Acta Appl Math (2017) 147:137–175
DOI 10.1007/s10440-016-0072-3

Functional Central Limit Theorems for Supercritical Superprocesses

Yan-Xia Ren[1] · **Renming Song**[2] · **Rui Zhang**[3]

Received: 25 August 2015 / Accepted: 9 September 2016 / Published online: 3 October 2016

Abstract In this paper, we establish some functional central limit theorems for a large class of general supercritical superprocesses with spatially dependent branching mechanisms satisfying a second moment condition. In the particular case when the state E is a finite set and the underlying motion is an irreducible Markov chain on E, our results are superprocess analogs of the functional central limit theorems of Janson (Stoch. Process. Appl. 110:177–245, 2004) for supercritical multitype branching processes. The results of this paper are refinements of the central limit theorems in Ren et al. (Stoch. Process. Appl. 125:428–457, 2015).

Keywords Functional central limit theorem · Supercritical superprocess · Excursion measures of superprocesses

Mathematics Subject Classification Primary 60J68 · Secondary 60F05 · 60G57 · 60J45

1 Introduction

Kesten and Stigum [22, 23] initiated the study of central limit theorems for supercritical branching processes. In these two papers, they established central limit theorems for su-

The research of Y.-X. Ren is supported by NSFC (Grant No. 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education. The research of R. Song is supported in part by a grant from the Simons Foundation (208236).

✉ R. Zhang
zhangrui27@cnu.edu.cn

Y.-X. Ren
yxren@math.pku.edu.cn

R. Song
rsong@math.uiuc.edu

1 LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, P.R. China

2 Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

3 School of Mathematical Sciences, Capital Normal University, Beijing 100048, P.R. China

percritical multitype Galton-Watson processes by using the Jordan canonical form of the mean matrix. Then in [5–7], Athreya proved central limit theorems for supercritical multitype continuous time branching processes, also using the Jordan canonical form of the mean matrix. Asmussen and Keiding [4] used martingale central limit theorems to prove central limit theorems for supercritical multitype branching processes. In [3], Asmussen and Hering established spatial central limit theorems for general supercritical branching Markov processes under a certain condition. In [21], Janson extended the results of [5–7, 22, 23] and established functional central limit theorems for multitype branching processes. In [21, Remark 4.1], Janson mentioned the possibility of extending his functional central limit theorems to the case of infinitely many types (with suitable assumptions). However, he ended this remark with the following sentence: "It is far from clear how such an extension should be formulated, and we have not pursued this".

The recent study of spatial central limit theorems for branching Markov processes started with [1]. In this paper, Adamczak and Miłoś proved some central limit theorems for supercritical branching Ornstein-Uhlenbeck processes with binary branching mechanism. In [2], Adamczak and Miłoś obtained a strong law of large numbers and central limit theorems of U-statistics of the OU branching system. In [31], Miłoś proved some central limit theorems for some supercritical super diffusions with branching mechanisms satisfying a fourth moment condition. In [32], we established central limit theorems for supercritical super Ornstein-Uhlenbeck processes with branching mechanisms satisfying only a second moment condition. More importantly, compared with the results of [1, 31], the central limit theorems in [32] are more satisfactory since our limit normal random variables are non-degenerate. In [33], we sharpened and generalized the spatial central limit theorems mentioned above, and obtained central limit theorems for a large class of general supercritical branching symmetric Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. In [34], we obtained central limit theorems for a large class of general supercritical superprocesses with symmetric spatial motions and with spatially dependent branching mechanisms satisfying only a second moment condition. Furthermore, we also obtained the covariance structure of the limit Gaussian field in [34]. In [35], we extended the results of [33] to supercritical branching nonsymmetric Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition.

The main purpose of this paper is to establish *functional* central limit theorems, for supercritical superprocesses with spatially dependent branching mechanisms satisfying only a second moment condition, similar to those of [21], for supercritical multitype branching processes. For critical branching Markov processes starting from a Poisson random field or an equilibrium distribution, and subcritical branching Markov processes with immigration, some functional central limit theorems for the occupation times were established in a series of papers, see, for instance, [8–10, 27–30] and reference therein. The first functional central limit theorem for the occupation times of critical superprocesses was given in Iscoe [19], and then generalized in [11]. The functional central limit theorem for the occupation time process of critical super α-stable processes, and the functional central limit theorem for the occupation time process of critical super-Brownian motion with immigration, where the immigration was governed by the Lebesgue measure or a super-Brownian motion, were established in [18, 26, 36]. However, up to now, no spatial *functional* central limit theorems have been established for general supercritical superprocesses. For simplicity, we will assume the spatial process is symmetric. One could combine the techniques of this paper with that of [35] to extend the results of this paper to the case when the spatial motion is not symmetric. We leave this to the interested reader.

The organization of this paper is as follows. In the remainder of this section, we spell out our assumptions and present our main result. Section 2 contains some preliminary results, while the proof of the main result is given in Sect. 3.

1.1 Spatial Process

Our assumptions on the underlying spatial process are the same as in [33]. In this subsection, we recall the assumptions on the spatial process.

E is a locally compact separable metric space and m is a σ-finite Borel measure on E with full support. ∂ is a point not contained in E and will be interpreted as the cemetery point. Every function f on E is automatically extended to $E_\partial := E \cup \{\partial\}$ by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is an m-symmetric Hunt process on E. The semigroup of ξ will be denoted by $\{P_t : t \geq 0\}$. We will always assume that there exists a family of continuous strictly positive symmetric functions $\{p_t(x, y) : t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p_t(x, y) f(y) m(dy).$$

It is well-known that for $p \geq 1$, $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $L^p(E, m)$.

Define $\widetilde{a}_t(x) := p_t(x, x)$. We will always assume that $\widetilde{a}_t(x)$ satisfies the following two conditions:

(a) For any $t > 0$, we have

$$\int_E \widetilde{a}_t(x) m(dx) < \infty.$$

(b) There exists $t_0 > 0$ such that $\widetilde{a}_{t_0}(x) \in L^2(E, m)$.

It is easy to check (see [33]) that condition (b) above is equivalent to

(b′) There exists $t_0 > 0$ such that for all $t \geq t_0$, $\widetilde{a}_t(x) \in L^2(E, m)$.

These two conditions are satisfied by a lot of Markov processes. In [33], we gave several classes of examples of Markov processes satisfying these two conditions.

1.2 Superprocesses

Our basic assumptions on the superprocess are the same as in [34]. In this subsection, we recall these assumptions. Let $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) be the set of (nonnegative) bounded Borel functions on E.

