Weak convergence of the extremes of branching Lévy processes with regularly varying tails

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Abstract

In this paper, we study the weak convergence of the extremes of supercritical branching Lévy processes $\{\mathbb{X}_t, t \geq 0\}$ whose spatial motions are Lévy processes with regularly varying tails. The result is drastically different from the case of branching Brownian motions. We prove that, when properly renormalized, $\mathbb{X}_t$ converges weakly. As a consequence, we obtain a limit theorem for the order statistics of $\mathbb{X}_t$.

AMS Subject Classifications (2020): Primary 60J80, 60F05; Secondary 60G57, 60G70

Keywords and Phrases: Branching Lévy process, extremal process, regularly varying, rightmost position.

1 Introduction

We consider a supercritical branching Lévy process. At time 0, we start with a single particle which moves according to a Lévy process $\{\xi_t, \mathrm{P}_x\}$ with Lévy exponent $\psi(\theta) = \log \mathrm{E}(e^{i\theta\xi_1})$. The lifetime of each particle is exponentially distributed with parameter β, then it splits into k new particles with probability p_k, $k \geq 0$. Once born, each particle will independently move (according to the same Lévy process) and split (according to the same offspring distribution). We use $\mathbb{P}_x$ to denote the law of the branching Lévy process when the initial particle starts at position x. The expectation with respect to $\mathbb{P}_x$ and P_x will be denoted by $\mathbb{E}_x$ and E_x., respectively. We write $\mathbb{P} := \mathbb{P}_0$, $\mathbb{E} := \mathbb{E}_0$, $\mathrm{P} := \mathrm{P}_0$ and $\mathrm{E} := \mathrm{E}_0$.

In this paper, we use ":=" as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$. We will label each particle using the classical Ulam-Harris system. We write $\mathbb{T}$ for the set of all the particles in the tree, o for the root of the tree. For each particle u, we introduce some notation.

- b_u and σ_u: the birth time and death time of u respectively.

*The research of this author is supported by the National Key R&D Program of China (No. 2020YFA0712900) and NSFC (Grant Nos. 12071011 and 11731009).

†Research supported in part by a grant from the Simons Foundation (#960480, Renming Song).

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- $\{\xi^u_t : t \in [b_u, \sigma_u]\}$: the spatial trajectory of u.
- $\tau_u := \sigma_u - b_u$ is the life length of u and $\tau_{u,t} := \sigma_u \wedge t - b_u \wedge t$ is the life length of u between $[0, t]$.
- $\mathcal{F}^{\mathbb{T}}_t := \sigma\{b_u \wedge t, \sigma_u \wedge t : u \in \mathbb{T}\}$.
- $X_u := \xi^u_{\sigma_u} - \xi^u_{b_u}$ and $X_{u,t} := \xi^u_{\sigma_u\wedge t} - \xi^u_{b_u\wedge t}$. Note that given $\mathcal{F}^{\mathbb{T}}_t$, $X_{u,t}, u \in \mathbb{T}$, are independent, and
$$X_{u,t} \stackrel{d}{=} \xi_{\tau_{u,t}}.$$
- I_v: the set of all the ancestors of v, including v itself.
- n^v_t : the number of particles in $I_v \setminus \{o\}$.
- $\mathcal{L}_t$ is the set of all particles alive at time t and Z_t is the number of particles alive at time t.

For $t \geq 0$, define $\mathbb{X}_t := \sum_{u\in\mathcal{L}_t} \delta_{\xi^u_t}$. The measure-valued process $\{\mathbb{X}_t, t \geq 0\}$ is called a branching Lévy process.

It is well known that $\{Z_t; t \geq 0\}$ is a continuous time branching process. In this paper, we consider the supercritical case, that is, $m := \sum_k kp_k > 1$. Then $\mathbb{P}(\mathcal{S}) > 0$, where $\mathcal{S}$ is the event of survival. The extinction probability $\mathbb{P}(\mathcal{S}^c)$ is the smallest root in $(0, 1)$ of the equation $\sum_k p_k s^k = s$, see, for instance, [5, Section III. 4]. The family $\{e^{-\lambda t} Z_t, t \geq 0\}$, where $\lambda = \beta(m - 1)$, is a non-negative martingale and hence

$$\lim_{t\to\infty} e^{-\lambda t} Z_t =: W \quad \text{exists a.s.}$$

For any two functions f and g on $[0, \infty)$, $f \sim g$ as $s \to 0_+$ means that $\lim_{s\downarrow 0} \frac{f(s)}{g(s)} = 1$. Similarly, $f \sim g$ as $s \to \infty$ means that $\lim_{s\to\infty} \frac{f(s)}{g(s)} = 1$. Throughout this paper we assume the following two conditions hold. The first condition is on the offspring distribution:

(H1) $\sum_{k\geq 1}(k \log k)p_k < \infty$.

Condition (H1) ensures that W is non-degenerate with $\mathbb{P}(W > 0) = \mathbb{P}(\mathcal{S})$. For more details, see [5, Section III.7]. The second condition is on the spatial motion:

(H2) There exist a complex constant c_* with $\Re(c_*) > 0$, $\alpha \in (0, 2)$ and a function $L(x)$: $\mathbb{R}_+ \to \mathbb{R}_+$ slowly varying at ∞ such that
$$\psi(\theta) \sim -c_*\theta^\alpha L(\theta^{-1}), \qquad \theta \to 0_+.$$

Since $e^{\psi(\theta)} = \mathrm{E}(e^{i\theta\xi_1})$, we have $\Re(\psi) \leq 0$ and $\psi(-\theta) = \overline{\psi(\theta)}$. Thus

$$\psi(\theta) \sim -\overline{c_*}|\theta|^\alpha L(|\theta|^{-1}), \quad \theta \to 0_-.$$

Under condition (H2), one can prove that (see Remark 2.3 below)

$$\mathrm{P}(|\xi_s| \geq x) \sim csx^{-\alpha} L(x), \quad x \to \infty,$$

that is, $|\xi_s|$ has regularly varying tails.

An important example satisfying (H2) is the strictly stable process.

Example 1.1 (Stable process.) *Let ξ be a strictly α-stable process, $\alpha \in (0,2)$, on $\mathbb{R}$ with Lévy measure*

$$n(dy) = c_1 x^{-(1+\alpha)} \mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + c_2 |x|^{-(1+\alpha)} \mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x,$$

where $c_1, c_2 \geq 0$, $c_1 + c_2 > 0$, and if $\alpha = 1$, $c_1 = c_2 = c$. For $\alpha \in (1,2)$, by [35, Lemma 14.11, (14.19)] and the fact $\Gamma(-\alpha) = -\alpha\Gamma(1-\alpha)$, we obtain that, for $\theta > 0$,

$$\int_0^\infty (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y) = -c_1 \alpha \Gamma(1-\alpha) e^{-i\pi\alpha/2} \theta^\alpha,$$

and taking conjugate on both sides of [35, Lemma 14.11 (14.19)], we have that

$$\int_{-\infty}^0 (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y) = -c_2 \alpha \Gamma(1-\alpha) e^{i\pi\alpha/2} \theta^\alpha.$$

Thus the Lévy exponent of ξ is given by: for $\theta > 0$,

$$\psi(\theta) = \int (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y) = -\alpha\Gamma(1-\alpha)(c_1 e^{-i\pi\alpha/2} + c_2 e^{i\pi\alpha/2})\theta^\alpha. \tag{1.1}$$

Similarly, by [35, Lemma 14.11 (14.18),(114.20)], we have for $\theta > 0$,

$$\psi(\theta) = \begin{cases} \int (e^{i\theta y} - 1) n(\mathrm{d}y) & \alpha \in (0,1); \\ \int (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y|\leq 1}) n(\mathrm{d}y) + ia\theta, & \alpha = 1 \end{cases} \tag{1.2}$$

$$= \begin{cases} -\alpha\Gamma(1-\alpha)(c_1 e^{-i\pi\alpha/2} + c_2 e^{i\pi\alpha/2})\theta^\alpha, & \alpha \in (0,1); \\ -c\pi\theta + ia\theta, & \alpha = 1, \end{cases} \tag{1.3}$$

where $a \in \mathbb{R}$ is a constant. It is clear that ψ satisfies (H2). For more details about the stable processes, we refer the readers to [35, Section 14].

In Section 4, we will give more examples satisfying condition (H2). Note that the non-symmetric 1-stable process does not satisfy (H2). However, in Example 4.3, we will show that our main result still holds for the non-symmetric 1-stable process.

The maximal position M_t of a branching Brownian motion has been studied intensively. Assume that $\beta = 1$, $p_0 = 0$ and $m = 2$. In the seminal paper [28], Kolmogorov, Petrovskii and Piskounov proved that $M_t/t \to \sqrt{2}$ in probability as $t \to \infty$. Bramson proved in [14] (see also [15]) that, under some moment conditions, $\mathbb{P}(M_t - m(t) \leq x) \to 1 - w(x)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ and $w(x)$ is a traveling wave solution. For more works on M_t, see [18, 19, 29, 34]. For inhomogeneous branching Brownian motions, many papers discussed the growth rate of the maximal position, see Bocharov and Harris [12, 13] and Bocharov [11] for the case with catalytic branching at the origin, Shiozawa [36], Nishimori et al. [33], Lalley and Sellke [30, 31] for the case with some general branching mechanisms.

Recently, the full statistics of the extremal configuration of branching Brownian motion were studied. Arguin et al. [3, 4] studied the limit property of the extremal process of branching Brownian motion. They proved that the random measure defined by

$$\mathcal{E}_t := \sum_{u \in \mathcal{L}_t} \delta_{\xi_t^u - m(t)}$$

converges weakly, and the limiting process is a (randomly shifted) Poisson cluster process. Almost at the same time, Aïdékon et al. [2] proved similar results using a totally different method.

For branching random walks, several authors have studied similar problems under an exponential moment assumption on the displacements of the offspring from the parent, see Aïdékon [1], Carmona and Hu [17], Hu and Shi [26], and Madaule [32]. When the displacements of the offspring from the parents are i.i.d. and have regularly varying tails, Durrett [22] studied the limit property of its maximum displacement M_n. More precisely, Durrett proved that $a_n^{-1} M_n$ converges weakly, where $a_n = m^{n/\alpha} L_0(m^n)$ and L_0 is slowly varying at ∞. Recently, the extremal processes of the branching random walks with regularly varying steps were studied by Bhattacharya et al. [8, 9]. In [8, 9], it was proved that the point random measures $\sum_{|v|=n} \delta_{a_n^{-1} S_v}$, where S_v is the position of v, converges weakly to a Cox cluster process, which are quite different from the case with exponential moments. See also [10, 24] for related works on branching random walks with heavy-tailed displacements.

