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$L \log L$ criterion for a class of multitype superdiffusions with non-local branching mechanisms

In Memory of Professor Kai Lai Chung on the 100th Anniversary of His Birth

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Abstract In this paper, we provide a pathwise spine decomposition for multitype superdiffusions with non-local branching mechanisms under a martingale change of measure. As an application of this decomposition, we obtain a necessary and sufficient condition (called the $L \log L$ criterion) for the limit of the fundamental martingale to be non-degenerate. This result complements the related results obtained in Kyprianou et al. (2012), Kyprianou and Murillo-Salas (2013) and Liu et al. (2009) for superprocesses with purely local branching mechanisms and in Kyprianou and Palau (2018) for super Markov chains.

Keywords multitype superdiffusion, non-local branching mechanism, switched diffusion, spine decomposition, martingale

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1 Introduction

1.1 Previous results

Suppose that $\{Z_n, n \geqslant 1\}$ is a Galton-Watson process with offspring distribution $\{p_n\}$, i.e., each particle lives for one unit of time; at the time of its death, it gives birth to k particles with probability p_k for $k = 0, 1, \ldots$; and Z_n is the total number of particles alive at time n. Let L be a random variable with distribution $\{p_n\}$ and $m := \sum_{n=1}^{\infty} n p_n$ be the expected number of offspring per particle. Then Z_n/m^n is a non-negative martingale. Let W be the limit of Z_n/m^n as $n \to \infty$. Kesten and Stigum [16] proved that, when $1 < m < \infty$ (i.e., in the supercritical case), W is non-degenerate (i.e., not almost surely zero) if and only if

$$\mathrm{E}[L \log^+ L] = \sum_{n=1}^{\infty} p_n n \log n < \infty. \tag{1.1}$$

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This result is usually called the Kesten-Stigum $L\log L$ criterion. In [1], Asmussen and Hering generalized this result to the case of branching Markov processes under some conditions.

In 1995, Lyons et al. [24] developed a martingale change of measure method to give a new proof for the Kesten-Stigum $L\log L$ criterion for (single type) branching processes. Later this approach was applied to prove the $L\log L$ criterion for multitype and general multitype branching processes in [3, 18].

In [23], the martingale change of the measure method was used to prove an $L\log L$ criterion for a class of superdiffusions. In this paper, we will establish a pathwise spine decomposition for multitype superdiffusions with purely non-local branching mechanisms. Our non-local branching mechanisms are special in the sense that the types of the offspring are different from their mother, but their spatial locations at birth are the same as their mother's spatial location immediately before her death. We will see below that, a multitype superdiffusion with a purely non-local branching mechanism given by (1.4) below can also be viewed as a superprocess having a switched diffusion as its spatial motion and $\widehat{\psi}(x,i;\cdot)$ defined in (1.20) as its (non-local) branching mechanism. Using a non-local Feynman-Kac transform, we prove that, under a martingale change of measure, the spine runs as a copy of an h-transformed switched-diffusion, which is a new switched diffusion. The non-local nature of the branching mechanism induces a different kind of immigration—the *switching-caused* immigration. That is to say, whenever there is a switching of types, new immigration happens and the newly immigrated particles choose their types according to a distribution π. The switching-caused immigration is a consequence of the non-local branching, and it does not occur when the branching mechanism is purely local. Note that in this paper we do not consider branching mechanism with a local term. It is interesting to consider superprocesses with a more general non-local branching mechanism and with a local branching mechanism. For this case, one can see the recent preprint [26], where the spine is a concatenation process.

Concurrently to our work, Kyprianou and Palau [21] considered super Markov chains with local and non-local branching mechanisms. Note that if particles do not move in space, our model reduces to the model considered in [21] with a purely non-local branching mechanism. Kyprianou and Palau [21] also found that immigration happens when particle jumps (they call this immigration *jump immigration*), which corresponds to our switching-caused immigration.

1.2 Model: Multitype superdiffusions

For integer $K \geqslant 2$, a K-type superdiffusion is defined as follows. Let $S := \{1, 2, \dots, K\}$ be the set of types. For each $k \in S$, $\mathcal{L}_k$ is a second order elliptic differential operator of divergence form

$$\mathcal{L}_k = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}^k \frac{\partial}{\partial x_j}\right) \quad \text{on} \quad \mathbb{R}^d, \tag{1.2}$$

with $A^k(x) = (a_{ij}^k(x))_{1\leqslant i,j\leqslant d}$ being a symmetric matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded:

$$\Lambda_1|v|^2 \leqslant \sum_{i,j=1}^{d} a_{i,j}^k(x) v_i v_j \leqslant \Lambda_2|v|^2 \quad \text{for all} \quad v \in \mathbb{R}^d \quad \text{and} \quad x \in \mathbb{R}^d$$

for some positive constants $0 < \Lambda_1 \leqslant \Lambda_2 < \infty$, where $a_{ij}^k(x) \in C^{2,\gamma}(\mathbb{R}^d), 1 \leqslant i,j \leqslant d$ for some $\gamma \in (0,1)$. Throughout this paper, for $i = 1, 2, \dots, C^{i,\gamma}(\mathbb{R}^d)$ stands for the space of i times continuously differentiable functions with all their ith order derivatives belonging to $C^{\gamma}(\mathbb{R}^d)$, the space of γ-Hölder continuous functions on $\mathbb{R}^d$.

Suppose that for each $i \in S$, $\xi^i := \{\xi_t^i, t \geqslant 0; \Pi_x^i, x \in \mathbb{R}^d\}$ is a diffusion process on $\mathbb{R}^d$ with generator $\mathcal{L}_i$, independent to each other. In this paper, we always assume that D is a domain of finite Lebesgue measure in $\mathbb{R}^d$. For $x \in D$, denote by $\xi^{i,D} := \{\xi_t^{i,D}, t \geqslant 0; \Pi_x^i, x \in D\}$ the subprocess of ξ^i killed upon exiting D, i.e.,

$$\xi_t^{i,D} = \begin{cases} \xi_t^i, & \text{if } t < \tau_D^i, \\ \partial, & \text{if } t \geqslant \tau_D^i, \end{cases}$$

where $\tau_D^i = \inf\{t \geqslant 0; \xi_t^i \notin D\}$ is the first exit time of D and ∂ is a cemetery point.

Let $\mathcal{M}_1(S)$ denote the set of all probability measures on S, and $\mathcal{M}_F(D \times S)$ denote the space of finite measures on $D \times S$. For any measurable set E, we use $B_b(E)$ (resp. $B_b^+(E)$) to denote the family of bounded (resp. bounded positive) $\mathcal{B}(E)$-measurable functions on E. Any function f on D is automatically extended to $D_\partial := D \cup \{\partial\}$ by setting $f(\partial) = 0$. Similarly, any function f on $D \times S$ is automatically extended to $D_\partial \times S$ by setting $f(\partial, i) = 0, i \in S$. If $f(t, x, i)$ is a function on $[0, +\infty) \times D \times S$, we say f is *locally bounded* if $\sup_{t \in [0,T]} \sup_{(x,i) \in D \times S} |f(t, x, i)| < +\infty$ for every finite $T > 0$. For a function $f(s, x, i)$ defined on $[0, +\infty) \times D \times S$ and a number $t \geqslant 0$, we denote by $f_t(\cdot)$ the function $(x, i) \mapsto f(t, x, i)$. For convenience we use the following convention throughout this paper: For any probability measure P, we also use P to denote the expectation with respect to P. When there is only one probability measure involved, we sometimes also use E to denote the expectation with respect to that measure.

We consider a multitype superdiffusion $\{\chi_t, t \geqslant 0\}$ on D, which is a strong Markov process taking values in $\mathcal{M}_F(D \times S)$. We can represent χ_t by $(\chi_t^1, \dots, \chi_t^K)$ with $\chi_t^i \in \mathcal{M}_F(D)$ for $1 \leqslant i \leqslant K$. For $f \in B_b^+(D \times S)$, we often use the convention

$$f(x) = (f(x, 1), \dots, f(x, K)) = (f_1(x), \dots, f_K(x)), \quad x \in D,$$

and $\langle f, \chi_t\rangle = \sum_{j=1}^K \langle f_j, \chi_t^j\rangle$. Suppose that $F(x, i; du)$ is a kernel from $D \times S$ to $(0, \infty)$ such that, for each $i \in S$, the function

$$m(x, i) := \int_0^\infty uF(x, i; du)$$

is bounded on D. Let n be a bounded Borel function on $D \times S$ such that $n(x, i) \geqslant m(x, i)$ for every $(x, i) \in D \times S$, and $p_j^{(i)}(x)$, $i, j \in S$, be non-negative Borel functions on D with $\sum_{j=1}^K p_j^{(i)}(x) = 1$. Define

$$\pi(x, i; \cdot) = \sum_{j=1}^K p_j^{(i)}(x)\delta_{(x,j)}(\cdot),$$

where $\delta_{(x,j)}$ denotes the unit mass at (x, j). Then $\pi(x, i; \cdot)$ is a Markov kernel on $D \times S$. For any $f \in B_b^+(D \times S)$, we write $\pi(x, i; f) = \sum_{j=1}^K p_j^{(i)}(x)f_j(x)$. Define

$$\zeta(x, i; f) = n(x, i)\pi(x, i; f) + \int_0^\infty (1 - \mathrm{e}^{-u\pi(x,i;f)} - u\pi(x, i; f))F(x, i; du).$$

Note that we can rewrite $\zeta(x, i; f)$ as

$$\zeta(x, i; f) = \widetilde{n}(x, i)\pi(x, i; f) + \int_0^\infty (1 - \mathrm{e}^{-u\pi(x,i;f)})F(x, i; du),$$

where

$$\widetilde{n}(x, i) := n(x, i) - m(x, i) \geqslant 0. \tag{1.3}$$

$\zeta(x, k; f)$ serves as the non-local branching mechanism, which is a special form of [8, (3.17)] with d (corresponding to n in the present paper) and n (corresponding to F in the present paper) independent of π, and $G(x, i; d\pi)$ being the unit mass at some $\pi(x, i; \cdot) \in \mathcal{M}_1(S)$, i.e., the non-locally displaced offspring born at $(x, i) \in D \times S$ choose their types independently according to the (non-random) distribution $\pi(x, i; \cdot)$. Suppose $b(x, i) \in B_b^+(D \times S)$. Put

$$\psi(x, i; f) = b(x, i)(f_i(x) - \zeta(x, i; f)), \quad (x, i) \in D \times S, \quad f \in B_b^+(D \times S). \tag{1.4}$$

Without loss of generality, we suppose that $p_i^{(i)}(x) = 0$ for all $(x, i) \in D \times S$, which means that ψ is a purely non-local branching mechanism. The Laplace-functional of χ is given by

$$\mathrm{P}_\mu \exp\langle -f, \chi_t\rangle = \exp\langle -u_t^f(\cdot), \mu\rangle, \tag{1.5}$$

where $u_t^f(x,i)$ is the unique locally bounded positive solution to the evolution equation

$$u_t^f(x,i) + \Pi_x^i\left[\int_0^t \psi(\xi_s^{i,D}, i; u_{t-s}^f)ds\right] = \Pi_x^i f_i(\xi_t^{i,D}), \quad \text{for} \quad t \geqslant 0, \tag{1.6}$$

where we use the convention that $u_t^f(x) = (u_t^f(x,1), \ldots, u_t^f(x,K))$. This process is called an $((\mathcal{L}_1, \ldots \mathcal{L}_K), \psi)$-multitype superdiffusion in D. It is well known (see, e.g., [14]) that for any non-negative bounded function f on $D \times S$, the $u_t^f(x,i)$ in (1.6) is a locally bounded positive solution to the following system of partial differential equations: for each $i \in S$,

$$\begin{cases} \dfrac{\partial u^f(t,x,i)}{\partial t} = \mathcal{L}_i(t,x,i) - \psi(x,i;u_t^f), & (t,x) \in (0,\infty) \times D, \\ u^f(0,x,i) = f_i(x), & x \in D, \\ u^f(t,x,i) = 0, & (t,x) \in (0,\infty) \times \partial D. \end{cases} \tag{1.7}$$

Multitype superdiffusions can be obtained as a scaling limit of a sequence of multitype branching diffusions (see [8] for details). The multitype superdiffusion χ considered in this paper are the scaling limits of multitype branching diffusions whose types can change only at branching times.