The superprocess $X = \{X_t : t \geq 0\}$ is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E satisfying the assumptions of the previous subsection, a branching rate function $\beta(x)$ on E which is a nonnegative bounded Borel function and a branching mechanism ψ of the form

$$\psi(x, \lambda) = -a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} \big(e^{-\lambda y} - 1 + \lambda y\big) n(x, dy), \quad x \in E,\ \lambda > 0, \tag{1.1}$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0, \infty)$ satisfying

$$\sup_{x \in E} \int_0^\infty y^2 n(x, dy) < \infty. \tag{1.2}$$

Let $\mathcal{M}_F(E)$ be the space of finite measures on E, equipped with topology of weak convergence. The superprocess X is a Markov process taking values in $\mathcal{M}_F(E)$. The existence of such superprocesses is well-known, see, for instance, [15] or [25]. As usual, $\langle f,\mu\rangle := \int f(x)\mu(dx)$ and $\|\mu\| := \langle 1,\mu\rangle$. According to [25, Theorem 5.12], there is a Borel right process $X = \{\Omega, \mathcal{G}, \mathcal{G}_t, X_t, \mathbb{P}_\mu\}$ taking values in $\mathcal{M}_F(E)$ such that for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu\big(e^{-\langle f, X_t\rangle}\big) = \big\langle u_f(\cdot,t),\mu\big\rangle, \tag{1.3}$$

where $u_f(x,t)$ is the unique positive solution to the equation

$$u_f(x,t) + \Pi_x\int_0^t \psi\big(\xi_s, u_f(\xi_s, t-s)\big)\beta(\xi_s)ds = \Pi_x f(\xi_t), \tag{1.4}$$

where $\psi(\partial,\lambda) = 0, \lambda > 0$. By the definition of Borel right processes (see [25, Definition A.18]), $(\mathcal{G},\mathcal{G}_t)_{t\geq 0}$ are augmented, $(\mathcal{G}_t : t \geq 0)$ is right continuous and X satisfies the Markov property with respect to $(\mathcal{G}_t : t \geq 0)$. Moreover, such a superprocess X has a Hunt realization in $\mathcal{M}_F(E)$, see [25, Theorem 5.12]. In this paper, the superprocess we deal with is always this Hunt realization.

Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and } A(x) := \beta(x)\left(2b(x) + \int_0^\infty y^2 n(x,dy)\right). \tag{1.5}$$

Then, by our assumptions, $\alpha(x) \in \mathcal{B}_b(E)$ and $A(x) \in \mathcal{B}_b(E)$. Thus there exists $K > 0$ such that

$$\sup_{x\in E}\big(|\alpha(x)| + A(x)\big) \leq K. \tag{1.6}$$

For any $f \in \mathcal{B}_b(E)$ and $(t,x) \in (0,\infty)\times E$, define

$$T_t f(x) := \Pi_x\big[e^{\int_0^t \alpha(\xi_s)ds} f(\xi_t)\big]. \tag{1.7}$$

It is well-known that $T_t f(x) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$ for every $x \in E$.

It is shown in [33] that there exists a family of continuous strictly positive symmetric functions $\{q_t(x,y), t > 0\}$ on $E\times E$ such that $q_t(x,y) \leq e^{Kt}p_t(x,y)$ and for any $f \in \mathcal{B}_b(E)$,

$$T_t f(x) = \int_E q_t(x,y) f(y) m(dy).$$

It follows immediately that, for any $p \geq 1$, $\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $L^p(E,m)$ and

$$\|T_t f\|_p \leq e^{Kt}\|f\|_p. \tag{1.8}$$

Define $a_t(x) := q_t(x,x)$. It follows from the assumptions (a) and (b) in the previous subsection that a_t enjoys the following properties:

(i) For any $t > 0$, we have

$$\int_E a_t(x) m(dx) < \infty.$$

(ii) There exists $t_0 > 0$ such that for all $t \geq t_0$, $a_t(x) \in L^2(E,m)$.

By Hölder's inequality, we get

$$q_t(x,y) = \int_E q_{t/2}(x,z)q_{t/2}(z,y)m(dz) \le a_t(x)^{1/2}a_t(y)^{1/2}.$$

Since $q_t(x,y)$ and $a_t(x)$ are continuous in $x \in E$, by the dominated convergence theorem, we get that, if $f \in L^2(E,m)$, $T_t f(\cdot)$ is continuous for any $t > 0$.

It follows from (i) above that, for any $t > 0$, T_t is a compact operator. The infinitesimal generator $\mathcal{L}$ of $\{T_t : t \ge 0\}$ in $L^2(E,m)$ has purely discrete spectrum with eigenvalues $-\lambda_1 > -\lambda_2 > -\lambda_3 > \cdots$. It is known that either the number of these eigenvalues is finite, or $\lim_{k\to\infty} \lambda_k = \infty$. The first eigenvalue $-\lambda_1$ is simple and the eigenfunction ϕ_1 associated with $-\lambda_1$ can be chosen to be strictly positive everywhere and continuous. We will assume that $\|\phi_1\|_2 = 1$. ϕ_1 is sometimes denoted as $\phi_1^{(1)}$. For $k > 1$, let $\{\phi_j^{(k)}, j = 1, 2, \dots n_k\}$ be an orthonormal basis of the eigenspace associated with $-\lambda_k$. It is well-known that $\{\phi_j^{(k)}, j = 1, 2, \dots n_k; k = 1, 2, \dots\}$ forms a complete orthonormal basis of $L^2(E,m)$ and all the eigenfunctions are continuous. For any $k \ge 1$, $j = 1, \dots, n_k$ and $t > 0$, we have $T_t\phi_j^{(k)}(x) = e^{-\lambda_k t}\phi_j^{(k)}(x)$ and

$$e^{-\lambda_k t/2}\left|\phi_j^{(k)}\right|(x) \le a_t(x)^{1/2}, \quad x \in E. \tag{1.9}$$

It follows from the relation above that all the eigenfunctions $\phi_j^{(k)}$ belong to $L^4(E,m)$. The basic facts recalled in this paragraph are well-known, for instance, one can refer to [13, Sect. 2].

In this paper, we always assume that the superprocess X is supercritical, that is, $\lambda_1 < 0$.

In this paper, we also assume that, for any $t > 0$ and $x \in E$,

$$\mathbb{P}_{\delta_x}\left\{\|X_t\| = 0\right\} \in (0,1). \tag{1.10}$$

Here is a sufficient condition for (1.10). Suppose that $\Phi(z) = \inf_{x\in E}(\psi(x,z)\beta(x))$ can be written in the form:

$$\Phi(z) = \widetilde{a}z + \widetilde{b}z^2 + \int_0^\infty \left(e^{-zy} - 1 + zy\right)\widetilde{n}(dy)$$

with $\widetilde{a} \in \mathbb{R}$, $\widetilde{b} \ge 0$ and $\widetilde{n}$ being a measure on $(0,\infty)$ satisfying $\int_0^\infty (y \wedge y^2)\widetilde{n}(dy) < \infty$. If $\widetilde{b} + \widetilde{n}(0,\infty) > 0$ and $\Phi(z)$ satisfies

$$\int^\infty \frac{1}{\Phi(z)}dz < \infty, \tag{1.11}$$

then (1.10) holds. For the last claim, see, for instance, [14, Lemma 11.5.1].