Shiozawa [37] studied branching symmetric stable processes with branching rate μ being a measure on $\mathbb{R}$ in a Kato class and offspring distribution $\{p_n(x), n \geq 0\}$ being spatially dependent. Under some conditions on μ and $\{p_n(x), n \geq 0\}$, Shiozawa [37] proved that the growth rate of the maximal displacement is exponential with rate given by the principal eigenvalue of the mean semigroup of the branching symmetric stable processes. In this paper, we study the extremes of branching Lévy processes with constant branching rate β and spatial motion having regularly varying tails (see condition (H2)). Since our branching rate β is not compactly supported, one can not get the growth rate of the maximal displacement from Shiozawa [37]. As a corollary of our extreme limit result, we get the growth rate of the maximal displacement, see Corollary 1.5 below.

The key idea of the proof in this paper is the "one large jump principle" which was inspired by [8, 9, 22]. Along the discrete times $n\delta$, the branching Lévy processes $\{\mathbb{X}_{n\delta}, n \geq 1\}$ is a branching random walk and the displacements from parents has the same law as $\mathbb{X}_\delta$. It is natural to think that one may get the results of this paper from the results for branching random walks directly. However we can not apply the results for branching random walks in [8, 9, 32] to $\{\mathbb{X}_{n\delta}, n \geq 1\}$. First, under condition (H2), the exponential moment assumption in [32] is not satisfied. Secondly, [8] assumes that the displacements are i.i.d., while the atoms of the random measure $\mathbb{X}_\delta$ are not independent. Lastly, although the displacements of offspring coming from the same parent are allowed to be dependent in [9], Assumption 2.5 in [9], where the displacements from parents are given by a special form (see [9, (2.9) and (2.10)]), seems to be very difficult to check for $\mathbb{X}_\delta$.

Branching Lévy processes are closely related to the Fisher-KPP equation when the classical Laplacian Δ is replaced by the infinitesimal generator of the corresponding Lévy process.

For any $g \in C_b^+(\mathbb{R})$, define $u_g(t,x) = \mathbb{E}_x\left(e^{-\int g(y)\mathbb{X}_t(\mathrm{d}y)}\right)$. By the Markov property and branching property, we have that

$$u_g(t,x) = \mathrm{E}_x\left(e^{-g(\xi_t)}\right) + \mathrm{E}_x\int_0^t \varphi(u_g(t-s,\xi_s))\mathrm{d}s,$$

where $\varphi(s) = \beta\left(\sum_k s^k p_k - s\right)$. Then $1-u_g$ is a mild solution to

$$\partial_t u - \mathcal{A}u = -\varphi(1-u), \tag{1.4}$$

with initial data $u(0,x) = 1 - e^{-g(x)}$, where $\mathcal{A}$ is the infinitesimal generator of ξ. In [16], Cabré and Roquejoffre proved that, under the assumption that the density of ξ is comparable to that of a symmetric α-stable process, the front position of $1-u$ is exponential in time. Using our main result, we give another proof of [16, Theorem 1.5] and also partially generalize it, see Remark 5.2.

1.1 Main results

Put $\mathbb{R}_0 = (-\infty,\infty)\setminus\{0\}$, and $\overline{\mathbb{R}}_0 = [-\infty,\infty]\setminus\{0\}$. Let $C_b^0(\mathbb{R})$ be the set of all bounded continuous functions vanishing in a neighborhood of 0. Let $C_c^+(\overline{\mathbb{R}}_0)$ be the set of all non-negative continuous functions on $\overline{\mathbb{R}}_0$ such that $g=0$ on $(-\delta,0)\cup(0,\delta)$ for some $\delta>0$. It is clear that if $g \in C_c^+(\overline{\mathbb{R}}_0)$, then $g^*(x) := \mathbf{1}_{\mathbb{R}_0}(x)g(x) \in C_b^0(\mathbb{R})$. Denote by $\mathcal{M}(\overline{\mathbb{R}}_0)$ the set of all Radon measures endowed with the topology of vague convergence (denoted by $\overset{v}{\to}$). Then $\mathcal{M}(\overline{\mathbb{R}}_0)$ is a metrizable space. For any $g \in \mathcal{B}_b^+(\overline{\mathbb{R}}_0)$, $\mu \in \mathcal{M}(\overline{\mathbb{R}}_0)$, we write $\mu(g) := \int_{\overline{\mathbb{R}}_0} g(x)\mu(\mathrm{d}x)$. A sequence of random elements ν_n in $\mathcal{M}(\overline{\mathbb{R}}_0)$ converges weakly to ν, denoted as $\nu_n \overset{d}{\to} \nu$, if and only if for all $g \in C_c^+(\overline{\mathbb{R}}_0)$, $\nu_n(g)$ converges weakly to $\nu(g)$. Note that, for any $a>0$, $[a,\infty]$ and $[-\infty,-a]$ are compact subsets of $\overline{\mathbb{R}}_0$.

We claim that there exists a non-decreasing function h_t with $h_t \uparrow \infty$ such that

$$\lim_{t\to\infty} e^{\lambda t} h_t^{-\alpha} L(h_t) = 1. \tag{1.5}$$

In fact, using [7, Theorem 1.5.4], there exists a non-increasing function g such that $g(x) \sim x^{-\alpha}L(x)$, as $x\to\infty$. Then $g(x)\to 0$ as $x\to\infty$. Define

$$h_t := \inf\{x>0 : g(x) \le e^{-\lambda t}\}.$$

It is clear that h_t is non-decreasing and $h_t \uparrow \infty$. By the definition of h_t, one has that, for any $\epsilon>0$,

$$g(h_t/(1+\epsilon)) \ge e^{-\lambda t} \ge g(h_t(1+\epsilon)),$$

which implies that

$$\begin{aligned}(1+\epsilon)^{-\alpha} &= (1+\epsilon)^{-\alpha}\lim_{t\to\infty}\frac{L(h_t)}{L(h_t/(1+\epsilon))} = \lim_{t\to\infty}\frac{g(h_t)}{g(h_t/(1+\epsilon))}\\ &\le \liminf_{t\to\infty} e^{\lambda t} g(h_t) \le \limsup_{t\to\infty} e^{\lambda t} g(h_t)\end{aligned}$$

$$\le \lim_{t\to\infty} \frac{g(h_t)}{g(h_t(1+\epsilon))} = (1+\epsilon)^\alpha \lim_{t\to\infty} \frac{L(h_t)}{L(h_t(1+\epsilon))} = (1+\epsilon)^\alpha.$$

Since ϵ is arbitrary, we get

$$\lim_{t\to\infty} e^{\lambda t} h_t^{-\alpha} L(h_t) = \lim_{t\to\infty} e^{\lambda t} g(h_t) = 1.$$

In particular, $h_t = e^{\lambda t/\alpha}$ if $L = 1$. In Lemma 2.1, we will prove that

$$e^{\lambda t}\mathrm{P}(h_t^{-1}\xi_s \in \cdot) \xrightarrow{v} s v_\alpha(\cdot),$$

where

$$v_\alpha(dx) = q_1 x^{-1-\alpha}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + q_2|x|^{-1-\alpha}\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x,$$

with q_1 and q_2 being nonnegative numbers, uniquely determined by the following equation: if $\alpha \neq 1$

$$c_* = \alpha\Gamma(1-\alpha)\left(q_1 e^{-i\pi\alpha/2} + q_2 e^{i\pi\alpha/2}\right),$$

and if $\alpha = 1$

$$q_1 = q_2 = \Re(c_*)/\pi.$$

Now we are ready to state our main result. Define a renormalized version of $\mathbb{X}_t$ by

$$\mathcal{N}_t := \sum_{v\in\mathcal{L}_t} \delta_{h_t^{-1}\xi_t^v}. \tag{1.6}$$

In this paper we will investigate the limit of $\mathcal{N}_t$ as $t\to\infty$.

Theorem 1.2 *Under $\mathbb{P}$, $\mathcal{N}_t$ converges weakly to a random measure $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$, defined on some extension $(\Omega, \mathcal{G}, P)$ of the probability space on which the branching Lévy process is defined, with Laplace transform given by*

$$E(e^{-\mathcal{N}_\infty(g)}) = \mathbb{E}\left(\exp\left\{-W\int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0} \mathbb{E}(1-e^{-Z_r g(x)})v_\alpha(dx)\mathrm{d}r\right\}\right), \quad g \in C_c^+(\overline{\mathbb{R}}_0). \tag{1.7}$$

Moreover, $\mathcal{N}_\infty = \sum_j T_j\delta_{e_j}$, where given W, $\sum_j \delta_{e_j}$ is a Poisson random measure with intensity $\vartheta W v_\alpha(dx)$, $\{T_j, j\ge 1\}$ is a sequence of i.i.d. random variables with common law:

$$P(T_j = k) = \vartheta^{-1}\int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r = k)\mathrm{d}r, \quad k\ge 1,$$

where $\vartheta = \int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r>0)\mathrm{d}r$, and $\sum_j \delta_{e_j}$ and $\{T_j, j\ge 1\}$ are independent.

Remark 1.3 *Write D_f for the set of discontinuity points of the function f. Then by Theorem 1.2, we have that $\mathcal{N}_t(f) \xrightarrow{d} \mathcal{N}_\infty(f)$ for any bounded measurable function f on $\overline{\mathbb{R}}_0$ with compact support satisfying $\mathcal{N}_\infty(D_f) = 0$ P-a.s. Furthermore, for any $k\ge 1$,*

$$(\mathcal{N}_t(B_1), \mathcal{N}_t(B_2), \cdots, \mathcal{N}_t(B_k)) \xrightarrow{d} (\mathcal{N}_\infty(B_1), \mathcal{N}_\infty(B_2), \cdots, \mathcal{N}_\infty(B_k)),$$

where $\{B_j\}$ are relatively compact subsets of $\overline{\mathbb{R}}_0$ satisfying $\mathcal{N}_\infty(\partial B_j) = 0$, $j = 1, \cdots, k$, P-a.s. See [27, Theorem 4.4] for a proof.

