MATHEMATICS

Absolute continuities of exit measures for superdiffusions

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Received May 13, 1999

Abstract Suppose $X=\{X_t, X_\tau, P_\mu\}$ is a superdiffusion in $\mathbf{R}^d$ with general branching mechanism ψ and general branching rate function A. We discuss conditions on A to guarantee that the exit measures X_{τ_D} of the superdiffusion X from bounded smooth domains in $\mathbf{R}^d$ have absolutely continuous states.

Keywords: exit measure, superdiffusion, absolutely continuous state.

Let L be a uniformly elliptic differential operator in $\mathbf{R}^d$, $\xi := \{\xi_s, \Pi_x, s \geqslant 0, x \in \mathbf{R}^d\}$ denote the diffusion in $\mathbf{R}^d$ with generator L, and let

$$\psi(x,z) = a(x)z + b(x)z^2 + \int_0^\infty (\mathrm{e}^{-uz} - 1 + uz)n(x,\mathrm{d}u), \tag{0.1}$$

where n is a kernel from $\mathbf{R}^d$ to $(0,\infty)$ and $a(x), b(x)$ and $\int_0^\infty u \wedge u^2 n(x,\mathrm{d}u)$ are positive bounded Borel functions on $\mathbf{R}^d$. Suppose A is a continuous additive function of ξ. Consider the d-dimensional superdiffusion $X=\{X_t, X_\tau, P_\mu\}$ with parameters (L, ψ, A) (enhanced model).

Let τ_D denote the first exit time of ξ from an open set D in $\mathbf{R}^d$, i.e. $\tau_D = \inf\{t: \xi_t \notin D\}$. In this paper we investigate the absolute continuities of the first exit measures X_{τ_D} with general branching mechanism ψ given above and general branching rate function A. Particularly, when $A(\mathrm{d}t) = \mathrm{d}t$ and $\psi(x,z) = z^{1+\beta}$, $0<\beta\leqslant 1$, the states of the random exit measures X_{τ_D} were studied by Sheu[1]. It was shown that, in the case $d < 1+\frac{2}{\beta}$, the states of X_{τ_D} are absolutely continuous with respect to the surface area on ∂D, whereas in the case $d > 1+\frac{2}{\beta}$, they are singular. Ren and Wang[2] showed that if the branching is restricted to a singular hyperplane, X_{τ_D} are absolutely continuous for any dimension $d \geqslant 1$. But if A is changed to a sufficient irregular branching rate function and the branching mechanism ψ is given by the general form (0.1), the question as to what properties the random exit measures X_{τ_D} have left open. Our purpose is to solve this problem in some sense.

1 Notations and main results

For every Borel-measurable space $(E, \mathscr{B}(E))$, we denote by $M(E)$ the set of all finite measures on $\mathscr{B}(E)$ endowed with the topology of weak convergence. The expression $\langle f, \mu \rangle$ stands for the integral of f with respect to μ. We write $f \in \mathscr{B}(E)$ if f is a $\mathscr{B}(E)$-measurable function. Writing $f \in p\mathscr{B}(E)(b\mathscr{B}(E))$ means that, in addition, f is positive (bounded).

We put $bp\mathscr{B}(E) = b\mathscr{B}(E) \cap p\mathscr{B}(E)$. If $E = \mathbf{R}^d$, we simply write $\mathscr{B}$ instead of $\mathscr{B}(E)$. For a bounded domain D, we write $\mu \in M_c(D)$ if $\mu \in M(D)$ and has a compact support in D, and write $\mu \in M_0(\partial D)$ if $\mu \in M(\partial D)$ and μ has a finite support in ∂D.

We denote by $\mathscr{T}$ the set of all exit times from open sets in $\mathbf{R}^d$. Set $\mathscr{F}_{\leqslant r} = \sigma(\xi_s,\ s \leqslant r)$, $\mathscr{F}_{>r} = \sigma(\xi_s,\ s > r)$ and $\mathscr{F}_\infty = \vee \{\mathscr{F}_{\leqslant r},\ r \geqslant 0\}$. For $\tau \in \mathscr{T}$, we put $F \in \mathscr{F}_{\geqslant \tau}$ if $F \in \mathscr{F}_\infty$ and if, for each r, $\{F,\ \tau > r\} \in \mathscr{F}_{>r}$.

