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Williams decomposition for superprocesses

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Abstract

We decompose the genealogy of a general superprocess with spatially dependent branching mechanism with respect to the last individual alive (Williams decomposition). This is a generalization of the main result of Delmas and Hénard [5] where only superprocesses with spatially dependent quadratic branching mechanism were considered. As an application of the Williams decomposition, we prove that, for some superprocesses, the normalized total measure will converge to a point measure at its extinction time. This partially generalizes a result of Tribe [27] in the sense that our branching mechanism is more general.

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1 Introduction

Let X be a superprocess with a spatially dependent branching mechanism. We assume that the extinction time H of X is finite. In this paper we study the genealogical structure of X. More precisely, we give a spinal decomposition of X involving the ancestral lineage of the last individual alive, conditioned on $H = h$ with $h > 0$ being a constant. This decomposition is called a Williams decomposition, in analogy with the terminology of Delmas and Hénard [5]. For a superprocess with spatially independent branching mechanism, the spatial motion is independent of the genealogical structure. As a consequence, the law of the ancestral lineage of the last individual alive does not differ from the original motion. Therefore, in this setting, the description of X conditioned on $H = h$ may be deduced from Abraham and Delmas [1] where no spatial motion is taken into account. On the contrary, for a superprocess with spatially dependent branching mechanism, the law of the ancestral lineage of the last individual alive should depend on the spatial motion and the extinction time h. Delmas and Hénard [5]

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gave a Williams decomposition for superprocesses with a spatially dependent quadratic branching mechanism given by

$$\Psi(x,z)=\beta(x)z+\alpha(x)z^2,$$

under some conditions $(H2)$ and $(H3)$ on $\beta(x)$ and $\alpha(x)$. Conditions $(H2)$ and $(H3)$ in [5] amount to saying that $1/\alpha$ belongs to the domain of the infinitesimal generator $\mathcal{L}$ of the spatial motion, and the function $\beta-\alpha\mathcal{L}(1/\alpha)$ is in the domain of $\mathcal{L}^{1/\alpha}$ where $\mathcal{L}^{1/\alpha}(u):=\alpha(\mathcal{L}(u/\alpha)-u\mathcal{L}(1/\alpha))$. In [5], the Williams decomposition was established for superprocesses with spatially dependent quadratic branching mechanism by using two transformations to change the branching mechanism $\Psi(x,z)$ to a spatially independent one, say Ψ_0, and then using the genealogy of superprocesses with branching mechanism Ψ_0 given by the Brownian snake. As mentioned in [5], the drawback of this approach is that one has to restrict to quadratic branching mechanisms with bounded and smooth parameters.

The goal of this paper is to establish a Williams decomposition for more general superprocesses. Our superprocesses are more general in two aspects: first the spatial motion can be a general Markov process and secondly the branching mechanism is general and spatially dependent (see (2.1) below). We will give conditions that guarantee our general superprocesses admit a Williams decomposition. The conditions are satisfied by a lot of superprocesses. We obtain a Williams decomposition by direct construction. For any fixed constant $h>0$, we first describe the motion of a spine up to time h and then construct three kinds of immigrations (continuous immigration, jump immigration and immigration at time 0) along the spine. We prove that, conditioned on $H=h$, the sum of the contributions of the three types of immigrations has the same distribution as X before time h, see Theorem 3.5 below. Note that for quadratic branching mechanisms, there is no jump immigration.

As an application of the Williams decomposition, we prove that, for some superprocesses, the normalized total measure will converge to a point measure at its extinction time, see Theorem 3.7 below. This partially generalizes a result of Tribe [27] in the sense that our branching mechanism is more general.

2 Preliminary

2.1 Superprocesses and assumptions

In this subsection, we describe the superprocesses we are going to work with and formulate our assumptions.

Suppose that E is a locally compact separable metric space. Let $E_\partial:=E\cup\{\partial\}$ be the one-point compactification of E. ∂ will be interpreted as the cemetery point. Any function f on E is automatically extended to E_∂ by setting $f(\partial)=0$.

Let $\mathbb{D}_E$ be the set of all the càdlàg functions from $[0,\infty)$ into E_∂ having ∂ as a trap. The filtration is defined by $\mathcal{F}_t=\mathcal{F}^0_{t+}$, where $\mathcal{F}^0_t$ is the natural canonical filtration, and $\mathcal{F}=\bigvee_{t\geq 0}\mathcal{F}_t$. Consider the canonical process ξ_t on $(\mathbb{D}_E,\{\mathcal{F}_t\}_{t\geq0})$. We will assume that $\xi=\{\xi_t,\Pi_x\}$ is a Hunt process on E and $\zeta:=\inf\{t>0:\xi_t=\partial\}$ is the lifetime of ξ. We will use $\{P_t:t\geq0\}$ to denote the semigroup of ξ. We will use $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) to denote the set of (non-negative) bounded Borel functions on E. We will use $\mathcal{M}_F(E)$ to denote the family of finite measures on E and $\mathcal{M}_F(E)^0$ to denote the family of non-zero finite measures on E.

Suppose that the branching mechanism is given by

$$\Psi(x,z)=-\alpha(x)z+b(x)z^2+\int_{(0,+\infty)}(e^{-zy}-1+zy)n(x,dy),\quad x\in E,\quad z>0,\qquad(2.1)$$

where $\alpha \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0,\infty)$ satisfying

$$\sup_{x\in E}\int_{(0,+\infty)} (y\wedge y^2)n(x,\,dy) < \infty. \tag{2.2}$$

Then there exists a constant $K>0$ such that

$$|\alpha(x)| + b(x) + \int_{(0,+\infty)} (y\wedge y^2)n(x,\,dy) \le K.$$

The boundedness assumption on α, b and the kernel n above is not absolutely necessary. For example, the boundedness of α can replaced by some kind of Kato class condition on α. However, under the Kato class condition, the argument will be more complicated, see [2, 9] for example. One might be able to get around of the boundedness assumption on b and (2.2) by changing the ds in (2.4) below by dA_s with A being an additive functional of ξ satisfying certain conditions. However, this would require that we rework most of the argument of this paper. Thus, in this paper, we will always assume that the boundedness assumption above is in force.

We equip $\mathcal{M}_F(E)$ with the topology of weak convergence. As usual, $\langle f,\mu\rangle := \int_E f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle$. Let $\mathbb{D}$ be the collection of the càdlàg functions from $[0,\infty)$ to $\mathcal{M}_F(E)$ having zero measure as a trap. Let X_t be the coordinate process on $\mathbb{D}$ and $(\mathcal{G},(\mathcal{G}_t)_{t\ge 0})$ the minimal augmented σ-fields on $\mathbb{D}$ generated by the coordinate process. According to [19, Theorem 5.12], there exist probability measures $\{\mathbb{P}_\mu : \mu \in M_F(E)\}$ such that $X = \{\mathbb{D},\mathcal{G},\mathcal{G}_t,X_t,\mathbb{P}_\mu\}$ is a Hunt process satisfying that, for every $f\in\mathcal{B}_b^+(E)$ and $\mu\in\mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu\left(e^{-\langle f,X_t\rangle}\right) = \langle u_f(t,\cdot),\mu\rangle, \tag{2.3}$$

where $u_f(t,x)$ is the unique non-negative solution to the equation

$$u_f(t,x) + \Pi_x\int_0^t \Psi(\xi_s, u_f(t-s,\xi_s))\,ds = \Pi_x f(\xi_t), \tag{2.4}$$

where $\Psi(\partial,z)=0$, $z>0$. $X=\{X_t : t\ge 0\}$ is called a superprocess with spatial motion $\xi=\{\xi_t,\Pi_x\}$ and branching mechanism Ψ, or sometimes a (Ψ,ξ)-superprocess. In this paper, the superprocess we deal with is always this Hunt realization. For the existence of X, see also [4] and [6]. For any $f\in\mathcal{B}_b(E)$ and $(t,x)\in(0,\infty)\times E$,

$$\mathbb{P}_{\delta_x}\langle f,X_t\rangle = \Pi_x\left[e^{\int_0^t \alpha(\xi_s)\,ds} f(\xi_t)\right].$$

Since $|\alpha(x)|\le K$, we have

$$|\mathbb{P}_{\delta_x}\langle f,X_t\rangle| \le \|f\|_\infty e^{Kt}. \tag{2.5}$$

Define $v(t,x) := -\log \mathbb{P}_{\delta_x}(\|X_t\|=0)$, and $H := \inf\{t\ge 0 : \|X_t\|=0\}$. It is obvious that $v(0,x)=\infty$. By the Markov property of X, we have, for any $h>0$,

$$e^{-v(h,x)} = \mathbb{P}_{\delta_x}\left(e^{-\langle v_{h-s},X_s\rangle}\right), \quad s\in[0,h), \tag{2.6}$$

where, for any $t\ge 0$, v_t denotes the function $x\to v(t,x)$. In this paper, we will consider the critical and subcritical case. More precisely, throughout this paper, we assume that X satisfies the following uniform global extinction property.

(H1) For any $t>0$,

$$\sup_{x\in E} v(t,x) < \infty \quad \text{and} \quad \lim_{t\to\infty} v(t,x) = 0. \tag{2.7}$$

Remark 2.1. Note that Assumption **(H1)** is equivalent to

$$\inf_{x\in E}\mathbb{P}_{\delta_x}(\|X_t\|=0)>0 \quad \text{for all } t>0 \quad \text{and} \quad \mathbb{P}_{\delta_x}(H<\infty)=\lim_{t\to\infty}\mathbb{P}_{\delta_x}(\|X_t\|=0)=1. \tag{2.8}$$

We also assume that

(H2) For any $x\in E$ and $t>0$,

$$w(t,x):=-\frac{\partial v}{\partial t}(t,x) \tag{2.9}$$

exists. Moreover, for any $0<r<t$,

$$\sup_{r\le s\le t}\sup_{x\in E}w(s,x)<\infty. \tag{2.10}$$

Note that, since $t\to v(t,x)$ is decreasing, we have $w(t,x)\ge 0$. We also use w_t to denote the function $x\to w(t,x)$.

Example 1. Assume that the spatial motion ξ is conservative, that is $P_t(1)\equiv 1$, and the branching mechanism is spatially independent, that is, there exist $a\ge 0$, $b\ge 0$ and a measure n on $(0,\infty)$ with $\int_0^\infty(y\wedge y^2)n(dy)<\infty$ such that

$$\Psi(x,z)=\widetilde{\Psi}(z):=az+bz^2+\int_0^\infty\left(e^{-yz}-1+yz\right)n(dy). \tag{2.11}$$

We also assume that $\widetilde{\Psi}$ satisfies the Grey condition (see [11]):

$$\widetilde{\Psi}(\infty)=\infty \text{ and } \int^\infty\frac{1}{\widetilde{\Psi}(z)}\,dz<\infty.$$

Then $\{\|X_t\|,t\ge 0\}$ is a continuous state branching process with branching mechanism $\widetilde{\Psi}(z)$. So $v(t,x)=v(t)<\infty$ does not depend on x, and $\lim_{t\to\infty}v(t)=0$ (see [11, 25]), thus Assumption **(H1)** holds immediately. Moreover, for $t>0$, we have that

$$w(t):=-\frac{\mathrm{d}}{\mathrm{d}t}v(t)=\widetilde{\Psi}(v(t)).$$

Thus Assumption **(H2)** is satisfied. See [14, Theorem 10.1] for more details.

Remark 2.2. Let $\widetilde{\Psi}(z)$ be a spatially independent branching mechanism satisfying the conditions in Example 1. Let $\widetilde{X}$ be a continuous state branching process with branching mechanism $\widetilde{\Psi}(z)$, and let $\tilde{v}(t)$ be its extinction probability at time t.

If $\Psi(x,z)\ge\widetilde{\Psi}(z)$, then one could show that (see the proof of [23, Lemma 2.3])

$$\sup_{x\in E}v(t,x)\le\tilde{v}(t)\to 0,\quad t\to\infty.$$

Thus Assumption **(H1)** holds.

In Section 3 we will give our Williams decomposition under conditions **(H1)** and **(H2)**, see Theorem 3.5 below. Note that Delmas and Hénard [5] gave a Williams decomposition under their conditions $(H1)$, $(H2)$ and $(H3)$. Our condition **(H1)** is similar to $(H1)$ in [5]. Conditions $(H2)$ and $(H3)$ in [5] are not easy to check. The only examples given in [5] are superdiffusions and multi-type Feller processes. It is easy to check that our Assumptions **(H1)** and **(H2)** hold for multi-type Feller processes. In Section 5, we will give more examples, including some class of superdiffusions, that satisfy Assumptions **(H1)** and **(H2)**. Our examples cover all examples considered in [5].

2.2 Excursion law of $\{X_t, t \geq 0\}$

We use W_0^+ to denote the collection of right continuous functions from $(0,\infty)$ to $\mathcal{M}_F(E)$ having zero measure as a trap. We use $(\mathcal{A}, \mathcal{A}_t)$ to denote the natural σ-fields on W_0^+ generated by the coordinate process.

Let $\{Q_t(\mu,\cdot) := \mathbb{P}_\mu(X_t \in \cdot) : t \geq 0,\ \mu \in \mathcal{M}_F(E)\}$ be the transition semigroup of X. Then by (2.3), we have

$$\int_{\mathcal{M}_F(E)} e^{-\langle f,\nu\rangle} Q_t(\mu, d\nu) = \exp\{-\langle V_t f, \mu\rangle\} \quad \text{for } \mu \in \mathcal{M}_F(E) \text{ and } t \geq 0,$$

where $V_t f(x) := u_f(t,x), x \in E$. This implies that $Q_t(\mu_1+\mu_2,\cdot) = Q_t(\mu_1,\cdot) * Q_t(\mu_2,\cdot)$ for any $\mu_1, \mu_2 \in \mathcal{M}_F(E)$, and hence $Q_t(\mu,\cdot)$ is an infinitely divisible probability measure on $\mathcal{M}_F(E)$. By the semigroup property of Q_t, V_t satisfies that

$$V_s V_t = V_{t+s} \quad \text{for all } s,t \geq 0.$$

Moreover, by the infinite divisibility of Q_t, each operator V_t has the representation

$$V_t f(x) = \lambda_t(x,f) + \int_{\mathcal{M}_F(E)^0} \left(1 - e^{-\langle f,\nu\rangle}\right) L_t(x,d\nu) \quad \text{for } t > 0,\ f \in \mathcal{B}_b^+(E), \tag{2.12}$$

where $\lambda_t(x,dy)$ is a bounded kernel on E and $(1 \wedge \nu(1))L_t(x,d\nu)$ is a bounded kernel from E to $\mathcal{M}_F(E)^0$. Let Q_t^0 be the restriction of Q_t to $\mathcal{M}_F(E)^0$. Let $E_0 := \{x \in E : \lambda_t(x,E) = 0 \text{ for all } t > 0\}$.

For $\lambda > 0$, we use $V_t\lambda$ to denote $V_t f$ when the function $f \equiv \lambda$. It then follows from (2.12) that for every $x \in E$ and $t > 0$,

$$V_t\lambda(x) = \lambda_t(x,E)\lambda + \int_{\mathcal{M}_F(E)^0} \left(1 - e^{-\lambda\langle 1,\nu\rangle}\right) L_t(x,d\nu).$$

The left hand side tends to $-\log \mathbb{P}_{\delta_x}(X_t = 0)$ as $\lambda \to +\infty$. Therefore, Assumption **(H1)** implies that $\lambda_t(x,E) = 0$ for all $t > 0$ and hence $x \in E_0$, which says that $E = E_0$.