Now we list the positions of all particles alive at time t in a decreasing order:

$$M_{t,1} \geq M_{t,2} \geq \cdots M_{t,Z_t},$$

and for $n > Z_t$, define $M_{t,n} := -\infty$. In particular, $M_{t,1} = \max_{v \in \mathcal{L}_t} \xi_t^v$ is the rightmost position of the particles alive at time t. We also order the atoms of $\mathcal{N}_\infty$ as $M_{(1)} \geq M_{(2)} \geq \cdots \geq M_{(k)} \geq \cdots$. Note that on the set $\mathcal{S}$, the number of the atoms of $\mathcal{N}_\infty$ is infinite, and thus $M_{(k)}, k \geq 1$, are well defined. On the set $\mathcal{S}^c$, $\mathcal{N}_\infty$ is null, then we define $M_{(k)} = -\infty$ for $k \geq 1$.

Define $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot|\mathcal{S})$ $(P^*(\cdot) := P(\cdot|\mathcal{S}))$ and let $\mathbb{E}^*(E^*)$ be the corresponding expectation.

Corollary 1.4 *For any $n \geq 1$,*

$$\left(h_t^{-1} M_{t,1}, h_t^{-1} M_{t,2}, \ldots, h_t^{-1} M_{t,n}; \mathbb{P}^*\right) \xrightarrow{d} \left(M_{(1)}, M_{(2)}, \ldots, M_{(n)}; P^*\right).$$

Moreover, $M_{(k)} > 0, k \geq 1$, P^*-a.s.

We write $R_t := M_{t,1} = \max_{v \in \mathcal{L}_t} \xi_t^v$.

Corollary 1.5

$$\left(h_t^{-1} \mathbb{R}_t; \mathbb{P}^*\right) \xrightarrow{d} \left(M_{(1)}; P^*\right),$$

where the law of $(M_{(1)}; P^*)$ is given by

$$P^*\left(M_{(1)} \leq x\right) = \begin{cases} \mathbb{E}^*\left(e^{-\alpha^{-1} q_1 \vartheta W x^{-\alpha}}\right), & x > 0; \\ 0, & x \leq 0. \end{cases}$$

Proof: Using Corollary 1.4, we get that $\left(h_t^{-1} \mathbb{R}_t; \mathbb{P}^*\right) \xrightarrow{d} \left(M_{(1)}; P^*\right)$, and $M_{(1)} > 0$ P^*-a.s. For any $x > 0$, we have that

$$\begin{aligned} P^*\left(M_{(1)} \leq x\right) =& P^*\left(\mathcal{N}_\infty(x, \infty) = 0\right) = P^*\left(\sum_j \mathbf{1}_{(x,\infty)}(e_j) = 0\right) \\ =& \mathbb{E}^*\left(e^{-\vartheta W v_\alpha(x,\infty)}\right) = \mathbb{E}^*\left(e^{-\alpha^{-1} q_1 \vartheta W x^{-\alpha}}\right). \end{aligned}$$

The proof is now complete. □

Remark 1.6 *Similarly, we can order the particles alive at time t in an increasing order: $L_{t,1} \leq L_{t,2} \leq \cdots \leq L_{t,Z_t}$. Then we can get the corresponding weak convergence of $(L_{t,1}, L_{t,2}, \cdots, L_{t,n})$.*

The rest of the paper is organized as follows. In Section 2, we introduce the one large jump principle and give the proof of Theorem 1.2 based on Proposition 2.6, which will be proved in Subsection 2.3. The proof of Corollary 1.4 will be given in Section 3. In Section 4, we will give more examples satisfying condition (H2) and conditions which are weaker than (H2), but sufficient for the main result of this paper. We will discuss the front position of the Fisher-KPP equation (1.4) in Section 5.

2 Proof of Theorem 1.2

2.1 Preliminaries

Recall that h_t is a function satisfying (1.5).

Lemma 2.1 *For any $g \in C_b^0$ and $s > 0$,*

$$\lim_{t\to\infty} e^{\lambda t} \mathrm{E}\left(g(h_t^{-1}\xi_s)\right) = s \int_{\mathbb{R}_0} g(x) v_\alpha(\mathrm{d}x). \tag{2.1}$$

Proof: Let ν_t be the law of $h_t^{-1}\xi_s$. Then by (H2), we have that, as $t \to \infty$,

$$\exp\left\{e^{\lambda t} \int_{\mathbb{R}} (e^{i\theta x} - 1)\nu_t(\mathrm{d}x)\right\} = \exp\left\{e^{\lambda t}\left(e^{s\psi(h_t^{-1}\theta)} - 1\right)\right\} \to \exp\left\{s\widetilde{\psi}(\theta)\right\}, \tag{2.2}$$

where

$$\widetilde{\psi}(\theta) = \begin{cases} -c_*\theta^\alpha, & \theta > 0; \\ -\overline{c_*}|\theta|^\alpha, & \theta \le 0. \end{cases}$$

Note that the left side of (2.2) is the characteristic function of an infinitely divisible random variable Y_t with Lévy measure $e^{\lambda t}\nu_t$, and by (1.2), $e^{s\widetilde{\psi}(\theta)}$ is the characteristic function of a strictly α-stable random variable Y with Lévy measure $sv_\alpha(dx)$. Thus Y_t weakly converges to Y. The desired result (2.1) follows immediately from [35, Theorem 8.7 (1)]. □

It is well known (see [7, Theorem 1.5.6] for instance) that, for any $\epsilon > 0$, there exists $a_\epsilon > 0$ such that for any $y > a_\epsilon$ and $x > a_\epsilon$,

$$\frac{L(y)}{L(x)} \le (1-\epsilon)^{-1} \max\{(y/x)^\epsilon, (y/x)^{-\epsilon}\}. \tag{2.3}$$

Lemma 2.2 *There exists $c_0 > 0$ such that for any $s > 0$ and $x > 2 + 2a_{0.5}$,*

$$G_s(x) := \mathrm{P}(|\xi_s| > x) \le c_0 s x^{-\alpha} L(x).$$

Proof: By [23, (3.3.1)], we have that, for any $x > 2$,

$$\begin{aligned} \mathrm{P}(|\xi_s| > x) &\le \frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}} (1 - e^{s\psi(\theta)})\,\mathrm{d}\theta \\ &\le s\frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}} \|\psi(\theta)\|\,\mathrm{d}\theta = s\int_0^2 \|\psi(\theta/x)\|\,\mathrm{d}\theta, \end{aligned}$$

where in the last equality, we use the symmetry of $\|\psi(\theta)\|$. By (H2), it is clear that there exists $c_1 > 0$ such that

$$\|\psi(\theta)\| \le c_1\theta^\alpha L(\theta^{-1}), \quad |\theta| \le 1.$$

Thus, for $x > 2 + 2a_{0.5}$, using (2.3) with $\epsilon = 0.5$, we get that

$$\mathrm{P}(|\xi_s| > x) \le c_1 s x^{-\alpha} \int_0^2 \theta^\alpha L(x/\theta)\,\mathrm{d}\theta \le 2c_1 s x^{-\alpha} L(x) \int_0^2 \theta^\alpha(\theta^{-1/2} + \theta^{1/2})\,\mathrm{d}\theta.$$

The proof is now complete. □

Remark 2.3 *It follows from Lemma 2.1 that*

$$\lim_{t\to\infty} e^{\lambda t}\mathrm{P}(|\xi_s| \ge h_t) = s\frac{q_1+q_2}{\alpha}, \tag{2.4}$$

which implies that

$$\mathrm{P}(|\xi_s| \ge x) \sim \frac{q_1+q_2}{\alpha}\, s\, x^{-\alpha}L(x), \quad x\to\infty. \tag{2.5}$$

Now we recall the many-to-one formula which is useful in computing expectations. Here we only list some special cases which will be used in our paper. See [25, Theorem 8.5] for general cases.

Lemma 2.4 (Many-to-one formula) *Let $\{n_t\}$ be a Poisson process with parameter β on some probability space $(\Omega, \mathcal{G}, P)$. Then for any $g \in \mathcal{B}_b^+(\mathbb{R})$,*

$$\mathbb{E}\left(\sum_{v\in\mathcal{L}_t} g(n_t^v)\right) = e^{\lambda t}E\left(g(n_t)\right),$$

and for any $0 \le s < t$,

$$\mathbb{E}\left(\sum_{v\in\mathcal{L}_t} \mathbf{1}_{b_v\le t-s}\right) = e^{\lambda t}P\left(n_t - n_{t-s} = 0\right) = e^{\lambda t}e^{-\beta s}.$$

2.2 Proof of the Theorem 1.2

Recall that, on some extension $(\Omega, \mathcal{G}, P)$ of the probability space on which the branching Lévy process is defined, given W, $\sum_j \delta_{e_j}$ is a Poisson random measure with intensity $\vartheta W v_\alpha(dx)$, $\{T_j, j\ge 1\}$ is a sequence of i.i.d. random variables with common law

$$P(T_j = k) = \vartheta^{-1}\int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r = k)\,\mathrm{d}r, \quad k \ge 1,$$

where $\vartheta = \int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r > 0)\,\mathrm{d}r$, and $\sum_j \delta_{e_j}$ and $\{T_j, j\ge 1\}$ are independent.