The process $X = \{X_t,\ X_\tau,\ P_\mu,\ t \geqslant 0,\ \tau \in \mathscr{T},\ \mu \in M(\mathbf{R}^d)\}$ is an $M(\mathbf{R}^d)$-valued Markov process. The transition measures of X_t, $t \geqslant 0$ are characterized through their Laplace transforms as

$$P_\mu \exp\langle -f, X_t\rangle = \exp\langle -v_t, \mu\rangle, \quad f \in bp\mathscr{B},\ \mu \in M(\mathbf{R}^d), \tag{1.1}$$

where v is the unique solution of the integral equation

$$v_t(x) + \Pi_x\left[\int_0^t \psi(\xi_s,\ v_{t-s}(\xi_s))\mathrm{d}s\right] = \Pi_x f(\xi_t). \tag{1.2}$$

Moreover, the transition measures of X_τ, $\tau \in \mathscr{T}$ are characterized as

$$P_\mu \exp\langle -f, X_{\tau_D}\rangle = \exp\langle -u,\ \mu\rangle, \quad f \in bp\mathscr{B},\ \mu \in M(\mathbf{R}^d), \tag{1.3}$$

where u is the unique solution of the integral equation

$$u(x) + \Pi_x\left[\int_0^\tau \psi(\xi_s,\ u(\xi_s))\mathrm{d}s\right] = \Pi_x f(\xi_\tau). \tag{1.4}$$

We call $X = \{X_t,\ X_\tau,\ P_\mu\}$ the superdiffusion with parameters $(L,\ \psi,\ A)$, ψ the branching mechanism of X and A the branching rate of X. The existence of X has been discussed in many papers (see, for example, Dynkin[3, 4]).

In the following we suppose D is a fixed bounded smooth domain in $\mathbf{R}^d$. Let $k(x,\ y)$ be the Poisson kernel of D. For $\nu \in M(\partial D)$ define

$$H_D\nu(x) = \int_D k(x, y)\nu(\mathrm{d}y). \tag{1.5}$$

The study of fundamental solutions of the following integral equation plays an important role in the investigation of the absolute continuity of X_{τ_D}.

$$u(x) + \Pi_x\int_0^{\tau_D} \psi(\xi_t, u(\xi_t))A(\mathrm{d}t) = H_D\nu(x), \quad x \in D, \tag{1.6}$$

with $\nu \in M(\partial D)$.

Let $S(\mathrm{d}z)$ be the surface area on the boundary ∂D of D. For $\nu = \sum_{i=1}^m \lambda_i \delta_{z_i}, z_1, \cdots, z_m \in \partial D$, $\lambda_i \in \mathbf{R}^+, i = 1, \cdots, m$, let

$$\nu_n(\mathrm{d}z) = \sum_{i=1}^m \lambda_i f_n^{z_i}(z) S(\mathrm{d}z), \tag{1.7}$$

where

$$f_n^{z_i}(z) = \begin{cases} \dfrac{1}{S(\partial D \cap B(z_i, 1/n))}, & z \in B(z_i,\ 1/n), \\ 0, & z \notin B(z_i, 1/n). \end{cases} \tag{1.8}$$

Clearly as $n \to \infty$, ν_n converges weakly to ν. ν_n is called the regularization of ν.

We now present the main results.

Theorem 1.1. Let A be a branching rate function of X satisfying

$$\Pi_x A(0,\ \tau_D) < \infty, \quad x \in D. \tag{1.9}$$

Assume that there exist a sequence of bounded smooth domains $\{D_n\}$ satisfying $D_n \uparrow D$ and a Borel subset N of surface measure 0 such that, for all $\nu \in M_0(\partial D)$ with finite support contained in $\partial D \setminus N$, ψ and A satisfy one of the following conditions (C1) and (C2):

(C1) ψ is given by (0.1) with $a(x) \equiv 0$.

$$\lim_{n\to\infty} \sup \Pi_x \int_{\tau_{D_k}}^{\tau_D} (H_D\nu_n(\xi_s))^2 A(\mathrm{d}s) \to 0 \quad (k \to \infty), \quad x \in D. \tag{1.10}$$

(C2) ψ is given by the particular form:

$$\psi(x,z) = \gamma(x) z^{1+\beta}, \quad 0 < \beta \leqslant 1, \quad \gamma \in bp\mathscr{B}, \tag{1.11}$$

$$\lim_{n\to\infty} \sup \Pi_x \int_{\tau_{D_k}}^{\tau_D} (H_D\nu_n(\xi_s))^{1+\beta} A(\mathrm{d}s) \to 0 \quad (k \to \infty), \quad x \in D, \tag{1.12}$$

where ν_n, the regularization of ν, is defined by (1.7).

Then we have:

(i) there exists a random measurable function x_D on ∂D such that

$$P_\mu(X_{\tau_D}(\mathrm{d}z) = x_D(z) S(\mathrm{d}z)) = 1, \quad \mu \in M_c(D); \tag{1.13}$$

(ii) for each finite collection $z_1, \cdots, z_k$ of points in $\partial D \setminus N$, the Laplace function of the random vector $[x_D(z_1), \cdots, x_D(z_k)]$ with respect to P_μ is given by

$$P_\mu \exp\left[-\sum_{i=1}^{k} \lambda_i x_D(z_i)\right] = \exp\langle -u, \mu\rangle, \quad \lambda_1, \cdots, \lambda_k \geqslant 0, \tag{1.14}$$

where $\mu \in M_c(D)$ and u is the unique positive solution of eq. (1.6) with $\nu(\mathrm{d}z) = \sum_{i=1}^{k} \lambda_i \delta_{z_i}(\mathrm{d}z)$.

