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Limiting Distributions for a Class of Super-Brownian Motions with Spatially Dependent Branching Mechanisms

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Abstract
In this paper, we consider a large class of super-Brownian motions in $\mathbb{R}$ with spatially dependent branching mechanisms. We establish the almost sure growth rate of the mass located outside a time-dependent interval $(-\delta t, \delta t)$ for $\delta > 0$. The growth rate is given in terms of the principal eigenvalue λ_1 of the Schrödinger-type operator associated with the branching mechanism. From this result, we see the existence of phase transition for the growth order at $\delta = \sqrt{\lambda_1/2}$. We further show that the super-Brownian motion shifted by $\sqrt{\lambda_1/2}\, t$ converges in distribution to a random measure with random density mixed by a martingale limit.

Keywords Super-Brownian motion · Spatially dependent branching mechanism · Growth rate · Convergence in distribution

Mathematics Subject Classification (2020) Primary 60J68 · 60G57 · Secondary 60F15 · 60F05

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1 Introduction and Main Results

1.1 Super-Brownian Motions

Let $\mathcal{M}(\mathbb{R})$ (resp. $\mathcal{M}_c(\mathbb{R})$) denote the set of finite (resp. finite and compactly supported) measures on $\mathbb{R}$. When μ is a measure on $\mathbb{R}$ and f is a measurable function, define $\langle f, \mu\rangle = \int_{\mathbb{R}} f(x)\mu(\mathrm{d}x)$ whenever the right-hand side makes sense. Sometimes we also write $\mu(f)$ for $\langle f, \mu\rangle$. Let $\left((B_t)_{t\geq 0}, \Pi_x, x \in \mathbb{R}\right)$ be a standard Brownian motion on $\mathbb{R}$ with $\Pi_x(B_0 = x) = 1$. The main process of interest in this paper is an $\mathcal{M}(\mathbb{R})$-valued Markov process $X = \{X_t : t \geq 0\}$ with evolution depending on two quantities P_t and ψ. Here, P_t is the semigroup of $\left((B_t)_{t\geq 0}, \Pi_x, x \in \mathbb{R}\right)$ and ψ is the so-called branching mechanism, which takes the form

$$\psi(x, \lambda) = -\beta(x)\lambda + \alpha(x)\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda u} - 1 + \lambda u\right) \pi(x, \mathrm{d}u) \quad x \in \mathbb{R}, \lambda \geq 0,$$

where $\beta \in C_c(\mathbb{R})$, $0 \not\equiv \alpha \in C_c^+(\mathbb{R})$, and π is a kernel from $\mathbb{R}$ to $(0, +\infty)$ such that

$$\int_{(0,+\infty)} u^2 \pi(x, \mathrm{d}u) \in C_c^+(\mathbb{R}).$$

The distribution of X is denoted by $\mathbb{P}_\mu$ if it is started at $\mu \in \mathcal{M}(\mathbb{R})$ at $t = 0$. X is called a (B_t, ψ)-superprocess or super-Brownian motion with branching mechanism ψ if for all $\mu \in \mathcal{M}(\mathbb{R})$, nonnegative bounded measurable function f and $t \geq 0$,

$$\mathbb{P}_\mu\left[e^{-\langle f, X_t\rangle}\right] = e^{-\langle u_f(t,\cdot),\mu\rangle}, \tag{1.1}$$

where $u_f(t, x) = -\log \mathbb{P}_{\delta_x}\left(e^{-\langle f, X_t\rangle}\right)$ is the unique nonnegative locally bounded solution to the following integral equation:

$$u_f(t, x) = P_t f(x) - \int_0^t P_s\left(\psi(\cdot, u_f(t-s, \cdot))\right)(x)\mathrm{d}s \quad \forall x \in \mathbb{R},\ t \geq 0.$$

The existence of such a process X is established in [12]. A closely related $\mathcal{M}(\mathbb{R})$-valued process is branching Brownian motion with the branching rate given by either a compactly supported measure or a function decaying sufficiently fast at infinity (see, e.g., [6, 7, 26, 27, 33, 34] and references therein).

The process X may be loosely described as a scaling limit of branching particle systems as follows. Let $\mu \in \mathcal{M}(\mathbb{R})$. Suppose that, at time zero, a random number of particles are set in $\mathbb{R}$, according to a Poisson random measure with intensity $N\mu$. The particles move independently according to the law of a standard Brownian motion in $\mathbb{R}$ from its starting point. A given particle lives an exponential amount of time with mean lifetime b_N and upon its death gives birth to a random number of offspring. The offspring wander and propagate in the same fashion. Offspring are born at the death site of their parents, and the distribution $(p_k^N(x); k \geq 0)$ of the number of offspring is allowed to depend on the death site x, and on the parameter N. The mass distribution of particles alive at time t may be viewed as a random measure $X_t^{(N)}$ (each particle being given weight $1/N$). Under suitable hypotheses, this sequence of measure-valued process converges in distribution, as $N \to \infty$, to a limit measure-valued Markov process X with $X_0 = \mu$. The typical conditions are $b_N \to 0$ and

$$\lim_{N\to\infty}\left[\sum_{k=0}^{\infty} p_k^{(N)}(x)(1-\lambda/N)^k - (1-\lambda/N)\right](N/b_N) = \psi(x,\lambda), \quad \lambda \geq 0.$$

Super-Brownian motion is a special type of superprocesses, which arise as stochastic models describing the evolution of a random mass distributed in space. For details on superprocesses as scaling limits of branching particle systems, we refer to Dynkin [12] and [24, Chapter 4].

Another link between superprocesses and branching Markov processes is provided by the so-called skeleton decomposition, which is developed by [10, 15, 22]. The skeleton decomposition provides a pathwise description of a superprocesses in terms of immigrations along a branching Markov process called the skeleton. We shall work with this skeleton construction in this paper, see Proposition 4.1.

1.2 Notation and Some Facts

Throughout this paper, we use ":=" to denote a definition. For functions f and g on $\mathbb{R}$, $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$ and $(f,g) := \int_{-\infty}^{+\infty} f(x)g(x)\mathrm{d}x$. For positive functions $f(x)$ and $g(x)$ on $(0,+\infty)$, we write $f(x) \sim g(x)\,(x \to +\infty)$ if $\lim_{x\to+\infty} f(x)/g(x) = 1$. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$, $a \vee b := \max\{a,b\}$. The letters c and C (with subscript) denote finite positive constants which may vary from place to place.

Let $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ denote the space of locally finite Borel measures on $\mathbb{R}$ with vague topology, which is generated by the integration maps $\pi_f : \mu \mapsto \mu(f)$ for all compactly supported bounded continuous functions f on $\mathbb{R}$. A random variable taking values in $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ is called a random measure on $\mathbb{R}$. We say random measures ξ_n converges in distribution to ξ if $\mathrm{E}[F(\xi_n)] \to \mathrm{E}[F(\xi)]$ for every bounded continuous function F on $\mathcal{M}_{\mathrm{loc}}(\mathbb{R})$. [21, Theorem 4.11] proves that ξ_n converges in distribution to ξ if and only if the random variables $\langle f, \xi_n\rangle$ converge in distribution to $\langle f, \xi\rangle$ for every $f \in C_c^+(\mathbb{R})$.

For a measurable function f, we set

$$e_f(t) := \exp\left\{\int_0^t f(\xi_s)\mathrm{d}s\right\}, \quad t \geq 0,$$

whenever it is well defined. We define the Feynman–Kac semigroup P_t^β by

$$P_t^\beta f(x) := \Pi_x\left[e_\beta(t) f(\xi_t)\right] \quad \text{for } f \in \mathcal{B}_b^+(\mathbb{R}).$$

Define

$$\gamma(x) := \alpha(x) + \frac{1}{2}\int_{(0,+\infty)} u^2\pi(x, \mathrm{d}u), \quad x \in \mathbb{R}. \tag{1.2}$$

It is known (cf. [12]) that for every $\mu \in \mathcal{M}(\mathbb{R})$ and $f \in \mathcal{B}_b^+(\mathbb{R})$, the first two moments of $\langle f, X_t\rangle$ exist and can be expressed as

$$\mathbb{P}_\mu\left[\langle f, X_t\rangle\right] = \langle P_t^\beta f, \mu\rangle, \tag{1.3}$$

and

$$\mathrm{Var}_\mu\left(\langle f, X_t\rangle\right) = \int_0^t \langle P_s^\beta\left(2\gamma\left(P_{t-s}^\beta f\right)^2\right), \mu\rangle \mathrm{d}s.$$

The spectrum of the operator $\mathcal{L} = \frac{1}{2}\Delta + \beta$, denoted by $\sigma(\mathcal{L})$, consists of $(-\infty, 0]$ and at most a finite number of nonnegative eigenvalues. Throughout this paper, we make the following assumption:

$$\lambda_1 := \sup(\sigma(\mathcal{L})) > 0. \tag{A1}$$

Then, λ_1 is simple and the corresponding eigenfunction (ground state) h can be taken to be strictly positive, bounded and continuous. We choose h that is normalized with $\int_{-\infty}^{+\infty} h^2(x)\mathrm{d}x = 1$. We remark here that (A1) is automatically satisfied when $\beta \geq 0$ is a nontrivial function. One has (see, for example, [27, Lemma 3.1])

$$h(x) = \int_{-\infty}^{+\infty} G_{\lambda_1}(x, y)\beta(y)h(y)\mathrm{d}y. \tag{1.4}$$

where $G_{\lambda_1}(x, y)$ denotes the λ_1-potential density of Brownian motion. Using the fact that

$$G_{\lambda_1}(x, y) \sim \frac{1}{\sqrt{2\lambda_1}}\mathrm{e}^{-\sqrt{2\lambda_1}|x-y|} \text{ as } |x - y| \to +\infty, \tag{1.5}$$

one can easily show that

$$h(x) \sim C_\mp \mathrm{e}^{-\sqrt{2\lambda_1}|x|} \text{ as } x \to \pm\infty, \tag{1.6}$$

where

$$C_\mp := \frac{1}{\sqrt{2\lambda_1}}\int_{-\infty}^{+\infty} \beta(y)h(y)\mathrm{e}^{\pm\sqrt{2\lambda_1}y}\mathrm{d}y. \tag{1.7}$$

Since $\mathrm{e}^{-\lambda_1 t}P_t^\beta h = h$ for all $t \geq 0$, one can show by the Markov property that

$$W_t^h(X) := \mathrm{e}^{-\lambda_1 t}\langle h, X_t\rangle, \quad \forall t \geq 0$$

is a nonnegative $\mathbb{P}_\mu$-martingale for every $\mu \in \mathcal{M}(\mathbb{R})$. Let $W^h_\infty(X)$ be the martingale limit. It then follows by [28, Theorem 3.2] that for every nontrivial $\mu \in \mathcal{M}_c(\mathbb{R})$,

$$\lim_{t\to+\infty} W^h_t(X) = W^h_\infty(X) \quad \mathbb{P}_\mu\text{-a.s. and in } L^2(\mathbb{P}_\mu).$$

Hence, $W^h_\infty(X)$ is nondegenerate in the sense that $\mathbb{P}_\mu\left(W^h_\infty(X) > 0\right) > 0$.

1.3 Main Results

For any $R \geq 0$, define

$$\mathcal{X}^R_t := \langle \mathbf{1}_{(-R,R)^c}, X_t \rangle.$$

Theorem 1.1 *For any $\delta > \sqrt{\lambda_1/2}$ and $\mu \in \mathcal{M}_c(\mathbb{R})$,*

$$\lim_{t\to+\infty} \mathcal{X}^{\delta t}_t = 0 \quad \mathbb{P}_\mu\text{-}a.s.$$

For any $0 \leq \delta < \sqrt{\lambda_1/2}$ and $\mu \in \mathcal{M}_c(\mathbb{R})$,

$$\lim_{t\to+\infty} \frac{\log \mathcal{X}^{\delta t}_t}{t} = \lambda_1 - \sqrt{2\lambda_1}\delta \quad \mathbb{P}_\mu\text{-}a.s.\ on\ \{W^h_\infty(X) > 0\}.$$

According to Theorem 1.1, for $\delta < \sqrt{\lambda_1/2}$, the mass outside $(-\delta t, \delta t)$ at time t grows exponentially with a positive rate $\lambda_1 - \sqrt{2\lambda_1}\delta$, while for $\delta > \sqrt{\lambda_1/2}$, it converges to 0. In the latter case, Proposition 3.8 shows that the upper bound of the mass outside $(-\delta t, \delta t)$ decreases exponentially with a negative rate. The version of Theorem 1.1 has been proved recently for branching Brownian motions with branching rate given by a compactly supported measure in [7, 33, 34]. The idea of our proof is similar to that of [34]: The upper bound and the lower bound of $\mathcal{X}^{\delta t}_t$ are considered separately, and the proofs for convergence follow two main steps. The first step is to obtain the limit along lattice times. This is done via a Borel–Cantelli argument and thus requires the asymptotics of the expectation of $\mathcal{X}^{\delta t}_t$ (Lemma 3.1). The second step is to extend the limits to all times. For the aforementioned class of branching Brownian motions, a key fact used in the proofs is that the particles alive at time t located in $(-\delta t, \delta t)^c$ are the children of the particles alive at time $\lfloor t \rfloor$. However, this kind of property fails for the super-Brownian motions. We overcome this difficulty by appealing to a stochastic integral representation of super-Brownian motions (eq. (3.6)). This representation enables us to decompose the super-Brownian motion in terms of martingale measures and hence providing useful structural properties of super-Brownian motions. Let us mention that the result, which corresponds to Theorem 1.1 for the aforementioned class of branching Brownian motions, implies that the supremum of the support of the process, denoted by R_t, grows linearly with rate $\sqrt{\lambda_1/2}$ as $t \to +\infty$ a.s. on the survival event. Recently, Nishimori and Shiozawa [27] proved that

$$R_t = \sqrt{\frac{\lambda_1}{2}}t + Y_t,$$

where the conditional distribution of Y_t on the survival event is convergent. [27] is a generalization of [23] and [8]. However, analogous result does not hold for the super-Brownian motions. As we show in Remark 4.10, for the (B_t, ψ)-superprocess, the conditional distributions of $R_t - \sqrt{\lambda_1/2}\,t$ are not even tight.

The growth order of $\mathcal{X}_t^{\delta t}$ undergoes the phase transition at $\delta = \sqrt{\lambda_1/2}$. We further obtain the limiting distributions of the super-Brownian motion at the critical phase in Theorem 1.2. For $\nu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$ and $x \in \mathbb{R}$, we use $\nu + x$ to denote the measure induced by the shift operator $y \mapsto x + y$, that is, $\int_{\mathbb{R}} f(y)(\nu + x)(\mathrm{d}y) = \int_{\mathbb{R}} f(y + x)\nu(\mathrm{d}y)$ for all $f \in \mathcal{B}^+(\mathbb{R})$.

Theorem 1.2 *For every $\mu \in \mathcal{M}(\mathbb{R})$, $((X_t \pm \sqrt{\lambda_1/2}\,t)_{t\geq 0}, \mathbb{P}_\mu)$ converges in distribution to $W_\infty^h(X)\eta_\pm(\mathrm{d}x)$, where $\eta_\pm(\mathrm{d}x)$ are (nonrandom) measures on $\mathbb{R}$ defined by $\eta_\pm(\mathrm{d}x) = C_\pm \mathrm{e}^{\pm\sqrt{2\lambda_1}x}\mathrm{d}x$ with $C_\pm$ being defined by (1.7).*

For branching Markov processes, results of the type of Theorem 1.2 have been established in recent years for various models. See, e.g., [2, 3, 5] for spatially homogeneous branching Brownian motions, [1, 19, 25] for branching random walks, [30] for branching Lévy processes, [4, 18] for multitype branching Brownian motions, and [6, 26] for spatially inhomogeneous branching Brownian motions. In contrast, there is much less work for superprocesses. Very recently, Ren et al. [29] showed that super-Brownian motion with a spatially independent branching mechanism translated by a centered term converges in distribution. Later, Ren et al. [31] represented the limiting process as the limit of a sequence of Poisson random measures in which each atom is decorated by an independent copy of an auxiliary measure. As far as the authors know, there are no references on the vague convergence for superprocesses with spatially dependent branching mechanisms. To prove Theorem 1.2, we appeal to the skeleton techniques for superprocesses. Intuitively, under suitable assumptions, for a given superprocess $(X_t)_{t\geq 0}$ there exists a related branching Markov process $(Z_t)_{t\geq 0}$, called the skeleton, such that at each fixed time $t \geq 0$, the law of Z_t may be coupled to the law of X_t in such a way that given X_t, Z_t has the law of a Poisson point process with random intensity determined by X_t. We exploit this fact and carry the long time behavior from the skeleton to the superprocess. Our idea is partly inspired by [31] where the skeleton techniques have been used successfully to establish the limiting distribution for super-Brownian motions with spatially independent branching mechanisms.

Theorem 1.2 yields the following result on the convergence of the mass at the critical phase.

Theorem 1.3 *For $\delta = \sqrt{\lambda_1/2}$ and $\mu \in \mathcal{M}_c(\mathbb{R})$, $(\mathcal{X}_t^{\delta t}, \mathbb{P}_\mu)$ converges in distribution to $\frac{1}{\sqrt{2\lambda_1}}(C_+ + C_-)W_\infty^h(X)$.*

We remark that results of this paper are restricted to the one-dimensional super-Brownian motion. For higher-dimensional case ($d \geq 2$), one may consider the growth rate of the mass located outside balls with time-dependent radius. For $d \geq 2$, Theorem 1.1 remains true by replacing $\mathcal{X}_t^{\delta t}$ with $X_t(B_{\delta t}^c)$ where $B_{\delta t}$ denotes the ball centered at the origin with radius δt. The argument in this paper can be applied with minor modifications to prove this statement. The growth rate at the critical phase $\delta = \sqrt{\lambda_1/2}$ might depend on the spatial dimension d. For branching Brownian motions with compactly supported branching rates, [34, Theorem 3.9] shows that the growth order of the population around the forefront depends on the spatial dimension. However, for our model the exact growth rate of population at the critical phase in higher dimensions remains open.

The rest of this paper is organized as follows. In Sect. 2, we derive the long time asymptotic properties of Feynman–Kac functionals related to the first and second moments of superprocesses. Section 3 is devoted to the proof of Theorem 1.1. The proofs of Theorems 1.2 and 1.3 are given in Sect. 4.

2 Estimates on the Feynman–Kac Functionals

In this section, we show two lemmas related to the Feynman–Kac functionals of Brownian motions, which will be used in the proofs of the main results.

Let $a(t)$ be a function on $[0, +\infty)$ with $a(t) = o(t)$ as $t \to +\infty$. For $\delta > 0$, define $R(t) := \delta t + a(t)$. Let A be a Borel set of $\mathbb{R}$ with $\inf A > -\infty$. Let $b : [0, +\infty) \to [0, +\infty)$ be a function with $b(t) = o(t)$ as $t \to +\infty$. For $r \in \mathbb{R}$ and $\Theta \subseteq \{\pm 1\}$, define $C_\Theta(r, A) := \{x \in \mathbb{R} : \ \theta x \in r + A \text{ for some } \theta \in \Theta\}$.

Lemma 2.1 *Suppose* $\delta \in (0, \sqrt{2\lambda_1})$.

(i) For any $a \in \left(0, 1 - \frac{\delta}{\sqrt{2\lambda_1}}\right)$, *there exist constants* $C_1, T_1 > 0$ *such that for* $t \geq T_1$, $s \in [0, at]$ *and* $|x| \leq b(t)$,

$$\left|\Pi_x\left[e_\beta(t-s),\, B_{t-s} \in C_\Theta(R(t), A)\right] - \mathrm{e}^{\lambda_1(t-s)} h(x) \int_{C_\Theta(R(t),A)} h(y)dy\right| \leq \mathrm{e}^{-C_1 t}\mathrm{e}^{\lambda_1(t-s) - \sqrt{2\lambda_1}R(t)}. \tag{2.1}$$

(ii) There exist constants $C_2 > 0$ *and* $T_2 > 1$ *such that for* $t \geq T_2$, $s \in [0, t-1]$ *and* $|x| \leq b(t)$,

$$\Pi_x\left[\int_0^{t-s} \gamma(B_r) e_\beta(r) \Pi_{B_r}\left[e_\beta(t-s-r),\, B_{t-s-r} \in C_\Theta(R(t), A)\right]^2 dr\right] \leq C_2 \mathrm{e}^{2\lambda_1(t-s) - 2\sqrt{2\lambda_1}R(t)}, \tag{2.2}$$

where γ *is defined by (1.2).*

We remark here that for the special case where $\Theta = \{\pm 1\}$, $A = (0, +\infty)$ (correspondingly $C_\Theta(R(t), A) = \{y \in \mathbb{R} : \ |y| > R(t)\}$) and $b(t) = b$ for some constant $b > 0$, the above two inequalities follow, respectively, from Lemma 3.8 and Lemma

3.9 of [27]. Here, we show the results for more general case where A can be any left-bounded Borel set and $b(t) = o(t)$. Our proofs are based on [27, Section 3.3].