Lemma 2.5 *Let $\mathcal{N}_\infty = \sum_j T_j\delta_{e_j}$. Then, $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$ and the Laplace transform of $\mathcal{N}_\infty$ is given by*

$$E(e^{-\mathcal{N}_\infty(g)}) = \mathbb{E}\left(\exp\left\{-W\int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}(1-e^{-Z_r g(x)})v_\alpha(\mathrm{d}x)\,\mathrm{d}r\right\}\right), \quad g \in C_c^+(\overline{\mathbb{R}}_0).$$

Proof: First note that for any $a > 0$, $\vartheta W v_\alpha([-\infty, -a]\cup[a,\infty]) < \infty$. Thus, given W, $\sum_j \mathbf{1}_{|e_j|\ge a}$ is Poisson distributed with parameter $\vartheta W v_\alpha([-\infty,-a]\cup[a,\infty])$, which implies that $\sum_j \mathbf{1}_{|e_j|\ge a} < \infty$, a.s. Thus by the definition of $\mathcal{N}_\infty$,

$$P(\mathcal{N}_\infty([-\infty,-a]\cup[a,\infty]) < \infty) = P\left(\sum_j \mathbf{1}_{|e_j|\ge a} < \infty\right) = 1.$$

So $\mathcal{N}_\infty \in \mathcal{M}(\overline{\mathbb{R}}_0)$. Note that

$$
\begin{aligned}
\phi(\theta) := E\left(e^{-\theta T_j}\right) &= \vartheta^{-1}\sum_{k\geq 1} e^{-\theta k}\int_0^\infty e^{-\lambda r}\mathbb{P}(Z_r = k)\,\mathrm{d}r \\
&= \vartheta^{-1}\int_0^\infty e^{-\lambda r}\mathbb{E}\left(e^{-\theta Z_r}, Z_r > 0\right)\,\mathrm{d}r \\
&= 1 - \vartheta^{-1}\int_0^\infty e^{-\lambda r}\mathbb{E}\left(1 - e^{-\theta Z_r}\right)\,\mathrm{d}r.
\end{aligned}
$$

Thus, for any $g \in C_c^+(\overline{\mathbb{R}}_0)$,

$$
\begin{aligned}
E\left(e^{-\mathcal{N}_\infty(g)}\right) = E\left(e^{-\sum_j T_j g(e_j)}\right) &= E\left(\prod_j \phi(g(e_j))\right) \\
&= \mathbb{E}\left(e^{-\vartheta W\int_{\mathbb{R}_0}(1-\phi(g(x)))v_\alpha(\mathrm{d}x)}\right) \\
&= \mathbb{E}\left(\exp\left\{-W\int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}(1-e^{-Z_r g(x)})v_\alpha(\mathrm{d}x)\,\mathrm{d}r\right\}\right).
\end{aligned}
$$

The proof is now complete. □

To prove Theorem 1.2, we use the idea of "one large jump", which has been used in [22] and [8, 9] for branching random walks. "One large jump" means that with large probability, for all $v \in \mathcal{L}_t$, at most one of the random variables $\{|X_{u,t}| : u \in I_v\}$ is bigger than $h_t\theta/t$ ($\theta > 0$). Thus to investigate the limit property of $\mathcal{N}_t$, defined by (1.6), we will consider another point process:

$$\widetilde{\mathcal{N}}_t := \sum_{v\in\mathcal{L}_t}\sum_{u\in I_v}\delta_{h_t^{-1}X_{u,t}}.$$

Proposition 2.6 *Under $\mathbb{P}$, as $t \to \infty$,*

$$\widetilde{\mathcal{N}}_t \xrightarrow{d} \mathcal{N}_\infty.$$

The proof of this proposition is postponed to the next subsection. The following lemma formalizes the well-known one large jump principle (see, e.g., Steps 3 and 4 in Section 2 of [22]) at the level of point processes. Because of Lemma 2.7 below, it is enough to investigate the weak convergence of $\widetilde{\mathcal{N}}_t$, which is much easier compared to that of $\mathcal{N}_t$.

Lemma 2.7 *Assume $g \in C_c^+(\overline{\mathbb{R}}_0)$. For any $\epsilon > 0$,*

$$\lim_{t\to\infty}\mathbb{P}\left(|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| > \epsilon\right) = 0.$$

Proof: Since $g \in C_c^+(\overline{\mathbb{R}}_0)$, we have $\mathrm{Supp}(g) \subset \{x : |x| > \delta\}$ for some $\delta > 0$.

Step 1: For any $\theta > 0$, let $A_t(\theta)$ denote the event that for all $v \in \mathcal{L}_t$, at most one of the random variables $\{|X_{u,t}| : u \in I_v\}$ is bigger than $h_t\theta/t$. We claim that

$$\mathbb{P}(A_t(\theta)^c) \to 0. \tag{2.6}$$

Note that

$$\mathbb{P}\left(A_t(\theta)^c|\mathcal{F}_t^{\mathbb{T}}\right) \le \sum_{v\in\mathcal{L}_t} \mathbb{P}\left(\sum_{u\in I_v} \mathbf{1}_{\{|X_{u,t}|>h_t\theta/t\}} \ge 2|\mathcal{F}_t^{\mathbb{T}}\right). \tag{2.7}$$

By Lemma 2.2 and (2.3) with $\epsilon = 0.5$, we have that for $h_t\theta/t > 2 + 2a_{0.5}$ and $h_t > a_{0.5}$,

$$\begin{aligned}\mathbb{P}\left(|X_{u,t}| > h_t\theta/t|\mathcal{F}_t^{\mathbb{T}}\right) &= \mathrm{P}\left(|\xi_s| > h_t\theta/t\right)|_{s=\tau_{u,t}} \le c_0\tau_{u,t}h_t^{-\alpha}t^{\alpha}\theta^{-\alpha}L(h_t\theta/t)\\ &\le 2c_0\theta^{-\alpha}t^{1+\alpha}h_t^{-\alpha}L(h_t)[(\theta/t)^{1/2} + (\theta/t)^{-1/2}] := p_t. \end{aligned}\tag{2.8}$$

Recall that the number of elements in I_v is $n_t^v + 1$. Since conditioned on $\mathcal{F}_t^{\mathbb{T}}$, $\{X_{u,t}, u \in I_v\}$ are independent, by (2.8), we get that

$$\begin{aligned}\mathbb{P}\left(\sum_{u\in I_v} \mathbf{1}_{\{|X_{u,t}|>h_t\theta/t\}} \ge 2|\mathcal{F}_t^{\mathbb{T}}\right) &\le \sum_{m=2}^{n_t^v+1} \binom{n_t^v+1}{m} p_t^m = p_t^2 \sum_{m=0}^{n_t^v-1} \binom{n_t^v+1}{m+2} p_t^m\\ &\le p_t^2 \sum_{m=0}^{n_t^v-1} n_t^v(n_t^v+1)\binom{n_t^v-1}{m} p_t^m\\ &= p_t^2 n_t^v(n_t^v+1)(1+p_t)^{n_t^v-1}.\end{aligned}$$

Thus by (2.7) and the many-to-one formula (Lemma 2.4), we get that

$$\begin{aligned}\mathbb{P}(A_t(\theta)^c) &= \mathbb{E}\left(\mathbb{P}(A_t(\theta)^c|\mathcal{F}_t^{\mathbb{T}})\right) \le e^{\lambda t}p_t^2 E\left(n_t(n_t+1)(1+p_t)^{n_t-1}\right)\\ &= e^{\lambda t}p_t^2(2\beta + (1+p_t)\beta^2)e^{\beta p_t},\end{aligned}\tag{2.9}$$

where n_t is a Poisson process with parameter β on some probability space $(\Omega, \mathcal{G}, P)$. Since $e^{\lambda t}h_t^{-\alpha}L(h_t) \to 1$, (2.6) follows from (2.8) and (2.9) immediately.

Step 2: Let $\varrho > \beta + 1$ to be chosen later. Let $B_t(\varrho)$ be the event that for all $v \in \mathcal{L}_t$, $n_t^v \le \varrho t$. Using the many-to-one formula, we have that

$$\begin{aligned}\mathbb{P}(B_t(\varrho)^c) &\le \mathbb{E}\Big(\sum_{v\in\mathcal{L}_t} \mathbf{1}_{n_t^v>\varrho t}\Big) = e^{\lambda t}P(n_t > \varrho t) \le e^{\lambda t}\inf_{r>0} e^{-r\varrho t}E(e^{rn_t})\\ &= e^{\lambda t}\inf_{r>0} e^{((e^r-1)\beta - r\varrho)t} = e^{\lambda t}e^{-(\varrho(\log\varrho - \log\beta)-\varrho+\beta)t}.\end{aligned}$$

Choose ϱ large enough so that $\varrho(\log\varrho - \log\beta) - \varrho + \beta > \lambda$, then

$$\lim_{t\to\infty} \mathbb{P}(B_t(\varrho)^c) = 0.$$

Step 3: Since $g \in C_c^+(\overline{\mathbb{R}}_0)$, g is uniformly continuous, that is for any $a > 0$, there exists $\eta > 0$ such that $|g(x_1) - g(x_2)| \le a$ whenever $|x_1 - x_2| < \eta$.

Now consider θ small enough such that $\varrho\theta < \eta \wedge (\delta/2)$. Let $v' \in I_v$ be such that $|X_{v',t}| = \max_{u\in I_v}\{|X_{u,t}|\}$. We note that, on the event $A_t(\theta)$, $|X_{u,t}| \le \theta h_t/t \le h_t\delta/2$ for any $u \in I_v \setminus \{v'\}$ and $t > 1$, and thus $g(X_{u,t}/h_t) = 0$, which implies that

$$\widetilde{\mathcal{N}}_t(g) = \sum_{v\in\mathcal{L}_t}\sum_{u\in I_v} g(X_{u,t}/h_t) = \sum_{v\in\mathcal{L}_t} g(X_{v',t}/h_t).$$

Thus it follows that, on the event $A_t(\theta)$,

$$|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| = \left| \sum_{v \in \mathcal{L}_t} \left[g(\xi_t^v/h_t) - g(X_{v',t}/h_t) \right] \right| \tag{2.10}$$

Since $\xi_t^v = \sum_{u \in I_v} X_{u,t}$, on the event $A_t(\theta) \cap B_t(\varrho)$, we have that

$$h_t^{-1}|\xi_t^v - X_{v',t}| = h_t^{-1} \left| \sum_{u \in I_v \setminus \{v'\}} X_{u,t} \right| \le \theta t^{-1} n_t^v \le \varrho\theta < \eta \wedge (\delta/2).$$

Note that if $|X_{v',t}/h_t| \le \delta/2$, then $|\xi_t^v|/h_t < \delta$, which implies that $g(\xi_t^v/h_t) - g(X_{v',t}/h_t) = 0$. Thus we get that

$$|g(\xi_t^v/h_t) - g(X_{v',t}/h_t)| = |g(\xi_t^v/h_t) - g(X_{v',t}/h_t)|\mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}} \le a\mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}}. \tag{2.11}$$

It follows from (2.10) and (2.11) that, on the event $A_t(\theta) \cap B_t(\varrho)$,

$$\begin{aligned} |\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| &\le a \sum_{v \in \mathcal{L}_t} \mathbf{1}_{\{|X_{v',t}|>h_t\delta/2\}} \le a \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v} \mathbf{1}_{\{|X_{u,t}|>h_t\delta/2\}} \\ &= a\widetilde{\mathcal{N}}_t \left\{ [-\infty, -\delta/2) \cup (\delta/2, \infty] \right\}. \end{aligned}$$

Let $f \in C_c^+(\overline{\mathbb{R}}_0)$ satisfy $f(x) = 1$, for $|x| \ge \delta/2$. Then

$$|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| \le a\widetilde{\mathcal{N}}_t(f)).$$

