sqrt{t}} e^{-\frac{(\beta-1)(1-\sigma^2)\lambda^2}{\sigma^2+(1-\sigma^2)/\sqrt{t}}}\,\mathrm{d}\lambda.$$

By the dominated convergence theorem, we get $\lim_{R\to\infty}\limsup_{t\to\infty}\mathbb{E}[Y_t(A,K,R)] = 0$ as desired.

Setp 2. Now we prove (4.2) by showing that for fixed $A, K, \epsilon > 0$, the expectation of

$$Y_t(A,K,\epsilon,R) := \sum_{u\in N_t^2} 1_{\{t-T_u \in [\epsilon\sqrt{t}, \epsilon^{-1}\sqrt{t}], X_u(r)\leq vr+K, \forall r\leq T_u\}} 1_{\{X_u(T_u)\notin \Gamma^R_{T_u,t}\}} 1_{\{X_u(t)\geq m_t^{1,3}-A\}}$$

vanishes as first $t\to\infty$ then $R\to\infty$. Thanks to Lemma 2.10(i),

$$\mathbb{E}\big[Y_t(A,K,\epsilon,R)\big] = \alpha\int_{t-\frac{1}{\epsilon}\sqrt{t}}^{t-\epsilon\sqrt{t}} e^{\beta s+(t-s)}\mathbf{P}\begin{pmatrix}\sigma B_s + B_t - B_s \geq m_t^{1,3} - A,\\ \big|\sigma B_s - vs - (\theta - v)(t-s)\big| > R\sqrt{t-s},\\ \sigma B_r \leq vr + K, \forall r \leq s\end{pmatrix} \mathrm{d}s. \tag{4.7}$$

Then conditioned on $\sigma B_s = vs - (\theta - v)(t-s) + x$, by the Markov property, and noting that $m_t^{1,3} - [vs - (\theta - v)(t - s) + x] = \theta(t-s) - x - \frac{1}{2\theta}\log t$,

$$
\begin{aligned}
&\mathbf{P}\begin{pmatrix}\sigma B_s + B_t - B_s \geq m_t^{1,3} - A,\\ \big|\sigma B_s - vs - (\theta - v)(t-s)\big| > R\sqrt{t-s},\\ \sigma B_r \leq vr + K, \forall r \leq s\end{pmatrix}\\
&\lesssim \int_{\substack{|x|>R\sqrt{t-s}\\ x\leq(\theta-v)(t-s)+K}} \frac{\mathrm{d}x}{\sqrt{s}} e^{-\frac{(vs-(\theta-v)(t-s)+x)^2}{2\sigma^2 s}} \mathbf{P}\Big(B_{t-s} > \theta(t-s) - x - \frac{1}{2\theta}\log t - A\Big)\\
&\quad\times \mathbf{P}\big(\sigma B_u \leq vu + K, \forall u \leq s \mid \sigma B_s = vs - (\theta - v)(t-s) + x\big)\\
&\lesssim \frac{e^{-\beta s + (\frac{\theta^2}{2} - \theta v)(t-s)}}{\sqrt{t-s}} \int_{|x|>R\sqrt{t-s}} \frac{t-s+|x|}{t} e^{-\frac{((\theta-v)(t-s)-x)^2}{2\sigma^2 s}} e^{-\frac{(x+\frac{\log t}{2\theta}+A)^2}{2(t-s)}}\,\mathrm{d}x,
\end{aligned}
\tag{4.8}
$$

where the last inequality follows from the following three items:

- $\exp\{-\frac{(vs-(\theta-v)(t-s)+x)^2}{2\sigma^2 s}\} = e^{-\beta s + (\theta^2 - \theta v)(t-s)} e^{-\theta x} \exp\{-\frac{((\theta-v)(t-s)-x)^2}{2\sigma^2 s}\}$.
- Gaussian tail bounds yield that for $s \in [t - \epsilon^{-1}\sqrt{t}, t - \epsilon\sqrt{t}]$ and $x \leq (\theta - v)(t-s) + K$,

$$
\mathbf{P}\Big(B_{t-s} > \theta(t-s) - x - \frac{\log t}{2\theta} - A\Big) \lesssim_A \frac{\sqrt{t}}{\sqrt{t-s}} e^{-\frac{\theta^2}{2}(t-s)} e^{\theta x} \exp\Big\{-\frac{(x + \frac{\log t}{2\theta} + A)^2}{2(t-s)}\Big\}.
$$

- Lemma 2.1 implies that for $s \in [t - \epsilon^{-1}\sqrt{t}, t - \epsilon\sqrt{t}]$ and $x \leq (\theta - v)(t-s) + K$,

$$
\mathbf{P}\big(\sigma B_u \leq vu + K, \forall u \leq s | \sigma B_s = vs - (\theta - v)(t-s) + x\big) \lesssim \big(t - s + |x|\big)/t.
$$

Substituting (4.8) into (4.7), noting that $\frac{\theta^2}{2} - \theta v + 1 = \frac{\beta}{\sigma^2} - 2\beta + 1 = 0$ and making change of variables $t - s = \lambda\sqrt{t}$, $y = x/\sqrt{t-s}$, we get

$$
\begin{aligned}
\mathbb{E}\big[Y_t(A, K, \epsilon, R)\big] &\lesssim \int_{t-\epsilon^{-1}\sqrt{t}}^{t-\epsilon\sqrt{t}} \mathrm{d}s \int_{|x|>R\sqrt{t-s}} \frac{t-s+|x|}{t} e^{-\frac{((\theta-v)(t-s)-x)^2}{2\sigma^2 s}} e^{-\frac{(x+\frac{\log t}{2\theta}+A)^2}{2(t-s)}} \frac{\mathrm{d}x}{\sqrt{t-s}}\\
&= \int_{\epsilon}^{\epsilon^{-1}} \mathrm{d}\lambda \int_{|y|>R} \frac{\lambda\sqrt{t} + t^{1/4}\lambda^{1/2}|y|}{\sqrt{t}} e^{-\frac{[(\theta-v)\lambda - y\lambda^{1/2}t^{-1/4}]^2}{2\sigma^2(1-\lambda t^{-1/2})}} e^{-\frac{1}{2}(y + \frac{\log t/(2\theta)+A}{\lambda^{1/2}t^{1/4}})^2}\,\mathrm{d}y.
\end{aligned}
$$

Letting $t \to \infty$, we have $\limsup_{t\to\infty} \mathbb{E}[Y_t(A, K, \epsilon, R)] \lesssim \int_{\epsilon}^{\epsilon^{-1}} \lambda e^{-\frac{(\theta-v)^2\lambda^2}{2\sigma^2}}\,\mathrm{d}\lambda \int_{|y|>R} e^{-\frac{1}{2}y^2}\,\mathrm{d}y$ by the dominated convergence theorem. Then letting $R \to \infty$, the desired result follows. □

5. Boundary between $\mathscr{C}_I$ and $\mathscr{C}_{II}$

In this section we always assume that $(\beta, \sigma^2) \in \mathscr{B}_{I,II}$, i.e., $\beta < 1$ and $\beta\sigma^2 = 1$. Recall that $m_t^{1,2} = m_t = \sqrt{2}t - \frac{3}{2\sqrt{2}}\log t$. As we have remarked, the driven mechanism of the asymptotic behavior of the extremal particles in this setting is the same as in the case $(\beta, \sigma^2) \in \mathscr{C}_{II}$. The outline of the proof is very similar to that of [13, Theorem 1.2], but more careful estimations are needed.

Firstly we show that each particle $u \in N_t^2$ above level $m_t^{1,2}$ at time t should satisfy $T_u = O(1)$.

Lemma 5.1. *For any $A > 0$, we have*

$$
\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\big(\exists u \in N_t^2 : T_u \geq R, X_u(t) \geq m_t^{1,2} - A\big) = 0.
$$

We postpone the proof of Lemma 5.1 in the end of this section. Lemma 5.1 implies that we can approximate $\widehat{\mathcal{E}}_t := \sum_{u\in N_t^2} \delta_{X_u(t) - m_t^{1,2}}$ by $\widehat{\mathcal{E}}_t^R := \sum_{u\in N_t^2} 1_{\{T_u \leq R\}} \delta_{X_u(t) - m_t^{1,2}}$. Indeed we have

$$
\lim_{R\to\infty} \limsup_{t\to\infty} \big|\mathbb{E}\big(e^{-\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle}\big) - \mathbb{E}\big(e^{-\langle \widehat{\mathcal{E}}_t, \varphi\rangle}\big)\big| = 0.
$$

By the results in [3,9], a simple computation gives the convergence of $\widehat{\mathcal{E}}_t^R$ as t to ∞. We only restate the results here, see [13, Lemma 5.2] for a proof.

Lemma 5.2. *For any $\varphi \in \mathcal{T}$, we have $\lim_{t\to\infty}\langle \widehat{\mathcal{E}}_t^R, \varphi\rangle = \langle \widehat{\mathcal{E}}_\infty^R, \varphi\rangle$ in law, where*

$$\widehat{\mathcal{E}}_\infty^R = \mathrm{DPPP}\big(C_\star \bar{Z}_R \sqrt{2} e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\big).$$

Here $\bar{Z}_R$ is defined as follows. Firstly, for each $u \in \mathcal{B}$, the convergence of derivative martingale for the standard BBM and the branching property implies the a.s. convergence of

$$Z_t^{(u)} := \sum_{\substack{u' \in N_t^2 \\ u' \succcurlyeq u}} \big(\sqrt{2}t - X_{u'}(t)\big) e^{\sqrt{2}(X_{u'}(t)-\sqrt{2}t)}.$$

Let $Z_\infty^{(u)} := \liminf_{t\to\infty} Z_t^{(u)}$. Then $Z_\infty^{(u)} \overset{\text{law}}{=} \mathsf{Z}_\infty e^{\sqrt{2}X_u(T_u)-2T_u} > 0$ a.s., where Z_∞ is independent of $\{X_u(t) : u \in N_t\}_{t\geq 0}$. We set $\bar{Z}_R := \sum_{u\in\mathcal{B}} 1_{\{T_u \leq R\}} Z_\infty^{(u)}$. The main ingredient is to show that $\widehat{\mathcal{E}}_\infty^R$ converges in law as $R \to \infty$, which is done as follows.