1.3 Main Result

Let $\mathcal{M}_C(E)$ be the space of finite measure on E with compact support. We will use $(\cdot,\cdot)_m$ to denote inner product in $L^2(E,m)$. Any $f \in L^2(E,m)$ admits the following expansion:

$$f(x) = \sum_{k=1}^\infty \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x),$$

where $a_j^k = (f, \phi_j^{(k)})_m$ and the series converges in $L^2(E, m)$. a_1^1 will sometimes be written as a_1. For $f \in L^2(E, m)$, define

$$\gamma(f) := \inf\{k \ge 1 : \text{ there exists } j \text{ with } 1 \le j \le n_k \text{ such that } a_j^k \ne 0\},$$

where we use the usual convention $\inf \varnothing = \infty$. We note that if $f \in L^2(E, m)$ is nonnegative and $m(x : f(x) > 0) > 0$, then $(f, \phi_1)_m > 0$, which implies $\gamma(f) = 1$.

Define

$$H_t^{k,j} := e^{\lambda_k t}\langle \phi_j^{(k)}, X_t\rangle, \quad t \ge 0.$$

In [34, Lemma 1.1], it has been proved that, for any nonzero $\mu \in \mathcal{M}_C(E)$, $H_t^{k,j}$ is a martingale under $\mathbb{P}_\mu$. Moreover, if $\lambda_1 > 2\lambda_k$, then $\sup_{t>3t_0} \mathbb{P}_\mu(H_t^{k,j})^2 < \infty$. Thus the limit

$$H_\infty^{k,j} := \lim_{t\to\infty} H_t^{k.j}$$

exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$.

In particular, we write $W_t := H_t^{1,1} = e^{\lambda_1 t}\langle \phi_1, X_t\rangle$ and $W_\infty := H_\infty^{1,1}$. $\{W_t : t \ge 0\}$ is a nonnegative martingale and

$$W_t \to W_\infty, \quad \mathbb{P}_\mu\text{-a.s. and in } L^2(\mathbb{P}_\mu).$$

Thus W_∞ is non-degenerate. Moreover, we have $\mathbb{P}_\mu(W_\infty) = \langle \phi_1, \mu\rangle$. Put $\mathcal{E} = \{W_\infty = 0\}$, then $\mathbb{P}_\mu(\mathcal{E}) < 1$. It is clear that $\mathcal{E}^c \subset \{X_t(E) > 0, \forall t \ge 0\}$.

When one considers limiting behaviors of X, the first question to ask is the behavior of $\langle f, X_t\rangle$ with f being some nonnegative bounded Borel function, especially the case $f = I_K$ with K being a compact subset of E. It follows from [34, Remark 1.3] that for $f \in L^2(E, m) \cap L^4(E, m)$,

$$\lim_{t\to\infty} e^{\lambda_1 t}\langle f, X_t\rangle = (f, \phi_1)_m W_\infty \quad \text{in } L^2(\mathbb{P}_\mu).$$

In particular, the convergence also holds in $\mathbb{P}_\mu$-probability. In [34, Theorem 1.4], we also discussed the central limit theorems of $\langle f, X_t\rangle$, see Lemma 1.1 below. Similar types of results were established for branching Markov processes in [33, 35]. For a branching Markov process Z_t, considering the proper scaling limit of $\langle f, Z_t\rangle$ as $t \to \infty$ is equivalent to considering the scaling limit of $\langle f, Z_{t+s}\rangle$ as $s \to \infty$ for any $t > 0$. Note that $Z_{t+s} = \sum_{u\in\mathcal{L}_t} Z_s^{u,t}$, where $\mathcal{L}_t$ is the set of particles alive at time t and $Z_s^{u,t}$ is the branching Markov process starting from the particle $u \in \mathcal{L}_t$. Thus, conditioned on Z_t, Z_{t+s} is the sum of a finite number of independent terms and so we are basically considering central limit theorems for sums of independent random variables. This is one of the reasons that the results of [33, 35] can be considered central limit theorems. In the case of superprocesses, even though the particle picture is less clear, the main results of [32, 34] can also be considered central limit theorems by analogy with those of [33, 35]. The purpose of this paper is to establish the functional version of the central limit theorems of [34], that is, *functional* central limit theorems.

The following three subspaces of $L^2(E, m)$ will be needed in the statement of the main result:

$$\mathcal{C}_l := \left\{ g(x) = \sum_{k:\lambda_1>2\lambda_k} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : b_j^k \in \mathbb{R} \right\},$$

$$\mathcal{C}_c := \left\{ g(x) = \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : 2\lambda_k = \lambda_1, b_j^k \in \mathbb{R} \right\}$$

and

$$\mathcal{C}_s := \left\{ g(x) \in L^2(E, m) \cap L^4(E, m) : \lambda_1 < 2\lambda_{\gamma(g)} \right\}.$$

The space $\mathcal{C}_l$ consists of the functions in $L^2(E, m)$ that only have nontrivial projections onto the eigen-spaces corresponding to those "large" eigenvalues $-\lambda_k$ satisfying $\lambda_1 > 2\lambda_k$. The space $\mathcal{C}_l$ is of finite dimension. The space $\mathcal{C}_c$ is the (finite dimensional) eigen-space corresponding to the "critical" eigenvalue $-\lambda_k$ with $\lambda_1 = 2\lambda_k$. Note that there may not be a critical eigenvalue and $\mathcal{C}_c$ is empty in this case. The space $\mathcal{C}_s$ consists of the functions in $L^2(E, m) \cap L^4(E, m)$ that only have nontrivial projections onto the eigen-spaces corresponding to those "small" eigenvalues $-\lambda_k$ satisfying $\lambda_1 < 2\lambda_k$. The space $\mathcal{C}_s$ is of infinite dimension in general.