Combining the three steps above, we get that

$$\begin{aligned} &\limsup_{t\to\infty} \mathbb{P}\left(|\mathcal{N}_t(g) - \widetilde{\mathcal{N}}_t(g)| > \epsilon \right) \\ &\le \limsup_{t\to\infty} \mathbb{P}\left(A_t(\theta)^c\right) + \mathbb{P}\left(B_t(\varrho)^c\right) + \mathbb{P}\left(\widetilde{\mathcal{N}}_t(h) > a^{-1}\epsilon \right) \\ &= \limsup_{t\to\infty} \mathbb{P}\left(\widetilde{\mathcal{N}}_t(f) > a^{-1}\epsilon \right) = P\left(\mathcal{N}_\infty(f) > a^{-1}\epsilon \right), \end{aligned}$$

where the final equality follows from Proposition 2.6 (the proof of Proposition 2.6 does not use the result in this lemma). Then letting $a \to 0$, we get the desired result. □

Proof of Theorem 1.2: Using Lemma 2.5, Proposition 2.6 and Lemma 2.7, the results of Theorem 1.2 follow immediately. □

2.3 Proof of Proposition 2.6

To prove the weak convergence of $\widetilde{\mathcal{N}}_t$, we first cut the tree at time $t - s$. We divide the particles born before time t into two parts: the particles born before time $t - s$ and after $t - s$. Define

$$\widetilde{\mathcal{N}}_{s,t} := \sum_{v \in \mathcal{L}_t} \sum_{u \in I_v, b_u > t-s} \delta_{h_t^{-1} X_{u,t}}.$$

Lemma 2.8 *For any $\epsilon > 0$ and $g \in C_c^+(\overline{\mathbb{R}}_0)$,*

$$\lim_{s\to\infty} \limsup_{t\to\infty} \mathbb{P}\left(|\widetilde{\mathcal{N}}_t(g) - \widetilde{\mathcal{N}}_{s,t}(g)| > \epsilon \right) = 0.$$

Proof: Since $g \in C_c^+(\overline{\mathbb{R}}_0)$, We have $\mathrm{Supp}(g) \subset \{x : |x| > \delta\}$ for some $\delta > 0$.

Let $J_{s,t}$ be the event that for all u with $b_u \le t-s$, $|X_{u,t}| \le h_t\delta/2$. On $J_{s,t}$, $\widetilde{\mathcal{N}}_t(g) - \widetilde{\mathcal{N}}_{s,t}(g) = 0$, thus we only need to show that

$$\lim_{s\to\infty} \limsup_{t\to\infty} \mathbb{P}(J_{s.t}^c) = 0. \tag{2.12}$$

Recall that $G_s(x) := \mathrm{P}(|\xi_s| > x)$. By Lemma 2.2, we have that, for t large enough so that $h_t\delta/2 \ge 2 + 2a_{0.5}$,

$$\begin{aligned}
\mathbb{P}(J_{s.t}^c) = 1 - \mathbb{P}(J_{s.t}) &= 1 - \mathbb{E}\left(\prod_{u:b_u \le t-s} (1 - G_{\tau_{u,t}}(h_t\delta/2)) \right) \\
&\le \mathbb{E}\Big(\sum_{u:b_u\le t-s} G_{\tau_{u,t}}(h_t\delta/2) \Big) \\
&\le c_0 h_t^{-\alpha} (\delta/2)^{-\alpha} L(h_t\delta/2) \mathbb{E}\Big(\sum_{u:b_u\le t-s} \tau_{u,t} \Big).
\end{aligned} \tag{2.13}$$

In the first inequality above, we used the inequality $1 - \prod_{i=1}^n ((1-x_i)) \le \sum_{i=1}^n x_i$, $x_i \in (0,1)$. By the definition of $\tau_{u,t}$,

$$\begin{aligned}
\sum_{u:b_u\le t-s} \tau_{u,t} &= \sum_{u:b_u\le t-s} \int_0^t \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r \\
&= \int_0^{t-s} \sum_u \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r + \int_{t-s}^t \sum_u \mathbf{1}_{b_u < t-s, \sigma_u > r}\,\mathrm{d}r.
\end{aligned}$$

For the first part, noting that $r \in (b_u, \sigma_u)$ is equivalent to $u \in \mathcal{L}_r$, we get

$$\mathbb{E}\int_0^{t-s} \sum_u \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r = \mathbb{E}\int_0^{t-s} Z_r\,\mathrm{d}r = \int_0^{t-s} e^{\lambda r}\,\mathrm{d}r = \lambda^{-1}(e^{\lambda(t-s)} - 1).$$

For the second part, using the many-to-one formula, we have that

$$\mathbb{E}\left(\sum_u \mathbf{1}_{b_u<t-s,\sigma_u>r} \right) = \mathbb{E}\left(\sum_{u\in\mathcal{L}_r} \mathbf{1}_{b_u<t-s} \right) = e^{\lambda r} e^{-\beta(r+s-t)}.$$

Thus,

$$\mathbb{E}\int_{t-s}^t \sum_u \mathbf{1}_{b_u<t-s,\sigma_u>r}\,\mathrm{d}r = \int_{t-s}^t e^{\lambda r} e^{-\beta(r-t+s)}\,\mathrm{d}r = e^{\lambda t}\frac{e^{-\beta s} - e^{-\lambda s}}{\lambda - \beta}.$$

Combining all the above, we get that

$$\mathbb{P}(J_{s.t}^c) \le c_0 (\delta/2)^{-\alpha} e^{\lambda t} h_t^{-\alpha} L(h_t\delta/2) \Big(\lambda^{-1} e^{-\lambda s} + \frac{e^{-\beta s} - e^{-\lambda s}}{\lambda - \beta} \Big). \tag{2.14}$$

Now first letting $t \to \infty$, and then $s \to \infty$, we get (2.12) immediately. The proof is now complete. $\square$

Now we consider the weak convergence of $\widetilde{\mathcal{N}}_{s,t}$. For $w \in \mathcal{L}_{t-s}$, let $\mathcal{L}_t^w := \{v \in \mathcal{L}_t : w \in I_v\}$ be the set of all the offspring of w at time t. We rewrite $\widetilde{\mathcal{N}}_{s,t}$ as follows:

$$\widetilde{\mathcal{N}}_{s,t} = \sum_{w \in \mathcal{L}_{t-s}} M_{s,t}^w, \tag{2.15}$$

where $M_{s,t}^w := \sum_{v \in \mathcal{L}_t^w} \sum_{u \in I_v, b_u > t-s} \delta_{h_t^{-1} X_{u,t}}$ are i.i.d. with common law

$$M_{s,t} := \sum_{v \in \mathcal{L}_s} \sum_{u \in I_v \setminus \{o\}} \delta_{h_t^{-1} X_{u,s}} = \sum_{u \in D_s} Z_s^u \delta_{h_t^{-1} X_{u,s}},$$

where Z_s^u is the number of the offspring of u at time s, and $D_s = \{u : b_u \le s\} \setminus \{o\}$.

Lemma 2.9 *For any $j = 1, \cdots, n$, let $\gamma_j(t)$ be a $(0,1]$-valued function on $(0,\infty)$. Suppose a_t is a positive function with $\lim_{t\to\infty} a_t = \infty$ such that $\lim_{t\to\infty} a_t(1-\gamma_j(t)) \to c_j < \infty$. Then*

$$\lim_{t\to\infty} a_t \left(1 - \prod_{j=1}^n \gamma_j(t)\right) \to \sum_{j=1}^n c_j.$$

Proof: Note that

$$1 - \prod_{j=1}^n \gamma_j(t) = \sum_{j=1}^n \prod_{k=1}^{j-1} \gamma_k(t)(1-\gamma_j(t)).$$

Since $\gamma_j(t) \to 1$, thus we get that, as $t \to \infty$,

$$a_t \left(1 - \prod_{j=1}^n \gamma_j(t)\right) = \sum_{j=1}^n \prod_{k=1}^{j-1} \gamma_k(t) a_t (1-\gamma_j(t)) \to \sum_{j=1}^n c_j.$$

$\square$

Proof of Proposition 2.6: By Lemma 2.8, we only need to consider the convergence of $\widetilde{\mathcal{N}}_{s,t}$. Assume that $\mathrm{Supp}(g) \subset \{x : |x| > \delta\}$ for some $\delta > 0$. Using the Markov property and the decomposition of $\widetilde{\mathcal{N}}_{s,t}$ in (2.15), we have that

$$\mathbb{E}\left(e^{-\widetilde{\mathcal{N}}_{s,t}(g)}\right) = \mathbb{E}\left([\mathbb{E}(e^{-M_{s,t}(g)})]^{Z_{t-s}} \right). \tag{2.16}$$

We claim that

$$\lim_{t\to\infty} \left(1 - \mathbb{E}(e^{-M_{s,t}(g)})\right) e^{\lambda t} = \int_{\mathbb{R}_0} \mathbb{E}\left[\sum_{u \in D_s} \tau_{u,s} 1 - e^{-Z_s^u g(x)} \right] v_\alpha(\mathrm{d}x). \tag{2.17}$$

By the definition of $M_{s,t}$, we have that

$$\left(1 - \mathbb{E}(e^{-M_{s,t}(g)} | \mathcal{F}_s^{\mathbb{T}})\right) e^{\lambda t} = e^{\lambda t} \left(1 - \prod_{u \in D_s} \mathbb{E}(e^{-Z_s^u g(h_t^{-1} X_{u,s})} | \mathcal{F}_s^{\mathbb{T}})\right).$$

Note that, given $\mathcal{F}_s^{\mathbb{T}}$, $X_{u,s} \stackrel{d}{=} \xi_{\tau_{u,s}}$. Thus by Lemma 2.1 (with s replaced by $\tau_{u,s}$ and g replaced by $1-e^{-Z_s^u g(x)}$) we get that

$$e^{\lambda t}\left(1-\mathbb{E}[e^{-Z_s^u g(h_t^{-1}X_{u,s})}|\mathcal{F}_s^{\mathbb{T}}]\right) \to \tau_{u,s}\int_{\mathbb{R}_0} 1-e^{-Z_s^u g(x)} \upsilon_\alpha(\mathrm{d}x),$$

as $t\to\infty$. Hence it follows from Lemma 2.9 that

$$\lim_{t\to\infty} e^{\lambda t}\left(1-\mathbb{E}[e^{-M_{s,t}(g)}|\mathcal{F}_s^{\mathbb{T}}]\right) = \int_{\mathbb{R}_0}\sum_{u\in D_s}\tau_{u,s}[1-e^{-Z_s^u g(x)}]\upsilon_\alpha(dx). \tag{2.18}$$