Proof of Lemma 2.1: (i) Let $p^\beta(t, x, y)$ and $p(t, x, y)$ be the transition densities of P_t^β and P_t, respectively. Let $q_t(x, y) := p^\beta(t, x, y) - p(t, x, y) - \mathrm{e}^{\lambda_1 t} h(x)h(y)$. We have

$$\begin{aligned}
&\Pi_x \left[e_\beta(t-s), B_{t-s} \in C_\Theta(R(t), A)\right] - \mathrm{e}^{\lambda_1(t-s)} h(x) \int_{C_\Theta(R(t),A)} h(y)\mathrm{d}y \\
&\quad = \Pi_x \left(B_{t-s} \in C_\Theta(R(t), A)\right) + \int_{C_\Theta(R(t),A)} q_{t-s}(x, y)\mathrm{d}y.
\end{aligned}$$

We note that for $R > 0$ large enough such that $R + \inf A > 0$, $C_\Theta(R, A) \subseteq \{y \in \mathbb{R} : |y| \geq R + \inf A\}$. Thus for t sufficiently large such that $R(t) - b(t) + \inf A > 0$ and $|x| \leq b(t)$,

$$\begin{aligned}
\Pi_x \left(B_{t-s} \in C_\Theta(R(t), A)\right) &= \Pi_0 \left(B_{t-s} + x \in C_\Theta(R(t), A)\right) \\
&\leq \Pi_0 \left(|B_{t-s}| \geq R(t) - b(t) - \inf A\right). \qquad (2.3)
\end{aligned}$$

On the other hand, it follows similarly as [27, equation (3.19)] that for any $t \geq 1$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&\int_{C_\Theta(R,A)} q_t(x, y)\mathrm{d}y \\
&\quad = \int_0^1 \Big(\int_{\mathbb{R}} p_s^\beta(x, z)\Pi_z\big(B_{t-s} \in C_\Theta(R, A)\big)\beta(z)\mathrm{d}z\Big)\mathrm{d}s \\
&\qquad + \int_1^t \Big[\int_{\mathbb{R}} \big(p_s^\beta(x, z) - \mathrm{e}^{\lambda_1 s} h(x)h(z)\big)\Pi_z\big(B_{t-s} \in C_\Theta(R, A)\big)\beta(z)\mathrm{d}z\Big]\mathrm{d}s \\
&\qquad - \mathrm{e}^{\lambda_1 t} h(x) \int_{t-1}^{+\infty} \mathrm{e}^{-\lambda_1 s}\Big(\int_{\mathbb{R}} h(z)\beta(z)\Pi_z \left(B_s \in C_\Theta(R, A)\right) \mathrm{d}z\Big)\mathrm{d}s.
\end{aligned}$$

Thus, we have

$$\left| \int_{C_\Theta(R,A)} q_t(x, y)\mathrm{d}y \right| \leq (I) + (II) + (III),$$

where

$$\begin{aligned}
(I) &= \int_0^1 \Big(\int_{\mathbb{R}} p_s^\beta(x, z)\Pi_z\big(|B_{t-s}| \geq R + \inf A\big)\beta(z)\mathrm{d}z\Big)\mathrm{d}s, \\
(II) &= \int_1^t \Big[\int_{\mathbb{R}} \big(p_s^\beta(x, z) - \mathrm{e}^{\lambda_1 s} h(x)h(z)\big)\Pi_z\big(|B_{t-s}| \geq R + \inf A\big)\beta(z)\mathrm{d}z\Big]\mathrm{d}s, \\
(III) &= \mathrm{e}^{\lambda_1 t} h(x) \int_{t-1}^{+\infty} \mathrm{e}^{-\lambda_1 s}\Big(\int_{\mathbb{R}} h(z)\beta(z)\Pi_z \left(|B_s| \geq R + \inf A\right) \mathrm{d}z\Big)\mathrm{d}s.
\end{aligned}$$

The upper bounds for (I), (II), (III) are established through Lemmas 3.5−3.7 of [27]. These yield that if $\mathrm{supp}\beta \subset [-k, k]$ for some $k \in (0, +\infty)$, then there exist constants $c, C > 0$ such that for all $x \in \mathbb{R}$, $t \geq 1$ and $R + \inf A > 2k$,

$$\left|\int_{C_\Theta(R,A)} q_t(x, y)\mathrm{d}y\right| \leq C\Big[h(x)\Pi_0\left(|B_t| > R + \inf A - k\right) \\ + I_c(t, R + \inf A) + h(x)J(t, R + \inf A)\Big]. \tag{2.4}$$

Here, I_c and J are defined by (3.15) and (3.16) of [27], respectively. Using (2.3) and (2.4), one can apply similar argument of [27, Lemma 3.8] to prove (2.1). We omit the details here.

(ii) Noting that for t large enough such that $R(t)+\inf A \geq 0$, $C_\Theta(R(t), A) \subseteq \{y \in \mathbb{R} : |y| \geq R(t) + \inf A\}$, we have

$$\Pi_x\left[\int_0^{t-s} \gamma(B_r)e_\beta(r)\Pi_{B_r}\big[e_\beta(t-s-r), B_{t-s-r} \in C_\Theta(R(t), A)\big]^2\mathrm{d}r\right] \\ \leq \Pi_x\left[\int_0^{t-s} \gamma(B_r)e_\beta(r)\Pi_{B_r}\big[e_\beta(t-s-r), |B_{t-s-r}| \geq R(t) + \inf A\big]^2\mathrm{d}r\right].$$

Using the argument of [27, Lemma 3.9] with minor modifications, one can prove (1.2). We omit the details. □

Lemma 2.2 *Suppose the assumptions of Lemma 2.1(i) hold. Then, there exist $T > 0$ and $\theta_\pm(t)$ such that for $t \geq T$, $s \in [0, at]$ and $|x| \leq b(t)$,*

$$\theta_-(t) \leq \frac{\Pi_x\big[e_\beta(t-s), B_{t-s} \in C_\Theta(R(t), A)\big]}{C_\Theta\left(\int_A \mathrm{e}^{-\sqrt{2\lambda_1}y}\mathrm{d}y\right)h(x)\mathrm{e}^{\lambda_1(t-s)-\sqrt{2\lambda_1}R(t)}} \leq \theta_+(t), \tag{2.5}$$

where $\theta_\pm(t) \to 1$ as $t \to +\infty$ and $C_\Theta = C_-$, C_+ and $(C_+ + C_-)$ accordingly as $\Theta = \{1\}$, $\{-1\}$ and $\{\pm 1\}$.

Proof Without loss of generality, we assume in addition that $b(t) \to +\infty$ as $t \to +\infty$. Noting (1.4), we have

$$\int_{C_\Theta(R(t),A)} h(y)\mathrm{d}y = \int_{C_\Theta(R(t),A)}\left(\int_{-\infty}^{+\infty} G_{\lambda_1}(y, z)\beta(z)h(z)\mathrm{d}z\right)\mathrm{d}y.$$

Using (1.5) and the fact that β is compactly supported, one can easily show by elementary calculation that

$$\int_{C_\Theta(R(t),A)} h(y)\mathrm{d}y \sim C_\Theta\eta(A)\mathrm{e}^{-\sqrt{2\lambda_1}R(t)} \quad \text{as } t \to +\infty,$$

where $\eta(A) = \int_A \mathrm{e}^{-\sqrt{2\lambda_1}y}\mathrm{d}y$. It then follows from Lemma 2.1(i) that there exist constants $c_1, T_1 > 0$ such that for $t \geq T_1$, $s \in [0, at]$ and $|x| \leq b(t)$,

$$\left|\frac{\Pi_x\left[e_\beta(t-s), B_{t-s} \in C_\Theta(R(t)+A)\right]}{C_\Theta \eta(A) h(x) \mathrm{e}^{\lambda_1(t-s)-\sqrt{2\lambda_1}R(t)}} - \frac{\int_{C_\Theta(R(t),A)} h(y)\mathrm{d}y}{C_\Theta \eta(A)\mathrm{e}^{-\sqrt{2\lambda_1}R(t)}}\right| \leq \frac{\mathrm{e}^{-c_1 t}}{C_\Theta \eta(A) h(x)}.$$

By (1.6), there is a constant $c_2 > 0$ such that $h(x) \geq c_2 \mathrm{e}^{-\sqrt{2\lambda_1}|x|}$ for all $x \in \mathbb{R}$. So one has $\inf_{|x|\leq b(t)} h(x) \geq c_2 \mathrm{e}^{-\sqrt{2\lambda_1}b(t)}$. Thus,

$$\frac{\mathrm{e}^{-c_1 t}}{C_\Theta \eta(A) h(x)} \leq c_3 \mathrm{e}^{-c_1 t + \sqrt{2\lambda_1}b(t)} \to 0 \quad \text{as } t \to +\infty.$$

Hence, we obtain (2.5) by setting $\theta_\pm(t) = \frac{\int_{C_\Theta(R(t),A)} h(y)\mathrm{d}y}{C_\Theta \eta(A)\mathrm{e}^{-\sqrt{2\lambda_1}R(t)}} \pm c_3 \mathrm{e}^{-c_1 t + \sqrt{2\lambda_1}b(t)}$. □

3 Proof of Theorem 1.1

3.1 Estimates on the First Moment

Put

$$\pi_t^R(x) := \Pi_x\left[e_\beta(t); |B_t| \geq R\right], \quad t \geq 0, x \in \mathbb{R}.$$

In this section, we derive some estimates for $\pi_t^R(x)$, which will be used in the proof of Theorem 1.1.

For any $\delta \geq 0$, we define

$$\Lambda_\delta := \begin{cases} -\lambda_1 + \sqrt{2\lambda_1}\delta & \text{if } 0 \leq \delta < \sqrt{2\lambda_1}, \\ \frac{\delta^2}{2} & \text{if } \delta \geq \sqrt{2\lambda_1}. \end{cases}$$

Obviously, $\Lambda_\delta < 0$, $\Lambda_\delta = 0$ and $\Lambda_\delta > 0$ accordingly as $0 \leq \delta < \sqrt{\lambda_1/2}$, $\delta = \sqrt{\lambda_1/2}$ and $\delta > \sqrt{\lambda_1/2}$.

Lemma 3.1 *Suppose $\delta > 0$. For any compact set $K \subset \mathbb{R}$,*

$$\lim_{t\to+\infty} \sup_{y\in K} \frac{\log \pi_t^{\delta t}(y)}{t} = \lim_{t\to+\infty} \inf_{y\in K} \frac{\log \pi_t^{\delta t}(x)}{t} = -\Lambda_\delta. \tag{3.1}$$

Proof For $\delta \geq \sqrt{2\lambda_1}$, (3.1) is proved by [34, Lemmas 4.4–4.5]. For $0 < \delta < \sqrt{2\lambda_1}$, noting that $\pi_t^{\delta t}(x) = \Pi_x\left[e_\beta(t), B_t \in C_{\{\pm 1\}}(\delta t, (0, +\infty))\right]$, we get by Lemma 2.2 that for every compact set K, when t is sufficiently large,

$$\theta_-(t) c_1 \mathrm{e}^{\lambda_1 t - \sqrt{2\lambda_1}\delta t} \leq \pi_t^{\delta t}(y) \leq \theta_+(t) c_1 \mathrm{e}^{\lambda_1 t - \sqrt{2\lambda_1}\delta t} \quad \forall y \in K,$$

where $c_1 = (C_+ + C_-) \int_0^{+\infty} \mathrm{e}^{-\sqrt{2\lambda_1} y} \mathrm{d}y$ and $\theta_\pm(t) \to 1$ as $t \to +\infty$. Thus, (3.1) follows immediately. □

Remark 3.2 We remark here that for any compact set $K \subset \mathbb{R}$ and $x \in \mathbb{R}$,

$$\lim_{t\to+\infty} \sup_{y\in K} \frac{\log \pi_t^0(y)}{t} = \lim_{t\to+\infty} \frac{\log \pi_t^0(x)}{t} = \lambda_1. \tag{3.2}$$

The second equality follows immediately by [34, Theorem A.2]. We shall show the first equality. Let $\epsilon > 0$. It follows by [34, Lemma 4.3] that there is $p^* > 1$ such that for all $p \in (1, p^*)$,

$$c_1(p) := \sup_{y\in\mathbb{R}} \Pi_x \left[\sup_{t\geq 0} \mathrm{e}^{-p(\lambda_1+\epsilon)t} e_{p\beta}(t) \right] < +\infty.$$

By this and Jensen's inequality, we have

$$\pi_t^0(y) \leq \Pi_y \left[\mathrm{e}_{p\beta}(t) \right]^{1/p} = \mathrm{e}^{(\lambda_1+\epsilon)t} \Pi_y \left[\mathrm{e}^{-p(\lambda_1+\epsilon)t} e_{p\beta}(t) \right] \leq c_1(p)^{1/p} \mathrm{e}^{(\lambda_1+\epsilon)t}$$

for every $t \geq 0$ and $y \in \mathbb{R}$. Thus,

$$\limsup_{t\to+\infty} \sup_{y\in K} \frac{\log \pi_t^0(y)}{t} \leq \lambda_1 + \epsilon.$$

This implies the first identity of (3.2).

Lemma 3.3 *(i) For every $\sigma > 0$, there exists a constant $C_3 = C_3(\sigma) > 0$, such that for any $0 < \theta < \delta < +\infty$,*

$$\pi_s^{\delta t}(x) \leq C_3 \cdot (\theta t)^{-1} \mathrm{e}^{-\frac{\theta^2 t^2}{2\sigma}}, \quad \forall s \in (0, \sigma],\ |x| \leq (\delta - \theta)t,$$

when t is sufficiently large.

(ii) For every $\delta \geq 0$ and $\sigma > 0$, there exists a constant $C_4 = C_4(\delta, \sigma) > 0$ such that for any $s \in (0, \sigma]$, $t \geq s$ and $|x| \geq \delta(t - s)$,

$$\pi_s^{\delta t}(x) \geq C_4.$$

(iii) If $\delta > \sqrt{\lambda_1/2}$, then $x \mapsto \int_0^{+\infty} \pi_s^{\delta s}(x)\mathrm{d}s$ is a locally bounded function on $\mathbb{R}$.

Proof (i) Note that for every $s \geq 0$,

$$e_\beta(s) = 1 + \int_0^s e_\beta(r)\beta(B_r)\mathrm{d}r.$$

We have

$$
\begin{aligned}
\pi_s^{\delta t}(x) &= \Pi_x\left[e_\beta(s); |B_s| \geq \delta t\right] \\
&= \Pi_x\left(|B_s| \geq \delta t\right) + \Pi_x\left[\int_0^s e_\beta(r)\beta(B_r)\mathbf{1}_{\{|B_s|\geq \delta t\}}\mathrm{d}r\right] \\
&=: \mathrm{I}(x,s,t) + \mathrm{II}(x,s,t).
\end{aligned}
$$

For $R \geq 0$, let

$$
G(R) := \Pi_0\left(|B_1| \geq R\right) = \sqrt{\frac{2}{\pi}}\int_R^{+\infty} \mathrm{e}^{-\frac{y^2}{2}}\mathrm{d}y.
$$

Then for $s \in (0,\sigma]$ and $|x| \leq (\delta-\theta)t$,

$$
\begin{aligned}
\mathrm{I}(x,s,t) = \Pi_0\left(|B_s + x| \geq \delta t\right) &\leq \Pi_0\left(|B_s| \geq \delta t - |x|\right) \\
&\leq \Pi_0\left(|B_1| \geq \frac{\theta t}{\sqrt{\sigma}}\right) = G\left(\frac{\theta t}{\sqrt{\sigma}}\right). \qquad (3.3)
\end{aligned}
$$

Suppose $\mathrm{supp}\beta \subset [-k,k]$ for some $k \in (0,+\infty)$. By Markov property, we have for $x \in \mathbb{R}$ and $s \in (0,\sigma]$,

$$
\begin{aligned}
\mathrm{II}(x,s,t) &= \Pi_x\left[\int_0^s e_\beta(r)\beta(B_r)\Pi_{B_r}\left(|B_{s-r}| \geq \delta t\right)\mathrm{d}r\right] \\
&= \Pi_x\left[\int_0^s e_\beta(r)\beta(B_r)1_{\{|B_r|\leq k\}}\Pi_{B_r}\left(|B_{s-r}| \geq \delta t\right)\mathrm{d}r\right] \\
&\leq \Pi_x\left[\int_0^s e_\beta(r)\beta^+(B_r)\Pi_0\left(|B_{s-r}| \geq \delta t - k\right)\mathrm{d}r\right] \\
&\leq \int_0^\sigma \mathrm{e}^{\|\beta^+\|_\infty r}\|\beta^+\|_\infty G\left(\frac{\delta t - k}{\sqrt{\sigma - r}}\right)\mathrm{d}r \\
&\leq c_1 G\left(\frac{\delta t - k}{\sqrt{\sigma}}\right) \qquad (3.4)
\end{aligned}
$$

for some $c_1 = c_1(\sigma) > 0$. Note that for $t \geq k/(\delta-\theta)$, $G((\delta t - k)/\sqrt{\sigma}) \leq G(\theta t/\sqrt{\sigma})$. It follows from (3.3) and (3.4) that for $t \geq k/(\delta-\theta)$, $|x| \leq (\delta-\theta)t$ and $s \in (0,\sigma]$,

$$
\pi_s^{\delta t}(x) \leq (1 + c_1)G\left(\frac{\theta t}{\sqrt{\sigma}}\right).
$$

Thus, (i) follows by the fact that $G(R) \sim \sqrt{2/\pi}R^{-1}\mathrm{e}^{-R^2/2}$ as $R \to +\infty$.

(ii) We have

$$
\pi_s^{\delta t}(x) = \Pi_x\left[e_\beta(s); |B_s| \geq \delta t\right] \geq \mathrm{e}^{-\|\beta^-\|_\infty s}\Pi_x\left(|B_s| \geq \delta t\right).
$$

Note that for $x \geq \delta(t-s)$,

$$\begin{aligned}\Pi_x\left(|B_s| \geq \delta t\right) \geq \Pi_x\left(B_s \geq \delta t\right) &= \Pi_0\left(B_s \geq \delta t - x\right) \\ &\geq \Pi_0\left(B_s \geq \delta s\right) = \Pi_0\left(B_1 \geq \delta\sqrt{s}\right).\end{aligned}$$

Similarly, one can show that $\Pi_x\left(|B_s| \geq \delta t\right) \geq \Pi_0\left(B_1 \leq -\delta\sqrt{s}\right)$ for $x \leq -\delta(t-s)$. Hence, we get (ii) by setting $C_4 = \mathrm{e}^{-\|\beta^-\|_\infty \sigma}\Pi_0\left(B_1 \geq \delta\sqrt{\sigma}\right)$.

(iii) By Lemma 3.1, for $\delta > \sqrt{\lambda_1/2}$ and any compact set $K \subset \mathbb{R}$,

$$\lim_{t\to+\infty} \sup_{x\in K} \frac{\log \pi_t^{\delta t}(x)}{t} = -\Lambda_\delta < 0.$$

So there is some $T > 0$ such that for all $s \geq T$, $\sup_{x\in K} \pi_s^{\delta s}(x) \leq \exp\{-\Lambda_\delta s/2\}$. We also note that $\pi_s^{\delta s}(x) = \Pi_x\left[e_\beta(s); |B_s| \geq \delta s\right] \leq \mathrm{e}^{\|\beta^+\|_\infty s}$ for all $x \in K$ and $s \geq 0$. Hence,

$$\sup_{x\in K} \int_0^{+\infty} \pi_s^{\delta s}(x)\mathrm{d}s \leq \int_0^T \mathrm{e}^{\|\beta^+\|_\infty s}\mathrm{d}s + \int_T^{+\infty} \mathrm{e}^{-\Lambda_\delta s/2}\mathrm{d}s < +\infty.$$

□

3.2 The Upper Bound of $\mathcal{X}_t^{\delta t}$

For $t \geq 0$, let $\mathcal{F}_t$ denote the σ-field generated by $\{X_s : 0 \leq s \leq t\}$. It follows immediately from Lemma 3.3(ii) that for any $n \in \mathbb{N}$ and $t \in [n\sigma, (n+1)\sigma)$,

$$\pi_{(n+1)\sigma-t}^{\delta(n+1)\sigma}(x) \geq C_4, \quad \forall x \in \mathbb{R} \text{ such that } |x| \geq \delta t.$$

Thus for $t \in [n\sigma, (n+1)\sigma)$ and $\mu \in \mathcal{M}(\mathbb{R})$,

$$\mathcal{X}_t^{\delta t} = \langle \mathbf{1}_{(-\delta t,\delta t)^c}, X_t\rangle \leq C_4^{-1}\langle \pi_{(n+1)\sigma-t}^{\delta(n+1)\sigma}, X_t\rangle = C_4^{-1}\mathbb{P}_\mu\left[\mathcal{X}_{(n+1)\sigma}^{\delta(n+1)\sigma}|\mathcal{F}_t\right] \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.5}$$

Here, the last equality follows by the Markov property of X_t and (1.3). Hence to get an upper bound for $\mathcal{X}_t^{\delta t}$, we only need to compute $\mathbb{P}_\mu\left[\mathcal{X}_{(n+1)\sigma}^{\delta(n+1)\sigma} \mid \mathcal{F}_t\right]$.

