# The extremal process of super-Brownian motion

Yan-Xia Ren[a,1], Renming Song[b,2], Rui Zhang[c,*,3]

[a] *LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing, 100871, PR China*
[b] *Department of Mathematics, University of Illinois, Urbana, IL 61801, USA*
[c] *School of Mathematical Sciences & Academy for Multidisciplinary Studies, Capital Normal University, Beijing, 100048, PR China*

## Abstract

In this paper, we establish limit theorems for the supremum of the support, denoted by $M_t$, of a supercritical super-Brownian motion $\{X_t, t \geq 0\}$ on $\mathbb{R}$. We prove that there exists an $m(t)$ such that $(X_t - m(t), M_t - m(t))$ converges in law, and give some large deviation results for $M_t$ as $t \to \infty$. We also prove that the limit of the extremal process $\mathcal{E}_t := X_t - m(t)$ is a Poisson random measure with exponential intensity in which each atom is decorated by an independent copy of an auxiliary measure. These results are analogues of the results for branching Brownian motions obtained in Arguin et al. (2013), Aïdékon et al. (2013) and Roberts (2013).
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\* Corresponding author.
*E-mail addresses:* yxren@math.pku.edu.cn (Y.-X. Ren), rsong@illinois.edu (R. Song), zhangrui27@cnu.edu.cn (R. Zhang).

# 1. Introduction

## 1.1. Super-Brownian motion

Let $\psi$ be a function of the form:

$$\psi(\lambda) = -\alpha\lambda + \beta\lambda^2 + \int_0^\infty \left(e^{-\lambda y} - 1 + \lambda y\right)n(dy), \quad \lambda \geq 0,$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $n$ is a $\sigma$-finite measure satisfying

$$\int_0^\infty (y^2 \wedge y)n(dy) < \infty.$$

$\psi$ is called a branching mechanism. We will always assume that $\lim_{\lambda\to\infty} \psi(\lambda) = \infty$. Let $\{B_t, \mathrm{P}_x\}$ be a standard Brownian motion, and $\mathrm{E}_x$ be the corresponding expectation. In this paper we will consider a super-Brownian motion $X$ on $\mathbb{R}$ with branching mechanism $\psi$.

Let $\mathcal{B}^+(\mathbb{R})$ (resp. $\mathcal{B}_b^+(\mathbb{R})$) be the space of non-negative (resp. non-negative bounded) Borel functions on $\mathbb{R}$, and let $\mathcal{M}_F(\mathbb{R})$ be the space of finite measures on $\mathbb{R}$, equipped with the topology of weak convergence. A super-Brownian motion $X$ with branching mechanism $\psi$ is a Markov process taking values in $\mathcal{M}_F(\mathbb{R})$. The existence of such superprocesses is well-known, see, for instance, [17,18] or [35]. For any $\mu \in \mathcal{M}_F(\mathbb{R})$, we denote the law of $X$ with initial configuration $\mu$ by $\mathbb{P}_\mu$, and the corresponding expectation by $\mathbb{E}_\mu$. As usual, we use the notation: $\langle \phi, \mu \rangle := \int_\mathbb{R} \phi(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu \rangle$. Then for all $\phi \in \mathcal{B}_b^+(\mathbb{R})$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{E}_\mu \left(e^{-\langle \phi, X_t \rangle}\right) = \langle u_\phi(t, \cdot), \mu \rangle, \tag{1.1}$$

where $u_\phi(t, x)$ is the unique positive solution to the equation

$$u_\phi(t, x) + \mathrm{E}_x \int_0^t \psi(u_\phi(t - s, B_s))ds = \mathrm{E}_x \phi(B_t). \tag{1.2}$$

Note that the integral equation (1.2) is equivalent to the equation:

$$\frac{\partial}{\partial t}u_\phi(t, x) - \frac{1}{2}\frac{\partial^2}{\partial x^2}u_\phi(t, x) = -\psi(u_\phi(t, x)), \quad t > 0, x \in \mathbb{R}, \tag{1.3}$$

with initial condition $u_\phi(0, x) = \phi(x)$. Moreover, $\lim_{t\to 0} u_\phi(t, x) = \phi(x)$, if $\phi$ is a nonnegative bounded continuous function on $\mathbb{R}$.

$X$ is called a supercritical (critical, subcritical) super-Brownian motion if $\alpha > 0 (= 0, < 0)$. In this paper, we only deal with the supercritical case, that is $\alpha > 0$.

## 1.2. Maximal position of super-Brownian motion

The maximal position $M_t$ of branching-Brownian motions has been studied intensively. Without loss of generality, we assume in this subsection that the branching rate is 1, and the offspring distribution $\{p_k\}$ satisfies $p_0 = 0$ and the mean of the offspring distribution is 2. Denote by $\mathbf{P}_{\delta_0}$ the law of branching Brownian motion starting from one point located at 0. In the seminal paper [28], Kolmogorov, Petrovskii and Piskounov proved that $M_t/t \to \sqrt{2}$ in probability, which implies that the leading order of $M_t$ is $\sqrt{2}t$. In [12], Bramson provided a log correction to the leading order of $M_t$. He proved in [12] (see also [13]) that, under some moment conditions, $\mathbf{P}_{\delta_0}(M_t - m(t) \leq x) \to 1 - w(x)$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$ and $w(x)$ is a traveling wave solution. In [31], Lalley and Sellke gave a probabilistic representation of the traveling wave solution in terms of the limit of the derivative

martingale of branching Brownian motion. In [42], Roberts gave another proof of Bramson's result and also an almost sure fluctuation result of $M_t$. Large deviation results for $M_t$ were obtained by Chauvin and Rouault in [15,16].

Beyond the behavior of the maximal displacement of branching Brownian motions, the full statistics of the extremal configurations was studied in Arguin et al. [3–5] and Aïdékon et al. [2]. Assume the particles alive at time $t$ are ordered decreasingly: $x_1^t \geq x_2^t \geq \cdots \geq x_{n(t)}^t$, where $n(t)$ is the number of particles alive at time $t$. It is clear that $x_1^t$ is the maximum position $M_t$ at time $t$. Arguin et al. [4,5] studied the limit property of the extremal process of branching Brownian motion, which is the random measure defined by

$$\mathcal{E}_t := \sum_{j=1}^{n(t)} \delta_{x_j^t - m(t)}.$$

Note that $\mathcal{E}_t = Y_t - m(t)$, where $Y_t$ is the measure corresponding to configuration of the positions of the particles alive at time $t$. In [5], using the results of [13], Arguin et al. first proved that $\mathcal{E}_t$ converges in law, which implies the weak convergence of $x_k^t$, the $k$th maximal displacement for each fixed integer $k \geq 1$, and then gave a rigorous characterization of the limiting extremal process. It was proved in [5] that the limiting process is a (randomly shifted) Poisson cluster process, where the positions of the clusters form a Poisson point process with an exponential intensity measure. The law of the individual clusters is characterized as a branching Brownian motion conditioned to perform unusually large displacements. Almost at the same time, Aïdékon et al. [2] proved similar results using a totally different method.

In the recent paper [7], Berestycki et al. studied the asymptotic behavior of the extremal particles of branching Ornstein–Uhlenbeck processes. For inhomogeneous branching Brownian motions, many papers discussed the growth rate of the maximal position, see Bocharov and Harris [9,10] and Bocharov [8] for the case with catalytic branching at the origin, Shiozawa [44], Nishimori et al. [39], Lalley and Sellke [32,33] for the case with some general branching mechanisms. For branching random walks, we refer the readers to Hu et al. [22], Aïdékon [1], Madaule [36] and Carmona et al. [14].

Unlike the case of branching Brownian motions or branching random walks, there are very few results for the supremum of super-Brownian motions, see [19,30]. Let $X_t$ be the super-Brownian motion in Section 1.1 and let $M_t$ be the supremum of the support of $X_t$. We will prove that, under some conditions, $\mathbb{P}_{\delta_0}(M_t - m(t) \leq x) \to e^{-w(x)}$ as $t \to \infty$ for all $x \in \mathbb{R}$, where $m(t) := \sqrt{2\alpha} t - \frac{3}{2\sqrt{2\alpha}} \log t$ and $w$ is a traveling wave solution. We also give some large deviation results for $M_t$. In analogy to the case of branching Brownian motions, we will call the random measure $\mathcal{E}_t := X_t - m(t)$ the extremal process of the super-Brownian motion $X$, which is simply the super-Brownian motion seen from the position $m(t)$. We will generalize the results in [5] to super-Brownian motions and study the limit of $\mathcal{E}_t$. We will give the precise statements of our main results in Section 1.4.

Our proofs depend heavily on the convergence of solutions of the Kolmogorov–Petrovsky–Piscounov (KPP) equation (1.3), with general initial conditions not necessarily bounded between 0 and 1, to traveling wave solutions.

### 1.3. KPP equation related to super-Brownian motion

The classical KPP equation is a semilinear equation of the form

$$u_t(t, x) - \frac{1}{2} u_{xx}(t, x) = f(u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}. \tag{1.4}$$

The KPP equation has been studied for many years analytically, see for example, Kolmogorov et al. [28], Fisher [20], Aronson et al. [6], Bramson [13], Lau [34], Volpert et al. [45].

In [13], the nonlinear function $f$ can be any function on [0, 1] satisfying

$$f \in C^1[0, 1], \ f(0) = f(1) = 0, \ f(u) > 0 \text{ for } u \in (0, 1);$$
$$f'(0) = 1, \ f'(u) \leq 1, \text{ for } 0 < u \leq 1, \tag{1.5}$$

and

$$1 - f'(u) = O(u^\rho) \quad (\text{as } u \to 0) \text{ for some } \rho > 0. \tag{1.6}$$

Kolmogorov et al. [28] showed that under condition (1.5) and with Heaviside initial condition $u(0, x) = 1_{(-\infty, 0)}(x)$, (1.4) has a unique solution $u(t, x)$ satisfying

$$\lim_{t \to \infty} u(t, m(t) + x) = w(x), \quad \text{uniformly in } x \in \mathbb{R}, \tag{1.7}$$

for some centering term $m(t)$, where $m(t)$ satisfies $m(t) = \sqrt{2}t + o(t)$ as $t \to \infty$, and $w$ is a traveling wave solution, which is a function solving the ordinary differential equation $\frac{1}{2}w_{xx} + \sqrt{2}w_x + f(w) = 0$, and satisfying $0 < w(x) < 1$, $\lim_{x \to \infty}(x) = 0$, $\lim_{x \to -\infty} w(x) = 1$. Bramson [13] improved the above result in two aspects: first the initial condition $u(0, x)$ is a general function between 0 and 1, not just the Heaviside initial condition $u(0, x) = 1_{(-\infty, 0)}(x)$; secondly he proved that if in addition $f$ satisfies (1.6) and the initial condition $u(0, x)$ satisfy some integrability condition, (1.7) holds with $m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$. Note that, since 0 and 1 are two special solutions, it follows from the maximum principle that any solution of (1.4), with initial condition bounded between 0 and 1, must be bounded between 0 and 1.

An interesting link between branching Brownian motions and partial differential equations was observed by McKean [38] (see also Ikeda, Nagasawa and Watanabe [23–25]): $u(t, x) := \mathbf{P}_{\delta_0}(M_t > x)$ solves the KPP equation (1.4) with initial condition $u(0, x) = 1_{(-\infty, 0)}(x)$ and with $f(u) = (1 - u) - \sum_{k=0}^{\infty} p_k(1 - u)^k$, where $\{p_k, k \geq 0\}$ is the offspring distribution and the branching rate is 1. Moreover, if $p_0 = 0$, $\sum_k kp_k = 2$, and $\sum_k k^{1+\rho} p_k < \infty$, then $f(u)$ satisfies conditions (1.5) and (1.6). In probabilistic language, (1.7) gives the convergence in distribution for $M_t - m(t)$. There are also some papers using branching Brownian motions to study traveling wave solutions to the KPP equation, see [21,29], for instance. For a nice exposition of extremal processes of branching Brownian motions and their applications, see the book [11].

It follows from (1.3) that the super-Brownian motion $X$ is related to the KPP equation with $f = -\psi$. It is natural to use this relationship to investigate the maximal position of super-Brownian motions. Let $\lambda^*$ be the largest root of the equation $\psi(\lambda) = 0$. Since $\psi'(0) = -\alpha < 0$, $\psi(\infty) = \infty$, it follows from the strict convexity of $\psi$ that $\lambda^* > 0$ exists. Note that 0 and $\lambda^*$ are two special solutions of (1.3). One might think that the role of 0 and $\lambda^*$ for the KPP (1.3) corresponding to super-Brownian motions is similar that of 0 and 1 for the KPP equation (1.4) corresponding to branching Brownian motions. However, for super-Brownian motions we need to consider general non-negative solutions of the corresponding KPP equation (1.3) with initial condition $u(0, x)$ not necessarily bounded between 0 and $\lambda^*$. In this paper, we will first generalize Bramson's results in [13] to general non-negative solutions of the KPP equation (1.3) associated with super-Brownian motions, with initial conditions not necessarily bounded between 0 and $\lambda^*$, see (1.18) for example. Let $u_\phi(t, x)$ be a non-negative solution to (1.3) with initial condition $\phi$. In this paper, we will prove that there also exists some function $m(t)$ such that, for general initial condition $\phi$, $u_\phi(t, m(t) + x)$ converges to some traveling wave solution. More precisely, we consider non-increasing traveling wave solutions $w$ with speed

$\sqrt{2\alpha}$ to Eq. (1.3) such that

$$\lim_{x\to\infty} w(x) = 0, \qquad \lim_{x\to-\infty} w(x) = \lambda^*.$$

By a non-increasing traveling wave solution with speed $\sqrt{2\alpha}$ to (1.3), we mean a non-negative non-increasing function $w$ such that $w(x - \sqrt{2\alpha}t)$ is a solution to (1.3). Clearly, $w$ satisfies

$$\frac{1}{2}w_{xx} + \sqrt{2\alpha}w_x - \psi(w) = 0.$$

We will give an exact asymptotic expression for $m(t)$. We will then use these results to study asymptotic properties of the supremum of the support and the extremal process of the super-Brownian motion $X$.

## 1.4. Main results

We will assume that $\psi$ satisfies the following two conditions:

**(H1)** There exists $\gamma > 0$ such that

$$\int_1^\infty y(\log y)^{2+\gamma} n(dy) < \infty. \tag{1.8}$$

**(H2)** $\psi$ satisfies

$$\int^\infty \frac{1}{\sqrt{\int_{\lambda^*}^\xi \psi(u)du}} d\xi < \infty. \tag{1.9}$$

Let $\mathcal{R}$ be the smallest closed set such that supp $X_t \subseteq \mathcal{R}$, $t \geq 0$. It is known (cf. [43]) that **(H2)** implies Grey's condition

$$\int^\infty \frac{1}{\psi(\lambda)} d\lambda < \infty \tag{1.10}$$

and that

$$\mathbb{P}_\mu(\mathcal{R} \text{ is compact}) = e^{-\lambda^*\|\mu\|}.$$

It is well known that $\{\|X_t\|\}$ is a continuous state branching process and that, under condition (1.10),

$$\mathbb{P}_\mu(\|X_t\| = 0) > 0 \tag{1.11}$$

and $\lim_{t\to\infty} \mathbb{P}_\mu(\|X_t\| = 0) = e^{-\lambda^*\|\mu\|}$. Denote $\mathcal{S} := \{\forall t \geq 0, \|X_t\| > 0\}$.

For some of our results, we also need the following stronger assumption:

**(H3)** There exist $\vartheta \in (0, 1]$ and $a > 0, b > 0$ such that

$$\psi(\lambda) \geq -a\lambda + b\lambda^{1+\vartheta}, \quad \lambda > 0.$$

Clearly, condition **(H3)** implies **(H2)**. In particular, **(H3)** holds if $\beta > 0$. Actually, condition **(H3)** is only used in proving Lemma 3.1.