If particularly, A has the formal structure:

$$A_\zeta(\mathrm{d}s) := \mathrm{d}s \int \zeta(\mathrm{d}y) \delta_y(\xi_s), \tag{1.15}$$

where ζ, a measure on $\mathbf{R}^d$, is called the kernel of A. The next theorem gives sufficient conditions on ζ such that the results of Theorem 1.1 hold.

Theorem 1.2. Let A_ζ be the branching rate function given by (1.15). Suppose that there exist a positive constant $C > 0$ such that ζ satisfies:

$$\zeta(\mathrm{d}y) \leqslant C \mathrm{d}y_{d-1} \zeta_1(\mathrm{d}y_1), \quad y = [y_{d-1}, y_1] \in \mathbf{R}^{d-1} \times \mathbf{R}, \tag{1.16}$$

with ζ_1 being a locally finite measure in $\mathbf{R}^d$. If there exists a Borel subset N of zero surface measure such that ψ and ζ satisfy one of the following conditions (C3) and (C4), then the results of Theorem 1.1 hold.

(C3) ψ is given by the general form (0.1) with $a(x) \equiv 0$, and for each collection $z_i \in D \setminus N$, $i = 1, \cdots, m$, there exists an integer n_0 such that

$$\max_i \sup_{z \in B(z_i, 1/n)} \int_D \| x - z \|^{-\alpha d + \alpha + 1} \zeta(\mathrm{d}x) < \infty \quad n \geqslant n_0, \tag{1.17}$$

holds for some $\alpha > 2$.

(C4) ψ is given by the form (1.11), and for each collection $z_i \in D \setminus N$, $i = 1, \cdots, m$, (1.17) holds for some $\alpha > 1 + \beta$.

Corollary 1.1. Let A_ζ be the branching rate function given by (1.15). Under one of the following conditions, the exit measure X_{τ_D} is absolutely continuous about the surface measure $S(\mathrm{d}z)$.

(C5) $\psi(x,z)$ is given by the general form (0.1) with $a(x) \equiv 0$, and $\zeta(\mathrm{d}y) =$

$\zeta_{d-1}(y_{d-1})\mathrm{d}y_{d-1}\zeta_1(\mathrm{d}y_1)$ with $\zeta_{d-1}\in bp\mathscr{B}(\mathbf{R}^{d-1})$ and $\zeta_1(\mathrm{d}y_1)=\delta_c(\mathrm{d}y_1)$, $c\in\mathbf{R}$.

(C6) $d<3$, $\psi(x,z)$ is given by the general form (0.1) with $a(x)\equiv 0$, and $\zeta(\mathrm{d}y)=\zeta(y)\mathrm{d}y$ with $\zeta(y)\in bp\mathscr{B}$.

(C7) $d<1+\frac{2}{\beta}$, $\psi(x,z)$ is given by the form (1.11), and $\zeta(\mathrm{d}y)=\zeta(y)\mathrm{d}y$ with $\zeta(y)\in bp\mathscr{B}$.

(C8) $\psi(x,z)$ is given by the general form (0.1) with $a(x)\equiv 0$, and $\zeta(\mathrm{d}y)=\zeta(y)\mathrm{d}y$ with $\zeta(y)\in bp\mathscr{B}$ satisfying

$$\zeta(y)\leqslant Cd(y,\partial D)^{d-3+\varepsilon}\ \text{for some}\ \varepsilon>0\ \text{and}\ C>0.$$

(C9) $\psi(x,z)$ is given by the form (1.11), and $\zeta(\mathrm{d}y)=\zeta(y)\mathrm{d}y$ with $\zeta(y)\in bp\mathscr{B}$ satisfying

$$\zeta(y)\leqslant Cd(y,\partial D)^{\beta d-2-\beta+\varepsilon}\ \text{for some}\ \varepsilon>0\ \text{and}\ C>0.$$

Ren and Wang[2], Sheu[1] and Zhao[5] proved the result of Corollary 1.1 under conditions (C5), (C7) and (C8), respectively.

Throughout this paper the notation C always denotes a constant which may change values from line to line.