Fix a $q > \max\{K, -2\lambda_1\}$. For any $p \geq 1$ and $f \in L^p(E, m)$, define

$$U_q|f|(x) := \int_0^\infty e^{-qs} T_s\big(|f|\big)(x) ds, \quad x \in E.$$

Then,

$$\left(\int_E \big(U_q|f|(x)\big)^p m(dx) \right)^{1/p} \leq \int_0^\infty e^{-qs} \big\| T_s\big(|f|\big) \big\|_p ds \leq \int_0^\infty e^{-qs} e^{Ks} ds \|f\|_p < \infty, \tag{1.12}$$

which implies that $U_q|f| \in L^p(E, m)$. Let f^+ and f^- be the positive part and negative part of f respectively. For any $x \in E$ with $U_q|f|(x) < \infty$, we define

$$U_q f(x) := \int_0^\infty e^{-qs} T_s f(x) ds = U_q\big(f^+\big)(x) - U_q\big(f^-\big)(x),$$

otherwise we define $U_q f(x)$ be an arbitrary real number. It follows from (1.12) that U_q is a bounded linear operator on $L^p(E, m)$. Notice that

$$U_q\big(\phi_j^{(k)}\big)(x) = (q + \lambda_k)^{-1} \phi_j^{(k)}(x).$$

One can easily check that, for $f \in L^2(E, m)$, $\gamma(U_q f) = \gamma(f)$. In fact, by Fubini's theorem, we have

$$\big(U_q f, \phi_j^{(k)}\big)_m = \int_0^\infty e^{-qu} \big(T_u f, \phi_j^{(k)}\big)_m du = (q + \lambda_k)^{-1} \big(f, \phi_j^{(k)}\big)_m. \tag{1.13}$$

For any $f \in L^2(E, m)$, the random variable $\langle U_q|f|, X_t \rangle \in [0, \infty]$ is well defined. Since μ has compact support and $T_t(U_q|f|)$ is continuous, $\mathbb{P}_\mu(\langle U_q|f|, X_t \rangle) = \langle T_t(U_q|f|), \mu \rangle < \infty$, and thus $\mathbb{P}_\mu(\langle U_q|f|, X_t \rangle < \infty) = 1$. Therefore, for $t \geq 0$, $\mathbb{P}_\mu(\langle U_q f, X_t \rangle \text{ is finite}) = 1$. In Sect. 2.3, we will give a stronger result: for any $\mu \in \mathcal{M}_C(E)$, and $f \in L^2(E, m)$, it holds that

$$\mathbb{P}_\mu\big(\langle U_q|f|, X_t \rangle < \infty, \forall t \geq 0\big) = \mathbb{P}_\mu\big(\langle U_q f, X_t \rangle \text{ is finite}, \forall t \geq 0\big) = 1.$$

We denote by $\mathbb{D}(\mathbb{R}^d)$ the space of all càdlàg functions from $[0, \infty)$ into $\mathbb{R}^d$, equipped with the Skorokhod topology. There is a metric δ on $\mathbb{D}(\mathbb{R}^d)$ which is compatible with the

Skorokhod topology. See, for instance, [20, Chapter VI, 1.26], for the definition of δ. In the present paper, we will consider weak convergence of processes in the Skorokhod space $\mathbb{D}(\mathbb{R}^d)$, which is stronger than convergence in finite dimensional distributions.

For $\tau \geq 0$ and $f \in \mathcal{C}_s$, we define

$$\sigma_{f,\tau} := e^{\lambda_1 \tau/2} \int_0^\infty e^{\lambda_1 s} \big(A(T_s f)(T_{s+\tau} f), \phi_1\big)_m ds. \tag{1.14}$$

We write $\sigma_{f,0}$ as σ_f^2. In this paper, τ will be used to denote a nonnegative number which is also served as a time parameter for various processes. τ will never be used to denote stopping times. For $h \in \mathcal{C}_c$, define

$$\rho_h^2 := \big(Ah^2, \phi_1\big)_m. \tag{1.15}$$

For $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, we put

$$I_u g(x) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} e^{\lambda_k u} b_j^k \phi_j^{(k)}(x), \quad x \in E,\ u \geq 0,$$

and

$$F_t(g) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} e^{-\lambda_k t} b_j^k H_\infty^{k,j}, \quad t \geq 0.$$

Define

$$\beta_{g,\tau} := e^{-\lambda_1 \tau/2} \int_0^\infty e^{-\lambda_1 s} \big(A(I_s g)(I_{s+\tau} g), \phi_1\big)_m ds. \tag{1.16}$$

We write $\beta_g^2 := \beta_{g,0}$. For $f \in \mathcal{C}_s$ and $g \in \mathcal{C}_l$, we define

$$\eta_{\tau_1,\tau_2}(f,g) := -e^{\lambda_1(\tau_1+\tau_2)/2} \int_{\tau_1}^{\tau_2} e^{-\lambda_1 u} \big(A(T_{\tau_2-u} f)(I_{u-\tau_1} g), \phi_1\big)_m du, \quad 0 \leq \tau_1 \leq \tau_2. \tag{1.17}$$

The following lemma is the spatial central limit theorem in [34].

Lemma 1.1 *Assume that $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$, $g \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_C(E)$. Then, under $\mathbb{P}_\mu$,*

$$\begin{aligned}&\Big(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, e^{\lambda_1 t/2}\big(\langle g, X_t\rangle - F_t(g)\big), t^{-1/2} e^{\lambda_1 t/2}\langle h, X_t\rangle, e^{\lambda_1 t/2}\langle f, X_t\rangle\Big)\\ &\xrightarrow{d} \Big(W_\infty, \sqrt{W_\infty} G_3(g), \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f)\Big),\end{aligned} \tag{1.18}$$

where $G_3(g) \sim \mathcal{N}(0, \beta_g^2)$, $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W_∞, $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent.

Recall that q is a fixed number larger than $\max\{K, -2\lambda_1\}$. Now we state our main result of the functional central limit theorem.

Theorem 1.2 *Assume that $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$, $g \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_C(E)$. For any $t > 0$, define*

$$Y_t^{1,f}(\tau) := e^{\lambda_1(t+\tau)/2}\langle f, X_{t+\tau}\rangle, \quad \tau \geq 0,$$

$$Y_t^{2,h}(\tau) := t^{-1/2} e^{\lambda_1(t+\tau)/2} \langle h, X_{t+\tau}\rangle, \quad \tau \geq 0,$$

and

$$Y_t^{3,g}(\tau) := e^{\lambda_1(t+\tau)/2}\big(\langle g, X_{t+\tau}\rangle - F_{t+\tau}(g)\big), \quad \tau \geq 0.$$