Note that super-Brownian motions have been used to study traveling wave solutions to the KPP equation (1.3), see [29,30], for instance. For convenience, we write $\mathbb{P} := \mathbb{P}_{\delta_0}$ and $\mathbb{E} := \mathbb{E}_{\delta_0}$. Define, for $t \geq 0$,

$$Z_t := \langle (\sqrt{2\alpha}t - \cdot)e^{-\sqrt{2\alpha}(\sqrt{2\alpha}t - \cdot)}, X_t \rangle.$$

It has been proven in [30] that $\{Z_t, t \geq 0\}$ is a martingale, which is called the derivative martingale, and that $Z_t$ has an almost sure non-negative limit $Z_\infty$ as $t \to \infty$. Furthermore, $Z_\infty$ is almost surely positive on $\mathcal{S}$ if and only if

$$\int_1^\infty y(\log y)^2 n(dy) < \infty. \tag{1.12}$$

Clearly, (1.8) implies (1.12). Thus $Z_\infty$ is almost surely positive on $\mathcal{S}$. The traveling wave solution with speed $\sqrt{2\alpha}$ to (1.3) is given by

$$w(x) = -\log \mathbb{E}\left[\exp\left\{-cZ_\infty e^{-\sqrt{2\alpha}x}\right\}\right] \tag{1.13}$$

and

$$\lim_{x \to \infty} \frac{w(x)}{xe^{-\sqrt{2\alpha}x}} = c. \tag{1.14}$$

For more details, we refer our readers to [30, Theorems 2.4 and 2.6]. Under condition (1.12), $Z_\infty = \Delta(\sqrt{2\alpha})$, where $\Delta(\sqrt{2\alpha})$ is defined in [30, (40)]. Eq. (1.14) follows from the last equality in the proof of [30, Theorem 2.1(iii) and (iv)].

Let $\mathcal{C}_c(\mathbb{R})(\mathcal{C}_c^+(\mathbb{R}))$ be the class of all the (nonnegative) continuous functions with compact support. Let $\mathcal{M}_R(\mathbb{R})$ be the space of all the Radon measures on $\mathbb{R}$ equipped with the vague topology, see [27, p. 12]. Recall that for any random measures $\mu_t, \mu \in \mathcal{M}_R(\mathbb{R})$, $\mu_t \to \mu$ in distribution if and only if for any $f \in \mathcal{C}_c(\mathbb{R})$, $\langle f, \mu_t \rangle \to \langle f, \mu \rangle$ in distribution, see [27, Lemma 4.11]. It follows from [26, Corollary 4.5] that, for random measures $\mu_t, \mu \in \mathcal{M}_R(\mathbb{R})$, $\mu_t \to \mu$ in distribution is equivalent to $\langle f, \mu_t \rangle \to \langle f, \mu \rangle$ in distribution for any $f \in \mathcal{C}_c^+(\mathbb{R})$.

For any $z \in \mathbb{R}$ and function $f$ on $\mathbb{R}$, we define the shift operator $\theta_z f$ by $\theta_z f(y) := f(y+z)$, and for $\mu \in \mathcal{M}_R(\mathbb{R})$, we define $\mathcal{T}_z \mu$ by $\int f(y) \mathcal{T}_z \mu(dy) := \int f(y+z)\mu(dy)$. Sometimes, we also write $\mathcal{T}_z \mu$ as $\mu + z$. We define the rightmost point $M(\mu)$ of $\mu \in \mathcal{M}_R(\mathbb{R})$ by $M(\mu) := \sup\{x : \mu(x, \infty) > 0\}$. Here we use the convention that $\sup \emptyset = -\infty$. The supremum $M_t$ of the support of our super-Brownian motion $X_t$ is simply $M(X_t)$.

For any $\phi \in \mathcal{B}_b^+(\mathbb{R})$, we define

$$U_\phi(t, x) := -\log \mathbb{E}\left[\exp\left\{-\int_{\mathbb{R}} \phi(y - x)X_t(dy)\right\}\right]; \tag{1.15}$$

$$V_\phi(t, x) := -\log \mathbb{E}\left[\exp\left\{-\int_{\mathbb{R}} \phi(y - x)X_t(dy)\right\}, M_t \leq x\right]; \tag{1.16}$$

$$V(t, x) := -\log \mathbb{P}(M_t \leq x). \tag{1.17}$$

By the spatial homogeneity of $X$, we have $U_\phi(t, x) = u_\phi(t, -x)$, and thus $U_\phi(t, x)$ is the unique positive solution to (1.3) with initial condition $U_\phi(0, x) = \phi(-x)$.

By the Markov property of $X$, we have

$$V_\phi(t + r, x) = -\log \mathbb{E}_{\delta_{-x}}\left[e^{-\int \phi(y)X_{t+r}(dy)}, M_{t+r} \leq 0\right]$$

$$= \lim_{\theta \to \infty} -\log \mathbb{E}_{\delta_{-x}}\left[e^{-\langle \phi + \theta \mathbf{1}_{(0,\infty)}, X_{t+r}\rangle}\right] = \lim_{\theta \to \infty} -\log \mathbb{E}_{\delta_{-x}}\left[e^{-\langle u_{\phi + \theta \mathbf{1}_{(0,\infty)}}(r), X_t\rangle}\right]$$

$$= -\log \mathbb{E}_{\delta_{-x}}\left[e^{-\int_{\mathbb{R}} V_\phi(r, -y), X_t(dy)}\right] = U_{V_\phi(r, -y)}(t, x).$$

Thus, for any $r > 0$, $(t, x) \to V_\phi(t + r, x)$ is a solution to (1.3) with initial condition $V_\phi(r, x)$. Thus $V_\phi(t, x)$ is a solution to (1.3) with initial condition

$$V_\phi(0, x) = \begin{cases} \phi(-x), & x \geq 0; \\ \infty, & x < 0. \end{cases} \tag{1.18}$$

The constants introduced in the next result will be used in the statements of our main results.

**Proposition 1.1.**  *Assume that $\phi \in \mathcal{B}^+(\mathbb{R})$ satisfies the following integrability condition at $-\infty$:*

$$\int_0^\infty y e^{\sqrt{2\alpha} y} \phi(-y) \, dy < \infty. \tag{1.19}$$

*(1) If (**H1**) and (**H2**) hold, and $\phi$ is bounded, then the limit*

$$C(\phi) = \lim_{r \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty U_\phi(r, \sqrt{2\alpha}r + y) y e^{\sqrt{2\alpha} y} \, dy$$

*exists, $C(\theta_z \phi) = C(\phi) e^{\sqrt{2\alpha} z}$ for all $z \in \mathbb{R}$ and*

$$\lim_{t \to \infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2\alpha}} \log t} U_\phi(t, \sqrt{2\alpha}t + x) = C(\phi) e^{-\sqrt{2\alpha} x}, \quad x \in \mathbb{R}. \tag{1.20}$$

*If $\phi$ is non-trivial, then $C(\phi) \in (0, \infty)$.*

*(2) If (**H1**) and (**H3**) hold, and if there exists $x_0 < 0$ such that $\phi$ is bounded on $(-\infty, x_0]$, then (1.20) holds, and the limit*

$$\tilde{C}(\phi) := \lim_{r \to \infty} \sqrt{\frac{2}{\pi}} \int_0^\infty V_\phi(r, \sqrt{2\alpha}r + y) y e^{\sqrt{2\alpha} y} \, dy \in (0, \infty)$$

*exists and*

$$\lim_{t \to \infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2\alpha}} \log t} V_\phi(t, \sqrt{2\alpha}t + x) = \tilde{C}(\phi) e^{-\sqrt{2\alpha} x}, \quad x \in \mathbb{R}. \tag{1.21}$$

It has been shown in [30] that $\frac{M_t}{t} \to \sqrt{2\alpha}$, $\mathbb{P}(\cdot | \mathcal{S})$-a.s. Next, we give some large deviation results for $M_t$.

**Theorem 1.2.**  *Under (**H1**) and (**H3**), the following hold:*

*(1)*

$$\lim_{t \to \infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2\alpha}} \log t} \mathbb{P}(M_t > \sqrt{2\alpha}t) = \tilde{C}_0, \tag{1.22}$$

*where $\tilde{C}_0$ is the constant $\tilde{C}(\phi)$ with $\phi = 0$.*

*(2) For any $\delta > 0$, the limit*

$$\hat{C}(\delta) := \lim_{r \to \infty} \sqrt{\frac{2}{\pi}} \delta e^{-\frac{1}{2} \delta^2 r} \int_0^\infty V(r, \sqrt{2\alpha}r + y) y e^{(\sqrt{2\alpha} + \delta)y} \, dy \in (0, \infty)$$

*exists and*

$$\lim_{t \to \infty} \sqrt{t} e^{(\delta^2/2 + \sqrt{2\alpha}\delta)t} \mathbb{P}(M_t > (\sqrt{2\alpha} + \delta)t) = \hat{C}(\delta). \tag{1.23}$$

The analogue of the above results for branching Brownian motions was given in [15,16].

In the remainder of this paper, we define

$$m(t) := \sqrt{2\alpha}t - \frac{3}{2\sqrt{2\alpha}} \log t. \tag{1.24}$$

**Theorem 1.3.** *Suppose that $\phi \in \mathcal{B}^+(\mathbb{R})$ satisfies the integrability condition* (1.19) *at $-\infty$. Let $x(\cdot)$ be a function on $\mathbb{R}$ satisfying $\lim_{t\to\infty} x(t) = x \in \mathbb{R}$.*

*(1) If* (**H1**) *and* (**H2**) *hold, and $\phi$ is bounded, then*

$$\lim_{t\to\infty} U_\phi(t, m(t) + x(t)) = -\log \mathbb{E}\left[\exp\{-C(\phi)Z_\infty e^{-\sqrt{2\alpha}x}\}\right]. \tag{1.25}$$

*(2) If* (**H1**) *and* (**H3**) *hold, and if there exists $x_0 < 0$ such that $\phi$ is bounded on $(-\infty, x_0]$, then* (1.25) *holds, and*

$$\lim_{t\to\infty} V_\phi(t, m(t) + x(t)) = -\log \mathbb{E}\left[\exp\{-\tilde{C}(\phi)Z_\infty e^{-\sqrt{2\alpha}x}\}\right].$$

**Remark 1.4.** In the case when the nonlinear function $f$ satisfies (1.5) and (1.6), Bramson [13] studied the uniform convergence of solutions of the KPP equation (1.4) to traveling wave solutions. More precisely, under the integrability condition (1.19) at $-\infty$ and another growth condition of $\phi$ at $+\infty$, he proved that $u(t, m(t) + x)$ converge uniformly in $x \in \mathbb{R}$, where $u(t, x)$ is the solution of the KPP equation (1.4) with initial condition $u(0, x) = \phi(-x)$. In this paper, our condition on the nonlinear function $-\psi$ is weaker, and we will not study uniform convergence of solutions of (1.3) to traveling wave solutions.

**Remark 1.5.** Applying Theorem 1.3(2) to $\phi = 0$, we get that

$$\lim_{t\to\infty} \mathbb{P}(M_t - m(t) \leq x) = \mathbb{E}(e^{-\tilde{C}_0 e^{-\sqrt{2\alpha}x} Z_\infty}), \quad x \in \mathbb{R}. \tag{1.26}$$

Using this, one can check that for any $x \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}(M_t - m(t) \leq x|\mathcal{S}) = \lim_{t\to\infty} \frac{\mathbb{P}(M_t - m(t) \leq x) - \mathbb{P}(\mathcal{S}^c)}{1 - \mathbb{P}(\mathcal{S}^c)} = \mathbb{E}(e^{-\tilde{C}_0 e^{-\sqrt{2\alpha}x} Z_\infty}|\mathcal{S}).$$

Thus, $M_t - m(t)|_\mathcal{S}$ converges in distribution to a random variable $M^*$.

Let $\mathcal{H}$ be the class of all the nonnegative bounded functions vanishing on $(-\infty, a)$ for some $a \in \mathbb{R}$. It is clear that the functions in $\mathcal{H}$ satisfy the integrability condition (1.19) at $-\infty$. In Lemma 3.3, we will prove that for any $\phi \in \mathcal{H}$, $C(\lambda\phi) \to 0$, $\tilde{C}(\lambda\phi) \to \tilde{C}_0$ as $\lambda \to 0$. Recall that, for any $t > 0$, $\mathcal{E}_t = \mathcal{T}_{-m(t)} X_t$ is the extremal process of $X_t$. Then $U_\phi(t, m(t)) = -\log \mathbb{E}[\exp\{-\langle \phi, \mathcal{E}_t \rangle\}]$. Using the above theorem, we get that, for any $\phi \in \mathcal{H}$,

   (1) under (**H1**) and (**H2**), $\langle \phi, \mathcal{E}_t \rangle$ converges in distribution;
   (2) under (**H1**) and (**H3**), $(\langle \phi, \mathcal{E}_t \rangle, M_t - m(t))|_\mathcal{S}$ jointly converges in distribution.

In Theorems 1.6 and 1.7, we will describe these limits.

In Proposition 3.4, we will prove that, conditioned on $\{M_t > \sqrt{2\alpha}t + z\}$, $X_t - M_t$ converges in distribution to a limit (independent of $z$) denoted by $\Delta$. Let $\Delta_i$, $i \geq 1$, be a sequence of independent, identically distributed random variables with the same law as $\Delta$. Given $Z_\infty$, let $\sum_{j=1}^\infty \delta_{e_j}$ be a Poisson random measure with intensity $\tilde{C}_0 Z_\infty \sqrt{2\alpha} e^{-\sqrt{2\alpha}x} \, dx$. Assume that $\{\Delta_i, i \geq 1\}$ and $\sum_{j=1}^\infty \delta_{e_j}$ are independent.

**Theorem 1.6.** *Assume that* (**H1**) *and* (**H2**) *hold. Then, as $t \to \infty$, $\mathcal{E}_t$ converges in law to a random Radon measure $\mathcal{E}_\infty$ with Laplace transform*

$$\mathbb{E}\left[\exp\left\{-\int \phi(y)\mathcal{E}_\infty(dy)\right\}\right] = \mathbb{E}\left[\exp\{-C(\phi)Z_\infty\}\right], \quad \phi \in \mathcal{H}. \tag{1.27}$$

*Moreover, if, in addition, (**H3**) holds, then*

$$\mathcal{E}_\infty \overset{d}{=} \sum_j \mathcal{T}_{e_j} \Delta_j.$$

For any $t > 0$, we define $\mathcal{E}_t^* = X_t - m(t) - \frac{1}{\sqrt{2}} \log Z_\infty$. Then we have the following result.

**Theorem 1.7.** *Assume that (**H1**) and (**H3**) hold. Conditioned on $\mathcal{S}$, $(\mathcal{E}_t^*, Z_t)$ converges jointly in distribution to $(\mathcal{E}_\infty^*, Z_\infty^*)$, where $Z_\infty^*$ has the same law as $Z_\infty$ conditioned on $\mathcal{S}$, $\mathcal{E}_\infty^*$ and $Z_\infty^*$ are independent, and the Laplace transform of $\mathcal{E}_\infty^*$ is given by*

$$\mathbb{E}[\exp\{-\langle \phi, \mathcal{E}_\infty^* \rangle\}] = \exp\{-C(\phi)\}, \quad \phi \in \mathcal{C}_c^+(\mathbb{R}). \tag{1.28}$$

*Moreover,*

$$\mathcal{E}_\infty^* \overset{d}{=} \sum_j \mathcal{T}_{e_j'} \Delta_j, \tag{1.29}$$

*where $\sum_j \delta_{e_j'}$ is a Poisson random measure with intensity measure $\tilde{C}_0 \sqrt{2\alpha} e^{-\sqrt{2\alpha} x} dx$, which is independent of $\{\Delta_j, j \geq 1\}$.*

Following the arguments in Hu and Shi [22] for branching random walks (see also Roberts [42] for branching Brownian motions), together with Lemmas 4.2 and 4.3, one can prove some almost sure fluctuation results $M_t$. We will not pursue this in this paper.

Note that it suffices to prove the results of this paper for the case $\alpha = 1$ and $\lambda^* = 1$. For the general case, let $v(t, x) = \frac{1}{\lambda^*} u(\alpha^{-1} t, \alpha^{-1/2} x)$. If $u$ is a non-negative solution of (1.3), then, $v$ is a non-negative solution of (1.3) with $\psi$ replaced by $\psi^*(x) = \frac{\psi(\lambda^* x)}{\alpha \lambda^*}$. It is clear that $-\psi^*$ satisfies condition (1.5). Therefore, in the remainder of this paper, we assume that $\alpha = 1$ and $\lambda^* = 1$.