Define

$$r_{il}(x) = n(x,i)p_l^{(i)}(x), \quad x \in D, \quad i,l \in S. \tag{1.8}$$

Let $v(t,x,i) = \mathrm{P}_{\delta_{(x,i)}}\langle f, \chi_t\rangle$. Using (1.5) and (1.6), we see that for all $(t,x,i) \in (0,\infty) \times D \times S$,

$$v_t(x,i) = \Pi_x^i f_i(\xi_t^{i,D}) + \Pi_x^i \int_0^t b(\xi_s,i)\left(\sum_{l=1}^K r_{il}(\xi_s^{i,D}) v_{t-s}(\xi_s^{i,D}, l) - v_{t-s}(\xi_s^{i,D}, i)\right)ds. \tag{1.9}$$

Then $v(t,x,i)$ is the unique locally bounded solution to the following linear system (see, e.g., [14]): for each $i \in S$,

$$\begin{cases} \dfrac{\partial v(t,x,i)}{\partial t} = \mathcal{L}_i v(t,x,i) + b(x,i)\displaystyle\sum_{l=1}^K (r_{il}(x) - \delta_{il}) v(t,x,l), & (t,x) \in (0,\infty) \times D, \\ v(0,x,i) = f_i(x), & x \in D, \\ v(t,x,i) = 0, & (t,x) \in (0,\infty) \times \partial D. \end{cases} \tag{1.10}$$

Letting $\boldsymbol{v}(t,x) = (v(t,x,1), \ldots, v(t,x,K))^{\mathrm{T}}$, we can rewrite the partial differential equations in (1.10) as

$$\frac{\partial}{\partial t}\boldsymbol{v}(t,x) = \boldsymbol{\mathcal{L}}\boldsymbol{v}(t,x) + B(x) \cdot (R(x) - I)\boldsymbol{v}(t,x), \tag{1.11}$$

where

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} \mathcal{L}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{L}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{L}_K \end{pmatrix},$$

$$B(x) = \begin{pmatrix} b(x,1) & 0 & \cdots & 0 \\ 0 & b(x,2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b(x,K) \end{pmatrix}$$

and

$$R(x) = \begin{pmatrix} r_{11}(x) & r_{12}(x) & \cdots & r_{1d}(x) \\ r_{21}(x) & r_{22}(x) & \cdots & r_{2d}(x) \\ \vdots & \vdots & \ddots & \vdots \\ r_{K1}(x) & r_{K2}(x) & \cdots & r_{KK}(x) \end{pmatrix}.$$

In this paper we assume that $B(x) \cdot R(x)$ is symmetric, i.e.,

$$b(x,i)n(x,i)p_j^{(i)}(x) = b(x,j)n(x,j)p_i^{(j)}(x), \quad \text{for all} \quad i,j \in S, \quad x \in D. \tag{1.12}$$

We remark that we assume the symmetry of $B(x) \cdot R(x)$ and the symmetry of the operators $\mathcal{L}_k$ (i.e., $\mathcal{L}_k$ is given by the divergence form (1.2)) for simplicity. If the $\mathcal{L}_k$'s are of non-divergence form and $B(x) \cdot R(x)$ is not symmetric, we can use the intrinsic ultracontractivity introduced in [17].

Note that

$$R(x) - I = R(x) - N(x) + (N(x) - I), \tag{1.13}$$

where

$$N(x) = \mathrm{diag}(n(x,1), \ldots, n(x,K)), \quad x \in D.$$

Then by (1.8) and (1.13),

$$B(x) \cdot (R(x) - I) = \widehat{B}(x) \cdot (P(x) - I) + B(x)(N(x) - I), \tag{1.14}$$

where

$$\widehat{B}(x) = \mathrm{diag}(b(x,1)n(x,1), \ldots, b(x,K)n(x,K))$$

and

$$P(x) = (p_{ij}(x))_{i,j\in S}, \quad p_{ij}(x) = p_j^{(i)}(x).$$

Put $Q(x) = (q_{ij}(x))_{i,j\in S} = \widehat{B}(x) \cdot (P(x) - I)$. We assume that the matrix Q is irreducible on D in the sense that for any two distinct $k, l \in S$, there exist $k_0, k_1, \ldots, k_r \in S$ with $k_i \neq k_{i+1}$, $k_0 = k$, $k_r = l$ such that $\{x \in D : q_{k_i k_{i+1}}(x) > 0\}$ has positive Lebesgue measure for each $0 \leqslant i \leqslant r-1$. Let $\{(X_t, Y_t), t \geqslant 0\}$ be a switched diffusion with generator $\mathcal{A} := \mathcal{L} + Q(x)$ killed upon exiting from $D \times S$, and $\Pi_{(x,i)}$ be its law starting from (x,i). $\{(X_t, Y_t), t \geqslant 0\}$ is a symmetric Markov process on $D \times S$ with respect to $dx \times di$, the product of the Lebesgue measure on D and the counting measure on S.

Define

$$\zeta_1(x,i;f) = n(x,i)\pi(x,i;f) = n(x,i)\sum_{i=1}^{K} p_l^{(i)}(x) f_l(x) = \sum_{l=1}^{K} r_{il}(x) f_l(x) \tag{1.15}$$

and

$$\zeta_2(x,i;f) = \int_0^\infty (1 - \mathrm{e}^{-u\pi(x,i;f)} - u\,\pi(x,i;f)) F(x,i;du). \tag{1.16}$$

Then

$$\zeta(x,i;f) = \zeta_1(x,i;f) + \zeta_2(x,i;f). \tag{1.17}$$

Letting

$$u^f(t,x) = (u^f(t,x,1), \ldots, u^f(t,x,K))^{\mathrm{T}} \quad \text{and} \quad \zeta_2(x,f) = (\zeta_2(x,1;f), \ldots, \zeta_2(x,K;f))^{\mathrm{T}},$$

in view of (1.4) we can rewrite the partial differential equation in (1.7) as

$$\frac{\partial}{\partial t} u^f(t,x) = \mathcal{L}u^f(t,x) + B(x) \cdot (R(x) - I)\, u^f(t,x) + B(x) \cdot \zeta_2(x, u_t^f), \tag{1.18}$$

which, by (1.13), is equivalent to

$$\begin{aligned}\frac{\partial}{\partial t} u^f(t,x) = \mathcal{L}u^f(t,x) &+ \widehat{B}(x) \cdot (P(x) - I) u^f(t,x) \\ &+ B(x) \cdot [(N(x) - I) u^f(t,x) + \zeta_2(x, u_t^f)].\end{aligned} \tag{1.19}$$

For $f \in B_b^+(\mathbb{R}^d \times S)$, define

$$\widehat{\psi}(x,i;f) := -b(x,i)n(x,i)f_i(x) + b(x,i)(f_i(x) - \zeta_2(x,i;f)). \tag{1.20}$$

Then applying the strong Markov property of the switched diffusion process (X, Y) at its first switching time and using the approach from [5] (see in particular [5, p. 296, Proposition 2.2 and Theorem 2.5]) and [14], one can verify using (1.19) that $u_t^f(x,i)$ satisfies

$$u_t^f(x,i)+\ \Pi_{(x,i)}\left[\int_0^t \widehat{\psi}(X_s,Y_s;u_{t-s}^f)ds\right] = \Pi_{(x,i)}f(X_t,Y_t), \quad t \geqslant 0. \tag{1.21}$$

This means that $\{\chi_t, t \geqslant 0\}$ can be viewed as a superprocess with the switched diffusion (X_t, Y_t) as its spatial motion on the space $D \times S$ and $\widehat{\psi}(x, i; \cdot)$ as its (non-local) branching mechanism. See [10, 11] for a definition of superprocesses with general non-local branching mechanisms.

2 Main result

It follows from [5, Theorem 5.3] that the switched diffusion $\{(X_t, Y_t), t \geqslant 0\}$ in D has a transition density $p(t,(x,k),(y,l))$, which is positive for all $x, y \in D$ and $k, l \in S$. Furthermore, for any $k, l \in S$ and $t > 0$, $(x, y) \mapsto p(t,(x,k),(y,l))$ is continuous. Let $\{P_t : t \geqslant 0\}$ be the transition semigroup of $\{(X_t, Y_t), t \geqslant 0\}$. For any $t > 0$, P_t is a compact self-adjoint operator. Let $\{e^{\nu_k t} : k = 1, 2, \ldots\}$ be all the eigenvalues of P_t arranged in decreasing order, each repeated according to its multiplicity. Then $\lim_{k\to\infty} \nu_k = -\infty$ and the corresponding eigenfunctions $\{\varphi_k\}$ can be chosen so that they form an orthonormal basis of $L^2(D \times S, dx \times di)$. All the eigenfunctions φ_k are continuous. The eigenspace corresponding to $e^{\nu_1 t}$ is of dimension 1 and φ_1 can be chosen to be strictly positive.

Let $\{P_t^{\mathcal{A}+B\cdot(N-I)}, t \geqslant 0\}$ be the Feynman-Kac semigroup defined by

$$P_t^{\mathcal{A}+B\cdot(N-I)}f(x,i) := \Pi_{(x,i)}\left[f(X_t,Y_t)\ \exp\left(\int_0^t b(X_s,Y_s)(n(X_s,Y_s)-1)ds\right)\right].$$

Then, by (1.13), $P_t^{\mathcal{A}+B\cdot(N-I)}f(x,i)$ is the unique solution to (1.10) and thus

$$\mathrm{P}_{\delta_{(x,i)}}\langle f, \chi_t\rangle = P_t^{\mathcal{A}+B\cdot(N-I)}f(x,i). \tag{2.1}$$

Under the assumptions above, $P_t^{\mathcal{A}+B\cdot(N-I)}$ admits a density $\widetilde{p}(t,(x,i),(y,j))$, which is jointly continuous in $(x, y) \in D \times D$, such that

$$P_t^{\mathcal{A}+B\cdot(N-I)}f(x,i) = \sum_{j\in S}\int_D \widetilde{p}(t,(x,i),(y,j))f(y,j)dy,$$

for every $f \in \mathcal{B}_b^+(D \times S)$. $\{P_t^{\mathcal{A}+B\cdot(D-I)}, t \geqslant 0\}$ can be extended to a strongly continuous semigroup on $L^2(D \times S, dx \times di)$. The semigroup $\{P_t^{\mathcal{A}+B\cdot(N-I)}, t \geqslant 0\}$ is symmetric in $L^2(D \times S, dx \times di)$, i.e.,

$$\sum_{i\in S}\int_D f(x,i)P_t^{\mathcal{A}+B\cdot(N-I)}g(x,i)dx = \sum_{i\in S}\int_D g(x,i)P_t^{\mathcal{A}+B\cdot(N-I)}f(x,i)dx$$

for $f, g \in L^2(D\times S, dx\times di)$. For any $t > 0$, $P_t^{\mathcal{A}+B\cdot(N-I)}$ is a compact self-adjoint operator. The generator of the semigroup $\{P_t^{\mathcal{A}+B\cdot(N-I)}\}$ is $\mathcal{A} + B \cdot (N - I) = \mathcal{L} + B \cdot (R - I)$.

Let $\{e^{\lambda_k t} : k = 1, 2, \ldots\}$ be all the eigenvalues of $P_t^{\mathcal{A}+B\cdot(N-I)}$ arranged in decreasing order, each repeated according to its multiplicity. Then $\lim_{k\to\infty} \lambda_k = -\infty$ and the corresponding eigenfunctions $\{\phi_k\}$ can be chosen so that they form an orthonormal basis of $L^2(D\times S, dx\times di)$. All the eigenfunctions ϕ_k are continuous. The eigenspace corresponding to $e^{\lambda_1 t}$ is of dimension 1 and ϕ_1 can be chosen to be strictly positive. For simplicity, in the remainder of this paper, we will denote ϕ_1 as ϕ.

Throughout this paper we assume that $\{\chi_t, t \geqslant 0\}$ is supercritical and ϕ is bounded on $D \times S$, i.e., we assume the following assumption.