For $x \in E$, we get from (2.12) that

$$V_t f(x) = \int_{\mathcal{M}_F(E)^0} \left(1 - e^{-\langle f,\nu\rangle}\right) L_t(x,d\nu) \quad \text{for } t > 0,\ f \in \mathcal{B}_b^+(E).$$

It then follows from [19, Proposition 2.8 and Theorem A.40] that for every $x \in E$, the family of measures $\{L_t(x,\cdot) : t > 0\}$ on $\mathcal{M}_F(E)^0$ constitutes an entrance law for the restricted semigroup $\{Q_t^0 : t \geq 0\}$. Then one can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of σ-finite measures $\{\mathbb{N}_x : x \in E\}$ defined on $(W_0^+, \mathcal{A})$ such that $\mathbb{N}_x(\{\mathbf{0}\}) = 0$ (where $\{\mathbf{0}\} = \{\omega \in W_0^+ : \omega_t = 0, \forall t > 0\}$), and, for every $0 < t_1 < \cdots < t_n < \infty$, and nonzero $\mu_1, \cdots, \mu_n \in M_F(E)$,

$$\begin{aligned} &\mathbb{N}_x(\omega_{t_1} \in d\mu_1, \cdots, \omega_{t_n} \in d\mu_n) \\ =&L_t(x,d\mu_1)\mathbb{P}_{\mu_1}(X_{t_2-t_1} \in d\mu_2)\cdots\mathbb{P}_{\mu_{n-1}}(X_{t_n-t_{n-1}} \in d\mu_n). \end{aligned} \tag{2.13}$$

Thus, we have that for $f \in \mathcal{B}_b^+(E)$ and $t > 0$,

$$\int_{W_0^+} (1 - e^{-\langle f,\omega_t\rangle})\mathbb{N}_x(d\omega) = \int_{\mathcal{M}_F(E)^0} (1 - e^{-\langle f,\nu\rangle}) L_t(x,d\nu) = -\log \mathbb{P}_{\delta_x}(e^{-\langle f,X_t\rangle}). \tag{2.14}$$

According to Theorem [19, Theorem 8.22], for $\mathbb{N}_x$-a.e. $w \in W_0^+$ we have $w_t \to 0$ and $\|w_t\|^{-1}w_t \to \delta_x$ in $M_F(E)$ as $t \to 0$. This measure $\mathbb{N}_x$ is called the *Kuznetsov* *measure* corresponding to the entrance law $\{L_t(x,\cdot) : t > 0\}$ or the *excursion law* for the superprocess X. For earlier work on excursion law of superprocesses, see [7, 12, 18].

It follows from (2.14) that for any $t > 0$,

$$\mathbb{N}_x(\|\omega_t\| \neq 0) = -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0) \in (0, \infty). \tag{2.15}$$

Therefore, for any $r > 0$, $\mathbb{N}_x(\|w_r\| > 0) < \infty$. We define a new probability measure $\mathbb{N}_x^{(r)} := \mathbb{N}_x(\cdot, \|\omega_r\| > 0)/\mathbb{N}_x(\|\omega_r\| > 0)$. By (2.13), $((\omega_t)_{t\geq r}, \mathbb{N}_x^{(r)})$ is a Markov process with transition semigroup $Q_t(\mu, d\nu)$. Thus $\{w(t), t \geq r; \mathbb{N}_x^{(r)}\}$ has a Hunt realization. Thus, by [19, Proposition A.7], $\{w(t), t \geq r; \mathbb{N}_x^{(r)}\}$ has a modification $\{\widetilde{\omega}_t, t \geq r\}$ which is a càdlàg process on $[r, \infty)$. Since ω_t is right continuous, thus, $\omega_t = \widetilde{\omega}_t, t \geq r$, a.s., which yields that, $\mathbb{N}_x$-a.e., on $(w(r) > 0)$, $w(\cdot)$ has left limits on (r, ∞). Since $r > 0$ is arbitrary, $\mathbb{N}_x$-a.e. $w(\cdot)$ has left limits on $(0, \infty)$. Therefore, for any $x \in E$, the Kuznetsov measure $\mathbb{N}_x$ is actually carried by càdlàg paths $w \in W_0^+$. Recall that $\mathbb{D}$ is the collection of the càdlàg functions from $[0, \infty)$ to $\mathcal{M}_F(E)$ having the zero measure as a trap. Thus we may regard $\mathbb{N}_x$ as a measure on $(\mathbb{D}, \mathcal{G}^0)$, where $\mathcal{G}^0$ is the natural σ-field on $\mathbb{D}$ generated by the coordinate process.

3 Main results

In this and the next section we will always assume that Assumptions **(H1)**-**(H2)** hold. Recall that $H := \inf\{t \geq 0 : \|X_t\| = 0\}$. Note that

$$F_H(t) := \mathbb{P}_\mu(H \leq t) = \mathbb{P}_\mu(\|X_t\| = 0) = e^{-\langle v_t, \mu\rangle}. \tag{3.1}$$

By the continuity of $v(t, x)$ with respect to $t \in (0, \infty)$, we get that for any $t > 0$,

$$\mathbb{P}_\mu(H < t) = \lim_{\epsilon\downarrow 0} \mathbb{P}_\mu(H \leq t - \epsilon) = \lim_{\epsilon\downarrow 0} e^{-\langle v_{t-\epsilon}, \mu\rangle} = e^{-\langle v_t, \mu\rangle} = \mathbb{P}_\mu(H \leq t). \tag{3.2}$$

Taking derivatives with respect to h on both sides of (2.6) gives

$$w(h, x)e^{-v(h,x)} = \mathbb{P}_{\delta_x}\left(\langle w_{h-s}, X_s\rangle e^{-\langle v_{h-s}, X_s\rangle}\right), \quad s \in [0, h).$$

Note that the left-hand side does not depend on s. This suggests that $\{\langle w_{h-s}, X_s\rangle e^{-\langle v_{h-s}, X_s\rangle}, s \in [0, h]\}$ is a martingale. In fact, for $h > 0$, define

$$M_t^h := \frac{\langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}}{\langle w_h, X_0\rangle e^{-\langle v_h, X_0\rangle}}, \qquad 0 \leq t < h. \tag{3.3}$$

Then, under $\mathbb{P}_\mu$, $\{M_t^h, 0 \leq t < h\}$ is a nonnegative martingale with mean one (see Lemma 4.2 below). Since the density of the distribution function F_H is given by $\langle w_t, \mu\rangle e^{-\langle v_t, \mu\rangle}$, this martingale change of measure would give the desired effect of conditioning on $H = h$. The following theorem says that this is indeed the case.

Theorem 3.1. *Suppose that Assumptions* **(H1)**-**(H2)** *hold. For any $h > 0$ and $t < h$,*

$$\lim_{\epsilon\downarrow 0} \mathbb{P}_\mu(A | h \leq H < h + \epsilon) = \mathbb{P}_\mu(\mathbf{1}_A M_t^h), \qquad \forall A \in \mathcal{G}_t.$$

We define, for each $h > 0$,

$$\mathbb{P}_\mu(\cdot | H = h) := \lim_{\epsilon\downarrow 0} \mathbb{P}_\mu(\cdot | h \leq H < h + \epsilon).$$

Then, by Theorem 3.1, $\{X_t, t < h; \mathbb{P}_\mu(\cdot | H = h)\}$ has the same law as $\{X_t, t < h; \mathbb{P}_\mu^h\}$, where $\mathbb{P}_\mu^h$ is a new measure defined via the martingale M_t^h:

$$\left.\frac{d\mathbb{P}_\mu^h}{d\mathbb{P}_\mu}\right|_{\mathcal{G}_t} = M_t^h, \qquad t < h.$$

Corollary 3.2. *Suppose that Assumptions* **(H1)**-**(H2)** *hold. For any* $A \in \mathcal{G}_t$*, we have*

$$\mathbb{P}_\mu(A \cap \{H > t\}) = \int_t^\infty \mathbb{P}_\mu^h(A) F_H(dh).$$

Proof. It follows from Fubini's theorem that

$$\begin{aligned}
\int_t^\infty \mathbb{P}_\mu^h(A) F_H(dh) &= \int_t^\infty \mathbb{P}_\mu(\mathbf{1}_A M_t^h) F_H(dh) \\
&= \int_t^\infty \mathbb{P}_\mu(\mathbf{1}_A \langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle})\, dh \\
&= \mathbb{P}_\mu\left(\mathbf{1}_A \int_t^\infty \langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}\, dh\right) \\
&= \mathbb{P}_\mu\left(\mathbf{1}_A \int_0^\infty \langle w_h, X_t\rangle e^{-\langle v_h, X_t\rangle}\, dh\right) \\
&= \mathbb{P}_\mu\left(A \cap \{X_t \neq 0\}\right) = \mathbb{P}_\mu\left(A \cap \{H > t\}\right),
\end{aligned}$$

where in the fifth equality we use the fact that

$$\int_0^\infty \langle w_h, X_t\rangle e^{-\langle v_h, X_t\rangle}\, dh = \lim_{h\to\infty} e^{-\langle v_h, X_t\rangle} - \lim_{h\to 0} e^{-\langle v_h, X_t\rangle} = \mathbf{1}_{\{X_t \neq 0\}}. \qquad \square$$

It can be proved that

$$w(t+s, x) = \Pi_x\left(\exp\left\{-\int_0^t \Psi'_z(\xi_u, v(t+s-u, \xi_u))\, du\right\} w(s, \xi_t)\right),$$

where $\Psi'_z(x, z) = \frac{\partial \Psi(x,z)}{\partial z}$, see (4.9) below. Thus, for any $h > 0$ and $t \in [0, h)$, using the equality above with t and s replaced by $h - t$ and t respectively, we get

$$w(h, x) = \Pi_x\left(\exp\left\{-\int_0^t \Psi'_z(\xi_u, v(h-u, \xi_u))\, du\right\} w(h-t, \xi_t)\right).$$

Note that the left-hand side of the equality above does not depend on t (in fact, $w(t-s, x)$ is harmonic with respect to the operator $\frac{\partial}{\partial s} + \mathcal{L} - \Psi'_z(x, v(h-s, x))$). This suggests that we can construct a martingale. For any $h > 0$ and $t \in [0, h)$, we define

$$Y_t^h := \frac{w(h-t, \xi_t)}{w(h, \xi_0)} e^{-\int_0^t \Psi'_z(\xi_u, v(h-u, \xi_u))\, du}.$$

Then we have the following result whose proof will be given in Section 4.

Lemma 3.3. *Suppose that Assumptions* **(H1)**-**(H2)** *hold. Under* Π_x*,* $\{Y_t^h, t < h\}$ *is a nonnegative martingale satisfying* $\Pi_x(Y_t^h) = 1$*.*

Remark 3.4. In Example 1, $w(t, x)$ and $v(t, x)$ do not depend on x, and for any $h > 0$ and $0 \le t < h$, $Y_t^h \equiv 1$. For the particular branching mechanism $\Psi(x, z) = \beta(x)z + \alpha(x)z^2$, it was proved in [5] that a martingale change of measure via the martingale $\{Y_t^h, t < h\}$ will lead to the motion of the last survivor. Our Williams decomposition (see Theorem 3.5 below) says that this is also true for general branching mechanism.

Now we state our main result: the Williams decomposition. We will construct a new process $\{\Lambda_t^h, t < h\}$ which has the same law as $\{X_t, t < h; \mathbb{P}_\mu(\cdot | H = h)\}$.

Let $\mathcal{F}_{h-} := \bigvee_{t<h} \mathcal{F}_t$. Now we define a new probability measure Π_x^h on $(\mathbb{D}_E, \mathcal{F}_{h-})$ by

$$\left.\frac{d\Pi_x^h}{d\Pi_x}\right|_{\mathcal{F}_t} := Y_t^h, \qquad t \in [0, h).$$

Under Π_x^h, $(\xi_t)_{0\le t<h}$ is a conservative Markov process. If ν is a probability measure on E, we define

$$\Pi_\nu^h := \int_E \Pi_x^h \, \nu(dx).$$

Then, under Π_ν^h, $(\xi_t)_{0\le t<h}$ is a Markov process with initial measure ν.

We put

$$H(\omega) := \inf\{t>0 : \|\omega_t\| = 0\}, \quad \omega\in\mathbb{D}.$$

Let $\xi^h := \{(\xi_t)_{0\le t<h}, \Pi_\nu^h\}$, where $\nu(dx) = \frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\,\mu(dx)$. Given the trajectory of ξ^h, we define three processes as follows:

Continuous immigration Suppose that $\mathcal{N}^{1,h}(ds,d\omega)$ is a Poisson random measure on $[0,h)\times\mathbb{D}$ with intensity measure $2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds$. Define, for $t\in[0,h)$,

$$X_t^{1,h,\mathbb{N}} := \int_{[0,t]}\int_{\mathbb{D}} \omega_{t-s}\mathcal{N}^{1,h}(ds,d\omega). \tag{3.4}$$

Jump immigration Suppose that $\mathcal{N}^{2,h}(ds,d\omega)$ is a Poisson random measure on $[0,h)\times\mathbb{D}$ with intensity measure $\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}\int_0^\infty yn(\xi_s,dy)\mathbb{P}_{y\delta_{\xi_s}}(X\in d\omega)\,ds$. Define, for $t\in[0,h)$,

$$X_t^{2,h,\mathbb{P}} := \int_{[0,t]}\int_{\mathbb{D}} \omega_{t-s}\mathcal{N}^{2,h}(ds,d\omega). \tag{3.5}$$

Immigration at time 0 Let $\{X_t^{0,h}, 0\le t<h\}$ be a process distributed according to the law $\mathbb{P}_\mu(X\in\cdot|H<h)$.

We assume that the three processes $X^{0,h}$, $X^{1,h,\mathbb{N}}$ and $X^{2,h,\mathbb{P}}$ are independent given the trajectory of ξ^h. Define

$$\Lambda_t^h := X_t^{0,h} + X_t^{1,h,\mathbb{N}} + X_t^{2,h,\mathbb{P}}. \tag{3.6}$$

We write the law of Λ^h as $\mathbf{P}_\mu^{(h)}$.

Theorem 3.5. *Suppose that Assumptions* **(H1)**-**(H2)** *hold. The process* $\{\Lambda_t^h, t<h\}$ *under* $\mathbf{P}_\mu^{(h)}$ *has the same finite dimensional distributions as* $\{X_t, t<H\}$ *under* $\mathbb{P}_\mu$ *conditioned on* $H=h$.

If we define $\Lambda_t^h = 0$, for any $t\ge h$, then we have the following result.