Lemma 3.4 *For any $\delta \geq 0$, $\sigma > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$,*

$$\limsup_{n\to+\infty} \frac{\log \mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma} \leq -\Lambda_\delta \quad \mathbb{P}_\mu\text{-}a.s.$$

Proof Let $\epsilon > 0$. By the Markov inequality and (1.3), we have

$$\begin{aligned}\mathbb{P}_\mu\left(\frac{\log \mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma} \geq -\Lambda_\delta + \epsilon\right) &= \mathbb{P}_\mu\left(\mathcal{X}_{n\sigma}^{\delta n\sigma} \geq \mathrm{e}^{(-\Lambda_\delta+\epsilon)n\sigma}\right) \\ &\leq \mathrm{e}^{(\Lambda_\delta-\epsilon)n\sigma}\langle \pi_{n\sigma}^{\delta n\sigma}, \mu\rangle \\ &= \mathrm{e}^{n\sigma\left(\frac{\log\langle \pi_{n\sigma}^{\delta n\sigma}, \mu\rangle}{n\sigma}+\Lambda_\delta-\epsilon\right)}.\end{aligned}$$

Since μ is compactly supported, (3.1) and (3.2) imply that

$$\limsup_{n\to+\infty} \frac{\log\langle \pi_{n\sigma}^{\delta n\sigma}, \mu\rangle}{n\sigma} \leq -\Lambda_\delta.$$

Thus, when n is large enough, we have

$$\mathbb{P}_\mu\left(\frac{\log \mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma} \geq -\Lambda_\delta + \epsilon\right) \leq \mathrm{e}^{-\epsilon n\sigma/2},$$

which in turn implies that $\sum_{n=0}^{+\infty} \mathbb{P}_\mu\left(\frac{\log \mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma} \geq -\Lambda_\delta + \epsilon\right) < +\infty$. Hence, this lemma follows immediately by Borel–Cantelli lemma. □

Fitzsimmons [17] (see also [24, Chapter 7]) studied the martingale problem of superprocesses and established a stochastic integral representation for the finite variance branching case. We recall from [17] and [24, Corollary 7.15] that for any $\varphi \in C_c^2(\mathbb{R})$, the process

$$M_t(\varphi) := \langle \varphi, X_t\rangle - \langle \varphi, X_t\rangle - \int_0^t \langle (\frac{1}{2}\Delta + \beta)\varphi, X_s\rangle \mathrm{d}s$$

is a square-integrable $\mathcal{F}_t$-martingale. These martingales then induce a (worthy) martingale measure $M(\mathrm{d}s, \mathrm{d}x)$ (see [24, Chapter 7] for the precise definition) satisfying that $M_t(\varphi) = \int_0^t \int_{\mathbb{R}} \varphi(x) M(\mathrm{d}s, \mathrm{d}x)$. By standard techniques, the martingale $M_t(g) := \int_0^t \int_{\mathbb{R}} g(s, x) M(\mathrm{d}s, \mathrm{d}x)$ can be defined formally for a large class of measurable functions $g(s, x)$ on $[0, +\infty) \times \mathbb{R}$. Then, [17, Corollary 2.18] (see also [24, Theorem 7.26]) proved that for any $f \in \mathcal{B}_b(\mathbb{R})$, $t \geq 0$ and $\mu \in \mathcal{M}(\mathbb{R})$,

$$\langle f, X_t\rangle = \langle P_t^\beta f, X_0\rangle + \int_0^t \int_{-\infty}^{+\infty} P_{t-s}^\beta f(x) M(\mathrm{d}s, \mathrm{d}x) \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.6}$$

where for every $T > 0$, $[0, T] \ni t \mapsto \int_0^t \int_{-\infty}^{+\infty} P_{t-s}^\beta f(x) M(\mathrm{d}s, \mathrm{d}x)$ is a square-integrable $\mathcal{F}_t$-martingale with quadratic variation $t \mapsto \int_0^t \langle 2\gamma(P_{t-s}^\beta f)^2, X_s\rangle \mathrm{d}s$.

Lemma 3.5 *Suppose $\delta \geq 0$, $\sigma > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$. Then for any $\epsilon > 0$, the following holds $\mathbb{P}_\mu$-a.s.*

$$\lim_{n\to+\infty} \mathrm{e}^{\left(\frac{1}{2}\Lambda_\delta-\epsilon\right)(n+1)\sigma} \sup_{t\in[n\sigma,(n+1)\sigma]} \left|\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right] - \mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\right| = 0.$$

Proof By (3.6), we have

$$\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma} = \langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma}, X_0\rangle + \int_0^{(n+1)\sigma}\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x) M(\mathrm{d}s,\mathrm{d}x),$$

where $[0,(n+1)\sigma] \ni t \mapsto \int_0^t\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x)M(\mathrm{d}s,\mathrm{d}x)$ is a square-integrable $\mathbb{P}_\mu$-martingale with quadratic variation $t \mapsto \int_0^t \langle 2\gamma(\pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s})^2, X_s\rangle \mathrm{d}s$. Thus,

$$\begin{aligned}
&\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right] - \mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\\
&= \mathbb{P}_\mu\left[\int_0^{(n+1)\sigma}\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x)M(\mathrm{d}s,\mathrm{d}x)\Big|\mathcal{F}_t\right]\\
&\quad - \mathbb{P}_\mu\left[\int_0^{(n+1)\sigma}\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x)M(\mathrm{d}s,\mathrm{d}x)\Big|\mathcal{F}_{n\sigma}\right]\\
&= \int_{n\sigma}^t\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x)M(\mathrm{d}s,\mathrm{d}x).
\end{aligned}$$

By this and the L^2-maximum inequality for martingales, we have

$$\begin{aligned}
&\mathbb{P}_\mu\left[\sup_{t\in[n\sigma,(n+1)\sigma]}\left|\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right] - \mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\right|^2\right]\\
&\leq 4\mathbb{P}_\mu\left[\left(\int_{n\sigma}^{(n+1)\sigma}\int_{-\infty}^{+\infty} \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x)M(\mathrm{d}s,\mathrm{d}x)\right)^2\right]\\
&= 4\mathbb{P}_\mu\left[\int_{n\sigma}^{(n+1)\sigma}\langle 2\gamma\left(\pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}\right)^2, X_s\rangle\mathrm{d}s\right]\\
&\leq 8\|\gamma\|_\infty \mathrm{e}^{\|\beta^+\|_\infty\sigma}\mathbb{P}_\mu\left[\int_{n\sigma}^{(n+1)\sigma}\langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}, X_s\rangle\mathrm{d}s\right]\\
&= 8\|\gamma\|_\infty \mathrm{e}^{\|\beta^+\|_\infty\sigma}\int_{n\sigma}^{(n+1)\sigma}\mathbb{P}_\mu\left[\langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}, X_s\rangle\right]\mathrm{d}s\\
&= 8\|\gamma\|_\infty \mathrm{e}^{\|\beta^+\|_\infty\sigma}\int_{n\sigma}^{(n+1)\sigma}\mathbb{P}_\mu\left[\langle 1_{(-\delta(n+1)\sigma,\delta(n+1)\sigma)^c}, X_{(n+1)\sigma}\rangle\right]\mathrm{d}s\\
&= 8\|\gamma\|_\infty \mathrm{e}^{\|\beta^+\|_\infty\sigma}\sigma\langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma}, \mu\rangle.
\end{aligned} \tag{3.7}$$

The second inequality is because

$$\pi^{\delta(n+1)\sigma}_{(n+1)\sigma-s}(x) \leq \Pi_x \left[e_\beta((n+1)\sigma - s) \right] \leq \mathrm{e}^{\|\beta^+\|_\infty((n+1)\sigma-s)}.$$

The third equality follows from (1.3) and the Markov property of X_t. Let K be the compact support of μ. Lemma 3.1 implies that for any $\epsilon > 0$, there is $T > 0$ such that

$$\sup_{x\in K} \pi^{\delta t}_t(x) \leq \mathrm{e}^{(-\Lambda_\delta+\epsilon)t} \quad \forall t \geq T.$$

Thus, one has

$$\langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma}, \mu\rangle \leq \mathrm{e}^{(-\Lambda_\delta+\epsilon)(n+1)\sigma}\langle 1, \mu\rangle$$

for n sufficiently large. Putting this back to (3.7), one gets

$$\begin{aligned}&\mathbb{P}_\mu\left[\left(\mathrm{e}^{(\frac{1}{2}\Lambda_\delta-\epsilon)(n+1)\sigma} \sup_{t\in[n\sigma,(n+1)\sigma]} \left|\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right] - \mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\right|\right)^2\right] \\ &\leq c_1 \mathrm{e}^{(\Lambda_\delta-2\epsilon)(n+1)\sigma}\langle \pi^{\delta(n+1)\sigma}_{(n+1)\sigma}, \mu\rangle \leq c_1\mathrm{e}^{-\epsilon(n+1)\sigma}\langle 1, \mu\rangle,\end{aligned}$$

for some constant $c_1 = c_1(\sigma) > 0$. This implies that the sum

$$\sum_{n=0}^{+\infty} \mathbb{P}_\mu\left[\left(\mathrm{e}^{(\frac{1}{2}\Lambda_\delta-\epsilon)(n+1)\sigma} \sup_{t\in[n\sigma,(n+1)\sigma]} \left|\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right] - \mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\right|\right)^2\right]$$

is finite. The lemma follows by Borel–Cantelli lemma. □

Lemma 3.6 *Suppose $\sigma > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$. There is a constant $C_5 = C_5(\sigma) > 0$ such that for any $0 < \theta < \delta < +\infty$ and $\epsilon > 0$ the following inequality holds $\mathbb{P}_\mu$-a.s. for n sufficiently large.*

$$\begin{aligned}&\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right] \\ &\quad\leq C_5\left[(\theta n\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma+(\lambda_1+\sqrt{2\lambda_1}(\delta-\theta))n\sigma}W^h_{n\sigma}(X) + \mathrm{e}^{(-\Lambda_{\delta-\theta}+\epsilon)n\sigma}\right].\end{aligned}$$

Proof Fix arbitrary $\sigma > 0$ and $\mu \in \mathcal{M}_c(\mathbb{R})$. By Markov property, we have for any $0 < \theta < \delta < +\infty$,

$$\begin{aligned}\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right] &= \langle \pi^{\delta(n+1)\sigma}_\sigma, X_{n\sigma}\rangle \\ &= \langle \pi^{\delta(n+1)\sigma}_\sigma \mathbf{1}_{(-(\delta-\theta)n\sigma,(\delta-\theta)n\sigma)}, X_{n\sigma}\rangle \\ &\quad + \langle \pi^{\delta(n+1)\sigma}_\sigma \mathbf{1}_{(-(\delta-\theta)n\sigma,(\delta-\theta)n\sigma)^c}, X_{n\sigma}\rangle\end{aligned}$$

$$=: \mathrm{I}(n,\delta,\theta)+\mathrm{II}(n,\delta,\theta).$$

It follows from Lemma 3.3(i) that when n is sufficiently large,

$$\pi_\sigma^{\delta(n+1)\sigma}(x)\le C_3(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma},\quad \forall x\in\mathbb{R}\text{ such that }|x|<(\delta-\theta)n\sigma,$$

where $C_3=C_3(\sigma)>0$. Thus, we have

$$\begin{aligned}\mathrm{I}(n,\delta,\theta)&\le C_3(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma}\langle \mathbf{1}_{(-(\delta-\theta)n\sigma,(\delta-\theta)n\sigma)},X_{n\sigma}\rangle\\ &\le C_3(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma}\mathrm{e}^{\lambda_1 n\sigma}\left\langle \mathrm{e}^{-\lambda_1 n\sigma}\frac{h}{\inf_{|x|<(\delta-\theta)n\sigma}h(x)},X_{n\sigma}\right\rangle\\ &= C_3(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma+\lambda_1 n\sigma}\left(\inf_{|x|<(\delta-\theta)n\sigma}h(x)\right)^{-1}W^h_{n\sigma}(X).\end{aligned}\quad(3.8)$$

The continuity of h together with (1.6) implies that $\inf_{|x|<(\delta-\theta)n\sigma}h(x)\ge c_1\mathrm{e}^{-\sqrt{2\lambda_1}(\delta-\theta)n\sigma}$ for n sufficiently large. Thus, we get by (3.8) that $\mathbb{P}_\mu$-a.s.

$$\mathrm{I}(n,\delta,\theta)\le c_2\big(\theta n\sigma\big)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma+(\lambda_1+\sqrt{2\lambda_1}(\delta-\theta))n\sigma}W^h_{n\sigma}(X)\quad(3.9)$$

for n sufficiently large, where $c_2=c_2(\sigma)>0$. On the other hand, by Lemma 3.4 we have $\limsup_{n\to+\infty}\log\mathcal{X}_{n\sigma}^{(\delta-\theta)n\sigma}/n\sigma\le-\Lambda_{\delta-\theta}$ $\mathbb{P}_\mu$-a.s. Thus for any $\epsilon>0$,

$$\mathbb{P}_\mu\left(\mathcal{X}_{n\sigma}^{(\delta-\theta)n\sigma}\le \mathrm{e}^{(-\Lambda_{\delta-\theta}+\epsilon)n\sigma}\text{ for }n\text{ sufficiently large}\right)=1.$$

Note that by definition $\mathrm{II}(n,\delta,\theta)\le\|\pi_\sigma^{\delta(n+1)\sigma}\|_\infty\mathcal{X}_{n\sigma}^{(\delta-\theta)n\sigma}\le \mathrm{e}^{\|\beta^+\|_\infty\sigma}\mathcal{X}_{n\sigma}^{(\delta-\theta)n\sigma}$ for every $n\in\mathbb{N}$. We get that

$$\mathbb{P}_\mu\left(\mathrm{II}(n,\delta,\theta)\le c_3e^{(-\Lambda_{\delta-\theta}+\epsilon)n\sigma}\text{ for }n\text{ sufficiently large}\right)=1\quad(3.10)$$

for $c_3=\mathrm{e}^{\|\beta^+\|_\infty\sigma}$. This lemma follows immediately by combining (3.9) and (3.10).□

Lemma 3.7 *For any $\delta\ge0$, $\sigma>0$ and $\mu\in\mathcal{M}_c(\mathbb{R})$,*

$$\limsup_{n\to+\infty}\frac{\log\mathbb{P}_\mu\left[\mathcal{X}_{(n+1)\sigma}^{\delta(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]}{n\sigma}\le-\Lambda_\delta\quad\mathbb{P}_\mu\text{-}a.s.$$

Proof First we consider $\delta>0$. It follows by Lemma 3.6 that for any $0<\theta<\delta$ and $\epsilon>0$, $\mathbb{P}_\mu$-a.s.

$$\mathrm{e}^{\Lambda_\delta n\sigma}\mathbb{P}_\mu\left[\mathcal{X}_{(n+1)\sigma}^{\delta(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\le C_5(\sigma)\Big[\mathrm{e}^{(\Lambda_\delta-\Lambda_{\delta-\theta}+\epsilon)n\sigma}$$

$$+\left(\theta n\sigma\right)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}n^2\sigma+(\Lambda_\delta+\lambda_1+\sqrt{2\lambda_1}(\delta-\theta))n\sigma}W^h_{n\sigma}(X)\Big]. \tag{3.11}$$

for n sufficiently large. Since $\delta\mapsto\Lambda_\delta$ is nondecreasing and continuous on $(0,+\infty)$ and that $\mathbb{P}_\mu\left(W^h_\infty(X)<+\infty\right)=1$, one can choose θ so small that $\Lambda_\delta-\Lambda_{\delta-\theta}<\epsilon$. We also note that for fixed δ and θ, the first term on the right-hand side of (3.11) converges to 0 $\mathbb{P}_\mu$-a.s. as $n\to+\infty$. Thus, we get by (3.11) that

$$\mathbb{P}_\mu\left(\mathrm{e}^{\Lambda_\delta n\sigma}\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]\le 2\mathrm{e}^{2\epsilon n\sigma}\text{ for } n \text{ sufficiently large}\right)=1.$$

Hence, we prove this lemma for $\delta>0$. Now, we suppose $\delta=0$. By Markov property, we have

$$\mathbb{P}_\mu\left[\mathcal{X}^0_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]=\langle\pi^0_\sigma,X_{n\sigma}\rangle\le\mathrm{e}^{\|\beta^+\|_\infty\sigma}\mathcal{X}^0_{n\sigma}.$$

It follows by Lemma 3.4 that

$$\limsup_{n\to+\infty}\frac{\log\mathbb{P}_\mu\left[\mathcal{X}^0_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]}{n\sigma}\le\limsup_{n\to+\infty}\frac{\log\mathcal{X}^0_{n\sigma}}{n\sigma}\le-\Lambda_0\quad\mathbb{P}_\mu\text{-a.s.}$$

Hence, we complete the proof. □

Proposition 3.8 *Suppose $\mu\in\mathcal{M}_c(\mathbb{R})$. For any $\delta>\sqrt{\lambda_1/2}$,*

$$\limsup_{t\to+\infty}\frac{\log\mathcal{X}^{\delta t}_t}{t}\le-\frac{1}{2}\Lambda_\delta\quad\mathbb{P}_\mu\text{-a.s.}, \tag{3.12}$$

and for any $0\le\delta\le\sqrt{\lambda_1/2}$,

$$\limsup_{t\to+\infty}\frac{\log\mathcal{X}^{\delta t}_t}{t}\le-\Lambda_\delta\quad\mathbb{P}_\mu\text{-a.s.} \tag{3.13}$$

Proof Let $\sigma>0$. By (3.5), we have for any $n\in\mathbb{N}$ and $t\in[n\sigma,(n+1)\sigma)$,

$$\mathcal{X}^{\delta_t}_t\le C_4^{-1}\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]\quad\mathbb{P}_\mu\text{-a.s.} \tag{3.14}$$

One can decompose $\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]$ as $\mathrm{I}(n,\sigma,t)+\mathrm{II}(n,\sigma)$, where $\mathrm{I}(n,\sigma,t):=\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]-\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]$ and $\mathrm{II}(n,\sigma):=\mathbb{P}_\mu\left[\mathcal{X}^{\delta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_{n\sigma}\right]$. It follows by Lemma 3.7 that for any $\epsilon>0$,

$$\mathbb{P}_\mu\left(\mathrm{II}(n,\sigma)\le\mathrm{e}^{(-\Lambda_\delta+\epsilon)n\sigma}\text{ for } n \text{ sufficiently large}\right)=1. \tag{3.15}$$

On the other hand, by Lemma 3.5 we have

$$\mathbb{P}_{\mu}\left(\sup_{t\in[n\sigma,(n+1)\sigma)} \mathrm{I}(n,\sigma,t) \le \epsilon e^{(-\frac{1}{2}\Lambda_\delta+\epsilon)(n+1)\sigma} \text{ for } n \text{ sufficiently large}\right) = 1. \tag{3.16}$$

Combining (3.14)–(3.16), we get that

$$\sup_{t\in[n\sigma,(n+1)\sigma)} \mathcal{X}_t^{\delta t} \le C_4^{-1}\left(\mathrm{e}^{(-\Lambda_\delta+\epsilon)n\sigma} + \epsilon e^{(-\frac{1}{2}\Lambda_\delta+\epsilon)(n+1)\sigma}\right)$$

for n sufficiently large $\mathbb{P}_\mu$-a.s. It follows immediately that

$$\limsup_{t\to+\infty} \frac{\log \mathcal{X}_t^{\delta t}}{t} \le (-\Lambda_\delta) \vee \left(-\frac{1}{2}\Lambda_\delta\right) \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.17}$$

If $0 \le \delta \le \sqrt{\lambda_1/2}$, then $-\Lambda_\delta \ge -\Lambda_\delta/2 \ge 0$, and (3.13) follows directly from (3.17). Otherwise if $\delta > \sqrt{\lambda_1/2}$, then $-\Lambda_\delta < -\frac{1}{2}\Lambda_\delta < 0$, and hence (3.12) follows. □

3.3 The Lower Bound of $\mathcal{X}_t^{\delta t}$

Let $p^\beta(t,x,y)$ be the transition density of P_t^β. It is easy to see that

$$\mathrm{e}^{-\|\beta^-\|_\infty t} p(t,x,y) \le p^\beta(t,x,y) \le \mathrm{e}^{\|\beta^+\|_\infty t} p(t,x,y) \quad \forall t \ge 0,\ x,y \in \mathbb{R}. \tag{3.18}$$

Here, $p(t,x,y)$ is the transition density of a Brownian motion on $\mathbb{R}$. Let P_t^h be the semigroup obtained from P_t^β through Doob's h-transform, that is,

$$P_t^h f(x) = \frac{\mathrm{e}^{-\lambda_1 t}}{h(x)} P_t^\beta(hf)(x) \quad \forall t \ge 0,\ x \in \mathbb{R},\ f \in \mathcal{B}^+(\mathbb{R}). \tag{3.19}$$

Then, P_t^h has a transition density with respect to the measure $h^2(y)\mathrm{d}y$, which is given by

$$p^h(t,x,y) = \frac{\mathrm{e}^{-\lambda_1 t} p^\beta(t,x,y)}{h(x)h(y)} \quad \forall t \ge 0,\ x,y \in \mathbb{R}. \tag{3.20}$$

It is proved in [9, eq. (2.14)] that there exists a constant $a > 0$ such that

$$\left|p^h(t,x,y) - 1\right| \le e^{-a(t-1)} p^h(1,x,x)^{1/2} p^h(1,y,y)^{1/2} \quad \forall t > 1,\ x,y \in \mathbb{R}.$$

This together with (3.18) and (3.20) implies that there is some constant $c_1 > 0$ such that

$$\left|p^h(t,x,y)-1\right| \le c_1 \mathrm{e}^{-at} h(x)^{-1} h(y)^{-1} \quad \forall t>1,\ x,y\in\mathbb{R}. \tag{3.21}$$

Lemma 3.9 *Suppose $\mu \in \mathcal{M}(\mathbb{R})$ and $\sigma > 0$. For any $f \in \mathcal{B}_b^+(\mathbb{R})$ such that f/h is bounded from above and that $\int_{-\infty}^{+\infty} f(x)h(x)\mathrm{d}x > 0$, we have*

$$\lim_{n\to+\infty} \frac{\log\langle f, X_{n\sigma}\rangle}{n\sigma} = \lambda_1 \quad \mathbb{P}_\mu\text{-a.s. on } \{W_\infty^h(X) > 0\}.$$

Proof Without loss of generality, we assume $0 \ne \mu \in \mathcal{M}(\mathbb{R})$. It follows by Proposition 3.8 that

$$\limsup_{n\to+\infty} \frac{\log\langle f, X_{n\sigma}\rangle}{n\sigma} \le \limsup_{n\to+\infty} \frac{\log\|f\|_\infty + \log \mathcal{X}_{n\sigma}^0}{n\sigma} \le \lambda_1 \quad \mathbb{P}_\mu\text{-a.s.}$$

Hence, we only need to show that

$$\liminf_{n\to+\infty} \frac{\log\langle f, X_{n\sigma}\rangle}{n\sigma} \ge \lambda_1 \quad \mathbb{P}_\mu\text{-a.s. on } \{W_\infty^h(X) > 0\},$$

or equivalently, for any $\epsilon > 0$,

$$\mathbb{P}_\mu\left(\mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle \ge \mathrm{e}^{-\epsilon n\sigma} \text{ for } n \text{ sufficiently large} \mid W_\infty^h(X) > 0\right) = 1. \tag{3.22}$$