2 Proof of Theorem 1.1

In this section we investigate conditions on A to guarantee the absolute continuity of the exit measure X_{τ_D} in the case that A is a general branching rate function. Let us first state a lemma on the integral equation (1.6). For $c\in p\mathscr{B}$ put

$$H^c(r_1,r_2)=\exp\left(-\int_{r_1}^{r_2}c(\xi_s)A(\mathrm{d}s)\right),\quad 0\leqslant r_1\leqslant r_2.$$

Lemma 2.1. Suppose $A(\mathrm{d}t)$ is a non-negative continuous additive function of the diffusion ξ. Let $\tau\in\mathscr{T}$, and $c,g\in bp\mathscr{B}$. Assume that $\omega\in\mathscr{B}$ and $F\in\mathscr{F}_{\geqslant\tau}$ satisfy

$$\Pi_x\int_0^\tau|\omega(\xi_s)|A(\mathrm{d}s)<\infty,\quad \Pi_x|F|<\infty,\quad x\in\mathbf{R}^d.$$

Then

$$g(x)=\Pi_x\left[H^c(0,\tau)F+\int_0^\tau H^c(0,s)\omega(\xi_s)A(\mathrm{d}s)\right]\tag{2.1}$$

if and only if

$$g(x)+\Pi_x\int_0^\tau(cg)(\xi_s)A(\mathrm{d}s)=\Pi_x\left[F+\int_0^\tau\omega(\xi_s)A(\mathrm{d}s)\right].\tag{2.2}$$

Proof. The proof is similar to that of Lemma 2.1 in Ren and Wang[2]. We omit the details.

Theorem 2.1 (Fundamental solutions). Let ψ be given by (0.1), A be a branching rate function of X and ν belong to $M(\partial D)$. Assume that there exist a sequence of bounded smooth domains $\{D_n\}$ and a sequence of functions $f_n\in bp\mathscr{B}(\partial D)$ such that

$$D_n\uparrow D;\quad \nu_n=:f_n(z)S(\mathrm{d}z)\xrightarrow{w}\nu\quad\text{weakly as } n\to\infty,\tag{2.3}$$

and A satisfies (1.9) and (1.10).

(i) (Existence and uniqueness). There is exactly one measurable non-negative function $U[A,\nu]$ defined on D which solves (1.6).

(ii) (Continuity of regularization). The solution $U[A, \nu]$ is continuous with respect to the operation of regulation of ν in the following sense:

$$U[A,\nu_n](\cdot)\xrightarrow{bp} U[A,\nu](\cdot) \quad (n\to\infty) \text{ in each compact subset } K \text{ of } D. \tag{2.4}$$

(iii) (First derivative with respect to small parameter). If $a(x)\equiv 0$ in (0.1), then

$$\lambda^{-1}U[A,\lambda\nu](\cdot)\xrightarrow{bp} H_D\nu(\cdot)\ (\lambda\to\infty) \text{ in each compact subset } K \text{ of } D. \tag{2.5}$$

Proof. For simplicity, we write τ and τ_n parallel with τ_D and τ_{D_n}, respectively.

Assume that, for each n, u_n is a non-negative solution of the cumulate equation (1.6) with ν replaced by ν_n. We want to show that, for any compact subset K of D, $\{u_n(\cdot);\ n\geqslant 1\}$ is a Cauchy sequence in the Banach space $b\mathscr{B}(K)$ endowed with the topology of bounded pointwise convergence. Note that there is a constant C depending only on D such that

$$k(x,z)\leqslant C\rho(x)\|x-z\|^{-d},\ x\in D,\quad z\in\partial D, \tag{2.6}$$

where $\rho(x)=d(x,\partial D)$. First of all, for a fixed compact subset K of D, we have the following domination:

$$0\leqslant u_n(x)\leqslant\int_{\partial D}k(x,z)f_n(z)S(\mathrm{d}z)\leqslant C\nu_n(\partial D)\leqslant C,\quad \text{for } x\in K,\quad n\geqslant 1. \tag{2.7}$$

Let

$$M_k=(\sup_{x\in D_k,\ n\geqslant 1}H_Df_n(x))\vee 1;\quad \lambda_k(x)=\Big[2b(x)+\int_0^\infty u\wedge u^2n(x,\mathrm{d}u)\Big]M_k;$$

$$c_k(x)=(a+\lambda_k)(x);\quad R_k(x,z)=c_k(x)z-\psi(x,z).$$

Then

$$[R_k(x,z)]'_z=$$

$$2b(x)(M_k-z)+\int_0^1 u(M_ku-1+\mathrm{e}^{-uz})n(x,\mathrm{d}u)+\int_1^\infty u(M_k-1+\mathrm{e}^{-uz})n(x,\mathrm{d}u),$$

which means for all $x\in\mathbf{R}^d$, $0\leqslant z\leqslant M_k$,

$$0\leqslant[R_k(x,z)]'_z\leqslant\lambda_k(x).$$

Consequently

$$|R_k(x,z_1)-R_k(x,z_2)|\leqslant\lambda_k(x)|z_1-z_2|,\quad x\in\mathbf{R}^d,\quad 0\leqslant z_2,\ z_1\leqslant M_k. \tag{2.8}$$