Corollary 3.6. *Assume that Assumptions* **(H1)**-**(H2)** *hold.* $\{X_t;\mathbb{P}_\mu\}$ *has the same finite dimensional distributions as*

$$\int_0^\infty \mathbf{P}_\mu^{(h)}(\Lambda^h\in\cdot)F_H(dh).$$

Proof. Let $f_k\in\mathcal{B}_b^+(E)$, $k=1,2,\cdots,n$ and $0=t_0<t_1<t_2<\cdots<t_n$. We put $t_{n+1}=\infty$ and define $(t_n,t_{n+1}] := (t_n,\infty)$. We will show that

$$\mathbb{P}_\mu\left(\exp\Big\{-\sum_{j=1}^n\langle f_j, X_{t_j}\rangle\Big\}\right) = \int_{(0,\infty)}\mathbf{P}_\mu^{(h)}\left(\exp\Big\{-\sum_{j=1}^n\langle f_j,\Lambda_{t_j}^h\rangle\Big\}\right)F_H(dh).$$

Since $\Lambda_t^h = 0$, for $t \geq h$, we get that

$$\begin{aligned}
&\int_{(0,\infty)} \mathbf{P}_\mu^{(h)} \left(\exp\Big\{ -\sum_{j=1}^n \langle f_j, \Lambda_{t_j}^h \rangle \Big\} \right) F_H(dh) \\
=&\sum_{r=0}^n \int_{(t_r, t_{r+1}]} \mathbf{P}_\mu^{(h)} \left(\exp\Big\{ -\sum_{j=1}^r \langle f_j, \Lambda_{t_j}^h \rangle \Big\} \right) F_H(dh) \\
=&\sum_{r=0}^n \int_{(t_r, t_{r+1}]} \mathbb{P}_\mu^h \left(\exp\Big\{ -\sum_{j=1}^r \langle f_j, X_{t_j} \rangle \Big\} \right) F_H(dh) \\
=&\sum_{r=0}^n \mathbb{P}_\mu \left(\exp\Big\{ -\sum_{j=1}^r \langle f_j, X_{t_j} \rangle \Big\}; t_r < H \leq t_{r+1} \right) \\
=&\mathbb{P}_\mu \left(\exp\Big\{ -\sum_{j=1}^n \langle f_j, X_{t_j} \rangle \Big\} \right),
\end{aligned}$$

where the second equality follows from Theorem 3.5, and the third equality follows from Corollary 3.2. The proof is now complete. □

The decomposition (3.6) is called a Williams decomposition or spinal decomposition of the supperprocess $\{X_t, t < h\}$ conditioned on $H = h$, and $\xi^h = \{(\xi_t)_{0 \leq t < h}, \Pi_\nu^h\}$ is called the spine of the decomposition. It gives us a tool to study the behavior of the superprocesses X near extinction, see Theorem 3.7 below. To state Theorem 3.7, we need the following assumption:

(H3) For any bounded open set $B \subset E$ and any $t > 0$, the function

$$x \to -\log \mathbb{P}_{\delta_x} \Big(\int_0^t X_s(B^c)\, ds = 0 \Big)$$

is finite for $x \in B$ and locally bounded.

Theorem 3.7. *Suppose that* **(H1)**-**(H3)** *hold and that for any* $\mu \in \mathcal{M}_F(E)$*,*

$$\text{the limit } \lim_{t \uparrow h} \xi_t \text{ exists } \Pi_\nu^h\text{-a.s.,} \tag{3.7}$$

where $\nu(dx) = \frac{w(h,x)}{\langle w(h,\cdot), \mu \rangle} \mu(dx)$*. Define* $\xi_{h-} := \lim_{t \uparrow h} \xi_t$*. Then there exists an E-valued random variable* Z *such that*

$$\lim_{t \uparrow H} \frac{X_t}{\|X_t\|} = \delta_Z, \quad \mathbb{P}_\mu\text{-a.s.,}$$

where the limit above is in the sense of weak convergence. Moreover, conditioned on $\{H = h\}$*,* Z *has the same distribution as* $\{\xi_{h-}, \Pi_\nu^h\}$*, that is, for any* $f \in \mathcal{C}_b^+(E)$*,*

$$\mathbb{P}_\mu f(Z) = \int_0^\infty \Pi_\nu^h(f(\xi_{h-})) F_H(dh). \tag{3.8}$$

Note that, if the martingale $\{Y_t^h, 0 \leq t < h\}$ is uniformly integrable, then condition (3.7) holds. In fact, under this uniform integrability condition, the almost sure limit $\lim_{t \uparrow h} Y_t^h =: Y_h^h$ exists, we also have $\Pi_x Y_h^h = 1$ and

$$\frac{d\Pi_x^h}{d\Pi_x}\Big|_{\mathcal{F}_{h-}} = Y_h^h.$$

Since $\{\lim_{t\uparrow h}\xi_t \text{ exists }\} \in \mathcal{F}_{h-}$ and ξ is a Hunt process under Π_x, we have that

$$\Pi^h_\nu(\lim_{t\uparrow h}\xi_t \text{ exists }) = \int_E \Pi_x(Y^h_h, \lim_{t\uparrow h}\xi_t \text{ exists })\,\nu(dx) = \int_E \Pi_x(Y^h_h)\,\nu(dx) = 1.$$

Assumption **(H3)** is a technical condition which is kind of strong. It would be interesting to weaken this condition. If $E=\mathbb{R}^d$, then **(H3)** is equivalent to the condition that, for any bounded open set $B\subset\mathbb{R}^d$ and any $t>0$,

$$\mathbb{P}_{\delta_x}\Big(\overline{\bigcup_{s\in[0,t]}\text{supp}(X_s)}\subset\overline{B}\Big)>0,$$

where supp(μ) denotes the support of the measure $\mu\in\mathcal{M}_f(\mathbb{R}^d)$. If the spatial motion is an α-stable-process, $\alpha\in(0,2]$, with a spatially independent branching mechanism cz^2, where c is a positive constant, then it is known that, if $\alpha\in(0,2)$, then for any $t>0$, supp$(X_t)=\emptyset$ or $\mathbb{R}^d$ almost surely, see [22, Example III.2.3]. Therefore, super-α-stable processes with $\alpha\in(0,2)$ do not satisfy Assumption **(H3)**. But Tribe [27] proved that Theorem 3.7 is true for super-α-stable processes with branching mechanism z^2. This also shows that Assumption **(H3)** is not really necessary. Super-Brownian motion in $\mathbb{R}^d$ (corresponding to $\alpha=2$) does satisfy condition **(H3)**. See Example 2 below for more general cases where Assumption **(H3)** holds.

Example 2. Assume that ξ is a diffusion on $\mathbb{R}^d$ with infinitesimal generator

$$L=\sum a_{ij}(x)\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}+\sum b_j(x)\frac{\partial}{\partial x_j},$$

which satisfies the following two conditions:

(A) (Uniform ellipticity) There exists a constant $\gamma>0$ such that

$$\sum a_{i,j}(x)u_iu_j\geq\gamma\sum u_j^2,\qquad x\in\mathbb{R}^d.$$

(B) a_{ij} and b_j are bounded Hölder continuous functions.

Suppose that the branching mechanism $\Psi(x,z)$ satisfies that, for some $\alpha\in(1,2]$ and $c>0$, $\Psi(x,z)\geq cz^\alpha$ for all $x\in\mathbb{R}^d$.

Let $\{X,\mathbb{P}_\mu\}$ and $\{\widetilde{X},\widetilde{\mathbb{P}}_\mu\}$ be a (ξ,Ψ)-superprocess and a (ξ,cz^α)-superprocess respectively. Then, for any open set $B\subset\mathbb{R}^d$,

$$-\log\mathbb{P}_{\delta_x}\left(\exp\left\{-\lambda\int_0^t X_s(B^c)\,ds\right\}\right)=u(t,x),$$

where $u(t,x)$ is the unique bounded positive solution on $[0,t]\times\mathbb{R}^d$ of

$$u(t,x)+\Pi_x\int_0^t\Psi(\xi_s,u(t-s,\xi_s))\,ds=\lambda\Pi_x\int_0^t I_{B^c}(\xi_s)\,ds.$$

Similarly

$$-\log\widetilde{\mathbb{P}}_{\delta_x}\left(\exp\left\{-\lambda\int_0^t \widetilde{X}_s(B^c)\,ds\right\}\right)=\tilde{u}(t,x),$$

where $\tilde{u}(t,x)$ is the unique bounded positive solution on $[0,t]\times\mathbb{R}^d$ of

$$\tilde{u}(t,x)+\Pi_x\int_0^t\widetilde{\Psi}(\xi_s,\tilde{u}(t-s,\xi_s))\,ds=\lambda\Pi_x\int_0^t I_{B^c}(\xi_s)\,ds.$$

Observe that $\tilde{u}$ is also the unique bounded solution of

$$\tilde{u}(t,x) + \Pi_x \int_0^t \Psi(\xi_s, \tilde{u}(t-s,\xi_s))\, ds = \lambda \Pi_x \int_0^t I_{B^c}(\xi_s) ds + \Pi_x \int_0^t g(t-s,\xi_s)\, ds,$$

where

$$g(s,x) := \Psi(x, \tilde{u}(s,x)) - \widetilde{\Psi}(x, \tilde{u}(s,x)), \quad s \in [0,t],\, x \in \mathbb{R}^d,$$

is a bounded positive Borel function on $[0,t] \times \mathbb{R}^d$. By [19, Theorem 5.16],

$$\tilde{u}(t,x) = -\log \mathbb{P}_{\delta_x} \left(\exp\left\{ -\int_0^t \left(\lambda X_s(B^c) + \langle g_{t-s}, X_s \rangle\right) \, ds \right\} \right),$$

where $g_s(x) = g(s,x), x \in E$. Therefore, $u(t,x) \le \tilde{u}(t,x)$, which is equivalent to

$$-\log \mathbb{P}_{\delta_x} \left(\exp\left\{ -\lambda \int_0^t X_s(B^c)\, ds \right\} \right) \le -\log \widetilde{\mathbb{P}}_{\delta_x} \left(\exp\left\{ -\lambda \int_0^t \widetilde{X}_s(B^c)\, ds \right\} \right).$$

Let $\mathcal{R}$ be the range of $\widetilde{X}$, the minimal closed subset of $\mathbb{R}^d$ which supports all the measures $\widetilde{X}_t$, $t \ge 0$. Then

$$-\log \widetilde{\mathbb{P}}_{\delta_x} \left(\exp\left\{ -\lambda \int_0^t \widetilde{X}_s(B^c)\, ds \right\} \right) \le -\log \widetilde{\mathbb{P}}_{\delta_x}(\mathcal{R} \subset B),$$

hence we have

$$-\log \mathbb{P}_{\delta_x} \left(\exp\left\{ -\lambda \int_0^t X_s(B^c)\, ds \right\} \right) \le -\log \widetilde{\mathbb{P}}_{\delta_x}(\mathcal{R} \subset B).$$

Thus, by the monotone convergence theorem, we have that

$$\begin{aligned} -\log \mathbb{P}_{\delta_x} \Big(\int_0^t X_s(B^c)\, ds = 0 \Big) &= \lim_{\lambda \to \infty} -\log \mathbb{P}_{\delta_x} \left(\exp\left\{ -\lambda \int_0^t X_s(B^c)\, ds \right\} \right) \\ &\le -\log \widetilde{\mathbb{P}}_{\delta_x}(\mathcal{R} \subset B). \end{aligned}$$

By [6, Theorem 8.1], $x \to -\log \widetilde{\mathbb{P}}_{\delta_x}(\mathcal{R} \subset B)$ is finite and continuous in $x \in B$. Therefore the superprocess X satisfies Assumption **(H3)**.

Remark 3.8. Now we consider the superprocess in Example 1. We assume that ξ is a diffusion in $\mathbb{R}^d$ satisfying the conditions in Example 2, and the branching mechanism $\Psi(z)$ satisfies that, for some $\alpha \in (1,2]$ and $c > 0$, $\Psi(z) \ge cz^\alpha$. Thus Assumption **(H3)** holds. Since $Y_t^h = 1$ and $\Pi_x^h = \Pi_x$, condition (3.7) holds automatically. Therefore, Theorem 3.7 holds and Z has the same law as ξ_H, where $\xi_0 \sim \nu(dx) = \mu(dx)/\|\mu\|$. Moreover, ξ and H are independent.

Compared with [27], the example above assumes that the spatial motion ξ is a diffusion, while in [27], the spatial motion is a Feller process. However, in [27], the branching mechanism is binary ($\Psi(z) = z^2$), while in the example above, the branching mechanisms is more general. Thus Theorem 3.7 is a partial generalization of the results in [27] and it does not cover the results in [27]. Our result is more general in the sense that we consider more general branching mechanism, while the spatial motion in [27] is more general than ours.

Recently, there are lots of work on spine decomposition (see [8, 15, 20, 24] for instance) and backbone decomposition (also called skeleton decomposition) (see [8, 16] for instance) for superprocesses with spatially dependent branching mechanism. Intuitively, on the survival set, the superprocess is decomposed into a 'thinner' process which almost surely survives and which is decorated with immigrations. For the spine decomposition, the 'thinner' process is a Markov process of one particle (the spatial motion of the spine), and for the backbone decomposition the 'thinner' process is a branching Markov process. The Williams decomposition gives a spinal decomposition of X conditioned on $H = h$ with $h > 0$ being a constant, where the 'thinner' process is the spatial motion process of the last individual alive. These decompositions are important tools for studying limit behaviors of superprocesses. The above Theorem 3.7 is one application of the Williams decomposition. It would be interesting to explore other applications.

4 Proofs of main results

We will use $\mathbb{P}_{r,\delta_x}$ to denote the law of X starting from the unit mass δ_x at time $r > 0$. Similarly, we will use $\Pi_{r,x}$ to denote the law of ξ starting from x at time $r > 0$. First, we give a useful lemma.

Lemma 4.1. *Suppose that $f \in \mathcal{B}_b^+(E)$ and $g_i \in \mathcal{B}_b^+(E)$, $i = 1, 2, \cdots, n$. For any $0 < t_1 \le t_2 \le \cdots \le t_n$ and $0 \le r \le t_n$, we have*

$$\mathbb{P}_{r,\mu}\left(\langle f, X_{t_n}\rangle \exp\Big\{-\sum_{j:t_j\ge r}\langle g_j, X_{t_j}\rangle\Big\}\right)$$
$$= \int_E \Pi_{r,x}\left(\exp\left\{-\int_r^{t_n}\Psi_z'(\xi_u, U_g(u,\xi_u))\,du\right\} f(\xi_{t_n})\right)\mu(dx)e^{-\langle U_g(r,\cdot),\mu\rangle}, \tag{4.1}$$

where

$$U_g(r,x) := -\log \mathbb{P}_{r,\delta_x}\Big(\exp\Big\{-\sum_{j:t_j\ge r}\langle g_j, X_{t_j}\rangle\Big\}\Big).$$

In particular, for any $f \in \mathcal{B}_b^+(E)$ and $g \in \mathcal{B}_b^+(E)$, we have

$$\mathbb{P}_{\delta_x}\left(\langle f, X_t\rangle e^{-\langle g, X_t\rangle}\right) = \Pi_x\left(\exp\left\{-\int_0^t \Psi_z'(\xi_u, u_g(t-u,\xi_u))\,du\right\} f(\xi_t)\right)e^{-u_g(t,x)}. \tag{4.2}$$

Proof. By [19, Proposition 5.14], we have that, for $0 \le r \le t_n$,

$$-\log \mathbb{P}_{r,\mu}\left(\exp\Big\{-\sum_{j:t_j\ge r}\langle g_j, X_{t_j}\rangle - \theta\langle f, X_{t_n}\rangle\Big\}\right) = \langle F_\theta(r,\cdot),\mu\rangle,$$

where $F_\theta(r,x)$ is the unique bounded positive solution on $[0,t_n]\times E$ of

$$F_\theta(r,x) + \Pi_{r,x}\int_r^{t_n}\Psi(\xi_u, F_\theta(u,\xi_u))\,du = \sum_{j:t_j\ge r}\Pi_{r,x} g_j(\xi_{t_j}) + \theta\Pi_{r,x} f(\xi_{t_n}). \tag{4.3}$$

Let $F_\theta'(r,x) := \frac{\partial}{\partial\theta}F_\theta(r,x)$. Then,

$$\mathbb{P}_{r,\mu}\left(\langle f, X_{t_n}\rangle \exp\Big\{-\sum_{j:t_j\ge r}\langle g_j, X_{t_j}\rangle\Big\}\right) = -\frac{\partial}{\partial\theta}e^{-\langle F_\theta(r,\cdot),\mu\rangle}\big|_{\theta=0+}$$
$$= \langle F_0'(r,\cdot),\mu\rangle e^{-\langle U_g(r,\cdot),\mu\rangle}.$$

Differentiating both sides of (4.3) with respect to θ and then letting $\theta \to 0$, we get that

$$F_0'(r,x) + \Pi_{r,x}\int_r^{t_n}\Psi_z'(\xi_u, U_g(u,\xi_u))F_0'(u,\xi_u)\,du = \Pi_{r,x} f(\xi_{t_n}).$$

Then, by [6, Lemma 1.5] with $\tau = t_n$, we get that

$$F_0'(r,x) = \Pi_{r,x}\Big[e^{-\int_r^{t_n} \Psi_z'(\xi_u, U_g(u,\xi_u))\,du} f(\xi_{t_n})\Big].$$

Therefore (4.1) holds. □

Recall that $v(t,x) := -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0)$ and $w(t,x) := -\frac{\partial v}{\partial t}(t,x) \geq 0$. Recall the definition of M_t^h in (3.3).