Lemma 5.3. *For all $\varphi \in \mathcal{T}$, we have $\lim_{R\to\infty}\langle \widehat{\mathcal{E}}_\infty^R, \varphi\rangle = \langle \widehat{\mathcal{E}}_\infty, \varphi\rangle$ in law, where*

$$\widehat{\mathcal{E}}_\infty = \mathrm{DPPP}\big(C_\star \bar{Z}_\infty \sqrt{2} e^{-\sqrt{2}x}\,\mathrm{d}x, \mathfrak{D}^{\sqrt{2}}\big),$$

and $\bar{Z}_\infty := \lim_{R\to\infty} \bar{Z}_R = \sum_{u\in\mathcal{B}} Z_\infty^{(u)} < \infty$ a.s..

Proof. As shown in the proof of [13, Lemma 5.3], it is sufficient to prove that

$$Y := \sum_{u\in\mathcal{B}} \big(1 + \big(\sqrt{2}T_u - X_u(T_u)\big)_+\big) e^{\sqrt{2}X_u(T_u)-2T_u} < \infty \quad \mathbb{P}\text{-a.s.}$$

We claim that it is enough to show that for each $K \in \mathbb{N}$,

$$Y_K = \sum_{\substack{u\in\mathcal{B} \\ X_u(s)\leq\sqrt{2}s+\sigma K, s\leq T_u}} \big(1 + \big[\sqrt{2}T_u - X_u(T_u)\big]_+\big) e^{\sqrt{2}X_u(T_u)-2T_u} < \infty \quad \mathbb{P}\text{-a.s.}$$

In fact, as $\beta\sigma^2 = 1$, by (2.4) we have $\mathbb{P}(\bigcup_{K\in\mathbb{N}}\{\forall t \geq 0, u \in N_t^1, X_u(t) \leq \sqrt{2}t + \sigma K\}) = 1$. Then for almost every realization ω, we can find $K = K(\omega) \in \mathbb{N}$ large enough so that

$$X_u(t) \leq \sqrt{2}t + \sigma K, \quad \forall t \geq 0, u \in N_t^1.$$

As a consequence we have $Y(\omega) = Y_{K(\omega)}(\omega) < \infty$ for almost all ω.

Next we compute $\mathbb{E}(Y_K)$ for fixed $K \in \mathbb{N}$. Applying Lemma 2.10(ii), we have

$$\mathbb{E}(Y_K) = \alpha \int_0^\infty e^{\beta s} \mathbf{E}\big(1_{\{\sigma B_u \leq \sqrt{2}u + \sigma K, u\leq s\}}\big[1 + (\sqrt{2}s - \sigma B_s)_+\big] e^{\sqrt{2}(\sigma B_s - \sqrt{2}s)}\big)\,\mathrm{d}s.$$

By Girsanov's theorem, we replace $(\sigma B_u - \sqrt{2}u)_{u\leq s}$ by $(\sigma B_u)_{u\leq s}$ (noting $\beta\sigma^2 = 1$) and get

$$\begin{aligned}\mathbb{E}(Y_K) &\leq \alpha \int_0^\infty \mathbf{E}\big(1_{\{B_u \leq K, u\leq s\}}\big(1 + \sigma|B_s|\big) e^{\sqrt{2}(\sigma - \sqrt{\beta})B_s}\big)\,\mathrm{d}s \\ &\leq \alpha \int_0^\infty \mathbf{E}\bigg(\sum_{n\geq 0} 1_{\{B_u \leq K, u\leq s; B_s - K \in [-n-1,-n]\}}\big(1 + \sigma|B_s|\big) e^{\sqrt{2}(\sigma-\sqrt{\beta})B_s}\bigg)\,\mathrm{d}s.\end{aligned}$$

Thanks to the Brownian estimate (2.1) and noticing that $\sigma = \frac{1}{\sqrt{\beta}} > \sqrt{\beta}$, we have

$$\mathbb{E}(Y_K) \lesssim \alpha \int_0^\infty \frac{K}{(1+s)^{3/2}} \sum_{n\geq 0} (1 + K + n)^2 e^{\sqrt{2}(\sigma-\sqrt{\beta})(K-n)}\,\mathrm{d}s < \infty,$$

which implies $Y_K < \infty$, $\mathbb{P}$-a.s. We complete the proof. □

Now it suffices to show Lemma 5.1.

Proof of Lemma 5.1. The proof consists of two step. For convenience we write $\varepsilon(t) = (\log t)^3$. Firstly we show that

$$\limsup_{t\to\infty} \mathbb{P}\big(\exists u \in N_t^2 : T_u \geq t - \varepsilon(t), X_u(t) \geq m_t^{1,2} - A\big) = 0; \tag{5.1}$$

and secondly, we prove

$$\lim_{R\to\infty} \limsup_{t\to\infty} \mathbb{P}\big(\exists u \in N_t^2 : T_u \in \big[R, t - \varepsilon(t)\big], X_u(t) \geq m_t^{1,2} - A\big) = 0. \tag{5.2}$$

Step 1 (Proof of (5.1)). Recall that $a_t = \frac{3}{2\theta}\log(t+1)$, $t \geq 0$. As shown in the proof of Lemma 3.1, thanks to the inequality (2.3), it suffices to show that for each $K \geq 1$, the mean of

$$Y_t(K) := \sum_{u\in N_t^2} 1_{\{X_u(t)\geq m_t^{1,2}-A\}} 1_{\{T_u > t-\varepsilon(t)\}} 1_{\{X_u(r)\leq \sqrt{2}r - a_t + a_{t-r} + K, r\leq T_u\}}$$

vanishes as $t \to \infty$. Applying Lemma 2.10(i) and the Markov property for Brownian motion,

$$\begin{aligned}
\mathbb{E}\big[Y_t(K)\big] &= \alpha \int_{t-\varepsilon(t)}^t e^{\beta s + t - s} \mathbf{P}\begin{pmatrix} \sigma B_s + B_t - B_s \geq m_t^{1,2} - A \\ \sigma B_r \leq \sqrt{2}r - a_t + a_{t-r} + K, r \leq s \end{pmatrix} \mathrm{d}s \\
&= \alpha \int_{t-\varepsilon(t)}^t e^{\beta s} \mathbf{E}\bigg(F\Big(t-s, \sigma B_s - \sqrt{2}s + \frac{3}{2\sqrt{2}}\log t + A\Big) 1_{\{\sigma B_r \leq \sqrt{2}r - a_t + a_{t-r} + K, r\leq s\}}\bigg) \mathrm{d}s,
\end{aligned}$$

where $F(r,x) = e^r \mathbf{P}(B_r \geq \sqrt{2}r - x)$. By Markov's inequality, $F(r,x) \leq e^{\sqrt{2}x}$. By Girsanov's theorem, we replace $\sigma B_s - \sqrt{2}s$ by σB_s, and then the integral above equals to

$$\int_{t-\varepsilon(t)}^t \mathbf{E}\bigg(e^{-\sqrt{2\beta}B_s} F\Big(t-s, \sigma B_s + \frac{3}{2\sqrt{2}}\log t + A\Big) 1_{\{\sigma B_r \leq a_{t-r} - a_t + K, r\leq s\}}\bigg) \mathrm{d}s.$$

Thus

$$\mathbb{E}\big[Y_t(K)\big] \lesssim_{A,\alpha} t^{3/2} \int_{t-\varepsilon(t)}^t \mathbf{E}\big(e^{\sqrt{2}(\sigma-\sqrt{\beta})B_s} 1_{\{\sigma B_r \leq a_{t-r} - a_t + K, r\leq s\}}\big)\, \mathrm{d}s.$$

By inequality (6.3) in [13] with $\lambda = \sqrt{2} - \theta$ (or making a change of measure to Bessel-3 process), we have

$$\mathbf{E}\big(e^{\sqrt{2}(\sigma-\sqrt{\beta})B_s} 1_{\{\sigma B_r \leq a_{t-r} - a_t + K, r\leq s\}}\big) \leq C e^{(\sqrt{2}-\theta)K} \bigg(\frac{t-s+1}{t+1}\bigg)^{\frac{3(\sqrt{2}-\theta)}{2\theta}} (s+1)^{-\frac{3}{2}}.$$

Since now $\theta < \sqrt{2}$, a simple computation yields

$$\mathbb{E}\big[Y_t(K)\big] \lesssim_{A,\alpha,K} t^{3/2} \int_{t-\varepsilon(t)}^t \bigg(\frac{t-s+1}{t+1}\bigg)^{\frac{3(\sqrt{2}-\theta)}{2\theta}} (s+1)^{-\frac{3}{2}}\, \mathrm{d}s \overset{t\to\infty}{\longrightarrow} 0.$$