Assumption 2.1. $\lambda_1 > 0$ *and its corresponding positive eigenfunction* ϕ *is bounded.*

Define

$$R^{\phi}(x) := (r^{\phi}_{ij}(x)), \quad r^{\phi}_{ij}(x) := r_{ij}(x)\frac{\phi(x,j)}{\phi(x,i)} = n(x,i)\frac{p^{(i)}_j(x)\phi(x,j)}{\phi(x,i)} \tag{2.2}$$

and

$$\pi(\phi)(x,i) := \pi(x,i;\phi) = \sum_{j=1}^{K} p^{(i)}_j(x)\phi(x,j), \quad (x,i)\in D\times S. \tag{2.3}$$

Let $\{\mathcal{E}_t; t\geqslant 0\}$ be the minimal augmented filtration generated by the switched diffusion (X,Y) in D. Define a measure $\Pi^{\phi}_{(x,i)}$ by

$$\frac{d\Pi^{\phi}_{(x,i)}}{d\Pi_{(x,i)}}\bigg|_{\mathcal{E}_t} = \mathrm{e}^{-\lambda_1 t}\frac{\phi(X_t,Y_t)}{\phi(x,i)}\exp\bigg(\int_0^t b(X_s,Y_s)(n(X_s,Y_s)-1)ds\bigg). \tag{2.4}$$

Then $\{(X,Y),\Pi^{\phi}_{(x,i)}\}$ is a conservative Markov process which is symmetric with respect to the measure $\phi^2(x,i)dx\times di$. The process $\{(X,Y),\Pi^{\phi}_{(x,i)}\}$ has a transition density $p^{\phi}(t,(x,i),(y,j))$ with respect to $dy\times dj$ given by

$$p^{\phi}(t,(x,i),(y,j)) = \frac{\mathrm{e}^{-\lambda_1 t}\phi(y,j)}{\phi(x,i)}\widetilde{p}(t,(x,i),(y,j)), \quad (x,i)\in D\times S.$$

Let $\{P^{\phi}_t : t\geqslant 0\}$ be the transition semigroup of (X,Y) under $\Pi^{\phi}_{(x,i)}$. Then ϕ^2 is the unique invariant probability density of $\{P^{\phi}_t : t\geqslant 0\}$, i.e., for any $f\in B^+_b(D\times S)$,

$$\sum_{i=1}^{K}\int_D \phi^2(x,i)P^{\phi}_t f(x,i)dx = \sum_{i=1}^{K}\int_D f(x,i)\phi(x,i)^2 dx.$$

Since the infinitesimal generator of $\{(X,Y),\Pi_{(x,i)}\}$ is $\mathcal{L}+\widehat{B}(x)\cdot(P(x)-I)$ with zero Dirichlet boundary condition on $\partial D\times S$, it follows from [25, Theorem 4.2] that the generator of $\{(X,Y),\Pi^{\phi}_{(x,i)}\}$ is

$$\begin{aligned}
&\frac{1}{\phi}[\mathcal{L}(\boldsymbol{u}\phi)+\widehat{B}(x)\cdot(P(x)-I)(\boldsymbol{u}\phi)-\boldsymbol{u}(\mathcal{L}(\phi)+\widehat{B}(x)\cdot(P(x)-I)\phi)]\\
&=\frac{1}{\phi}[\mathcal{L}(\boldsymbol{u}\phi)+\widehat{B}(x)\cdot(P(x)-I)(\boldsymbol{u}\phi)+B(x)\cdot(N(x)-I)(\mathbf{u}\phi)-\lambda_1\boldsymbol{u}\phi]\\
&=\frac{1}{\phi}[\mathcal{L}(\boldsymbol{u}\phi)+B(x)\cdot(R(x)-I)(\boldsymbol{u}\phi)]-\lambda_1\boldsymbol{u}\\
&=\mathcal{L}^{\phi}\boldsymbol{u}+B(x)\cdot(R^{\phi}(x)-I)\boldsymbol{u}-\lambda_1\boldsymbol{u},
\end{aligned}$$

where in the first equality above we used the fact that ϕ is an eigenfunction of $P^{\mathcal{A}+B\cdot(N-I)}_t$ and (1.14).

Define

$$\widetilde{p}_{ij}(x) = \frac{\phi(x,i)}{n(x,i)\pi(x,i;\phi)}r^{\phi}_{ij}(x) = \frac{p^{(i)}_j(x)\phi(x,j)}{\pi(x,i;\phi)}, \quad i,j\in S, \quad x\in D$$

and

$$\widetilde{P}(x) = (\widetilde{p}_{ij}(x))_{i,j\in S}.$$

Note that

$$\begin{aligned}
B(x)\cdot(R^{\phi}-I)-\lambda_1 &= \mathrm{diag}\bigg(\frac{bn\pi(\phi)}{\phi}(x,1),\ldots,\frac{bn\pi(\phi)}{\phi}(x,K)\bigg)(\widetilde{P}(x)-I)\\
&\quad + B(x)\bigg[\mathrm{diag}\bigg(\frac{n\pi(\phi)}{\phi}(x,1),\ldots,\frac{n\pi(\phi)}{\phi}(x,K)\bigg)-I\bigg]-\lambda_1\\
&= \mathrm{diag}\bigg(\frac{bn\pi(\phi)}{\phi}(x,1),\ldots,\frac{bn\pi(\phi)}{\phi}(x,K)\bigg)(\widetilde{P}(x)-I).
\end{aligned}$$

Thus the generator of $\{(X,Y),\Pi^{\phi}_{(x,i)}\}$ is

$$\mathcal{L}^{\phi}+\operatorname{diag}\bigg(\frac{bn\pi(\phi)}{\phi}(x,1),\ldots,\frac{bn\pi(\phi)}{\phi}(x,K)\bigg)(\widetilde{P}(x)-I), \tag{2.5}$$

which is the generator of a new switched diffusion. Here,

$$\mathcal{L}^{\phi}=\begin{pmatrix}\mathcal{L}_1^{\phi(\cdot,1)} & 0 & \cdots & 0\\ 0 & \mathcal{L}_2^{\phi(\cdot,2)} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathcal{L}_K^{\phi(\cdot,K)}\end{pmatrix}$$

with

$$\mathcal{L}_k^{\phi(\cdot,k)}v(x):=\frac{1}{\phi(x,k)}\mathcal{L}_k(\phi(x,k)v(x)).$$

For any measure μ on $D\times S$ such that $\langle\phi,\mu\rangle<\infty$, define

$$\Pi^{\phi}_{\phi\mu}=\frac{1}{\langle\phi,\mu\rangle}\int\phi(x,i)\Pi^{\phi}_{(x,i)}d\mu.$$

By (2.5), the jumping intensity of (X,Y) under $\Pi^{\phi}_{\phi\mu}$ is $\frac{bn\pi(\phi)}{\phi}(x,i)$ at $(x,i)\in D\times S$.

Throughout this paper, we assume the following assumption.

Assumption 2.2. *The semigroup $\{P_t:t\geqslant 0\}$ is intrinsically ultracontractive, i.e., for any $t>0$, there exists $c_t>0$ such that*

$$p(t,(x,k),(y,l))\leqslant c_t\phi(x,k)\phi(y,l),\quad x,y\in D,\quad k,l\in S.$$

It follows from [7, Theorem 3.4] that the semigroup $\{P_t^{\mathcal{A}+B\cdot(N-I)}:t\geqslant 0\}$ is also intrinsically ultracontractive, i.e., for any $t>0$, there exists $c_t>0$ such that

$$\widetilde{p}(t,(x,k),(y,l))\leqslant c_t\phi(x,k)\phi(y,l),\quad x,y\in D,\quad k,l\in S.$$

As a consequence, one can easily show (see, for example, [2]) that for any $t_0>0$, there exists $c>0$ such that for all $t\geqslant t_0$,

$$\left|\frac{\mathrm{e}^{-\lambda_1 t}\widetilde{p}(t,(x,k),(y,l))}{\phi(x,k)\phi(y,l)}-1\right|\leqslant c\mathrm{e}^{(\lambda_2-\lambda_1)t},\quad x,y\in D,\quad k,l\in S.$$

Hence for any $\delta\in(0,1)$, there exists $t_0>0$ such that for all $t\geqslant t_0$,

$$\left|\frac{\mathrm{e}^{-\lambda_1 t}\widetilde{p}(t,(x,k),(y,l))}{\phi(x,k)\phi(y,l)}-1\right|\leqslant \delta,\quad x,y\in D,\quad k,l\in S.$$

Thus for any $f\in B_b(D\times S)$, $t>t_0$ and $(x,i)\in D\times S$,

$$\left|P_t^{\phi}f(x,i)-\int_{D\times S}f(y,j)\phi(y,j)^2dydj\right|\leqslant\delta\int_{D\times S}f(y,j)\phi(y,j)^2dydj. \tag{2.6}$$

It follows from (2.6) that for any $f\in B_b^+(D\times S)\cap L^1(\phi^2(x,i)\,dx\times di)$, $t>t_0$ and $(x,i)\in D\times S$,

$$(1-\delta)\int_{D\times S}f(y,j)\phi(y,j)^2dydj\leqslant P_t^{\phi}f(x,i)\leqslant(1+\delta)\int_{D\times S}f(y,j)\phi(y,j)^2dydj. \tag{2.7}$$

Lemma 2.3. *Define*

$$W_t(\phi):=\mathrm{e}^{-\lambda_1 t}\langle\phi,\chi_t\rangle. \tag{2.8}$$

Then $\{W_t(\phi),t\geqslant 0\}$ is a non-negative P_μ-martingale for each nonzero $\mu\in M_F(D\times S)$ and therefore there exists a limit $W_\infty(\phi)\in[0,\infty)$ P_μ-a.s.

Proof. By the Markov property of χ and (2.1), and using the fact that ϕ is an eigenfunction corresponding to λ_1, we get that for any nonzero $\mu \in M_F(D \times S)$,

$$
\begin{aligned}
\mathrm{P}_\mu[W_{t+s}(\phi) \,|\, \mathcal{F}_t] &= \frac{1}{\langle \phi, \mu \rangle} \mathrm{e}^{-\lambda_1 t} \mathrm{P}_{\chi_t}[\mathrm{e}^{-\lambda_1 s} \langle \phi, \chi_s \rangle] \\
&= \frac{1}{\langle \phi, \mu \rangle} \mathrm{e}^{-\lambda_1 t} \langle \mathrm{e}^{-\lambda_1 s} P_s^{\mathcal{A}+B\cdot(N-I)} \phi, \chi_t \rangle \\
&= \frac{1}{\langle \phi, \mu \rangle} \mathrm{e}^{-\lambda_1 t} \langle \phi, \, \chi_t \rangle = W_t(\phi).
\end{aligned}
$$

This proves that $\{W_t(\phi), t \geqslant 0\}$ is a non-negative P_μ-martingale, so it has an almost sure limit $W_\infty(\phi) \in [0, \infty)$ as $t \to \infty$. $\square$

We define a new kernel $F^{\pi(\phi)}(x, i; dr)$ from $D \times S$ to $(0, \infty)$ such that for any non-negative measurable function f on $(0, \infty)$,

$$
\int_0^\infty f(r) F^{\pi(\phi)}(x, i; dr) = \int_0^\infty f(\pi(x, i; \phi) r) F(x, i; dr), \quad (x, i) \in D \times S.
$$

Define

$$
l(x, i) := \int_0^\infty r \log^+(r) F^{\pi(\phi)}(x, i; dr). \tag{2.9}
$$

The main result of this paper is the following theorem.

Theorem 2.4. *Suppose that $\{\chi_t; t \geqslant 0\}$ is a multitype superdiffusion and that Assumptions* 2.1 *and* 2.2 *hold. Assume that $\mu \in M_F(D \times S)$ is non-trivial. Then $W_\infty(\phi)$ is non-degenerate under P_μ if and only if*

$$
\int_D \phi(x, i) b(x, i) l(x, i) dx < \infty \quad \textit{for every} \quad i \in S, \tag{2.10}
$$

where l is defined in (2.9)*. Moreover, when* (2.10) *is satisfied, $W_t(\phi)$ converges to $W_\infty(\phi)$ in L^1 under P_μ.*

Since (2.10) does not depend on μ, it is also equivalent to that $W_\infty(\phi)$ is non-degenerate under P_μ for every non-trivial measure $\mu \in M_F(D \times S)$.

The proof of this theorem is accomplished by combining the ideas from [24] with the "spine decomposition" of [12, 23]. The new feature here is that we consider a different type of branching mechanisms. The new type of branching mechanisms considered here is non-local as opposed to the local branching mechanisms in [12, 23]. The non-local branching mechanisms we consider here result in a kind of *non-local* immigration, as opposed to the local immigration in [23].

In the next section, we show that when D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, Assumption 2.2 holds. In Section 4, we give our spine decomposition of the superdiffusion χ under a martingale change of measure with the help of Poisson point processes. In Section 5, we use this spine decomposition to prove Theorem 2.4.

3 Intrinsic ultracontractivity

In this section, we show that when D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$, Assumption 2.2 holds, i.e., the semigroup $\{P_t : t \geqslant 0\}$ is intrinsically ultracontractive and the first eigenfunction is bounded.

Throughout this section, we assume that D is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Let $p_0(t, x, y)$ be the transition density of the killed Brownian motion in D. For each $i \in S$, let $p_i(t, x, y)$ be the transition density of $\xi_t^{i,D}$, the process obtained by killing the diffusion with generator $\mathcal{L}_i$ upon exiting from D.