Lemma 4.2. *Suppose that Assumptions* **(H1)-(H2)** *hold. Under* $\mathbb{P}_\mu$, $\{M_t^h, t < h\}$ *is a nonnegative martingale with* $\mathbb{P}_\mu(M_t^h) = 1$.

Proof. For any $h > 0$ and $0 \leq t < h$, by Assumption **(H2)** and the dominated convergence theorem, we get that

$$\begin{aligned}\mathbb{P}_\mu\Big[\langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}\Big] &= \frac{\partial}{\partial h}\mathbb{P}_\mu e^{-\langle v_{h-t}, X_t\rangle} \\ &= \frac{\partial}{\partial h} e^{-\langle v_h, \mu\rangle} = \langle w_h, \mu\rangle e^{-\langle v_h, \mu\rangle},\end{aligned} \tag{4.4}$$

where in the second equality, we used the Markov property of X. Thus, it follows that $\mathbb{P}_\mu(M_t^h) = 1$.

By the Markov property of X, we obtain that, for $s < t < h$,

$$\begin{aligned}\mathbb{P}_\mu\Big[\langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}\Big|\mathcal{G}_s\Big] &= \mathbb{P}_{X_s}\Big[\langle w_{h-t}, X_{t-s}\rangle e^{-\langle v_{h-t}, X_{t-s}\rangle}\Big] \\ &= \langle w_{h-s}, X_s\rangle e^{-\langle v_{h-s}, X_s\rangle},\end{aligned}$$

which implies that, under $\mathbb{P}_\mu$, $\{M_t^h, t < h\}$ is a nonnegative martingale. The proof is complete. □

Proof of Theorem 3.1: For any $A \in \mathcal{G}_t$, by the Markov property of X,

$$\begin{aligned}\mathbb{P}_\mu(A|h \leq H < h+\epsilon) &= \frac{\mathbb{P}_\mu(A \cap \{h \leq H < h+\epsilon\})}{\mathbb{P}_\mu(h \leq H < h+\epsilon)} \\ &= \frac{\mathbb{P}_\mu(\mathbf{1}_A \mathbb{P}_{X_t}(h-t \leq H < h-t+\epsilon))}{e^{-\langle v_{h+\epsilon}, \mu\rangle} - e^{-\langle v_h, \mu\rangle}} \\ &= \frac{\mathbb{P}_\mu\Big(\mathbf{1}_A\big(e^{-\langle v_{h-t+\epsilon}, X_t\rangle} - e^{-\langle v_{h-t}, X_t\rangle}\big)\Big)}{e^{-\langle v_{h+\epsilon}, \mu\rangle} - e^{-\langle v_h, \mu\rangle}}.\end{aligned}$$

By Assumption **(H2)**, we get that

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\Big(e^{-\langle v_{h+\epsilon}, \mu\rangle} - e^{-\langle v_h, \mu\rangle}\Big) = \langle w_h, \mu\rangle e^{-\langle v_h, \mu\rangle} \tag{4.5}$$

and

$$\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}\Big(e^{-\langle v_{h-t+\epsilon}, X_t\rangle} - e^{-\langle v_{h-t}, X_t\rangle}\Big) = \langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}. \tag{4.6}$$

Note that, for $0 < \epsilon < 1$,

$$\begin{aligned}&\frac{1}{\epsilon}\Big(e^{-\langle v_{h-t+\epsilon}, X_t\rangle} - e^{-\langle v_{h-t}, X_t\rangle}\Big) \leq \frac{1}{\epsilon}\Big(1 - \exp\{-\langle v_{h-t} - v_{h-t+\epsilon}, X_t\rangle\}\Big) \\ &\leq \frac{\langle v_{h-t} - v_{h-t+\epsilon}, X_t\rangle}{\epsilon} \leq \sup_{h-t \leq s \leq h-t+1}\sup_{x \in E} w(s,x)\langle 1, X_t\rangle.\end{aligned}$$

By Assumption **(H2)** and (2.5),

$$\mathbb{P}_\mu\left(\sup_{h-t\le s\le h-t+1}\sup_{x\in E} w(s,x)\langle 1, X_t\rangle\right) < \infty.$$

Thus, it follows from the dominated convergence theorem that

$$\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{P}_\mu\Big(\mathbf{1}_A\big(e^{-\langle v_{h-t+\epsilon}, X_t\rangle} - e^{-\langle v_{h-t}, X_t\rangle}\big)\Big) = \mathbb{P}_\mu\big(\mathbf{1}_A\langle w_{h-t}, X_t\rangle e^{-\langle v_{h-t}, X_t\rangle}\big). \tag{4.7}$$

Thus, by (4.5) and (4.7), we have that

$$\lim_{\epsilon\downarrow 0}\mathbb{P}_\mu(A|h\le H < h+\epsilon) = \mathbb{P}_\mu(\mathbf{1}_A M^h_t).$$

The proof is now complete. □

Proof of Lemma 3.3: By the Markov property of X, we get that,

$$e^{-v(t+s,x)} = \mathbb{P}_{\delta_x}(X_{t+s}=0) = \mathbb{P}_{\delta_x}(\mathbb{P}_{X_t}(X_s=0)) = \mathbb{P}_{\delta_x}(e^{-\langle v_s, X_t\rangle}), \tag{4.8}$$

which implies that $u_{v_s}(t,x) = v(t+s,x)$. By (4.4) with $h = t+s$ and $\mu = \delta_x$, we get that

$$\begin{aligned} w(t+s,x)e^{-v(t+s,x)} &= \mathbb{P}_{\delta_x}(\langle w_s, X_t\rangle e^{-\langle v_s, X_t\rangle}) \\ &= \Pi_x\left(\exp\left\{-\int_0^t \Psi_z'(\xi_u, v(t+s-u,\xi_u))\,du\right\} w(s,\xi_t)\right) e^{-v(t+s,x)}, \end{aligned}$$

where in the last equality we used Lemma 4.1 and the fact that $u_{v_s}(t,x) = v(t+s,x)$. Thus, it follows immediately that

$$w(t+s,x) = \Pi_x\left(\exp\left\{-\int_0^t \Psi_z'(\xi_u, v(t+s-u,\xi_u))\,du\right\} w(s,\xi_t)\right). \tag{4.9}$$

For $0<s<t$, by the Markov property of ξ, we have that

$$\begin{aligned} &\Pi_x\Big(w(h-t,\xi_t)e^{-\int_0^t \Psi_z'(\xi_u, v(h-u,\xi_u))\,du}|\mathcal{F}_s\Big) \\ &= e^{-\int_0^s \Psi_z'(\xi_u, v(h-u,\xi_u))\,du}\Pi_x\Big(w(h-t,\xi_t)e^{-\int_s^t \Psi_z'(\xi_u, v(h-u,\xi_u))\,du}|\mathcal{F}_s\Big) \\ &= e^{-\int_0^s \Psi_z'(\xi_u, v(h-u,\xi_u))\,du}\Pi_{\xi_s}\Big(w(h-t,\xi_{t-s})e^{-\int_0^{t-s} \Psi_z'(\xi_u, v(h-s-u,\xi_u))\,du}\Big) \\ &= e^{-\int_0^s \Psi_z'(\xi_u, v(h-u,\xi_u))\,du} w(h-s,\xi_s), \end{aligned}$$

where the last equality above follows from (4.9). The proof is now complete. □

4.1 Williams decomposition

Proof of Theorem 3.5: Let $f_k\in\mathcal{B}^+_b(E)$, $k=1,2,\cdots,n$ and $0=t_0<t_1<t_2<\cdots<t_n=t<h$. We will show that

$$\mathbb{P}^h_\mu\left(\exp\left\{-\sum_{j=1}^n\langle f_j, X_{t_j}\rangle\right\}\right) = \mathbf{P}^{(h)}_\mu\left(\exp\left\{-\sum_{j=1}^n\langle f_j, \Lambda^h_{t_j}\rangle\right\}\right).$$

By the definition of Λ^h_t, we have

$$\begin{aligned} &\mathbf{P}^{(h)}_\mu\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j, \Lambda^h_{t_j}\rangle\Big\}\Big) \\ &= \int_E \frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\,\mu(dx)\Pi^h_x\Big[\mathbf{P}^{(h)}_\mu\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j, \Lambda^h_{t_j}\rangle\Big\}|\xi^h\Big)\Big]. \end{aligned} \tag{4.10}$$

By the construction of Λ_t^h, we have

$$\begin{aligned}
&\mathbf{P}_\mu^{(h)}\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,\Lambda_{t_j}^h\rangle\Big\}|\xi^h\Big)\\
=&\mathbb{P}_\mu\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,X_{t_j}\rangle\Big\}|H<h\Big)\times\mathbf{P}_\mu^{(h)}\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,X_{t_j}^{1,h,\mathbb{N}}\rangle\Big\}|\xi^h\Big)\\
&\times\mathbf{P}_\mu^{(h)}\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,X_{t_j}^{2,h,\mathbb{P}}\rangle\Big\}|\xi^h\Big)\\
=:&(I)\times(II)\times(III). \qquad (4.11)
\end{aligned}$$

Define, for $s<h$,

$$J_s(h,x):=-\log\mathbb{P}_{\delta_x}\left[e^{-\sum_{j=1}^n\langle f_j,X_{t_j-s}\rangle\mathbf{1}_{s\le t_j}};\|X_{h-s}\|=0\right]. \qquad (4.12)$$

We first deal with part (I). By (4.12), we have

$$J_0(h,x)=-\log\mathbb{P}_{\delta_x}\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,X_{t_j}\rangle\Big\};\|X_h\|=0\Big). \qquad (4.13)$$

By (3.2), $\mathbb{P}_\mu(H<h)=\mathbb{P}_\mu(H\le h)=e^{-\langle v_h,\mu\rangle}$. Thus we have

$$(I)=e^{\langle v(h,\cdot),\mu\rangle}e^{-\langle J_0(h,\cdot),\mu\rangle}. \qquad (4.14)$$

Next we deal with part (II). By the definition of $X^{1,h,\mathbb{N}}$ and Fubini's theorem, we have

$$\begin{aligned}
\sum_{j=1}^n\langle f_j,X_{t_j}^{1,h,\mathbb{N}}\rangle&=\sum_{j=1}^n\int_0^t\int_{\mathbb{D}}\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}\mathcal{N}^{1,h}(ds,d\omega)\\
&=\int_0^t\int_{\mathbb{D}}\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}\,\mathcal{N}^{1,h}(ds,d\omega). \qquad (4.15)
\end{aligned}$$

Therefore,

$$\begin{aligned}
(II)&=\mathbf{P}_\mu^{(h)}\left(\exp\left\{-\int_0^t\int_{\mathbb{D}}\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}\,\mathcal{N}^{1,h}(ds,d\omega)\right\}|\xi^h\right)\\
&=\exp\left\{-\int_0^t 2b(\xi_s)\,ds\int_{\mathbb{D}}\Big(1-e^{-\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}}\Big)\mathbf{1}_{H(\omega)<h-s}\mathbb{N}_{\xi_s}(d\omega)\right\}.
\end{aligned}$$

By the dominated convergence theorem, we obtain that, for $s\neq t_j$, $j=1,2,\cdots,n$,

$$\begin{aligned}
&\int_{\mathbb{D}}\Big(1-e^{-\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}}\Big)\mathbf{1}_{H(\omega)<h-s}\mathbb{N}_{\xi_s}(d\omega)\\
=&\int_{\mathbb{D}}\Big(1-e^{-\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}}\Big)\mathbf{1}_{\|\omega_{h-s}\|=0}\mathbb{N}_{\xi_s}(d\omega)\\
=&\lim_{\theta\to\infty}\int_{\mathbb{D}}\Big(1-e^{-\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}}\Big)e^{-\theta\|\omega_{h-s}\|}\mathbb{N}_{\xi_s}(d\omega)\\
=&\lim_{\theta\to\infty}\int_{\mathbb{D}}\Big(1-e^{-\sum_{j=1}^n\langle f_j,\omega_{t_j-s}\rangle\mathbf{1}_{s<t_j}-\theta\|\omega_{h-s}\|}\Big)\mathbb{N}_{\xi_s}(d\omega)-\int_{\mathbb{D}}\Big(1-e^{-\theta\|\omega_{h-s}\|}\Big)\mathbb{N}_{\xi_s}(d\omega)\\
=&\lim_{\theta\to\infty}-\log\mathbb{P}_{\delta_{\xi_s}}e^{-\sum_{j=1}^n\langle f_j,X_{t_j-s}\rangle\mathbf{1}_{s<t_j}-\theta\|X_{h-s}\|}+\log\mathbb{P}_{\delta_{\xi_s}}e^{-\theta\|X_{h-s}\|}\\
=&-\log\mathbb{P}_{\delta_{\xi_s}}\left[e^{-\sum_{j=1}^n\langle f_j,X_{t_j-s}\rangle\mathbf{1}_{s<t_j}};\|X_{h-s}\|=0\right]+\log\mathbb{P}_{\delta_{\xi_s}}\left(\|X_{h-s}\|=0\right)\\
=&J_s(h,\xi_s)-v(h-s,\xi_s).
\end{aligned}$$

Hence,

$$(II) = \exp\left\{-\int_0^t 2b(\xi_s)\Big(J_s(h,\xi_s) - v(h-s,\xi_s)\Big)\,ds\right\}. \tag{4.16}$$

Now we deal with (III). Using arguments similar to those leading to (4.15), we get that

$$\sum_{j=1}^n \langle f_j, X_{t_j}^{2,h,\mathbb{P}}\rangle = \int_0^t \int_{\mathbb{D}} \sum_{j=1}^n \langle f_j, \omega_{t_j-s}\rangle \mathbf{1}_{s\le t_j}\,\mathcal{N}^{2,h}(ds,d\omega).$$

Thus,

$$\begin{aligned}(III) &= \mathbf{P}_\mu^{(h)}\left(\exp\left\{-\int_0^t \int_{\mathbb{D}} \sum_{j=1}^n \langle f_j, \omega_{t_j-s}\rangle \mathbf{1}_{s\le t_j}\,\mathcal{N}^{2,h}(ds,d\omega)\right\}|\xi^h\right)\\ &= \exp\left\{-\int_0^t ds\int_0^\infty yn(\xi_s,dy)\mathbb{P}_{y\delta_{\xi_s}}\left[\left(1-e^{-\sum_{j=1}^n\langle f_j,X_{t_j-s}\rangle \mathbf{1}_{s\le t_j}}\right)\mathbf{1}_{H<h-s}\right]\right\}\\ &= \exp\left\{-\int_0^t ds\int_0^\infty yn(\xi_s,dy)\Big(e^{-yv(h-s,\xi_s)} - e^{-yJ_s(h,\xi_s)}\Big)\right\}. \end{aligned}\tag{4.17}$$