The rest of the paper is organized as follows. In Section 2, we generalize some results in [13] to the case when the nonlinear term satisfies a weaker condition and to general initial conditions. In Section 3.1, we give the proofs of the large deviation results, including Proposition 1.1 and Theorem 1.2. In Section 3.2, we study the convergence of the extremal process. In Section 4, we give the proof of Lemma 3.1.

## 2. Some results on the KPP equation (1.3)

It follows from the Feyman–Kac formula that, if $u$ is a non-negative solution to (1.3), then, for any $0 \leq r < t$,

$$u(t, x) = \mathrm{E}_x \left[ u(r, B_{t-r}) \exp\left\{ \int_0^{t-r} k(u(t - s, B_s)) \, ds \right\} \right], \tag{2.1}$$

where $k(\lambda) = \frac{-\psi(\lambda)}{\lambda}$. Recall that we always assume that $\alpha = 1$ and $\lambda^* = 1$. Note that $k(\lambda)$ is decreasing and $k(\lambda) \leq 1$ for all $\lambda > 0$. We first give some basic results on non-negative solutions $u$ of the KPP equation (1.3) with initial conditions $u(0, \cdot)$ not necessarily bounded between 0 and 1.

**Lemma 2.1.** *Assume that $u(t, x)$ is a solution to (1.3) with initial condition $u(0, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$ satisfying the following integrability condition at $\infty$:*

$$\int_0^\infty y e^{\sqrt{2} y} u(0, y) \, dy < \infty. \tag{2.2}$$

Then for any $t > 0$, $u(t, \cdot)$ is a bounded function also satisfying the integrability condition (2.2) at $\infty$ and

$$\int_0^\infty u(r, \sqrt{2}r + y)ye^{\sqrt{2}y}\,dy < \infty. \tag{2.3}$$

**Proof.** By (2.1) with $r = 0$, we have that $u(t, x) \leq e^t E_x(u(0, B_t))$. Thus, for any $t > 0$, $u(t, \cdot)$ is a bounded function. To finish the proof, it suffices to show that $E_x(u(0, B_t))$ satisfies (2.2). Note that

$$\int_0^\infty E_y(u(0, B_t))ye^{\sqrt{2}y}\,dy = \int_\mathbb{R} \int_0^\infty u(0, x + y)ye^{\sqrt{2}y}\,dy P_0(B_t \in dx)$$

$$= \int_\mathbb{R} \int_x^\infty u(0, y)(y - x)e^{\sqrt{2}(y-x)}\,dy P_0(B_t \in dx).$$

If $x > 0$, we have

$$\int_x^\infty u(0, y)(y - x)e^{\sqrt{2}(y-x)}\,dy \leq \int_0^\infty u(0, y)ye^{\sqrt{2}y}\,dy,$$

and if $x \leq 0$, we have

$$\int_x^\infty u(0, y)(y - x)e^{\sqrt{2}(y-x)}\,dy$$

$$= \int_x^0 u(0, y)(y - x)e^{\sqrt{2}(y-x)}\,dy + \int_0^\infty u(0, y)(y - x)e^{\sqrt{2}(y-x)}\,dy$$

$$\leq \|u_0\|_\infty |x|e^{-x\sqrt{2}} + \int_0^\infty u(0, y)ye^{\sqrt{2}y}\,dy e^{-x\sqrt{2}} + |x|e^{-x\sqrt{2}} \int_0^\infty u(0, y)e^{\sqrt{2}y}\,dy$$

$$\leq c(1 + |x|)e^{-x\sqrt{2}}.$$

Thus,

$$\int_0^\infty ye^{\sqrt{2}y}u(t, y)\,dy \leq e^t \int_0^\infty E_y(u(0, B_t))ye^{\sqrt{2}y}\,dy < \infty.$$

Using a similar argument, we can get (2.3). The proof is now complete. $\quad\square$

**Lemma 2.2** (*Maximum Principle*). *Let $v_1(t, x)$ and $v_2(t, x)$ be non-negative functions satisfying*

$$\frac{\partial}{\partial t}v_2(t, x) - \frac{1}{2}\frac{\partial^2}{\partial x^2}v_2(t, x) + \psi(v_2(t, x)) \geq \frac{\partial}{\partial t}v_1(t, x) - \frac{1}{2}\frac{\partial^2}{\partial x^2}v_1(t, x) + \psi(v_1(t, x)),$$
$$t > 0, x \in (a, b),$$

*and*

$$v_1(0, x) \leq v_2(0, x), \quad x \in (a, b),$$

*where $-\infty \leq a < b \leq \infty$. Moreover, if $a > -\infty$, we assume $v_1(t, a) \leq v_2(t, a)$ for all $t > 0$, and if $b < \infty$, we assume $v_1(t, b) \leq v_2(t, b)$ for all $t > 0$. Then we have that*

$$v_1(t, x) \leq v_2(t, x), \quad t > 0, x \in (a, b).$$

**Proof.** The proof is a slight modification of the proof of [13, Proposition 3.1], using [41, Theorem 3.4]. See also the proof of [11, Proposition 6.4]. We omit the details. $\quad\square$

**Lemma 2.3.** *Assume that $u_1$, $u_2$ and $u_3$ are solutions to* (1.3) *with non-negative bounded initial conditions.*

(1) *If for some $c > 1$, $u_1(0, \cdot) \leq cu_2(0, \cdot)$, then $u_1(t, x) \leq cu_2(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$.*

(2) *If $u_3(0, \cdot) \leq u_1(0, \cdot) + u_2(0, \cdot)$, then $u_3(t, x) \leq u_1(t, x) + u_2(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$.*

**Proof.** (1) Let $v(t, x) = cu_2(t, x)$. Then

$$v_t - \frac{1}{2}v_{xx} = -c\psi(u_2) \geq -\psi(v),$$

here we used the fact that $\psi'(\lambda)$ is increasing. Applying the maximum principle in Lemma 2.2, we get that

$$u_1(t, x) \leq cu_2(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}.$$

(2) Let $v(t, x) := u_1(t, x) + u_2(t, x)$. Since $\psi'(\lambda)$ is increasing, for any $\theta > 0$, the function $\lambda \to \psi(\lambda + \theta) - \psi(\lambda) - \psi(\theta)$ is increasing, which implies that $\psi(\lambda + \theta) \geq \psi(\lambda) + \psi(\theta)$. Then

$$v_t - \frac{1}{2}v_{xx} = -\psi(u_1) - \psi(u_2) \geq -\psi(v).$$

Applying the maximum principle in Lemma 2.2, we get that $u_3(t, x) \leq u_1(t, x) + u_2(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}$. $\quad\square$

For any $\lambda < 1$ and $y > e^{2+\gamma}$, one can easily check that $(\lambda y \wedge 1) \leq |\log \lambda|^{-2-\gamma}(\log y)^{2+\gamma}$. Thus, for any $\lambda < 1$,

$$0 \leq 1 + \psi'(\lambda) = 2\beta\lambda + \int_0^\infty y(1 - e^{-\lambda y})n(dy)$$

$$\leq \left(2\beta + \int_0^{e^{2+\gamma}} y^2 n(dy)\right)\lambda + |\log \lambda|^{-(2+\gamma)} \int_{e^{2+\gamma}}^\infty y(\log y)^{2+\gamma} n(dy) \leq c_1 |\log \lambda|^{-(2+\gamma)},$$

where $\gamma$ is the constant in **(H1)** and $c_1 > 0$ is a constant. Thus **(H1)** implies

$$1 + \psi'(\lambda) \leq c_1 |\log \lambda|^{-(2+\gamma)} \quad \text{for } \lambda < 1. \tag{2.4}$$

Since $\psi'(\lambda)$ is increasing, we have $-k(\lambda) = \psi(\lambda)/\lambda \leq \psi'(\lambda)$. Thus

$$0 \leq 1 - k(\lambda) \leq c_1 |\log \lambda|^{-(2+\gamma)}, \quad \text{for } \lambda < 1. \tag{2.5}$$

In the remainder of this section, we will generalize [13, Proposition 8.3] to non-negative solutions of (1.3) with initial conditions not necessarily bounded between 0 and 1. The main idea of the proof is similar to that of [13]. For the KPP equation (1.3), $-\psi$ plays the role of $f$ in [13]. Condition (1.6) is translated to the following condition on $\psi$:

$$1 + \psi'(\lambda) = O(\lambda^\rho) \quad \text{as } \lambda \to 0, \text{ for some } \rho > 0. \tag{2.6}$$

However, many results in [13] still hold under the weaker condition (2.4). We will clearly spell out the reason when we apply results from [13] under this weaker condition.

In the rest of this section, we use $u(t, x)$ to denote the solution to (1.3) with initial condition $u(0, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$. Let $\tilde{u}(t, x)$ be the solution to (1.3) with $\tilde{u}(0, \cdot) = u(0, \cdot) \wedge 1$. Then, it is clear that

$$\tilde{u}(0, x) \leq u(0, x) \leq s_u \tilde{u}(0, x), \quad x \in \mathbb{R},$$

where $s_u = \sup_x u(0, x) \vee 1$. It follows from Lemma 2.3 that

$$\tilde{u}(t, x) \leq u(t, x) \leq s_u \tilde{u}(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}. \tag{2.7}$$

Since $\tilde{u}(t, x) \in [0, 1]$, we have

$$0 \leq u(t, x) \leq s_u. \tag{2.8}$$

(2.7) and (2.8) will play key roles later in this paper, they allow us to "scale" solutions to KPP equations for Bramson's results to carry over. Let $\tilde{m}(t)$ be the median of $\tilde{u}$, that is

$$\tilde{m}(t) := \sup\{x : \tilde{u}(t, x) \geq 1/2\}.$$

It was proved in [13, (3.22')], without using condition (1.6) (equivalently, (2.6)), that

$$\tilde{m}(t)/t \to \sqrt{2}, \quad t \to \infty.$$

Now we recall some notation from [13], see [13, (6.11)–(6.14), (7.6)–(7.9), (7.42), (7.44)]. In the list of notation below, $\delta \in (\frac{1}{2+\gamma}, 1/2)$, $r > 1$ and $t > 3r$.

- If $L$ is a function on $[0, t]$, define

$$\theta_{r,t} \circ L(s) := \begin{cases} L(s + s^\delta) + 4s^\delta, & r \leq s \leq t/2; \\ L(s + (t - s)^\delta) + 4(t - s)^\delta, & t/2 \leq s \leq t - 2r; \\ L(s), & \text{otherwise.} \end{cases}$$

- The inverse of $\theta_{r,t}$ is defined by

$$\theta_{r,t}^{-1} \circ L := \inf\{l : \theta_{r,t} \circ l \geq L\},$$

  that is

$$\theta_{r,t}^{-1} \circ L(s) = \begin{cases} -\infty, & r \leq s < r + r^\delta; \\ L(u) - 4u^\delta, & r + r^\delta \leq s \leq t/2 + (t/2)^\delta; \\ L(u) - 4(t - u)^\delta, & t/2 + (t/2)^\delta \leq s \leq t - 2r; \\ (L(u) - 4(t - u)^\delta) \vee L(s), & t - 2r < s < t - 2r + (2r)^\delta; \\ L(s), & \text{otherwise,} \end{cases} \tag{2.9}$$

  where for $r + r^\delta \leq s \leq t/2 + (t/2)^\delta$, $u$ is determined by $s = u + u^\delta$; for $t/2 + (t/2)^\delta \leq s \leq t - 2r + (2r)^\delta$, $u$ is determined by $s = u + (t - u)^\delta$.
- $L_{r,t}(s) := \tilde{m}(s) - \frac{s}{t}\tilde{m}(t) + \frac{t-s}{t}\log r, \quad 0 \leq s \leq t$.
- $\underline{\mathcal{L}}_{r,t}(s) := \theta_{r,t}^{-1} \circ L_{r,t}(s)$.
- $\overline{\mathcal{L}}_{r,t}(s) := \theta_{r,t} \circ L_{r,t}(s) \vee \underline{\mathcal{L}}_{r,t}(s) \vee L_{r,t}(s)$.
- For any $x$, define

$$\overline{\mathcal{M}}_{r,t}^x(s) := \begin{cases} \overline{\mathcal{L}}_{r,t}(s) + \frac{s}{t}\tilde{m}(t) - \frac{t-s}{t}\log r, & 0 \leq s \leq t - 2r; \\ \frac{x + \tilde{m}(t)}{2}, & t - 2r < s \leq t. \end{cases} \tag{2.10}$$

- 

$$\underline{\mathcal{M}}_{r,t}'(s) := \begin{cases} \underline{\mathcal{L}}_{r,t}(s) + \frac{s}{t}\tilde{m}(t) - \frac{t-s}{t}\log r, & r + r^\delta \leq s \leq t - 2r; \\ -\infty, & \text{otherwise.} \end{cases} \tag{2.11}$$

- 

$$n_{r,t}(s) = \sqrt{2}r + \frac{s - r}{t - r}(m(t) - \sqrt{2}r), \quad s \in [r, t]. \tag{2.12}$$

The following lemma says that [13, Proposition 7.2] still holds without condition (2.6).

**Lemma 2.4.** *Assume that* **(H1)** *holds. Let* $u(t, x)$ *be a solution to* (1.3) *with initial condition* $u(0, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$ *satisfying the integrability condition* (2.2) *at* $\infty$. *For all* $t > 3r > 0$, *and continuous function* $x(s)$ *with* $x(s) > \overline{\mathcal{M}}_{r,t}^x(t-s)$ *in* $[2r, t-r]$, *we have that*

$$e^{3r-t} \int_{2r}^{t-r} k(u(t-s, x(s))) \, ds \to 1, \quad r \to \infty, \tag{2.13}$$

*uniformly in* $t$.