It is known (see [6]) that there exist positive constants C_i, $i = 1, 2, 3, 4$, such that for all $t \in (0, 1]$, $j = 0, 1, \ldots, K$ and $x, y \in D$,

$$
p_j(t, x, y) \geqslant C_1 \left(\frac{\delta_D(x)}{\sqrt{t}} \wedge 1 \right) \left(\frac{\delta_D(y)}{\sqrt{t}} \wedge 1 \right) t^{-d/2} \mathrm{e}^{-\frac{C_2 |x-y|^2}{t}}, \tag{3.1}
$$

$$p_j(t,x,y) \leqslant C_3\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right)\left(\frac{\delta_D(y)}{\sqrt{t}}\wedge 1\right)t^{-d/2}\mathrm{e}^{-\frac{C_4|x-y|^2}{t}}. \tag{3.2}$$

Using these we can see that there exists $C_5>0$ such that for any $t\in(0,C_4/C_2]$ and $x,y\in D$,

$$p_j(t,x,y)\leqslant C_5 p_0(C_2t/C_4,x,y). \tag{3.3}$$

It follows from [5, Theorem 5.3] that for any $x,y\in D$ and $k,l\in S$,

$$\begin{aligned}
p(t,(x,k),(y,l)) = {} & \delta_{kl}p_k(t,x,y)\\
& + \sum_{n=0}^{\infty}\sum_{\substack{1\leqslant l_1,l_2,\dots,l_n\leqslant K\\ l_1\neq k, l_n\neq l, l_i\neq l_{i+1}}}\int\cdots\int_{0<t_1<t_2<\cdots<t_n<t}\int_D\cdots\int_D p_k(t_1,x,y_1)q_{kl_1}(y_1)\\
& \times p_{l_1}(t_2-t_1,y_1,y_2)q_{l_1l_2}(y_2)\cdots q_{l_nl}(y_n)\\
& \times p_l(t-t_n,y_n,y)dy_n\cdots dy_1dt_n\cdots dt_1.
\end{aligned} \tag{3.4}$$

Let $M>0$ be such that

$$|q_{kl}(x)|\leqslant M, \quad x\in D, \quad k,l\in S.$$

Then it follows from (3.3) and (3.4) that for $t\in(0,C_4/C_2]$, $x,y\in D$ and $k,l\in S$,

$$\begin{aligned}
p(t,(x,k),(y,l)) \leqslant {} & C_5p_0(C_2t/C_4,x,y)\\
& + \sum_{n=0}^{\infty}(MKC_5)^n\int\cdots\int_{0<t_1<t_2<\cdots<t_n<t}\int_D\cdots\int_D p_0(C_2t_1/C_4,x,y_1)\\
& \times p_0(C_2(t_2-t_1)/C_4,y_1,y_2)\cdots p_0(C_2(t_-t_n)/C_4,y_n,y)dy_n\cdots dy_1dt_n\cdots dt_1\\
\leqslant {} & C_5p_0(C_2t/C_4,x,y)+\sum_{n=0}^{\infty}\frac{(MKC_5t)^n}{n}p_0(C_2t/C_4,x,y).
\end{aligned}$$

Thus there exists $t_0\in(0,C_4/C_2)$ such that for $t\in(0,t_0]$, $x,y\in D$ and $k,l\in S$,

$$p(t,(x,k),(y,l))\leqslant C_6p_0(C_2t/C_4,x,y) \tag{3.5}$$

for some $C_6>0$.

Now we prove a similar lower bound. It follows from (3.4) that for any $t\in(0,1]$, $x,y\in D$ and $k\in S$,

$$p(t,(x,k),(y,k))\geqslant p_k(t,x,y). \tag{3.6}$$

Now suppose $k\neq l$. Let $l_0,l_1,\dots,l_n\in S$ with $l_i\neq l_{i+1}$, $l_0=k$, $l_n=l$ such that $\{x\in D: q_{l_il_{i+1}}(x)>0\}$ has positive Lebesgue measure for $i=0,1,\dots,n-1$. Then it follows from (3.4) that

$$\begin{aligned}
p(t,(x,k),(y,l))\geqslant {} & \int\cdots\int_{0<t_1<t_2<\cdots<t_n<t}\int_D\cdots\int_D p_k(t_1,x,y_1)q_{kl_1}(y_1)\\
& \times p_{l_1}(t_2-t_1,y_1,y_2)q_{l_1l_2}(y_2)\cdots q_{l_nl}(y_n)\\
& \times p_l(t-t_n,y_n,y)dy_n\cdots dy_1dt_n\cdots dt_1.
\end{aligned}$$

Thus it follows from (3.1) that there exists $C_7>0$ such that for any $t\in(0,1]$, $x,y\in D$,

$$p(t,(x,k),(y,l))\geqslant C_7\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right). \tag{3.7}$$

Combining (3.6) and (3.7) we get that for any $t\in(0,1]$, $x,y\in D$ and $k,l\in S$,

$$p(t,(x,k),(y,l))\geqslant C_8\left(\frac{\delta_D(x)}{\sqrt{t}}\wedge 1\right) \tag{3.8}$$

for some $C_8 > 0$.

It follows from (3.5) and (3.8) that there exist positive constants $C_9 < C_{10}$ such that for all $(x, k) \in D \times S$,

$$C_9 \delta_D(x) \leqslant \phi(x, k) \leqslant C_{10} \delta_D(x).$$

Combining this with (3.5), and using the semigroup property, we immediately get the intrinsic ultarcontractivity of $\{P_t : t \geqslant 0\}$. The boundedness of ϕ is an immediate consequence of the display above.

4 Spine decomposition

Let $\{\mathcal{F}_t; t \geqslant 0\}$ be the minimal augmented filtration generated by $\{\chi_t, t \geqslant 0\}$. We define a probability measure $\widetilde{\mathrm{P}}_\mu$ by

$$\frac{d\widetilde{\mathrm{P}}_\mu}{d\mathrm{P}_\mu}\bigg|_{\mathcal{F}_t} = \frac{1}{\langle \phi, \mu \rangle} W_t(\phi). \tag{4.1}$$

The purpose of this section is to give a spine decomposition of $\{\chi_t, t \geqslant 0\}$ under $\widetilde{\mathrm{P}}_\mu$. This decomposition will play an important role in proving Theorem 2.4

The spine decomposition is roughly as follows: Under $\widetilde{\mathrm{P}}_\mu$, $\{\chi_t, t \geqslant 0\}$ has the same law as the sum of the following two independent measured-valued processes: the first process is a copy of χ under P_μ, and the second process is, roughly speaking, obtained by taking an "immortal particle" that moves according to the law of $\{(X, Y), \Pi^\phi_{\phi\mu}\}$ and spins off pieces of mass that continue to evolve according to the dynamics of χ.

Define

$$\eta(x, i; \lambda) = \int_0^\infty \mathrm{e}^{-u\lambda} u F(x, i; du), \quad \lambda \geqslant 0, \quad (x, i) \in D \times S. \tag{4.2}$$

We first give a formula for the one-dimensional distribution of χ under $\widetilde{\mathrm{P}}_\mu$.

Theorem 4.1. *Suppose $\mu \in M_F(D \times S)$ and $g \in B_b^+(D \times S)$. Let D_J be the set of jump times of (X, Y). Then*

$$\begin{aligned}
&\widetilde{\mathrm{P}}_\mu(\exp\langle -g, \chi_t \rangle) \\
&\quad = \mathrm{P}_\mu\big(\exp\langle -g, \chi_t \rangle\big) \Pi^\phi_{\phi\mu}\left[\exp\left(\sum_{s \in D_J, 0 < s \leqslant t} \log\left(\frac{\eta(X_s, Y_s; \pi(X_s, Y_s; u^g_{t-s}))}{n(X_s, Y_s)} + \frac{\widetilde{n}(X_s, Y_s)}{n(X_s, Y_s)}\right)\right)\right],
\end{aligned} \tag{4.3}$$

where u^g_{t-s} is the unique locally bounded positive solution of (1.6) with f replaced by g.

Proof. By (4.1),

$$\begin{aligned}
\widetilde{\mathrm{P}}_\mu(\exp\langle -g, \chi_t \rangle) &= \frac{\mathrm{e}^{-\lambda_1 t}}{\langle \phi, \mu \rangle} \mathrm{P}_\mu(\langle \phi, \chi_t \rangle \exp\langle -g, \chi_t \rangle) \\
&= \frac{\mathrm{e}^{-\lambda_1 t}}{\langle \phi, \mu \rangle} \frac{\partial}{\partial \theta} \mathrm{P}_\mu(\exp\langle -g - \theta\phi, \chi_t \rangle)\,|_{\theta=0} \\
&= \frac{\mathrm{e}^{-\lambda_1 t}}{\langle \phi, \mu \rangle} \frac{\partial}{\partial \theta} \exp\langle -u^{g+\theta\phi}_t, \mu \rangle\,|_{\theta=0} \\
&= \frac{\mathrm{e}^{-\lambda_1 t}}{\langle \phi, \mu \rangle} \exp\langle -u^g_t, \mu \rangle \left\langle \frac{\partial}{\partial \theta} u^{g+\theta\phi}_t \bigg|_{\theta=0}, \mu \right\rangle.
\end{aligned} \tag{4.4}$$

Note that $\exp\langle -u^g_t, \mu \rangle = \mathrm{P}_\mu \exp\langle -g, \chi_t \rangle$, and $u^{g+\theta\phi}_t$ is the unique locally bounded positive solution of the integral equation

$$u^{g+\theta\phi}_t(x, i) + \Pi_{(x,i)}\left[\int_0^t \widehat{\psi}(X_s, Y_s; u^{g+\theta\phi}_{t-s}) ds\right] = \Pi_{(x,i)}[(g + \theta\phi)(X_t, Y_t)], \quad t \geqslant 0.$$

Taking derivative with respect to θ on both sides of the above equation, and then letting $\theta = 0$, we have that $v_t(x,i) := \frac{\partial}{\partial\theta} u_t^{g+\theta\phi}|_{\theta=0}$ satisfies

$$\begin{aligned}
&v_t(x,i) - \Pi_{(x,i)} \int_0^t b(X_s,Y_s)(n(X_s,Y_s)-1)v_{t-s}(X_s,Y_s)ds \\
&\quad + \Pi_{(x,i)} \int_0^t b(X_s,Y_s)(m(X_s,Y_s) - \eta(X_s,Y_s;\pi(X_s,Y_s;u^g_{t-s}))) \sum_{j=1}^K p_j^{(Y_s)}(X_s)v_{t-s}(X_s,j)ds \\
&\quad = \Pi_{(x,i)}[\phi(X_t,Y_t)].
\end{aligned} \tag{4.5}$$

Let

$$J((x,k),d(y,l)) = \delta(x-y)q_{kl}(x)1_{\{k\neq l\}}dydl, \quad (x,k) \in D\times S, \tag{4.6}$$

where dl stands for the counting measure on S. Then $(J((x,k),d(y,l)),t)$ is a Lévy system of (X,Y). Define

$$F(t-s,(x,i),(y,j)) := \log\left(\frac{\eta(x,i;\pi(x,i;u^g_{t-s})}{n(x,i)} - \frac{m(x,i)}{n(x,i)} + 1\right) 1_{i\neq j}. \tag{4.7}$$

Clearly, $F \leqslant 0$. We would like to apply Lemma A.1 with $\xi = (X,Y)$, $q(t-s,(x,i)) = b(x,i)(n(x,i)-1)$, J given by (4.6) and F given by (4.7). Since $q_{ij}(x)$, $i,j\in S$, are bounded in D and D has finite Lebesgue measure, we have $\sup_{(x,i)\in D\times S} J((x,k),D\times S) < \infty$. By Remark A.2(iii), (A.1) and (A.2) are satisfied. Thus we can apply Lemma A.1 to get

$$\begin{aligned}
v_t(x,i) &= \Pi_{(x,i)}\Bigg[\exp\Bigg\{\sum_{s\in D_J, 0<s\leqslant t} \log\left(\frac{\eta(X_s,Y_s;\pi(X_s,Y_s;u^g_{t-s}))}{n(X_s,Y_s)} - \frac{m(X_s,Y_s)}{n(X_s,Y_s)} + 1\right) \\
&\qquad + \int_0^t b(X_s,Y_s)(n(X_s,Y_s)-1)ds\Bigg\}\phi(X_t,Y_t)\Bigg] \\
&= \mathrm{e}^{\lambda_1 t}\phi(x,i)\Pi^{\phi}_{(x,i)}\left[\exp\left\{\sum_{s\in D_J, 0<s\leqslant t} \log\left(\frac{\eta(X_s,Y_s;\pi(X_s,Y_s;u^g_{t-s}))}{n(X_s,Y_s)} + \frac{\widetilde{n}(X_s,Y_s)}{n(X_s,Y_s)}\right)\right\}\right].
\end{aligned} \tag{4.8}$$

Combining (4.4) and (4.8), we obtain

$$\begin{aligned}
\widetilde{\mathrm{P}}_\mu(\exp\langle -g,\chi_t\rangle) &= \mathrm{P}_\mu(\exp\langle -g,\chi_t\rangle) \\
&\quad \times \Pi^{\phi}_{\phi\mu}\left[\exp\left\{\sum_{s\in D_J, 0<s\leqslant t} \log\left(\frac{\eta(X_s,Y_s;\pi(X_s,Y_s;u^g_{t-s}))}{n(X_s,Y_s)} + \frac{\widetilde{n}(X_s,Y_s)}{n(X_s,Y_s)}\right)\right\}\right].
\end{aligned}$$

This completes the proof. □

Define

$$\widetilde{F}(x,i;du) = \frac{1}{n(x,i)}(\widetilde{n}(x,i)\delta_0 + I_{(0,\infty)}u\,F(x,i;du)). \tag{4.9}$$

Then, by (1.3) and (4.2), $\widetilde{F}(x,i;\cdot)$ is a probability measure on $[0,\infty)$ for any $(x,i)\in D\times S$ and

$$\frac{\eta(x,i;\lambda)}{n(x,i)} + \frac{\widetilde{n}(x,i)}{n(x,i)} = \int_{[0,\infty)} \mathrm{e}^{-u\lambda}\widetilde{F}(x,i;du) \quad \text{for every} \quad \lambda \geqslant 0.$$

Thus we may rewrite (4.3) as

$$\begin{aligned}
&\widetilde{\mathrm{P}}_\mu(\exp\langle -g,\chi_t\rangle) \\
&\quad = \mathrm{P}_\mu(\exp\langle -g,\chi_t\rangle)\cdot \Pi^{\phi}_{\phi\mu}\left[\prod_{s\in D_J, 0<s\leqslant t} \int_0^\infty \exp(-u\pi(X_s,Y_s;u^g_{t-s}))\widetilde{F}(X_s,Y_s;du)\right].
\end{aligned} \tag{4.10}$$

From (4.10) we see that the superdiffusion $\{\chi_t, t\geqslant 0; \widetilde{\mathrm{P}}_\mu\}$ can be decomposed into two independent parts. The first part is a copy of the original superdiffusion and the second part is an immigration process.