Using Lemma 2.1 with $c=c_k$, $F=f_n(\xi_\tau)-\int_{\tau_k}^\tau\psi(\xi_s,\ u_n(\xi_s))A(\mathrm{d}s)$, $g(x)=u_n(x)$ and $\omega(x)=c_k(x)u_n(x)-\psi(x,u_n(x))=R_k(x,u_n(x))$ we get

$$(x)=\Pi_xH^{c_k}(0,\tau_k)\Big[f_n(\xi_\tau)-\int_{\tau_k}^\tau\psi(\xi_s,\ u_n(\xi_s))A(\mathrm{d}s)\Big]+\Pi_x\int_0^{\tau_k}A(\mathrm{d}s)H^{c_k}(0,s)R_k(\xi_s,\ u_n(\xi_s)$$

For $x\in\mathbf{R}^d$, put

$$g_{m,n}(x)=\psi(x,H_Df_m(x))+\psi(x,H_Df_n(x)). \tag{2.9}$$

Note that $u_n(x)\leqslant M_k$ for $x\in D_k$. By (2.8) and the strong Markov property of ξ we have, for $m,n\geqslant 1,x\in D$ and sufficiently large k (satisfying $x\in D_k$),

$$|u_m-u_n|(x)\leqslant\Pi_xH^{c_k}(0,\tau_k)\Big|\Pi_{\xi_{\tau_k}}(f_n-f_m)(\xi_\tau)\Big|+\Pi_xH^{c_k}(0,\tau_k)\Pi_{\xi_{\tau_k}}\int_0^\tau g_{m,n}(\xi_s)A(\mathrm{d}s)$$

$$+\Pi_x\int_0^{\tau_k}H^{c_k}(0,s)(\lambda_k|u_m-u_n|)(\xi_s)A(\mathrm{d}s).$$

Iterating the above inequality $l \geqslant 1$ times yields

$$|u_m - u_n|(x) \leqslant \Pi_x \left| \Pi_{\xi_{\tau_k}} (f_n - f_m)(\xi_\tau) \right| + \Pi_x \int_{\tau_k}^{\tau} g_{m,n}(\xi_s) A(\mathrm{d}s)$$

$$+ CM_k \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) \frac{\left(\int_0^s \lambda_k(\xi_r) A(\mathrm{d}r) \right)^l}{l!} A(\mathrm{d}s). \tag{2.10}$$

Noticing that for fixed k,

$$\Pi_x f_n(\xi_\tau) \xrightarrow{bp} \int_{\partial D} k(x,z) \nu(\mathrm{d}z) \quad (n \to \infty), \quad \text{for } x \in D_k.$$

By dominated convergence theorem and condition (1.9), we obtain

$$\lim_{m,n \to \infty} \Pi_x \left| \Pi_{\xi_{\tau_k}} (f_n - f_m)(\xi_\tau) \right| = 0, \tag{2.11}$$

and

$$\lim_{l \to \infty} \Pi_x \int_0^{\tau_k} H^{c_k}(0,s) \frac{\left(\int_0^s \lambda_k(\xi_r) A(\mathrm{d}r) \right)^l}{l!} A(\mathrm{d}s) = 0. \tag{2.12}$$

Note that (see Ren and Wang[2])

$$\psi(x,z) \leqslant C(z + z^2), \quad z \geqslant 0, \quad x \in \mathbf{R}^d. \tag{2.13}$$

Thus by (1.9), (1.10), (2.13) and Hölder inequality, for every $x \in D$,

$$\limsup_{m,n \to \infty} \Pi_x \int_{\tau_k}^{\tau} g_{m,n}(\xi_s) A(\mathrm{d}s) \to 0 \quad (k \uparrow \infty). \tag{2.14}$$

Combining (2.10), (2.11), (2.12) and (2.14), we have

$$\limsup_{m,n \to \infty} |u_n(x) - u_m(x)| = 0, \quad x \in D.$$

Therefore there exists a non-negative measurable function u in D such that, for each compact subset $K \subset D$,

$$u_n(x) \xrightarrow{bp} u(x) \quad (n \to \infty), \quad x \in K.$$

By Fatou's lemma, (1.10) also holds for $\varepsilon_n \equiv 0$. Repeating the procedure from the beginning with this u instead of u_m we conclude that u solves eq. (1.6). By similar arguments we conclude that u is uniquely determined by the equation. Summarizing the above, we now have proved the statements (i) and (ii) of Theorem 2.1.