Step 2 (Proof of (5.2)). As in Step 1, it suffices to show that for each $A, R, K > 0$, the expectation of

$$Y_t(A,R,K) := \sum_{u\in\mathcal{B}} 1_{\{T_u \in [R, t-\varepsilon(t)]\}} 1_{\{M_t^u \geq m_t^{1,2} - A\}} 1_{\{X_u(r)\leq \sqrt{2}r + \sigma K, r\leq T_u\}}$$

converges to 0 as first $t \to \infty$ then $R \to \infty$. Let $F_t(r,x) := \mathsf{P}(x + \mathsf{M}_r > m_t^{1,2} - A)$, where M_r is the maximum at time r of a standard BBM. By the branching Markov property, Lemma 2.10(ii), and Girsanov's theorem

$$\begin{aligned}\mathbb{E}\big[Y_t(A,R,K)\big] &= \mathbb{E}\bigg[\sum_{u\in\mathcal{B}} 1_{\{T_u\in[R,t-\varepsilon(t)]\}} 1_{\{X_u(r)\le\sqrt{2}r+\sigma K, r\le T_u\}} F_t\big(t-T_u, X_u(T_u)\big)\bigg]\\ &= \alpha\int_R^{t-\varepsilon(t)} e^{\beta s}\mathbf{E}\big[F_t(t-s,\sigma B_s)1_{\{B_r\le\sqrt{2\beta}r+K, r\le s\}}\big]\,\mathrm{d}s\\ &= \alpha\int_R^{t-\varepsilon(t)} \mathbf{E}\big[e^{-\sqrt{2\beta}B_s}F_t(t-s,\sigma B_s+\sqrt{2}s)1_{\{B_r\le K, r\le s\}}\big]\,\mathrm{d}s.\end{aligned}$$

Applying (2.5), we have for all $s \le t-\varepsilon(t)$ and $x\in\mathbb{R}$,

$$\begin{aligned}F_t(t-s,x) &= \mathsf{P}\bigg(\mathsf{M}_{t-s} \ge m_{t-s}^{1,2} + \sqrt{2}s + \frac{3}{2\sqrt{2}}\log\frac{t-s}{t} - A - x\bigg)\\ &\lesssim_A \bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\bigg(1+|\sqrt{2}s-x|+\log\frac{t}{t-s}\bigg)e^{-\sqrt{2}(\sqrt{2}s-x)},\end{aligned}$$

We now get upper bound the expectation:

$$\mathbb{E}\big[Y_t(A,R,K)\big] \lesssim_A \alpha\int_R^{t-\varepsilon(t)}\bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\mathbf{E}\bigg[\bigg(1+|B_s|+\log\frac{t}{t-s}\bigg)e^{\sqrt{2}(\sigma-\sqrt{\beta})B_s}1_{\{B_r\le K, r\le s\}}\bigg]\,\mathrm{d}s.$$

Applying (2.1), as in the proof of Lemma 5.3, we have

$$\begin{aligned}&\mathbf{E}\bigg[\bigg(1+|B_s|+\log\frac{t}{t-s}\bigg)e^{\sqrt{2}(\sigma-\sqrt{\beta})B_s}1_{\{B_r\le K, r\le s\}}\bigg]\\ &\le \sum_{n\ge 0}\bigg(2+K+n+\log\frac{t}{t-s}\bigg)e^{\sqrt{2}(\sigma-\sqrt{\beta})(K-n)}\frac{CKn}{s^{3/2}} \lesssim_{\beta,K} \frac{(1+\log\frac{t}{t-s})}{s^{3/2}}.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}\big[Y_t(A,R,K)\big] &\lesssim_{\alpha,\beta,A,K} \int_R^{t-\varepsilon(t)}\bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\frac{(1+\log\frac{t}{t-s})}{s^{3/2}}\,\mathrm{d}s\\ &= \bigg(\int_R^{t/2}+\int_{t/2}^{t-\varepsilon(t)}\bigg)\bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\frac{(1+\log\frac{t}{t-s})}{s^{3/2}}\,\mathrm{d}s =: (I)+(II).\end{aligned}$$

For (I), since $\frac{t}{t-s}\le 2$ when $s\in[R,t/2]$, we have

$$\int_R^{t/2}\bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\frac{(1+\log\frac{t}{t-s})}{s^{3/2}}\,\mathrm{d}s \le 100\int_R^{\infty}\frac{1}{s^{3/2}}\,\mathrm{d}s \overset{R\to\infty}{\longrightarrow} 0.$$

For (II), making a change of variable $t-s=\lambda t$, we have

$$\begin{aligned}\int_{t/2}^{t-\varepsilon(t)}\bigg(\frac{t}{t-s}\bigg)^{\frac{3}{2}}\frac{(1+\log\frac{t}{t-s})}{s^{3/2}}\,\mathrm{d}s &= \frac{1}{\sqrt{t}}\int_{\varepsilon(t)/t}^{1/2}\frac{1}{\lambda^{3/2}}\bigg(1+\log\frac{1}{\lambda}\bigg)\frac{1}{(1-\lambda)^{3/2}}\,\mathrm{d}\lambda\\ &\lesssim \frac{1+\log t}{\sqrt{t}}\int_{\varepsilon(t)/t}^{1/2}\frac{1}{\lambda^{3/2}}\,\mathrm{d}\lambda \lesssim \frac{1+\log t}{\sqrt{t}}\sqrt{\frac{t}{\varepsilon(t)}} \overset{t\to\infty}{\longrightarrow} 0.\end{aligned}$$

We now complete the proof. □

6. Spine decomposition and Proofs of Propositions 2.5 and 2.6

6.1. *Proof of Proposition 2.5*

Proof. It is well-known that, for $\eta \in [0, \sqrt{2})$, the additive martingale

$$\mathsf{W}_t(\eta) = \sum_{u\in\mathsf{N}_t} e^{\eta \mathsf{X}_u(t) - (\eta^2/2+1)t}$$

is uniformly integrable and converges to a non-trivial limit $\mathsf{W}_\infty(\eta)$ as $t \to \infty$. Moreover, the law of the BBM under the probability measure tilted by $\mathsf{W}_t(\eta)$, can be described naturally by introducing a spine (see for example [41]). We identify the spine as a distinguished genealogical line of descent from the initial ancestor and denote the individual on the spine at time t by ξ_t. More precisely, there exists a branching diffusion $(\{\mathsf{X}_u(t) : u \in \mathsf{N}_t, \xi_t\}_{t\geq 0}, \widehat{\mathsf{P}})$ with distinguished and randomized spine, defined on some enlarged probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathsf{P}})$, having the following properties:

(i) the diffusion along the spine begins from the origin of space at time 0 and moves according to a Brownian motion with drift η,
(ii) points of fission along the spine are independent of its motion and occur with rate 2,
(iii) at each fission time of the spine, the spine gives birth to 2 offspring, and the spine is chosen randomly so that at each fission point, the next individual to represent the spine is chosen with uniform probability from the two offsprings,
(iv) offspring of individuals on the spine which are not part of the spine initiate P-branching Brownian motions at their space-time point of creation.

Let $\mathcal{F}_t$ be the σ-algebra generated by the BBM up to time t, i.e., $\mathcal{F}_t = \sigma(\{\mathsf{X}_u(s) : s \leq t, u \in \mathsf{N}_t\})$. It was established in [41] that

$$\mathrm{d}\widehat{\mathsf{P}}|_{\mathcal{F}_t} = \mathsf{W}_t(\eta)\,\mathrm{d}\mathsf{P}|_{\mathcal{F}_t} \quad \text{and} \quad \widehat{\mathsf{P}}(\xi_t = u|\mathcal{F}_t) = \frac{e^{\eta \mathsf{X}_u(t) - (\eta^2/2+1)t}}{\mathsf{W}_t(\eta)} 1_{\{u\in\mathsf{N}_t\}}.$$

Denote $\Xi(t) := \mathsf{X}_{\xi_t}(t)$, i.e., Ξ_t is the position of the individual on the spine at time t. We now have, for every $t \geq 0$ and measurable function f,

$$\begin{aligned}\widehat{\mathsf{E}}\left[f\left(\frac{\Xi(t)}{t}\right)\middle|\mathcal{F}_t\right] &= \widehat{\mathsf{E}}\left[\sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t)}{t}\right)1_{\{\xi_t=u\}}\middle|\mathcal{F}_t\right] \\ &= \sum_{u\in\mathsf{N}_t} f\left(\frac{\mathsf{X}_u(t)}{t}\right)\frac{e^{\eta \mathsf{X}_u(t) - (\eta^2/2+1)t}}{\mathsf{W}_t(\eta)} = \frac{W_t^f(\eta)}{W_t(\eta)}.\end{aligned}$$

Since under $\widehat{\mathsf{P}}$, Ξ is a Brownian motion with drift η, and the function $x \mapsto f(x)$ is bounded continuous, $f(\frac{\Xi(t)}{t})$ converges to $f(\eta)$ in $L^1(\widehat{\mathsf{P}})$. Thanks to the Jensen's inequality we have

$$\lim_{t\to\infty} \frac{W_t^f(\eta)}{W_t(\eta)} = \lim_{t\to\infty} \widehat{\mathsf{E}}\left[f\left(\frac{\Xi(t)}{t}\right)\middle|\mathcal{F}_t\right] \to f(\eta) \quad \text{in } L^1(\widehat{\mathsf{P}}).$$

Thus

$$\mathsf{E}\big[\big|W_t^f(\eta) - f(\eta)W_t(\eta)\big|\big] = \widehat{\mathsf{E}}\left[\left|\frac{W_t^f(\eta)}{W_t(\eta)} - f(\eta)\right|\right] \to 0,$$

yielding the desired result. □

6.2. *Proof of Proposition 2.6*

The proof is inspired by [4]. Fix an arbitrarily $K > 0$. Define the *truncated derivative martingale* $\mathsf{D}_t^{(K)}$ as the following:

$$\mathsf{D}_t^{(K)} := \sum_{u\in\mathsf{N}_t} \big(\sqrt{2}t - \mathsf{X}_u(t) + K\big)e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))} 1_{\{\sqrt{2}s - \mathsf{X}_u(s) \geq -K, \forall s\leq t\}}. \tag{6.1}$$

It was proved in [41] that $(\mathsf{D}_t^{(K)})_{t>0}$ is a uniformly integrable martingale and $\lim_{t\to\infty} \mathsf{D}_t^{(K)} = \mathsf{Z}_\infty$ a.s. on the event $\{\mathsf{X}_u(t) \leq \sqrt{2}t + K, \forall t \geq 0, u \in \mathsf{N}_t\}$, whose probability tends to 1 when $K \to \infty$. Moreover, using this martingale to do

a change of measure, we can also get a "spine decomposition" of the BBM: Let $\mathcal{F}_t$ be the σ-algebra generated by the branching Brownian motion up to time t. Let $\mathsf{Q}^{(K)}$ be the probability measure such that

$$\mathrm{d}\mathsf{Q}^{(K)}|_{\mathcal{F}_t} = \frac{\mathsf{D}_t^{(K)}}{K} \cdot \mathrm{d}\mathsf{P}|_{\mathcal{F}_t}. \tag{6.2}$$

Similar to the spine decomposition in Section 6.1, we can identify distinguished genealogical lines of descent from the initial ancestor each of which shall be referred to as a spine. Denoting by ξ_t the individual belonging to the spine, and by $\Xi(t) = \mathsf{X}_{\xi_t}(t)$ the position of this individual. The process $(\mathsf{X}_u(t) : u \in \mathsf{N}_t)_{t\geq 0}$ under $\mathsf{Q}^{(K)}$ has similar properties as in Section 6.1 with (i) changed to (i′) below and other items (ii) to (iv) unchanged:

(i′) The spatial motion $(\Xi(t))_{t\geq 0}$ of the spine is such that $R_K(t) := \sqrt{2}t - \Xi(t) + K$ is a 3-dim Bessel process started at K.