Moreover, for $h_t\delta \geq 2+2a_{0.5}$,

$$\begin{aligned} e^{\lambda t}\left(1-\mathbb{E}[e^{-M_{s,t}(g)}|\mathcal{F}_s^{\mathbb{T}}]\right) &\leq e^{\lambda t}\mathbb{E}\left(M_{s,t}(g)|\mathcal{F}_s^{\mathbb{T}}\right) \leq \|g\|_\infty e^{\lambda t}\sum_{u\in D_s} Z_s^u G_{\tau_{u,s}}(h_t\delta) \\ &\leq c_0\|g\|_\infty \delta^{-\alpha} e^{\lambda t} h_t^{-\alpha} L(h_t\delta)\sum_{u\in D_s}\tau_{u,s}Z_s^u \\ &\leq C\sum_{u\in D_s}\tau_{u,s}Z_s^u, \end{aligned} \tag{2.19}$$

where C is a constant not depending on t. The third inequality follows from Lemma 2.2 and the final inequality from the fact $e^{\lambda t}h_t^{-\alpha}L(h_t\delta)\to 1$. Since $\tau_{u,s} = \int_0^s \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r$, we have that

$$\begin{aligned} \mathbb{E}\left(\sum_{u\in D_s}\tau_{u,s}Z_s^u\right) &= \int_0^s \mathbb{E}\left(\sum_{u\in D_s}\mathbf{1}_{(b_u,\sigma_u)}(r)Z_s^u\right)\mathrm{d}r \\ &= \int_0^s \mathbb{E}\left(\sum_{u\in\mathcal{L}_r-\{o\}} Z_s^u\right)\mathrm{d}r \leq \int_0^s \mathbb{E}(Z_s)\,\mathrm{d}r = se^{\lambda s} < \infty. \end{aligned}$$

Thus by (2.18), (2.19) and the dominated convergence theorem, the claim (2.17) holds.

By (2.17) and the fact that $\lim_{t\to\infty} e^{-\lambda t}Z_{t-s} = e^{-\lambda s}W$, we have

$$\lim_{t\to\infty}\left[\mathbb{E}\left(e^{-M_{s,t}(g)}\right)\right]^{Z_{t-s}} = \exp\left\{-e^{-\lambda s}W\int_{\mathbb{R}_0}\mathbb{E}\Big[\sum_{u\in D_s}\tau_{u,s}(1-e^{-Z_s^u g(x)})\Big]\upsilon_\alpha(\mathrm{d}x)\right\}.$$

Thus by (2.16) and the bounded convergence theorem, we get that

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\widetilde{N}_{s,t}(g)}\right) = \mathbb{E}\left(\exp\left\{-e^{-\lambda s}W\int_{\mathbb{R}_0}\mathbb{E}\Big[\sum_{u\in D_s}\tau_{u,s}(1-e^{-Z_s^u g(x)})\Big]\upsilon_\alpha(\mathrm{d}x)\right\}\right).$$

By the definition of $\tau_{u,s}$, we have that

$$\sum_{u\in D_s}\tau_{u,s}\left(1-e^{-Z_s^u g(x)}\right) = \sum_{u\in D_s}\int_0^s \mathbf{1}_{(b_u,\sigma_u)}(r)\,\mathrm{d}r\left(1-e^{-Z_s^u g(x)}\right) = \int_0^s\sum_{u\in\mathcal{L}_r\setminus\{o\}}\left(1-e^{-Z_s^u g(x)}\right)\mathrm{d}r.$$

Using the Markov property, and the branching property, $Z_s^u, u \in \mathcal{L}_r$ are i.i.d. with the same distribution as Z_{s-r}, and independent with $\mathcal{L}_r$. Thus,

$$\begin{aligned}\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\left(1-e^{-Z_s^u g(x)}\right) &= \int_0^s \mathbb{E}\left(Z_r - \mathbf{1}_{\{o\in\mathcal{L}_r\}}\right)\mathbb{E}\left(1-e^{-Z_{s-r}g(x)}\right)\,\mathrm{d}r \\ &= \int_0^s \left(e^{\lambda r}-e^{-\beta r}\right)\mathbb{E}\left(1-e^{-Z_{s-r}g(x)}\right)\,\mathrm{d}r, \end{aligned} \tag{2.20}$$

which implies that

$$e^{-\lambda s}\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\left(1-e^{-Z_s^u g(x)}\right)\to\int_0^\infty e^{-\lambda r}\mathbb{E}\left(1-e^{-Z_r g(x)}\right)\,\mathrm{d}r,$$

and

$$e^{-\lambda s}\mathbb{E}\sum_{u\in D_s}\tau_{u,s}\left(1-e^{-Z_s^u g(x)}\right)\le\int_0^\infty e^{-\lambda r}\mathbb{E}\left(1-e^{-Z_r g(x)}\right)\,\mathrm{d}r\le\lambda^{-1}\mathbf{1}_{\{|x|>\delta\}}.$$

The final inequality follows from the fact that $\mathrm{Supp}(g)\subset\{x:|x|>\delta\}$. Since $v_\alpha(\mathbf{1}_{\{|x|>\delta\}})<\infty$, using the dominated convergence theorem we get that

$$\lim_{s\to\infty}e^{-\lambda s}\int_{\mathbb{R}_0}\mathbb{E}\Big[\sum_{u\in D_s}\tau_{u,s}(1-e^{-Z_s^u g(x)})\Big]v_\alpha(\mathrm{d}x)=\int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}\left(1-e^{-Z_r g(x)}\right)v_\alpha(\mathrm{d}x)\,\mathrm{d}r,$$

which implies that

$$\lim_{s\to\infty}\lim_{t\to\infty}\mathbb{E}\left(e^{-\widetilde{\mathcal{N}}_{s,t}(g)}\right)=\mathbb{E}\left(\exp\left\{-W\int_0^\infty e^{-\lambda r}\int_{\mathbb{R}_0}\mathbb{E}\left(1-e^{-Z_r g(x)}\right)v_\alpha(\mathrm{d}x)\,\mathrm{d}r\right\}\right).$$

By Lemmas 2.8 and 2.5, we get that

$$\lim_{t\to\infty}\mathbb{E}\left(e^{-\widetilde{\mathcal{N}}_t(g)}\right)=E\left(e^{-\mathcal{N}_\infty(g)}\right).$$

The proof is now complete. $\square$

3 Joint convergence of the order statistics

Proof of Corollary 1.4: First, we will show that $M_{(k)}>0$, a.s.. Recall that, given W, $\sum_j\delta_{e_j}$ is a Poisson random measure with intensity $\vartheta W v_\alpha$. Since $v_\alpha(0,\infty)=\infty$, we have that $P^*\left(\sum_j\mathbf{1}_{(0,\infty)}(e_j)=\infty\right)=1$, which implies that $P^*\left(\mathcal{N}_\infty(0,\infty)=\infty\right)=1$. Thus $M_{(k)}>0$, P^*-a.s.

Note that, for any $x\in\overline{\mathbb{R}}_0$, $\mathcal{N}_\infty(\{x\})=0$, a.s.. Since $\{M_{t,k}\le h_t x\}=\{\mathcal{N}_t(x,\infty)\le k-1\}$ for any $x>0$, by Remark 1.3 with $B_k=(x_k,\infty)$, we have that for any $n\ge1$ and $x_1,x_2,x_3,\cdots,x_n>0$,

$$\mathbb{P}\left(M_{t,1}\le h_t x_1, M_{t,2}\le h_t x_2, M_{t,3}\le h_t x_3,\cdots,M_{t,n}\le h_t x_n\right)$$

$$
\begin{aligned}
&= \mathbb{P}\left(\mathcal{N}_t((x_k,\infty) \le k-1, k=1,\cdots,n\right) \\
&\to P(\mathcal{N}_\infty((x_k,\infty)) \le k-1, k=1,\cdots,n) \\
&= P(M_{(1)} \le x_1, M_{(2)} \le x_2, M_{(3)} \le x_3, \cdots, M_{(n)} \le x_n) \quad \text{as } t\to\infty.
\end{aligned}
$$

Thus, as $t \to \infty$,

$$
\begin{aligned}
&\mathbb{P}^*\left(M_{t,1} \le h_t x_1, M_{t,2} \le h_t x_2, \cdots, M_{t,n} \le h_t x_n\right) \\
=&\mathbb{P}(\mathcal{S})^{-1}\left[\mathbb{P}(M_{t,k} \le h_t x_k, k=1,\cdots,n) - \mathbb{P}(M_{t,k} \le h_t x_k, k=1,\cdots,n,\mathcal{S}^c)\right] \\
\to&\mathbb{P}(\mathcal{S})^{-1}[P(M_{(k)} \le x_k, k=1,\cdots,n) - \mathbb{P}(\mathcal{S}^c)] \\
=&P^*(M_{(k)} \le x_k, k=1,\cdots,n),
\end{aligned}
\tag{3.1}
$$

where in the final equality, we used the fact that on the event of extinction, $M_{(k)} = -\infty$, $k \ge 1$.

Now we consider the case $x_1, \cdots, x_n \in \mathbb{R}$ with $x_i \le 0$ for some i, and $x_j > 0$, $j \ne i$. By (3.1), we get that, for any $\epsilon > 0$

$$
\begin{aligned}
&\limsup_{t\to\infty} \mathbb{P}^*(M_{t,1} \le h_t x_1, M_{t,2} \le h_t x_2, \cdots, M_{t,n} \le h_t x_n) \\
\le&\lim_{t\to\infty} \mathbb{P}^*(M_{t,j} \le h_t x_j, j \ne i, M_{t,i} \le h_t \epsilon) \\
=&P^*(M_{(j)} \le x_j, j \ne i, M_{(i)} \le \epsilon).
\end{aligned}
$$

The right hand side of the display above tends to 0 as $\epsilon \to 0$ since $M_{(i)} > 0$ a.s.. Thus

$$
\lim_{t\to\infty} \mathbb{P}^*(M_{t,k} \le h_t x_k, k=1,\cdots,n) = 0 = P^*(M_{(k)} \le x_k, k=1,\cdots,n). \tag{3.2}
$$

Similarly, we can get (3.2) holds for any $x_1, \cdots, x_n \in \mathbb{R}$.

The proof is now complete. □

4 Examples and an extension

In this section, we first give more examples satisfying (H2).