**Proof.** First note that the proofs of [13, (7.16) and (7.18)] did not use (1.6) (equivalently (2.6)). Thus there exists a constant $C > 0$ such that for $r$ large enough, $s \in [r, t-2r]$ and $y > \tilde{m}(s + (s \wedge (t-s))^\delta)$,

$$\tilde{u}(s, y) \leq C e^{-(s \wedge (t-s))^\delta}.$$

It follows immediately from the key inequality (2.7) that

$$u(s, y) \leq c_2 e^{-(s \wedge (t-s))^\delta}. \tag{2.14}$$

For $r$ large enough and $s \in [r, t-2r]$, by the definition of $\overline{\mathcal{M}}_{r,t}^x$, we have

$$\overline{\mathcal{M}}_{r,t}^x(s) \geq \theta_{r,t} \circ L_{r,t}(s) + \frac{s}{t}\tilde{m}(t) - \frac{t-s}{t} \log r$$

$$= L_{r,t}(s + (s \wedge (t-s))^\delta) + 4(s \wedge (t-s))^\delta + \frac{s}{t}\tilde{m}(t) - \frac{t-s}{t} \log r$$

$$= \tilde{m}(s + (s \wedge (t-s))^\delta) + (s \wedge (t-s))^\delta(4 - \log r/t - \tilde{m}(t)/t) \geq \tilde{m}(s + (s \wedge (t-s))^\delta),$$

where in the last inequality, we use the fact that $4 - \log r/t - \tilde{m}(t)/t \geq 4 - \log t/t - \tilde{m}(t)/t \to 4 - \sqrt{2} > 0$ as $t \to \infty$. Thus, by (2.14), for $r$ large enough, we have

$$e^{t-3r} \geq \exp\left\{ \int_{2r}^{t-r} k(u(t-s, x(s))) \, ds \right\}$$

$$\geq e^{t-3r} \exp\left\{ -\int_{2r}^{t-r} (1 - k(c_2 e^{-(s \wedge (t-s))^\delta})) \, ds \right\}$$

$$\geq e^{t-3r} \exp\left\{ -2 \int_r^\infty (1 - k(c_2 e^{-s^\delta})) \, ds \right\}. \tag{2.15}$$

For $\delta > \frac{1}{2+\gamma}$, by (2.5),

$$\int_r^\infty (1 - k(c_2 e^{-s^\delta})) \, ds \leq \int_r^\infty c_1 |s^\delta - \log c_2|^{-(2+\gamma)} \, ds \to 0, \quad r \to \infty.$$

Thus, the desired result follows immediately. $\square$

The lemma above implies that, under **(H1)**, (7.12) in [13, Proposition 7.2] is valid for $\tilde{u}(t, x)$. Since in the proofs of [13, Proposition 8.1, Corollary 1 on p. 125, Proposition 8.2, Corollary 1 on p. 130 and Corollary 2 on p. 133], only [13, (7.12)] was used, these results hold for $\tilde{u}(t, x)$ under **(H1)**. Thus,

$$\tilde{m}(t) = m(t) + O(1), \tag{2.16}$$

where

$$m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t. \tag{2.17}$$

**Proposition 2.5.** *Assume that $u(t, x)$ is a solution to* (1.3) *with initial condition $u(0, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$ satisfying the integrability condition* (2.2) *at $\infty$. Define, for $0 \leq r \leq t$,*

$$\Psi(r, t, x) := \frac{e^{-\sqrt{2}(x-\sqrt{2}t)}}{\sqrt{2\pi(t-r)}} \int_0^\infty u(r, \sqrt{2}r + y) e^{\sqrt{2}y} e^{-\frac{(x-\sqrt{2}t-y)^2}{2(t-r)}} (1 - e^{-2(x-m(t))y/(t-r)}) \, dy, \tag{2.18}$$

*where $m(t)$ is defined in* (2.17). *Then for $r$ large enough, $t \geq 8r$ and $x \geq m(t) + 9r$,*

$$\gamma(r)^{-1} \Psi(r, t, x) \leq u(t, x) \leq \gamma(r) \Psi(r, t, x), \tag{2.19}$$

*where $\gamma(r) \downarrow 1$ as $r \to \infty$.*

To prove the proposition above, we need the following lemma whose proof is similar to that of [13, (8.62)]. Let $(B_{x,y}^t, P)$ be a Brownian bridge starting from $x$ and ending at $y$ at time $t$, and $E$ be the expectation with respect to $P$.

**Lemma 2.6.** *Assume that $u(t, x)$ solves the KPP equation* (1.3) *with initial condition $u(0, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$ satisfying the integrability condition* (2.2) *at $\infty$. Then for large $r$, $t > 8r$ and $x \geq \tilde{m}(t) + 8r$,*

$$u(t, x) \geq \psi_1(r, t, x)$$

$$:= C_1(r) e^{t-r} \int_{-\infty}^\infty u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} P\left[ B_{x,y}^{t-r}(s) > \bar{\mathcal{M}}_{r,t}^x(t-s), s \in [0, t-r] \right] dy \tag{2.20}$$

*and*

$$u(t, x) \leq \psi_2(r, t, x)$$

$$:= C_2(r) e^{t-r} \int_{-\infty}^\infty u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} P\left[ B_{x,y}^{t-r}(s) > \underline{\mathcal{M}}_{r,t}'(t-s), s \in [0, t-r] \right] dy \tag{2.21}$$

*with $C_1(r) \to 1$, $C_2(r) \to 1$, as $r \to \infty$. Moreover,*

$$1 \leq \frac{\psi_2(r, t, x)}{\psi_1(r, t, x)} \leq \gamma(r), \tag{2.22}$$

*with $\gamma(r) \downarrow 1$, as $r \to \infty$.*

**Proof.** Let

$$\mathcal{A} := \left\{ B_{x,y}^{t-r}(s) > \overline{\mathcal{M}}_{r,t}^x(t-s), s \in [0, t-r] \right\}.$$

It follows from (2.1) that

$$u(t, x) = \int_{-\infty}^\infty u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} E\left[ \exp\left\{ \int_0^{t-r} k(u(t-s, B_{x,y}^{t-r}(s))) \, ds \right\} \right] dy$$

$$\geq \int_{-\infty}^\infty u(r, y) \frac{e^{-\frac{(x-y)^2}{2(t-r)}}}{\sqrt{2\pi(t-r)}} E\left[ \exp\left\{ \int_0^{t-r} k(u(t-s, B_{x,y}^{t-r}(s))) \, ds \right\}, \mathcal{A} \right] dy. \tag{2.23}$$

For $r$ large enough, $t > 8r$, $s \in [0, 2r]$ and $x \geq \tilde{m}(t) + 8r$, it holds that

$$\overline{\mathcal{M}}_{r,t}^x(t - s) = (x + \tilde{m}(t))/2 \geq \tilde{m}(t) + 4r \geq \tilde{m}(t - s) + r,$$

where in the last inequality, we used the fact that $\tilde{m}(t) - \tilde{m}(t - s) = m(t) - m(t - s) + O(1)$ is bounded from below, because $m(t)$ is increasing on $t \geq 1$. Thus, by (2.7) first, and then applying [13, Proposition 8.2] to $\tilde{u}$, we get that on $\mathcal{A}$,

$$u(t - s, B_{x,y}^{t-r}(s)) \leq s_u \tilde{u}(t - s, B_{x,y}^{t-r}(s)) \leq c_3 r e^{-\sqrt{2}r}.$$

It follows from (2.5) that for $r$ large enough,

$$E\left[\exp\{\int_0^{2r} k(u(t - s, B_{x,y}^{t-r}(s))) \, ds\}, \mathcal{A}\right]$$
$$\geq e^{2r} \exp\left\{-2rc_1 |\log(c_3 r e^{-\sqrt{2}r})|^{-(2+\gamma)}\right\} P(\mathcal{A}). \tag{2.24}$$

Note that $2r |\log(c_3 r e^{-\sqrt{2}r})|^{-(2+\gamma)} \to 0$ as $r \to \infty$. By Lemma 2.4, we have

$$E\left[\exp\{\int_{2r}^{t-r} k(u(t - s, B_{x,y}^{t-r}(s))) \, ds\}, \mathcal{A}\right]$$
$$\geq e^{t-3r} \int_r^\infty c_1 |s^\delta - \log c_2|^{-(2+\gamma)} \, ds \, P(\mathcal{A}). \tag{2.25}$$

Combining (2.23)–(2.25), we immediately get (2.20). The proof of (2.21) is similar to that of [13, Proposition 8.3(b)] and the proof of (2.22) is similar to that of [13, (8.62)]. Here we omit the details.  $\square$

**Proof of Proposition 2.5.**  Recall that $n_{r,t}(\cdot)$ is defined by (2.12). First, we claim that, for $s \in [r, t]$,

$$\underline{\mathcal{M}}_{r,t}'(s) \leq n_{r,t}(s) \leq \overline{\mathcal{M}}_{r,t}^x(s). \tag{2.26}$$

It has been proved in [13, Lemma 2.2] that for $y > \sqrt{2}r$ and $x > m(t)$,

$$P\left[B_{x,y}^{t-r}(s) > n_{r,t}(t - s), s \in [0, t - r]\right]$$
$$= P\left[B_{0,0}^{t-r}(s) > -\frac{s}{t-r}(y - \sqrt{2}r) - \frac{t-r-s}{t-r}(x - m(t)), s \in [0, t - r]\right]$$
$$= 1 - \exp\left\{-\frac{2(x - m(t))(y - \sqrt{2}r)}{t - r}\right\},$$

and for $y \leq \sqrt{2}r$, $P\left[B_{x,y}^{t-r}(s) > n_{r,t}(t - s), s \in [0, t - r]\right] = 0$. Thus, combining Lemma 2.6 and (2.26), the desired result follows immediately.

Now we prove the claim. For $r$ large enough, $s \in [r + r^\delta, t/2]$ and $u$ determined by $s = u + u^\delta$,

$$\underline{\mathcal{M}}_{r,t}'(s) = L_{r,t}(u) - 4u^\delta + \frac{s}{t}\tilde{m}(t) - \frac{t-s}{t}\log r$$
$$= \tilde{m}(u) - u^\delta(4 - \frac{\log r + \tilde{m}(t)}{t}) \leq m(u) \leq m(s),$$

where in the last inequality we used that fact $m(t)$ is increasing for $t$ large enough. Similarly, for $s \in [t/2, t - 2r]$, $\underline{\mathcal{M}}_{r,t}'(s) \leq m(s)$. Thus, for all $s \in [0, t]$, $\underline{\mathcal{M}}_{r,t}'(s) \leq m(s)$.

By the definition of $n_{r,t}(s)$, for $r$ large enough,

$$n_{r,t}(s) = \sqrt{2}r + \frac{s-r}{t-r}(m(t) - \sqrt{2}r) = \sqrt{2}s - \frac{s-r}{t-r}\frac{3}{2\sqrt{2}}\log t \geq m(s),$$

where we used the fact that for large $r$, $t \to \log t/(t-r)$ is decreasing. Thus, we get that $\underline{\mathcal{M}}'_{r,t}(s) \leq n_{r,t}(s)$, $s \in [r,t]$.

Now we deal with $\overline{\mathcal{M}}^x_{r,t}(s)$. For $r$ large enough, $s \in [r, t/2]$,

$$\overline{\mathcal{M}}^x_{r,t}(s) \geq m(s + s^\delta) \geq \sqrt{2}s \geq n_{r,t}(s).$$

For $r$ large enough, $s \in [t/2, t-2r]$,

$$\overline{\mathcal{M}}^x_{r,t}(s) \geq m(s + (t-s)^\delta) \geq \sqrt{2}s + \sqrt{2}(t-s)^\delta - \frac{3}{2\sqrt{2}}\log t \geq n_{r,t}(s).$$

For $r$ large enough, $s \in [t-2r, t]$ and $x \geq m(t) + 9r$,

$$\overline{\mathcal{M}}^x_{r,t}(s) = \frac{\tilde{m}(t) + x}{2} \geq m(t) \geq n_{r,t}(s),$$

where the last inequality follows from the fact that, for $r$ large enough, $\sqrt{2}r \leq m(t)$. The proof is now complete.  □

## 3. Proof of main results

We first give a useful lemma. The proof of this lemma will be given in Section 4.

**Lemma 3.1.** *Assume that* (**H1**) *and* (**H3**) *hold. Then, for any $t > 0$ and $\theta > 0$, we have that $V(t, \cdot) \in \mathcal{B}_b^+(\mathbb{R})$, and*

$$\int_0^\infty V(t,x)x e^{\theta x}\, dx < \infty.$$

**Corollary 3.2.** *Assume that* (**H1**) *and* (**H3**) *hold. If $\phi \in \mathcal{B}^+(\mathbb{R})$ satisfies the integrability condition* (1.19) *at $-\infty$ and if there exists $x_0 < 0$ such that $\phi$ is bounded on $(-\infty, x_0]$, then for any $t > 0$, we have that $U_\phi(t, \cdot)$ and $V_\phi(t, \cdot)$ are bounded functions satisfying the integrability condition* (2.2) *at $\infty$.*

**Proof.** First, we assume that $\phi \in \mathcal{B}_b^+(\mathbb{R})$ and satisfies the integrability condition (1.19) at $-\infty$. Note that $U_\phi(t, x)$ satisfies the KPP equation (1.3) with $U_\phi(0, x) = \phi(-x) \in \mathcal{B}_b^+(\mathbb{R})$ satisfying the integrability condition (2.2) at $\infty$, thus by Lemma 2.1, $U_\phi(t, x)$ is bounded and satisfies (2.2). For $V_\phi(t, x)$, it is clear that, by Lemma 2.3(2),

$$V_\phi(t, x) = \lim_{\theta \to \infty} U_{\phi + \theta \mathbf{1}_{(0,\infty)}}(t, x)$$
$$\leq U_\phi(t, x) + \lim_{\theta \to \infty} U_{\theta \mathbf{1}_{(0,\infty)}}(t, x) \leq U_\phi(t, x) + V(t, x),$$

where $V$ is defined in (1.17). By Lemma 3.1, $V(t, x)$ is bounded and satisfies the integrability condition (2.2) at $\infty$. Thus $V_\phi$ is a bounded function satisfying the integrability condition (2.2) at $\infty$, when $\phi \in \mathcal{B}_b^+(\mathbb{R})$ and satisfies the integrability condition (1.19) at $-\infty$.

Now we assume that $\phi \in \mathcal{B}^+(\mathbb{R})$ satisfies the integrability condition (1.19) at $-\infty$ and that there exists $x_0 < 0$ such that $\phi$ is bounded on $(-\infty, x_0]$. Let $\tilde{\phi}(x) := \phi(x)\mathbf{1}_{x \leq x_0} \in \mathcal{B}_b^+(\mathbb{R})$. Note

that

$$V_\phi(t, x) \leq -\log \mathbb{E}\Big[\exp\Big\{-\int_{\mathbb{R}} \tilde{\phi}(y-x)X_t(dy)\Big\}, M_t \leq x + x_0\Big] = V_{\tilde{\phi}(\cdot + x_0)}(t, x + x_0).$$

Thus $V_\phi$ is a bounded function satisfying the integrability condition (2.2) at $\infty$. Since $U_\phi(t, x) \leq V_\phi(t, x)$, $U_\phi$ is also a bounded function satisfying the integrability condition (2.2) at $\infty$. $\square$

### 3.1. Large deviation results

**Proof of Proposition 1.1.** (1) Let $\Psi$ be defined by (2.18) with $u$ replaced by $U_\phi$. We claim that

$$\lim_{t\to\infty} e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \Psi(r, t, \sqrt{2}t + x) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_\phi(r, \sqrt{2}r + y)y e^{\sqrt{2}y} \, dy =: C(\phi, r).$$

(3.1)

Note that, for $r > 1$ and $t \geq 8r$,

$$e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \frac{e^{-\sqrt{2}x}}{\sqrt{2\pi(t-r)}} U_\phi(r, \sqrt{2}r + y)e^{\sqrt{2}y} e^{-\frac{(x-y)^2}{2(t-r)}} \left(1 - e^{-2(x+\frac{3}{2\sqrt{2}} \log t)y/(t-r)}\right)$$

$$\leq c(1 + |x|)U_\phi(r, \sqrt{2}r + y)y e^{\sqrt{2}y}.$$

By (2.3), the right hand side of the inequality above is integrable. So by the dominated convergence theorem, we get that the claim is true and that $C(\phi, r) \in (0, \infty)$.