For any $t > 0$ and any $u \in \mathsf{N}_t$, we have

$$\mathsf{Q}^{(K)}(\xi_t = u|\mathcal{F}_t) = \frac{(\sqrt{2}t - \mathsf{X}_u(t) + K)e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))}}{\mathsf{D}_t^{(K)}} 1_{\{\sqrt{2}s - \mathsf{X}_u(s) \geq -K, \forall s \leq t\}}. \tag{6.3}$$

Throughout this section, we use $(R_t, \mathbf{P}_x^{\mathrm{Bes}})$ to denote a Bessel-3 process starting from x. The expectation with respect to $\mathbf{P}_x^{\mathrm{Bes}}$ is denoted as $\mathbf{E}_x^{\mathrm{Bes}}$. We truncate $\mathsf{W}_t^{F_t}(\sqrt{2})$ as in (6.1), that is to say, we define

$$\mathsf{W}_t^{(K),F_t}(\sqrt{2}) := \sum_{u\in\mathsf{N}_t} F_t\left(\frac{\sqrt{2}t - \mathsf{X}_u(t)}{\sqrt{t}}\right) e^{-\sqrt{2}(\sqrt{2}t - \mathsf{X}_u(t))} 1_{\{\sqrt{2}s - \mathsf{X}_u(s) \geq -K, \forall s \leq t\}}. \tag{6.4}$$

Proposition 6.1. *For each fixed $K \in \mathbb{N}$ and $\lambda > 0$,*

$$\mathsf{E}_{\mathsf{Q}^{(K)}}\left|\frac{\sqrt{t}}{\langle F_t, \mu\rangle}\frac{\mathsf{W}_t^{(K),F_t}(\sqrt{2})}{\mathsf{D}_t^{(K)}} - \sqrt{\frac{2}{\pi}}\right| \to 0, \quad \text{as } t \to \infty. \tag{6.5}$$

Proposition 2.6 is an easy consequence of Proposition 6.1.

Proof of Proposition 2.6 admitting Proposition 6.1. (i) Applying (6.2) and (6.5), we have, for any $K > 0$,

$$\mathsf{E}\left|\frac{\sqrt{t}}{\langle F_t, \mu\rangle}\mathsf{W}_t^{(K),F_t}(\sqrt{2}) - \sqrt{\frac{2}{\pi}}\mathsf{D}_t^{(K)}\right| = K\,\mathsf{E}_{\mathsf{Q}^{(K)}}\left|\frac{\sqrt{t}}{\langle F_t, \mu\rangle}\frac{\mathsf{W}_t^{(K),F_t}(\sqrt{2})}{\mathsf{D}_t^{(K)}} - \sqrt{\frac{2}{\pi}}\right| \overset{t\to\infty}{\longrightarrow} 0.$$

Since $\mathsf{D}_t^{(K)}$ is a uniformly integrable martingale, writing $\mathsf{D}_\infty^{(K)} := \lim_t \mathsf{D}_t^{(K)}$, we have $\mathsf{D}_t^{(K)} \overset{t\to\infty}{\longrightarrow} \mathsf{D}_\infty^{(K)}$ in $L^1(\mathsf{P})$. Combined with the equation above, we have

$$\mathsf{E}\left[\left|\frac{\sqrt{t}}{\langle F_t, \mu\rangle}\mathsf{W}_t^{(K),F_t}(\sqrt{2}) - \sqrt{\frac{2}{\pi}}\mathsf{D}_\infty^{(K)}\right|\right] \overset{t\to\infty}{\longrightarrow} 0. \tag{6.6}$$

On the event $\{\mathsf{X}_u(t) \leq \sqrt{2}t + K, \forall t \geq 0, u \in \mathsf{N}_t\}$, which has high probability when K is large (see (2.4)), we have $\mathsf{W}_t^{(K),F_t}(\sqrt{2}) = \mathsf{W}_t^{F_t}(\sqrt{2})$ and $\mathsf{D}_\infty^{(K)} = \mathsf{Z}_\infty$. Therefore, $\frac{\sqrt{t}}{\langle F_t, \mu\rangle}\mathsf{W}_t^{F_t}(\sqrt{2})$ converges to $\sqrt{\frac{2}{\pi}}\mathsf{Z}_\infty$ in probability P as $t \to \infty$.

(ii) We now prove (2.7). Note that for $F_{t,\lambda} = G((z-\lambda)t^{1/4})$, $\langle F_{t,\lambda}, \mu\rangle \sim C_\lambda t^{-1/4}$ uniformly in $\lambda \in I$, where $C_\lambda = \lambda e^{-\frac{\lambda^2}{2}} \int G(y)\,\mathrm{d}y$, thus

$$\mathsf{W}_{t-\lambda\sqrt{t}}^{(K),F_{t,\lambda}}(\sqrt{2}) = \mathsf{W}_{\tilde{t}}^{(K),\widetilde{F}_{\tilde{t},\lambda}}(\sqrt{2})$$

with $\tilde{t} = t - \lambda\sqrt{t}$, $\widetilde{F}_{\tilde{t},\lambda} = G((z-\lambda)\tilde{h}_{\tilde{t}})$, and $\tilde{h}_{\tilde{t}} = t^{-1/4}$. Using (6.6) with t replaced by $\tilde{t}$ and h_t replaced by $\tilde{h}_{\tilde{t}}$, we get

$$\mathsf{E}\left|\int_I t^{3/4}\mathsf{W}_{t-\lambda\sqrt{t}}^{(K),F_{t,\lambda}}(\sqrt{2}) - \sqrt{\frac{2}{\pi}}C_\lambda\mathsf{D}_\infty^{(K)}\,\mathrm{d}\lambda\right| \leq \int_I \mathsf{E}\left|t^{3/4}\mathsf{W}_{t-\lambda\sqrt{t}}^{(K),F_{t,\lambda}}(\sqrt{2}) - \sqrt{\frac{2}{\pi}}C_\lambda\mathsf{D}_\infty^{(K)}\right|\mathrm{d}\lambda \to 0,$$

$t \to \infty$, where we used the dominated convergence theorem. In fact, (6.11) below implies that $\mathsf{E}[\frac{\sqrt{t}}{\langle F_{t,\lambda},\mu\rangle}\mathsf{W}^{(K),F_{t,\lambda}}_{t-\lambda\sqrt{t}}(\sqrt{2})] \le 1$ for all $\lambda \in I$ and for large t, which guarantees the existence of a dominating function. Then the argument below (6.6) yields that we can remove the truncation in $\mathsf{W}^{(K),F_{t,\lambda}}_{t-\lambda\sqrt{t}}(\sqrt{2})$ and $\mathsf{D}^{(K)}_{\infty}$ and get the desired convergence in probability result. □

The remaining part in this section is devoted to proving Proposition 6.1. The parameter K is always fixed. For simplicity, $\mathsf{W}^{(K),F_t}_t(\sqrt{2})$ is abbreviated as $\mathsf{W}^{(K),F}_t$. Recall that $R_K(t) = \sqrt{2}t - \Xi(t) + K$. We set

$$U^{(K)}_t := \frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{\mathsf{W}^{(K),F}_t}{\mathsf{D}^{(K)}_t}, \tag{6.7}$$

and

$$V^{(K)}_t := \frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{1}{R_K(t)}F_t\left(\frac{R_K(t)-K}{\sqrt{t}}\right).$$

To prove Proposition 6.1, we only need to prove that

$$\lim_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}\left|U^{(K)}_t - \sqrt{2/\pi}\right| = 0. \tag{6.8}$$

Lemma 6.2. *For each $t > 0$, we have*

$$U^{(K)}_t = \mathsf{E}_{\mathsf{Q}^{(K)}}\left[V^{(K)}_t|\mathcal{F}_t\right]. \tag{6.9}$$

Proof. By (6.3), we have

$$\begin{aligned}\mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{1}{R_K(t)}F_t\left(\frac{R_K(t)-K}{\sqrt{t}}\right)\Big|\mathcal{F}_t\right] &= \sum_{u\in\mathsf{N}_t}\mathsf{Q}^{(K)}[\xi_t = u|\mathcal{F}_t]\frac{F_t([\sqrt{2}t - \mathsf{X}_u(t)]/\sqrt{t})}{\sqrt{2}t - \mathsf{X}_u(t) + K}\\ &= \frac{1}{\mathsf{D}^{(K)}_t}\sum_{u\in\mathsf{N}_t}F_t\left(\frac{\sqrt{2}t-\mathsf{X}_u(t)}{\sqrt{t}}\right)e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}1_{\{\sqrt{2}s-\mathsf{X}_u(s)\ge -K,\forall s\le t\}} = \frac{\mathsf{W}^{(K),F}_t}{\mathsf{D}^{(K)}_t}\end{aligned}$$

as desired. □

Lemma 6.3. *For $p \in [1,2]$, define*

$$I_p(t) := \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\left(U^{(K)}_t\right)^p\right]. \tag{6.10}$$

Then

$$\lim_{t\to\infty} I_1(t) = \lim_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}\left[V^{(K)}_t\right] = \sqrt{\frac{2}{\pi}}, \tag{6.11}$$

and $\limsup_{t\to\infty}(r_t^2h_t)^{p-1}I_p(t) < \infty$.