For any $n \in \mathbb{N}$ and $\sigma > 0$, we have

$$\mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle = \mathrm{I}(n,\sigma) + \mathrm{II}(n,\sigma) + \mathrm{III}(n,\sigma),$$

where

$$\begin{aligned}
\mathrm{I}(n,\sigma) &= \mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle - \mathbb{P}_\mu\left[\mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle | \mathcal{F}_{n\sigma/2}\right],\\
\mathrm{II}(n,\sigma) &= \mathbb{P}_\mu\left[\mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle | \mathcal{F}_{n\sigma/2}\right] - (f,h) W_{n\sigma/2}^h(X),\\
\mathrm{III}(n,\sigma) &= (f,h) W_{n\sigma/2}^h(X).
\end{aligned}$$

Since $\lim_{t\to+\infty} W_t^h(X) = W_\infty^h(X)$ $\mathbb{P}_\mu$-a.s., we have

$$\mathbb{P}_\mu\left(\mathrm{III}(n,\sigma) \ge \frac{1}{2}(f,h) W_\infty^h(X) > 0 \text{ for } n \text{ sufficiently large} \mid W_\infty^h(X) > 0\right) = 1. \tag{3.23}$$

Let $\phi(x) := f(x)/h(x)$ for $x \in \mathbb{R}$. By Markov property and (3.19), we have

$$\begin{aligned}\mathrm{II}(n,\sigma) &= \mathrm{e}^{-\lambda_1 n\sigma}\langle P^{\beta}_{n\sigma/2}(\phi h), X_{n\sigma/2}\rangle - (\phi h, h)\mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\langle h, X_{n\sigma/2}\rangle \\ &= \mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\langle hP^{h}_{n\sigma/2}(\phi), X_{n\sigma/2}\rangle - (\phi h, h)\mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\langle h, X_{n\sigma/2}\rangle \\ &= \mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\Big\langle h\int_{-\infty}^{+\infty}\left(p^h(n\sigma/2,\cdot,y)-1\right)\phi(y)h^2(y)\mathrm{d}y, X_{n\sigma/2}\Big\rangle.\end{aligned}$$

It follows by (3.21) that for $n \in \mathbb{N}$ with $n\sigma > 1$,

$$\begin{aligned}|\mathrm{II}(n,\sigma)| &\leq \mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\langle h\int_{-\infty}^{+\infty}\left|p^h(n\sigma/2,\cdot,y)-1\right|\phi(y)h^2(y)\mathrm{d}y, X_{n\sigma/2}\rangle \\ &\leq c_1\mathrm{e}^{-an\sigma/2}(\phi,h)\mathrm{e}^{-\frac{1}{2}\lambda_1 n\sigma}\mathcal{X}^0_{n\sigma/2}.\end{aligned}$$

This together with (3.13) yields that

$$\mathbb{P}_\mu\left(\lim_{n\to+\infty}|\mathrm{II}(n,\sigma)|=0\right)=1. \tag{3.24}$$

By (3.6), we have

$$\mathrm{e}^{-\lambda_1 n\sigma}\langle f, X_{n\sigma}\rangle = \langle hP^h_{n\sigma}\phi, X_0\rangle + \int_0^{n\sigma}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 s}h(x)P^h_{n\sigma-s}\phi(x)M(\mathrm{d}s,\mathrm{d}x).$$

Here, $[0, n\sigma] \ni t \mapsto \int_0^t\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 s}h(x)P^h_{n\sigma-s}\phi(x)M(\mathrm{d}s,\mathrm{d}x)$ is a square-integrable martingale with quadratic variation $t \mapsto \int_0^t\langle 2\gamma\mathrm{e}^{-2\lambda_1 s}h^2\left(P^h_{n\sigma-s}\phi\right)^2, X_s\rangle\mathrm{d}s$. Hence,

$$\mathrm{I}(n,\sigma) = \int_{n\sigma/2}^{n\sigma}\int_{-\infty}^{+\infty}\mathrm{e}^{-\lambda_1 s}h(x)P^h_{n\sigma-s}\phi(x)M(\mathrm{d}s,\mathrm{d}x).$$

Moreover, by (3.19) we have

$$\begin{aligned}\mathbb{P}_\mu\left[\mathrm{I}(n,\sigma)^2\right] &= \mathbb{P}_\mu\left[\int_{n\sigma/2}^{n\sigma}\langle 2\gamma\mathrm{e}^{-2\lambda_1 s}h^2\left(P^h_{n\sigma-s}\phi\right)^2, X_s\rangle\mathrm{d}s\right] \\ &\leq 2\|\gamma\|_\infty\|\phi\|_\infty\|h\|_\infty\int_{n\sigma/2}^{n\sigma}\mathrm{e}^{-2\lambda_1 s}\mathbb{P}_\mu\left[\langle hP^h_{n\sigma-s}\phi, X_s\rangle\right]\mathrm{d}s \\ &= c_2\int_{n\sigma/2}^{n\sigma}\mathrm{e}^{-2\lambda_1 s}\langle P^\beta_s\left(hP^h_{n\sigma-s}\phi\right), \mu\rangle\mathrm{d}s \\ &= c_2\int_{n\sigma/2}^{n\sigma}\mathrm{e}^{-\lambda_1 s}\langle hP^h_{n\sigma}\phi, \mu\rangle\mathrm{d}s \\ &\leq c_2\|\phi\|_\infty\langle h,\mu\rangle\int_{n\sigma/2}^{n\sigma}\mathrm{e}^{-\lambda_1 s}\mathrm{d}s\end{aligned}$$

$$= c_2\|\phi\|_\infty\langle h,\mu\rangle\lambda_1^{-1}\mathrm{e}^{-\lambda_1 n\sigma/2}\left(1-\mathrm{e}^{-\lambda_1 n\sigma/2}\right).$$

Immediately $\sum_{n=0}^{+\infty}\mathbb{P}_\mu\left[\mathrm{I}(n,\sigma)^2\right] < +\infty$. Hence by the Fubini theorem,

$$\mathbb{P}_\mu\left(\lim_{n\to+\infty}|\mathrm{I}(n,\sigma)|=0\right)=1.$$

This together with (3.23) and (3.24) yields (3.22). Hence, we complete the proof. □

Lemma 3.10 *Suppose* $0<\delta<\sqrt{\lambda_1/2}$ *and* $\sigma>0$. *For any nontrivial* $\mu\in\mathcal{M}_c(\mathbb{R})$,

$$\liminf_{n\to+\infty}\frac{\log\mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma}\geq-\Lambda_\delta\quad\mathbb{P}_\mu\text{-a.s. on }\{W_\infty^h(X)>0\}.$$

Proof We define a quadratic branching mechanism $\tilde{\psi}$ by

$$\tilde{\psi}(x,\lambda):=-\beta(x)\lambda+\gamma(x)\lambda^2,\quad\forall x\in\mathbb{R},\lambda\geq0,$$

where γ is defined in (1.2). Let $((\tilde{X}_t)_{t\geq0},\mathbb{P}_{\delta_x})$ be a $(B_t,\tilde{\psi})$-superprocess started from Dirac measure at x. For any $R,t\geq0$ and $x\in\mathbb{R}$, let $\tilde{u}^R(t,x):=-\log\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\tilde{X}((-R,R)^c)}\right]$ and $u^R(t,x):=-\log\mathbb{P}_{\delta_x}\left[\mathrm{e}^{-\mathcal{X}_t^R}\right]$. Noting that $\psi\leq\tilde{\psi}$, we have by [24, Corollary 5.18] that

$$u^R(t,x)\geq\tilde{u}^R(t,x)\quad\forall t,R\geq0,x\in\mathbb{R}. \tag{3.25}$$

It is known that $(t,x)\mapsto\tilde{u}^R(t,x)$ is the unique nonnegative locally bounded solution to the following integral equation.

$$\tilde{u}^R(t,x)=\Pi_x\left(|B_t|\geq R\right)+\Pi_x\left[\int_0^t\beta(B_s)\tilde{u}^R(t-s,B_s)-\gamma(B_s)\tilde{u}^R(t-s,B_s)^2\mathrm{d}s\right].$$

By [24, Proposition 2.9], $\tilde{u}^R(t,x)$ also satisfies that

$$\tilde{u}^R(t,x)=\Pi_x\left[\exp\left\{\int_0^t\left(\beta(B_s)-\gamma(B_s)\tilde{u}^R(t-s,B_s)\right)\mathrm{d}s\right\}\mathbf{1}_{\{|B_t|\geq R\}}\right]. \tag{3.26}$$

Immediately, we have

$$\tilde{u}^R(t,x)\leq\Pi_x\left[e_\beta(t),|B_t|\geq R\right]=\pi_t^R(x)\quad\forall x\in\mathbb{R},\ t\geq0. \tag{3.27}$$

Let $q\in(0,1)$ and $p=1-q$. By (3.26) and (3.27), one has for all $n\in\mathbb{N}$ and $x\in\mathbb{R}$,

$$\tilde{u}^{\delta n\sigma}(nq\sigma,x)$$

$$
\begin{aligned}
&= \Pi_x \left[\exp\left\{ \int_0^{nq\sigma} \left(\beta(B_s) - \gamma(B_s) \tilde{u}^{\delta n\sigma}(nq\sigma - s, B_s) \right) \mathrm{d}s \right\} \mathbf{1}_{\{|B_{nq\sigma}| \geq \delta n\sigma\}} \right] \\
&\geq \Pi_x \left[\exp\left\{ \int_0^{nq\sigma} \left(\beta(B_s) - \gamma(B_s) \pi^{\delta n\sigma}_{nq\sigma - s}(B_s) \right) \mathrm{d}s \right\} \mathbf{1}_{\{|B_{nq\sigma}| \geq \delta n\sigma\}} \right] \\
&\geq \Pi_x \left[\exp\left\{ \int_0^{nq\sigma} \left(\beta(B_s) - \gamma(B_s) \pi^{\frac{\delta}{q}(nq\sigma - s)}_{nq\sigma - s}(B_s) \right) \mathrm{d}s \right\} \mathbf{1}_{\{|B_{nq\sigma}| \geq \delta n\sigma\}} \right] \\
&= \Pi_x \left[\exp\left\{ - \int_0^{nq\sigma} \gamma(B_{nq\sigma - r}) \pi^{\frac{\delta}{q} r}_r (B_{nq\sigma - r}) \mathrm{d}r \right\} e_\beta(nq\sigma) \mathbf{1}_{\{|B_{nq\sigma}| \geq \delta n\sigma\}} \right]. \qquad (3.28)
\end{aligned}
$$

The final equality follows from the changes of variables. By choosing $q \in (0, \delta/\sqrt{2\lambda_1})$, we have $\delta/q > \sqrt{2\lambda_1}$. Since γ is compactly supported, it follows from Lemma 3.3(iii) that

$$
c_1 := \sup_{x \in \mathbb{R}^d} \gamma(x) \int_0^{+\infty} \pi_s^{\frac{\delta}{q} s}(x) \mathrm{d}s < +\infty.
$$

Putting this back to (3.28), one gets

$$
\tilde{u}^{\delta n\sigma}(nq\sigma, x) \geq \mathrm{e}^{-c_1} \Pi_x \left[e_\beta(nq\sigma) \mathbf{1}_{\{|B_{nq\sigma}| \geq \delta n\sigma\}} \right] = \mathrm{e}^{-c_1} \pi_{nq\sigma}^{\frac{\delta}{q} \cdot nq\sigma}(x). \qquad (3.29)
$$

Let K be a compact set of $\mathbb{R}$ with $\int_K h(x)\mathrm{d}y > 0$. Since $\frac{\delta}{q} > \sqrt{2\lambda_1}$, we have by Lemma 3.1 that

$$
\lim_{n \to +\infty} \inf_{x \in K} \frac{\log \pi^{\delta n\sigma}_{nq\sigma}(x)}{nq\sigma} = -\Lambda_{\frac{\delta}{q}} = -\frac{\delta^2}{2q^2}.
$$

Let $\epsilon \in (0, -\Lambda_\delta/8)$. The above equation combined with (3.25) and (3.29) implies that there is some constant $c_2 > 0$ such that

$$
\inf_{x \in K} u^{\delta n\sigma}(nq\sigma, x) \geq c_2 \mathrm{e}^{-\frac{\delta^2}{2q} n\sigma - \frac{1}{2}\epsilon nq\sigma} \qquad (3.30)
$$

for n sufficiently large. By the Markov property, we have

$$
\begin{aligned}
&\mathbb{P}_\mu \left(\mathcal{X}^{\delta n\sigma}_{n\sigma} \leq \mathrm{e}^{\left(-\Lambda_\delta - \frac{3}{2}\epsilon\right) n\sigma};\ X_{np\sigma}(K) \geq \mathrm{e}^{\left(\lambda_1 - \frac{1}{2}\epsilon\right) np\sigma} \right) \\
&= \mathbb{P}_\mu \left[\mathbb{P}_{X_{np\sigma}} \left[\mathrm{e}^{-\mathcal{X}^{\delta n\sigma}_{nq\sigma}} \geq \mathrm{e}^{-\mathrm{e}^{(-\Lambda_\delta - \frac{3}{2}\epsilon) n\sigma}} \right]; X_{np\sigma}(K) \geq \mathrm{e}^{(\lambda_1 - \frac{1}{2}\epsilon) np\sigma} \right] \\
&\leq \exp\{\mathrm{e}^{\left(-\Lambda_\delta - \frac{3}{2}\epsilon\right) n\sigma}\} \mathbb{P}_\mu \left[\mathrm{e}^{-\langle u^{\delta n\sigma}(nq\sigma, \cdot), X_{np\sigma}\rangle};\ X_{np\sigma}(K) \geq \mathrm{e}^{\left(\lambda_1 - \frac{1}{2}\epsilon\right) np\sigma} \right] \\
&\leq \exp\{\mathrm{e}^{\left(-\Lambda_\delta - \frac{3}{2}\epsilon\right) n\sigma}\} \mathbb{P}_\mu \left[\exp\{-c_2 \mathrm{e}^{-\frac{\delta^2}{2q} n\sigma - \frac{1}{2}\epsilon nq\sigma} X_{np\sigma}(K)\};\ X_{np\sigma}(K) \geq \mathrm{e}^{\left(\lambda_1 - \frac{1}{2}\epsilon\right) np\sigma} \right] \\
&\leq \exp\{\mathrm{e}^{\left(-\Lambda_\delta - \frac{3}{2}\epsilon\right) n\sigma} - c_2 \mathrm{e}^{-\frac{\delta^2}{2q} n\sigma - \frac{1}{2}\epsilon nq\sigma + (\lambda_1 - \frac{1}{2}\epsilon) np\sigma}\} \mathbb{P}_\mu \left[X_{np\sigma}(K) \geq \mathrm{e}^{\left(\lambda_1 - \frac{1}{2}\epsilon\right) np\sigma} \right]
\end{aligned}
$$

$$= \exp\left\{-\mathrm{e}^{\left(-\Lambda_\delta-\frac{3}{2}\epsilon\right)n\sigma}\left(c_2\mathrm{e}^{n\sigma\left(\sqrt{2\lambda_1}\delta+\epsilon-\lambda_1 q-\frac{\delta^2}{2q}\right)}-1\right)\right\}\mathbb{P}_\mu\left[X_{np\sigma}(K)\geq \mathrm{e}^{\left(\lambda_1-\frac{1}{2}\epsilon\right)np\sigma}\right].$$

The first and second inequalities follow from Chebyshev's inequality and (3.30), respectively. We choose $q\in\left(\left(\frac{\delta}{\sqrt{2\lambda_1}}+\frac{\epsilon-\sqrt{\epsilon^2+2\sqrt{2\lambda_1}\delta\epsilon}}{2\lambda_1}\right)\vee 0,\frac{\delta}{\sqrt{2\lambda_1}}\right)$ then $\sqrt{2\lambda_1}\delta+\epsilon-\lambda_1 q-\frac{\delta^2}{2q}>0$. Note that $-\Lambda_\delta-\frac{3}{2}\epsilon>-13\Lambda_\delta/16>0$ for $0<\delta<\sqrt{\lambda_1/2}$ and $\epsilon\in(0,-\Lambda_\delta/8)$. The above inequality implies that

$$\mathbb{P}_\mu\left(\mathcal{X}_{n\sigma}^{\delta n\sigma}\leq \mathrm{e}^{\left(-\Lambda_\delta-\frac{3}{2}\epsilon\right)n\sigma};\ X_{np\sigma}(K)\geq \mathrm{e}^{\left(\lambda_1-\frac{1}{2}\epsilon\right)np\sigma}\right)$$

decreases faster than exponentially as $n\to+\infty$. Thus,

$$\sum_{n=0}^{+\infty}\mathbb{P}_\mu\left(\mathcal{X}_{n\sigma}^{\delta n\sigma}\leq \mathrm{e}^{\left(-\Lambda_\delta-\frac{1}{2}\epsilon\right)n\sigma};\ X_{np\sigma}(K)\geq \mathrm{e}^{\left(\lambda_1-\frac{1}{2}\epsilon\right)np\sigma}\right)<+\infty,$$

and by Borel–Cantelli lemma,

$$\text{either }\frac{\log\mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma}>-\Lambda_\delta-\frac{1}{2}\epsilon\text{ or }\frac{\log X_{np\sigma}(K)}{np\sigma}<\lambda_1-\frac{1}{2}\epsilon \tag{3.31}$$

for n sufficiently large $\mathbb{P}_\mu$-a.s. Note that by Lemma 3.9

$$\mathbb{P}_\mu\left(\lim_{n\to+\infty}\frac{\log X_{np\sigma}(K)}{np\sigma}=\lambda_1\middle|W_\infty^h(X)>0\right)=1.$$

We get by (3.31) that

$$\mathbb{P}_\mu\left(\frac{\log\mathcal{X}_{n\sigma}^{\delta n\sigma}}{n\sigma}>-\Lambda_\delta-\frac{1}{2}\epsilon\quad\text{for } n \text{ sufficiently large}\middle|W_\infty^h(X)>0\right)=1.$$

This lemma follows by letting $\epsilon\downarrow 0$. □

Proposition 3.11 *For $0\leq\delta<\sqrt{\lambda_1/2}$ and $\mu\in\mathcal{M}_c(\mathbb{R})$,*

$$\liminf_{t\to+\infty}\frac{\log\mathcal{X}_t^{\delta t}}{t}\geq-\Lambda_\delta\quad \mathbb{P}_\mu\text{-a.s. on }\{W_\infty^h(X)>0\}.$$

Proof First we consider $\delta=0$. Since

$$W_t^h(X)=\mathrm{e}^{-\lambda_1 t}\langle h,X_t\rangle\leq\|h\|_\infty\mathrm{e}^{-\lambda_1 t}\mathcal{X}_t^0,$$

we have

$$\frac{\log\mathcal{X}_t^0}{t}\geq\frac{\log W_t^h(X)}{t}-\frac{\log\|h\|_\infty}{t}+\lambda_1.$$

Since $W_t^h(X) \to W_\infty^h(X)$ $\mathbb{P}_\mu$-a.s., we get that

$$\liminf_{t\to+\infty} \frac{\log \mathcal{X}_t^0}{t} \geq \lambda_1 = -\Lambda_0 \quad \mathbb{P}_\mu\text{-a.s. on } \{W_\infty^h(X) > 0\}.$$

Now, suppose $0 < \delta < \sqrt{\lambda_1/2}$. Let $\sigma > 0$. We take $\theta > 0$ small such that $\delta_\theta := \delta + \theta < \sqrt{\lambda_1/2}$. By (3.5), we have

$$\mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} \middle| \mathcal{F}_{n\sigma} \right] \geq c_1 \mathcal{X}_{n\sigma}^{\delta_\theta n\sigma}$$

for some constant $c_1 = c_1(\theta, \sigma) > 0$. It then follows by Lemma 3.10 that

$$\mathbb{P}_\mu \left(\liminf_{n\to+\infty} \frac{\log \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right]}{n\sigma} \geq -\Lambda_{\delta_\theta} \middle| W_\infty^h(X) > 0 \right) = 1. \tag{3.32}$$

Let $0 < \epsilon < -\Lambda_{\delta_\theta}/4$. We have

$$\begin{aligned}
&\sup_{t\in[n\sigma,(n+1)\sigma)} e^{\Lambda_{\delta_\theta} t} \left| \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] - \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \right| \\
&\leq \sup_{t\in[n\sigma,(n+1)\sigma)} e^{\Lambda_{\delta_\theta} n\sigma} \left| \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] - \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \right| \\
&= e^{-\left(-\frac{1}{2}\Lambda_{\delta_\theta} - \epsilon\right) n\sigma - \left(\frac{1}{2}\Lambda_{\delta_\theta} - \epsilon\right)\sigma} \\
&\quad \times \sup_{t\in[n\sigma,(n+1)\sigma)} e^{\left(\frac{1}{2}\Lambda_{\delta_\theta} - \epsilon\right)(n+1)\sigma} \left| \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] - \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \right|.
\end{aligned}$$

By Lemma 3.5, the final term in the right-hand side converges to 0 as $n \to +\infty$. Thus, we get

$$\lim_{n\to+\infty} \sup_{t\in[n\sigma,(n+1)\sigma)} e^{\Lambda_{\delta_\theta} t} \left| \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] - \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \right| = 0 \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.33}$$

Note that for any $t \in [n\sigma, (n+1)\sigma)$,

$$\begin{aligned}
&\mathrm{e}^{\Lambda_{\delta_\theta} t} \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] \\
&\geq \mathrm{e}^{\Lambda_{\delta_\theta} (n+1)\sigma} \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \\
&\quad - \sup_{t\in[n\sigma,(n+1)\sigma)} e^{\Lambda_{\delta_\theta} t} \left| \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_t \right] - \mathbb{P}_\mu \left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta (n+1)\sigma} | \mathcal{F}_{n\sigma} \right] \right|
\end{aligned}$$

Hence, by (3.32) and (3.33) we get that

$$\mathbb{P}_\mu\left(\liminf_{t\to+\infty}\frac{\log\mathbb{P}_\mu\left[\mathcal{X}^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]}{t}\geq-\Lambda_{\delta_\theta}\middle|W^h_\infty(X)>0\right)=1. \tag{3.34}$$