Recall that

$$\Psi_z'(x,z) = -\alpha(x) + 2b(x)z + \int_0^\infty y(1-e^{-yz})n(x,dy).$$

Combining (4.16) and (4.17), we get that

$$\begin{aligned}&(II)\times(III)\\ &=\exp\left\{-\int_0^t \Big(2b(\xi_s)J_s(h,\xi_s) + \int_0^\infty y\Big(1-e^{-yJ_s(h,\xi_s)}\Big)n(\xi_s,dy)\Big)\,ds\right\}\\ &\quad\times\exp\left\{\int_0^t \Big(2b(\xi_s)v(h-s,\xi_s) - \int_0^\infty y\Big(1-e^{-yv(h-s,\xi_s)}\Big)n(\xi_s,dy)\Big)\,ds\right\}\\ &=\exp\left\{-\int_0^t \Psi_z'(\xi_s,J_s(h,\xi_s))\,ds\right\}\times\exp\left\{\int_0^t \Psi_z'(\xi_s,v(h-s,\xi_s))\,ds\right\}. \end{aligned}\tag{4.18}$$

By (4.11), (4.14) and (4.18), we get that, for $h>t$,

$$\begin{aligned}&\Pi_x^h\Big[\mathbf{P}_\mu^{(h)}\left(\exp\left\{-\sum_{j=1}^n\langle f_j,\Lambda_{t_j}^h\rangle\right\}|\xi^h\right)\Big]\\ &=e^{\langle v(h,\cdot),\mu\rangle}e^{-\langle J_0(h,\cdot),\mu\rangle}\\ &\qquad\times\Pi_x^h\Big[\exp\left\{-\int_0^t \Psi_z'(\xi_s,J_s(h,\xi_s))\,ds\right\}\times\exp\left\{\int_0^t \Psi_z'(\xi_s,v(h-s,\xi_s))\,ds\right\}\Big]\\ &=e^{\langle v(h,\cdot),\mu\rangle}e^{-\langle J_0(h,\cdot),\mu\rangle}\Pi_x\Big[\frac{w(h-t,\xi_t)}{w(h,x)}\exp\left\{-\int_0^t \Psi_z'(\xi_s,J_s(h,\xi_s))\,ds\right\}\Big].\end{aligned}$$

So, by (4.10), we obtain that

$$\begin{aligned}\mathbf{P}_\mu^{(h)}\left(\exp\left\{-\sum_{j=1}^n\langle f_j,\Lambda_{t_j}^h\rangle\right\}\right) &= \frac{e^{\langle v(h,\cdot),\mu\rangle}}{\langle w_h,\mu\rangle}e^{-\langle J_0(h,\cdot),\mu\rangle}\\ &\quad\times\int_E \Pi_x\Big[w(h-t,\xi_t)\exp\left\{-\int_0^t\Big(\Psi_z'(\xi_s,J_s(h,\xi_s))\Big)\,ds\right\}\Big]\,\mu(dx).\end{aligned}\tag{4.19}$$

Now we calculate $J_s(h,x)$ defined in (4.12). For $0\le s<t<h$, by the Markov property of X, we have that

$$\begin{aligned} J_s(h,x) &= -\log \mathbb{P}_{\delta_x}\left[e^{-\sum_{j=1}^n \langle f_j, X_{t_j-s}\rangle \mathbf{1}_{s\le t_j}}\mathbb{P}_{X_{t-s}}(\|X_{h-t}\|=0)\right] \\ &= -\log \mathbb{P}_{\delta_x}\left[e^{-\sum_{j=1}^n \langle f_j, X_{t_j-s}\rangle \mathbf{1}_{s\le t_j}-\langle v(h-t,\cdot),X_{t-s}\rangle}\right] \\ &= -\log \mathbb{P}_{s,\delta_x}\left[e^{-\sum_{j=1}^n \langle f_j, X_{t_j}\rangle \mathbf{1}_{s\le t_j}-\langle v(h-t,\cdot),X_{t}\rangle}\right]. \end{aligned} \tag{4.20}$$

Using Lemma 4.1 with $r=0$, we have that

$$\begin{aligned} & e^{-\langle J_0(h,\cdot),\,\mu\rangle}\int_E \Pi_x\Big[w(h-t,\xi_t)\exp\Big\{-\int_0^t \Big(\Psi_z'(\xi_s, J_s(h,\xi_s))\Big)\,ds\Big\}\Big]\,\mu(dx) \\ =&\mathbb{P}_\mu\Big[\langle w(h-t,\cdot),X_t\rangle \exp\Big\{-\sum_{j=1}^n\langle f_j, X_{t_j}\rangle-\langle v(h-t,\cdot),X_t\rangle\Big\}\Big]. \end{aligned}$$

Thus, by (4.19), we get that

$$\mathbf{P}_\mu^{(h)}\Big(\exp\Big\{-\sum_{j=1}^n\langle f_j,\Lambda_{t_j}^h\rangle\Big\}\Big)=\mathbb{P}_\mu\Big[\exp\Big\{-\sum_{j=1}^n\langle f_j, X_{t_j}\rangle\Big\}M_t^h\Big].$$

Now, the proof is complete. □

4.2 The behavior of X_t near extinction

Recall that, for any $\mu\in\mathcal{M}_F(E)$, $\xi^h=\{(\xi_t)_{0\le t<h},\Pi_\nu^h\}$, where $\nu(dx)=\frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\mu(dx)$.

Theorem 4.3. *Suppose that Assumptions* **(H1)**-**(H3)**) *hold and that for any* $\mu\in\mathcal{M}_F(E)$,

$$\textit{the limit } \lim_{t\uparrow h}\xi_t \textit{ exists } \Pi_\nu^h\text{-a.s.},$$

where $\nu(dx)=\frac{w(h,x)}{\langle w(h,\cdot),\mu\rangle}\mu(dx)$. *Define* $\xi_{h-}:=\lim_{t\uparrow h}\xi_t$. *Then, for any* $h>0$,

$$\lim_{t\uparrow h}\frac{\Lambda_t^h}{\|\Lambda_t^h\|}=\delta_{\xi_{h-}},\qquad \mathbf{P}_\mu^{(h)}\text{-a.s.}$$

Proof. By the decomposition (3.6), we have

$$\Lambda_t^h:=X_t^{0,h}+X_t^{1,h,\mathbb{N}}+X_t^{2,h,\mathbb{P}}.$$

Define

$$H_0:=\inf\{t\ge 0: X_t^{0,h}=0\}\quad \textit{and}\quad H(\Lambda^h):=\inf\{t\ge0:\Lambda_t^h=0\}.$$

Then by the definition of $X^{0,h}$, we have $H_0<h$. By Theorem 3.5, $H(\Lambda^h)=h$. It follows that

$$\lim_{t\uparrow h}\frac{X_t^{0,h}}{\|\Lambda_t^h\|}=0,\qquad \mathbf{P}_\mu^{(h)}\text{-a.s.} \tag{4.21}$$

Note that E_∂ is a compact separable metric space. According to [26, Exercise 9.1.16 (iii)], $C_b(E_\partial;\mathbb{R})$, the space of bounded continuous $\mathbb{R}$-valued functions f on E_∂, is separable. Therefore, $C_b^+(E)$, the space of nonnegative bounded continuous $\mathbb{R}$-valued functions f on E, is also a separable space. It suffices to prove that, for any $f\in C_b^+(E)$,

$$\mathbf{P}_\mu^{(h)}\left(\lim_{t\uparrow h}\frac{\langle f_h, X_t^{1,h,\mathbb{N}}\rangle+\langle f_h, X_t^{2,h,\mathbb{P}}\rangle}{\|\Lambda_t^h\|}=0\right)=1, \tag{4.22}$$

where $f_h(x) = f(x) - f(\xi_{h-})$. Note that

$$\begin{aligned}
&\mathbf{P}_\mu^{(h)}\left(\lim_{t\uparrow h}\frac{\langle f_h, X_t^{1,h,\mathbb{N}}\rangle + \langle f_h, X_t^{2,h,\mathbb{P}}\rangle}{\|\Lambda_t^h\|} = 0\right)\\
&= \mathbf{P}_\mu^{(h)}\left[\mathbf{P}_\mu^{(h)}\left(\lim_{t\uparrow h}\frac{\langle f_h, X_t^{1,h,\mathbb{N}}\rangle + \langle f_h, X_t^{2,h,\mathbb{P}}\rangle}{\|\Lambda_t^h\|} = 0\big|\xi^h\right)\right].
\end{aligned}$$

Therefore, it suffices to prove that, for any $f \in C_b^+(E)$,

$$\mathbf{P}_\mu^{(h)}\left(\lim_{t\uparrow h}\frac{\langle f_h, X_t^{1,h,\mathbb{N}}\rangle + \langle f_h, X_t^{2,h,\mathbb{P}}\rangle}{\|\Lambda_t^h\|} = 0\big|\xi^h\right) = 1, \quad \mathbf{P}_\mu^{(h)}\text{-a.s.} \tag{4.23}$$

Step 1 We first prove that given ξ^h,

$$\lim_{t\uparrow h}\frac{\langle f_h, X_t^{1,h,\mathbb{N}}\rangle}{\|\Lambda_t^h\|} = 0, \quad \mathbf{P}_\mu^{(h)}\text{-a.s.} \tag{4.24}$$

Note that given ξ^h,

$$\langle f_h, X_t^{1,h,\mathbb{N}}\rangle := \int_0^t\int_{\mathbb{D}}\langle f_h, \omega_{t-s}\rangle\mathcal{N}^{1,h}(ds, d\omega),$$

where $\mathcal{N}^{1,h}(ds, d\omega)$ is a Poisson random measure on $[0, h) \times \mathbb{D}$ with intensity measure

$$2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds.$$

Let I_1 be the support of the measure $\mathcal{N}^{1,h}$. Note that I_1 is a random subset of $[0, h) \times \mathbb{D}$.

In the remainder of this proof, we always assume that ξ^h is given. Since $f \in C_b^+(E)$, for any $\epsilon > 0$, there exists $\delta_1 > 0$, depending on ξ_{h-}, such that $|f(x) - f(\xi_{h-})| \le \epsilon$ for all $|x - \xi_{h-}| \le \delta_1$. It follows from the fact that $\xi_{h-} = \lim_{s\uparrow h}\xi_s$ there exists $\delta_2 \in (0, h)$, depending on ξ_{h-}, such that $|\xi_s - \xi_{h-}| < \delta_1/2$ for all $s \in (h - \delta_2, h)$. Let $B := B(\xi_{h-}, \delta_1) = \{x \in E : |x - \xi_{h-}| < \delta_1\}$. Then, for any $t \in (h - \delta_2/2, h)$, we have

$$\begin{aligned}
|\langle f_h, X_t^{1,h,\mathbb{N}}\rangle| =& |\langle f_h\mathbf{1}_{\bar{B}}, X_t^{1,h,\mathbb{N}}\rangle + \langle f_h\mathbf{1}_{\bar{B}^c}, X_t^{1,h,\mathbb{N}}\rangle|\\
\le& \epsilon\langle 1, X_t^{1,h,\mathbb{N}}\rangle + 2\|f\|_\infty\langle\mathbf{1}_{\bar{B}^c}, X_t^{1,h,\mathbb{N}}\rangle\\
\le& \epsilon\langle 1, \Lambda_t^h\rangle + 2\|f\|_\infty\int_0^{h-\delta_2}\int_{\mathbb{D}}\langle 1, \omega_{t-s}\rangle\mathcal{N}^{1,h}(ds, d\omega)\\
&+ 2\|f\|_\infty\int_{h-\delta_2}^t\int_{\mathbb{D}}\langle\mathbf{1}_{\bar{B}^c}, \omega_{t-s}\rangle\mathcal{N}^{1,h}(ds, d\omega)\\
=:& \epsilon\langle 1, \Lambda_t^h\rangle + 2\|f\|_\infty J_1(t) + 2\|f\|_\infty J_2(t).
\end{aligned} \tag{4.25}$$

It follows that

$$\frac{|\langle f_h, X_t^{1,h,\mathbb{N}}\rangle|}{\|\Lambda_t^h\|} \le \epsilon + 2\|f\|_\infty\frac{J_1(t)}{\|\Lambda_t^h\|} + 2\|f\|_\infty\frac{J_2(t)}{\|\Lambda_t^h\|}. \tag{4.26}$$

First we deal with J_1. For $s \in (0, h - \delta_2)$ and $t \in (h - \delta_2/2, h)$, we have $t - s > \delta_2/2$. Thus, for $t \in (h - \delta_2/2, h)$, we have

$$J_1(t) = \int_0^{h-\delta_2}\int_{\omega(\delta_2/2)\neq 0,\, H(\omega)<h-s}\langle 1, \omega_{t-s}\rangle\mathcal{N}^{1,h}(ds, d\omega) = \sum_{(s,\,\omega)\in(I_1\cap S_1)}\langle 1, \omega_{t-s}\rangle,$$

where

$$S_1 := \{(s, \omega) : s \in [0, h - \delta_2),\, w(\delta_2/2) \neq 0 \quad\text{and}\quad H(\omega) < h - s\}. \tag{4.27}$$

Notice that

$$\begin{aligned}&\int_{S_1} 2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds\\&\leq 2K\int_0^{h-\delta_2}\mathbb{N}_{\xi_s}(w(\delta_2/2)\neq 0)ds\\&=2K\int_0^{h-\delta_2}v(\delta_2/2,\xi_s)ds\leq 2Kh\|v_{\delta_2/2}\|_\infty<\infty,\end{aligned}\tag{4.28}$$

which implies that given ξ^h,

$$\mathcal{N}^{1,h}(S_1)<\infty,\quad \mathbf{P}_\mu^{(h)}\text{-a.s.}$$

That is, given ξ^h, $\sharp\{I_1\cap S_1\}<\infty$, $\mathbf{P}_\mu^{(h)}$-a.s. For any $(s,\omega)\in(I_1\cap S_1)$, we have $s+H(\omega)<h$, which implies that $H_1:=\max_{(s,\omega)\in(I_1\cap S_1)}(s+H(\omega))<h$. Thus, for any $t\in(H_1,h)$, $J_1(t)=0$, which implies that given ξ^h,

$$\lim_{t\uparrow h}\frac{J_1(t)}{\|\Lambda_t^h\|}=0,\quad \mathbf{P}_\mu^{(h)}\text{-a.s.}\tag{4.29}$$

To deal with J_2, we define

$$\mathbb{D}_1:=\{\omega:\exists u\in(0,\delta_2),\text{such that }\langle\mathbf{1}_{\bar{B}^c},\omega_u\rangle>0\},\quad\text{and}\quad S_2=[h-\delta_2,h)\times\mathbb{D}_1.\tag{4.30}$$

Then,

$$J_2(t)=\sum_{(s,\,\omega)\in(I_1\cap S_2)}\langle\mathbf{1}_{\bar{B}^c},\omega_{t-s}\rangle\mathbf{1}_{s<t}\,.$$

We claim that $\sharp\{I_1\cap S_2\}<\infty$. Then using arguments similar to those leading to (4.29), we can get that given ξ^h,

$$\lim_{t\uparrow h}\frac{J_2(t)}{\|\Lambda_t^h\|}=0,\quad \mathbf{P}_\mu^{(h)}\text{-a.s.}\tag{4.31}$$

Now we prove the claim. It suffices to prove that given ξ^h

$$\int_{S_2}2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds<\infty.\tag{4.32}$$

Note that

$$\int_{S_2}2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds\leq 2K\int_{h-\delta_2}^h\mathbb{N}_{\xi_s}(\mathbb{D}_1)ds.$$

For $\omega\in\mathbb{D}$, we have

$$\begin{aligned}\mathbb{D}_1=&\{\omega\in\mathbb{D}:\exists u\in(0,\delta_2),\text{such that }\langle\mathbf{1}_{\bar{B}^c},\omega_u\rangle>0\}\\=&\left\{\omega\in\mathbb{D}:\int_0^{\delta_2}\langle\mathbf{1}_{\bar{B}^c},\omega_u\rangle\,du>0\right\}\\\subset&\left\{\omega\in\mathbb{D}:\int_0^{\delta_2}\langle\mathbf{1}_{B^c},\omega_u\rangle\,du>0\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{N}_x(\mathbb{D}_1) \leq & \mathbb{N}_x\left(\int_0^{\delta_2}\langle \mathbf{1}_{B^c}, \omega_u\rangle\, du > 0\right)\\ =& \lim_{\lambda\to\infty}\mathbb{N}_x\left(1-\exp\left\{-\lambda\int_0^{\delta_2}\langle \mathbf{1}_{B^c}, \omega_u\rangle\, du\right\}\right)\\ =& \lim_{\lambda\to\infty} -\log \mathbb{P}_{\delta_x}\left(\exp\left\{-\lambda\int_0^{\delta_2}\langle \mathbf{1}_{B^c}, X_u\rangle\, du\right\}\right)\\ =& -\log \mathbb{P}_{\delta_x}\left(\int_0^{\delta_2}\langle \mathbf{1}_{B^c}, \omega_u\rangle\, du = 0\right).\end{aligned} \tag{4.33}$$

Combining (4.33) and Assumption **(H3)**, we get

$$\begin{aligned}&\int_{S_2} 2\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}b(\xi_s)\mathbb{N}_{\xi_s}(d\omega)ds\\ \leq & 2K\delta_2 \sup_{x\in B(\xi_{h-},\delta_1/2)}\Big[-\log \mathbb{P}_{\delta_x}\left(\int_0^{\delta_2}\langle \mathbf{1}_{B^c}, \omega_u\rangle\, du = 0\right)\Big] < \infty.\end{aligned}$$

Combining (4.26), (4.29) and (4.31), we get (4.24).