Proof. Noticing that $(R_K(t),\mathsf{Q}^{(K)})$ is a Bessel-3 process starting from K, by (6.9), we have

$$I_1(t) = \mathsf{E}_{\mathsf{Q}^{(K)}}\left(V^{(K)}_t\right) = \mathbf{E}^{\mathrm{Bes}}_K\left(\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{1}{R_t}F_t\left(\frac{R_t-K}{\sqrt{t}}\right)\right). \tag{6.12}$$

For $p \in (1,2]$, since $x \mapsto x^p$ is convex, by (6.9) and Jensen's inequality,

$$I_p(t) \le \mathsf{E}_{\mathsf{Q}^{(K)}}\left(\left[V^{(K)}_t\right]^p\right) = \mathbf{E}^{\mathrm{Bes}}_K\left(\left[\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{1}{R_t}F_t\left(\frac{R_t-K}{\sqrt{t}}\right)\right]^p\right). \tag{6.13}$$

Recall that $F_t(z) = G([z-r_t]h_t^{-1})$ and our assumption that $r_t + yh_t = \Theta(r_t)$ uniformly for $y \in \mathrm{supp}(G)$, simple computations yield that

$$\langle F_t,\mu\rangle = \int_0^\infty G\left(\frac{z-r_t}{h_t}\right)ze^{-\frac{z^2}{2}}\,\mathrm{d}z = h_t\int_{\mathbb{R}}G(y)(r_t+yh_t)e^{-\frac{(r_t+yh_t)^2}{2}}\,\mathrm{d}y = \Theta(r_th_t). \tag{6.14}$$

We claim that, for $p \in [1,2]$ and any $\epsilon_0 > 0$,

$$\mathbf{E}_u^{\mathrm{Bes}}\left(\left[\frac{\sqrt{t}}{R_t}F_t\left(\frac{R_t-K}{\sqrt{t}}\right)\right]^p\right) \sim \sqrt{\frac{2}{\pi}}h_t\int_{\mathbb{R}} G(y)^p(r_t+yh_t)^{2-p}e^{-\frac{(r_t+yh_t)^2}{2}}\,\mathrm{d}y = \Theta\big(r_t^{2-p}h_t\big) \tag{6.15}$$

uniformly for $u \in (0, t^{\frac{1}{2}-\epsilon_0}]$. Then taking $p=1$ in (6.15) and combining (6.12), (6.14), we get (6.11). Taking $p \in (1,2]$ in (6.15) and using (6.13), (6.14), we get $I(t) \lesssim (r_t^2 h_t)^{1-p}$.

Now we show the claim. Suppose G is supported on $[-A, A]$ for some constant $A > 0$. First we recall the density formula for 3-dimensional Bessel process (see [53, Section VI.3])

$$\mathbf{P}_x^{\mathrm{Bes}}(R_s \in \mathrm{d}z) = \sqrt{\frac{2}{\pi}}z^2 e^{-\frac{(z-x)^2}{2s}}\frac{1-e^{-\frac{2xz}{s}}}{2xz\sqrt{s}}\,\mathrm{d}z, \quad \text{for } x, z > 0. \tag{6.16}$$

Using the scaling property for Bessel process, we have

$$\begin{aligned}\mathbf{E}_u^{\mathrm{Bes}}\left(\left[\frac{\sqrt{t}}{R_t}F_t\left(\frac{R_t-K}{\sqrt{t}}\right)\right]^p\right) &= \mathbf{E}^{\mathrm{Bes}}_{\frac{u}{\sqrt{t}}}\left[\frac{1}{R_1^p}F_t(R_1-K/\sqrt{t})^p\right] \\ &= \sqrt{\frac{2}{\pi}}\int_{|z-r_t-K/\sqrt{t}|\le Ah_t} G\left(\frac{z-r_t-K/\sqrt{t}}{h_t}\right)^p z^{2-p}e^{-\frac{(z-u/\sqrt{t})^2}{2s}}\frac{1-e^{-2zu/\sqrt{t}}}{2zu/\sqrt{t}}\,\mathrm{d}z \\ &= h_t\sqrt{\frac{2}{\pi}}\int_{-A}^{A} G(y)^p z_y^{2-p}e^{-\frac{1}{2}(z_y-\frac{u}{\sqrt{t}})^2}\frac{1-e^{-2z_yu/\sqrt{t}}}{2z_yu/\sqrt{t}}\,\mathrm{d}y,\end{aligned} \tag{6.17}$$

where in the last equality, we substituted z by $z_y = r_t + K/\sqrt{t} + yh_t$. By the dominated convergence theorem, we get that, for any $\epsilon_0 > 0$,

$$\int_{-A}^{A} G(y)^p z_y^{2-p}e^{-\frac{1}{2}(z_y-\frac{u}{\sqrt{t}})^2}\frac{1-e^{-2z_yu/\sqrt{t}}}{2z_yu/\sqrt{t}}\,\mathrm{d}y \sim \int_{-A}^{A} G(y)^p(r_t+yh_t)^{2-p}e^{-\frac{(r_t+yh_t)^2}{2}}\,\mathrm{d}y$$

uniformly for $u \in (0, t^{\frac{1}{2}-\epsilon_0}]$ as $t \to \infty$, and (6.15) follows. □

The naive bound $I_p(t) \lesssim (r_t^2 h_t)^{1-p}$ in Lemma 6.3 is not good enough to apply the second moment method (which needs to show that $\limsup_{t\to\infty} I_2(t) \le (\frac{2}{\pi})$). To overcome this difficulties, we need to further truncate $W_t^{(K),F}$, by only counting particles with typical behavior in the summation in (6.4). The arguments are borrowed from Aïdékon, Shi [4] and Pain [51]. Before giving the typical behavior of the particles we want, we need to introduce some notation as follows.

Recall that ξ_s represents the individual on the spine at time s. We write $\mathsf{N}_t^{\xi_s}$ as the descendants of ξ_s at time t. The points of fission along the spine form a Poisson process with rate 2, which is denoted by $\Lambda(\mathrm{d}s)$. We then define, for a interval $[c_1, c_2] \subset [0, t]$,

$$\mathsf{D}_t^{(K),[c_1,c_2]} := \int_{[c_1,c_2]}\sum_{u\in\mathsf{N}_t^{\xi_s}}\big(\sqrt{2}t - \mathsf{X}_u(t) + K\big)e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}1_{\{\sqrt{2}s-\mathsf{X}_u(s)\ge -K, \forall s\le t\}}\Lambda(\mathrm{d}s),$$

which is the contribution for the martingale $\mathsf{D}_t^{(K)}$ of particles who are descendants of individual on spine at time interval $[c_1, c_2]$. Then $\mathsf{D}_t^{(K)} = \mathsf{D}_t^{(K),[0,t]} = \mathsf{D}_t^{(K),[0,c_1)} + \mathsf{D}_t^{(K),[c_1,t]}$. Similarly, we define

$$W_t^{(K),F,[c_1,c_2]} := \int_{[c_1,c_2]}\sum_{u\in\mathsf{N}_t^{\xi_s}} e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}F_t\left(\frac{\sqrt{2}t-\mathsf{X}_u(t)}{\sqrt{t}}\right)1_{\{\sqrt{2}s-\mathsf{X}_u(s)\ge -K, \forall s\le t\}}\Lambda(\mathrm{d}s).$$

Put $k_t = t^\gamma$ for some $\gamma \in (0,1)$, and for $0 < a < \frac{1}{2} < b < 1$ (γ, a, b will be chosen later), let

$$\begin{aligned}G_t^1 &:= \big\{k_t^a \le R_K(k_t) \le k_t^b\big\} \cap \big\{R_K(s) \ge \log^3 t, s \in [k_t, t]\big\}, \\ G_t^2 &:= \big\{\mathsf{D}_t^{(K),[k_t,t]} \le e^{-\log^2 t}\big\}.\end{aligned}$$

We define

$$G_t := G_t^1 \cap G_t^2. \tag{6.18}$$

Now we describe the road to (6.8). As said before, by Lemma 6.3, the first moment of $U_t^{(K)}$ converges, but it's not trivial to show this is the case for the second moment. A good idea then is to prove that $\mathsf{E}_{\mathsf{Q}^{(K)}}[U_t^{(K)} 1_{G_t}] \to \sqrt{2/\pi}$ and $\mathsf{E}_{\mathsf{Q}^{(K)}}[(U_t^{(K)})^2 1_{G_t}] \to 2/\pi$ as $t \to \infty$. The first assertion is easy, but the second one is not (at least for the authors). However, using the idea in [4] and thanks to (6.9), we rewrite $I_2(t)$ as

$$I_2(t) = \mathsf{E}_{\mathsf{Q}^{(K)}}\big[U_t^{(K)} V_t^{(K)}\big].$$

We expect that the integration of $U_t^{(K)} V_t^{(K)}$ on the good event G_t has the desired limit $2/\pi$. Specifically, we are going to show that

$$J(t) := \mathsf{E}_{\mathsf{Q}^{(K)}}\big[U_t^{(K)} V_t^{(K)} 1_{G_t}\big] \overset{t\to\infty}{\longrightarrow} 2/\pi. \tag{6.19}$$