It remains to verify the asymptotic property (2.5). By (1.6) (with ν replaced by $\lambda\nu$)

$$|\lambda^{-1} U[A, \lambda\nu] - H_D\nu|(x) \leqslant \Pi_x \int_0^{\tau} \lambda^{-1} \psi(\xi_s, \lambda H_D\nu(\xi_s)) A(\mathrm{d}s). \tag{2.15}$$

By (2.13),

$$\lambda^{-1} \psi(x, \lambda H_D\nu(x)) \leqslant C[(H_D\nu)^2 + H_D\nu](x), \quad \text{for } x \in \mathbf{R}^d, \quad 0 < \lambda \leqslant 1.$$

Domination (2.7), condition (1.10), and Fatou's lemma imply that $\Pi_x \int_0^{\tau} (H_D\nu)^2(\xi_s) A(\mathrm{d}s) < \infty$, and therefore by (1.9) and Hölder inequality, $\Pi_x \int_0^{\tau} [(H_D\nu)^2 + H_D\nu](\xi_s) A(\mathrm{d}s) < \infty$.

Letting $\lambda \downarrow 0$ in (2.15), by Fatou's lemma,

$$\limsup_{\lambda \downarrow 0} |\lambda^{-1} U[A, \lambda\nu] - H_D\nu|(x) \leqslant \Pi_x \int_0^{\tau} A(\mathrm{d}s) \limsup_{\lambda \downarrow 0} \lambda^{-1} \psi(\xi_s, \lambda H_D\nu(\xi_s)).$$

It is easy to prove that for fixed $z \geqslant 0$, $x \in \mathbf{R}^d$, $\lim_{\lambda \downarrow 0} \lambda^{-1} \psi(x, \lambda z) = 0$. So we obtain

$$\lambda^{-1} U[A, \lambda\nu](r, \cdot) \to H_D\nu(\cdot) \text{ pointwisely as } \lambda \downarrow 0.$$

But $\lambda^{-1} U[A, \lambda\nu](\cdot)$ are all dominated by the same function $H_D\nu(\cdot)$ and $H_D\nu(\cdot)$ is bounded in any compact subset K of D. The statement (iii) follows. We finish the proof of Theorem 2.1.

Checking the above proof we find the following remark.

Remark 2.1. If $\psi(x, z)$ is given by the particular form (1.11) and if condition (1.10) is replaced by condition (1.12), the results of Theorem 2.1 also hold.

Proof of Theorem 1.1. Using Theorem 2.1, Remark 2.1, and the same argument of Theorem 2.1 in Ren and Wang[2], we can prove the results of Theorem 1.1 hold. We omit the details.

3 Proof of Theorem 1.2 and Corollary 1.1

In this section we discuss that if particularly A has a branching rate kernel, ζ, say, under what conditions the related superdiffusion has absolutely continuous exit measures. We first study requirement (1.9).

Lemma 3.1. Let A_ζ be the branching rate function given by (1.15) with kernel ζ satisfying (1.16). If ζ_1 is a locally finite measure on $\mathbf{R}^d$, then $\Pi_x A_\zeta(0, \tau_D)$ is bounded in D.

Proof.

$$\Pi_x A_\zeta(0, \tau_D) = \Pi_x \int_0^{\tau_D} \mathrm{d}s \int \zeta(\mathrm{d}y) \delta_y(\xi_s) = \int_D G_D(x, y) \zeta(\mathrm{d}y) \leqslant C \int_D g(x, y) \zeta(\mathrm{d}y),$$

where $G_D(x, y)$ denotes the Green function of ξ in D and

$$g(x, y) = \begin{cases} \| x - y \|^{2-d}, & d \geqslant 3, \\ \log^+ \| x - y \|^{-1} + 1, & d = 2. \end{cases}$$

Since D is bounded, there exists a constant $M > 1$ such that $D \subset B(0, M)$, Thus $D \subset B(x, 2M)$ for all $x \in D$.

In the case $d \geqslant 3$,

$$\begin{aligned} \Pi_x A_\zeta(0, \tau_D) &\leqslant \int_{B(x,2M) \cap D} g(x, y) \zeta_{d-1}(y_{d-1}) \mathrm{d}y_{d-1} \zeta_1(\mathrm{d}y_1) \\ &\leqslant C \int_{(-M,M)} \zeta_1(\mathrm{d}y_1) \int_{\| y_{d-1} - x_{d-1} \| < 2M} (\| y_{d-1} - x_{d-1} \|)^{2-d} \mathrm{d}y_{d-1} \\ &\leqslant C \int_{(-M,M)} \zeta_1(\mathrm{d}y_1) \int_0^{2M} r^{d-2} r^{2-d} \mathrm{d}r \leqslant C. \end{aligned}$$