Step 2 Next we prove that given ξ^h,

$$\lim_{t\uparrow h}\frac{\langle f_h, X_t^{2,h,\mathbb{P}}\rangle}{\|\Lambda_t^h\|} = 0, \quad \mathbf{P}_\mu^{(h)}\text{-a.s.} \tag{4.34}$$

Note that given ξ^h,

$$\langle f_h, X_t^{2,h,\mathbb{P}}\rangle := \int_0^t\int_{\mathbb{D}}\langle f_h, \omega_{t-s}\rangle \mathcal{N}^{2,h}(ds, d\omega),$$

where $\mathcal{N}^{2,h}(ds, d\omega)$ is a Poisson random measure on $[0,h)\times\mathbb{D}$ with intensity measure

$$\mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s}\int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X\in d\omega)ds.$$

Let I_2 be the support of the measure $\mathcal{N}^{2,h}$. Note that I_2 is a random countable subset of $[0,h)\times\mathbb{D}$. Using arguments similar to those leading to (4.25), we get that

$$\begin{aligned}\langle f_h, X_t^{2,h,\mathbb{P}}\rangle \leq & \epsilon\langle 1, \Lambda_t^h\rangle + 2\|f\|_\infty\int_0^{h-\delta_2}\int_{\mathbb{D}}\langle 1, \omega_{t-s}\rangle\mathcal{N}^{2,h}(ds,d\omega)\\ &+ 2\|f\|_\infty\int_{h-\delta_2}^t\int_{\mathbb{D}}\langle \mathbf{1}_{\bar{B}^c}, \omega_{t-s}\rangle\mathcal{N}^{2,h}(ds,d\omega)\\ =&\epsilon\langle 1, \Lambda_t^h\rangle + 2\|f\|_\infty\sum_{(s,\omega)\in(I_2\cap S_1)}\langle 1, \omega_{t-s}\rangle + 2\|f\|_\infty\sum_{(s,\omega)\in(I_2\cap S_2)}\langle \mathbf{1}_{\bar{B}^c}, \omega_{t-s}\rangle\\ =:&\epsilon\langle 1, \Lambda_t^h\rangle + 2\|f\|_\infty J_3(t) + 2\|f\|_\infty J_4(t),\end{aligned}$$

where S_1 and S_2 are the sets defined in (4.27) and (4.30) respectively. It follows that

$$\frac{|\langle f_h, X_t^{2,h,\mathbb{P}}\rangle|}{\|\Lambda_t^h\|} \leq \epsilon + 2\|f\|_\infty\frac{J_3(t)}{\|\Lambda_t^h\|} + 2\|f\|_\infty\frac{J_4(t)}{\|\Lambda_t^h\|}. \tag{4.35}$$

So, to prove (4.34), we only need to prove that

$$\lim_{t\uparrow h}\frac{J_3(t)}{\|\Lambda_t^h\|} = 0, \quad \mathbf{P}_\mu^{(h)}\text{-a.s.,} \tag{4.36}$$

and

$$\lim_{t\uparrow h} \frac{J_4(t)}{\|\Lambda_t^h\|} = 0, \quad \mathbf{P}_\mu^{(h)}\text{-a.s.} \tag{4.37}$$

Note that

$$\begin{aligned}
&\int_{S_1} \mathbf{1}_{[0,h)}(s)\mathbf{1}_{H(\omega)<h-s} \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X \in d\omega)ds \\
\le &\int_0^{h-\delta_2} \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X_{\delta_2/2} \neq 0)ds \\
\le &\int_0^{h-\delta_2} v(\delta_2/2, \xi_s) \int_0^1 y^2 n(\xi_s, dy)ds + \int_0^{h-\delta_2} \int_1^\infty yn(\xi_s, dy)\, ds \\
\le &Kh(\|v_{\delta_2/2}\|_\infty + 1),
\end{aligned} \tag{4.38}$$

where in the second inequality we used the fact that

$$\mathbb{P}_{y\delta_{\xi_s}}(X_{\delta_2/2} \neq 0) = 1 - \mathbb{P}_{y\delta_{\xi_s}}(X_{\delta_2/2} = 0) = 1 - e^{-yv(\delta_2/2,\xi_s)} \le yv(\delta_2/2, \xi_s).$$

Thus, $\mathcal{N}^{2,h}(S_1) < \infty$, a.s., which implies that (4.36).

To prove (4.37) we only need to show that, given ξ^h,

$$\int_{S_2} \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X \in d\omega)ds < \infty. \tag{4.39}$$

In fact,

$$\begin{aligned}
&\int_{S_2} \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}(X \in d\omega)ds \\
\le &\int_{h-\delta_2}^h \int_0^\infty yn(\xi_s, dy)\mathbb{P}_{y\delta_{\xi_s}}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, X_u\rangle\, du > 0\Big)ds \\
\le &\int_{h-\delta_2}^h \int_1^\infty yn(\xi_s, dy)ds + \int_{h-\delta_2}^h \Big(-\log \mathbb{P}_{\delta_{\xi_s}}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, X_u\rangle\, du = 0\Big)\Big) \int_0^1 y^2 n(\xi_s, dy)ds \\
\le &Kh + Kh \sup_{x\in B(\xi_{h-},\delta_1/2)} \Big[-\log \mathbb{P}_{\delta_x}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, X_u\rangle\, du = 0\Big)\Big] < \infty,
\end{aligned}$$

where in the second inequality, we used the fact that

$$\begin{aligned}
\mathbb{P}_{y\delta_{\xi_s}}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, X_u\rangle\, du > 0\Big) =&1 - \exp\Big\{y \log \mathbb{P}_{\delta_{\xi_s}}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, \omega_u\rangle\, du = 0\Big)\Big\} \\
\le &- y \log \mathbb{P}_{\delta_{\xi_s}}\Big(\int_0^{\delta_2} \langle \mathbf{1}_{B^c}, X_u\rangle\, du = 0\Big).
\end{aligned}$$

The proof is now complete. □

Proof of Theorem 3.7: Since $\{X_t, t \ge 0\}$ is a Hunt process, $t \to X_t$ is right continuous, which implies that

$$\Big\{\lim_{t\uparrow H} \frac{X_t}{\|X_t\|} \quad \text{exists}\Big\} = \Big\{\lim_{t\in\mathbb{Q}\uparrow H} \frac{X_t}{\|X_t\|} \quad \text{exists}\Big\},$$

where $\mathbb{Q}$ is the set of all rational numbers in $[0, \infty)$. And, note that

$$H = \inf\{t \in \mathbb{Q} : \|X_t\| = 0\}.$$

Thus, by Corollary 3.6 and Theorem 4.3, we get that

$$\mathbb{P}_\mu\Big[\lim_{t\in\mathbb{Q}\uparrow H}\frac{X_t}{\|X_t\|}\text{ exists}\Big]=\int_0^\infty \mathbf{P}^{(h)}_\mu\Big[\lim_{t\in\mathbb{Q}\uparrow h}\frac{\Lambda^h_t}{\|\Lambda^h_t\|}\text{ exists}\Big]\,F_H(dh)=1.$$

Let $V:=\lim_{t\uparrow H}\frac{X_t}{\|X_t\|}$. Then, for any $f\in\mathcal{B}^+_b(E)$, by Theorem 4.3,

$$\begin{aligned}\mathbb{P}_\mu[\exp\{-\langle f,V\rangle\}]=&\mathbb{P}_\mu\Big[\lim_{t\in\mathbb{Q}\uparrow H}\exp\Big\{-\frac{\langle f,X_t\rangle}{\|X_t\|}\Big\}\Big]\\ =&\int_0^\infty\lim_{t\in\mathbb{Q}\uparrow h}\mathbf{P}^{(h)}_\mu\left[\exp\Big(-\frac{\langle f,\Lambda^h_t\rangle}{\|\Lambda^h_t\|}\Big)\right]\,F_H(dh)\\ =&\int_0^\infty\Pi^h_\nu[\exp(-f(\xi_{h-}))]\,F_H(dh).\end{aligned}$$

Thus, V is a Dirac measure of the form $V=\delta_Z$ and the law of Z satisfies (3.8). The proof is now complete. □

5 Examples

In this section, we will list some examples that satisfy Assumptions **(H1)** and **(H2)**. The purpose of these examples is to show that Assumptions **(H1)** and **(H2)** are satisfied in a lot of cases. We will not try to give the most general examples possible.

5.1 Examples in Delmas and Hénard [5]

Example 3. Suppose that P_t is conservative and preserves $C_b(E)$. Let $\mathcal{L}$ be the infinitesimal generator of P_t in $C_b(E)$ and $\mathcal{D}(\mathcal{L})$ be the domain of $\mathcal{L}$. Also assume that

$$\Psi(x,z)=-\alpha(x)z+b(x)z^2,$$

where $\sup_{x\in E}\alpha(x)\le 0$ and $\inf_{x\in E}b(x)>0$ and $1/b\in\mathcal{D}(\mathcal{L})$. Then by Remark 2.2, we know that Assumption **(H1)** is satisfied. One can check that

$$\Big(\frac{b^{-1}(\xi_t)}{b^{-1}(x)}e^{-\int_0^t(b(\xi_s)\mathcal{L}(1/b)(\xi_s))\,ds},t\ge 0\Big)$$

is a positive martingale under Π_x. Thus we define another probability measure $\Pi^{1/b}_x$ by

$$\frac{\Pi^{1/b}_x}{\Pi_x}\Big|_{\mathcal{F}_t}=\frac{b^{-1}(\xi_t)}{b^{-1}(x)}e^{-\int_0^t(b(\xi_s)\mathcal{L}(1/b)(\xi_s))\,ds},\quad t\ge 0.$$

Let $\mathcal{L}^{1/b}$ be the infinitesimal generator of ξ under $\Pi^{1/b}$. If $-\alpha(x)-b(x)\mathcal{L}(1/b)(x)\in\mathcal{D}(\mathcal{L}^{1/b})$, then it follows from [5, (3.10) and Lemma 4.9] that $w(t,x)$ exists and satisfies

$$w(t,x)\le\frac{1}{\inf_{x\in E}b(x)}e^{ct}\frac{\beta_0^2e^{\beta_0t}}{(e^{\beta_0t}-1)^2},$$

where c,β_0 are positive constants. Using this, one can check that Assumption **(H2)** is satisfied. This example shows that our result covers Delmas and Hénard [5, Corollary 4.14]. Since $\Psi(x,z)\ge cz^2$, where $c=\inf_{x\in E}b(x)$, we have seen in Example 2 that if $\mathcal{L}=L$ with L being given in Example 2, then Assumption **(H3)** is satisfied.

Now we give some examples of superprocesses, with general branching mechanisms, satisfying Assumptions **(H1)** and **(H2)**. We will see that Assumption **(H3)** is satisfied by some examples.

Recall that the general form of branching mechanism is given by

$$\Psi(x,z) = -\alpha(x)z + b(x)z^2 + \int_0^\infty (e^{-yz} - 1 + yz)n(x,dy).$$

By (2.2), there exists $K > 0$, such that

$$|\alpha(x)| + b(x) + \int_0^\infty (y \wedge y^2)n(x,dy) \le K.$$

Thus we have

$$|\Psi(x,z)| \le 3K(z+z^2), \qquad x \in \mathbb{R}^d. \tag{5.1}$$

5.2 Examples of some superdiffusions

In the next two examples, we always assume that $E = \mathbb{R}^d$ and that Ψ satisfies the following condition:

(C1) Ψ satisfies the conditions in Remark 2.2 and is Hölder continuous in the first variable, locally uniformly in the second variable, in the sense that for any $M > 0$, there exist $c > 0$ and $\gamma_0 \in (0,1]$ such that

$$|\Psi(x,z) - \Psi(y,z)| \le c|x-y|^{\gamma_0}, \quad x,y \in \mathbb{R}^d,\ z \in [0,M]. \tag{5.2}$$

By Remark 2.2, Assumption **(H1)** is satisfied. Therefore, in the following examples, we only need to check that Assumption **(H2)** and Assumption **(H3)** are satisfied.

Example 4. Assume that the spatial motion ξ is a diffusion on $\mathbb{R}^d$ satisfying the conditions in Example 2. The branching mechanism Ψ is of the form (2.1) and satisfies condition **(C1)**. Then the (ξ,Ψ)-superprocess X satisfies Assumptions **(H1)** and **(H2)**. We have seen in Example 2 that under the condition that there exist $\alpha \in (1,2]$ and $c > 0$ such that $\Psi(x,z) \ge cz^\alpha$ for all $x \in \mathbb{R}^d$, Assumption **(H3)** is satisfied.

We now proceed to prove that Assumption **(H2)** holds for this example. The main result is as follows:

Proposition 5.1. *Assume the conditions in Example 4 hold. The function $t \to v_t(x)$ is differentiable in $(0,\infty)$, and for any $s > 0$ and $t \in [0,1/2)$, $w(t+s,x) = -\frac{\partial}{\partial t}v_{t+s}(x)$ satisfies that*

$$w(t+s,x) = -\frac{\partial}{\partial t}P_t(v_s)(x) + \int_0^t \frac{\partial}{\partial t}P_{t-u}(\Psi_{s+u})(x)\,du + \Psi_{t+s}(x). \tag{5.3}$$

Moreover, $t \to w(t,x)$ is continuous and for any $s_0 > 0$, $\sup_{s>s_0}\sup_{x\in\mathbb{R}^d} w(s,x) < \infty$.

We will prove Proposition 5.1 through several lemmas.