By Markov property, for any $t\in[n\sigma,(n+1)\sigma)$,

$$\mathbb{P}_\mu\left[\mathcal{X}^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]=\langle\pi^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma-t},X_t\rangle.$$

So we have

$$\begin{aligned}\mathrm{I}(\delta_\theta,t)&:=\langle\pi^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma-t}\mathbf{1}_{(-\delta t,\delta t)^c},X_t\rangle\\&=\mathbb{P}_\mu\left[\mathcal{X}^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]-\langle\pi^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma-t}\mathbf{1}_{(-\delta t,\delta t)},X_t\rangle\\&=:\mathbb{P}_\mu\left[\mathcal{X}^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma}|\mathcal{F}_t\right]-\mathrm{II}(\delta_\theta,t).\end{aligned}\tag{3.35}$$

Lemma 3.3(i) implies that there is a constant $c_2>0$ independent of δ_θ and θ such that

$$\pi^{\delta_\theta(n+1)\sigma}_{(n+1)\sigma-t}(y)\leq c_2(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma}\quad\forall t\in[n\sigma,(n+1)\sigma),\ |y|<\delta t,$$

when n is sufficiently large. Hence, we get that for $t\in[n\sigma,(n+1)\sigma)$,

$$\begin{aligned}\mathrm{II}(\delta_\theta,t)&\leq c_2(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma}\langle\mathbf{1}_{(-\delta t,\delta t)},X_t\rangle\\&\leq\frac{c_2(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma+\lambda_1 t}}{\inf_{|y|<\delta(n+1)\sigma}h(y)}\langle\mathrm{e}^{-\lambda_1 t}h\mathbf{1}_{(-\delta t,\delta t)},X_t\rangle\\&\leq\frac{c_2(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma+\lambda_1 t}}{\inf_{|y|<\delta(n+1)\sigma}h(y)}W^h_t(X).\end{aligned}$$

By (1.6), there is a constant $c_3>0$ such that when n is sufficiently large,

$$\mathrm{II}(\delta_\theta,t)\leq c_3(\theta(n+1)\sigma)^{-1}\mathrm{e}^{-\frac{\theta^2}{2}(n+1)^2\sigma+(\lambda_1+\sqrt{2\lambda_1}\delta)(n+1)\sigma}W^h_t(x)$$

for all $t\in[n\sigma,(n+1)\sigma)$. This implies that

$$\lim_{n\to+\infty}\sup_{t\in[n\sigma,(n+1)\sigma)}\mathrm{e}^{\Lambda_{\delta_\theta}t}\mathrm{II}(\delta_\theta,t)=0\quad\mathbb{P}_\mu\text{-a.s.}\tag{3.36}$$

Note that by (3.35)

$$\mathrm{e}^{\Lambda_{\delta_\theta}t}\mathrm{I}(\delta_\theta,t)=\mathrm{e}^{t\left(\log\mathrm{I}(\delta_\theta,t)/t+\Lambda_{\delta_\theta}\right)}$$

$$= \mathrm{e}^{t\left(\log \mathbb{P}_\mu\left[\mathcal{X}_{(n+1)\sigma}^{\delta_\theta(n+1)\sigma}\middle|\mathcal{F}_t\right]/t+\Lambda_{\delta_\theta}\right)} - \mathrm{e}^{\Lambda_{\delta_\theta}t}\mathrm{II}(\delta_\theta, t).$$

This together with (3.34) and (3.36) implies that

$$\mathbb{P}_\mu\left(\liminf_{t\to+\infty}\frac{\log \mathrm{I}(\delta_\theta,t)}{t} \geq -\Lambda_{\delta_\theta}\middle|W_\infty^h(X)>0\right) = 1. \tag{3.37}$$

Note that by definition

$$\mathrm{I}(\delta_\theta,t) \leq \|\pi_{(n+1)\sigma-t}^{\delta_\theta(n+1)\sigma}\|_\infty \mathcal{X}_t^{\delta t} \leq \mathrm{e}^{\|\beta^+\|_\infty\sigma}\mathcal{X}_t^{\delta t} \quad \forall t\in[n\sigma,(n+1)\sigma).$$

By (3.37), we have

$$\mathbb{P}_\mu\left(\liminf_{t\to+\infty}\frac{\log \mathcal{X}_t^{\delta t}}{t} \geq -\Lambda_{\delta_\theta}|W_\infty^h(X)>0\right) = 1.$$

□

Proof of Theorem 1.1: Theorem 1.1 follows immediately from Propositions 3.8 and 3.11. □

4 Proofs of Theorem 1.2 and Theorem 1.3

4.1 Skeleton Decomposition

In this subsection, we shall establish the skeleton space for the (B_t,ψ)-superprocess. The following condition is fundamental for the skeleton construction.

There is a locally bounded function $w>0$ on $\mathbb{R}$ satisfying that

$$\mathbb{P}_\mu\left[\mathrm{e}^{-\langle w,X_t\rangle}\right] = \mathrm{e}^{-\langle w,\mu\rangle} \quad \forall\mu\in\mathcal{M}_c(\mathbb{R}). \tag{A2}$$

This locally bounded martingale function w assures that

$$\left(w(B_t)\exp\{-\int_0^t\frac{\psi(B_s,w(B_s))}{w(B_s)}\mathrm{d}s\}\right)_{t\geq0}$$

is a Π_x-(super)martingale. Thus, one can define a family of (sub)probability measures $\{\Pi_x^w, x\in\mathbb{R}\}$ by

$$\left.\frac{\mathrm{d}\Pi_x^w}{\mathrm{d}\Pi_x}\right|_{\sigma(B_s:s\in[0,t])} := \frac{w(B_t)}{w(x)}\exp\left\{-\int_0^t\frac{\psi(B_s,w(B_s))}{w(B_s)}\mathrm{d}s\right\} \quad \forall t\geq0.$$

We denote the process $((B_t)_{t\geq0}, \Pi_x^w, x\in\mathbb{R})$ by $(B_t^w)_{t\geq0}$.

An integer-valued locally finite random measure ξ on $\mathbb{R}$ is called a point process. If there is a locally finite measure λ on $\mathbb{R}$ such that $\xi(B)$ is Poisson distributed with mean $\lambda(B)$ for any Borel set B, and that $\xi(B_1), \cdots, \xi(B_n)$ are independent for any disjoint Borel sets $B_1, \cdots, B_n$, $n \geq 2$, then ξ is called *Poisson point process* with intensity λ. If we randomize by replacing the fixed measure λ by a random measure Λ on $\mathbb{R}$, then we get a *Cox process* directed by Λ. More precisely, given Λ, ξ is conditionally Poisson with intensity Λ almost surely.

Proposition 4.1 *Assume (A2) holds. For every $\mu \in \mathcal{M}(\mathbb{R})$, there exists a probability space with probability measure P_μ that carries two processes $(Z_t)_{t\geq 0}$ and $(\widehat{X}_t)_{t\geq 0}$ satisfying the following conditions.*

(i) *$((Z_t)_{t\geq 0}, \mathrm{P}_\mu)$ is branching Markov process with Z_0 being a Poisson point process with intensity $w(x)\mu(\mathrm{d}x)$, in which each particle moves independently as a copy of $(B_t^w)_{t\geq 0}$, and a particle at location x dies at rate $q(x)$ and is replaced by a random number of offspring with distribution $\{p_k(x) : k \geq 2\}$ uniquely identified by*

$$G(x,s) := q(x)\sum_{k=2}^{+\infty} p_k(x)(s^k - s) = \frac{1}{w(x)}\left[\psi(x, w(x)(1-s)) - (1-s)\psi(x, w(x))\right].$$

(ii) *$((\widehat{X}_t)_{t\geq 0}, \mathrm{P}_\mu)$ has the same distribution as $(X, \mathbb{P}_\mu)$.*
(iii) *For every $t \geq 0$, Z_t is a Cox process directed by $w\widehat{X}_t$.*

We show in the next proposition that the martingale function w in (A2) exists for the (B_t, ψ)-superprocess. Recall that $W_t^h(X) := \mathrm{e}^{-\lambda_1 t}\langle h, X_t\rangle, \quad \forall t \geq 0$.

Proposition 4.2 *Let $\mathcal{E} := \{W_\infty^h(X) = 0\}$ and $w(x) := -\log \mathbb{P}_{\delta_x}(\mathcal{E})$ for $x \in \mathbb{R}$. Then, w is a bounded positive function satisfying (A2). Moreover, $w'(x) = 0$ for $|x|$ sufficiently large.*

Proof Since $W_\infty^h(X)$ is nondegenerate under $\mathbb{P}_{\delta_x}$, $w(x) = -\log \mathbb{P}_{\delta_x}(\mathcal{E})$ takes values in $(0, +\infty]$. By (1.1), for any $\mu \in \mathcal{M}(\mathbb{R})$, $t \geq 0$ and $\lambda > 0$,

$$\mathbb{P}_\mu\left[\mathrm{e}^{-\lambda W_t^h(X)}\right] = \mathrm{e}^{-\left\langle u_{\lambda \mathrm{e}^{-\lambda_1 t} h}(t,\cdot), \mu\right\rangle},$$

where for $x \in \mathbb{R}$, $u_{\lambda \mathrm{e}^{-\lambda_1 t} h}(t,x) = -\log \mathbb{P}_{\delta_x}\left[\exp\{-\lambda W_t^h(X)\}\right]$. By letting $t \to +\infty$ and then $\lambda \to +\infty$ in the above equation, we get

$$\mathbb{P}_\mu(\mathcal{E}) = \mathrm{e}^{-\langle w, \mu\rangle} \quad \forall \mu \in \mathcal{M}(\mathbb{R}).$$

Using this and the Markov property of X_t, we have

$$\mathbb{P}_\mu\left[\mathrm{e}^{-\langle w, X_t\rangle}\right] = \mathbb{P}_\mu\left[\mathbb{P}_{X_t}(\mathcal{E})\right] = \mathbb{P}_\mu(\mathcal{E}) = \mathrm{e}^{-\langle w, \mu\rangle}.$$

So w satisfies (A2). We only need to show that w is a bounded function on $\mathbb{R}$ since the second assertion is a direct result of Lemma A.1 and the boundedness of w.

Let $\mathcal{E}_{\text{ext}} := \{\|X_t\| = 0 \text{ for } t \text{ large enough}\}$ and $w_{\text{ext}}(x) := -\log \mathbb{P}_{\delta_x}(\mathcal{E}_{\text{ext}})$ for $x \in \mathbb{R}$. Since $\psi(x,\lambda) \geq -\beta(x)\lambda + \alpha(x)\lambda^2 =: \hat{\psi}(x,\lambda)$ for $x \in \mathbb{R}$ and $\lambda \geq 0$, it follows by [24, Corollary 5.18] that the extinction probability of the (B_t, ψ)-superprocess is larger than that of the $(B_t, \hat{\psi})$-superprocess. Let $\hat{w}_{\text{ext}}$ be the log-Laplace exponent of the $(B_t, \hat{\psi})$-superprocess. Let $\emptyset \neq \mathcal{O} := \{x \in \mathbb{R} : \alpha(x) > 0\}$. By [16, Lemma 7.1], $\hat{w}_{\text{ext}}(x)$ is a locally bounded function on $\mathcal{O}$. Since $\mathcal{E}_{\text{ext}} \subseteq \mathcal{E}$, one has $w(x) \leq w_{\text{ext}}(x) \leq \hat{w}_{\text{ext}}(x) < +\infty$ for all $x \in \mathcal{O}$. In the remaining of this proof, we fix an arbitrary $c \in \mathcal{O}$ and $x_0 \in \mathbb{R} \setminus \mathcal{O}$. Without loss of generality, we assume $x_0 > c$.

We recall from [12] (see also [13, 14]) that the (B_t, ψ)-superprocess $(X_t)_{t \geq 0}$ can also be modeled as a system of exit measures from time-space open sets. In particular, branching property and Markov properties of such systems are established there. From this perspective, X_t can be viewed as the projection of the exit measure from $(0,t) \times \mathbb{R}$ on $\{t\} \times \mathbb{R}$. Let $\mathcal{H}$ be the set of nonnegative bounded functions on $[0,+\infty) \times \mathbb{R}$ satisfying that there is some S such that $f(s,y) = 0$ for all $(s,y) \in [S,+\infty) \times \mathbb{R}$. By [12, Theorem I.1.1 and Theorem I.1.2], for $t \in (0,+\infty)$ and $Q_t := (0,t) \times (c,+\infty)$, there exists a finite random measure X^{Q_t}, called the exit measure from Q_t, which is supported on the boundary of Q_t and satisfies that for every $f \in \mathcal{H}$,

$$\mathbb{P}_\mu\left[\mathrm{e}^{-\langle f, X^{Q_t}\rangle}\right] = \mathrm{e}^{-\langle U_f(t,\cdot),\mu\rangle},$$

where $U_f(t,x)$ is a solution to the following integral equation

$$U_f(t,x) + \Pi_x\left[\int_0^{\tau_c \wedge t} \psi(B_s, U_f(t-s, B_s))\right] = \Pi_x\left[f(\tau_c \wedge t, B_{\tau_c \wedge t})\right]. \quad (4.1)$$

Here, τ_c denotes the first exit time of $(B_t)_{t \geq 0}$ from $(c,+\infty)$. Let $X_t^c(A) := X^{Q_t}(\{t\} \times (A \cap (c,+\infty)))$ for any $A \subset \mathbb{R}$. This definition implies that X_t^c is the projection of X^{Q_t} on $\{t\} \times (c,+\infty)$. Let $u_g^c(t,x) := -\log \mathbb{P}_{\delta_x}\left[\exp\{-\langle g, X_t^c\rangle\}\right]$ for $g \in \mathcal{B}_b^+(\mathbb{R})$, $t \geq 0$ and $x \in \mathbb{R}$. By (4.1), $u_g^c(t,x)$ satisfies the following integral equation.

$$u_g^c(t,x) + \Pi_x\left[\int_0^t \psi(B_s^c, u_g^c(t-s, B_s^c))\mathrm{d}s\right] = \Pi_x\left[g(B_t^c)\right],$$

where $(B_t^c)_{t \geq 0}$ denotes the Brownian motion killed outside $(c,+\infty)$. This implies that $(X_t^c)_{t \geq 0}$ is a (B_t^c, ψ)-superprocess. Note that

$$\begin{aligned}\mathbb{P}_{\delta_{x_0}}\left[\mathrm{e}^{-\lambda_1 t}\langle h, X_t^c\rangle\right] &= \mathrm{e}^{-\lambda_1 t}\Pi_{x_0}\left[e_\beta(t)h(B_t); t < \tau_c\right] \\ &= \Pi_{x_0}^h(t < \tau_c).\end{aligned}$$

Here, $\Pi^h_{x_0}$ is the probability measure defined by

$$\frac{\mathrm{d}\Pi^h_{x_0}}{\mathrm{d}\Pi_{x_0}}\bigg|_{\sigma(B_s:s\le t)} = \mathrm{e}^{-\lambda_1 t} e_\beta(t)\frac{h(B_t)}{h(x_0)} \quad \forall t \ge 0.$$

It is known that (cf. [9, Section 3]) $((B_t)_{t\ge 0}, \Pi^h_{x_0})$ is a recurrent diffusion on $\mathbb{R}$. So $\Pi^h_{x_0}(t < \tau_c) \to 0$ as $t \to +\infty$. This implies that $\mathrm{e}^{-\lambda_1 t}\langle h, X^c_t\rangle$ converges to 0 in $L^1(\mathbb{P}_{\delta_{x_0}})$, and so there is a subsequence of $\mathrm{e}^{-\lambda_1 t}\langle h, X^c_t\rangle$ which converges to 0 $\mathbb{P}_{\delta_{x_0}}$-a.s.

On the other hand, we note that $\|X^{Q_t}((0,+\infty)\times\{c\})\|$ denotes the total mass of the projection of X^{Q_t} on $(0,t]\times\{c\}$. For $\lambda, t \ge 0$ and $y \in \mathbb{R}$, let $v^c_\lambda(t,y) := -\log \mathbb{P}_{\delta_y}\left[\exp\{-\lambda\|X^{Q_t}((0,+\infty)\times\{c\})\|\}\right]$. It follows by (4.1) that

$$\begin{aligned} v^c_\lambda(t,x) &= \lambda\Pi_x(\tau_c \le t) - \Pi_x\left[\int_0^{\tau_c\wedge t} \psi(B_s, v^c_\lambda(t-s, B_s))\mathrm{d}s\right] \\ &= \lambda\Pi_x\left[e_\beta(\tau_c\wedge t); \tau_c \le t\right] - \Pi_x\left[\int_0^{\tau_c\wedge t} e_\beta(s)\psi_0(B_s, v^c_\lambda(t-s, B_s))\mathrm{d}s\right], \end{aligned} \tag{4.2}$$

where $\psi_0(x,\lambda) = \psi(x,\lambda) + \beta(x)\lambda$. The second equation follows from [15, Lemma A.1].

Let $\|X^{\{c\}}\|$ be the limit of the nondecreasing sequence $\{\|X^{Q_t}((0,+\infty)\times\{c\})\| : t \ge 0\}$ and $v^c_\lambda(x) := -\log \mathbb{P}_{\delta_x}\left[\exp\{-\lambda\|X^{\{c\}}\|\}\right] = \lim_{t\to+\infty} v^c_\lambda(t,x)$ for $\lambda \ge 0$ and $x \ge c$. By (4.2), one has $v^c_\lambda(t,x) \le \lambda\Pi_x\left[e_\beta(\tau_c); \tau_c \le t\right]$, and so $v^c_\lambda(x) \le \lambda\Pi_x\left[e_\beta(\tau_c)\right]$. Since β is compactly supported, by [11, Theorem 9.22] $x \mapsto \Pi_x\left[e_\beta(\tau_c)\right]$ is a bounded function on $[c,+\infty)$. Thus for every $\lambda \ge 0$, $x \mapsto v^c_\lambda(x)$ is a bounded function on $[c,+\infty)$. We note that $\mathcal{E} = \{W^h_\infty(X) = 0\} = \{\exists\, t_n \to +\infty$ such that $\mathrm{e}^{-\lambda_1 t_n}\langle h, X_{t_n}\rangle \to 0\}$. We have

$$\begin{aligned} \mathrm{e}^{-w(x_0)} = \mathbb{P}_{\delta_{x_0}}(\mathcal{E}) &= \mathbb{P}_{\delta_{x_0}}\left[\mathbb{P}_{\delta_{x_0}}\left(\mathcal{E}|X^{Q_t}\right)\right] \\ &\ge \mathbb{P}_{\delta_{x_0}}\left[\mathbb{P}_{\|X^{\{c\}}\|\delta_c}(\mathcal{E}_{\mathrm{ext}})\right] \\ &= \mathbb{P}_{\delta_{x_0}}\left[\mathrm{e}^{-w_{\mathrm{ext}}(c)\|X^{\{c\}}\|}\right] = \mathrm{e}^{-v^c_{w_{\mathrm{ext}}(c)}(x_0)}. \end{aligned}$$

Thus, one gets $w(x_0) \le v^c_{w_{\mathrm{ext}}(c)}(x_0)$ and so w is a bounded function on $[c,+\infty)$. $\square$

4.2 Limiting Distributions for the Skeleton

Since $(\widehat{X}; \mathrm{P}_\mu)$ is equal in distribution to the (B_t, ψ)-superprocess, we may work on this skeleton space whenever it is convenient. For notational simplification, we will abuse the notation and denote $\widehat{X}$ by X. We will refer to $(Z_t)_{t\ge 0}$ as the skeleton branching diffusion (skeleton) of X. We use $u \in Z_t$ to denote a particle of the skeleton which is alive at time t, and $z_u(t)$ for its spatial location. We use $\|Z_t\|$ to denote the total number of particles alive at time t.

In this section, we shall show that the skeleton branching diffusion Z_t shifted by $\sqrt{\lambda_1/2}\,t$ converges in distribution to a Cox process directed by a random measure which has a random intensity mixed by the limit of an additive martingale (see Proposition 4.7). Our proof follows the same approach as [6] (see, also [26]): First, we represent the population moments in terms of Feynman–Kac functionals associated to Brownian motions, see (4.3) and (4.4). Using the estimates established in Sect. 2, we show in Lemma 4.3 that the second-order moment is asymptotically the same as the first-order moment. Combining this with the Chebyshev and Paley–Zygmund inequalities, we compute the asymptotic behavior of the distributions of particles near $\sqrt{\lambda_1/2}\,t$ in Lemma 4.4. We can then follow the argument of [6] to establish Proposition 4.6.

Recall that $w'(x) = 0$ when $|x|$ is large. So we assume that there are constants $M, w_\pm > 0$ such that $w(x) = w_-$ for $x \geq M$ and $w(x) = w_+$ for $x \leq -M$.

In what follows, we always assume the following:

(1) $R(t) = \delta t + a(t)$ where $\delta \in (0, \sqrt{2\lambda_1})$ and $a(t) = o(t)$ as $t \to +\infty$.
(2) For some $a \in \left(0, 1 - \frac{\delta}{\sqrt{2\lambda_1}}\right)$, $0 \leq s(t) < at$ for all $t \geq 0$ and $s(t) = o(t)$ as $t \to +\infty$.
(3) $b(t) \geq 0$ for all $t \geq 0$ and $b(t) = o(t)$ as $t \to +\infty$.
(4) $x(\cdot) : [0, +\infty) \to \mathbb{R}$ satisfies $|x(t)| \leq b(t)$ for all large $t > 0$.
(5) A is a Borel set of $\mathbb{R}$ with $\inf A > -\infty$.