To describe the second part precisely, we need to introduce another measure-valued process $\{\widehat{\chi}_t, t \geqslant 0\}$. Now we construct the measure-valued process $\{\widehat{\chi}_t, t \geqslant 0\}$ as follows:

(i) Suppose that $(\widehat{X}, \widehat{Y}) = \{(\widehat{X}_t, \widehat{Y}_t), t \geqslant 0\}$ is defined on some probability space $(\Omega, \mathrm{P}_{\mu,\phi})$, and $(\widehat{X}, \widehat{Y})$ has the same law as $((X, Y); \Pi^{\phi}_{\phi\mu})$. $(\widehat{X}, \widehat{Y})$ serves as the spine or the immortal particle, which visits every part of $D \times S$ for large times since it is an ergodic process. Let D_J be the set of jump points of $(\widehat{X}, \widehat{Y})$. D_J is countable.

(ii) Conditioned on $s \in D_J$, a measure-valued process χ^s started at $m_s \delta_{(\widehat{X}_s, l)} (l \in S)$ is immigrated at the space position $\widehat{X}_s$ and the new immigrated particles choose their types independently according to the (nonrandom) distribution $\pi(x, i; \cdot)$. We suppose $\{m_s; s \in D_J\}$ is also defined on $(\Omega, \mathrm{P}_{\mu,\phi})$ such that, given $s \in D_J$ and $(\widehat{X}_s, \widehat{Y}_s)$, the distribution of m_s is $\widetilde{F}(\widehat{X}_s, \widehat{Y}_s; dr)$.

(iii) Once the particles are in the system, they begin to move and branch according to the $((X, Y), \widehat{\psi}(x, i, \cdot))$-superprocess independently.

We use $(\chi^s_t,\ t \geqslant s)$ to denote the measure-valued process generated by the mass immigrated at time s and spatial position $\widehat{X}_s$. Conditional on $\{(\widehat{X}_t, \widehat{Y}_t), t \geqslant 0; m_s, s \in D_J\}$, $\{\chi^s_t, t \geqslant s\}$ for different $s \in D_J$ are independent $((X, Y), \widehat{\psi}(x, i, \cdot))$-superprocesses. Set

$$\widehat{\chi}_t = \sum_{s \in (0,t] \cap D_J} \chi^s_t. \tag{4.11}$$

The Laplace functional of $\widehat{\chi}_t$ is described in the following proposition.

Proposition 4.2. *The Laplace functional of $\widehat{\chi}_t$ under $\mathrm{P}_{\mu,\phi}$ is equal to*

$$\Pi^{\phi}_{\phi\mu}\bigg\{ \prod_{s \in (0,t] \cap D_J} \int_{[0,\infty)} \exp(-r\pi(X_s, Y_s; u^g_{t-s})) \widetilde{F}(X_s, Y_s; dr) \bigg\}.$$

Proof. For any $g \in B^+_b(D \times S)$, using (1.5), we have

$$\begin{aligned}
\mathrm{P}_{\mu,\phi}[\exp(-\langle g, \widehat{\chi}_t \rangle)] &= \mathrm{P}_{\mu,\phi}\bigg\{ \mathrm{P}_{\mu,\phi}\bigg[\exp\bigg(- \sum_{\sigma \in (0,t] \cap D_J} \langle g, \chi^{\sigma}_t \rangle \bigg) \bigg| \sigma((\widehat{X}, \widehat{Y}), m) \bigg] \bigg\} \\
&= \mathrm{P}_{\mu,\phi}\bigg[\prod_{s \in (0,t] \cap D_J} \exp(-m_s \pi(\widehat{X}_s, \widehat{Y}_s, u^g_{t-s})) \bigg] \\
&= \mathrm{P}_{\mu,\phi}\bigg\{ \mathrm{P}_{\mu,\phi}\bigg[\prod_{s \in (0,t] \cap D_J} \exp(-m_s \pi(\widehat{X}_s, \widehat{Y}_s, u^g_{t-s})) \bigg| \sigma((\widehat{X}, \widehat{Y})) \bigg] \bigg\} \\
&= \Pi^{\phi}_{\phi\mu}\bigg\{ \prod_{s \in (0,t] \cap D_J} \int_{[0,\infty)} \exp(-r\pi(X_s, Y_s, u^g_{t-s})) \widetilde{F}(X_s, Y_s; dr) \bigg\}.
\end{aligned}$$

This completes the proof. □

Without loss of generality, we suppose $\{\chi_t, t \geqslant 0; \mathrm{P}_{\mu,\phi}\}$ is a multitype superdiffusion defined on $(\Omega, \mathrm{P}_{\mu,\phi})$, having the same law as $\{\chi_t, t \geqslant 0; \mathrm{P}_{\mu}\}$ and independent of $\widehat{\chi} = \{\widehat{\chi}_t, t \geqslant 0\}$. Proposition 4.2 says that we have the following decomposition of $\{\chi_t, t \geqslant 0\}$ under $\widetilde{\mathrm{P}}_{\mu}$: for any $t > 0$,

$$(\chi_t, \widetilde{\mathrm{P}}_{\mu}) = (\chi_t + \widehat{\chi}_t,\ \mathrm{P}_{\mu,\phi}) \quad \text{in distribution.} \tag{4.12}$$

Since $\{\chi_t, t \geqslant 0; \widetilde{\mathrm{P}}_{\mu}\}$ is generated from the time-homogeneous Markov process $\{\chi_t, t \geqslant 0; \mathrm{P}_{\mu}\}$ via a non-negative martingale multiplicative functional, $\{\chi_t, t \geqslant 0; \widetilde{\mathrm{P}}_{\mu}\}$ is also a time-homogeneous Markov process (see [27, Section 62]). From the construction of $\{\widehat{\chi}_t, t \geqslant 0; \mathrm{P}_{\mu,\phi}\}$ we see that $\{\widehat{\chi}_t, t \geqslant 0; \mathrm{P}_{\mu,\phi}\}$ is a time-homogeneous Markov process. For a rigorous proof of $\{\widehat{\chi}_t, t \geqslant 0; \mathrm{P}_{\mu,\phi}\}$ being a time-homogeneous Markov process, we refer our readers to [13]. Although the paper [13] dealt with the representation of the superprocess conditioned to stay alive forever, one can check that the arguments there work in our case. Therefore, (4.12) implies the following theorem.

Theorem 4.3. *It holds that*

$$\{\chi_t, t \geqslant 0; \widetilde{\mathrm{P}}_{\mu}\} = \{\chi_t + \widehat{\chi}_t, t \geqslant 0; \mathrm{P}_{\mu,\phi}\} \quad \textit{in law.} \tag{4.13}$$

5 $L\log L$ criterion

In this section, we give a proof of the main result of this paper, Theorem 2.4. First, we make some preparations.

Proposition 5.1. *Let* $h(x,i)=\frac{1}{\phi(x,i)}\mathrm{P}_{\delta_{(x,i)}}(W_\infty(\phi))$. *Then*

(i) *h is a non-negative invariant function for the process* $((X,Y);\Pi^{\phi}_{(x,i)})$.

(ii) *Either W_∞ is non-degenerate under P_μ for all nonzero $\mu\in M_F(D\times S)$ or W_∞ is degenerate under P_μ for all $\mu\in M_F(D\times S)$.*

Proof. (i) By the Markov property of χ,

$$\begin{aligned} h(x,i)&=\frac{1}{\phi(x,i)}\mathrm{P}_{\delta_{(x,i)}}\Big[\lim_{s\to\infty}\langle \mathrm{e}^{-\lambda_1(t+s)}\phi,\chi_{t+s}\rangle\Big]\\ &=\frac{\mathrm{e}^{-\lambda_1 t}}{\phi(x,i)}\mathrm{P}_{\delta_{(x,i)}}\Big[\mathrm{P}_{\chi_t}\Big(\lim_{s\to\infty}\langle \mathrm{e}^{-\lambda_1 s}\phi,\chi_s\rangle\Big)\Big]\\ &=\frac{\mathrm{e}^{-\lambda_1 t}}{\phi(x,i)}\mathrm{P}_{\delta_{(x,i)}}[\mathrm{P}_{\chi_t}(W_\infty)]\\ &=\frac{\mathrm{e}^{-\lambda_1 t}}{\phi(x,i)}\mathrm{P}_{\delta_{(x,i)}}[\langle (h\phi),\chi_t\rangle]\\ &=\frac{\mathrm{e}^{-\lambda_1 t}}{\phi(x,i)}P_t^{\mathcal{A}+B\cdot(N-I)}(h\phi),\quad x\in D. \end{aligned}$$

By the definition of $\Pi^{\phi}_{(x,i)}$, we get that $h(x,i)=\Pi^{\phi}_{(x,i)}[h(X_t,Y_t)]$. So h is an invariant function of the process $((X,Y);\Pi^{\phi}_{(x,i)})$. The non-negativity of h is obvious.

(ii) Since h is non-negative and invariant, if there exists $(x_0,i)\in D\times S$ such that $h(x_0,i)=0$, then $h\equiv 0$ on $D\times S$. Since $\mathrm{P}_\mu(W_\infty(\phi))=\langle h\phi,\mu\rangle$, we then have $\mathrm{P}_\mu(W_\infty(\phi))=0$ for any $\mu\in M_F(D\times S)$. If $h>0$ on $D\times S$, then $\mathrm{P}_\mu(W_\infty(\phi))>0$ for any nonzero $\mu\in M_F(D\times S)$. □

Using Proposition 5.1 we see that, to prove Theorem 2.4, we only need to consider the case $d\mu=\phi(x,i)dxdi$, where di is the counting measure on S. So in the remaining part of this paper we always suppose that $d\mu=\phi(x,i)dxdi$.

Recall from (2.3) and (2.9) that

$$\pi(x,i;\phi)=\sum_{j=1}^{K}p_j^{(i)}(x)\phi(x,j),\quad (x,i)\in D\times S$$

and

$$l(x,i)=\int_0^\infty r\log^+(r)F^{\pi(\phi)}(x,i;dr)=\int_0^\infty r\pi(x,i,\phi)\log^+(r\pi(x,i,\phi))F(x,i;dr).$$

Lemma 5.2. *Let $(m_t;\ t\in D_J)$ be the Poisson point process constructed in Section 4, given the path of $(\widehat{X}_s,\widehat{Y}_s),s\geqslant 0$. Define*

$$\sigma_0=0,\quad \sigma_i=\inf\{s\in D_J;s>\sigma_{i-1},m_s\pi(\widehat{X}_s,\widehat{Y}_s;\phi)>1\},\quad \eta_i=m_{\sigma_i},\quad i=1,2,\dots$$

(i) *If* $\sum_{i=1}^K\int_D\phi(y,i)b(y,i)l(y,i)dy<\infty$, *then*

$$\sum_{s\in D_J}\mathrm{e}^{-\lambda_1 s}m_s\pi(\widehat{X}_s,\widehat{Y}_s;\phi)<\infty,\quad \mathrm{P}_{\mu,\phi}\text{-}a.s. \tag{5.1}$$

(ii) *If* $\sum_{i=1}^K\int_D\phi(y,i)b(y,i)l(y,i)dy=\infty$, *then*

$$\limsup_{i\to\infty}\mathrm{e}^{-\lambda_1\sigma_i}\eta_i\pi(\widehat{X}_{\sigma_i},\widehat{Y}_{\sigma_i};\phi)=\infty,\quad \mathrm{P}_{\mu,\phi}\text{-}a.s. \tag{5.2}$$

Proof. Since ϕ is bounded from above, σ_i is strictly increasing with respect to i.