Lemma 5.2. *For $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $t \to P_t f(x)$ is differentiable on $(0,\infty)$. Furthermore, there exists a constant c such that for any $t \in (0,1]$, $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$\left|\frac{\partial}{\partial t}P_t f(x)\right| \le c\|f\|_\infty t^{-1}. \tag{5.4}$$

Proof. For $t \in (n,n+1]$, $P_t f(x) = P_{t-n}(P_n f)(x)$. Thus, we only need to prove the differentiability for $t \in (0,1]$. It follows from [17, IV.(13.1)] that

$$\left|\frac{\partial}{\partial t}p(t,x,y)\right| \le c_1 t^{-\frac{d}{2}-1} e^{-\frac{c_2|x-y|^2}{t}}. \tag{5.5}$$

Thus by the dominated convergence theorem we have that for all $t \in (0,1]$ and $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t} P_t f(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t,x,y) f(y)\, dy,$$

and that for all $t \in (0,1]$, $x \in \mathbb{R}^d$ and bounded Borel functions f on $\mathbb{R}^d$,

$$\begin{aligned} \left| \frac{\partial}{\partial t} P_t f(x) \right| \le & c_1 \|f\|_\infty \int_{\mathbb{R}^d} t^{-\frac{d}{2}-1} e^{-\frac{c_2|x-y|^2}{t}}\, dy \\ = & c_3 \|f\|_\infty t^{-1} \int_0^\infty u^{\frac{d}{2}-1} e^{-c_2 u}\, du = c_4 \|f\|_\infty t^{-1}, \end{aligned}$$

where the first equality above is due to a simple change of variables. The proof is now complete. □

Lemma 5.3. *Assume that $f_s(x)$ is uniformly bounded in $(s,x) \in [0,1] \times \mathbb{R}^d$, that is, there is a constant $L > 0$ so that, for all $s \in [0,1]$ and $x \in \mathbb{R}^d$, $|f_s(x)| \le L$. Then there is a constant c such that for any $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,*

$$\left| \int_0^t P_{t-s} f_s(x)\, ds - \int_0^t P_{t-s} f_s(x')\, ds \right| \le cL(|x-x'| \wedge 1).$$

Proof. It follows from [17, IV.(13.1)] that there exist constants $c_1, c_2 > 0$ such that for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,

$$|\nabla_x p(t,x,y)| \le c_1 t^{-\frac{d+1}{2}} e^{-\frac{c_2|x-y|^2}{t}}. \tag{5.6}$$

Thus

$$|p(t,x,y) - p(t,x',y)| \le c_3((t^{-1/2}|x-x'|) \wedge 1) t^{-d/2} \left(e^{-\frac{c_4|x-y|^2}{t}} + e^{-\frac{c_4|x'-y|^2}{t}} \right). \tag{5.7}$$

Hence for any $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,

$$\left| \int_0^t P_{t-s} f_s(x)\, ds - \int_0^t P_{t-s} f_s(x')\, ds \right| \le c_5 L \int_0^1 s^{-1/2} |x-x'|\, ds \le c_6 L |x-x'|. \tag{5.8}$$

□

Lemma 5.4. *Assume that $f_s(x)$ satisfies the following conditions:*

(i) There is a constant L so that, for all $(s,x) \in [0,1] \times \mathbb{R}^d$, $|f_s(x)| \le L$.
(ii) For any $t_0 \in [0,1]$, $\lim_{s \to t_0} \sup_{x \in \mathbb{R}^d} |f_s(x) - f_{t_0}(x)| = 0$.
(iii) There exist constants $s_0 \in (0,1)$, $C > 0$ and $\gamma \in (0,1]$ such that for all $s \in [0, s_0]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| \le 1$,

$$|f_s(x) - f_s(x')| \le C|x-x'|^\gamma. \tag{5.9}$$

Then, $t \to \int_0^t P_{t-s} f_s(x)\, ds$ is differentiable on $(0, s_0)$, and for $t \in [0, s_0)$,

$$\frac{\partial}{\partial t} \int_0^t P_{t-s} f_s(x)\, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x)\, ds + f_t(x). \tag{5.10}$$

Proof. Let $G(t,x) := \int_0^t P_{t-s} f_s(x)\, ds$. First, we will show that for any $x \in \mathbb{R}^d$,

$$\lim_{t \downarrow 0} t^{-1} \int_0^t P_{t-s} f_s(x)\, ds = f_0(x). \tag{5.11}$$

Since $f_0 \in C_b(\mathbb{R}^d)$, we have $\lim_{s\to 0} P_s f_0(x) = f_0(x)$, which implies that

$$\lim_{t\to 0} t^{-1}\int_0^t P_{t-s}f_0(x)\,ds = \lim_{t\to 0} t^{-1}\int_0^t P_s f_0(x)\,ds = f_0(x).$$

Thus, it suffices to prove that

$$\lim_{t\to 0} t^{-1}\int_0^t P_{t-s}(f_s - f_0)(x)\,ds = 0. \tag{5.12}$$

Notice that

$$t^{-1}\int_0^t |P_{t-s}(f_s - f_0)(x)|\,ds \le \sup_{s\le t}\|f_s - f_0\|_\infty \to 0,$$

as $t \to 0$. Thus, (5.11) is valid.

For any $0 < t < t+r < s_0$, by the definition of $G(t,x)$,

$$\begin{aligned}
&\frac{1}{r}\left(G(t+r,x) - G(t,x)\right)\\
=&\frac{1}{r}\int_0^t \Big(P_{t+r-s}f_s(x) - P_{t-s}f_s(x)\Big)\,ds + \frac{1}{r}\int_t^{t+r} P_{t+r-s}f_s(x)\,ds\\
=&\int_0^t \frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r}\,ds + \frac{1}{r}\int_0^r P_{r-s}f_{t+s}(x)\,ds\\
=:&(I) + (II).
\end{aligned}$$

By (5.11), we have

$$\lim_{r\downarrow 0}(II) = f_t(x). \tag{5.13}$$

Now we deal with part (I). For $0 < t < t+r < s_0$, using (5.28), we obtain that

$$\begin{aligned}
&\left|\frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r}\right|\\
=&\left|\int_{\mathbb{R}^d} \frac{p(t+r-s,x,y) - p(t-s,x,y)}{r}(f_s(y) - f_s(x))\,dy\right|\\
\le& c_3\int_{\mathbb{R}^d} |f_s(y) - f_s(x)|(t-s)^{-\frac{d}{2}-1}e^{-\frac{c_4|x-y|^2}{t-s}}\,dy\\
\le& c_5\int_{\mathbb{R}^d} |x-y|^\gamma(t-s)^{-\frac{d}{2}-1}e^{-\frac{c_4|x-y|^2}{t-s}}\,dy\\
\le& c_6(t-s)^{\gamma/2-1}.
\end{aligned} \tag{5.14}$$

Thus, using the dominated convergence theorem, we get that, for any $0 \le t < t+r < s_0$,

$$\lim_{r\downarrow 0}(I) = \int_0^t \lim_{r\downarrow 0}\frac{P_{t+r-s}f_s(x) - P_{t-s}f_s(x)}{r}\,ds = \int_0^t \frac{\partial}{\partial t}P_{t-s}f_s(x)\,ds. \tag{5.15}$$

Combining (5.13) and (5.15), we get that

$$\lim_{r\downarrow 0}\frac{G(t+r,x) - G(t,x)}{r} = \int_0^t \frac{\partial}{\partial t}P_{t-s}f_s(x)\,ds + f_t(x).$$

Using similar arguments, we can also show that

$$\lim_{r\downarrow 0}\frac{G(t,x) - G(t-r,x)}{r} = \int_0^t \frac{\partial}{\partial t}P_{t-s}f_s(x)\,ds + f_t(x).$$

Thus, (5.10) follows immediately. The proof is now complete. □

Recall that $v(s, \cdot)$ is a bounded function and

$$v(t+s,x) + \int_0^t P_{t-u}(\Psi_{s+u})(x)\,du = P_t(v_s)(x),$$

where

$$\Psi_u(x) = \Psi(x, v(u,x)). \tag{5.16}$$

Lemma 5.5. *For any $s > 0$, there is a constant $c(s)$ such that for $t \in [0, 1/2)$ and $x, y \in \mathbb{R}^d$,*

$$|v_{t+s}(x) - v_{t+s}(y)| \le c(s)|x-y|.$$

Moreover, $c(s)$ is decreasing in $s > 0$.

Proof. Let $e(s) := \frac{1\wedge s}{2}$. Note that $t + e(s) \in (e(s), 1)$.

$$v(t+s,x) + \int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)+u}(\cdot)))(x)\,du = P_{t+e(s)}(v_{s-e(s)})(x).$$

It follows from (5.7) that there exists a constant c_1 such that for all $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &|P_{t+e(s)}(v_{s-e(s)})(x) - P_{t+e(s)}(v_{s-e(s)})(y)| \\ \le & c\|v_{s-e(s)}\|_\infty((t+e(s))^{-1/2}|x-y|) \wedge 1) \\ \le & c\|v_{s-e(s)}\|_\infty(t+e(s))^{-1/2}|x-y| \\ \le & c\|v_{s/2}\|_\infty(e(s))^{-1/2}|x-y|. \end{aligned} \tag{5.17}$$

Since $v(s-e(s)+u, x) \le v(s-e(s), x) \le v(s/2, x)$, we have for $u > 0$,

$$\|\Psi(\cdot, v_{s-e(s)+u}(\cdot))\|_\infty \le 3K(\|v_{s/2}\|_\infty + \|v_{s/2}\|_\infty^2).$$

Applying Lemma 5.3, we get that there is a constant $c_2 > 0$ such that for $t \in [0, 1/2)$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned} &\left|\int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)+u}(\cdot)))(x)\,du - \int_0^{t+e(s)} P_{t+e(s)-u}(\Psi(\cdot, v_{s-e(s)+u}(\cdot)))(y)\,du\right| \\ \le & c_2 3K(\|v_{s/2}\|_\infty + \|v_{s/2}\|_\infty^2)(|x-y| \wedge 1). \end{aligned} \tag{5.18}$$

The conclusions of the lemma now follow immediately from (5.17) and (5.18). □

Lemma 5.6. *The function $\Psi_u(x)$ given by* (5.16) *satisfies the following two properties:*

(1) For any $u_0 > 0$,

$$\lim_{u\to u_0} \sup_{x\in\mathbb{R}^d} |\Psi_u(x) - \Psi_{u_0}(x)| = 0;$$

(2) For $t_0 \in (0,1)$, there exists a constant $c > 0$ such that for any $|x-x'| \le 1$, $s > t_0$ and $t \in [0, 1/2]$,

$$|\Psi_{s+t}(x) - \Psi_{s+t}(x')| \le c|x-x'|^{\gamma_0}.$$

Proof. (1) For $z_1 < z_2 \in [0, a]$, we can easily check that

$$\begin{aligned} &|\Psi(x,z_1) - \Psi(x,z_2)| \\ \le & |\alpha(x)||z_1 - z_2| + b(x)|z_1^2 - z_2^2| + \int_0^\infty \left|e^{-yz_1} + yz_1 - e^{-yz_2} - yz_2\right| n(x,dy) \\ \le & K(1+2a)|z_1 - z_2| + \int_0^\infty (2 \wedge (ya))y|z_1 - z_2| n(x,dy) \le K(3+3a)|z_1 - z_2|, \end{aligned} \tag{5.19}$$

where in the second inequality above we use the fact that

$$|\frac{d}{dx}(e^{-x}+x)| = 1-e^{-x} \le 2\wedge x.$$

Thus, for $|u-u_0|\le u_0/2$, we have that

$$|\Psi_u(x)-\Psi_{u_0}(x)| \le 3K(1+\|v_{u_0/2}\|_\infty)|v_u(x)-v_{u_0}(x)|. \tag{5.20}$$

Thus, it suffices to show that $t\mapsto v_t(x)$ is continuous on $(0,\infty)$ uniformly in x.

It follows from Lemma 5.5 that, for any $t>0$, $x\mapsto v_t(x)$ is uniformly continuous, thus

$$\lim_{r\downarrow 0}\|P_r v_t - v_t\|_\infty = 0.$$

For $r>0$ and $t>0$, we have that

$$\begin{aligned}|v_t(x)-v_{t+r}(x)| \le& |P_r v_t(x)-v_t(x)| + |\int_0^r P_{r-u}(\Psi_{t+u})(x)\,du|\\ \le& \|P_r v_t - v_t\|_\infty + 3K(\|v_t\|_\infty + \|v_t\|_\infty^2)r \to 0, \quad r\downarrow 0,\end{aligned}$$

where in the last inequality we used (5.1) and the fact that $v_{t+u}(x)\le v_t(x)$.

The proof of $\lim_{r\downarrow 0}\|v_t - v_{t-r}\|_\infty = 0$ is similar and omitted. The proof of part (1) is now complete.

(2) For any $s>t_0$, and $t\in[0,1/2]$, $v(t+s,x)\le \|v_{t_0}\|_\infty$. By our assumption on Ψ, there exist $c_1>0$ and $\gamma_0\in(0,1]$ such that for $|x-y|\le 1$, $s>t_0$ and $t\in[0,1/2]$,

$$|\Psi(x,v_{s+t}(x)) - \Psi(y,v_{s+t}(x))| \le c_1|x-y|^{\gamma_0}.$$

By Lemma 5.5, there exists $c_2=c_2(t_0)$ such that for $s>t_0$ and $t\in[0,1/2]$,

$$|v_{s+t}(x)-v_{s+t}(y)| \le c_2|x-y|.$$

Thus, for $|x-y|\le 1$, $s>t_0$, and $t\in[0,1/2]$,

$$\begin{aligned}&|\Psi_{s+t}(x)-\Psi_{s+t}(y)|\\ \le& |\Psi(x,v_{s+t}(x))-\Psi(y,v_{s+t}(x))| + |\Psi(y,v_{s+t}(x))-\Psi(y,v_{s+t}(y))|\\ \le& |\Psi(x,v_{s+t}(x))-\Psi(y,v_{s+t}(x))| + 3K(1+\|v_{t_0}\|_\infty)|v_{s+t}(x)-v_{s+t}(y)|\\ \le& c_1|x-y|^{\gamma_0} + 3K(1+\|v_{t_0}\|_\infty)c_2|x-y|\\ \le& c_3|x-y|^{\gamma_0}.\end{aligned} \tag{5.21}$$

The proof of (2) is now complete. □

Proof of Proposition 5.1: For any $t,s>0$,

$$v(t+s,x) + \int_0^t P_{t-u}(\Psi_{s+u})(x)\,du = P_t(v_s)(x).$$

Thus, combining Lemmas 5.2, 5.4 and 5.6, (5.3) follows immediately.

For fixed $t\in(0,1/2)$, we deal with the three parts on right hand side of (5.3) separately.

Since $t\to v(t,x)$ is continuous, the function $s\to\Psi_{t+s}(x)=\Psi(x,v(t+s,x))$ is continuous and, by (5.1),

$$\sup_{s>t_0}|\Psi_{t+s}(x)| \le 3K(\|v_{t_0}\|_\infty + \|v_{t_0}\|_\infty^2) < \infty. \tag{5.22}$$

By (5.4),

$$\sup_{s>t_0} |\frac{\partial}{\partial t} P_t(v_s)(x)| \le c_4 \|v_{t_0}\|_\infty t^{-1} < \infty. \tag{5.23}$$

By (5.14) and Lemma 5.6 (2), we get that, for any $s > t_0$,

$$\sup_{s>t_0} \sup_{x\in\mathbb{R}^d} |\int_0^t \frac{\partial}{\partial t} P_{t-u}(\Psi_{s+u})(x)\, du| < \infty. \tag{5.24}$$

Combining (5.22) -(5.24), we get that, for $t_0 > 0$,

$$\sup_{s>t_0} \sup_{x\in\mathbb{R}^d} w(t+s, x) < \infty,$$

which implies that, for any $s_0 > 0$, $\sup_{s>s_0} \sup_{x\in\mathbb{R}^d} w(s,x) < \infty$. □

Let L be as in Example 2. Let E be a bounded smooth domain in $\mathbb{R}^d$ and let $p(t,x,y)$ be the Dirichlet heat kernel of L in E. It follows from [10, Theorem 2.1, p. 247] that there exist $c_i > 0, i = 1,2,3,4$, such that for all $t \in (0,1]$,

$$\left|\frac{\partial}{\partial t} p(t,x,y)\right| \le c_1 t^{-\frac{d}{2}-1} e^{-\frac{c_2|x-y|^2}{t}}, \quad \text{and}$$

$$|\nabla_x p(t,x,y)| \le c_3 t^{-\frac{d+1}{2}} e^{-\frac{c_4|x-y|^2}{t}}.$$

Using these instead of (5.5) and (5.6), and repeating the arguments for Example 4, we can get the following example.