Since $U_\phi$ is the solution to (1.3) with $U_\phi(0, x) = \phi(-x)$ satisfying the integrability condition (2.2) at $\infty$, by Proposition 2.5, for $r$ large enough, $t \geq 8r$ and $x \geq -\frac{3}{2\sqrt{2}} \log t + 9r$,

$$\gamma(r)^{-1} \Psi(r, t, \sqrt{2}t + x) \leq U_\phi(t, \sqrt{2}t + x) \leq \gamma(r)\Psi(r, t, \sqrt{2}t + x).$$

Thus, by (3.1), we have

$$\gamma(r)^{-1}C(\phi, r) \leq \liminf_{t\to\infty} e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} U_\phi(t, \sqrt{2}t + x)$$

$$\leq \limsup_{t\to\infty} e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} U_\phi(t, \sqrt{2}t + x) \leq \gamma(r)C(\phi, r).$$

(3.2)

Letting $r \to \infty$, by the fact that $\lim_{r\to\infty} \gamma(r) = 1$, we get that

$$\limsup_{r\to\infty} C(\phi, r) \leq \liminf_{t\to\infty} e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} U_\phi(t, \sqrt{2}t + x)$$

$$\leq \limsup_{t\to\infty} e^{\sqrt{2}x} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} U_\phi(t, \sqrt{2}t + x) \leq \liminf_{r\to\infty} C(\phi, r).$$

It follows that $C(\phi) := \lim_{r\to\infty} C(\phi, r)$ exists, and then (1.20) follows immediately. Now we show that $C(\phi) \in (0, \infty)$ if $\phi$ is non-trivial. In fact, by (3.2), we have

$$0 < \gamma(r)^{-1}C(\phi, r) \leq C(\phi) \leq \gamma(r)C(\phi, r) < \infty.$$

For any $z$, it is clear that $U_{\theta_{-z}\phi}(t, x) = U_\phi(t, x + z)$, which implies that

$$C(\phi)e^{-\sqrt{2}z}e^{-\sqrt{2}x} = \lim_{t\to\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} U_\phi(t, \sqrt{2}t + x + z) = C(\theta_{-z}\phi)e^{-\sqrt{2}x},$$

which further implies that $C(\theta_{-z}\phi) = C(\phi)e^{-\sqrt{2}z}$, that is

$$C(\theta_{-z}\phi) = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty U_\phi(r, \sqrt{2}r + y + z)ye^{\sqrt{2}y}\, dy = C(\phi)e^{-\sqrt{2}z}. \tag{3.3}$$

(2) Recall that in this part we assume that **(H1)** and **(H3)** hold, and $\phi \in \mathcal{B}^+(\mathbb{R})$ satisfies the integrability condition (1.19) at $-\infty$ and that there exists $x_0 < 0$ such that $\phi$ is bounded on $(-\infty, x_0]$. Note that, for $t_0 > 0$, $U_\phi(t + t_0, x + \sqrt{2}t_0)$ and $V_\phi(t + t_0, x + \sqrt{2}t_0)$ are the solution to (1.3) with initial data $U_\phi(t_0, x + \sqrt{2}t_0)$ and $V_\phi(t_0, x + \sqrt{2}t_0)$ respectively. By Corollary 3.2, we have that $U_\phi(t_0, x + \sqrt{2}t_0)$ and $V_\phi(t_0, x + \sqrt{2}t_0)$ are bounded and satisfy the integrability condition (2.2) at $\infty$. It follows from (1.20) (under the assumption of part (1)) that

$$\lim_{t\to\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} V_\phi(t, x + \sqrt{2}t) = \lim_{t\to\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} V_\phi(t + t_0, x + \sqrt{2}t_0 + \sqrt{2}t) = Ce^{-\sqrt{2}x},$$

where

$$C = \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty V_\phi(r + t_0, \sqrt{2}r + \sqrt{2}t_0 + y)ye^{\sqrt{2}y}\, dy$$

$$= \lim_{r\to\infty} \sqrt{\frac{2}{\pi}} \int_0^\infty V_\phi(r, \sqrt{2}r + y)ye^{\sqrt{2}y}\, dy := \tilde{C}(\phi).$$

Similarly, we can show that

$$\lim_{t\to\infty} \frac{t^{3/2}}{\frac{3}{2\sqrt{2}}\log t} U_\phi(t, x + \sqrt{2}t) = C(\phi)e^{-\sqrt{2}x},$$

i.e., (1.20) also holds under the assumption of part (2). The proof is now complete. $\quad\square$

**Proof of Theorem 1.2.** It is clear that (1.22) follows from (1.21) with $\phi = 0$. Now we prove (1.23). For $t_0 > 0$, using Proposition 2.5 with $u(0, x) = V(t_0, \sqrt{2}t_0 + x)$, we get that

$$\gamma(r)^{-1}\Psi(r, t, x) \le V(t_0 + t, \sqrt{2}t_0 + x) \le \gamma(r)\Psi(r, t, x).$$

By Lemma 3.1 and the dominated convergence theorem, we have that

$$\lim_{t\to\infty} \sqrt{t}e^{(\delta^2/2+\sqrt{2}\delta)t} \Psi(r, t, (\sqrt{2} + \delta)t + x)$$

$$= \sqrt{\frac{2}{\pi}}\delta e^{-(\delta+\sqrt{2})x}e^{-\frac{1}{2}\delta^2 r} \int_0^\infty V(t_0 + r, \sqrt{2}r + \sqrt{2}t_0 + y)ye^{(\sqrt{2}+\delta)y}\, dy \in (0, \infty).$$

Now, using arguments similar to that used in the proof of Proposition 1.1(1), we get that

$$\lim_{t\to\infty} \sqrt{t}e^{(\delta^2/2+\sqrt{2}\delta)t} V(t_0 + t, (\sqrt{2} + \delta)t + \sqrt{2}t_0 + x)$$

$$= \sqrt{\frac{2}{\pi}}\delta e^{-(\delta+\sqrt{2})x} \lim_{r\to\infty} e^{-\frac{1}{2}\delta^2 r} \int_0^\infty V(t_0 + r, \sqrt{2}r + \sqrt{2}t_0 + y)ye^{(\sqrt{2}+\delta)y}\, dy$$

$$= \sqrt{\frac{2}{\pi}}\delta e^{\frac{1}{2}\delta^2 t_0}e^{-(\delta+\sqrt{2})x} \lim_{r\to\infty} e^{-\frac{1}{2}\delta^2 r} \int_0^\infty V(r, \sqrt{2}r + y)ye^{(\sqrt{2}+\delta)y}\, dy \in (0, \infty),$$

where the limit above exists. Letting $x = \delta t_0$, we get that

$$\lim_{t\to\infty} \sqrt{t} e^{(\delta^2/2+\sqrt{2}\delta)t} V(t, (\sqrt{2}+\delta)t)$$

$$= \sqrt{\frac{2}{\pi}}\delta \lim_{r\to\infty} e^{-\frac{1}{2}\delta^2 r} \int_0^\infty V(r, \sqrt{2}r + y)y e^{(\sqrt{2}+\delta)y}\, dy := \hat{C}(\delta).$$

It follows that

$$\lim_{t\to\infty} \sqrt{t} e^{(\delta^2/2+\sqrt{2}\delta)t} \mathbb{P}(M_t > (\sqrt{2}+\delta)t) = \hat{C}(\delta). \quad \square$$

### 3.2. The extremal process

In this subsection we give the proofs of our main results–Theorems 1.3, 1.6 and 1.7. Recall that $m(t)$ is defined in (2.17).

#### 3.2.1. Proofs of Theorems 1.3 and 1.6

**Proof of Theorem 1.3.** (1) In this part, we assume that $\phi$ is bounded and satisfies the integrability condition (1.19) at $-\infty$. Define

$$w_\phi(x) := -\log \mathbb{E}(e^{-C(\phi)Z_\infty e^{-\sqrt{2}x}}).\tag{3.4}$$

Recall that (cf. (1.13)–(1.14)) $w$ is a traveling wave to (1.3) and satisfies

$$\lim_{x\to\infty} \frac{w_\phi(x)}{xe^{-\sqrt{2}x}} = C(\phi).\tag{3.5}$$

Let $\Psi$ be defined by (2.18) with $u$ replaced by $U_\phi$. We claim that, for any positive function $z(t)$ with $\lim_{t\to\infty} z(t) = z > 0$,

$$\lim_{t\to\infty} \Psi(r, t, z(t) + m(t)) = C(\phi, r)z e^{-\sqrt{2}z}.\tag{3.6}$$

In fact, for any $t \geq 8r$ and $y \geq 0$,

$$z(t)^{-1} \frac{t^{3/2}}{\sqrt{2\pi(t-r)}} U_\phi(r, \sqrt{2}r + y)e^{\sqrt{2}y} e^{-\frac{(z(t)+m(t)-\sqrt{2}t-y)^2}{2(t-r)}} \left(1 - e^{-2z(t)y/(t-r)}\right)$$

$$\leq c U_\phi(r, \sqrt{2}r + y)y e^{\sqrt{2}y}.\tag{3.7}$$

Thus, we can apply the dominated convergence theorem to get that

$$\lim_{t\to\infty} z(t)^{-1} e^{\sqrt{2}z(t)} \Psi(r, t, z(t) + m(t)) = C(\phi, r),$$

which is the same as (3.6). Put

$$f^{(r,t)}(x) := |\Psi(r, t, x + m(t)) - C(\phi, r)x e^{-\sqrt{2}x}|, \quad x > 0.$$

It follows from the key inequality (2.8) that $U_\phi(t, x) \leq (\|\phi\|_\infty \vee 1)$. Applying Proposition 2.5, we get that, for $r$ large enough, $x > 9r$ and $t \geq 8r$,

$$U_\phi(t, m(t) + x) \leq \gamma(r)\Psi(r, t, x + m(t)).$$

Thus, for any $t \geq 8r$,

$$U_\phi(t, m(t) + x) \leq \gamma(r)C(\phi, r)x e^{-\sqrt{2}x}\mathbf{1}_{x>9r} + \gamma(r)f^{(r,t)}(x)\mathbf{1}_{x>9r} + (\|\phi\|_\infty \vee 1)\mathbf{1}_{x\leq 9r}.$$

Let $u_{1,r}(s, x)$, $u_{2,r}(s, x)$ and $v_{r,t}(s, x)$ be the solutions to the KPP equation (1.3) with initial conditions $C(\phi)xe^{-\sqrt{2}x}1_{x>9r}$, $(\|\phi\|_\infty \vee 1)1_{x\leq 9r}$ and $\gamma(r)f^{(r,t)}(x)1_{x>9r}$ respectively. Then, by Lemma 2.3, we have

$$U_\phi(t + s, m(t) + x) \leq \left(\frac{\gamma(r)C(\phi, r)}{C(\phi)} \vee 1\right) u_{1,r}(s, x) + u_{2,r}(s, x) + v_{r,t}(s, x).$$

Let $a(r) := \frac{\gamma(r)C(\phi,r)}{C(\phi)} \vee 1$. Applying (2.1) with $r = 0$ and using the fact that $k(\lambda) \leq 1$, we get that

$$v_{r,t}(s, x) \leq e^s\gamma(r)E_x(f^{(r,t)}(B_s)1_{B_s>9r}).$$

Thus,

$$\begin{aligned}
&U_\phi(t + s, m(t + s) + x(t + s)) \\
&\leq a(r)u_{1,r}(s, x(t + s) + m(t + s) - m(t)) + u_{2,r}(s, x(t + s) + m(t + s) - m(t)) \\
&\quad + e^s\gamma(r)E_0(f^{(r,t)}(m(t + s) - m(t) + x(t + s) + B_s), m(t + s) - m(t) \\
&\quad + x(t + s) + B_s > 9r).
\end{aligned}$$

Letting $t \to \infty$ and using (3.6), we get

$$\limsup_{t\to\infty} U_\phi(t, m(t) + x(t)) \leq a(r)u_{1,r}(s, x + \sqrt{2}s) + u_{2,r}(s, x + \sqrt{2}s).$$

Since $(\|\phi\|_\infty \vee 1)1_{x<9r}$ satisfies the integrability condition (2.2) at $\infty$, we have by Proposition 1.1(1) that $u_{2,r}(s, x + \sqrt{2}s) \to 0$ as $s \to \infty$. Since $C(\phi)xe^{-\sqrt{2}x}/w_\phi(x) \to 1$ as $x \to \infty$, by [13, Lemma 3.4] (Note that our (1.14), which is exactly [13, (1.13)], holds under **(H1)–(H2)**, thus [13, Lemma 3.4] holds under **(H1)–(H2)**), we get that

$$u_{1,r}(s, x + \sqrt{2}s) \to w_\phi(x), \quad s \to \infty.$$

Now letting $s \to \infty$ and then $r \to \infty$, we get that

$$\limsup_{t\to\infty} U_\phi(t, m(t) + x(t)) \leq \lim_{r\to\infty} a(r)w_\phi(x) = w_\phi(x).$$

On the other hand,

$$\gamma(r)^{-1}C(\phi, r)xe^{-\sqrt{2}x}1_{x>9r} \leq U_\phi(t, m(t) + x) + \gamma(r)^{-1}f^{(r,t)}(x)1_{x>9r}.$$

Using arguments similar as above, we can get that

$$\liminf_{t\to\infty} U_\phi(t, m(t) + x(t)) \geq w_\phi(x).$$

Therefore, we have

$$\lim_{t\to\infty} U_\phi(t, m(t) + x(t)) = w_\phi(x). \tag{3.8}$$

(2) Recall that, in this part, $\phi$ is not necessarily bounded. Applying (3.8) to $(t, x) \to V_\phi(t + t_0, \sqrt{2}t_0 + x)$ and Proposition 1.1(2), we get that

$$\lim_{t\to\infty} V_\phi(t + t_0, m(t) + \sqrt{2}t_0 + x(t)) = -\log\mathbb{E}(e^{-\tilde{C}(\phi)Z_\infty e^{-\sqrt{2}x}}).$$

Since $m(t + t_0) - m(t) - \sqrt{2}t_0 + x(t) \to x$, we get that

$$\lim_{t\to\infty} V_\phi(t + t_0, m(t + t_0) + x(t)) = -\log\mathbb{E}(e^{-\tilde{C}(\phi)Z_\infty e^{-\sqrt{2}x}}),$$

which implies the desired result. Similarly, applying Corollary 3.2 and (3.8) to $(t, x) \rightarrow U_\phi(t + t_0, \sqrt{2}t_0 + x)$, we can show that (3.8) also holds for $\phi$ satisfying the weaker condition of this part. The proof is now complete.  □

Using Theorem 1.3, we get the convergence of the Laplace transforms. To obtain weak convergence, we need to show the continuity of $C(\phi)$ and $\tilde{C}(\phi)$.

**Lemma 3.3.**  *Assume that* (**H1**) *and* (**H2**) *hold. Then for any* $\phi \in \mathcal{H}$,

$$\lim_{\lambda \downarrow 0} C(\lambda \phi) = C(0) = 0. \tag{3.9}$$

*If, in addition,* (**H3**) *holds, then for any* $\phi \in \mathcal{H}$,

$$\lim_{\lambda \downarrow 0} \tilde{C}(\lambda \phi) = \tilde{C}_0. \tag{3.10}$$

**Proof.**  For any $\phi \in \mathcal{H}$, choose $m_\phi$ such that $\phi(x) = 0$ for all $x < m_\phi$. Then we have for all $N > 0$,

$$\mathbb{E}(\exp\{-\lambda\langle\phi, \mathcal{E}_t\rangle\}) \geq \mathbb{E}(\exp\{-\lambda\|\phi\|_\infty \mathcal{E}_t(m_\phi, \infty)\}) \geq e^{-\lambda\|\phi\|_\infty N}\mathbb{P}(\mathcal{E}_t(m_\phi, \infty) \leq N).$$

Letting $t \rightarrow \infty$, $\lambda \rightarrow 0$ and then $N \rightarrow \infty$, by Theorem 1.3 we see that, to prove (3.9), it suffices to show that

$$\lim_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}(\mathcal{E}_t(m_\phi, \infty) > N) = 0. \tag{3.11}$$

Let $g(x) = \mathbf{1}_{(0,\infty)}(x)$, then $u_g(t, x) = -\log \mathbb{E}(\exp\{-X_t(-x, \infty)\})$ is increasing on $x$. For any $n \geq 1$,

$$\mathbb{E}(\mathcal{E}_t(m_\phi, \infty) > N, \exp\{-\langle\theta_{-n}g, \mathcal{E}_{t+1}\rangle\})$$
$$= \mathbb{E}\left(\mathcal{E}_t(m_\phi, \infty) > N, \mathbb{E}_{X_t}\left(\exp\left\{-\int g(y - n - m(t+1))X_1(dy)\right\}\right)\right)$$
$$= \mathbb{E}\left(\mathcal{E}_t(m_\phi, \infty) > N, \exp\left\{-\int u_g(1, x - n - m(t+1) + m(t))\mathcal{E}_t(dx)\right\}\right)$$
$$\leq \mathbb{E}\left(\mathcal{E}_t(m_\phi, \infty) > N, \exp\left\{-u_g(1, m_\phi - n - m(t+1) + m(t))\mathcal{E}_t(m_\phi, \infty)\right\}\right)$$
$$\leq \exp\{-u_g(1, m_\phi - n - m(t+1) + m(t))N\}.$$

Thus, we get that

$$\limsup_{t \rightarrow \infty} \mathbb{P}(\mathcal{E}_t(m_\phi, \infty) > N)$$
$$\leq \limsup_{t \rightarrow \infty} \mathbb{E}(\mathcal{E}_t(m_\phi, \infty) > N, \exp\{-\langle\theta_{-n}g, \mathcal{E}_{t+1}\rangle\})$$
$$+ 1 - \lim_{t \rightarrow \infty} \mathbb{E}(\exp\{-\langle\theta_{-n}g, \mathcal{E}_{t+1}\rangle\})$$
$$\leq \exp\{-u_g(1, m_\phi - n - \sqrt{2})N\} + 1 - \mathbb{E}(\exp\{-C(g)e^{-\sqrt{2}n}Z_\infty\}).$$

Letting $N \rightarrow \infty$ and then $n \rightarrow \infty$, (3.11) follows immediately. Thus (3.9) is valid.