Although this equation does not implies that $\mathsf{E}_{\mathsf{Q}^{(K)}}[(U_t^{(K)})^2 1_{G_t}] \to 2/\pi$, but this is the case if we replace $U_t^{(K)} = \mathsf{E}_{\mathsf{Q}^{(K)}}[V_t^{(K)} \mid \mathcal{F}_t]$ by

$$\overline{U}_t^{(K)} := \mathsf{E}_{\mathsf{Q}^{(K)}}\big[V_t^{(K)} 1_{G_t} | \mathcal{F}_t\big].$$

Indeed, observing that $\overline{U}_t^{(K)} \le U_t^{(K)}$, we have

$$\limsup_{t\to\infty} \mathsf{E}_{\mathsf{Q}^{(K)}}\big[\big(\overline{U}_t^{(K)}\big)^2\big] \le \lim_{t\to\infty} \mathsf{E}_{\mathsf{Q}^{(K)}}\big[\overline{U}_t^{(K)} U_t^{(K)}\big] = \lim_{t\to\infty} \mathsf{E}_{\mathsf{Q}^{(K)}}\big[U_t^{(K)} V_t^{(K)} 1_{G_t}\big] = 2/\pi. \tag{6.20}$$

Then we shall show that $\overline{U}_t^{(K)}$ is a good approximation for $U_t^{(K)}$, in the sense that

$$\mathsf{E}_{\mathsf{Q}^{(K)}}\big|U_t^{(K)} - \overline{U}_t^{(K)}\big| = \mathsf{E}_{\mathsf{Q}^{(K)}}\big(V_t^{(K)} 1_{G_t^c}\big) \overset{t\to\infty}{\longrightarrow} 0. \tag{6.21}$$

Firstly this yields that $\mathsf{E}_{\mathsf{Q}^{(K)}}[\overline{U}_t^{(K)}] \overset{t\to\infty}{\longrightarrow} \sqrt{2/\pi}$ since $\mathsf{E}_{\mathsf{Q}^{(K)}}[U_t^{(K)}] \overset{t\to\infty}{\longrightarrow} \sqrt{2/\pi}$ by Lemma 6.3. Combining this with (6.20) it follows that $\overline{U}_t^{(K)}$ converges in $L^2(\mathsf{Q}^{(K)})$, and hence in $L^1(\mathsf{Q}^{(K)})$ to $\sqrt{2/\pi}$. Again, using (6.21) we get (6.8).

In conclusion, now it suffices to prove (6.19) *and* (6.21)*!*

Before giving our proof, we state a lemma first:

Lemma 6.4. *Let $k_t = t^\gamma$ for some $\gamma \in (0,1)$, and let $0 < a < \frac{1}{2} < b < 1$ such that $k_t^b \le t^{\frac{1}{2}-\epsilon_0}$ for some $\epsilon_0 > 0$. For any fixed $A > 0$, the following assertions hold.*

(i) $\lim_{t\to\infty} \mathsf{Q}^{(K)}(G_t) = 1$ *and* $\lim_{t\to\infty} \inf_{u\in[k_t^a, k_t^b]} \mathsf{Q}^{(K)}(G_t | R_K(k_t) = u) = 1$.

(ii) $\mathsf{Q}^{(K)}(\{|R_K(t) - r_t\sqrt{t}| \le A\sqrt{t}h_t\} \cap G_t^c) \lesssim (r_t^2 h_t)\log^3 t\,((r_t\sqrt{t})^{-1} + k_t^{-1/2} + k_t^{-3(\frac{1}{2}-a)})$.

It not hard to see that we can choose γ, a, b satisfying the condition in Lemma 6.4. Now fix these three parameters. Based on Lemma 6.4, whose proof is postponed to the end of this section, we now prove (6.21) and (6.19).

Proof of (6.21)**.** By the definition of $V_t^{(K)}$

$$\mathsf{E}_{\mathsf{Q}^{(K)}}\big(V_t^{(K)} 1_{G_t^c}\big) = \frac{\sqrt{t}}{\langle F_t, \mu\rangle} \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{1}{R_K(t)} F_t\left(\frac{R_K(t) - K}{\sqrt{t}}\right) 1_{G_t^c}\right].$$

When $F_t(\frac{R_K(t)-K}{\sqrt{t}}) > 0$, using the assumption that $r_t + yh_t = \Theta(r_t)$ uniformly in $y \in \mathrm{supp}(G)$, we have $R_K(t) = \Theta(r_t\sqrt{t})$. Since F_t is bounded and $\sqrt{t}r_t \gg \log^3 t$, applying (6.14) and Lemma 6.4(ii),

$$\begin{aligned}\mathsf{E}_{\mathsf{Q}^{(K)}}\big(V_t^{(K)}1_{G_t^c}\big) &\lesssim_{\beta,\sigma^2,\bar{r},G} \frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{1}{r_t\sqrt{t}}\mathsf{Q}^{(K)}\big(\big\{\big|R_K(t)-r_t\sqrt{t}\big|\le 2A\sqrt{t}h_t\big\}\cap G_t^c\big)\\ &\lesssim \frac{1}{r_t^2h_t}\big(r_t^2h_t\big)\log^3 t\bigg(\frac{1}{r_t\sqrt{t}}+\frac{1}{k_t^{1/2}}+\frac{1}{k_t^{3/4}}\bigg).\end{aligned}$$

Since by our assumption $r_t\sqrt{t} \gg \log^3(t)$ and $k_t = t^\gamma$, we get that $\mathsf{E}_{\mathsf{Q}^{(K)}}(V_t^{(K)}1_{G_t^c}) \overset{t\to\infty}{\longrightarrow} 0$ as desired. □

Proof of (6.19). Recall that

$$J(t) = \mathsf{E}_{\mathsf{Q}^{(K)}}\big[U_t^{(K)}V_t^{(K)}1_{G_t}\big] = \bigg(\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\bigg)^2\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\mathsf{W}_t^{(K),F}}{\mathsf{D}_t^{(K)}}\frac{1}{R_K(t)}F_t\bigg(\frac{R_K(t)-K}{\sqrt{t}}\bigg)1_{G_t}\bigg].$$

It suffices to show that $\limsup_{t\to\infty} J(t) \le 2/\pi$, because similar to (6.20), $\liminf_{t\to\infty} J(t) \ge \liminf_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}[(\overline{U}_t^{(K)})^2]$ and by (6.21) and Lemma 6.3, $\liminf_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}[(\overline{U}_t^{(K)})^2] \ge 2/\pi$. The rest of the proof is divided into four steps:

- We claim that $J^{(1)}(t) := (\frac{\sqrt{t}}{\langle F_t,\mu\rangle})^2\mathsf{E}_{\mathsf{Q}^{(K)}}[\frac{\mathsf{W}_t^{(K),F,[k_t,t]}}{\mathsf{D}_t^{(K)}}\frac{1}{R_K(t)}F_t(\frac{R_K(t)-K}{\sqrt{t}})1_{G_t}] = o(1)$. To see this, noting that $F_t(\frac{\sqrt{2}t-X_u(t)}{\sqrt{t}}) > 0$ only if $\sqrt{2}t - X_u(t) = \Theta(r_t\sqrt{t})$, we have $W_t^{(K),F,[c_1,c_2]} \le \mathsf{D}_t^{(K),[c_1,c_2]}$. So on G_t, we have $\mathsf{W}_t^{(K),F,[k_t,t]} \le \mathsf{D}_t^{(K),[k_t,t]} \le e^{-\log^2 t}$, and $\frac{1}{R_K(t)}F_t(\frac{R_K(t)-K}{\sqrt{t}}) \le 1$. Hence by (6.14) and (6.2), $J^{(1)}(t) \le (\sqrt{t}r_t^{-1}h_t^{-1})^2 \times e^{-\log^2 t} = o(1)$ by (6.14).
- Let $J^{(2)}(t) = (\frac{\sqrt{t}}{\langle F_t,\mu\rangle})^2\mathsf{E}_{\mathsf{Q}^{(K)}}[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}\frac{1}{R_K(t)}F_t(\frac{R_K(t)-K}{\sqrt{t}})1_{G_t}]$. Using that $\frac{1}{\mathsf{D}_t^{(K)}} \le \frac{1}{\mathsf{D}_t^{(K),[0,k_t)}}$, we have $J(t) \le J^{(1)}(t) + J^{(2)}(t)$. Then by the branching Markov property, we get

$$J^{(2)}(t) \le \bigg(\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\bigg)^2\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{\{R_K(k_t)\in[k_t^a,k_t^b]\}}\bigg]\sup_{u\in[k_t^a,k_t^b]}\mathbf{E}_u^{\mathrm{Bes}}\bigg[\frac{F_t(\frac{R_{t-k_t}-K}{\sqrt{t}})}{R_{t-k_t}}\bigg].$$

  By doing the same computation as in (6.17), we have uniformly for $u \in [k_t^a, k_t^b]$, $\mathbf{E}_u^{\mathrm{Bes}}[\frac{F_t(\frac{R_{t-k_t}-K}{\sqrt{t}})}{R_{t-k_t}}] \sim \sqrt{\frac{2}{\pi}}\frac{\langle F_t,\mu\rangle}{\sqrt{t}}$. And this yields that

$$\limsup_{t\to\infty} J(t) \le \sqrt{\frac{2}{\pi}}\limsup_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{\{R_K(k_t)\in[k_t^a,k_t^b]\}}\bigg]. \tag{6.22}$$

- We claim that

$$\limsup_{t\to\infty} J(t) \le \sqrt{\frac{2}{\pi}}\limsup_{t\to\infty}\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{G_t}\bigg]. \tag{6.23}$$

  To see this, using the branching Markov property and part (i) of Lemma 6.4, we have

$$\begin{aligned}&\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{G_t}\bigg]\\ &\ge \mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{\{R_K(k_t)\in[k_t^a,k_t^b]\}}\bigg]\inf_{u\in[k_t^a,k_t^b]}\mathsf{Q}^{(K)}\big(G_t|R_K(k_t)=u\big)\\ &= \big(1-o(1)\big)\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}}1_{\{R_K(k_t)\in[k_t^a,k_t^b]\}}\bigg].\end{aligned}$$

  Substituting into (6.22), the inequality (6.23) follows.