Then, for each fixed $t \in [0,\infty)$, $(W_t, Y_t^{1,U_q f}(\cdot), Y_t^{2,h}(\cdot), Y_t^{3,g}(\cdot))$ *is a* $\mathbb{D}(\mathbb{R}^4)$*-valued random variable under* $\mathbb{P}_\mu$, *where* W_t *is regarded as a constant process. Furthermore, under* $\mathbb{P}_\mu$,

$$\big(W_t, Y_t^{1,U_q f}(\cdot), Y_t^{2,h}(\cdot), Y_t^{3,g}(\cdot)\big) \xrightarrow{d} \big(W_\infty, \sqrt{W_\infty} G^{1,U_q f}(\cdot), \sqrt{W_\infty} G^{2,h}, \sqrt{W_\infty} G^{3,g}(\cdot)\big),$$
$$\text{as } t \to \infty, \tag{1.19}$$

in $\mathbb{D}(\mathbb{R}^4)$. *Here* $G^{2,h} \sim \mathcal{N}(0, \rho_h^2)$ *is a constant process, and* $\{(G^{1,U_q f}(\tau), G^{3,g}(\tau)) : \tau \geq 0\}$ *is a continuous* $\mathbb{R}^2$*-valued Gaussian process, on some probability space* $(\hat{\Omega}, \mathcal{F}, P)$, *with mean* 0 *and covariance functions given by*

$$P\big(G^{1,U_q f}(\tau_1) G^{1,U_q f}(\tau_2)\big) = \sigma_{U_q f, \tau_2-\tau_1}, \quad \text{for } 0 \leq \tau_1 \leq \tau_2, \tag{1.20}$$

$$P\big(G^{3,g}(\tau_1) G^{3,g}(\tau_2)\big) = \beta_{g, \tau_2-\tau_1}, \quad \text{for } 0 \leq \tau_1 \leq \tau_2, \tag{1.21}$$

and

$$P\big(G^{3,g}(\tau_1) G^{1,U_q f}(\tau_2)\big) = \begin{cases} \eta_{\tau_1,\tau_2}(U_q f, g), & \text{if } 0 \leq \tau_1 < \tau_2, \\ 0, & \text{if } \tau_1 \geq \tau_2 \geq 0. \end{cases} \tag{1.22}$$

Moreover, W_∞, $G^{2,h}$ *and* $(G^{1,U_q f}, G^{3,g})$ *are independent.*

For $f \in L^2(E,m)$, we define

$$f_{(s)}(x) := \sum_{k:\lambda_1 > 2\lambda_k} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x),$$

$$f_{(l)}(x) := \sum_{k:\lambda_1 < 2\lambda_k} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x),$$

$$f_{(c)}(x) := f(x) - f_{(s)}(x) - f_{(l)}(x).$$

Then $f_{(l)} \in \mathcal{C}_s$, $f_{(c)} \in \mathcal{C}_c$ and $f_{(s)} \in \mathcal{C}_l$.

Remark 1.3 Assume that $g = U_q f$ for some $f \in L^2(E,m) \cap L^4(E,m)$ satisfying $\lambda_1 \geq 2\lambda_{\gamma(f)}$. Then $g_{(l)} = U_q f_{(l)}$, $g_{(c)} = U_q f_{(c)}$ and $g_{(s)} = U_q f_{(s)}$. In particular, if $\lambda_1 = 2\lambda_{\gamma(f)}$ then $g_{(s)} = 0$.

If $f_{(c)} = 0$, then $g = g_{(l)} + g_{(s)}$, thus we have

$$e^{\lambda_1(t+\tau)/2}\big(\langle g, X_{t+\tau}\rangle - F_{t+\tau}(g_{(s)})\big) = Y_t^{1,g_{(l)}}(\tau) + Y_t^{3,g_{(s)}}(\tau).$$

Using the convergence of the first, second and fourth components in Theorem 1.2, we get for any nonzero $\mu \in \mathcal{M}_C(E)$, it holds under $\mathbb{P}_\mu$ that, as $t \to \infty$,

$$\big(W_t, e^{\lambda_1(t+\cdot)/2}\big(\langle g, X_{t+\cdot}\rangle - F_{t+\cdot}(g_{(s)})\big)\big) \xrightarrow{d} \big(W_\infty, \sqrt{W_\infty}\big(G^{1,g_{(l)}} + G^{3,g_{(s)}}\big)\big), \tag{1.23}$$

where $G^{1,g_{(l)}} + G^{3,g_{(s)}}$ is a continuous Gaussian process, on some probability space $(\hat{\Omega}, \mathcal{F}, P)$, with mean 0 and covariance function

$$P\big[\big(G^{1,g_{(l)}}(\tau_1) + G^{3,g_{(s)}}(\tau_1)\big)\big(G^{1,g_{(l)}}(\tau_2) + G^{3,g_{(s)}}(\tau_2)\big)\big]$$
$$= \sigma_{g_{(l)},\tau_2-\tau_1} + \eta_{\tau_1,\tau_2}(g_{(l)}, g_{(s)}) + \beta_{g_{(s)},\tau_2-\tau_1}, \quad 0 \le \tau_1 \le \tau_2.$$

If $f_{(c)} \neq 0$, then

$$t^{-1/2}e^{\lambda_1(t+\tau)/2}\big(\langle g, X_{t+\tau}\rangle - F_{t+\tau}(g_{(s)})\big) = t^{-1/2}\big(Y_t^{1,g_{(l)}}(\tau) + Y_t^{3,g_{(s)}}(\tau)\big) + Y_t^{2,g_{(c)}}(\tau).$$

By (1.23), we get

$$t^{-1/2}\big(Y_t^{1,g_{(l)}}(\cdot) + Y_t^{3,g_{(s)}}(\cdot)\big) \xrightarrow{d} 0.$$

Thus using the convergence of the first and third components in Theorem 1.2, we get

$$\big(W_t, t^{-1/2}e^{\lambda_1(t+\cdot)/2}\big(\langle g, X_{t+\cdot}\rangle - F_{t+\cdot}(g_{(s)})\big)\big) \xrightarrow{d} \big(W_\infty, \sqrt{W_\infty}G^{2,g_{(c)}}\big),$$

where $G^{2,g_{(c)}} \sim \mathcal{N}(0, \rho^2_{g_{(c)}})$ is a constant process. Moreover, W_∞ and $G^{2,g_{(c)}}$ are independent. Note that, if $\lambda_1 = 2\lambda_{\gamma(f)}$, then $F_{t+\cdot}(g_{(s)}) = 0$, and thus we have $(W_t, t^{-1/2}e^{\lambda_1(t+\cdot)/2}\langle g, X_{t+\cdot}\rangle) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G^{2,g_{(c)}})$.

2 Preliminaries

In this section, we give some useful results and facts. In the remainder of this paper we will use the following notation: for two positive functions f and g on E, $f(x) \lesssim g(x)$ means that there exists a constant $c > 0$ such that $f(x) \le cg(x)$ for all $x \in E$.