Lemma 4.1 *Assume that L^* is a positive function on $(0,\infty)$ slowly varying at ∞ such that $l_\epsilon(x) := \sup_{y\in(0,x]} y^\epsilon L^*(y) < \infty$ for any $\epsilon > 0$ and $x > 0$. Then, for any $\epsilon > 0$, there exist $c_\epsilon, C_\epsilon > 0$ such that for any $y > 0$ and $a > c_\epsilon$,*

$$
\frac{L^*(ay)}{L^*(a)} \le C_\epsilon(y^\epsilon + y^{-\epsilon}). \tag{4.1}
$$

Proof: By [7, Theorem 1.5.6], for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that for any $a \ge c_\epsilon$ and $y \ge a^{-1}c_\epsilon$,

$$
\frac{L^*(ay)}{L^*(a)} \le (1-\epsilon)^{-1} \max\{y^\epsilon, y^{-\epsilon}\}. \tag{4.2}
$$

Thus for any $a > c_\epsilon$,

$$\frac{L^*(c_\epsilon)}{L^*(a)} \le (1-\epsilon)^{-1}(a/c_\epsilon)^\epsilon. \tag{4.3}$$

Hence for $a > c_\epsilon$ and $0 < y \le a^{-1}c_\epsilon$, we have that

$$\frac{L^*(ay)}{L^*(a)} \le l_\epsilon(c_\epsilon)(ay)^{-\epsilon}/L^*(a) \le \frac{l_\epsilon(c_\epsilon)}{L^*(c_\epsilon)(1-\epsilon)c_\epsilon^\epsilon} y^{-\epsilon}. \tag{4.4}$$

Combining (4.2) and (4.4), there exists $C_\epsilon > 0$ such that for any $y > 0$ and $a > c_\epsilon$,

$$\frac{L^*(ay)}{L^*(a)} \le C_\epsilon(y^\epsilon + y^{-\epsilon}).$$

□

Example 4.2 *Let*

$$n(\mathrm{d}y) = c_1 x^{-(1+\alpha)} L^*(x) \mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + c_2|x|^{-(1+\alpha)} L^*(|x|)\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x,$$

where $\alpha \in (0,2), c_1, c_2 \ge 0$, $c_1 + c_2 > 0$ *and* L^* *is a positive function on* $(0,\infty)$ *slowly varying at* ∞ *such that* $\sup_{y\in(0,x]} y^\epsilon L^*(y) < \infty$ *for any* $\epsilon > 0$ *and* $x > 0$.

(1) For $\alpha \in (0,1)$, *assume that the Lévy exponent of* ξ *has the following form:*

$$\psi(\theta) = ia\theta - b^2\theta^2 + \int (e^{i\theta y} - 1) n(\mathrm{d}y),$$

where $a \in \mathbb{R}, b \ge 0$. *Using Lemma 4.1 with* $\epsilon \in (0, (1-\alpha)\wedge\alpha)$, *we have that, by the dominated convergence theorem, as* $\theta \to 0_+$,

$$\begin{aligned}\int_0^\infty (e^{i\theta y} - 1) n(\mathrm{d}y) &= \theta^\alpha \int_0^\infty (e^{iy} - 1) y^{-1-\alpha} L^*(\theta^{-1}y)\,\mathrm{d}y \\ &\sim \theta^\alpha L^*(\theta^{-1}) \int_0^\infty (e^{iy} - 1) y^{-1-\alpha}\,\mathrm{d}y = -\alpha\Gamma(1-\alpha)e^{-i\pi\alpha/2}\theta^\alpha L^*(\theta^{-1}),\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^0 (e^{i\theta y} - 1) n(\mathrm{d}y) &= \theta^\alpha \int_0^\infty (e^{-iy} - 1) y^{-1-\alpha} L^*(\theta^{-1}y)\,\mathrm{d}y \\ &\sim \theta^\alpha L^*(\theta^{-1}) \int_0^\infty (e^{-iy} - 1) y^{-1-\alpha}\,\mathrm{d}y = -\alpha\Gamma(1-\alpha)e^{i\pi\alpha/2}\theta^\alpha L^*(\theta^{-1}).\end{aligned}$$

Thus as $\theta \to 0_+$,

$$\psi(\theta) \sim -\alpha\Gamma(1-\alpha)(e^{-i\pi\alpha/2}c_1 + e^{i\pi\alpha/2}c_2)\theta^\alpha L^*(\theta^{-1}).$$

(2) For $\alpha \in (1,2)$, *assume that the Lévy exponent of* ξ *has the following form:*

$$\psi(\theta) = -b^2\theta^2 + \int (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y),$$

where $b \geq 0$. Using Lemma 4.1 with $\epsilon \in (0,(2-\alpha)\wedge(\alpha-1))$, we have that, by the dominated convergence theorem, as $\theta \to 0_+$,

$$\begin{aligned}\int_0^\infty (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y) &= \theta^\alpha \int_0^\infty (e^{iy} - 1 - iy) y^{-1-\alpha} L^*(\theta^{-1}y)\,\mathrm{d}y \\ &\sim \theta^\alpha L^*(\theta^{-1}) \int_0^\infty (e^{iy} - 1 + iy) y^{-1-\alpha}\,\mathrm{d}y \\ &= -\alpha\Gamma(1-\alpha) e^{-i\pi\alpha/2} \theta^\alpha L^*(\theta^{-1}),\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^0 (e^{i\theta y} - 1 - i\theta y) n(\mathrm{d}y) &= \theta^\alpha \int_0^\infty (e^{-iy} - 1 + iy) y^{-1-\alpha} L^*(\theta^{-1}y)\,\mathrm{d}y \\ &\sim \theta^\alpha L^*(\theta^{-1}) \int_0^\infty (e^{-iy} - 1 + iy) y^{-1-\alpha}\,\mathrm{d}y \\ &= -\alpha\Gamma(1-\alpha) e^{i\pi\alpha/2} \theta^\alpha L^*(\theta^{-1}).\end{aligned}$$

Thus as $\theta \to 0_+$,

$$\psi(\theta) \sim -\alpha\Gamma(1-\alpha)(e^{-i\pi\alpha/2}c_1 + e^{i\pi\alpha/2}c_2)\theta^\alpha L^*(\theta^{-1}).$$

(3) For $\alpha = 1$, assume that $c_1 = c_2$ and the Lévy exponent of ξ has the following form:

$$\psi(\theta) = ia\theta - b^2\theta^2 + \int (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y|\leq 1}) n(\mathrm{d}y),$$

where $a \in \mathbb{R}, b \geq 0$. Since $c_1 = c_2$, we have

$$\int_{-\infty}^\infty (e^{i\theta y} - 1 - i\theta y \mathbf{1}_{|y|\leq 1}) n(\mathrm{d}y) = -2c_1\theta \int_0^\infty (1 - cosy) y^{-2} L^*(\theta^{-1}y)\,\mathrm{d}y.$$

Using Lemma 4.1 with $\epsilon \in (0,1)$, we have that, by the dominated convergence theorem,

$$\lim_{\theta\to 0_+} L^*(\theta^{-1})^{-1} \int_0^\infty (1 - cosy) y^{-2} L^*(\theta^{-1}y)\,\mathrm{d}y = \int_0^\infty (1 - cosy) y^{-2}\,\mathrm{d}y = \pi/2,$$

which implies that as $\theta \to 0_+$,

$$\psi(\theta) \sim -(c_1\pi - ia)\theta L^*(\theta^{-1}).$$

□

Remark 4.3 (An extension) *Checking the proof of Theorem 1.2, we see that Theorem 1.2 holds for more general branching Lévy processes with spatial motions satisfying the following assumptions:*

(A1) *There exist a non-increasing function h_t with $h_t \uparrow \infty$ and a measure $\pi(\mathrm{d}x) \in \mathcal{M}(\overline{\mathbb{R}}_0)$ such that*

$$\lim_{t\to\infty} e^{\lambda t} \mathrm{E}(g(h_t^{-1}\xi_s)) = s \int_{\mathbb{R}_0} g(x)\pi(\mathrm{d}x), \quad g \in C_c^+(\overline{\mathbb{R}}_0).$$

(A2) $e^{\lambda t}p_t^2 \to 0$, *where* $p_t := \sup_{s\le t} \mathrm{P}(|\xi_s| > h_t\theta/t)$.

(A3) *For any* $\theta > 0$,

$$\sup_{t>1}\sup_{s\le t} s^{-1}e^{\lambda t}\mathrm{P}(|\xi_s| > h_t\theta) < \infty.$$

First, (H2) implies (A1)-(A3). Next we explain that Theorem 1.2 holds under Assumptions (A1)-(A3). Checking the proof of Lemma 2.7, we see that Lemma 2.7 holds under conditions (A1)-(A3). In fact, we may replace Lemma 2.2 by (A2) to get (2.6) *(see* (2.8) *and* (2.9)*). For the proof of Lemma 2.8, using (A3), we get that*

$$\mathbb{P}(J_{s,t}^c) \le Ce^{-\lambda t}\mathbb{E}\sum_{u:b_u\le t-s}\tau_{u,t},$$

which says that (2.13) *holds. Thus* (2.12) *holds using the same arguments in Lemma 2.8. Replacing Lemma 2.1 by (A1), we see that Proposition 2.6 holds with* v_α *replaced by* $\pi(\mathrm{d}x)$*. So under (A1)-(A3), Theorem 1.2 holds with* v_α *replaced by* $\pi(\mathrm{d}x)$*.*

An easy example which satisfies (A1)-(A3), but does not satisfy (H2) is the non-symmetric 1-stable process. Assume ξ *is a non-symmetric 1-stable process with Lévy measure*

$$n(dx) = c_1x^{-2}\mathbf{1}_{(0,\infty)}(x)\mathrm{d}x + c_2|x|^{-2}\mathbf{1}_{(-\infty,0)}(x)\mathrm{d}x,$$

where $c_1, c_2 \ge 0$, $c_1 + c_2 > 0$, *and* $c_1 \ne c_2$*. The Lévy exponent of* ξ *is given by, for* $\theta > 0$

$$\psi(\theta) = -\frac{\pi}{2}(c_1+c_2)\theta - i(c_1-c_2)\theta\log\theta + ia(c_1-c_2)\theta \sim -i(c_1-c_2)\theta\log\theta, \quad \theta \to 0+,$$

where a *is constant. Thus* $c_* = i(c_1 - c_2)$*. So* $\psi(\theta)$ *does not satisfy (H2) since* $\Re(c_*) = 0$*.*