- By (6.23), we have $\limsup_{t\to\infty} J(t) \le \sqrt{\frac{2}{\pi}} \limsup_{t\to\infty} J^{(3)}(t)$, where $J^{(3)}(t) := \frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{E}_{\mathsf{Q}^{(K)}}[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}} 1_{G_t}]$. So it suffices to show that $J^{(3)}(t) = (1+o(1))\sqrt{\frac{2}{\pi}}$. By assumption, there is a constant $q>0$ such that $t^{-q} \lesssim h_t$ for large t. Set $\eta := q+2$. Then note that

$$J^{(3)}(t) \le \frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}} 1_{G_t\cap\{\mathsf{D}_t^{(K)}\ge t^{-\eta}\}}\right] + \frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{Q}^{(K)}\big(\mathsf{D}_t^{(K)} < t^{-\eta}\big). \tag{6.24}$$

On the one hand, the Markov inequality and (6.14) yields

$$\frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{Q}^{(K)}\big(\mathsf{D}_t^{(K)} < t^{-\eta}\big) = \frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{Q}^{(K)}\left(\frac{1}{\mathsf{D}_t^{(K)}} > t^{\eta}\right) \lesssim \frac{\sqrt{t}}{t^{\eta} r_t h_t} \mathsf{E}_{\mathsf{Q}^{(K)}}\left(\frac{1}{\mathsf{D}_t^{(K)}}\right) = o(1). \tag{6.25}$$

On the other hand, on $G_t\cap\{\mathsf{D}_t^{(K)}\ge t^{-\eta}\}$, we have $\mathsf{D}_t^{(K),[k_t,t]} \le e^{-\log^2 t} \le \frac{1}{t}\mathsf{D}_t^{(K)}$. Then $\mathsf{D}_t^{(K),[0,k_t)} \ge (1-t^{-1})\mathsf{D}_t^{(K)}$ and hence

$$\begin{aligned} \frac{\sqrt{t}}{\langle F_t,\mu\rangle} \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K),[0,k_t)}} 1_{G_t\cap\{\mathsf{D}_t^{(K)}\ge t^{-\eta}\}}\right] &\le \frac{1}{1-\frac{1}{t}} \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{\mathsf{W}_t^{(K),F,[0,k_t)}}{\mathsf{D}_t^{(K)}}\right] \\ &\le \frac{1}{1-\frac{1}{t}} \mathsf{E}_{\mathsf{Q}^{(K)}}\left[\frac{\sqrt{t}}{\langle F_t,\mu\rangle}\frac{\mathsf{W}_t^{(K),F}}{\mathsf{D}_t^{(K)}}\right] = \big(1+o(1)\big)\sqrt{\frac{2}{\pi}}, \end{aligned} \tag{6.26}$$

where the last equality follows from (6.11). Combining (6.24), (6.25) and (6.26), we finally get $J^{(3)}(t) = (1+o(1))\sqrt{\frac{2}{\pi}}$. We now complete the proof. □

In the end, we give the proof of Lemma 6.4.

Proof of Lemma 6.4. Firstly, we show that $\lim_{t\to\infty} \mathsf{Q}^{(K)}(G_t^1) = 1$. In fact, by the Markov property,

$$\mathsf{Q}^{(K)}\big(G_t^1\big) \ge \mathbf{P}_K^{\mathrm{Bes}}\big(R_K(k_t) \in \big[k_t^a, k_t^b\big]\big) \inf_{u\in[k_t^a,k_t^b]} \mathbf{P}_u^{\mathrm{Bes}}\big(R(s) \ge \log^3 t, \forall s\in[0,t-k_t]\big).$$

The following estimates (6.27) and (6.28) tell us that $\lim_{t\to\infty} \mathbf{P}_K^{\mathrm{Bes}}(R_K(k_t)\in[k_t^a,k_t^b]) = 1$. Then combining with (6.29) below, we get $\lim_{t\to\infty} \mathbb{Q}^{(K)}(G_t^1) = 1$.

- Given some small constant ϵ_0, for fixed $a < \frac{1}{2} < b$, using (6.16) and the fact $1-e^{-y} \le y$ for all $y>0$, we have uniformly for $x \le s^{\frac{1}{2}-\epsilon_0}$,

$$\mathbf{P}_x^{\mathrm{Bes}}\big(R_s \le s^a\big) = \mathbf{P}_{x/\sqrt{s}}^{\mathrm{Bes}}\big(R_1 \le s^{a-1/2}\big) \lesssim \int_0^{s^{a-\frac{1}{2}}} e^{-\frac{(z-x/\sqrt{s})^2}{2}} z^2\,\mathrm{d}z \lesssim \Theta\big(s^{-3(\frac{1}{2}-a)}\big); \tag{6.27}$$

and

$$\mathbf{P}_x^{\mathrm{Bes}}\big(R_s \ge s^b\big) = \mathbf{P}_{x/\sqrt{s}}^{\mathrm{Bes}}\big(R_1 \ge s^{b-1/2}\big) \lesssim \int_{s^{b-\frac{1}{2}}}^{\infty} e^{-\frac{(z-x/\sqrt{s})^2}{2}} z^2\,\mathrm{d}z = s^{-g_1(s)}, \tag{6.28}$$

where g_1 satisfies that $\lim_{s\to\infty} g_1(s) = \infty$.
- For $u\in[k_t^a,k_t^b]$, let $\tau = \inf\{s>0 : R_s < \log^3 t\}$. By the hitting probability of 3-dim Brownian motion (see e.g. [34, (9.1.5)]), we have

$$\mathbf{P}_u^{\mathrm{Bes}}\big(R(s)\ge\log^3 t, \forall s\in[0,t]\big) \ge \mathbf{P}_u^{\mathrm{Bes}}(\tau=\infty) = 1 - \frac{\log^3 t}{u}. \tag{6.29}$$

Secondly, we show that $\mathsf{Q}^{(K)}(G_t^1\backslash G_t^2) \to 0$. Let $\mathcal{G}_\infty$ be the sigma-algebra generated by the spacial motion of the spine $(\Xi(t))_{t\ge0}$ and the Poisson point process Λ which represents the birth times along the spine. We know that, under $\mathsf{Q}^{(K)}$,

given $\mathcal{G}_\infty$, the processes $(\{\mathsf{X}_u(t) : u \in \mathsf{N}_t^{\xi_s}, t \geq s\}, s \in \operatorname{supp}(\Lambda))$ are independent BBMs starting from $\Xi(s)$. Therefore,

$$\begin{aligned}\mathsf{E}_{\mathsf{Q}^{(K)}}\big[\mathsf{D}_t^{(K),[k_t,t]}|\mathcal{G}_\infty\big] &\leq \int_{[k_t,t]} \mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\sum_{u\in\mathsf{N}_t^{\xi_s}} \big(\sqrt{2}t - \mathsf{X}_u(t) + K\big)e^{-\sqrt{2}(\sqrt{2}t-\mathsf{X}_u(t))}\Big|\mathcal{G}_\infty\bigg]\Lambda(\mathrm{d}s)\\ &= \int_{[k_t,t]} \big(\sqrt{2}s - \Xi(s) + K\big)e^{-\sqrt{2}(\sqrt{2}s-\Xi(s))}\Lambda(\mathrm{d}s).\end{aligned}$$

By the definition of G_t^1, we have $\mathsf{E}_{\mathsf{Q}^{(K)}}[\mathsf{D}_t^{(K),[k_t,t]}|\mathcal{G}_\infty]1_{G_t^1} \leq e^{-\sqrt{2}(\log^3 t-K)}\int_{k_t}^t R_K(s)\Lambda(\mathrm{d}s)$. By the Markov inequality, we get

$$\begin{aligned}\mathsf{Q}^{(K)}\big(G_t^1 \cap \big\{\mathsf{D}_t^{(K),F,[k_t,t]} \geq e^{-\log^2 t}\big\}\big) &\leq \mathsf{E}_{\mathsf{Q}^{(K)}}\big[e^{\log^2 t}\mathsf{D}_t^{(K),F,[k_t,t]}1_{G_t^1}\big]\\ &= e^{\log^2 t}\mathsf{E}_{\mathsf{Q}^{(K)}}\big(\mathsf{E}_{\mathsf{Q}^{(K)}}\big[\mathsf{D}_t^{(K),F,[k_t,t]}|\mathcal{G}_\infty\big]1_{G_t^1}\big)\\ &\lesssim e^{-\sqrt{2}\log^3 t+\log^2 t}\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\int_{k_t}^t R_K(s)\,\mathrm{d}s\bigg] = t^{-g_2(t)},\end{aligned} \tag{6.30}$$

where g_2 satisfies that $\lim_{t\to\infty} g_2(t) = \infty$, and we used the fact that Λ and R_K are independent under $\mathsf{Q}^{(K)}$, and $\mathsf{E}_{\mathsf{Q}^{(K)}}[\int_{k_t}^t R_K(s)\Lambda(\mathrm{d}s)] = 2\mathsf{E}_{\mathsf{Q}^{(K)}}[\int_{k_t}^t R_K(s)\,\mathrm{d}s]$. Therefore, $\mathsf{Q}^{(K)}(G_t^1\backslash G_t^2) \to 0$, and then $\lim_{t\to\infty}\mathsf{Q}^{(K)}(G_t) = 1$.