(i) Suppose that $\sum_{i=1}^K \int_D \phi(y,i)b(y,i)l(y,i)dy < \infty$. For any $\varepsilon > 0$, we write the sum in (5.1) as

$$\begin{aligned}\sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \pi(\widehat{X}_s, \widehat{Y}_s; \phi) &= \sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \pi(\widehat{X}_s, \widehat{Y}_s; \phi) 1_{\{m_s\pi(\widehat{X}_s,\widehat{Y}_s;\phi)\leqslant \mathrm{e}^{\varepsilon s}\}} \\ &\quad + \sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \pi(\widehat{X}_s, \widehat{Y}_s; \phi) 1_{\{m_s\pi(\widehat{X}_s,\widehat{Y}_s;\phi)> \mathrm{e}^{\varepsilon s}\}} \\ &= \sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \pi(\widehat{X}_s, \widehat{Y}_s; \phi) 1_{\{\pi(\widehat{X}_s,\widehat{Y}_s;\phi)m_s\leqslant \mathrm{e}^{\varepsilon s}\}} \\ &\quad + \sum_{i=1}^{\infty} \mathrm{e}^{-\lambda_1 \sigma_i} \eta_i \pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi) 1_{\{\eta_i\pi(\widehat{X}_{\sigma_i},\widehat{Y}_{\sigma_i};\phi)> \mathrm{e}^{\varepsilon \sigma_i}\}} \\ &=: I + II. \end{aligned} \tag{5.3}$$

Note that the jumping intensity of $\{(\widehat{X}, \widehat{Y}), \mathrm{P}_{\mu,\phi}\}$ is $\frac{bn\pi(\phi)}{\phi}(x,i)$ at $(x,i) \in D\times S$. Thus

$$\begin{aligned}&\sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}(\eta_i \pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi) > \mathrm{e}^{\varepsilon\sigma_i}) \\ &\quad = \sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}[\mathrm{P}_{\mu,\phi}(\eta_i \pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi) > \mathrm{e}^{\varepsilon\sigma_i} \mid \sigma(\widehat{X}, \widehat{Y}))] \\ &\quad = \mathrm{P}_{\mu,\phi}\left[\mathrm{P}_{\mu,\phi}\left(\sum_{i=1}^{\infty} 1_{\{\eta_i > \mathrm{e}^{\varepsilon\sigma_i}\pi(\widehat{X}_{\sigma_i},\widehat{Y}_{\sigma_i};\phi)^{-1}\}} \,\middle|\, \sigma(\widehat{X}, \widehat{Y})\right)\right] \\ &\quad = \Pi^{\phi}_{\phi\mu}\left[\int_0^{\infty} (bn\pi(\phi)/\phi)(X_s, Y_s)\left(\int_{\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\varepsilon s}}^{\infty} \widetilde{F}(X_s, Y_s; dr)\right) ds\right].\end{aligned}$$

Recall that under $\Pi^{\phi}_{\phi\mu}$, (X,Y) starts at the invariant measure $\phi^2(x,i)dxdi$. By the definition of $\widetilde{F}$ given in (4.9), we get

$$\begin{aligned}&\sum_{i=1}^{\infty} \mathrm{P}_{\mu,\phi}(\eta_i \pi(X_{\sigma_i}, Y_{\sigma_i}; \phi) > \mathrm{e}^{\varepsilon\sigma_i}) \\ &\quad = \int_0^{\infty} ds \sum_{j=1}^{K} \int_D dy (b\phi)(y,j) \int_{\pi(y,j;\phi)^{-1}\mathrm{e}^{\varepsilon s}}^{\infty} \pi(y,j;\phi) r\, F(y,j;dr) \\ &\quad = \sum_{j=1}^{K} \int_D (b\phi)(y,j) dy \int_{\pi(y,j;\phi)^{-1}}^{\infty} \pi(y,j;\phi) r\, F(y,j;dr) \int_0^{\frac{\log(r\pi(y,j;\phi))}{\varepsilon}} ds \\ &\quad = \varepsilon^{-1} \sum_{j=1}^{K} \int_D (b\phi)(y,j) l(y,j) dy.\end{aligned}$$

By the assumption that $\sum_{j=1}^K \int_D (b\phi)(y,j)l(y,j)dy < \infty$ and the Borel-Cantelli lemma, we get

$$\mathrm{P}_{\mu,\phi}(\eta_i \pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi) > \mathrm{e}^{\varepsilon\sigma_i} \text{ i.o.}) = 0 \tag{5.4}$$

for all $\varepsilon > 0$, which implies that

$$II < \infty, \quad \mathrm{P}_{\mu,\phi}\text{-a.s.} \tag{5.5}$$

Meanwhile for $\varepsilon < \lambda_1$,

$$\begin{aligned}\mathrm{P}_{\mu,\phi} I &= \mathrm{P}_{\mu,\phi}\left[\sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \pi(\widehat{X}_s, \widehat{Y}_s; \phi) 1_{\{m_s \leqslant \mathrm{e}^{\varepsilon s}\pi(\widehat{X}_s,\widehat{Y}_s;\phi)^{-1}\}}\right] \\ &= \Pi^{\phi}_{\phi\mu} \int_0^{\infty} dt \mathrm{e}^{-\lambda_1 t} \int_0^{\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\varepsilon t}} \frac{bn\pi(\phi)}{\phi}(X_t, Y_t) \pi(X_t, Y_t; \phi) r \widetilde{F}(X_t, Y_t; dr)\end{aligned}$$

$$\leqslant \Pi^{\phi}_{\phi\mu} \int_0^{\infty} dt e^{-(\lambda_1-\varepsilon)t} \int_0^{\infty} \frac{b\pi(\phi)}{\phi}(X_t, Y_t) rF(X_t, Y_t; dr),$$

where for the inequality above we used the fact that $r \leqslant \pi(X_t, Y_t, \phi)^{-1} e^{\varepsilon t}$ implies that $r\pi(X_t, Y_t, \phi) \leqslant e^{\varepsilon t}$. By the assumption that $\sup_{(x,i)\in D\times S} \int_0^{\infty} rF(x,i,dr) < \infty$, we have

$$\begin{aligned} P_{\mu,\phi} I &\leqslant \frac{1}{\lambda_1 - \varepsilon} \sum_{i=1}^{K} \int_D b(y,i)\pi(y,i;\phi)\phi(y,i) \int_0^{\infty} rF(y,i,dr)dy \\ &\leqslant \frac{1}{\lambda_1 - \varepsilon} \left\| \int_0^{\infty} rF(y,i,dr) \right\|_{\infty} \|b\|_{\infty} < \infty. \end{aligned}$$

Thus

$$I < \infty, \quad P_{\mu,\phi}\text{-a.s.} \tag{5.6}$$

Combining (5.3), (5.5) and (5.6), we obtain (5.1).

(ii) Next, we assume $\sum_{i=1}^{K} \int_D (b\phi)(y,i) l(y,i)dy = \infty$. To establish (5.2), it suffices to show that for any $L > 0$,

$$\limsup_{i\to\infty} e^{-\lambda_1\sigma_i} \eta_i \pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi) > L, \quad P_{\mu,\phi}\text{-a.s.} \tag{5.7}$$

Put $L_0 := 1 \vee (\max_{(x,i)\in D\times S} \phi(x,i))$. Then for $L \geqslant L_0$,

$$L \inf_{(x,i)\in D\times S} \phi(x,i)^{-1} \geqslant 1.$$

Note that for any $T \in (0,\infty)$, conditional on $\sigma(\widehat{X}, \widehat{Y})$,

$$\sharp\{i : \sigma_i \in (0,T]; \eta_i > L\pi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}; \phi)^{-1} e^{\lambda_1\sigma_i}\}$$

is a Poisson random variable with parameter

$$\int_0^T dt (b\pi(\phi)/\phi)(\widehat{X}_t, \widehat{Y}_t) \int_{L\pi(\widehat{X}_t, \widehat{Y}_t;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(\widehat{X}_t, \widehat{Y}_t; dr).$$

Since $(\widehat{X}, \widehat{Y}; P_{\mu,\phi})$ has the same law as $(X, Y; \Pi^{\phi}_{\mu\phi})$, we have

$$\begin{aligned} &P_{\mu,\phi} \int_0^T dt \frac{b\pi(\phi)}{\phi}(\widehat{X}_t, \widehat{Y}_t) \int_{L\pi(\widehat{X}_t, \widehat{Y}_t;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(\widehat{X}_t, \widehat{Y}_t; dr) \\ &= \int_0^T dt \sum_{j=1}^{K} \int_D dy (b\pi(\phi)\phi)(y,j) \int_{L\pi(y,j;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(y,j;dr) < \infty. \end{aligned}$$

Thus

$$\int_0^T dt \frac{b\pi(\phi)}{\phi}(\widehat{X}_t, \widehat{Y}_t) \int_{L\pi(\widehat{X}_t, \widehat{Y}_t;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(\widehat{X}_t, \widehat{Y}_t; dr) < \infty, \quad P_{\mu,\phi}\text{-a.s.}$$

Consequently, we have

$$\sharp\{i : \sigma_i \in (0,T]; \eta_i > L\pi(\widehat{X}_t, \widehat{Y}_t; \phi)^{-1} e^{\lambda_1\sigma_i}\} < \infty, \quad P_{\mu,\phi}\text{-a.s.} \tag{5.8}$$

So, to prove (5.7), we need to prove

$$\int_0^{\infty} dt \frac{b\pi(\phi)}{\phi}(\widehat{X}_t, \widehat{Y}_t) \int_{L\pi(\widehat{X}_t, \widehat{Y}_t;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(\widehat{X}_t, \widehat{Y}_t; dr) = \infty, \quad P_{\mu,\phi}\text{-a.s.},$$

which is equivalent to

$$\int_0^{\infty} dt \frac{b\pi(\phi)}{\phi}(X_t, Y_t) \int_{L\pi(X_t, Y_t;\phi)^{-1} e^{\lambda_1 t}}^{\infty} rF(X_t, Y_t; dr) = \infty, \quad \Pi^{\phi}_{\phi\mu}\text{-a.s.} \tag{5.9}$$

For this purpose we first prove that

$$\Pi^{\phi}_{\phi\mu}\left[\int_0^{\infty} dt \frac{b\pi(\phi)}{\phi}(X_t,Y_t)\int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty} rF(X_t,Y_t;dr)\right]=\infty. \tag{5.10}$$

Applying Fubini's theorem, we get

$$\begin{aligned}
&\Pi^{\phi}_{\phi\mu}\left[\int_0^{\infty} dt \frac{b\pi(\phi)}{\phi}(X_t,Y_t)\int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty} rF(X_t,Y_t;dr)\right]\\
&=\sum_{j=1}^{K}\int_D b(y,j)\pi(y,j;\phi)\phi(y,j)dy\int_0^{\infty} dt\int_{L\pi(y,j;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty} rF(y,j;dr)\\
&=\sum_{j=1}^{K}\int_D b(y,j)\pi(y,j;\phi)\phi(y,j)dy\int_{L\pi(y,j;\phi)^{-1}}^{\infty} rF(y,j;dr)\int_0^{\frac{1}{\lambda_1}\log(\frac{r\pi(y,j;\phi)}{L})} dt\\
&=\sum_{j=1}^{K}\frac{1}{\lambda_1}\int_D b(y,j)\pi(y,j;\phi)\phi(y,j)dy\int_{L\pi(y,j;\phi)^{-1}}^{\infty}(\log[r\pi(y,j;\phi)]-\log L)rF(y,j;dr)\\
&\geqslant\sum_{j=1}^{K}\frac{1}{\lambda_1}\int_D b(y,j)\pi(y,j;\phi)\phi(y,j)dy\left[\int_{L\pi(y,j;\phi)^{-1}}^{\infty} r\log[r\pi(y,j;\phi)]F(y,j;dr)-A\right]\\
&=\sum_{j=1}^{K}\frac{1}{\lambda_1}\int_D (b\phi)(y,j)dy\int_L^{\infty} r\log r\, F^{\pi(\phi)}(y,j;dr)-\sum_{j=1}^{K}\frac{A}{\lambda_1}\int_D (b\phi)(y,j)\pi(y,j;\phi)dy,
\end{aligned}$$

for some constant $A>0$, where in the inequality we used the facts that $L\pi(y,j;\phi)^{-1}>1$ for any $(y,j)\in D\times S$ and $\sup_{(y,j)\in D\times S}\int_1^{\infty} rF(y,j;dr)<\infty$. It is easy to see that

$$\sum_{j=1}^{K}\frac{A}{\lambda_1}\int_D (b\phi)(y,j)\pi(y,j;\phi)dy\leqslant\frac{A}{\lambda_1}\|b\|_{\infty}<\infty.$$

Since

$$\sum_{j=1}^{K}\int_D (b\phi)(y,j)dy\int_1^{\infty} r\log r\, F^{\pi(\phi)}(y,j;dr)=\infty$$

and

$$\begin{aligned}
&\sum_{j=1}^{K}\int_D (b\phi)(y,j)dy\int_1^{L} r\log r\, F^{\pi(\phi)}(y,j;dr)\\
&\qquad\leqslant L\log L\sum_{j=1}^{K}\int_D (b\phi)(y,j)F(y,j;[\|\phi\|_{\infty}^{-1},\infty))dy<\infty,
\end{aligned}$$

we get that

$$\sum_{j=1}^{K}\int_D (b\phi)(y,j)dy\int_L^{\infty} r\log r\, F^{\pi(\phi)}(y,j;dr)=\infty,$$

and therefore, (5.10) holds.