We use $\mathbf{P}_\nu$ to denote the probability measure where the branching Markov process $(Z_t)_{t\geq 0}$ started from the integer-valued measure ν. For every $x \in \mathbb{R}$, the first two moments of $((Z_t)_{t\geq 0}, \mathbf{P}_{\delta_x})$ can be expressed by the spatial motion and the branching rate (cf. [32, Lemma 3.3]): For $f \in \mathcal{B}_b^+(\mathbb{R})$,

$$\mathbf{P}_{\delta_x}\left[\langle f, Z_t\rangle\right] = \Pi_x^w\left[\mathrm{e}^{\int_0^t \frac{\partial}{\partial s}G(B_r^w,1)\mathrm{d}r} f(B_t^w)\right] = \frac{1}{w(x)}P_t^\beta(wf)(x), \tag{4.3}$$

and

$$\begin{aligned}
&\mathbf{P}_{\delta_x}\left[\langle f, Z_t\rangle^2\right] \\
&= \mathbf{P}_{\delta_x}\left[\langle f^2, Z_t\rangle\right] + \Pi_x^w\left[\int_0^t \mathrm{e}^{\int_0^r \frac{\partial}{\partial s}G(B_u^w,1)\mathrm{d}u}\frac{\partial^2}{\partial s^2}G(B_r^w,1)\mathbf{P}_{\delta_{B_r^w}}\left[\langle f, Z_{t-r}\rangle\right]^2\mathrm{d}r\right] \\
&= \frac{1}{w(x)}P_t^\beta(wf^2)(x) + \frac{1}{w(x)}\int_0^t P_s^\beta\left[2\gamma\left(P_{t-s}^\beta(wf)\right)^2\right](x)\mathrm{d}s.
\end{aligned} \tag{4.4}$$

One can easily show by (4.3) that

$$W_t^{h/w}(Z) := \mathrm{e}^{-\lambda_1 t}\left\langle \frac{h}{w}, Z_t\right\rangle, \quad t \geq 0,$$

is a nonnegative $\mathbf{P}_{\delta_x}$-martingale for every $x \in \mathbb{R}$, and a nonnegative P_μ-martingale for every $\mu \in \mathcal{M}(\mathbb{R})$. We use $W_\infty^{h/w}(Z)$ to denote the martingale limit. It is proved by

[15, Proposition 1.1] that

$$W_\infty^{h/w}(Z) = W_\infty^h(X) \quad \mathrm{P}_\mu\text{-a.s.} \tag{4.5}$$

for all $\mu \in \mathcal{M}(\mathbb{R})$.

Lemma 4.3 *(i) There exist $T_1 > 0$ and $\theta_i(t)$ $(i = 1, 2)$ such that for $t \geq T_1$,*

$$\theta_1(t) \leq \frac{\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right]}{w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\mathrm{e}^{\lambda_1(t-s(t))-\sqrt{2\lambda_1}R(t)}} \leq \theta_2(t), \tag{4.6}$$

where $\eta_-(dx) = C_-\mathrm{e}^{-\sqrt{2\lambda_1}x}\mathrm{d}x$, and for $i = 1, 2$, $\theta_i(t) \to 1$ as $t \to +\infty$.
(ii) There exist $T_2, C > 0$ such that for $t \geq T_2$,

$$\begin{aligned}\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right] &\leq \mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))^2\right] \\ &\leq \mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right] + C\frac{h(x(t))}{w(x(t))}\mathrm{e}^{2\lambda_1(t-s(t))-2\sqrt{2\lambda_1}R(t)}.\end{aligned} \tag{4.7}$$

Proof (i) We have

$$\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right] = \frac{1}{w(x(t))}P^\beta_{t-s(t)}\left(w\mathbf{1}_{A+R(t)}\right)(x(t)).$$

Note that for t large enough such that $R(t) + (\inf A \wedge 0) \geq M$, $w(x) = w_-$ for all $x \in A + R(t)$. It follows that

$$\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right] = \frac{w_-}{w(x(t))}P^\beta_{t-s(t)}\mathbf{1}_{A+R(t)}(x(t)).$$

Thus, (4.6) follows immediately from Lemma 2.2.

(ii) The first inequality of (4.7) is obvious since, by (4.4),

$$\begin{aligned}\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))^2\right] &= \mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A + R(t))\right] \\ &\quad + \frac{2}{w(x(t))}\int_0^{t-s(t)} P^\beta_r\left[\gamma P^\beta_{t-s(t)-r}\left(w\mathbf{1}_{A+R(t)}\right)^2\right](x(t))\mathrm{d}r.\end{aligned} \tag{4.8}$$

Suppose supp$\gamma \subset [-k, k]$ for some $0 < k < +\infty$. Let σ_k be the first hitting time of $[-k, k]$ by the Brownian motion. Noting that $s(t) \leq at < t - 1$ for t sufficiently large, we have

$$\begin{aligned}&\int_0^{t-s(t)} P^\beta_r\left[\gamma\left(P^\beta_{t-s(t)-r}\mathbf{1}_{A+R(t)}\right)^2\right](x(t))\mathrm{d}r \\ &= \Pi_{x(t)}\left[\int_0^{t-s(t)} e_\beta(r)\gamma(B_r)\left(P^\beta_{t-s(t)-r}\mathbf{1}_{A+R(t)}(B_r)\right)^2\mathrm{d}r\right]\end{aligned}$$

$$
\begin{aligned}
&= \Pi_{x(t)}\left[\int_{\sigma_k}^{t-s(t)} e_\beta(r)\gamma(B_r)\left(P^\beta_{t-s(t)-r}\mathbf{1}_{A+R(t)}(B_r)\right)^2\mathrm{d}r;\, \sigma_k\le t-s(t)\right]\\
&= \Pi_{x(t)}\Big[e_\beta(\sigma_k)\ \Pi_{B_u}\left[\int_0^{t-s(t)-u} e_\beta(r)\gamma(B_r)\left(P^\beta_{t-s(t)-u-r}\mathbf{1}_{A+R(t)}(B_r)\right)^2\mathrm{d}r\right]\Big|_{u=\sigma_k};\\
&\qquad \sigma_k\le t-s(t)\Big]\\
&\le c_1\Pi_{x(t)}\left[e_\beta(\sigma_k)\frac{h(B_{\sigma_k})}{\inf_{x\in[-k,k]}h(x)}\mathrm{e}^{2\lambda_1(t-s(t)-\sigma_k)-2\sqrt{2\lambda_1}R(t)};\, \sigma_k\le t-s(t)\right]\\
&= c_2\mathrm{e}^{2\lambda_1(t-s(t))-2\sqrt{2\lambda_1}R(t)}\Pi_{x(t)}\left[e_\beta(\sigma_k)h(B_{\sigma_k})\mathrm{e}^{-2\lambda_1\sigma_k};\,\sigma_k\le t-s(t)\right].
\end{aligned}\tag{4.9}
$$

The above inequality follows from Lemma 2.1(ii). Since $\mathrm{e}^{-\lambda_1 t}\mathrm{e}_\beta(t)h(B_t)$ is a martingale, by the optional stopping theorem, the last term in (4.9) is no larger than $h(x(t))$. So we get that

$$
\int_0^{t-s(t)} P^\beta_r\left[\gamma\left(P^\beta_{t-s(t)-r}\mathbf{1}_{A+R(t)}\right)^2\right](x(t))\mathrm{d}r \le c_3\mathrm{e}^{2\lambda_1(t-s(t))-2\sqrt{2\lambda_1}R(t)}h(x(t)).
$$

We also note that for t large enough, $w(x)=w_-$ for all $x\in A+R(t)$. Thus,

$$
\begin{aligned}
&\int_0^{t-s(t)} P^\beta_r\left[\gamma P^\beta_{t-s(t)-r}\left(w\mathbf{1}_{A+R(t)}\right)^2\right](x(t))\mathrm{d}r\\
&= w_-^2\int_0^{t-s(t)} P^\beta_r\left[\gamma\left(P^\beta_{t-s(t)-r}\mathbf{1}_{A+R(t)}\right)^2\right](x(t))\mathrm{d}r\\
&\le c_3 w_-^2\mathrm{e}^{2\lambda_1(t-s(t))-2\sqrt{2\lambda_1}R(t)}h(x(t)).
\end{aligned}
$$

Putting this back to (4.8), we get (4.7). □

Lemma 4.4 *Assume that $\delta=\sqrt{\lambda_1/2}$ and that $\lambda_1 s(t)+\sqrt{2\lambda_1}a(t)\to+\infty$ as $t\to+\infty$. Then, there exist $C,T>0$ and $\theta_i(t)$ ($i=4,5,6,7$) such that for $t\ge T$,*

$$
\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right)\le 1-\theta_4(t)w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t), \tag{4.10}
$$

$$
\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right)\ge 1-\theta_5(t)w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t), \tag{4.11}
$$

$$
\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=1\right)\le \theta_6(t)w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t), \tag{4.12}
$$

$$
\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=1\right)\ge \theta_7(t)w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t), \tag{4.13}
$$

$$
\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))\ge 2\right)\le C\frac{h(x(t))}{w(x(t))}\Theta^2(t). \tag{4.14}
$$

where $\Theta(t)=\mathrm{e}^{-\lambda_1 s(t)-\sqrt{2\lambda_1}a(t)}$ and $\theta_i(t)\to 1$ as $t\to+\infty$ for $i=4,5,6,7$.

Proof We note that if Z is an integer-valued random variable, then

$$\frac{E[Z]^2}{E[Z^2]} \le P(Z>0) = P(Z\ge 1) \le E[Z], \tag{4.15}$$

and

$$P(Z\ge 2) \le E[Z(Z-1); Z\ge 2] = E[Z(Z-1)] = E[Z^2]-E[Z]. \tag{4.16}$$

It is easy to see that (4.14) follows immediately from (4.16) and Lemma 4.3(ii). Since $1-\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right) = \mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))>0\right)$, we have by (4.15) and Lemma 4.3 that for t large enough,

$$\begin{aligned} 1-\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right) &\le \mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A+R(t))\right] \\ &\le \theta_2(t) w_- \eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t), \end{aligned}$$

and

$$\begin{aligned} &1-\mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right) \\ &\ge \frac{\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A+R(t))\right]^2}{\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A+R(t))^2\right]} \\ &\ge \frac{\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A+R(t))\right]^2}{\mathbf{P}_{\delta_{x(t)}}\left[Z_{t-s(t)}(A+R(t))\right] + C\frac{h(x(t))}{w(x(t))}\Theta^2(t)} \\ &\ge \frac{\left[\theta_1(t) w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t)\right]^2}{\theta_2(t) w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t) + C\frac{h(x(t))}{w(x(t))}\Theta^2(t)} \\ &= \frac{\theta_1(t)^2}{\theta_2(t)+Cw_-^{-1}\eta_-^{-1}(A)\Theta(t)} w_-\eta_-(A)\frac{h(x(t))}{w(x(t))}\Theta(t). \end{aligned}$$

Since $\theta_i(t)\to 1$ for $i=1,2$ and $\Theta(t)\to 0$ as $t\to+\infty$, $\frac{\theta_1(t)^2}{\theta_2(t)+Cw_-^{-1}\eta_-^{-1}(A)\Theta(t)}\to 1$ as $t\to+\infty$. Hence, we prove (4.10) and (4.11).

We note that

$$\begin{aligned} \mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=1\right) = 1 &- \mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))=0\right) \\ &- \mathbf{P}_{\delta_{x(t)}}\left(Z_{t-s(t)}(A+R(t))\ge 2\right). \end{aligned}$$

Thus, (4.12) and (4.13) follow immediately from (4.10), (4.11) and (4.14). $\square$

Lemma 4.5 *(i) For every $\delta\in(\sqrt{\lambda_1/2},\sqrt{2\lambda_1})$ and $x\in\mathbb{R}$,*

$$\lim_{t\to+\infty}\mathbf{P}_{\delta_x}\left(\max_{u\in Z_t}|z_u(t)|<\delta t\right)=1.$$

(ii) For every $x \in \mathbb{R}$,

$$\liminf_{t\to+\infty} \mathrm{e}^{-\lambda_1 t}\|Z_t\| > 0 \quad \mathbf{P}_{\delta_x}\text{-a.s.}$$

Proof (i) We have

$$\begin{aligned}\mathbf{P}_{\delta_x}\left(\max_{u\in Z_t}|z_u(t)| < \delta t\right) &= \mathbf{P}_{\delta_x}\left(Z_t((-\delta t, \delta t)^c) = 0\right)\\ &= 1 - \mathbf{P}_{\delta_x}\left(Z_t((-\delta t, \delta t)^c) \geq 1\right)\\ &\geq 1 - \mathbf{P}_{\delta_x}\left[Z_t((-\delta t, \delta t)^c)\right].\end{aligned}$$

So it suffices to show that

$$\lim_{t\to+\infty} \mathbf{P}_{\delta_x}\left[Z_t((-\delta t, \delta t)^c)\right] = 0. \tag{4.17}$$

Note that for t large enough such that $\delta t > M$,

$$\begin{aligned}\mathbf{P}_{\delta_x}\left[Z_t((-\delta t, \delta t)^c)\right] &= \frac{1}{w(x)} P_t^\beta \left(w\mathbf{1}_{(-\delta t,\delta t)^c}\right)(x)\\ &\leq \frac{w_+ \vee w_-}{w(x)} P_t^\beta \mathbf{1}_{(-\delta t,\delta t)^c}(x) = \frac{w_+ \vee w_-}{w(x)}\pi_t^{\delta t}(x).\end{aligned}$$

It follows by Lemma 3.1 that $\lim_{t\to+\infty}\log \pi_t^{\delta t}(x)/t = -\Lambda_\delta < 0$ for any $\delta \in (\sqrt{\lambda_1/2}, \sqrt{2\lambda_1})$. So we get $\pi_t^{\delta t}(x) \to 0$ as $t \to +\infty$ and (4.17) follows immediately.

(ii) We note that for $t \geq 0$,

$$\mathrm{e}^{-\lambda_1 t}\|Z_t\| = \mathrm{e}^{-\lambda_1 t}\left\langle \frac{w}{h}\cdot\frac{h}{w}, Z_t\right\rangle \geq \|\frac{h}{w}\|_\infty^{-1} W_t^{h/w}(Z),$$

where $\|h/w\|_\infty < +\infty$. So it suffices to show that for every $x \in \mathbb{R}$,

$$\mathbf{P}_{\delta_x}\left(W_\infty^{h/w}(Z) > 0\right) = 1. \tag{4.18}$$

We have

$$\begin{aligned}\mathrm{P}_{\delta_x}\left(W_\infty^{h/w}(Z) = 0\right) &= \mathrm{P}_{\delta_x}\left[\prod_{u\in Z_0}\mathbf{P}_{\delta_{z_u(0)}}\left(W_\infty^{h/w}(Z) = 0\right)\right]\\ &= \mathrm{e}^{-w(x)\left(1-\mathbf{P}_{\delta_x}\left(W_\infty^{h/w}(Z)=0\right)\right)}.\end{aligned} \tag{4.19}$$

The final equality is because $(Z_0, \mathrm{P}_{\delta_x})$ is a Poisson point process with intensity $w\delta_x$. On the other hand, by (4.5) we have

$$\mathrm{P}_{\delta_x}\left(W_\infty^{h/w}(Z) = 0\right) = \mathrm{P}_{\delta_x}\left(W_\infty^{h}(X) = 0\right) = \mathrm{e}^{-w(x)}.$$

Combining this with (4.19), we get that $\mathbf{P}_{\delta_x}\left(W_\infty^{h/w}(Z)=0\right)=0$ and (4.18) follows immediately. $\square$

Proposition 4.6 *For every $x\in\mathbb{R}$, $((Z_t\pm\sqrt{\lambda_1/2}\,t)_{t\geq 0},\mathbf{P}_{\delta_x})$ converges in distribution to a Cox process directed by $w_\pm W_\infty^{h/w}(Z)\eta_\pm(\mathrm{d}x)$, where $W_\infty^{h/w}(Z)$ is the martingale limit of $((W_t^{h/w}(Z))_{t\geq 0},\mathbf{P}_{\delta_x})$.*

Proof Take $R(t)=\sqrt{\lambda_1/2}\,t$ and fix a function $s(\cdot)$ such that $s(t)\to+\infty$ and $s(t)=o(t)$ as $t\to+\infty$. For notational simplicity, in the proof we shall write $s(t)$ as s. We only consider $Z_t-R(t)$. The result for $Z_t+R(t)$ can be proved similarly.

In view of Lemma A.2, to show the result for $\{Z_t-R(t):t\geq 0\}$, it suffices to show that for any subsequence $\{Z_{t_n}-R(t_n):n\geq 1\}$ with $t_n\to+\infty$, conditions (i) and (ii) are satisfied when taking $\xi_n=Z_{t_n}-R(t_n)$ and $\eta(\mathrm{d}x)=w_-W_\infty^{h/w}(Z)\eta_-(\mathrm{d}x)$. It follows from Lemma 4.3(i) that for any Borel set A with $\inf A>-\infty$,

$$\mathbf{P}_{\delta_x}\left[Z_t(A+R(t))\right]\sim w_-\eta_-(A)\frac{h(x)}{w(x)}\ \text{as } t\to+\infty, \tag{4.20}$$

which implies that condition (ii) of Lemma A.2 is satisfied with $\xi_n=Z_{t_n}-R(t_n)$. Hence, we only need to verify condition (i).

Take $m\in\mathbb{N}$, $k_1,\cdots,k_m\in\mathbb{Z}^+$ and mutually disjoint Borel sets $A_1,\cdots,A_m$ in $\mathbb{R}$ with $\inf A_i>-\infty$ for $i=1,\cdots,m$. Put $k:=k_1+\cdots+k_m$ and $A:=\bigcup_{i=1}^m A_i$. Let $\mathcal{G}_s$ be the σ-field generated by $\{Z_r:r\in[0,s]\}$. It suffices to show that

$$\begin{aligned}&\mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^m\{Z_t(A_i+R(t))=k_i\}\mid\mathcal{G}_s\right)\\&\quad\to \mathrm{e}^{-w_-W_\infty^{h/w}(Z)\sum_{i=1}^m\eta_-(A_i)}\prod_{i=1}^m\frac{\left(w_-W_\infty^{h/w}(Z)\eta_-(A_i)\right)^{k_i}}{k_i!}.\end{aligned} \tag{4.21}$$

in probability as $t\to+\infty$.[1] For $u\in Z_s$, let $Z_{t-s}^{(u)}$ be the point process of the locations of the particles alive at time t whose ancestor is u. Take a constant $\kappa>\sqrt{\lambda_1/2}$. Define $\mathcal{E}_t^1:=\left\{\max_{u\in Z_s}|z_u(s)|\leq\kappa s,\ \|Z_s\|\geq k\right\}$ and $\mathcal{E}_t^2:=\left\{Z_{t-s}^{(u)}(A+R(t))\leq 1\quad\forall u\in Z_s\right\}$. It follows from Lemma 4.5 that $\mathbf{P}_{\delta_x}\left(\mathcal{E}_t^1\right)\to 1$ as $t\to+\infty$. Since $\mathcal{E}_t^1\in\mathcal{G}_s$, we get

$$\mathbf{P}_{\delta_x}\left((\mathcal{E}_t^1)^c\mid\mathcal{G}_s\right)=\mathbf{1}_{(\mathcal{E}_t^1)^c}\to 0 \text{ as } t\to+\infty\quad\text{in probability.} \tag{4.22}$$

[1] Actually (4.21) is a bit stronger than what one needs for the proof of Proposition 4.6. The proof can be shortened by applying [20, Proposition 16.17] In fact by the aforementioned result, one only needs to show that (i) (4.20) holds for all relatively compact sets $A\subset\mathbb{R}$, and (ii) $\lim_{t\to+\infty}\mathbf{P}_{\delta_x}(Z_t(A+R(t))=0)=\mathbf{P}_{\delta_x}\left[\exp\{-w_-W_\infty^{h/w}(Z)\eta_-(A)\}\right]$ for all compact sets A. However, since (4.21) further yields the limit of the order statistics of Z_t (see Proposition 4.8 and the remark below), we present it here for the sake of being more self-contained.

On the event $\mathcal{E}_t^1$, we have

$$\mathbf{P}_{\delta_x}\left((\mathcal{E}_t^2)^c \,|\, \mathcal{G}_s\right) = \mathbf{P}_{\delta_x}\left(Z_{t-s}^{(u)}(A+R(t)) \geq 2 \text{ for some } u \in Z_s \,|\, \mathcal{G}_s\right)$$
$$\leq \sum_{u\in Z_s} \mathbf{P}_{\delta_{z_u(s)}}\left(Z_{t-s}^{(u)}(A+R(t)) \geq 2\right). \tag{4.23}$$

By Lemma 4.4, for t large enough, on the event $\mathcal{E}_t^1$,

$$\mathbf{P}_{\delta_{z_u(s)}}\left(Z_{t-s}^{(u)}(A+R(t)) \geq 2\right) \leq c_1 \frac{h(z_u(s))}{w(z_u(s))} \mathrm{e}^{-2\lambda_1 s}.$$

Hence, we get by (4.23) that on $\mathcal{E}_t^1$

$$\mathbf{P}_{\delta_x}\left((\mathcal{E}_t^2)^c \,|\, \mathcal{G}_s\right) \leq c_1 \sum_{u\in Z_s} \frac{h(z_u(s))}{w(z_u(s))} \mathrm{e}^{-2\lambda_1 s} \leq c_1 \mathrm{e}^{-\lambda_1 s} W_s^{h/w}(Z).$$

This yields $\mathbf{1}_{\mathcal{E}_t^1} \mathbf{P}_{\delta_x}\left((\mathcal{E}_t^2)^c \,|\, \mathcal{G}_s\right) \to 0$ $\mathbf{P}_{\delta_x}$-a.s. Consequently by (4.22) we have

$$\mathbf{P}_{\delta_x}\left((\mathcal{E}_t^2)^c \,|\, \mathcal{G}_s\right) \to 0 \quad \text{in probability as } t \to +\infty. \tag{4.24}$$

By (4.22) and (4.24), we have

$$\mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^m \{Z_t(A_i+R(t)) = k_i\} \,|\, \mathcal{G}_s\right)$$
$$= \mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^m \{Z_t(A_i+R(t)) = k_i\}, \mathcal{E}_t^1, \mathcal{E}_t^2 \,|\, \mathcal{G}_s\right) + \epsilon_t^1$$

for some $\epsilon_t^1 \to 0$ in probability.