In [33, (2.25)], we have proved that

$$\int_0^{t_0} T_s(a_{2t_0})(x)ds \lesssim a_{t_0}(x)^{1/2}. \tag{2.1}$$

2.1 Estimates on the Moments of X

In this subsection, we will recall some results about the moments of $\langle f, X_t\rangle$. The first result is [33, Lemma 2.1].

Lemma 2.1 *For any $f \in L^2(E, m)$, $x \in E$ and $t > 0$, we have*

$$T_t f(x) = \sum_{k=\gamma(f)}^{\infty} e^{-\lambda_k t} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x) \tag{2.2}$$

and

$$\lim_{t\to\infty} e^{\lambda_{\gamma(f)}t} T_t f(x) = \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)} \phi_j^{(\gamma(f))}(x), \tag{2.3}$$

where the series in (2.2) *converges absolutely and uniformly in any compact subset of* E. *Moreover, for any* $t_1 > 0$,

$$\sup_{t>t_1} e^{\lambda_{\gamma(f)}t}\left|T_t f(x)\right| \le e^{\lambda_{\gamma(f)}t_1}\|f\|_2\left(\int_E a_{t_1/2}(x)m(dx)\right)a_{t_1}(x)^{1/2}, \tag{2.4}$$

$$\sup_{t>t_1} e^{(\lambda_{\gamma(f)+1}-\lambda_{\gamma(f)})t}\left|e^{\lambda_{\gamma(f)}t}T_t f(x) - f^*(x)\right|$$
$$\le e^{\lambda_{\gamma(f)+1}t_1}\|f\|_2\left(\int_E a_{t_1/2}(x)m(dx)\right)\left(a_{t_1}(x)\right)^{1/2}, \tag{2.5}$$

where $f^* = \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)}\phi_j^{(\gamma(f))}$.

We now recall the second moments of the superprocess $\{X_t : t \ge 0\}$ (see, for example, [34]): for $f \in L^2(E,m)\cap L^4(E,m)$ and $\mu \in \mathcal{M}_C(E)$, we have for any $t > 0$,

$$\mathbb{P}_\mu\langle f, X_t\rangle^2 = \left(\mathbb{P}_\mu\langle f, X_t\rangle\right)^2 + \int_E\int_0^t T_s\left[A(T_{t-s}f)^2\right](x)ds\mu(dx). \tag{2.6}$$

Thus,

$$\mathbb{V}\mathrm{ar}_\mu\langle f, X_t\rangle = \left\langle \mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f, X_t\rangle, \mu\right\rangle = \int_E\int_0^t T_s\left[A(T_{t-s}f)^2\right](x)ds\mu(dx), \tag{2.7}$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$. Moreover, for $f \in L^2(E,m)\cap L^4(E,m)$,

$$\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle \le e^{Kt}T_t\left(f^2\right)(x) \in L^2(E,m). \tag{2.8}$$

The next result is [34, Lemma 2.6].

Recall that t_0 is the constant in condition (b) in Sect. 1.1.

Lemma 2.2 *Assume that* $f \in L^2(E,m)\cap L^4(E,m)$.

(1) *If* $\lambda_1 < 2\lambda_{\gamma(f)}$, *then for any* $x \in E$,

$$\lim_{t\to\infty} e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle = \sigma_f^2\phi_1(x). \tag{2.9}$$

Moreover, for $(t,x) \in (3t_0,\infty)\times E$, *we have*

$$e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle \lesssim a_{t_0}(x)^{1/2}. \tag{2.10}$$

(2) *If* $\lambda_1 = 2\lambda_{\gamma(f)}$, *then for any* $(t,x) \in (3t_0,\infty)\times E$,

$$\left|t^{-1}e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \rho_{f^*}^2\phi_1(x)\right| \lesssim t^{-1}a_{t_0}(x)^{1/2}, \tag{2.11}$$

where $f^* = \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)}\phi_j^{(\gamma(f))}$.

(3) *If* $\lambda_1 > 2\lambda_{\gamma(f)}$, *then for any* $x \in E$,

$$\lim_{t\to\infty} e^{2\lambda_{\gamma(f)}t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle = \int_0^\infty e^{2\lambda_{\gamma(f)}s}T_s\left(A\left(f^*\right)^2\right)(x)ds. \tag{2.12}$$

Moreover, for any $(t,x) \in (3t_0,\infty)\times E$,

$$e^{2\lambda_{\gamma(f)}t}\mathbb{P}_{\delta_x}\langle f, X_t\rangle^2 \lesssim a_{t_0}(x)^{1/2}. \tag{2.13}$$

2.2 Excursion Measures of X

We use $\mathbb{D}$ to denote the space of $\mathcal{M}_F(E)$-valued right continuous functions $t \mapsto \omega_t$ on $(0, \infty)$ having zero as a trap. We use $(\mathcal{A}, \mathcal{A}_t)$ to denote the natural σ-algebras on $\mathbb{D}$ generated by the coordinate process.

It is known (see [25, Sect. 8.4]) that one can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of σ-finite measures $\{\mathbb{N}_x : x \in E\}$ defined on $(\mathbb{D}, \mathcal{A})$ such that $\mathbb{N}_x(\{0\}) = 0$,

$$\int_{\mathbb{D}} \left(1 - e^{-\langle f,\omega_t\rangle}\right)\mathbb{N}_x(d\omega) = -\log \mathbb{P}_{\delta_x}\left(e^{-\langle f,X_t\rangle}\right), \quad f \in \mathcal{B}_b^+(E),\ t > 0, \tag{2.14}$$

and, for every $0 < t_1 < \cdots < t_n < \infty$, and nonzero $\mu_1, \ldots, \mu_n \in \mathcal{M}_F(E)$,

$$\begin{aligned}&\mathbb{N}_x(\omega_{t_1} \in d\mu_1, \ldots, \omega_{t_n} \in d\mu_n)\\ &\quad = \mathbb{N}_x(\omega_{t_1} \in d\mu_1)\mathbb{P}_{\mu_1}(X_{t_2-t_1} \in d\mu_2)\cdots \mathbb{P}_{\mu_{n-1}}(X_{t_n-t_{n-1}} \in d\mu_n).\end{aligned} \tag{2.15}$$

For earlier work on excursion measures of superprocesses, see [16, 17, 24].