Example 5. Assume that E be is bounded smooth domain in $\mathbb{R}^d$ and that the spatial motion is ξ^E, which is the diffusion ξ of Example 2 killed upon exiting E. The branching mechanism Ψ is of the form in (2.1) and satisfies **(C1)**. Then the (ξ^E, Ψ)-superprocess X satisfies Assumptions **(H1)** and **(H2)**. Using the same argument as in Example 2, one can see that under the condition that there exist $\alpha \in (1,2]$ and $c > 0$ such that $\Psi(x,z) \ge cz^\alpha$ for all $x \in E$, Assumption **(H3)** is satisfied.

5.3 Examples of some superprocesses with discontinuous spatial motion

Now we give an example of a superprocess with discontinuous spatial motion and general branching mechanism such that Assumptions **(H1)** and **(H2)** are satisfied.

Example 6. Suppose that $B = \{B_t\}$ is a Brownian motion in $\mathbb{R}^d$ and $S = \{S_t\}$ is an independent subordinator with Laplace exponent φ, that is

$$\mathbb{E} e^{-\lambda S_t} = e^{-t\varphi(\lambda)}, \qquad t > 0,\ \lambda > 0.$$

The process $\xi_t = B_{S_t}$ is called a subordinate Brownian motion in $\mathbb{R}^d$. Subordinate Brownian motions form a large class of Lévy processes. When S is an $(\alpha/2)$-stable subordinator, that is, $\varphi(\lambda) = \lambda^{\alpha/2}$, ξ is a symmetric α-stable process in $\mathbb{R}^d$. Suppose that Ψ is of the form (2.1) and satisfies **(C1)**. Suppose further that φ satisfies the following conditions:

1. $\int_0^1 \frac{\varphi(r^2)}{r} dr < \infty$.
2. There exist constants $\delta \in (0,2]$ and $a_1 \in (0,1)$ such that

$$a_1 \lambda^{\delta/2} \varphi(r) \le \varphi(\lambda r), \qquad \lambda \ge 1, r \ge 1. \tag{5.25}$$

then X satisfies Assumptions **(H1)** and **(H2)**.

Condition (5.25) can be rewritten in the form

$$\frac{\varphi(\lambda r)}{\varphi(r)} \geq a_1 \lambda^{\delta/2}, \qquad \lambda \geq 1, r \geq 1,$$

and so it is a very weak lower scaling condition at infinity for φ.

As we have seen in the paragraph before Example 2, Example 6 does not satisfy Assumption **(H3)**.

Proposition 5.7. *Assume that the conditions in Example 6 hold. The function $t \to v_t(x)$ is differentiable in $(0, \infty)$, and for any $s > 0$ and $t \in [0, 1/2)$, $w(t+s, x) = -\frac{\partial}{\partial t} v_{t+s}(x)$ satisfies that*

$$w(t+s, x) = -\frac{\partial}{\partial t} P_t(v_s)(x) + \int_0^t \frac{\partial}{\partial t} P_{t-u}(\Psi_{s+u})(x)\, du + \Psi_{t+s}(x). \tag{5.26}$$

Moreover, $t \to w(t, x)$ is continuous and for any $s_0 > 0$, $\sup_{s>s_0} \sup_{x \in \mathbb{R}^d} w(s, x) < \infty$.

In the following, we will give several lemmas which are similar to those in the proof of Proposition 5.1.

Now we proceed to prove the second assertion of the example above. The arguments are similar to that for the second assertion of Example 4. Without loss of generality, we will assume that $\varphi(1) = 1$. First we introduce some notation. Put $\Phi(r) = \varphi(r^2)$ and let Φ^{-1} be the inverse function of Φ. For $t > 0$ and $x \in \mathbb{R}^d$, we define

$$\rho(t, x) := \Phi\left(\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-1}\right)\left(\frac{1}{\Phi^{-1}(t^{-1})} + |x|\right)^{-d}.$$

For $t > 0$, $x \in \mathbb{R}^d$ and $\beta, \gamma \in \mathbb{R}$, we define

$$\rho_\gamma^\beta(t, x) := \Phi^{-1}(t^{-1})^{-\gamma}(|x|^\beta \wedge 1)\rho(t, x), \quad t > 0, x \in \mathbb{R}^d.$$

Let $p(t, x, y) = p(t, x-y)$ be the transition density of ξ and let $\{P_t : t \geq 0\}$ be the transition semigroup of ξ. It is well known that $\{P_t : t \geq 0\}$ satisfies the strong Feller property, that is, for any $t > 0$, P_t maps bounded Borel functions on $\mathbb{R}^d$ to bounded continuous functions on $\mathbb{R}^d$.

Now we list some other properties of the semigroup $\{P_t : t \geq 0\}$ which will be used later.

Lemma 5.8. *For $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the function $t \to P_t f(x)$ is differentiable on $(0, \infty)$. Furthermore, there exists a constant c such that for any $t \in (0, 1]$, $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$,*

$$\left|\frac{\partial}{\partial t} P_t f(x)\right| \leq c\|f\|_\infty t^{-1}. \tag{5.27}$$

Proof. For $t \in (n, n+1]$, $P_t f(x) = P_{t-n}(P_n f)(x)$. Thus, we only need to prove the differentiability for $t \in (0, 1]$. It follows from [13, Lemma 3.1(a) and Theorem 3.4] that

$$\left|\frac{\partial}{\partial t} p(t, x)\right| \leq c_1 \rho(t, x). \tag{5.28}$$

By [13, Lemma 2.6(a)], we have

$$\int_{\mathbb{R}^d} \rho(t, x) dx < c_2 t^{-1}, \qquad t \in (0, 1]. \tag{5.29}$$

Thus by the dominated convergence theorem we have that for all $t \in (0, 1]$ and $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t} P_t f(x) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} p(t, x, y) f(y)\, dy,$$

and that for all $t \in (0,1]$, $x \in \mathbb{R}^d$ and bounded Borel function f on $\mathbb{R}^d$,

$$\left| \frac{\partial}{\partial t} P_t f(x) \right| \le c_3 \|f\|_\infty t^{-1}.$$

The proof is now complete. □

Lemma 5.9. *Assume that $f_s(x)$ is uniformly bounded in $(s,x) \in [0,1] \times \mathbb{R}^d$. Then there is a constant c such that for any $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,*

$$\left| \int_0^t P_{t-s} f_s(x)\, ds - \int_0^t P_{t-s} f_s(x')\, ds \right| \le cL(|x-x'|^{\delta/2} \wedge 1).$$

Proof. It follows from [13, Proposition 3.3] that there exists a constant $c_1 > 0$ such that for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,

$$|p(t,x) - p(t,x')| \le c_1 \big((\Phi^{-1}(t^{-1})|x-x'|) \wedge 1 \big) t \left(\rho(t,x) + \rho(t,x') \right) . \tag{5.30}$$

Thus using (5.29) we get that for any $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$,

$$\left| \int_0^t P_{t-s} f_s(x)\, ds - \int_0^t P_{t-s} f_s(x')\, ds \right| \le c_2 L \int_0^t \big((\Phi^{-1}(s^{-1})|x-x'|) \wedge 1 \big)\, ds. \tag{5.31}$$

When $|x-x'| < 1$, $\Phi(|x-x'|^{-1}) \ge \Phi(1) = 1$. Thus,

$$\int_0^t \big((\Phi^{-1}(s^{-1})|x-x'|) \wedge 1 \big)\, ds \le |x-x'| \int_{(\Phi(|x-x'|^{-1}))^{-1}}^1 \Phi^{-1}(s^{-1}) ds + \big(\Phi(|x-x'|^{-1}) \big)^{-1}.$$

It is well known that φ, the Laplace exponent of a subordinator, satisfies

$$\varphi(\lambda r) \le \lambda \varphi(r), \qquad \lambda \ge 1, r > 0.$$

Using this, we immediately get that

$$\Phi^{-1}(\lambda r) \ge \lambda^{1/2} \Phi^{-1}(r), \qquad \lambda \ge 1, r > 0.$$

For $s \in [(\Phi(|x-x'|^{-1})^{-1}, 1]$, by taking $r = s^{-1}$ and $\lambda = s\Phi(|x-x'|^{-1})$ in the display above, we get

$$\Phi^{-1}(s^{-1}) \le |x-x'|^{-1} s^{-1/2} (\Phi(|x-x'|^{-1}))^{-1/2}.$$

Therefore

$$\begin{aligned} &|x-x'| \int_{(\Phi(|x-x'|^{-1}))^{-1}}^1 \Phi^{-1}(s^{-1}) ds \\ &\le (\Phi(|x-x'|^{-1}))^{-1/2} \int_{(\Phi(|x-x'|^{-1}))^{-1}}^1 s^{-1/2} ds \le c_3 (\Phi(|x-x'|^{-1}))^{-1/2}. \end{aligned}$$

Consequently for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| < 1$, we have

$$\int_0^t ((\Phi^{-1}(t^{-1})|x-x'|) \wedge 1)\, ds \le c_4 (\Phi(|x-x'|^{-1}))^{-1/2}.$$

By taking $r = 1$ and $\lambda = |x-x'|^{-1}$ in (5.25), we get

$$a_1 |x-x'|^{-\delta} \le \Phi(|x-x'|^{-1}).$$

Thus for all $t \in (0,1]$ and $x, x' \in \mathbb{R}^d$ with $|x-x'| < 1$, we have

$$\int_0^t ((\Phi^{-1}(s^{-1})|x-x'|) \wedge 1)\, ds \le c_4 a_1^{-1/2} |x-x'|^{\delta/2}. \tag{5.32}$$

Combining (5.31) and (5.32), we immediately get the desired conclusion. □

Lemma 5.10. *Assume that $f_s(x)$ satisfies the assumptions of Lemma 5.4 with $\gamma \in (0, \delta/2]$. Then, $t \to \int_0^t P_{t-s} f_s(x)\, ds$ is differentiable on $(0, s_0)$, and for $0 \le t < s_0$,*

$$\frac{\partial}{\partial t} \int_0^t P_{t-s} f_s(x)\, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x)\, ds + f_t(x). \tag{5.33}$$

Proof. Let $G(t,x) := \int_0^t P_{t-s} f_s(x)\, ds$. For any $0 < t < t + r < s_0$, by the definition of $G(t,x)$,

$$\begin{aligned}
&\frac{1}{r}\left(G(t+r,x) - G(t,x)\right) \\
=&\frac{1}{r}\int_0^t \Big(P_{t+r-s} f_s(x) - P_{t-s} f_s(x)\Big)\, ds + \frac{1}{r}\int_t^{t+r} P_{t+r-s} f_s(x)\, ds \\
=&\int_0^t \frac{P_{t+r-s} f_s(x) - P_{t-s} f_s(x)}{r}\, ds + \frac{1}{r}\int_0^r P_{r-s} f_{t+s}(x)\, ds \\
=:&(I) + (II).
\end{aligned}$$

Using the same arguments as those leading to (5.11), we get

$$\lim_{t\downarrow 0} t^{-1} \int_0^t P_{t-s} f_s(x)\, ds = f_0(x),$$

which implies that

$$\lim_{r\downarrow 0} (II) = f_t(x). \tag{5.34}$$

Now we deal with part (I). For $0 < t < t + r < s_0$, using (5.28), we obtain that

$$\begin{aligned}
&\left|\frac{P_{t+r-s} f_s(x) - P_{t-s} f_s(x)}{r}\right| \\
=&\left|\int_{\mathbb{R}^d} \frac{p(t+r-s,x,y) - p(t-s,x,y)}{r} (f_s(y) - f_s(x))\, dy\right| \\
\le& c_3 \int_{\mathbb{R}^d} |f_s(y) - f_s(x)| \rho(t-s, x-y)\, dy \\
\le& c_4 \int_{\mathbb{R}^d} \rho_0^\gamma(t-s, x-y)\, dy \\
\le& c_5 (t-s)^{-1} \Phi^{-1}((t-s)^{-1})^{-\gamma},
\end{aligned} \tag{5.35}$$

where in the last inequality we used [13, Lemma 2.6(a)]. It follows from [13, Lemma 2.3] that

$$\int_0^t (t-s)^{-1} \Phi^{-1}((t-s)^{-1})^{-\gamma} ds \le c_6 \Phi^{-1}(t^{-1})^{-\gamma}. \tag{5.36}$$

Thus, using the dominated convergence theorem, we get that, for any $0 \le t < t + r < s_0$,

$$\lim_{r\downarrow 0} (I) = \int_0^t \lim_{r\downarrow 0} \frac{P_{t+r-s} f_s(x) - P_{t-s} f_s(x)}{r}\, ds = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x)\, ds. \tag{5.37}$$

Combining (5.34) and (5.37), we get that

$$\lim_{r\downarrow 0} \frac{G(t+r,x) - G(t,x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x)\, ds + f_t(x).$$

Using similar arguments, we can also show that

$$\lim_{r\downarrow 0} \frac{G(t,x) - G(t-r,x)}{r} = \int_0^t \frac{\partial}{\partial t} P_{t-s} f_s(x)\, ds + f_t(x).$$

Thus, (5.33) follows immediately. The proof is now complete. □

Lemma 5.11. *For any $s > 0$, there is a constant $c(s)$ such that for $t \in [0, 1/2)$ and $x, y \in \mathbb{R}^d$,*

$$|v_{t+s}(x) - v_{t+s}(y)| \le c(s)|x - y|^{\delta/2}.$$

Moreover, $c(s)$ is decreasing in $s > 0$.

Proof. The proof of this lemma is similar as that of Lemma 5.5. We use Lemma 5.9 instead of Lemma 5.3. Here we omit the details. □

Lemma 5.12. *The function $\Psi_u(x)$ satisfies the following two properties:*

(1) For any $u_0 > 0$,

$$\lim_{u \to u_0} \sup_{x \in \mathbb{R}^d} |\Psi_u(x) - \Psi_{u_0}(x)| = 0;$$

(2) For $t_0 \in (0, 1)$, there exists a constant $c > 0$ and $\gamma_1 \in (0, \delta/2]$ such that for any $|x - x'| \le 1$, $s > t_0$ and $t \in [0, 1/2]$,

$$|\Psi_{s+t}(x) - \Psi_{s+t}(x')| \le c|x - x'|^{\gamma_1}.$$

Proof. The proof of part (1) is exactly the same as that of part (1) of Lemma 5.6.

Using arguments similar to that in the proof of part (2) of Lemma 5.6 and using Lemma 5.11 instead of Lemma 5.5, we can get the result in part (2). Here we omit the details. □

Proof of Proposition 5.7: Combining Lemmas 5.8, 5.10 and 5.12, and using arguments similar to that in the proof of Proposition 5.1, Proposition 5.7 follows immediately. □

Remark 5.13. Actually, by the same arguments and the results from [13], one checks that in the example above, we could have replaced the subordinate Brownian motion by the non-symmetric jump process considered there, which contains the non-symmetric stable-like process discussed in [3].

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