We note that on the event $\mathcal{E}_t^2$, $\{Z_{t-s}^{(u)}(A+R(t)) : u \in Z_s\}$ are Bernoulli random variables. So we have

$$\mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^m \{Z_t(A_i+R(t)) = k_i\}, \mathcal{E}_t^1, \mathcal{E}_t^2 \,|\, \mathcal{G}_s\right)$$
$$= \frac{1}{k_1! \cdots k_m!} \times \mathbf{P}_{\delta_x}\Bigg(\bigcup_{(u_1,\cdots,u_k)\subset Z_s} \Big\{ \bigcap_{j=1}^{k_1} \{Z_{t-s}^{(u_j)}(A_1+R(t)) = 1\},$$
$$\cdots, \bigcap_{j=k-k_m+1}^{k} \{Z_{t-s}^{(u_j)}(A_m+R(t)) = 1\},$$
$$\bigcap_{u\in Z_s\setminus\{u_1,\cdots,u_k\}} \{Z_{t-s}^{(u)}(A+R(t)) = 0\}\Big\}, \mathcal{E}_t^1, \mathcal{E}_t^2 \,|\, \mathcal{G}_s\Bigg). \tag{4.25}$$

Here, $\bigcup_{(u_1,\cdots,u_k)\subset Z_s}$ is the union over all k-permutations of Z_s, and $(u_1,\cdots,u_k)\subset Z_s$ means that $u_1\in Z_s, u_2\in Z_s : u_2\neq u_1,..., u_k\in Z_s : u_k\neq u_j, j=1,\cdots,k-1$. By (4.24) and the fact that $\mathcal{E}_t^1\in\mathcal{G}_s$, the conditional probability in the right-hand side of (4.25) equals

$$\mathbf{1}_{\mathcal{E}_t^1}\mathbf{P}_{\delta_x}\Big(\bigcup_{(u_1,\cdots,u_k)\subset Z_s}\{\bigcap_{j=1}^{k_1}\{Z_{t-s}^{(u_j)}(A_1+R(t))=1\},\cdots,$$
$$\bigcap_{j=k-k_m+1}^{k}\{Z_{t-s}^{(u_j)}(A_m+R(t))=1\},\bigcap_{u\in Z_s\backslash\{u_1,\cdots,u_k\}}\{Z_{t-s}^{(u)}(A+R(t))=0\}\}\,|\,\mathcal{G}_s\Big)+\epsilon_t^2,$$

where $\epsilon_t^2\to 0$ in probability. Since $\bigcup_{(u_1,\cdots,u_k)\subset Z_s}\{\cdots\}$ is a union of mutually disjoint events, we have

$$\begin{aligned}
&\mathbf{P}_{\delta_x}\Big(\bigcup_{(u_1,\cdots,u_k)\subset Z_s}\{\bigcap_{j=1}^{k_1}\{Z_{t-s}^{(u_j)}(A_1+R(t))=1\},\cdots,\bigcap_{j=k-k_m+1}^{k}\{Z_{t-s}^{(u_j)}(A_m+R(t))=1\},\\
&\qquad\bigcap_{u\in Z_s\backslash\{u_1,\cdots,u_k\}}\{Z_{t-s}^{(u)}(A+R(t))=0\}\}\,|\,\mathcal{G}_s\Big)\\
&=\sum_{(u_1,\cdots,u_k)\subset Z_s}\mathbf{P}_{\delta_x}\big(\bigcap_{j=1}^{k_1}\{Z_{t-s}^{(u_j)}(A_1+R(t))=1\},\cdots,\bigcap_{j=k-k_m+1}^{k}\{Z_{t-s}^{(u_j)}(A_m+R(t))=1\},\\
&\qquad\bigcap_{u\in Z_s\backslash\{u_1,\cdots,u_k\}}\{Z_{t-s}^{(u)}(A+R(t))=0\}\,|\,\mathcal{G}_s\big)\\
&=\sum_{(u_1,\cdots,u_k)\subset Z_s}\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=1\big)\times\cdots\times\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_m+R(t))=1\big)\\
&\quad\times\prod_{u\in Z_s\backslash\{u_1,\cdots,u_k\}}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)\\
&=\Big[\prod_{u\in Z_s}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)\Big]\times\Big[\sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\\
&\quad\times\cdots\times\frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_m+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\Big].
\end{aligned}$$

The second equality follows by the Markov branching property. So far we have proved that

$$\begin{aligned}
&\mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^{m}\{Z_t(A_i+R(t))=k_i\}\,|\,\mathcal{G}_s\right)\\
&=\epsilon_t^1+\epsilon_t^2+\frac{1}{k_1!\cdots k_m!}\mathbf{1}_{\mathcal{E}_t^1}\Big[\prod_{u\in Z_s}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)\Big]
\end{aligned}$$

$$\times\Big[\sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\times\cdots\times\frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_m+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\Big].$$

Hence to prove (4.21), it suffices to prove that

$$\lim_{t\to+\infty}\mathbf{1}_{\mathcal{E}_t^1}\prod_{u\in Z_s}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)=\mathrm{e}^{-w_-W_\infty^{h/w}(Z)\eta_-(A)} \tag{4.26}$$

in probability and

$$\begin{aligned}&\lim_{t\to+\infty}\mathbf{1}_{\mathcal{E}_t^1}\sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\times\cdots\\&\quad\times\frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_m+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\\&=\prod_{i=1}^m\Big(w_-W_\infty^{h/w}(Z)\eta_-(A_i)\Big)^{k_i}\qquad\text{in probability.}\end{aligned}\tag{4.27}$$

(i) We first prove (4.26). It follows from (4.10) that for t large enough, on the event $\mathcal{E}_t^1$,

$$\begin{aligned}\prod_{u\in Z_s}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)&\le\prod_{u\in Z_s}\left(1-\theta_4(t)w_-\eta_-(A)\frac{h(z_u(s))}{w(z_u(s))}\mathrm{e}^{-\lambda_1 s}\right)\\&\le\prod_{u\in Z_s}\exp\{-\theta_4(t)w_-\eta_-(A)\frac{h(z_u(s))}{w(z_u(s))}\mathrm{e}^{-\lambda_1 s}\}\\&=\exp\{-\theta_4(t)w_-\eta_-(A)W_s^{h/w}(Z)\}.\end{aligned}\tag{4.28}$$

The second inequality is from the fact that $1-x\le\mathrm{e}^{-x}$ for all $x\ge0$. For the lower bound, it follows from (4.11) that for t large enough, on $\mathcal{E}_t^1$

$$\begin{aligned}&\prod_{u\in Z_s}\mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)\\&\ge\prod_{u\in Z_s}\left(1-\theta_5(t)w_-\eta_-(A)\frac{h(z_u(s))}{w(z_u(s))}\mathrm{e}^{-\lambda_1 s}\right)\\&=\exp\left\{\sum_{u\in Z_s}\log\left(1-\theta_5(t)w_-\eta_-(A)\frac{h(z_u(s))}{w(z_u(s))}\mathrm{e}^{-\lambda_1 s}\right)\right\}.\end{aligned}\tag{4.29}$$

Note that $c := \sup_{y\in\mathbb{R}} h(y)/w(y) \le \|h\|_\infty / \inf_{y\in\mathbb{R}} w(y) < +\infty$. Using the fact that

$$\log(1-x) \ge \frac{\log(1-x^*)}{x^*} x \quad \forall x^* \in (0,1),\ x \in [0, x^*],$$

we get by (4.29) that on $\mathcal{E}_t^1$,

$$\begin{aligned}
&\prod_{u\in Z_s} \mathbf{P}_{\delta_{z_u(s)}}\big(Z_{t-s}(A+R(t))=0\big)\\
&\ge \exp\left\{\frac{\log\left(1-\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}\right)}{\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}} \sum_{u\in Z_s} \theta_5(t)w_-\eta_-(A)\frac{h(z_u(s))}{w(z_u(s))}\mathrm{e}^{-\lambda_1 s}\right\}\\
&= \exp\left\{\frac{\log\left(1-\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}\right)}{\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}} \theta_5(t)w_-\eta_-(A)W_s^{h/w}(Z)\right\}.
\end{aligned} \tag{4.30}$$

It is easy to see that the final terms of (4.28) and (4.30) converge to $\mathrm{e}^{-w_-\eta_-(A)W_\infty^{h/w}(Z)}$ almost surely. Thus, (4.26) follows immediately.

(ii) Now, we prove (4.27). We use $\theta_i^{(j)}(t)$ to denote the functions $\theta_i(t)$ in Lemma 4.4 corresponding to the set A_j. It follows by (4.11) and (4.12) that, on the event $\mathcal{E}_t^1$,

$$\begin{aligned}
&\sum_{(u_1,\cdots,u_k)\subset Z_s} \frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)} \times\cdots\times \frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_m+R(t))=1\big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\big(Z_{t-s}(A_1+R(t))=0\big)}\\
&\le \sum_{(u_1,\cdots,u_k)\subset Z_s} \frac{\theta_6^{(1)}(t)w_-\eta_-(A_1)\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}}{1-\theta_5(t)w_-\eta_-(A)\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}} \times\cdots\times \frac{\theta_6^{(m)}(t)w_-\eta_-(A_m)\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s}}{1-\theta_5(t)w_-\eta_-(A)\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s}}\\
&\le \frac{\prod_{i=1}^m \theta_6^{(i)}(t)^{k_i}}{\left(1-\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}\right)^k}\\
&\quad\times\left(\sum_{(u_1,\cdots,u_k)\subset Z_s} w_-\eta_-(A_1)\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times w_-\eta_-(A_m)\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s}\right)\\
&\le \frac{\prod_{i=1}^m \theta_6^{(i)}(t)^{k_i}}{\left(1-\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}\right)^k}\\
&\quad\times\left(\sum_{u_1\in Z_s} w_-\eta_-(A_1)\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\right)\times\cdots\times\left(\sum_{u_k\in Z_s} w_-\eta_-(A_m)\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s}\right)\\
&= \frac{\prod_{i=1}^m \theta_6^{(i)}(t)^{k_i}}{\left(1-\theta_5(t)w_-\eta_-(A)c\mathrm{e}^{-\lambda_1 s}\right)^k}\prod_{i=1}^m\left(w_-\eta_-(A_i)W_s^{h/w}(Z)\right)^{k_i}.
\end{aligned} \tag{4.31}$$

For the lower bound, we have by (4.12) that on $\mathcal{E}_t^1$,

$$
\begin{aligned}
&\sum_{(u_1,\cdots,u_k)\subset Z_s} \frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\Big(Z_{t-s}(A_1+R(t))=1\Big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\Big(Z_{t-s}(A_1+R(t))=0\Big)} \times\cdots\times \frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\Big(Z_{t-s}(A_m+R(t))=1\Big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\Big(Z_{t-s}(A_1+R(t))=0\Big)} \\
&\geq \sum_{(u_1,\cdots,u_k)\subset Z_s} \mathbf{P}_{\delta_{z_{u_1}(s)}}\Big(Z_{t-s}(A_1+R(t))=1\Big)\times\cdots\times \mathbf{P}_{\delta_{z_{u_k}(s)}}\Big(Z_{t-s}(A_m+R(t))=1\Big) \\
&\geq \sum_{(u_1,\cdots,u_k)\subset Z_s} \theta_7^{(1)}(t)w_-\eta_-(A_1)\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\theta_7^{(m)}(t)w_-\eta_-(A_m)\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s} \\
&= \Big[\prod_{i=1}^m \Big(\theta_7^{(i)}(t)w_-\eta_-(A_i)\Big)^{k_i}\Big]\times\Big[\sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s}\Big].
\end{aligned}
\tag{4.32}
$$

Note that the sum $\sum_{u_1,\cdots,u_k\in Z_s}$ is no larger than the sum of $\sum_{(u_1,\cdots,u_k)\subset Z_s}$ and $\sum_{1\leq i<j\leq k}\sum_{u_1,\cdots,u_k\in Z_s,u_i=u_j}$, and that

$$
\begin{aligned}
&\sum_{u_1,\cdots,u_k\in Z_s,u_i=u_j}\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s} \\
&\leq c_1\mathrm{e}^{-\lambda_1 s}\sum_{u_1,\cdots,u_{k-1}\in Z_s}\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\frac{h(z_{u_{k-1}}(s))}{w(z_{u_{k-1}}(s))}\mathrm{e}^{-\lambda_1 s} \\
&\leq c_1\mathrm{e}^{-\lambda_1 s}W_s^{h/w}(Z)^{k-1}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
W_s^{h/w}(Z)^k &= \sum_{u_1,\cdots,u_k\in Z_s}\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s} \\
&\leq \sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{h(z_{u_1}(s))}{w(z_{u_1}(s))}\mathrm{e}^{-\lambda_1 s}\times\cdots\times\frac{h(z_{u_k}(s))}{w(z_{u_k}(s))}\mathrm{e}^{-\lambda_1 s} \\
&\quad + c_2\mathrm{e}^{-\lambda_1 s}W_s^{h/w}(Z)^{k-1}.
\end{aligned}
\tag{4.33}
$$

Putting this back to (4.32), we get that on $\mathcal{E}_t^1$,

$$
\begin{aligned}
&\sum_{(u_1,\cdots,u_k)\subset Z_s}\frac{\mathbf{P}_{\delta_{z_{u_1}(s)}}\Big(Z_{t-s}(A_1+R(t))=1\Big)}{\mathbf{P}_{\delta_{z_{u_1}(s)}}\Big(Z_{t-s}(A_1+R(t))=0\Big)}\times\cdots \\
&\qquad\times\frac{\mathbf{P}_{\delta_{z_{u_k}(s)}}\Big(Z_{t-s}(A_m+R(t))=1\Big)}{\mathbf{P}_{\delta_{z_{u_k}(s)}}\Big(Z_{t-s}(A_1+R(t))=0\Big)} \\
&\geq \Big[\prod_{i=1}^m\Big(\theta_7^{(i)}(t)w_-\eta_-(A_i)\Big)^{k_i}\Big]\times\Big[W_s^{h/w}(Z)^k-c_2\mathrm{e}^{-\lambda_1 s}W_s^{h/w}(Z)^{k-1}\Big].
\end{aligned}
\tag{4.34}
$$

We note that the final terms of (4.31) and (4.34) converge to

$$\prod_{i=1}^{m}\left(w_{-}\eta_{-}(A_i)W_{\infty}^{h/w}(Z)\right)^{k_i}$$

almost surely. Thus, (4.27) follows immediately. Therefore, we complete the proof.□

Proposition 4.7 *For every $\mu \in \mathcal{M}(\mathbb{R})$, $((Z_t \pm \sqrt{\lambda_1/2}\,t)_{t\geq 0}, \mathrm{P}_\mu)$ converges in distribution to a Cox process directed by $w_{\pm}W_{\infty}^{h/w}(Z)\eta_{\pm}(\mathrm{d}x)$, where $W_{\infty}^{h/w}(Z)$ is the martingale limit of $((W_t^{h/w}(Z))_{t\geq 0}, \mathrm{P}_\mu)$.*

Proof For any $f \in C_c^+(\mathbb{R})$ and $\mu \in \mathcal{M}(\mathbb{R})$,

$$\begin{aligned}
\mathrm{P}_\mu\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle}\right] &= \mathrm{P}_\mu\left[\mathrm{P}_\mu\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle} | Z_0\right]\right] \\
&= \mathrm{P}_\mu\left[\prod_{u\in Z_0}\mathbf{P}_{\delta_{z_u(0)}}\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle}\right]\right] \\
&= \exp\left\{-\int_{\mathbb{R}}\left(1-\mathbf{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle}\right]\right)w(x)\mu(\mathrm{d}x)\right\}.
\end{aligned}$$

The final equality is because (Z_0, P_μ) is a Poisson point process with intensity $w\mu$. Similarly, one can prove that for every $\lambda \geq 0$,

$$\mathrm{P}_\mu\left[\mathrm{e}^{-\lambda W_{\infty}^{h/w}(Z)}\right] = \exp\left\{-\int_{\mathbb{R}}\left(1-\mathbf{P}_{\delta_x}\left[\mathrm{e}^{-\lambda W_{\infty}^{h/w}(Z)}\right]\right)w(x)\mu(\mathrm{d}x)\right\}.$$

Since by Proposition 4.6

$$\lim_{t\to+\infty}\mathbf{P}_{\delta_x}\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle}\right] = \mathbf{P}_{\delta_x}\left[\mathrm{e}^{-w_{\pm}W_{\infty}^{h/w}(Z)\langle 1-\mathrm{e}^{-f}, \eta_{\pm}\rangle}\right]$$

for all $x \in \mathbb{R}$, we get by the bounded convergence theorem that

$$\lim_{t\to+\infty}\mathrm{P}_\mu\left[\mathrm{e}^{-\langle f, Z_t \pm \sqrt{\frac{\lambda_1}{2}}t\rangle}\right] = \mathrm{P}_\mu\left[\mathrm{e}^{-w_{\pm}W_{\infty}^{h/w}(Z)\langle 1-\mathrm{e}^{-f}, \eta_{\pm}\rangle}\right].$$

Hence, we prove this proposition. □

For $t \geq 0$, let $\max Z_t := \max\{z_u(t) : u \in Z_t\}$ be the maximum displacement of the skeleton branching diffusion.

Proposition 4.8 *For any $\mu \in \mathcal{M}_c(\mathbb{R})$ and $y \in \mathbb{R}$,*

$$\lim_{t\to+\infty}\mathrm{P}_\mu\left(\max Z_t - \sqrt{\frac{\lambda_1}{2}}t \leq y \,\Big|\, W_{\infty}^{h}(X) > 0\right)$$

$$= \mathrm{P}_\mu \left[\exp\left\{ -\frac{w_- C_-}{\sqrt{2\lambda_1}} \mathrm{e}^{-\sqrt{2\lambda_1}y} W^h_\infty(X) \right\} \Big| \, W^h_\infty(X) > 0 \right]. \tag{4.35}$$

This implies that conditioned on $\{W^h_\infty(X) > 0\}$, the maximal displacement of the skeleton branching diffusion centered by $\sqrt{\lambda_1/2}\,t$ converges in distribution to a randomly shifted Gumbel distribution.

Proof Fix an arbitrary $y \in \mathbb{R}$. If we set $A = A_1 = (y, +\infty)$ and $k = k_1 = 0$ in (4.21), then we get

$$\mathbf{P}_{\delta_x} \left(Z_t \Big(\sqrt{\frac{\lambda_1}{2}} t + y, +\infty \Big) = 0 \, \Big| \, \mathcal{G}_s \right) \to \mathrm{e}^{-\frac{w_- C_-}{\sqrt{2\lambda_1}} \mathrm{e}^{-\sqrt{2\lambda_1}y} W^{h/w}_\infty(Z)}$$

in probability as $t \to +\infty$. It follows immediately that

$$\lim_{t\to+\infty} \mathbf{P}_{\delta_x} \left(\max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y \right) = \mathbf{P}_{\delta_x} \left[\mathrm{e}^{-\frac{w_- C_-}{\sqrt{2\lambda_1}} \mathrm{e}^{-\sqrt{2\lambda_1}y} W^{h/w}_\infty(Z)} \right].$$

Using this and the branching property, we can show that for any $\mu \in \mathcal{M}_c(\mathbb{R})$,

$$\lim_{t\to+\infty} \mathrm{P}_\mu \left(\max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y \right) = \mathrm{P}_\mu \left[\mathrm{e}^{-\frac{w_- C_-}{\sqrt{2\lambda_1}} \mathrm{e}^{-\sqrt{2\lambda_1}y} W^{h}_\infty(X)} \right]. \tag{4.36}$$

We note that

$$\begin{aligned}
&\mathrm{P}_\mu \left(\max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y, \ W^h_\infty(X) = 0 \right) \\
&= \mathrm{P}_\mu \left(\max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y, \ W^{h/w}_\infty(Z) = 0 \right) \\
&= \mathrm{P}_\mu \left(\|Z_0\| = 0 \right) \\
&\quad + \mathrm{P}_\mu \left(\mathbf{P}_{Z_t} \left(W^{h/w}_\infty(Z) = 0 \right); \|Z_0\| \neq 0, \ \max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y \right).
\end{aligned} \tag{4.37}$$

Noting (4.18), one has $\mathbf{P}_{Z_t} \left(W^{h/w}_\infty(Z) = 0 \right) = 0$ P_μ-a.s. on $\{\|Z_0\| \neq 0\}$. Thus, the second term in the right-hand side of (4.37) equals 0, and one gets

$$\mathrm{P}_\mu \left(\max Z_t - \sqrt{\frac{\lambda_1}{2}} t \le y, \ W^h_\infty(X) = 0 \right) = \mathrm{P}_\mu \left(\|Z_0\| = 0 \right) = \mathrm{e}^{-\langle w, \mu \rangle}.$$

Thus, we have

$$
\begin{aligned}
&\mathrm{P}_\mu\left(\max Z_t-\sqrt{\frac{\lambda_1}{2}}t\le y\,\Big|\, W^h_\infty(X)>0\right)\\
&\quad-\mathrm{P}_\mu\left[\exp\left\{-\frac{w_-C_-}{\sqrt{2\lambda_1}}\mathrm{e}^{-\sqrt{2\lambda_1}y}W^h_\infty(X)\right\}\Big|\, W^h_\infty(X)>0\right]\\
&=\frac{\mathrm{P}_\mu\left(\max Z_t-\sqrt{\frac{\lambda_1}{2}}t\le y,\ W^h_\infty(X)>0\right)}{\mathrm{P}_\mu\left(W^h_\infty(X)>0\right)}\\
&\quad-\frac{\mathrm{P}_\mu\left[\exp\left\{-\frac{w_-C_-}{\sqrt{2\lambda_1}}\mathrm{e}^{-\sqrt{2\lambda_1}y}W^h_\infty(X)\right\},\, W^h_\infty(X)>0\right]}{\mathrm{P}_\mu\left(W^h_\infty(X)>0\right)}\\
&=\frac{\mathrm{P}_\mu\left(\max Z_t-\sqrt{\frac{\lambda_1}{2}}t\le y\right)-\mathrm{e}^{-\langle w,\mu\rangle}}{\mathrm{P}_\mu\left(W^h_\infty(X)>0\right)}\\
&\quad-\frac{\mathrm{P}_\mu\left[\exp\left\{-\frac{w_-C_-}{\sqrt{2\lambda_1}}\mathrm{e}^{-\sqrt{2\lambda_1}y}W^h_\infty(X)\right\}\right]-\mathrm{P}_\mu\left(W^h_\infty(X)=0\right)}{\mathrm{P}_\mu\left(W^h_\infty(X)>0\right)}.
\end{aligned}
$$

Hence, (4.35) follows by (4.36). □

Remark 4.9 One can order the positions of the particles alive at time t in a nonincreasing order: $R_{t,1}\ge R_{t,2}\ge\cdots\ge R_{t,\|Z_t\|}$. Then similarly as in Proposition 4.8, one can get the weak convergence of $(R_{t,1},R_{t,2},\cdots,R_{(t,n)})$.