Now we prove (3.10) under the additional assumption (**H3**). It is clear that

$$0 \leq \mathbb{P}(M_t - m(t) \leq 0) - \mathbb{E}(\exp\{-\lambda\langle\phi, \mathcal{E}_t\rangle\}, M_t - m(t) \leq 0) \leq 1 - \mathbb{E}(\exp\{-\lambda\langle\phi, \mathcal{E}_t\rangle\}).$$

Thus, by (3.9) and Theorem 1.3(2) with $x(t) = 0$, we get that

$$\mathbb{E}(\exp\{-\lim_{\lambda \to 0} \tilde{C}(\lambda \phi) Z_\infty\}) = \mathbb{E}(\exp\{-\tilde{C}_0 Z_\infty\}).$$

Now (3.10) follows immediately. The proof is now complete. □

For any $t > 0$, we define $\bar{\mathcal{E}}_t := \mathcal{T}_{-\sqrt{2}t} X_t$.

**Proposition 3.4.** *Assume that* (**H1**) *and* (**H3**) *hold. For any* $z \in \mathbb{R}$, *conditioned on* $\{M_t > \sqrt{2}t + z\}$, $(\bar{\mathcal{E}}_t - z, M_t - \sqrt{2}t - z)$ *converges in distribution to a limit (independent of $z$)* $(\bar{\mathcal{E}}_\infty, Y)$, *where $Y$ is an exponential random variable with parameter $\sqrt{2}$ and for any $\phi \in \mathcal{C}_c^+(\mathbb{R})$,*

$$\mathbb{E}\left[\exp\left\{-\int_{\mathbb{R}} \phi(y) \bar{\mathcal{E}}_\infty(dy)\right\}, Y > x\right] = \frac{\tilde{C}(\theta_x \phi) e^{-\sqrt{2}x} - C(\phi)}{\tilde{C}_0}, \quad x > 0.$$

*Moreover,*

$$(X_t - M_t, M_t - \sqrt{2}t - z)|_{M_t > \sqrt{2}t + z}$$

*converges in law to* $(\Delta, Y)$, *where the random measure $\Delta = \bar{\mathcal{E}}_\infty - Y$ is independent of $Y$.*

**Remark 3.5.** Define $Y_t := X_t(\sqrt{2}t, \infty) = \langle h, \bar{\mathcal{E}}_t \rangle$, where $h(x) = 1_{(0,\infty)}(x)$. It follows from Proposition 3.4 that $Y_t|_{Y_t > 0}$ converges weakly.

**Proof of Proposition 3.4.** First, we show that $M_t - \sqrt{2}t - z|_{M_t > \sqrt{2}t + z}$ converges in distribution to an exponential random variable with parameter $\sqrt{2}$. In fact, by (1.21) with $\phi = 0$, we get that for any $x > 0$,

$$\lim_{t \to \infty} \mathbb{P}(M_t - \sqrt{2}t - z > x | M_t > \sqrt{2}t + z) = \lim_{t \to \infty} \frac{\frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathbb{P}(M_t > \sqrt{2}t + z + x)}{\frac{t^{3/2}}{\frac{3}{2\sqrt{2}} \log t} \mathbb{P}(M_t > \sqrt{2}t + z)} = e^{-\sqrt{2}x}.$$

For any $x > 0$ and $\phi \in \mathcal{B}_b^+(\mathbb{R})$ satisfying the integrability condition (1.19) at $-\infty$, applying Proposition 1.1 several times, we get

$$\lim_{t \to \infty} \mathbb{E}\left(e^{-\int_{\mathbb{R}} \phi(y - \sqrt{2}t - z) X_t(dy)}, M_t > \sqrt{2}t + z + x | M_t > \sqrt{2}t + z\right)$$

$$= \lim_{t \to \infty} \frac{\mathbb{E}\left(e^{-\int_{\mathbb{R}} \phi(y - \sqrt{2}t - z) X_t(dy)}, M_t > \sqrt{2}t + z + x\right)}{\mathbb{P}(M_t > \sqrt{2}t + z)}$$

$$= \lim_{t \to \infty} \frac{1 - \mathbb{E}(e^{-\int_{\mathbb{R}} \phi(y - \sqrt{2}t - z) X_t(dy)}, M_t \le \sqrt{2}t + z + x)}{\mathbb{P}(M_t > \sqrt{2}t + z)}$$

$$\quad - \lim_{t \to \infty} \frac{1 - \mathbb{E}(e^{-\int_{\mathbb{R}} \phi(y - \sqrt{2}t - z) X_t(dy)})}{\mathbb{P}(M_t > \sqrt{2}t + z)}$$

$$= \lim_{t \to \infty} \frac{V_{\theta_x \phi}(t, \sqrt{2}t + z + x)}{V(t, \sqrt{2}t + z)} - \lim_{t \to \infty} \frac{U_\phi(t, \sqrt{2}t + z)}{V(t, \sqrt{2}t + z)}$$

$$= \frac{\tilde{C}(\theta_x \phi) e^{-\sqrt{2}x} - C(\phi)}{\tilde{C}_0},$$

where in the second to last equality above, we used L'Hospital's rule and the facts that $V_{\theta_x \phi}(t, \sqrt{2}t + z + x) \to 0$ and $U_\phi(t, \sqrt{2}t + z) \to 0$ (which are consequences of (1.21) and (1.20) respectively). Now applying Lemma 3.3, we get that for any $\phi \in \mathcal{C}_c^+(\mathbb{R})$,

$$(\langle \phi, \bar{\mathcal{E}}_t - z \rangle, M_t - \sqrt{2}t - z)|_{M_t > \sqrt{2}t + z}$$

converges jointly in distribution. Thus the limit has the form $(\langle \phi, \bar{\mathcal{E}}_\infty \rangle, Y)$ (independent of $z$), where the random measure $\bar{\mathcal{E}}_\infty \in \mathcal{M}_R(\mathbb{R})$.

It follows by [5, Lemma 4.13] that, conditioned on $M_t > \sqrt{2}t + z$, $X_t - M_t$ converges in law to $\bar{\mathcal{E}}_\infty - Y$. Thus,

$$\mathbb{E}\left(e^{-\int_\mathbb{R} \phi(y - M_t) X_t(dy)}, M_t - \sqrt{2}t - z > x | M_t > \sqrt{2}t + z\right)$$
$$= \mathbb{E}\left(e^{-\int_\mathbb{R} \phi(y - M_t) X_t(dy)} | M_t > \sqrt{2}t + z + x\right) \mathbb{P}(M_t > \sqrt{2}t + z + x | M_t > \sqrt{2}t + z)$$
$$\to \mathbb{E}(e^{-\int_\mathbb{R} \phi(y - Y) \bar{\mathcal{E}}_\infty(dy)}) \mathbb{P}(Y > x).$$

The desired independence result follows immediately. $\square$

**Proof of Theorem 1.6.** The weak convergence of $\mathcal{E}_t$ and (1.27) follow immediately from Theorem 1.3(1) and Lemma 3.3. Now we assume that (**H3**) also holds and prove the second assertion of Theorem 1.6. For any $\phi \in \mathcal{C}_c^+(\mathbb{R})$, choose $m_\phi$ such that $\phi(y) = 0$ for all $y < m_\phi$. Then we have

$$\frac{C(\phi)}{\tilde{C}_0} = \lim_{t \to \infty} \frac{1 - \mathbb{E}\left[e^{-\int \phi(y - \sqrt{2}t) X_t(dy)}\right]}{\mathbb{P}(M_t > \sqrt{2}t)}$$
$$= \lim_{t \to \infty} \frac{\mathbb{E}\left[1 - e^{-\int \phi(y - \sqrt{2}t) X_t(dy)}, M_t > \sqrt{2}t + m_\phi\right]}{\mathbb{P}(M_t > \sqrt{2}t)}$$
$$= \lim_{t \to \infty} \mathbb{E}\left[1 - e^{-\int \phi(y - \sqrt{2}t) X_t(dy)} | M_t > \sqrt{2}t + m_\phi\right] \frac{\mathbb{P}(M_t > \sqrt{2}t + m_\phi)}{\mathbb{P}(M_t > \sqrt{2}t)}$$
$$= e^{-\sqrt{2} m_\phi} \lim_{t \to \infty} \mathbb{E}\left[1 - e^{-\int \phi(y + m_\phi - \sqrt{2}t - m_\phi) X_t(dy)} | M_t > \sqrt{2}t + m_\phi\right]$$
$$= e^{-\sqrt{2} m_\phi} \mathbb{E}\left[1 - e^{-\int \phi(y + m_\phi) \bar{\mathcal{E}}_\infty(dy)}\right]$$
$$= e^{-\sqrt{2} m_\phi} \int_0^\infty \sqrt{2} e^{-\sqrt{2}x} \mathbb{E}\left[1 - e^{-\int \phi(y + m_\phi + x) \Delta(dy)}\right] dx$$
$$= \int_{-\infty}^\infty \sqrt{2} e^{-\sqrt{2}x} \mathbb{E}\left[1 - e^{-\int \phi(y + x) \Delta(dy)}\right] dx, \tag{3.12}$$

where in the first, fifth and sixth equalities we used Proposition 3.4, and in the fourth equality we used Proposition 1.1. By the definition of $\sum_j \Delta_j(dx + e_j)$, we deduce that

$$\mathbb{E}\left(e^{-\sum_j \phi(y + e_j) \Delta_j(dy)}\right) = \mathbb{E} \prod_j \left[\mathbb{E}\left(e^{-\int \phi(y + x) \Delta(dy)}\right)\right]_{x = e_j}$$
$$= \exp\left\{-\int \left(1 - \mathbb{E}\left(e^{-\int \phi(y + x) \Delta(dy)}\right)\right) \tilde{C}_0 Z_\infty \sqrt{2} e^{-\sqrt{2}x} dx\right\} = \exp\{-C(\phi) Z_\infty\}. \tag{3.13}$$

The proof is now complete. $\square$

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### 3.2.2. Proof of [Theorem 1.7](#)

**Lemma 3.6.** *Assume that* **(H1)** *and* **(H3)** *hold, and that* $\phi \in \mathcal{B}^+(\mathbb{R})$ *satisfies the integrability condition* [(1.19)](#) *at* $-\infty$ *and that there exists* $x_0 < 0$ *such that* $\phi$ *is bounded on* $(-\infty, x_0]$. *Then*

$$\lim_{s \to \infty} \lim_{t \to \infty} \mathbb{E}\left[ e^{-\int \phi(y - m(t) - \frac{1}{\sqrt{2}} \log Z_s) X_t(dy)} e^{-\theta Z_s}, Z_s > 0 \right] = e^{-C(\phi)} \mathbb{E}\left[ e^{-\theta Z_\infty}, Z_\infty > 0 \right].$$

**Proof.** By the Markov property, we have for $s < t$,

$$\mathbb{E}\left[ \exp\left\{ -\int \phi(y - m(t) - \frac{1}{\sqrt{2}} \log Z_s) X_t(dy) \right\} \exp\{-\theta Z_s\}, Z_s > 0 \right]$$

$$= \mathbb{E}\left[ \exp\left\{ -\int U_\phi(t - s, m(t) + \frac{1}{\sqrt{2}} \log Z_s - y) X_s(dy) \right\} \exp\left\{-\theta Z_s\right\}, Z_s > 0 \right].$$

Now applying [Theorem 1.3](#)(2) and [(3.4)](#), we get that as $t \to \infty$,

$$\mathbb{E}\left[ \exp\left\{ -\int \phi(y - m(t) - \frac{1}{\sqrt{2}} \log Z_s) X_t(dy) \right\} \exp\{-\theta Z_s\}, Z_s > 0 \right]$$

$$\to \mathbb{E}\left[ \exp\left\{ -\int w_\phi(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) X_s(dy) \right\} \exp\left\{-\theta Z_s\right\}, Z_s > 0 \right].$$

For any $L > 0$, define $A(s, L) := \{Z_s > 0, \log Z_s \in [-L, L], M_s \leq \sqrt{2}s - \log s\}$. Then

$$\mathbb{E}\left[ \exp\left\{ -\int w_\phi(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) X_s(dy) \right\} \exp\left\{-\theta Z_s\right\}, Z_s > 0 \right]$$

$$\leq \mathbb{E}\left[ \exp\left\{ -\int w_\phi(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) X_s(dy) \right\} \exp\left\{-\theta Z_s\right\}, A(s, L) \right]$$

$$+ \mathbb{P}(Z_s > 0, |\log Z_s| > L) + \mathbb{P}(M_s > \sqrt{2}s - \log s) := (I) + (II) + (III).$$

Since $\{Z_\infty = 0\} = \mathcal{S}^c$ and $\{Z_\infty > 0\} = \mathcal{S}$ a.s., we have

$$\lim_{L \to \infty} \lim_{s \to \infty} \sup \mathbb{P}(Z_s > 0, |\log Z_s| > L)$$

$$\leq \lim_{L \to \infty} \lim_{s \to \infty} \sup \left( \mathbb{P}(Z_s > 0, |\log Z_s| > L, \mathcal{S}) + \mathbb{P}(X_s \neq 0, \mathcal{S}^c) \right)$$

$$\leq \lim_{L \to \infty} \mathbb{P}(|\log Z_\infty| \geq L, \mathcal{S}) = 0. \tag{3.14}$$

By [(1.26)](#), we have

$$\lim_{s \to \infty} \mathbb{P}(M_s > \sqrt{2}s - \log s) = 0. \tag{3.15}$$

Now we consider $(I)$. Since $\frac{w_\phi(x)}{xe^{-\sqrt{2}x}} \to C(\phi)$, as $x \to \infty$, and on $A(s, L)$, for $y \in \text{supp } X_s$, $\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y \geq \log s - L/\sqrt{2}$, thus for any $\epsilon > 0$, there exists $N$ such that for $s > N$,

$$(1 - \epsilon)C(\phi) \int (\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) e^{-\sqrt{2}(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y)} X_s(dy)$$

$$\leq \int w_\phi(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) X_s(dy)$$

$$\leq (1 + \epsilon)C(\phi) \int (\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y) e^{-\sqrt{2}(\sqrt{2}s + \frac{1}{\sqrt{2}} \log Z_s - y)} X_s(dy).$$

Note that on $A(s, L)$, for $s$ large enough, $\frac{|\log Z_s|}{\sqrt{2}(\sqrt{2}s-y)} \leq \frac{L}{\sqrt{2}\log s} \leq \epsilon$. Thus $(I)$ is less than or equal to

$$\mathbb{E}\left[\exp\left\{-(1-\epsilon)^2 C(\phi)(Z_s)^{-1}\int(\sqrt{2}s-y)e^{-\sqrt{2}(\sqrt{2}s-y)}X_s(dy)\right\}\exp\left\{-\theta Z_s\right\}, A(s, L)\right]$$
$$\leq \exp\left\{-(1-\epsilon)^2 C(\phi)\right\}\mathbb{E}\left[\exp\left\{-\theta Z_s\right\}, Z_s > 0\right]. \tag{3.16}$$

Similarly,

$$(I) \geq \exp\left\{-(1+\epsilon)^2 C(\phi)\right\}\mathbb{E}\left[\exp\left\{-\theta Z_s\right\}, A(s, L)\right]. \tag{3.17}$$

Combining (3.14)–(3.17), letting $s \to \infty$, then $L \to \infty$, and then $\epsilon \to 0$, we get that

$$\lim_{s\to\infty}\mathbb{E}\left[\exp\left\{-\int w_\phi(\sqrt{2}s + \frac{1}{\sqrt{2}}\log Z_s - y)X_s(dy)\right\}\exp\left\{-\theta Z_s\right\}, Z_s > 0\right]$$
$$= \exp\left\{-C(\phi)\right\}\mathbb{E}\left[\exp\left\{-\theta Z_\infty\right\}, Z_\infty > 0\right].$$

The proof is now complete. $\square$

**Proof of Theorem 1.7.** Using arguments similar to that leading to (3.13), we get that for any $\phi \in \mathcal{C}_c^+(\mathbb{R})$,

$$\mathbb{E}\left(e^{-\sum_j \phi(y+e'_j)\Delta_j(dy)}\right) = \exp\left\{-\int\left(1 - \mathbb{E}\left(e^{-\int \phi(y+x)\Delta(dy)}\right)\right)\tilde{C}_0\sqrt{2}e^{-\sqrt{2}x}\,dx\right\}$$
$$= \exp\{-C(\phi)\}.$$

Since $Z_t \to Z_\infty$, we only need to prove that, for any $\phi \in \mathcal{C}_c^+(\mathbb{R})$ and $\theta \geq 0$,

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-\int \phi(y)\mathcal{E}_t^*(dy)}e^{-\theta Z_\infty}, Z_\infty > 0\right] = \exp\{-C(\phi)\}\mathbb{E}[\exp\{-\theta Z_\infty\}, Z_\infty > 0]. \tag{3.18}$$

**Step 1** Define, for any $b > 1$,

$$g_b(x) := \begin{cases} 0, & |x| > b; \\ 1, & |x| < b-1; \\ \text{linear}, & \text{otherwise.} \end{cases}$$

It is clear that $|g_b(x) - g_b(y)| \leq |x - y|$.