For any $\mu \in \mathcal{M}_C(E)$, let $N(d\omega)$ be a Poisson random measure on the space $\mathbb{D}$ with intensity $\int_E \mathbb{N}_x(d\omega)\mu(dx)$, in a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbf{P}_\mu)$. We define another process $\{\Lambda_t : t \geq 0\}$ by $\Lambda_0 = \mu$ and

$$\Lambda_t := \int_{\mathbb{D}} \omega_t N(d\omega), \quad t > 0.$$

Let $\widetilde{\mathcal{F}}_t$ be the σ-algebra generated by $\{N(A) : A \in \mathcal{A}_t\}$. Then, $\{\Lambda, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \mathbf{P}_\mu\}$ has the same law as $\{X, (\mathcal{G}_t)_{t\geq 0}, \mathbb{P}_\mu\}$, see [25, Theorem 8.24]. Thus,

$$\begin{aligned}\mathbb{P}_\mu\left[\exp\left\{i\theta\langle f, X_{t+s}\rangle\right\}|X_t\right] &= \mathbf{P}_{X_t}\left[\exp\left(i\theta\langle f, \Lambda_s\rangle\right)\right]\\ &= \exp\left\{\int_E \int_{\mathbb{D}} \left(e^{i\theta\langle f,\omega_s\rangle} - 1\right)\mathbb{N}_x(d\omega)X_t(dx)\right\}.\end{aligned} \tag{2.16}$$

The proposition below contains some useful properties of $\mathbb{N}_x$. The proofs are similar to those in [16, Corollary 1.2, Proposition 1.1].

Proposition 2.3 *If* $\mathbb{P}_{\delta_x}|\langle f, X_t\rangle| < \infty$, *then*

$$\int_{\mathbb{D}} \langle f, \omega_t\rangle \mathbb{N}_x(d\omega) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle. \tag{2.17}$$

If $\mathbb{P}_{\delta_x}\langle f, X_t\rangle^2 < \infty$, *then*

$$\int_{\mathbb{D}} \langle f, \omega_t\rangle^2 \mathbb{N}_x(d\omega) = \mathbb{V}ar_{\delta_x}\langle f, X_t\rangle. \tag{2.18}$$

2.3 Potential Functions

Recall that $q > \max\{K, -2\lambda_1\}$. For any $x \in E$ such that $U_q|f|(x) < \infty$, we have

$$U_q f(x) = \int_0^\infty e^{-qs} T_s f(x) ds. \tag{2.19}$$

Lemma 2.4 *If $f \in L^2(E, m)$, then for any $\mu \in \mathcal{M}_C(E)$,*

$$\mathbb{P}_\mu\big\{\langle U_q|f|, X_t\rangle < \infty, \ \forall t \geq 0\big\} = 1. \tag{2.20}$$

Moreover, $\langle U_q f, X_t\rangle$ is finite and right continuous, $\mathbb{P}_\mu$-a.s.

Proof First, we claim that, if f is nonnegative and bounded, $e^{-qt}\langle U_q f, X_t\rangle$ is a nonnegative right continuous supermartingale with respect to $\{\mathcal{G}_t : t \geq 0\}$. In fact, since $T_t f(x) \leq \|f\|_\infty e^{Kt}$, we have

$$U_q f(x) \leq \|f\|_\infty \int_0^\infty e^{-qt} e^{Kt} dt = (q-K)^{-1}\|f\|_\infty < \infty.$$

Since $T_t f(x)$ is continuous, by the dominated convergence theorem, we get that $U_q f$ is continuous. Thus, $U_q f$ is a bounded and continuous function on E. Since X is a right continuous process in $\mathcal{M}_F(E)$, we get that $t \mapsto \langle U_q f, X_t\rangle$ is right continuous. By Fubini's theorem, we have, for any $x \in E$ and $t \geq 0$,

$$T_t[U_q f](x) = \int_0^\infty e^{-qs} T_{t+s} f(x) ds = e^{qt} \int_t^\infty e^{-qs} T_s f(x) ds \leq e^{qt} U_q f(x).$$

By the Markov property of X, we have, for $t > s$,

$$\mathbb{P}_\mu\big(e^{-qt}\langle U_q f, X_t\rangle | \mathcal{G}_s\big) = e^{-qt}\big\langle T_{t-s}(U_q f), X_s\big\rangle \leq e^{-qs}\langle U_q f, X_s\rangle.$$

Thus, $e^{-qt}\langle U_q f, X_t\rangle$ is a supermartingale.

Now, if $f \in L^2(E, m)$ is nonnegative, then $f_M(x) := f(x)\mathbf{1}_{f(x)\leq M}$ is bounded. Therefore $e^{-qt}\langle U_q(f_M), X_t\rangle$ is a nonnegative right continuous supermartingale with respect to $\{\mathcal{G}_t : t \geq 0\}$, and, as $M \to \infty$,

$$\forall t \geq 0 : e^{-qt}\big\langle U_q(f_M), X_t\big\rangle \uparrow e^{-qt}\langle U_q f, X_t\rangle.$$

Since $U_q f \in L^2(E, m)$, $\mathbb{P}_\mu\langle U_q f, X_t\rangle = \langle T_t(U_q f), \mu\rangle < \infty$. Thus, by [12, Sect. 1.4, Theorem 5], $e^{-qt}\langle U_q f, X_t\rangle$ is a right continuous supermartingale. By [12, Sect. 1.4, Corollary 1], $e^{-qt}\langle U_q f, X_t\rangle$ is bounded on each finite interval, $\mathbb{P}_\mu$-a.s., which implies that for any $N > 0$,

$$\mathbb{P}_\mu\big(e^{-qt}\langle U_q f, X_t\rangle < \infty, \ t \in [0, N]\big) = 1.$$

Thus, we have

$$\mathbb{P}_\mu\big(\langle U_q f, X_t\rangle < \infty, \ t \in [0, \infty)\big) = 1.$$

Finally, we consider general $f \in L^2(E, m)$. Let

$$\begin{aligned}\Omega_0 := &\big\{\langle U_q|f|, X_t\rangle < \infty, \forall t \geq 0\big\}\\ &\cap \big\{\omega : \big\langle U_q\big(f^+\big), X_t(\omega)\big\rangle \text{ and } \big\langle U_q\big(f^-\big), X_t(\omega)\big\rangle \text{ are right continuous}\big\}.\end{aligned}$$

We have proved that, for any $\mu \in \mathcal{M}_F(E)$, $\mathbb{P}_\mu(\Omega_0) = 1$. It follows that, for $\omega \in \Omega_0$,

$$\big\langle U_q f, X_t(\omega)\big\rangle = \big\langle U_q\big(f^+\big), X_t(\omega)\big\rangle - \big\langle U_q\big(f^-\big), X_t(\omega)\big\rangle$$

is well defined and right continuous. The proof is now complete. □

2.4 Martingale Problem of X

In this subsection, we recall the martingale problem of superprocesses. For our superprocess X, there exists a worthy $(\mathcal{G}_t)$-martingale measure $\{M_t(B) : t \geq 0; B \in \mathcal{B}(E)\}$ with dominating measure

$$\nu(ds, dx, dy) := ds \int_E A(z)\delta_z(dx)\delta_z(dy)X_s(dz) \tag{2.21}$$

such that for $t \geq 0$, $f \in \mathcal{B}_b(E)$ and $\mu \in \mathcal{M}_C(E)$, we have, $\mathbb{P}_\mu$-a.s.,

$$\langle f, X_t\rangle = \langle T_t f, \mu\rangle + \int_0^t \int_E T_{t-s} f(z) M(ds, dz). \tag{2.22}$$

For the validity of (2.22), see [25, Theorem 7.26]. Recall that, roughly speaking, a martingale measure is called worthy if it admits a dominating measure. The second term on the right-hand side of (2.22) stands for the stochastic integral of $T_{t-s} f(z)$ with respect to the worthy martingale measure M. For the precise definition of worthy martingale measures and stochastic integrals with respect to worthy martingale measures, we refer our readers to [25, Sect. 7.3].