By (2.7), there exists a constant $t_0>0$ such that for any $t>t_0$ and any $f\in B_b^+(D\times S)$,

$$\begin{aligned}
\frac{1}{2}\int_{D\times S}\phi^2(y,j)f(y,j)dydi&\leqslant\int_{D\times S} p^{\phi}(t,(x,i),(y,j))f(y,j)dydi\\
&\leqslant 2\int_{D\times S}\phi^2(y,j)f(y,j)dydi
\end{aligned} \tag{5.11}$$

holds for any $(x,i)\in D\times S$. For $T>t_0$, we define

$$\xi_T=\int_0^T dt\frac{b\pi(\phi)}{\phi}(X_t,Y_t)\int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty} rF(X_t,Y_t;dr)$$

and

$$A_T=\sum_{j=1}^K\int_{t_0}^T dt\int_D (b\phi)(y,j)dy\int_{L\mathrm{e}^{\lambda_1 t}}^{\infty} rF^{\pi(\phi)}(y,j;dr).$$

Our goal is to prove (5.9), which is equivalent to

$$\xi_\infty:=\int_0^\infty dt\frac{b\pi(\phi)}{\phi}(X_t,Y_t)\int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty} rF(X_t,Y_t;dr)=\infty,\quad \Pi^{\phi}_{\phi\mu}\text{-a.s.}\tag{5.12}$$

Since $\{\xi_\infty=\infty\}$ is an invariant event, by the ergodic property of $\{(X,Y),\Pi^{\phi}_{\phi\mu}\}$, it is enough to prove

$$\Pi^{\phi}_{\phi\mu}(\xi_\infty=\infty)>0.\tag{5.13}$$

Note that

$$\Pi^{\phi}_{\phi\mu}\xi_T=\sum_{j=1}^K\int_0^T dt\int_D (b\phi)(y,j)dy\int_{L\mathrm{e}^{\lambda_1 t}}^{\infty} rF^{\pi(\phi)}(y,j;dr)\geqslant A_T\tag{5.14}$$

and

$$\begin{aligned}\lim_{T\to\infty}\Pi^{\phi}_{\phi\mu}\xi_T\geqslant A_\infty&=\sum_{j=1}^K\int_{t_0}^\infty dt\int_D (b\phi)(y,j)dy\int_{L\mathrm{e}^{\lambda_1 t}}^{\infty} rF^{\pi(\phi)}(y,j;dr)\\&=\sum_{j=1}^K\int_D (b\phi)(y,j)dy\int_{L\mathrm{e}^{\lambda_1 t_0}}^{\infty}\left(\frac{1}{\lambda_1}(\log r-\log L)-t_0\right)rF^{\pi(\phi)}(y,j;dr)\\&\geqslant c\sum_{j=1}^K\int_D (b\phi)(y,j)l(y,j)dy=\infty,\end{aligned}\tag{5.15}$$

where c is a positive constant. By [9, Exercise 1.3.8],

$$\Pi^{\phi}_{\phi\mu}\left(\xi_T\geqslant\frac{1}{2}\Pi^{\phi}_{\phi\mu}\xi_T\right)\geqslant\frac{(\Pi^{\phi}_{\phi\mu}\xi_T)^2}{4\Pi^{\phi}_{\phi\mu}(\xi_T^2)}.\tag{5.16}$$

If we can prove that there is a constant $\widehat{c}>0$ such that for all $T>t_0$,

$$\frac{(\Pi^{\phi}_{\phi\mu}\xi_T)^2}{4\Pi^{\phi}_{\phi\mu}(\xi_T^2)}\geqslant\widehat{c},\tag{5.17}$$

then by (5.16) we would get

$$\Pi^{\phi}_{\phi\mu}\left(\xi_T\geqslant\frac{1}{2}\Pi^{\phi}_{\phi\mu}\xi_T\right)\geqslant\widehat{c},$$

and therefore

$$\Pi^{\phi}_{\phi\mu}\left(\xi_\infty\geqslant\frac{1}{2}\Pi^{\phi}_{\phi\mu}\xi_T\right)\geqslant\Pi^{\phi}_{\phi\mu}\left(\xi_T\geqslant\frac{1}{2}\Pi^{\phi}_{\phi\mu}\xi_T\right)\geqslant\widehat{c}>0.$$

Since $\lim_{T\to\infty}\Pi^{\phi}_{\phi\mu}\xi_T=\infty$ (see (5.15)), the above inequality implies (5.13). Now we only need to prove (5.17). For this purpose we first estimate $\Pi^{\phi}_{\phi\mu}(\xi_T^2)$:

$$\Pi^{\phi}_{\phi\mu}\xi_T^2=\Pi^{\phi}_{\phi\mu}\int_0^T dt\int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^{\infty}\frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr)$$

$$
\begin{aligned}
&\times \int_0^T ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(X_s,Y_s)uF(X_s,Y_s;du) \\
&= 2\Pi_{\phi\mu}^\phi \int_0^T dt \int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty \frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr) \\
&\quad\times \int_t^T ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(\widehat{X}_s,\widehat{Y}_s)uF(X_s,Y_s;du) \\
&= 2\Pi_{\phi\mu}^\phi \int_0^T dt \int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty \frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr) \\
&\quad\times \int_t^{(t+t_0)\wedge T} ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(X_s,Y_s)uF(X_s,Y_s;du) \\
&\quad + 2\Pi_{\phi\mu}^\phi \int_0^T dt \int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty \frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr) \\
&\quad\times \int_{(t+t_0)\wedge T}^T ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(X_s,Y_s)uF(X_s,Y_s;du) \\
&=: III + IV,
\end{aligned}
$$

where

$$
\begin{aligned}
III = 2\Pi_{\phi\mu}^\phi &\int_0^T dt \int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty \frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr) \\
&\times \int_t^{(t+t_0)\wedge T} ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(X_s,Y_s)uF(X_s,Y_s;du)
\end{aligned}
$$

and

$$
\begin{aligned}
IV &= 2\Pi_{\phi\mu}^\phi \int_0^T dt \int_{L\pi(X_t,Y_t;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty \frac{b\pi(\phi)}{\phi}(X_t,Y_t)rF(X_t,Y_t;dr) \\
&\quad\times \int_{(t+t_0)\wedge T}^T ds \int_{L\pi(X_s,Y_s;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \frac{b\pi(\phi)}{\phi}(X_s,Y_s)uF(X_s,Y_s;du) \\
&= 2\sum_{j=1}^K \int_0^T dt \int_D (b\phi)(y,j)dy \int_{L\pi(y,j;\phi)^{-1}\mathrm{e}^{\lambda_1 t}}^\infty r\pi(y,j;\phi)F(y,j;dr) \\
&\quad\times \int_{(t+t_0)\wedge T}^T ds \int_D p^\phi(s-t,(y,j),(z,k))\frac{b\pi(\phi)}{\phi}(z,k)dz \int_{L\pi(z,k;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty uF(z,k;du).
\end{aligned}
$$

By our assumption we have $\| \int_1^\infty rF(\cdot;dr)\|_\infty < \infty$. Since $L\inf_{(x,i)\in D\times S}\phi(x,j)^{-1} \geqslant 1$, we have

$$
III \leqslant c_1 \Pi_{\phi\mu}^\phi \xi_T,
$$

for some positive constant c_1 which does not depend on T. Using (5.11) and the definition of $n^{\pi(\phi)}$, we get that

$$
\begin{aligned}
&\int_{(t+t_0)\wedge T}^T ds \int_D p^\phi(s-t,(y,j),(z,k))\frac{b\pi(\phi)}{\phi}(z,k)dz \int_{L\pi(z,k;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty uF(z,k;du) \\
&\quad\leqslant 2\int_{(t+t_0)\wedge T}^T ds \int_D (b\phi)(z,k)dz \int_{L\phi(z,k;\phi)^{-1}\mathrm{e}^{\lambda_1 s}}^\infty \pi(z,k;\phi)uF(z,k;du) \\
&\quad\leqslant 2\int_{t_0}^T ds \int_D (b\phi)(z,k)dz \int_{L\mathrm{e}^{\lambda_1 s}}^\infty rF^{\pi(\phi)}(z,k;dr) \\
&\quad= 2\sum_{k=1}^K \int_{t_0}^T ds \int_D (b\phi)(z,k)dz \int_{L\mathrm{e}^{\lambda_1 s}}^\infty rF^{\pi(\phi)}(z,k;dr) = 2A_T.
\end{aligned}
$$

Then using (5.14), we have

$$IV \leqslant 4A_T \Pi^{\phi}_{\phi\mu}\xi_T \leqslant 4(\Pi^{\phi}_{\phi\mu}\xi_T)^2.$$

Combining the estimates above on III and IV, we get that there exists a $c_2 > 0$ independent of T such that for $T > t_0$,

$$\Pi^{\phi}_{\phi\mu}(\xi_T^2) \leqslant 4(\Pi^{\phi}_{\phi\mu}(\xi_T))^2 + c_1\Pi^{\phi}_{\phi\mu}(\xi_T) \leqslant c_2(\Pi^{\phi}_{\phi\mu}(\xi_T))^2.$$

Then we have (5.17) with $\widehat{c} = 1/c_2$, and the proof of the theorem is now completed. □

Definition 5.3. Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, $\{\mathcal{F}_t, t \geqslant 0\}$ is a filtration on $(\Omega, \mathcal{F})$ and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. A real valued process U_t on $(\Omega, \mathcal{F}, \mathrm{P})$ is called a $\mathrm{P}(\cdot\,|\,\mathcal{G})$-martingale (resp. submartingale, supermartingale) with respect to $\{\mathcal{F}_t, t \geqslant 0\}$ if (i) it is adapted to $\{\mathcal{F}_t \vee \mathcal{G}, t \geqslant 0\}$; (ii) for any $t \geqslant 0$, $\mathrm{E}(|U_t|\,|\,\mathcal{G}) < \infty$ and (iii) for any $t > s$,

$$\mathrm{E}(U_t\,|\,\mathcal{F}_s \vee \mathcal{G}) = (\text{resp.}\ \ \geqslant, \leqslant)\ U_s, \quad \text{a.s.}$$

We need the following result. For its proof, see [23, Lemma 3.3].

Lemma 5.4. *Suppose that $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, $\{\mathcal{F}_t, t \geqslant 0\}$ is a filtration on $(\Omega, \mathcal{F})$ and $\mathcal{G}$ is a σ-field of $\mathcal{F}$. If U_t is a $\mathrm{P}(\cdot\,|\,\mathcal{G})$-submartingale with respect to $\{\mathcal{F}_t, t \geqslant 0\}$ satisfying*

$$\sup_{t\geqslant 0} \mathrm{E}(|U_t|\,|\,\mathcal{G}) < \infty \quad a.s., \tag{5.18}$$

then there exists a finite random variable U_∞ such that U_t converges a.s. to U_∞.

We are now in the position to prove the main result of this paper.

Proof of Theorem 2.4. Recall that, by Proposition 5.1, to prove Theorem 2.4, we only need to consider the case $d\mu = \phi(x,i)dxdi$, where di is the counting measure on S.