4.3 Proofs of Theorem 1.2 and Theorem 1.3

The main idea of the proof for Theorem 1.2 is from [21, Lemma 4.17]: Suppose $\xi_1,\xi_2,\cdots$ are Cox processes on $\mathbb{R}$ directed by some random measures $\eta_1,\eta_2,\cdots$. Then, ξ_n converges in distribution to some ξ if and only if η_n converges in distribution to some η, in which case ξ is distributed as a Cox process directed by η.

Proof of Theorem 1.2: Fix $\mu\in\mathcal{M}(\mathbb{R})$. Proposition 4.1(iii) implies that $(Z_t\pm\sqrt{\lambda_1/2}\,t,\mathrm{P}_\mu)$ is distributed as a Cox process directed by the random measure $w(x\mp\sqrt{\lambda_1/2}\,t)\left(X_t\pm\sqrt{\lambda_1/2}\,t\right)(\mathrm{d}x)$. It then follows by [21, Lemma 4.17] and Proposition 4.7 that the latter converges in distribution to $w_\pm W^h_\infty(X)\eta_\pm(\mathrm{d}x)$. This implies that $\int_{\mathbb{R}}f(x)w(x\mp\sqrt{\lambda_1/2}\,t)\left(X_t\pm\sqrt{\lambda_1/2}\,t\right)(\mathrm{d}x)$ converges in distribution to $w_\pm W^h_\infty(X)\langle f,\eta_\pm\rangle$ for every $f\in C_c^+(\mathbb{R})$. Recall that for $x\ge M$, $w=w_-$ and for $x\le -M$, $w=w_+$. Note that for t large enough such that $x+\sqrt{\lambda_1/2}\,t\ge M$ and $x-\sqrt{\lambda_1/2}\,t\le -M$ for all $x\in\operatorname{supp}f$, $\int_{\mathbb{R}}f(x)w(x\mp\sqrt{\lambda_1/2}\,t)\left(X_t\pm\sqrt{\lambda_1/2}\,t\right)(\mathrm{d}x)=w_\pm\langle f,X_t\pm\sqrt{\lambda_1/2}\,t\rangle$. Thus, one gets that $\langle f,X_t\pm\sqrt{\lambda_1/2}\,t\rangle$ converges in distribution to $W^h_\infty(X)\langle f,\eta_\pm\rangle$. This implies that $X_t\pm\sqrt{\lambda_1/2}\,t$ converges in distribution to $W^h_\infty(X)\eta_\pm(\mathrm{d}x)$. □

Remark 4.10 (i) Theorem 1.2 implies that for any bounded and compactly supported measurable function f on $\mathbb{R}$ whose set of discontinuous points has zero Lebesgue measure, $\langle f, X_t \pm \sqrt{\lambda_1/2}\,t\rangle$ converges in distribution to $W^h_\infty(X)\langle f, \eta_\pm\rangle$. In particular for any compact set $B \subset \mathbb{R}$ whose boundary has zero Lebesgue measure, $X_t\left(\mp\sqrt{\lambda_1/2}\,t + B\right)$ converges in distribution to $W^h_\infty(X)\eta_\pm(B)$.
(ii) We use $\max X_t$ to denote the supremum of the support of X_t, i.e., $\max X_t := \sup\{x : X_t(x, +\infty) > 0\}$. Let $m > 0$ and $y \in \mathbb{R}$. We have

$$\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) \geq \mathbb{P}_\mu\left(\left\langle 1_{(y,y+m)}, X_t - \sqrt{\frac{\lambda_1}{2}}t\right\rangle > 0\right). \tag{4.38}$$

We note that $\left\langle 1_{(y,y+m)}, X_t - \sqrt{\frac{\lambda_1}{2}}t\right\rangle$ converges in distribution to $W^h_\infty(X)\langle 1_{(y,y+m)}, \eta_-\rangle$. Hence, letting $t \to +\infty$ in (4.38), we get that

$$\liminf_{t\to+\infty} \mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) \geq \mathbb{P}_\mu\left(W^h_\infty(X) > 0\right). \tag{4.39}$$

Note that

$$\begin{aligned}
&\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y \,\middle|\, W^h_\infty(X) > 0\right)\\
&= \frac{\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) - \mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y,\, W^h_\infty(X) = 0\right)}{\mathbb{P}_\mu\left(W^h_\infty(X) > 0\right)}\\
&= \frac{\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) - \mathbb{P}_\mu\left(\mathrm{e}^{-\langle w, X_t\rangle}; \max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right)}{\mathbb{P}_\mu\left(W^h_\infty(X) > 0\right)}\\
&\geq \frac{\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) - \mathbb{P}_\mu\left(\mathrm{e}^{-\langle w, X_t\rangle}\right)}{\mathbb{P}_\mu\left(W^h_\infty(X) > 0\right)}\\
&= \frac{\mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y\right) - \mathrm{e}^{-\langle w, \mu\rangle}}{\mathbb{P}_\mu\left(W^h_\infty(X) > 0\right)}.
\end{aligned}$$

Hence by (4.39), we have for any $y \in \mathbb{R}$,

$$\liminf_{t\to+\infty} \mathbb{P}_\mu\left(\max X_t - \sqrt{\frac{\lambda_1}{2}}t > y \,\middle|\, W^h_\infty(X) > 0\right) \geq 1 - \frac{\mathrm{e}^{-\langle w,\mu\rangle}}{1 - \mathrm{e}^{-\langle w,\mu\rangle}} > 0.$$

So conditioned on $\{W^h_\infty(X) > 0\}$, the distributions of $\{\max X_t - \sqrt{\lambda_1/2}\,t : t \geq 0\}$ are not tight. This is very different from the behavior we observe in Proposition 4.8 for the skeleton. Loosely speaking, supremum of the support of super-Brownian motion may grow much faster than that of the embedded skeleton. Similar phenomenon has been observed for spatially independent branching super-Brownian motions, see, for example, [31, Remark 2.12]. This is partly because of the effect of the infinitesimal branching of super-Brownian motions.

Proof of Theorem 1.3: We take $\delta = \sqrt{\lambda_1/2}$ and $\mu \in \mathcal{M}_c(\mathbb{R})$. Suppose supp$\mu \subset [-k, k]$ for some $0 < k < +\infty$. We have

$$\mathbb{P}_\mu\left[\mathcal{X}_t^{\delta t}\right] = \int_{\mathbb{R}} \Pi_x\left[e_\beta(t), |B_t| \geq \delta t\right]\mu(\mathrm{d}x).$$

Since by Lemma 2.2 for t large enough

$$\Pi_x\left[e_\beta(t), |B_t| \geq \delta t\right] \leq \theta_+(t)\frac{1}{\sqrt{2\lambda_1}}(C_+ + C_-)h(x) \quad \forall x \in [-k, k],$$

where $\theta_+(t) \to 1$ as $t \to +\infty$, we have

$$\mathbb{P}_\mu\left[\mathcal{X}_t^{\delta t}\right] \leq \theta_+(t)\frac{1}{\sqrt{2\lambda_1}}(C_+ + C_-)\int_{\mathbb{R}} h(x)\mu(\mathrm{d}x).$$

This implies that

$$\sup_{t\geq 0}\mathbb{P}_\mu\left(\mathcal{X}_t^{\delta t} > \lambda\right) \leq \sup_{t\geq 0}\frac{\mathbb{P}_\mu\left[\mathcal{X}_t^{\delta t}\right]}{\lambda} \to 0 \quad \text{as } \lambda \to +\infty.$$

So the distributions of $\{\mathcal{X}_t^{\delta t} : t \geq 0\}$ are tight.

Applying similar argument as in the proof of Proposition 4.6, one can show that for any $x \in \mathbb{R}$, integers $m, n \geq 0$, integers $k_1, \cdots, k_m, l_1, \cdots, l_n \geq 0$ and Borel sets $A_1, \cdots, A_m, B_1, \cdots, B_n$ such that $\inf A_i > -\infty$ and $\sup B_j < +\infty$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$,

$$\begin{aligned}
&\mathbf{P}_{\delta_x}\left(\bigcap_{i=1}^m\{Z_t(\delta t + A_i) = k_i\}, \bigcap_{j=1}^n\{Z_t(-\delta t + B_j) = l_j\} \,|\, \mathcal{G}_s\right)\\
&\to \exp\{-w_- W_\infty^{h/w}(Z)\sum_{i=1}^m \eta_-(A_i) - w_+ W_\infty^{h/w}(Z)\sum_{j=1}^n \eta_+(B_j)\}\\
&\prod_{i=1}^m \frac{\left(w_- W_\infty^{h/w}(Z)\eta_-(A_i)\right)^{k_i}}{k_i!}\prod_{j=1}^n \frac{\left(w_+ W_\infty^{h/w}(Z)\eta_+(B_j)\right)^{l_j}}{l_j!}
\end{aligned} \tag{4.40}$$

in probability as $t \to +\infty$. This implies that the point process $((Z_t - \delta t) + (Z_t + \delta t), \mathbf{P}_{\delta_x})$ converges in distribution to a Cox process directed by the random measure $W_\infty^{h/w}(Z)(w_-\eta_-(\mathrm{d}x) + w_+\eta_+(\mathrm{d}x))$. Applying similar argument as in the proof of

Theorem 1.2, one can further show that the random measure $((X_t-\delta t)+(X_t+\delta t), \mathbb{P}_\mu)$ converges in distribution to $W^h_\infty(X)(\eta_-(\mathrm{d}x)+\eta_+(\mathrm{d}x))$. On the other hand, by taking $n=m=1$ and $A_1=-B_1=[0,+\infty)$ in (4.40), one gets

$$\begin{aligned}&\mathbf{P}_{\delta x}\left(Z_t([\delta t,+\infty))=k_1,\ Z_t((-\infty,-\delta t])=l_1 \mid \mathcal{G}_s\right)\\ &\to \mathrm{e}^{-\frac{1}{\sqrt{2\lambda_1}}W^{h/w}_\infty(Z)(w_-C_-+w_+C_+)}\frac{\left(w_-W^{h/w}_\infty(Z)/\sqrt{2\lambda_1}\right)^{k_1}}{k_1!}\frac{\left(w_+W^{h/w}_\infty(Z)/\sqrt{2\lambda_1}\right)^{l_1}}{l_1!}\end{aligned}$$

in probability as $t\to+\infty$. Using similar computations as in the proof of Proposition 4.7, one gets that for all $\lambda_1,\lambda_2\geq 0$,

$$\begin{aligned}&\lim_{t\to+\infty}\mathrm{P}_\mu\left[\mathrm{e}^{-\lambda_1 Z_t([\delta t,+\infty))-\lambda_2 Z_t((-\infty,-\delta t])}\right]\\ &=\mathrm{P}_\mu\left[\mathrm{e}^{-\left(1-\mathrm{e}^{-\lambda_1}\right)\frac{1}{\sqrt{2\lambda_1}}W^h_\infty(X)w_-C_--\left(1-\mathrm{e}^{-\lambda_2}\right)\frac{1}{\sqrt{2\lambda_1}}W^h_\infty(X)w_+C_+}\right].\end{aligned}\tag{4.41}$$

Recall that given X_t, Z_t is a Poisson point process with intensity wX_t. Thus for t sufficiently large,

$$\begin{aligned}&\mathrm{P}_\mu\left[\mathrm{e}^{-\lambda_1 Z_t([\delta t,+\infty))-\lambda_2 Z_t((-\infty,-\delta t])}\right]\\ &=\mathrm{P}_\mu\left[\mathrm{e}^{-\left\langle\left(1-\mathrm{e}^{-\lambda_1}\right)\mathbf{1}_{[\delta t,+\infty)}+\left(1-\mathrm{e}^{-\lambda_2}\right)\mathbf{1}_{(-\infty,-\delta t]},wX_t\right\rangle}\right]\\ &=\mathrm{P}_\mu\left[\mathrm{e}^{-\left(1-\mathrm{e}^{-\lambda_1}\right)w_-X_t([\delta t,+\infty))-\left(1-\mathrm{e}^{-\lambda_2}\right)w_+X_t((-\infty,-\delta t])}\right].\end{aligned}$$

Hence, by (4.41) we have

$$\begin{aligned}&\lim_{t\to+\infty}\mathrm{P}_\mu\left[\mathrm{e}^{-\left(1-\mathrm{e}^{-\lambda_1}\right)w_-X_t([\delta t,+\infty))-\left(1-\mathrm{e}^{-\lambda_2}\right)w_+X_t((-\infty,-\delta t])}\right]\\ &=\mathrm{P}_\mu\left[\mathrm{e}^{-\left(1-\mathrm{e}^{-\lambda_1}\right)\frac{1}{\sqrt{2\lambda_1}}W^h_\infty(X)w_-C_--\left(1-\mathrm{e}^{-\lambda_2}\right)\frac{1}{\sqrt{2\lambda_1}}W^h_\infty(X)w_+C_+}\right].\end{aligned}$$

For $0\leq\lambda<w_-\wedge w_+$, taking λ_1,λ_2 such that $(1-\mathrm{e}^{-\lambda_1})w_-=(1-\mathrm{e}^{-\lambda_2})w_+=\lambda$, one gets that

$$\lim_{t\to+\infty}\mathbb{P}_\mu\left[\mathrm{e}^{-\lambda\mathcal{X}^{\delta t}_t}\right]=\mathbb{P}_\mu\left[\mathrm{e}^{-\lambda(C_++C_-)\frac{1}{\sqrt{2\lambda_1}}W^h_\infty(X)}\right].\tag{4.42}$$

Suppose $(\mathcal{X}^{\delta t}_t,\mathbb{P}_\mu)$ converges in distribution to ξ along a subsequence $\{t_n:n\geq 1\}\subset[0,+\infty)$, for some random variable ξ. Let F_1 and F_2 be the distribution functions of ξ and $(C_++C_-)W^h_\infty(X)/\sqrt{2\lambda_1}$, respectively. It suffices to show that $F_1=F_2$. Let D_1 be the set of continuous points of F_1. We note that $\{\mathcal{X}^{\delta t}_t\leq x\}\subseteq\{X_t((\delta t,\delta t+$

$N)) + X_t((-\delta t - N, -\delta t)) \leq x\}$ for all $x, N \in \mathbb{R}$. Thus for any $y \in D_1$,

$$\begin{aligned} F_1(y) &= \lim_{n\to+\infty} \mathbb{P}_\mu\left(\mathcal{X}_{t_n}^{\delta t_n} \leq y\right) \\ &\leq \limsup_{n\to+\infty} \mathbb{P}_\mu\left(X_{t_n}((\delta t_n, \delta t_n + N)) + X_{t_n}((-\delta t_n - N, -\delta t_n)) \leq y\right) \\ &\leq \mathbb{P}_\mu\left(W_\infty^h(X)\left(\eta_-((0,N)) + \eta_+((-N,0))\right) \leq y\right) \\ &= F_2\left(\frac{\eta_-((0,+\infty)) + \eta_+((-\infty,0))}{\eta_-((0,N)) + \eta_+((-N,0))}\, y\right). \end{aligned}$$

By letting $N \to +\infty$, one gets that $F_1(y) \leq F_2(y)$. If $F_1(y) < F_2(y)$ for some $y \in D_1$, then there is some $\epsilon > 0$ such that $F_1(x) < F_2(x)$ for all $x \in (y, y+\epsilon)$. This yields that for any $\lambda > 0$,

$$\mathrm{E}\left[\mathrm{e}^{-\lambda\xi}\right] - \mathbb{P}_\mu\left[\mathrm{e}^{-\lambda(C_+ + C_-)\frac{1}{\sqrt{2\lambda_1}} W_\infty^h(X)}\right] = \lambda \int_0^{+\infty} \mathrm{e}^{-\lambda x}\left(F_1(x) - F_2(x)\right) \mathrm{d}x < 0,$$

which contradicts (4.42). Thus, we have $F_1(x) = F_2(x)$ for all $x \in D_1$ and hence for all $x \in \mathbb{R}$. □

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Declarations

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Appendix A

Lemma A.1 *The martingale function w in (A2) is a solution to the following equation.*

$$\frac{1}{2}w''(x) - \psi(x, w(x)) = 0, \quad \forall x \in \mathbb{R}. \tag{A1}$$

Proof It is proved by [15, Lemma 2.1] that the martingale function w which satisfies (A2) is continuous on $\mathbb{R}$. Moreover, the argument leading to [15, (2.4)] shows that for any compact set D of $\mathbb{R}$,

$$w(x) = \Pi_x \left[w(B_{t\wedge\tau_D})\right] - \Pi_x \left[\int_0^{t\wedge\tau_D} \psi(B_s, w(B_s))\mathrm{d}s\right], \quad \forall t \geq 0,\ x \in \mathbb{R},$$

where τ_D denotes the first exit time of Brownian motion from D. Since w is continuous and locally bounded, letting $t \to +\infty$ in the above equation, we get by the bounded convergence theorem that

$$w(x) = \Pi_x \left[w(B_{\tau_D})\right] - \Pi_x \left[\int_0^{\tau_D} \psi(B_s, w(B_s))\mathrm{d}s\right], \quad x \in D.$$

Applying similar argument as in the last paragraph of Page 708 in [16], one can show that w is a solution to (A1). □

Lemma A.2 *Suppose $\{\xi_n : n \geq 1\}$ is a sequence of point processes on $\mathbb{R}$, and η is a locally finite random measure on $\mathbb{R}$. Then, ξ_n converges in distribution to a Cox process directed by η if the following conditions hold.*

(i) For $m \in \mathbb{N}$, mutually disjoint bounded Borel sets $A_1, \cdots, A_m$ of $\mathbb{R}$ and $k_1, \cdots, k_m \in \mathbb{Z}^+$,

$$\lim_{n\to+\infty} \mathrm{P}\left(\xi_n(A_1) = k_1, \cdots, \xi_n(A_m) = k_m\right) = \mathrm{E}\left[\mathrm{e}^{-\sum_{i=1}^m \eta(A_i)} \prod_{i=1}^m \frac{\eta(A_i)^{k_i}}{k_i!}\right].$$

(ii) For any bounded Borel set A of $\mathbb{R}$, $\sup_n \mathrm{E}\left[\xi_n(A)\right] < +\infty$.

Proof We need to show that for all $f \in C_c^+(\mathbb{R})$,

$$\mathrm{E}\left[\mathrm{e}^{-\langle f, \xi_n\rangle}\right] \to \mathrm{E}\left[\mathrm{e}^{-\langle 1-\mathrm{e}^{-f}, \eta\rangle}\right] \quad \text{as } n \to +\infty. \tag{A2}$$

It is easy to deduce from (i) that (A2) holds if f is a nonnegative compactly supported simple function. For an arbitrary $f \in C_c^+(\mathbb{R})$ with $\mathrm{supp} f \subset A$ where A is a bounded Borel set of $\mathbb{R}$, one can find a nondecreasing sequence of nonnegative compactly supported simple functions $\{f_k : k \geq 1\}$ such that f_n converges uniformly to f. We note that for $k, n \geq 1$,

$$\begin{aligned}
\left|\mathrm{E}\left[\mathrm{e}^{-\langle f_k, \xi_n\rangle}\right] - \mathrm{E}\left[\mathrm{e}^{-\langle f, \xi_n\rangle}\right]\right| &\leq \mathrm{E}\left[\left|\mathrm{e}^{-\langle f_k, \xi_n\rangle} - \mathrm{e}^{-\langle f, \xi_n\rangle}\right|\right] \\
&\leq \mathrm{E}\left[\left|\langle f_k, \xi_n\rangle - \langle f, \xi_n\rangle\right|\right] \\
&\leq \mathrm{E}\left[\langle |f_k - f|, \xi_n\rangle\right] \\
&\leq \|f_k - f\|_\infty \mathrm{E}\left[\xi_n(A)\right].
\end{aligned}$$

It follows by (ii) that $\sup_n \left|\mathrm{E}\left[\mathrm{e}^{-\langle f_k, \xi_n\rangle}\right] - \mathrm{E}\left[\mathrm{e}^{-\langle f, \xi_n\rangle}\right]\right| \to 0$ as $k \to +\infty$. So we have

$$\lim_{n\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle f, \xi_n\rangle}\right] = \lim_{n\to+\infty} \lim_{k\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle f_k, \xi_n\rangle}\right]$$

$$
\begin{aligned}
&= \lim_{k\to+\infty} \lim_{n\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle f_k,\xi_n\rangle}\right] \\
&= \lim_{k\to+\infty} \mathrm{E}\left[\mathrm{e}^{-\langle 1-\mathrm{e}^{-f_k},\eta\rangle}\right] \\
&= \mathrm{E}\left[\mathrm{e}^{-\langle 1-\mathrm{e}^{-f},\eta\rangle}\right].
\end{aligned}
$$

The first and final equalities are from the bounded convergence theorem. □

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