In the case $d = 2$,

$$\begin{aligned} \Pi_x A_\zeta(0, \tau_D) &\leqslant C \int_{B(x,2M) \cap D} (\log^+ \| y - x \|^{-1} + 1) \mathrm{d}y_{d-1} \zeta_1(\mathrm{d}y_1) \\ &= C \int_{B(x,1) \cap D} -\log \| y - x \| \mathrm{d}y_{d-1} \zeta_1(\mathrm{d}y_1) + C \int_D \mathrm{d}y_{d-1} \zeta_1(\mathrm{d}y_1) \\ &\leqslant C \int_{(-1,1)} \zeta_1(\mathrm{d}y_1) \int_{B(x_{d-1},1)} -\log \| y_{d-1} - x_{d-1} \| \mathrm{d}y_{d-1} + C\zeta_1(-M, M) \\ &\leqslant C \int_{(-1,1)} \zeta_1(\mathrm{d}y_1) \int_0^1 -r \log r \mathrm{d}r + C\zeta_1(-M, M) \leqslant C. \end{aligned}$$

We conclude that $\Pi_x A_\zeta(0, \tau_D)$ is bounded in D.

Now let us discuss the requirements (1.10) and (1.12) in Theorem 1.1 in the case $A =$

A_ζ as indicated in (1.15).

Lemma 3.2. Suppose ζ satisfies (1.16) with ζ_1 being locally finite. Let $\nu = \sum_{i=1}^{m} \lambda_i \delta_{z_i}$ for some fixed $z_1, \cdots, z_m \in \partial D, \lambda_1, \cdots, \lambda_m \geqslant 0$ and let ν_n be defined by (1.7).

(i) If (1.17) holds for some $\alpha > 2$, then the corresponding branching rate function A_ζ satisfies condition (1.10).

(ii) If (1.17) holds for some $\alpha > 1 + \beta$, then the corresponding branching rate function A_ζ satisfies condition (1.12).

Proof. We only prove (i). Result (ii) can be proved similarly. Suppose $\{D_n\}$ is a sequence of bounded smooth domains in $\mathbf{R}^d$ satisfying $D_n \uparrow D$ as $n \uparrow \infty$. Recall that we use τ_n to denote the first exit time from D_n and τ the first exit time from D. Fix $x \in D$. For any integer k, the bounded convergence theorem implies that

$$\lim_{n\to\infty} \Pi_x \int_0^{\tau_k} (H_D \nu_n)^2(\xi_s) A(\mathrm{d}s) = \Pi_x \int_0^{\tau_k} (H_D \nu)^2(\xi_s) A(\mathrm{d}s) < \infty .$$

Consequently condition (1.10) is satisfied if

$$\lim_{n\to\infty} \Pi_x \int_0^{\tau_D} (H_D \nu_n)^2(\xi_s) A(\mathrm{d}s) = \Pi_x \int_0^{\tau_D} (H_D \nu)^2(\xi_s) A(\mathrm{d}s) < \infty ,$$

which is equivalent to

$$\lim_{n\to\infty} \int_D G_D(x,y)(H_D \nu_n)^2(y) \zeta(\mathrm{d}y) = \int_D G_D(x,y)(H_D \nu)^2(y) \zeta(\mathrm{d}y) < \infty . \quad (3.1)$$

Let K be an arbitrary compact set of D. Note that $\int_K G_D(x,y)\zeta(\mathrm{d}y) \leqslant \int_D G_D(x,y)\zeta(\mathrm{d}y) = \Pi_x A_\zeta(0, \tau_D) < \infty$. The dominated convergence theorem implies that

$$\lim_{n\to\infty} \int_K G_D(x,y)(H_D \nu_n)^2(y) \zeta(\mathrm{d}y) = \int_K G_D(x,y)(H_D \nu)^2(y) \zeta(\mathrm{d}y) < \infty .$$

So, to prove (3.1) it suffices to show that there exists an integer n_0 such that

$$\sup_{n \geqslant n_0} \int_{D \setminus D_k} G_D(x,y)(H_D \nu_n)^2(y) \zeta(\mathrm{d}y) \to 0 \quad (k \uparrow \infty). \quad (3.2)$$

Assume that x belongs to D_k, $k \geqslant 1$. Since $G_D(x,y)/\rho(y)$ is bounded in $D \setminus D_k$, where $\rho(y) = d(y, \partial D)$ is the distance from y to ∂D, to prove (3.2) it is enough to show that there exists an integer n_0 such that

$$\sup_{n \geqslant n_0} \int_{D \setminus D_k} \rho(y)(H_D \nu_n)^2(y) \zeta(\mathrm{d}y) \to 0 \quad (k \uparrow \infty). \quad (3.3)$$