We first prove that if $\sum_{i=1}^K \int_D \phi(x,i)b(x,i)l(x,i)dx < \infty$, then W_∞ is non-degenerate under P_μ. Since $W_t(\phi)$ is a non-negative martingale, to show it is a closed martingale, it suffices to prove $\mathrm{P}_\mu(W_\infty(\phi)) = \mathrm{P}_\mu(W_0(\phi)) = \langle \phi, \mu\rangle$. Since $W_t^{-1}(\phi)$ is a positive supermartingale under $\widetilde{\mathrm{P}}_\mu$, $W_t(\phi)$ converges to some non-negative random variable $W_\infty(\phi) \in (0, \infty]$ under $\widetilde{\mathrm{P}}_\mu$. By [9, Theorem 5.3.3], we only need to prove that

$$\widetilde{\mathrm{P}}_\mu\left(W_\infty(\phi) < \infty\right) = 1. \tag{5.19}$$

By (4.12), $(\chi_t, t \geqslant 0; \widetilde{\mathrm{P}}_\mu)$ has the same law as $(\chi_t + \hat{\chi}_t, t \geqslant 0; \mathrm{P}_{\mu,\phi})$, where $\{\chi_t, t \geqslant 0; \mathrm{P}_{\mu,\phi})$ is a copy of $(\chi_t, t \geqslant 0; \mathrm{P}_\mu)$, and $\hat{\chi}_t = \sum_{s\in(0,t]\cap D_J} \chi_t^s$. Put

$$M_t(\phi) := \sum_{s\in(0,t]\cap\mathcal{D}_J} \langle \phi, \chi_t^s\rangle\ \mathrm{e}^{-\lambda_1 t}. \tag{5.20}$$

Then

$$(W_t(\phi), t \geqslant 0; \widetilde{\mathrm{P}}_\mu) = (W_t(\phi) + M_t(\phi), t \geqslant 0; \mathrm{P}_{\mu,\phi}) \quad \text{in law}, \tag{5.21}$$

where $\{W_t(\phi), t \geqslant 0\}$ is a copy of the martingale defined in (2.8) and is independent of $M_t(\phi)$. Let $\mathcal{G}$ be the σ-field generated by $\{Y_t, m_t, t \geqslant 0\}$. Then, conditional on $\mathcal{G}$, $(\chi_t^s, t \geqslant s, \mathrm{P}_{\mu,\phi})$ has the same law as $(\chi_{t-s}, t \geqslant s, \mathrm{P}_{m_s\delta_{\widehat{Y}_s}})$ and $(\chi_t^s, t \geqslant s, \mathrm{P}_{\mu,\phi})$ are independent for $s \in \mathcal{D}_J$. Then we have

$$M_t(\phi) \overset{d}{=} \sum_{s\in(0,t]\cap\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} W_{t-s}^s(\phi), \tag{5.22}$$

where for each $s \in \mathcal{D}_J$, $W_t^s(\phi)$ is a copy of the martingale defined by (2.8) with $\mu = m_s\delta_{\widehat{Y}_s}$, and conditional on $\mathcal{G}$, $\{W_t^s(\phi), t \geqslant 0\}$ are independent for $s \in \mathcal{D}_J$. To prove (5.19), by (5.21), it suffices to show that

$$\mathrm{P}_{\mu,\phi}\Big(\lim_{t\to\infty}[W_t(\phi) + M_t(\phi)] < \infty\Big) = 1.$$

Since $(W_t(\phi), t \geqslant 0)$ is a non-negative martingale under the probability $\mathrm{P}_{\mu,\phi}$, it converges $\mathrm{P}_{\mu,\phi}$ almost surely to a finite random variable $W_\infty(\phi)$ as $t \to \infty$. So we only need to prove

$$\mathrm{P}_{\mu,\phi}\Big(\lim_{t\to\infty} M_t(\phi) < \infty\Big) = 1. \tag{5.23}$$

Define $\mathcal{H}_t := \mathcal{G} \vee \sigma(\chi^\sigma_{(s-\sigma)}; \sigma \in [0,t] \cap \mathcal{D}_m, s \in [\sigma, t])$. Then $(M_t(\phi))$ is a $\mathrm{P}_{\mu,\phi}(\cdot \,|\, \mathcal{G})$-non-negative submartingale with respect to $(\mathcal{H}_t)$. By (5.22) and Lemma 5.2,

$$\begin{aligned}\sup_{t\geqslant 0} \mathrm{P}_{\mu,\phi}(M_t(\phi) \,|\, \mathcal{G}) &= \sup_{t\geqslant 0} \sum_{s\in[0,t]\cap\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \phi(\widehat{X}_s, \widehat{Y}_s) \\ &\leqslant \sum_{s\in\mathcal{D}_J} \mathrm{e}^{-\lambda_1 s} m_s \phi(\widehat{X}_s, \widehat{Y}_s) < \infty, \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}\end{aligned}$$

Then by Lemma 5.4, $M_t(\phi)$ converges $\mathrm{P}_{\mu,\phi}$-a.s. to $M_\infty(\phi)$ as $t \to \infty$ and $\mathrm{P}_{\mu,\phi}(M_\infty(\phi) < \infty) = 1$, which establishes (5.23).

Now we prove the other direction. Assume that $\sum_{i=1}^K \int_D \phi(y,i) b(y,i) l(y,i) dy = \infty$. We are going to prove that $W_\infty(\phi) := \lim_{t\to\infty} W_t(\phi)$ is degenerate with respect to P_μ. By [15, Proposition 2], $\frac{1}{W_t(\phi)}$ is a supermartingale under $\widetilde{\mathrm{P}}_\mu$, and thus $1/[M_t(\phi) + W_t(\phi)]$ is a non-negative supermartingale under $\mathrm{P}_{\mu,\phi}$. Recall that $W_t(\phi)$ is a non-negative martingale under $\mathrm{P}_{\mu,\phi}$. Then the limits $\lim_{t\to\infty} W_t(\phi)$ and $1/\lim_{t\to\infty}[M_t(\phi) + W_t(\phi)]$ exist and finite $\mathrm{P}_{\mu,\phi}$-a.s. Therefore $\lim_{t\to\infty} M_t(\phi)$ exists in $[0,\infty]$ $\mathrm{P}_{\mu,\phi}$-a.s. Recall the definition of $(\eta_i, \sigma_i; i = 1, 2, \ldots)$ in Lemma 5.2, and note that $\lim_{i\to\infty} \sigma_i = \infty$. By Lemma 5.2,

$$\limsup_{t\to\infty} M_t(\phi) \geqslant \limsup_{i\to\infty} M_{\sigma_i}(\phi) \geqslant \limsup_{i\to\infty} \mathrm{e}^{-\lambda_1\sigma_i} \eta_i \phi(\widehat{X}_{\sigma_i}, \widehat{Y}_{\sigma_i}) = \infty, \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

So we have

$$\lim_{t\to\infty} M_t(\phi) = \infty, \quad \mathrm{P}_{\mu,\phi}\text{-a.s.}$$

By (5.21),

$$\widetilde{\mathrm{P}}_\mu(W_\infty(\phi) = \infty) = 1.$$

It follows from [9, Theorem 5.3.3] that $\mathrm{P}_\mu(W_\infty = 0) = 1$. □

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Appendix A Non-local Feynman-Kac transform

In this appendix, we establish a result on time-dependent non-local Feynman-Kac transform, which has been used in the proof of Theorem 4.1.

Let E be a Lusin space and $\mathcal{B}(E)$ be the Borel σ-field on E, and let m be a σ-finite measure on $\mathcal{B}(E)$ with $\mathrm{supp}[m]=E$. Let $\{\xi_t, t\geqslant 0; \Pi_x\}$ be an m-symmetric Borel standard process on E with the Lévy system (J,t), where $J(x,dy)$ is a kernel from $(E,\mathcal{B}(E))$ to $(E\cup\{\partial\},\mathcal{B}(E\cup\{\partial\}))$.

Lemma A.1. *Suppose that $\{\xi_t, t\geqslant 0; \Pi_x\}$ is an m-symmetric Borel standard process on E with Lévy system (J,t). Assume that q is a locally bounded function on $[0,\infty)\times E$ and that F is a non-positive, $\mathcal{B}([0,\infty)\times E\times E)$-measurable function vanishing on the diagonal of $E\times E$ so that for any $x\in E$,*

$$\sum_{0<s\leqslant t} F(t-s,\xi_{s-},\xi_s) > -\infty \quad \textit{for every} \quad t>0, \quad \Pi_x\textit{-a.s.}, \tag{A.1}$$

and

$$\sup_{x\in E}\Pi_x\left[\int_0^t\int_{E_\partial}(1-\mathrm{e}^{F(t-s,\,\xi_s,\,y)})J(\xi_s,dy)ds\right]<\infty \quad \textit{for every} \quad t>0. \tag{A.2}$$

For any $x\in E$, $t\geqslant 0$ and $f\in B_b^+(E)$, define

$$h(t,x):=\Pi_x[\mathrm{e}^{\int_0^t q(t-s,\xi_s)ds+\sum_{0<s\leqslant t}F(t-s,\,\xi_{s-},\,\xi_s)}f(\xi_t)]. \tag{A.3}$$

Then h is the unique locally bounded positive solution of the following integral equation:

$$h(t,x)=\Pi_x f(\xi_t)+\Pi_x\int_0^t q(t-s,\xi_s)h(t-s,\xi_s)ds$$

$$+\Pi_x\left[\int_0^t\int_E(\mathrm{e}^{F(t-s,\,\xi_s,\,y)}-1)h(t-s,y)J(\xi_s,dy)ds\right]. \tag{A.4}$$

Proof. Note that under the locally boundedness assumption of $q(t,x)$ and (A.1), the function h of (A.3) is well defined and positive, and there exists $c>0$ such that

$$h(t,x)\leqslant \mathrm{e}^{ct}\Pi_x[f(\xi_t)].$$

Thus $h(t,x)$ is bounded on $[0,T]\times E$ for any $T>0$. The assumption (A.2) implies that the last term of (A.4) is absolutely convergent and defines a bounded function on $[0,T]\times E$ for every $T>0$. For $s\leqslant t$, define

$$A_{s,t}=\int_s^t q(t-r,\xi_r)dr+\sum_{s<r\leqslant t}F(t-r,\xi_{r-},\xi_r),$$

which is right continuous and has left limits as a function of s. Note that

$$\begin{aligned}
\mathrm{e}^{A_{0,t}}-1&=-(\mathrm{e}^{A_{t,t}}-\mathrm{e}^{A_{0,t}})\\
&=\int_0^t \mathrm{e}^{A_{s-,t}}q(t-s,\xi_s)ds-\sum_{0<s\leqslant t}(\mathrm{e}^{A_{s,t}}-\mathrm{e}^{A_{s-,t}})\\
&=\int_0^t \mathrm{e}^{A_{s,t}}q(t-s,\xi_s)ds+\sum_{0<s\leqslant t}\mathrm{e}^{A_{s,t}}(\mathrm{e}^{F(t-s,\,\xi_{s-},\,\xi_s))}-1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\Pi_x[(\mathrm{e}^{A_{0,t}}-1)f(\xi_t)]\\
&\quad=\Pi_x\left[\int_0^t \mathrm{e}^{A_{s,t}}q(t-s,\xi_s)f(\xi_t)ds\right]+\Pi_x\left[\sum_{0<s\leqslant t}\mathrm{e}^{A_{s,t}}(\mathrm{e}^{F(t-s,\,\xi_{s-},\,\xi_s)}-1)f(\xi_t)\right].
\end{aligned}$$

By the Markov property of ξ and the fact that

$$A_{s,t}=\left(\int_0^{t-s}q(t-s-r,\xi_r)dr+\sum_{0<r\leqslant t-s}F(t-s-r,\,\xi_{r-},\,\xi_r)\right)\circ\theta_s,$$

we have

$$\begin{aligned}
h(t,x)&=\Pi_x f(\xi_t)+\Pi_x\left[\int_0^t q(t-s,\xi_s)\Pi_{\xi_s}(\mathrm{e}^{\int_0^{t-s}q(t-s-r,\xi_r)dr+\sum_{0<r\leqslant t-s}F(t-s-r,\,\xi_{r-},\,\xi_r)}f(\xi_{t-s}))\right]\\
&\quad+\Pi_x\left[\sum_{0<s\leqslant t}(\mathrm{e}^{F(t-s,\,\xi_{s-},\,\xi_s)}-1)\Pi_{\xi_s}[\mathrm{e}^{\int_0^{t-s}q(t-s-r,\xi_r)dr+\sum_{0<r\leqslant t-s}F(t-s-r,\,\xi_{r-},\,\xi_r)}f(\xi_{t-s})]\right]\\
&=\Pi_x f(\xi_t)+\Pi_x\int_0^t q(t-s,\xi_s)h(t-s,\xi_s)ds+\Pi_x\left[\sum_{0<s\leqslant t}(\mathrm{e}^{F(t-s,\,\xi_{s-},\,\xi_s)}-1)h(t-s,\xi_s)\right]\\
&=\Pi_x f(\xi_t)+\Pi_x\int_0^t q(t-s,\xi_s)h(t-s,\xi_s)ds\\
&\quad+\Pi_x\left[\int_0^t\int_E(\mathrm{e}^{F(t-s,\,\xi_s,\,y)}-1)h(t-s,z)J(\xi_s,dy)ds\right].
\end{aligned}$$

Thus $h(t,x)$ defined by (A.3) is a locally bounded positive solution of (A.4).

It follows from [22, Proposition 2.15] that (A.4) has a unique locally bounded positive solution. □

Remark A.2. (i) Lemma A.1 can be easily extended to signed F (with the same argument) by replacing (A.1) and (A.2) by

$$\sum_{0<s\leqslant t}F^-(t-s,\,\xi_{s-},\,\xi_s)<\infty\quad\text{for every}\quad t>0,\quad \Pi_x\text{-a.s.},\tag{A.1$'$}$$

and

$$\sup_{x\in E}\Pi_x\left[\int_0^t\int_{E_\partial}|1-\mathrm{e}^{F(t-s,\,\xi_s,\,y)}|(\xi_s,dy)ds\right]<\infty \quad \text{for every} \quad t>0. \tag{A.2$'$}$$

(ii) If F does not depend on t, the above result follows easily from the results of [4].

(iii) If $\sup_{x\in E} J(x, E\cup\{\partial\})<\infty$, or if

$$\sup_{x\in E}\Pi_x\left[\int_0^t\int_{E_\partial}|F(t-s,\xi_s,y)|J(\xi_s,dy)ds\right]<\infty \quad \text{for every} \quad t>0, \tag{A.5}$$

then (A.1) and (A.2) are satisfied.