By [6, Section 1.5, Exercise 1], we have that

$$\frac{1}{t}\mathrm{P}(\xi_t \in \cdot) \xrightarrow{v} n(\mathrm{d}x), \qquad \text{as } t \to 0.$$

Since $e^{-\lambda t}\xi_s \overset{d}{=} \xi_{se^{-\lambda t}} + (c_1 - c_2)s\lambda te^{-\lambda t}$ *for* $s, t > 0$*, we have that*

$$e^{\lambda t}\mathrm{P}(e^{-\lambda t}\xi_s \in \cdot) \xrightarrow{v} s\, n(\mathrm{d}x), \quad \text{as } t \to \infty.$$

So (A1) holds with $h_t = e^{\lambda t}$*. We claim that, for any* $x > 0$ *and* $s > 0$*,*

$$\mathrm{P}(|\xi_s| > x) \le c(sx^{-1} + s^2x^{-2} + s^2x^{-2}(\log x)^2), \tag{4.5}$$

where c *is a constant. Thus it is easy to prove that (A2) and (A3) hold.*

In fact, for any $x > 0$

$$\mathrm{P}(|\xi_s| > x) \le \frac{x}{2}\int_{-2x^{-1}}^{2x^{-1}}(1 - e^{s\psi(\theta)})\,\mathrm{d}\theta = x\int_0^{2x^{-1}}(1 - \Re(e^{s\psi(\theta)}))\,\mathrm{d}\theta.$$

Note that

$$1 - \Re(e^{s\psi(\theta)}) = 1 - e^{s\Re(\psi(\theta))}\cos[s\Im(\psi(\theta))]$$

$$
\begin{aligned}
&= 1 - e^{s\Re(\psi(\theta))} + e^{s\Re(\psi(\theta))}(1 - \cos[s\Im(\psi(\theta))]) \\
&\le -s\Re(\psi(\theta)) + s^2[\Im(\psi(\theta))]^2 \\
&= \frac{\pi}{2}(c_1 + c_2)s\theta + (c_1 - c_2)^2 s^2 (a - \log\theta)^2\theta^2.
\end{aligned}
$$

Thus we have that

$$
\begin{aligned}
\mathrm{P}(|\xi_s| > x) &\le \pi(c_1 + c_2)s\, x^{-1} + (c_1 - c_2)^2 s^2 x^{-2} \int_0^2 (a - \log\theta + \log x)^2\theta^2 \, \mathrm{d}\theta \\
&\le \pi(c_1 + c_2)s\, x^{-1} + 2(c_1 - c_2)^2 s^2 x^{-2} \int_0^2 [(a - \log\theta)^2 + (\log x)^2]\theta^2 \, \mathrm{d}\theta \\
&\le c(sx^{-1} + s^2x^{-2} + s^2x^{-2}(\log x)^2),
\end{aligned}
$$

which proves the claim (4.5).

5 Front position of Fisher-KPP equation

Recall that $u_g(t,x) = \mathbb{E}_{\delta_x}\left(e^{-\mathbb{X}_t(g)}\right) = \mathbb{E}\left(e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}\right)$. Then $1-u_g(t,x)$ is a mild solution to (1.4). For $\theta \in (0,1)$, the level set $\{x \in \mathbb{R} : 1-u_g(t,x) = \theta\}$ is also called the front of $1-u_g$. The evolution of the front of $1-u_g$ as time goes to ∞ is of considerable interest. Using analytic method, [16, Theorem 1.5] proved that if the density of ξ is comparable to that of a symmetric α-stable process, the front position is exponential in time, which is in contrast with branching Brownian motion where it is linear in time. In this paper, we provide a probabilistic proof of [16, Theorem 1.5] using Corollary 1.5, and also partially generalize it.

Proposition 5.1 *(1) Assume that a_t satisfies $a_t/h_t \to \infty$ as $t \to \infty$, and that g is a non-negative function satisfying*

$$
e^{\lambda t} \sup_{x \le -a_t/2} g(x) \to 0, \qquad \text{as } t \to \infty. \tag{5.1}
$$

Then

$$
\lim_{t\to\infty} \sup_{x \le -a_t} (1 - u_g(t,x)) = 0.
$$

(2) Assume that c_t satisfies $c_t/h_t \to 0$ as $t \to \infty$, and that g is a non-negative function satisfying $a_0 := \liminf_{x\to\infty} g(x) > 0$. Then

$$
\lim_{t\to\infty} \sup_{x \ge -c_t} |u_g(t,x) - \mathbb{P}(\mathcal{S}^c)| = 0.
$$

Proof: (1) Let $g^*(x) = \sup_{y \le -x} g(y)$. Note that, for $x \le -a_t$,

$$
\begin{aligned}
1 - u_g(t,x) &= \mathbb{E}\left(1 - e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}\right) \\
&\le \mathbb{P}(R_t \ge a_t/2) + \mathbb{E}\left(1 - e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}; R_t < a_t/2\right)
\end{aligned}
$$

$$\leq \mathbb{P}\left(R_t \geq a_t/2\right) + \mathbb{E}(1 - e^{-g^*(a_t/2)Z_t})$$
$$\leq \mathbb{P}(R_t \geq a_t/2) + e^{\lambda t} g^*(a_t/2), \tag{5.2}$$

where in the second inequality, we use the fact that, on the event $\{R_t < a_t/2\}$, $\xi_t^v + x < a_t/2 - a_t = -a_t/2$ and $g(\xi_t^v + x) \leq g^*(a_t/2)$. By the assumption (5.1), $e^{\lambda t} g^*(a_t/2) \to 0$. By Corollary 1.5, one has that $\mathbb{P}^*(R_t \geq a_t/2) \to 0$. Thus

$$\mathbb{P}(R_t \geq a_t/2) \leq \mathbb{P}^*(R_t \geq a_t/2)\mathbb{P}(\mathcal{S}) + \mathbb{P}(\|X_t\| > 0, \mathcal{S}^c) \to 0,$$

as $t \to \infty$. Thus by (5.2),

$$\lim_{t\to\infty} \sup_{x \leq -a_t} (1 - u_g(t,x)) = 0.$$

(2) Note that

$$|u_g(t,x) - \mathbb{P}(\mathcal{S}^c)| \leq \mathbb{E}\left(e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}; \mathcal{S}\right) + \mathbb{E}\left(1 - e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}; \mathcal{S}^c\right). \tag{5.3}$$

Noticing that on the event $Z_t = 0$, $1 - e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)} = 0$, we get that , for any $x \in \mathbb{R}$

$$\mathbb{E}\left(1 - e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}; \mathcal{S}^c\right) \leq \mathbb{P}(Z_t > 0, \mathcal{S}^c) \to 0,$$

as $t \to \infty$. Let $g_*(x) = \inf_{y \geq x} g(y)$. Since $c_t/h_t \to 0$, for any $\epsilon > 0$, there exists $t_\epsilon > 0$ such that $c_t \leq \epsilon h_t$ for $t > t_\epsilon$. For any $t > t_\epsilon$ and $x \geq -c_t$, we have that

$$\begin{aligned}
\mathbb{E}\left(e^{-\sum_{v\in\mathcal{L}_t} g(\xi_t^v + x)}; \mathcal{S}\right) &\leq \mathbb{E}\left(e^{-g_*(c_t)\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\xi_t^v > 2c_t}}; \mathcal{S}\right) \\
&\leq \mathbb{E}\left(e^{-g_*(c_t)\sum_{v\in\mathcal{L}_t} \mathbf{1}_{\xi_t^v > 2\epsilon h_t}}; \mathcal{S}\right) \\
&= \mathbb{E}\left(e^{-g_*(c_t)\mathcal{N}_t(2\epsilon,\infty)}; \mathcal{S}\right).
\end{aligned}$$

Thus

$$\limsup_{t\to\infty} \sup_{x \geq -c_t} |u_g(t,x) - \mathbb{P}(\mathcal{S}^c)| \leq E\left(e^{-a_0 \mathcal{N}_\infty(2\epsilon,\infty)}, \mathcal{S}\right). \tag{5.4}$$

Since on the event $\mathcal{S}$, $\vartheta W v_\alpha(0,\infty) = \infty$, thus $\mathcal{N}_\infty(0,\infty) = \infty$. Now letting $\epsilon \to 0$ in (5.4), we get the desired result. □

Remark 5.2 *Proposition 5.1 is a slight generalization of [16, Theorem 1.5]. Assume that $p_0 = 0$, which ensures that $\mathbb{P}(\mathcal{S}^c) = 0$. If $L = 1$, then $h_t = e^{\lambda t/\alpha}$, and we have the following results:*

(1) Let g be a non-negative measurable function satisfying

$$g(x) \leq C|x|^{-\alpha}, \quad x < 0. \tag{5.5}$$

Then for any $\gamma > \lambda/\alpha$

$$e^{\lambda t} g^*(-e^{\gamma t}/2) \leq C 2^\alpha e^{\lambda t} e^{-\alpha\gamma t} \to 0.$$

Thus by Proposition 5.1, we have that

$$\lim_{t\to\infty} \sup_{x \leq -e^{\gamma t}} (1 - u_g(t,x)) = 0.$$

(2) *Assume that g is a non-negative function satisfying $a_0 := \liminf_{x\to\infty} g(x) > 0$. For any $\gamma < \lambda/\alpha$, by Proposition 5.1, we have that*

$$\lim_{t\to\infty} \sup_{x\geq -e^{\gamma t}} u_g(t,x) = 0.$$

Note that in the notation of [16], $\sigma^{**} = \lambda/\alpha$, and our condition (5.5) is equivalent to

$$1 - e^{-g(x)} \leq C|x|^{-\alpha}, \quad x < 0, \tag{5.6}$$

for some constant C. If g is nondecreasing, it is clear that $\liminf_{x\to\infty} g(x) > 0$. Thus when the Lévy process ξ satisfies (H2) with $L = 1$, we can get that the conclusion of [16, Theorem 1.5] holds from Proposition 5.1. Note that the independent sum of Brownian motion and a symmetric α-stable process satisfies (H2) with $L = 1$, but its transition density is not comparable with that of the symmetric α-stable process, see [21, 38]. Note also that the independent sum of a symmetric α-stable process and a symmetric β-stable process, $0 < \alpha < \beta < 2$, also satisfies (H2) with $L = 1$, but its transition density is not comparable with that of the symmetric α-stable process, see [20]. Note that in this paper we do not need to assume that g is nondecreasing. Thus Proposition 5.1 partially generalizes [16, Theorem 1.5].

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