For $M > 0$, we have

$$\int_{D \setminus D_k} \rho(y)(H_D \nu_n)^2(y) \zeta(\mathrm{d}y) \leqslant M^2 \int_{D \setminus D_k} \rho(y) \zeta(\mathrm{d}y) + \int_{D \cap (H_D \nu_n > M)} (H_D \nu_n)^2 \rho(y) \zeta(\mathrm{d}y) \quad (3.4)$$

and

$$\int_{D \cap (H_D \nu_n > M)} (H_D \nu_n)^2 \rho(y) \zeta(\mathrm{d}y) = - \int_M^\infty \lambda^2 \mathrm{d}\beta_n(\lambda), \quad (3.5)$$

where $\beta_n(\lambda) = \int_{D \cap (H_D \nu_n > \lambda)} \rho(y) \zeta(\mathrm{d}y)$, $\lambda > 0$. For simplicity, we set $h_n(x) = H_D \nu_n(x)$. Then estimate

$$\lambda\beta_n(\lambda) \leqslant \int_{D\cap(h_n>\lambda)} h_n(y)\rho(y)\zeta(\mathrm{d}y) = \int_{\partial D}\nu_n(\mathrm{d}z)\int_{D\cap(h_n>\lambda)} k(y,z)\rho(y)\zeta(\mathrm{d}y)$$
$$\leqslant C \max_i \sup_{z\in B(z_i,1/n)} \int_{D\cap(h_n>\lambda)} k(y,z)\rho(y)\zeta(\mathrm{d}y).$$

Choose $\alpha>2$, by Hölder's inequality, we have

$$\lambda\beta_n(\lambda) \leqslant C \max_i \sup_{z\in B(z_i,1/n)} (A(z))^{\frac{1}{\alpha}}(\beta_n(\lambda))^{\frac{1}{\alpha'}},$$

where $A(z) = \int_D k(y,z)^\alpha \rho(y)\zeta(\mathrm{d}y), \frac{1}{\alpha}+\frac{1}{\alpha'} = 1$. By (2.6) we have

$$A(z) \leqslant C\int_D \rho(y)^{\alpha+1}\| y-z\|^{-\alpha d}\zeta(\mathrm{d}y) \leqslant C\int_D \| y-z\|^{-\alpha d+\alpha+1}\zeta(\mathrm{d}y).$$

The assumption (1.17) implies that there exists an integer n_0 such that

$$\lambda\beta_n(\lambda) \leqslant C\beta_n(\lambda)^{\frac{1}{\alpha'}}, \quad n \geqslant n_0,$$

and so

$$\beta_n(\lambda) \leqslant C\lambda^{-\alpha}, \quad \text{for all } \lambda > 0,\ n \geqslant n_0. \tag{3.6}$$

Since $\alpha>2$, we have, by integration by parts and (3.6),

$$-\int_M^\infty \lambda^2 \mathrm{d}\beta_n(\lambda) = M^2\beta_n(M) + 2\int_M^\infty \beta_n(\lambda)\lambda\,\mathrm{d}\lambda \leqslant CM^{2-\alpha}. \tag{3.7}$$

Condition (3.3) follows easily from (3.4), (3.5) and (3.7). Thus we finish the proof of Lemma 3.2.

Proof of Theorem 1.2. Summarizing Lemmas 3.1 and 3.2, the conditions in Theorem 1.1 hold, and therefore the results of Theorem 1.1 hold.

Proof of Corollary 1.1. We only prove the result under condition (C8). The others can be proved similarly. It is clear that under condition (C8), ζ satisfies (1.16).

$$\sup_{z\in\partial D}\int_D \| x-z\|^{-\alpha d+\alpha+1}\zeta(x)\mathrm{d}x \leqslant C\sup_{z\in\partial D}\int_D \| x-z\|^{-\alpha d+\alpha+1} d(x,\partial D)^{d-3+\varepsilon}\mathrm{d}x$$
$$\leqslant C\sup_{z\in\partial D}\int_D \| x-z\|^{-\alpha d+\alpha+1+d-3+\varepsilon}\mathrm{d}x \leqslant C\int_0^{\mathrm{diam}(D)} r^{(2-\alpha)(d-1)+\varepsilon-1}\mathrm{d}r.$$

If $(2-\alpha)(d-1)+\varepsilon>0$, then $\sup_{z\in\partial D}\int_D \| x-z\|^{-\alpha d+\alpha+1}\zeta(x)\mathrm{d}x < \infty$. Since when $\alpha=2$, $(2-\alpha)(d-1)+\varepsilon=\varepsilon>0$, there exists $\alpha>2$, such that

$$\sup_{z\in\partial D}\int_D \| x-z\|^{-\alpha d+\alpha+1}\zeta(x)\mathrm{d}x < \infty.$$

Thus, by Theorem 1.2, under condition (C8), X_{τ_D} is absolutely continuous.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 19801019) and the Postdoctoral Foundation of China.

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