First, we consider $\phi(x) = |f(x)|g_b(x)$ where $f(x) = \sum_{i=1}^n \theta_i e^{\beta_i x}$ and $\theta_i, \beta_i \in \mathbb{R}$. Let $\overline{f}(x) := \sum_{i=1}^n |\theta_i|e^{\beta_i x}$. It is clear that $\phi \in \mathcal{C}_c^+(\mathbb{R})$. By Lemma 3.6, to prove (3.18), it suffices to show that

$$\lim_{s\to\infty}\limsup_{t\to\infty}\left|\mathbb{E}\left[e^{-\int \phi(y-m(t)-\frac{1}{\sqrt{2}}\log Z_\infty)X_t(dy)}e^{-\theta Z_\infty}, Z_\infty > 0\right]\right.$$
$$\left. - \mathbb{E}\left[e^{-\int \phi(y-m(t)-\frac{1}{\sqrt{2}}\log Z_s)X_t(dy)}e^{-\theta Z_s}, Z_s > 0\right]\right| = 0. \tag{3.19}$$

For any $K > 0$ and $M > 0$, let $\overline{f}^M(y) = \overline{f}(y)\mathbf{1}_{|y|\leq M+b}$ and

$$A(s, t, K, M) = \{\langle \overline{f}^M, \mathcal{E}_t\rangle > K\} \cup \left\{\frac{1}{\sqrt{2}}|\log Z_\infty| > M\right\} \cup \left\{\frac{1}{\sqrt{2}}|\log Z_s| > M\right\}.$$

Since $\langle \overline{f}^M, \mathcal{E}_t\rangle$ converges weakly and $Z_s \to Z_\infty$, a.s., for any $\epsilon > 0$, there exist $K, M$ such that

$$\lim_{s\to\infty}\limsup_{t\to\infty}\mathbb{P}(A(s, t, K, M), Z_\infty > 0) < \epsilon. \tag{3.20}$$

Note that, if $|y| > |x_1| \vee |x_2| + b$, $\phi(y - x_1) - \phi(y - x_2) = 0$; otherwise,

$$
|\phi(y - x_1) - \phi(y - x_2)|
$$
$$
\leq |f(y - x_1) - f(y - x_2)| + |f(y - x_2)||g_b(y - x_1) - g_b(y - x_2)|
$$
$$
\leq \overline{f}(y) \left[ \sum_j |e^{-\beta_j x_1} - e^{-\beta_j x_2}| + \sum_j e^{-\beta_j x_2}|x_1 - x_2| \right] =: \overline{f}(y) H(x_1, x_2).
$$

By the inequality $|e^{-x} - e^{-y}| \leq 1 - e^{-|x-y|}$ for any $x, y > 0$, we get that on $A(s, t, K, M)^c \cap \{Z_s > 0, Z_\infty > 0\}$,

$$
\left| e^{-\int \phi(y - m(t) - \frac{1}{\sqrt{2}} \log Z_\infty) X_t(dy)} e^{-\theta Z_\infty} - e^{-\int \phi(y - m(t) - \frac{1}{\sqrt{2}} \log Z_s) X_t(dy)} e^{-\theta Z_s} \right|
$$
$$
\leq 1 - \exp\left\{ -\theta|Z_s - Z_\infty| - \langle \overline{f}^M, \mathcal{E}_t \rangle H \left( \frac{1}{\sqrt{2}} \log Z_\infty, \frac{1}{\sqrt{2}} \log Z_s \right) \right\}
$$
$$
\leq 1 - \exp\left\{ -\theta|Z_s - Z_\infty| - KH \left( \frac{1}{\sqrt{2}} \log Z_\infty, \frac{1}{\sqrt{2}} \log Z_s \right) \right\}.
$$

Since $Z_s \to Z_\infty$, the left hand side of (3.19) is no more than

$$
\lim_{s \to \infty} \limsup_{t \to \infty} \left[ \mathbb{P}(Z_s \leq 0, Z_\infty > 0) + \mathbb{P}(A(s, t, K, M), Z_\infty > 0) \right.
$$
$$
\left. + \mathbb{E} \left( 1 - \exp\left\{ -\theta|Z_s - Z_\infty| - KH \left( \frac{1}{\sqrt{2}} \log Z_\infty, \frac{1}{\sqrt{2}} \log Z_s \right) \right\}, Z_s > 0, Z_\infty > 0 \right) \right]
$$
$$
\leq \epsilon.
$$

Now (3.19) follows immediately. Thus, the result holds for $\phi(x)$ of the form specified at the beginning of this step.

**Step 2** We will show that (3.18) holds for $\phi \in C_c^+(\mathbb{R})$. Choose $b > 1$ such that $\phi(x) = 0$ for $|x| > b - 1$. According to the Stone–Weierstrass theorem, for any $n \geq 1$, there exists a polynomial $Q_{n,b}$ such that

$$
\sup_{y \in [e^{-b}, e^b]} |Q_{n,b}(y) - \phi(\log y)| \leq n^{-1},
$$

which is equivalent to that

$$
\sup_{y \in [-b, b]} |Q_{n,b}(e^y) - \phi(y)| \leq n^{-1}.
$$

Let $\phi_{n,b}(y) := |Q_{n,b}(e^y)|g_b(y)$, it is clear that all the functions $\phi_{n,b}$ satisfy the conditions in Step 1, and $|\phi_{n,b}(y) - \phi(y)| \leq n^{-1} g_b(y)$. Thus

$$
\left| \mathbb{E}\left[ e^{-\int \phi(y)\mathcal{E}_t^*(dy)} e^{-\theta Z_\infty}, Z_\infty > 0 \right] - \mathbb{E}\left[ e^{-\int \phi_{n,b}(y)\mathcal{E}_t^*(dy)} e^{-\theta Z_\infty}, Z_\infty > 0 \right] \right|
$$
$$
\leq \mathbb{E}\left[ 1 - e^{-\int |\phi(y) - \phi_{n,b}(y)|\mathcal{E}_t^*(dy)}, Z_\infty > 0 \right]
$$
$$
\leq \mathbb{E}\left[ 1 - e^{-n^{-1} \int g_b(y)\mathcal{E}_t^*(dy)}, Z_\infty > 0 \right].
$$

In Step 1, we have shown that,

$$
\lim_{n \to \infty} \lim_{t \to \infty} \mathbb{E}\left[ 1 - e^{-n^{-1} \int g_b(y)\mathcal{E}_t^*(dy)}, Z_\infty > 0 \right] = \lim_{n \to \infty} (1 - \exp\{-C(n^{-1}g_b)\})\mathbb{P}(Z_\infty > 0) = 0.
$$

Thus we have

$$\lim_{t\to\infty}\mathbb{E}\Big[e^{-\int\phi(y)\mathcal{E}_t^*(dy)}e^{-\theta Z_\infty}, Z_\infty > 0\Big] = \lim_{n\to\infty}\lim_{t\to\infty}\mathbb{E}\Big[e^{-\int\phi_{n,b}(y)\mathcal{E}_t^*(dy)}e^{-\theta Z_\infty}, Z_\infty > 0\Big]$$
$$= \lim_{n\to\infty}\exp\{-C(\phi_{n,b})\}\mathbb{E}\Big[e^{-\theta Z_\infty}, Z_\infty > 0\Big].$$

Since $|\phi_{n,b}(y) - \phi(y)| \le n^{-1}g_b(y)$, by Lemmas 2.3 and 3.3, we have

$$|C(\phi_{n,b}) - C(\phi)| \le C(n^{-1}g_b) \to 0, \quad n \to \infty.$$

Thus, (3.18) is valid for all $\phi \in \mathcal{C}_c^+(\mathbb{R})$. □

In fact, it is not surprising that $\mathcal{E}_\infty^*$ has the decomposition (1.29). A random measure $M$ is said to be exp-$\sqrt{2}$-stable if any $a, b$ satisfying $e^{\sqrt{2}a} + e^{\sqrt{2}b} = 1$, it holds that

$$\mathcal{T}_a M + \mathcal{T}_b \hat{M} \overset{d}{=} M,$$

where $\hat{M}$ is an independent copy of $M$. The following proposition shows that $\mathcal{E}_\infty^*$ satisfies the exp-$\sqrt{2}$-stability:

**Proposition 3.7.** *Under* (**H1**) *and* (**H3**), $\mathcal{E}_\infty^*$ *satisfies the exp-$\sqrt{2}$-stability.*

**Proof.** The Laplace transform of $\mathcal{T}_a \mathcal{E}_\infty^*$ is given by

$$\mathbb{E}(\exp\{-\langle\phi, \mathcal{T}_a\mathcal{E}_\infty^*\rangle\}) = \mathbb{E}(\exp\{-\langle\theta_a\phi, \mathcal{E}_\infty^*\rangle\}) = \exp\{-C(\theta_a\phi)\} = \exp\{-C(\phi)e^{\sqrt{2}a}\},$$
$$\phi \in \mathcal{C}_c^+(\mathbb{R}).$$

Therefore, the desired result follows. □

**Remark 3.8.** Let $M_1, \ldots, M_n$ be a sequence of i.i.d. random measures with the same law as $\mathcal{T}_{-\log n/\sqrt{2}}\mathcal{E}_\infty^*$. Then, by Proposition 3.7, $\mathcal{E}_\infty^*$ is equal in law to $M_1 + \cdots + M_n$. Thus $\mathcal{E}_\infty^*$ is infinitely divisible. Applying [37, Theorem 3.1], we get that for any $\phi \in \mathcal{C}_c^+(\mathbb{R})$,

$$C(\phi) = -\log\mathbb{E}[\exp\{-\langle\phi, \mathcal{E}_\infty^*\rangle\}] = c\int_{\mathbb{R}}\phi(x)e^{-\sqrt{2}x}\,dx$$
$$+ \int_{\mathbb{R}}e^{-\sqrt{2}x}\int_{\mathcal{M}_R(\mathbb{R})\setminus\{0\}}[1 - \exp\{-\langle\phi, \mu\rangle\}]$$
$$\times \mathcal{T}_x\Lambda(d\mu)\,dx,$$

for some constant $c > 0$ and some measure $\Lambda$ on $\mathcal{M}_R(\mathbb{R})\setminus\{0\}$ with the property that for every bounded Borel set $A \subset \mathbb{R}$,

$$\int_{\mathbb{R}}e^{-\sqrt{2}x}\int_{\mathcal{M}_R(\mathbb{R})\setminus\{0\}}[1 \wedge \mu(A - x)]\Lambda(d\mu)\,dx < \infty.$$

Now we choose a function $\phi \in \mathcal{C}_c^+(\mathbb{R})$ such that $\phi(x) = 0$ for any $x < 0$. It is clear that $U_{\lambda\phi}(t, x) \le V(t, x)$. Under (**H1**) and (**H3**), it holds that $C(\lambda\phi) \le \tilde{C}_0 \in (0, \infty)$ for any $\lambda > 0$. This implies that $c = 0$. Thus

$$\mathcal{E}_\infty^* \overset{d}{=} \sum_j \mathcal{T}_{\xi_j} D_j,$$

where $\sum_j \delta_{(x_j, D_j)}$ is Poisson point process with intensity measure $e^{-\sqrt{2}x}\,dx\,\Lambda(d\mu)$. Theorem 1.7 says that $\Lambda(d\mu) = \sqrt{2}\tilde{C}_0\mathbb{P}(\Delta \in d\mu)$ where $\Delta$ is the limit of $X_t - M_t$ conditioned on $\{M_t > \sqrt{2}t\}$.

## 4. Proof of Lemma 3.1

In this section, we will give an upper estimate for $-\log \mathbb{P}_{\delta_x}(X_s([-A, A]^c) = 0, 0 \le s \le t)$, which implies Lemma 3.1. Pinsky [40] has proved a similar result for super-Brownian motions with quadratic branching mechanism. Here we use the idea of [40] to generalize the result to super-Brownian motions with more general branching mechanisms.

**Lemma 4.1** (*Maximum Principle*). *Let $\tilde{\psi}(\lambda) := -a\lambda + b\lambda^{1+\vartheta}$, where $a > 0, b > 0, \vartheta > 0$. Assume that $v_1(x)$ and $v_2(x)$ are two functions defined on $(a_1, a_2)$ such that $v_i(x) \ge (ab^{-1})^{1/\vartheta}$, $i = 1, 2$, $v_1(a_i) \le v_2(a_i)$, $i = 1, 2$, and that*

$$\frac{1}{2} \frac{d^2}{dx^2} v_2(x) - \tilde{\psi}(v_2(x)) \le \frac{1}{2} \frac{d^2}{dx^2} v_1(x) - \tilde{\psi}(v_1(x)), \quad x \in (a_1, a_2).$$

*Then we have that*

$$v_1(x) \le v_2(x), \quad x \in (a_1, a_2).$$

**Proof.** The proof is a slight modification of the proof of [13, Proposition 3.1], using [41, Theorem 3.4]. See also the proof of [11, Proposition 6.4]. We omit the details. $\square$

**Lemma 4.2.** *Let $\tilde{\psi}(\lambda) := -a\lambda + b\lambda^{1+\vartheta}$, where $a > 0, b > 0, \vartheta \in (0, 1]$. For any $A > 0$, there exists an even function $h_A(x)$ on $(-A, A)$ such that*

$$\frac{1}{2} \Delta h_A(x) = \tilde{\psi}(h_A(x)), \quad |x| < A, \tag{4.1}$$

*and that $\lim_{x \to A} h_A(x) = \lim_{x \to -A} h_A(x) = \infty$. Moreover, there exist positive constants $c_1 = c_1(a, b, \vartheta)$, $c_2 = c_2(a, b, \vartheta)$ and $c_3 = c_3(a, b, \vartheta)$ such that*

*(1) $\max\{(ab^{-1})^{1/\vartheta}, c_2 A^{2/\vartheta}(A^2 - x^2)^{-2/\vartheta}\} \le h_A(x) \le (ab^{-1})^{1/\vartheta}(1 + c_1 A^{2/\vartheta}(A^2 - x^2)^{-2/\vartheta})$ for $|x| < A$;*

*(2) $\frac{|h'_A(x)|}{h_A(x)} \le \frac{c_3}{A - |x|}$, for $|x| < A$.*

**Proof. Step 1:** First, for any $m > (ab^{-1})^{1/\vartheta}$, let $h_m(x)$ be the solution to the problem:

$$\frac{1}{2} \Delta h_m(x) = \tilde{\psi}(h_m(x)), \quad |x| < A, \tag{4.2}$$

$$h_m(A) = h_m(-A) = m. \tag{4.3}$$

Clearly $h_m$ is even. Since $(ab^{-1})^{1/\vartheta}$ is a solution of $-a\lambda + b\lambda^{1+\vartheta} = 0$, the maximum principle in Lemma 4.1 implies that $h_m(x) \ge (ab^{-1})^{1/\vartheta}$ for $|x| < A$.