Thirdly, the same computations as in (6.30) yields

$$\begin{aligned}\mathsf{Q}^{(K)}\big(G_t^1\backslash G_t^2|R_K(k_t) = u\big) &= \mathsf{Q}^{(K)}\big(G_t^1 \cap \big\{\mathsf{D}_t^{(K),F,[k_t,t]} \geq e^{-\log^2 t}\big\}|R_K(k_t) = u\big)\\ &\leq e^{-\sqrt{2}\log^3 t+\log^2 t}\mathsf{E}_{\mathsf{Q}^{(K)}}\bigg[\int_{k_t}^t R_K(s)\,\mathrm{d}s\Big|R(k_t) = u\bigg] \to 0, \quad \text{uniformly in } u \leq k_t^b.\end{aligned}$$

Also (6.29) implies that $\inf_{u\in[k_t^a,k_t^b]}\mathsf{Q}^{(K)}(G_t^1|R_K(k_t) = u) \to 1$. Thus Lemma 6.4(i) follows.

Finally, we give the upper bound of $\mathsf{Q}^{(K)}(\{|R_K(t) - r_t\sqrt{t}| \leq A\sqrt{t}h_t\}\backslash G_t)$. Recall that in (6.30) we have shown that $\mathsf{Q}^{(K)}(G_t^1\backslash G_t^2) = t^{-g_2(t)}$ with $g_2(t) \to \infty$. So it suffices to consider $\mathsf{Q}^{(K)}(\{|R_K(t) - r_t\sqrt{t}| \leq A\sqrt{t}h_t\}\backslash G_t^1)$.

- The estimates (6.27) and (6.28) imply that

$$\begin{aligned}&\mathsf{Q}^{(K)}\big(R_K(k_t) \notin \big[k_t^a, k_t^b\big], \big|R_K(t) - r_t\sqrt{t}\big| \leq A\sqrt{t}h_t\big)\\ &\quad\leq \mathbf{P}_K^{\mathrm{Bes}}\big(R(k_t) \leq k_t^a\big)\sup_{u\leq k_t^a}\mathbf{P}_u^{\mathrm{Bes}}\big(\big|R(t) - r_t\sqrt{t}\big| \leq A\sqrt{t}h_t\big) + \mathbf{P}_K^{\mathrm{Bes}}\big(R(k_t) \geq k_t^b\big)\\ &\quad\lesssim \mathbf{P}_K^{\mathrm{Bes}}\big(R(k_t) \leq k_t^a\big)\sup_{x\leq k_t^a/\sqrt{t}}\mathbf{P}_x^{\mathrm{Bes}}\big(|R_1 - r_t| \leq Ah_t\big) + k_t^{-g_1(k_t)} \lesssim k_t^{-3(\frac{1}{2}-a)}\big(r_t^2h_t\big).\end{aligned} \tag{6.31}$$

- Note that by the Markov property,

$$\begin{aligned}&\mathsf{Q}^{(K)}\big(\big\{R_K(k_t) \in \big[k_t^a, k_t^b\big], \big|R_K(t) - r_t\sqrt{t}\big| \leq A\sqrt{t}h_t\big\}\backslash G_t^1\big)\\ &\quad= \int_{k_t^a}^{k_t^b}\mathbf{P}_K^{\mathrm{Bes}}\big(R(k_t) \in \mathrm{d}u\big)\mathbf{P}_u^{\mathrm{Bes}}\big(\tau < t - k_t; |R_{t-k_t} - r_t\sqrt{t}| \leq A\sqrt{t}h_t\big),\end{aligned}$$

where $\tau = \inf\{s > 0 : R_s < \log^3 t\}$. It is known that $(\tau, \mathbf{P}_u^{\mathrm{Bes}})$ has a probability density function (see e.g., [20, page 339, 2.0.2])

$$p_\tau(s;u) = \frac{\log^3 t}{u}\frac{(u-\log^3 t)}{\sqrt{2\pi}s^{3/2}}\exp\bigg\{-\frac{(u-\log^3 t)^2}{2s}\bigg\}\,\mathrm{d}s \lesssim \frac{\log^3 t}{s^{3/2}}\exp\bigg\{-\frac{u^2}{4s}\bigg\}\,\mathrm{d}s.$$

Let $t' = t - k_t$. Using the strong Markov property, $\mathbf{P}_u^{\mathrm{Bes}}(\tau < t'; |R_{t'} - r_t\sqrt{t}| \le A\sqrt{t}h_t)$ equals

$$\begin{aligned}
&\int_0^{t'} \mathbf{P}^{\mathrm{Bes}}_{\log^3 t}\big(|R_s - r_t\sqrt{t}| \le Ah_t\sqrt{t}\big)p_\tau\big(t'-s;u\big)\,\mathrm{d}s\\
&\lesssim \int_0^{t'} r_t^2\frac{t}{s}e^{-cr_t^2\frac{t'}{s}}h_t\sqrt{\frac{t}{s}}\frac{\log^3 t}{(t'-s)^{3/2}}e^{-\frac{u^2}{4(t'-s)}}\,\mathrm{d}s\\
&\lesssim \big(r_t^2h_t\big)\frac{\log^3 t}{\sqrt{t}}\int_0^1 \frac{1}{[\lambda(1-\lambda)]^{3/2}}e^{-cr_t^2\frac{1}{\lambda}}e^{-\frac{u^2}{4t'}\frac{1}{1-\lambda}}\,\mathrm{d}\lambda \lesssim r_t^2h_t\log^3 t\bigg(\frac{1}{r_t\sqrt{t}}+\frac{1}{u}\bigg),
\end{aligned}$$

where in the first inequality we used the domination $e^{-(r_t\sqrt{t}-Ah_t\sqrt{t}-\log^3 t)^2/2s} \le e^{-cr_t^2t'/s}$ with $c>0$ being some constant and in the third inequality we used the domination $\int_0^1 \lambda^{-3/2}e^{-cr_t^2/\lambda}\,\mathrm{d}\lambda \le \int_0^\infty \frac{1}{r_t\sqrt{\eta}}e^{-c\eta}\,\mathrm{d}\eta \lesssim \frac{1}{r_t}$. Thus

$$\begin{aligned}
&\mathsf{Q}^{(K)}\big(\big\{R_K(k_t)\in\big[k_t^a,k_t^b\big],\big|R_K(t)-r_t\sqrt{t}\big|\le A\sqrt{t}h_t\big\}\backslash G_t^1\big)\\
&\lesssim r_t^2h_t\log^3 t\int_{k_t^a}^{k_t^b}\bigg(\frac{1}{r_t\sqrt{t}}+\frac{1}{u}\bigg)\mathbf{P}_K^{\mathrm{Bes}}(R_{k_t}\in\mathrm{d}u) \le r_t^2h_t\log^3 t\bigg(\frac{1}{r_t\sqrt{t}}+\frac{1}{k_t^{1/2}}\bigg).
\end{aligned}\tag{6.32}$$

Combining (6.31) and (6.32), we get Lemma 6.4(ii). □

Appendix: Proof of Lemma 2.2

Proof. Denote by Pr the probability in the left hand side of (2.2). Noting that the Gaussian process $(\sigma B_r - \frac{\sigma^2 r}{\sigma^2 s+t-s}[\sigma B_s + B_t - B_s])_{r\in[0,s]}$ is independent of $\sigma B_s + B_t - B_s$ (checking their covariance), we have

$$\begin{aligned}
\Pr &= \mathbf{P}\bigg(\sigma B_r - \frac{\sigma^2 r}{\sigma^2 s+t-s}(\sigma B_s + B_t - B_s) \le vr - \frac{\sigma^2 r}{\sigma^2 s+(t-s)}(\widetilde{m}_t + x) + K, \forall r\le s\bigg)\\
&= \mathbf{P}\bigg(\sigma B_r - \frac{r}{s}\sigma B_s \le \bar{Z} + vr - \frac{\sigma^2 r}{\sigma^2 s+(t-s)}(\widetilde{m}_t + x) + K, \forall r\le s\bigg),
\end{aligned}$$

where $\bar{Z} := \frac{\sigma^2 r}{\sigma^2 s+t-s}(\sigma B_s + B_t - B_s) - \frac{r}{s}\sigma B_s$. Simple computation yields

$$vr - \frac{\sigma^2 r}{\sigma^2 s+(t-s)}(\widetilde{m}_t + x) = \frac{(1-\sigma^2)(t-s)v + \sigma^2(w_t - x)}{\sigma^2 s+(t-s)}r.$$

Let $Z := \frac{\sigma^2 s+(t-s)}{r}\bar{Z} = \sigma^2(B_t - B_s) - \frac{t-s}{s}\sigma B_s$. We now have

$$\Pr = \mathbf{P}\bigg(\sigma B_r - \frac{r}{s}\sigma B_s \le \frac{Z + (1-\sigma^2)(t-s)v + \sigma^2(w_t - x)}{\sigma^2 s+(t-s)}r + K, \forall r\le s\bigg).$$

We bound the probability that Z is large. Observe that Z is a Gaussian and $\mathrm{Var}(Z) = \sigma^2(t-s)(\sigma^2 + \frac{t-s}{s}) \le 2(t-s)$ as $\sigma^2 \le 1$. Applying the Gaussian tail bound, we have for large t, $\mathbf{P}(|Z| > 10(t-s+\log t)) \le \frac{1}{t^2}$ for all $s\in[t-\sqrt{t}\log t, t]$. Moreover, Z is independent of the Brownian bridge $(B_r - \frac{r}{s}B_s)_{r\le s}$. By formula of total probability with partition $\{Z \le 10(t-s+\log t)\}$ and its complement, using again $\sigma\le 1$, we have

$$\begin{aligned}
\Pr &\le \mathbf{P}\bigg(\sigma B_r - \frac{r}{s}\sigma B_s \le \frac{(10+v)(t-s+w_t+|x|)}{\sigma^2 s+(t-s)}r + K, \forall r\le s\bigg) + \frac{1}{t^2}\\
&\le 3\bigg[\frac{(10+v)(t-s+w_t+|x|)s}{\sigma(\sigma^2 s+(t-s))} + \frac{K}{\sigma}\bigg]\frac{K}{\sigma} + \frac{1}{t^2} \lesssim_{K,\beta,\sigma} \frac{t-s+w_t+|x|}{t},
\end{aligned}$$

where in the second inequality we used Lemma 2.1. □

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