**Step 2** We claim that there exists $c_1 > 0$ such that the function $g(x) = (ab^{-1})^{1/\vartheta}(1 + c_1 A^{2/\vartheta}(A^2 - x^2)^{-2/\vartheta})$ satisfies

$$\frac{1}{2} \Delta g(x) \le \tilde{\psi}(g(x)) = -ag(x) + bg(x)^{1+\vartheta}, \quad |x| < A.$$

Given this claim, we can use the maximum principle in Lemma 4.1 and the fact $\lim_{x \to A} g(x) = \lim_{x \to -A} g(x) = \infty$ to get

$$g(x) \ge h_m(x), \quad |x| < A.$$

Now we prove the claim. Since $\lim_{\lambda \downarrow 0} \frac{-(1+\lambda)+(1+\lambda)^{1+\vartheta}}{\lambda^{1+\vartheta}} = \infty$ and $\lim_{\lambda \to \infty} \frac{-(1+\lambda)+(1+\lambda)^{1+\vartheta}}{\lambda^{1+\vartheta}} = 1$, we have

$$c_4 := \inf_{\lambda \geq 0} \frac{-(1 + \lambda) + (1 + \lambda)^{1+\vartheta}}{\lambda^{1+\vartheta}} \in (0, \infty).$$

Thus, we have that

$$-ag(x) + bg(x)^{1+\vartheta} \geq c_4 a (ab^{-1})^{1/\vartheta} c_1^{1+\vartheta} A^{2+2/\vartheta} (A^2 - x^2)^{-2(1+\vartheta)/\vartheta}.$$

It is clear that, for any $x \in (-A, A)$,

$$\frac{1}{2} \Delta g(x) = (ab^{-1})^{1/\vartheta} c_1 A^{2/\vartheta} 2\vartheta^{-1} [A^2 + (4\vartheta^{-1} + 1)x^2](A^2 - x^2)^{-2-2/\vartheta}$$

$$\leq (ab^{-1})^{1/\vartheta} c_1 2\vartheta^{-1} (4\vartheta^{-1} + 2) A^{2+2/\vartheta} (A^2 - x^2)^{-2-2/\vartheta}.$$

Therefore it suffices to choose

$$c_1 = \left( 4c_4^{-1} a^{-1} \vartheta^{-1} (\frac{2}{\vartheta} + 1) \right)^{1/\vartheta}.$$

**Step 3** For any $\delta > 0$, define $g_\delta(x) := \frac{c_2 A^{2/\vartheta}}{((A+\delta)^2 - x^2)^{2/\vartheta}}$, where $c_2 > (ab^{-1})^{1/\vartheta}$ is a constant. We claim that there exists $c_2 = c_2(a, b, \vartheta) > 0$ such that

$$\frac{1}{2} \Delta g_\delta(x) \geq -ag_\delta(x) + bg_\delta(x)^{1+\vartheta}, \quad |x| < A + \delta. \tag{4.4}$$

Given this claim, we can apply the maximum principle in Lemma 4.1 to get that, for $m$ large enough,

$$h_m(x) \geq g_\delta(x), \quad |x| < A.$$

Now we prove the claim. In fact,

$$\frac{1}{2} \Delta g_\delta(x) \geq c_2 2\vartheta^{-1} A^{2+2/\vartheta} ((A + \delta)^2 - x^2)^{-2-2/\vartheta}$$

and

$$-ag_\delta(x) + bg_\delta(x)^{1+\vartheta} \leq bg_\delta(x)^{1+\vartheta} = bc_2^{1+\vartheta} A^{2/\vartheta+2} ((A + \delta)^2 - x^2)^{-2-2/\vartheta}.$$

Thus we only need to choose

$$c_2 = (2b^{-1}\vartheta^{-1})^{1/\vartheta}.$$

**Step 4** By the maximum principle in Lemma 4.1, $h_m$ is non-decreasing in $m$, thus $h_A(x) := \lim_{m \to \infty} h_m(x)$ exists. Hence for any $\delta > 0$,

$$g_\delta(x) \leq h_A(x) \leq g(x).$$

Letting $\delta \to 0$, we have that, for any $|x| < A$,

$$\frac{c_2 A^{2/\vartheta}}{(A^2 - x^2)^{2/\vartheta}} \leq h_A(x) \leq (ab^{-1})^{1/\vartheta} (1 + c_1 A^{2/\vartheta} (A^2 - x^2)^{-2/\vartheta}).$$

Clearly $\lim_{x \to A} h_A(x) = \lim_{x \to -A} h_A(x) = \infty$.

**Step 5** Now we show that $h_A$ satisfies (4.1). By (4.2), we have that for any $0 < A' < A$,

$$h_m(x) = -E_x \int_0^{\tau_{A'}} \tilde{\psi}(h_m(B_s)) \, ds + E_x(h_m(B_{\tau_{A'}})), \quad x \in (-A', A'),$$

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where $\tau_{A'}$ is the exit time of $B$ from $(-A', A')$. Letting $m \to \infty$ and applying the dominated convergence theorem, we get that

$$h_A(x) = -\mathrm{E}_x \int_0^{\tau_{A'}} \tilde{\psi}(h(B_s)) \, ds + \mathrm{E}_x(h(B_{\tau_{A'}})), \quad x \in (-A', A'),$$

which implies that $h_A$ satisfies (4.1) for $x \in (-A', A')$. Since $A' \in (0, A)$ is arbitrary, $h_A$ satisfies (4.1) for $x \in (-A, A)$.

**Step 6** Finally, we prove that $\frac{|h'_A(x)|}{h_A(x)} \leq \frac{c_3}{A-|x|}$, $\quad |x| < A$. Since $h_A$ is an even function, we have $\frac{|h'_A(x)|}{h_A(x)} = \frac{|h'_A(|x|)|}{h_A(|x|)}$. To prove the desired result, we only need to consider $x \geq 0$. Since $h_A(x) \geq (ab^{-1})^{1/\vartheta}$ and

$$\frac{1}{2} \Delta h_A(x) = \tilde{\psi}(h_A(x)) \geq 0, \quad |x| < A,$$

we know that $h'_A(x)$ is increasing on $(-A, A)$. Since $h_A$ is an even function, we have $h'_A(0) = 0$. Thus, $h'_A(x) \geq 0$ for $x \in [0, A)$, which implies that

$$\frac{h'_A(x)}{h_A(x)} \geq 0, \quad x \in [0, A). \tag{4.5}$$

Define $w_1(x) = \frac{2a(c_1)^\vartheta}{A-x} - \frac{h'_A(x)}{h_A(x)}$, for $x \in [0, A)$. Then, for any $x \in (0, A)$,

$$w_1'(x) = \frac{2(c_1)^\vartheta}{(A-x)^2} - 2(bh_A(x)^\vartheta - a) + \left(\frac{h'_A(x)}{h_A(x)}\right)^2 \geq 0,$$

where the last inequality follows from the fact that

$$bh_A(x)^\vartheta - a \leq a(1 + c_1 A^{2/\vartheta}(A^2 - x^2)^{-2/\vartheta})^\vartheta - a \leq ac_1^\vartheta A^2 (A^2 - x^2)^{-2} \leq \frac{a(c_1)^\vartheta}{(A-x)^2}.$$

Since $h'_A(0) = 0$, then $w_1(0) > 0$. Thus for any $x \in (0, A)$, $w_1(x) \geq w_1(0) > 0$, that is

$$\frac{2a(c_1)^\epsilon}{A-x} \geq \frac{h'_A(x)}{h_A(x)}, \quad x \in [0, A). \tag{4.6}$$

Combining (4.5) and (4.6), we get the desired result. $\quad \square$

**Lemma 4.3.** *Assume* (**H1**) *and* (**H3**) *hold. Then, there exist positive constants* $c_4 = c_4(a, b, \vartheta)$ *and* $c_5 = c_5(a, b, \vartheta)$ *such that for any* $A > 0$ *and* $|x| < A$,

$$-\log \mathbb{P}_{\delta_x}(X_s([-A, A]^c) = 0, \forall s \in [0, t)) \leq h_A(x) \exp\left\{-\left(c_4 \frac{(A-|x|)^2}{t} - at - c_5\right)\right\}. \tag{4.7}$$

**Proof.** Let $\tilde{X}$ be a super-Brownian motion with branching mechanism $\tilde{\psi}(\lambda) := -a\lambda + b\lambda^{1+\vartheta}$. Define

$$h(t, x) := -\log \mathbb{P}_{\delta_x}(\tilde{X}_s([-A, A]^c) = 0, \forall s \leq t)$$

and

$$h_m(t, x) := -\log \mathbb{E}_{\delta_x}\left[\exp\left\{-\int_0^t \langle \phi_m, \tilde{X}_s \rangle \, ds\right\}\right],$$

where $\phi_m \in C^\infty(\mathbb{R})$ satisfies

$$\phi_m(y) = 0, \quad |y| < A, |y| > A + m + 1,$$

$$\phi_m(y) = m, \quad A + \frac{1}{m} \le |y| \le A + m.$$

Then $h(t, x) = \lim_{m \to \infty} h_m(t, x)$ and $h_m(t, x)$ satisfies the equation

$$h_m(t, x) + \mathrm{E}_x \int_0^t \tilde{\psi}(h_m(t - s, B_s))\, ds = \mathrm{E}_x \int_0^t \phi_m(B_s)\, ds.$$

For the display above we refer the readers to [35, Corollay 5.17]. Thus,

$$\frac{\partial h_m}{\partial t}(t, x) - \frac{1}{2}\Delta h_m(t, x) = -\tilde{\psi}(h(t, x)) + \phi_m(x), \quad t > 0,$$

which implies that

$$\frac{\partial h_m}{\partial t}(t, x) - \frac{1}{2}\Delta h_m(t, x) = -\tilde{\psi}(h(t, x)), \quad |x| < A, t > 0.$$

Since $\psi \ge \tilde{\psi}$, then using arguments similar to that used in [35, Corollary 5.18], we get that

$$-\log \mathbb{E}_{\delta_x}\left[\exp\left\{-\int_0^t \langle \phi_m, X_s \rangle\, ds\right\}\right] \le h_m(t, x).$$

Letting $m \to \infty$, we get

$$-\log \mathbb{P}_{\delta_x}(X_s([-A, A]^c) = 0, \forall s \in [0, t]) \le h(t, x),$$

so it suffices to show that the result holds for $h(t, x)$.

Let $f$ be an even function satisfying

$$f \in C^2([-1, 1]), \quad f(y) > 0, \text{ if } -1 < y < 1;$$

$$f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0, \quad f'(1) = 0, \quad f''(1) > 0.$$

$$\sup_{y \in [0,1]} \frac{(f'(y))^2}{f(y)} < \infty. \tag{4.8}$$

It has been proved in the proof of [40, Theorem 1] that such $f$ exists. Define

$$v(t, x) := h_A(x) \exp\left\{c_5 + at - \frac{\delta A^2}{t} f(\frac{x}{A})\right\}, \quad |x| < A,$$

where $c_5, \delta > 0$ are to be fixed later. It is clear that $\lim_{t \to 0} v(t, x) = 0$, $\lim_{|x| \to A} v(t, x) = \infty$, since $\lim_{|x| \to A} h_A(x) = \infty$.

To prove the result, we want to find suitable $c_5, \delta$ such that

$$\frac{\partial v}{\partial t}(t, x) - \frac{1}{2}\Delta v(t, x) \ge av(t, x) - bv(t, x)^{1+\vartheta}, \quad |x| < A. \tag{4.9}$$

Once we have this, we can apply the maximum principle in Lemma 2.2 to get $h_m(t, x) \le v(t, x)$. Letting $m \to \infty$, we arrive at

$$h(t, x) \le v(t, x), \quad |x| < A, t > 0.$$

Since $f''(1) > 0$, we have $\inf_{y \in [0,1]} \frac{f(y)}{(1-y)^2} > 0$. Thus,

$$h(t, x) \le v(t, x) \le h_A(x) \exp\left\{c_5 + at - \frac{c_4(A - |x|)^2}{t}\right\},$$

where $c_4 = \delta \inf_{y \in [0,1]} \frac{f(y)}{(1-y)^2} > 0$.

Now we prove (4.9). Note that, by (4.1), (4.9) is equivalent to, for $x \in [0, A)$, $y = x/A$,

$$a + \frac{\delta A^2}{t^2} f(y) - \frac{\delta^2 A^2}{2t^2}(f'(y))^2 + \frac{h'_A(x)}{h_A(x)} \frac{\delta A}{t} f'(y) + \frac{\delta}{2t} f''(y) \geq -bv(t, x)^\vartheta + bh_A(x)^\vartheta.$$

Note that $\frac{(f'(y))^2}{f(y)}$, $\frac{|f'(y)|}{1-y}$, and $f''(y)$ are all bounded. Let $K$ be the common upper bound. By Lemma 4.2, $\frac{|h'_A(x)|}{h_A(x)} \leq c_3(A - x)^{-1}$. Choose $\delta \in (0, K^{-1})$. It suffices to show that

$$a + \frac{\delta A^2}{2t^2} f(y) - \frac{c_3 K \delta}{t} - \frac{K\delta}{2t} \geq -bv(t, x)^\vartheta + bh_A(x)^\vartheta. \tag{4.10}$$

If $\frac{\delta A^2}{t} f(y) \geq c_5/2$, then the left hand side of (4.10) is bigger than

$$a + \frac{c_5}{4t} - \frac{c_3}{t} - \frac{1}{2t},$$

and by Lemma 4.2, the right hand side of (4.10) is less than

$$bh_A(x)^\vartheta \leq a(1 + c_1 A^{2/\vartheta}(A^2 - x^2)^{-2/\vartheta})^\vartheta \leq a(1 + c_1^\vartheta A^{-2}(1 - y^2)^{-2})$$

$$= a + ac_1^\vartheta(f(y))^{-1}A^{-2}\frac{f(y)}{(1 - y^2)^2} \leq a + a\frac{2\delta c_1^\vartheta K}{c_5 t} \leq a + \frac{2ac_1^\vartheta}{c_5 t}.$$

Thus, when we choose $c_5$ large enough, (4.10) is true.

If $c_5/2 \geq \frac{\delta A^2}{t} f(y)$, then the left hand side of (4.10) is bigger than

$$a - \frac{c_3}{t} - \frac{1}{2t},$$

and the right hand side of (4.10) is less than

$$bh_A(x)^\vartheta(1 - e^{\vartheta c_5/2}) \leq -(e^{\vartheta c_5/2} - 1)bc_2^\vartheta A^2(A^2 - x^2)^{-2}$$

$$= -bc_2^\vartheta(e^{\vartheta c_5/2} - 1)\frac{1}{f(y)A^2} \frac{f(y)}{(1 - y)^2(1 + y)^2}$$

$$\leq -\frac{bc_2^\vartheta \delta(e^{\vartheta c_5/2} - 1)}{2c_5} \inf_{y \in [0,1]} \frac{f(y)}{(1 - y)^2} \frac{1}{t}.$$

Since $\inf_{y \in [0,1]} \frac{f(y)}{(1-y)^2} > 0$, we can choose $c_5$ large enough such that (4.10) is true. The proof is now complete. $\square$

**Proof of Lemma 3.1.** It is clear that

$$\mathbb{E}\left[\exp\left\{-\int_\mathbb{R} \phi(y - x)X_t(dy)\right\}, M_t \leq x\right] \geq \mathbb{P}(\|X_t\| = 0) > 0,$$

where the last inequality follows from (1.11). Thus

$$V(t, x) \leq -\log \mathbb{P}(\|X_t\| = 0) < \infty,$$

which implies that $V(t, \cdot)$ is a bounded function. For any $x > 1$, it follows from Lemma 4.3 that

$$V(t, x) \leq -\log \mathbb{P}(X_s([-x, x]^c) = 0, s \leq t) \leq h_x(0) \exp\{-\frac{c_4}{t}x^2 + at + c_5\}$$

$$\leq c(t)e^{-c_4 x^2/t},$$

where $c(t)$ is a constant which may depend on $t$. Thus, the desired result follows. $\square$

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Acknowledgment

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