Let $\mathcal{L}^2_\nu(E)$ be the space of two-parameter predictable processes $h_s(x)$ such that for all $T > 0$ and $\mu \in \mathcal{M}_C(E)$,

$$\begin{aligned}\mathbb{P}_\mu\left[\int_0^T \int_{E^2} h_s(x)h_s(y)\nu(ds, dx, dy)\right] &= \mathbb{P}_\mu\left[\int_0^T \int_E A(z)h_s(z)^2 X_s(dz)ds\right] \\ &= \int_E \int_0^T T_s\big[Ah_s^2\big](z)ds\mu(dz) < \infty.\end{aligned}$$

Then, for $h \in \mathcal{L}^2_\nu(E)$,

$$M_t(h) := \int_0^t \int_E h_s(z)M(ds, dz)$$

is well defined and it is a square-integrable càdlàg $\mathcal{G}_t$-martingale under $\mathbb{P}_\mu$, for each $\mu \in \mathcal{M}_C(E)$, with

$$\big\langle M(h)\big\rangle_t = \int_0^t \big\langle Ah_s^2, X_s\big\rangle ds. \tag{2.23}$$

For $f \in L^2(E, m) \cap L^4(E, m)$ and $\mu \in \mathcal{M}_C(E)$, we have

$$\int_E \int_0^t T_s\big[A(T_{t-s} f)^2\big](z)ds\mu(dz) = \mathbb{V}ar_\mu\langle f, X_t\rangle < \infty,$$

which implies that

$$\int_0^t \int_E T_{t-s} f(z)M(ds, dz)$$

is well defined. Now, using a routine limit argument, we can show that (2.22) holds for all $f \in L^2(E, m) \cap L^4(E, m)$ and $\mu \in \mathcal{M}_C(E)$.

For $f \in L^2(E,m) \cap L^4(E,m)$, $U_q f \in L^2(E,m) \cap L^4(E,m)$. By (2.22), for $t > 0$ and $\mu \in \mathcal{M}_C(E)$, we have, $\mathbb{P}_\mu$-a.s.,

$$\begin{aligned}\langle U_q f, X_t\rangle &= \langle T_t(U_q f), \mu\rangle + \int_0^t \int_E T_{t-s}(U_q f)(z) M(ds, dz)\\&= \langle T_t(U_q f), \mu\rangle + \int_0^t \int_E \int_0^\infty e^{-qu} T_{u+t-s} f(z) du M(ds, dz)\\&= \langle T_t(U_q f), \mu\rangle + e^{qt} \int_0^t \int_E \int_t^\infty e^{-qu} T_{u-s} f(z) du M(ds, dz)\\&= \langle T_t(U_q f), \mu\rangle + e^{qt} \int_t^\infty e^{-qu} du \int_0^t \int_E T_{u-s} f(z) M(ds, dz)\\&=: J_1^f(t) + e^{qt} J_2^f(t),\end{aligned} \tag{2.24}$$

where the fourth equality follows from the stochastic Fubini's theorem for martingale measures (see, for instance, [25, Theorem 7.24]). Thus, for $t > 0$ and $\mu \in \mathcal{M}_C(E)$,

$$\mathbb{P}_\mu\big(\langle U_q f, X_t\rangle = J_1^f(t) + e^{qt} J_2^f(t)\big) = 1. \tag{2.25}$$

For any $u > 0$ and $0 \le T \le u$, we define

$$M_T^{(u)} := \int_0^T \int_E T_{u-s} f(x) M(ds, dx).$$

Then, for any $\mu \in \mathcal{M}_C(E)$, $\{M_T^{(u)}, 0 \le T \le u\}$ is a càdlàg square-integrable martingale under $\mathbb{P}_\mu$ with

$$\langle M^u\rangle_T = \int_0^T \langle A(T_{u-s} f)^2, X_s\rangle ds. \tag{2.26}$$

Note that

$$\mathbb{P}_\mu\big(M_u^{(u)}\big)^2 = \mathbb{P}_\mu\langle M^u\rangle_u = \mathbb{V}ar_\mu\langle f, X_u\rangle. \tag{2.27}$$

Lemma 2.5 *If $f \in L^2(E,m) \cap L^4(E,m)$ and $\mu \in \mathcal{M}_C(E)$, then $t \mapsto \langle U_q f, X_t\rangle$ is a càdlàg process on* $[0, \infty)$, $\mathbb{P}_\mu$*-a.s. Moreover,*

$$\mathbb{P}_\mu\big(\langle U_q f, X_t\rangle = J_1^f(t) + e^{qt} J_2^f(t), \forall t > 0\big) = 1. \tag{2.28}$$

Proof Since $\langle U_q f, X_t\rangle$ is right continuous, $\mathbb{P}_\mu$-a.s., in light of (2.25), to prove (2.28), it suffices to prove that $J_1^f(t)$ and $J_2^f(t)$ are all càdlàg in $(0, \infty)$, $\mathbb{P}_\mu$-a.s.

For $J_1^f(t)$, by Fubini's theorem, for $t > 0$,

$$J_1^f(t) = e^{qt} \int_t^\infty e^{-qs} \langle T_s f, \mu\rangle ds.$$

Thus, it is easy to see that $J_1^f(t)$ is continuous in $t \in (0, \infty)$.

Now, we consider $J_2^f(t)$. We claim that, for any $t_1 > 0$,

$$\mathbb{P}_\mu\big(J_2^f(t) \text{ is càdlàg in } [t_1, \infty)\big) = 1. \tag{2.29}$$

By the definition of J_2^f, for $t \geq t_1$,

$$J_2^f(t) = \int_{t_1}^{\infty} e^{-qu} M_t^{(u)} \mathbf{1}_{t<u} du. \tag{2.30}$$

Since $t \mapsto M_t^{(u)} \mathbf{1}_{t<u}$ is right continuous, by the dominated convergence theorem, to prove (2.29), it suffices to show that

$$\mathbb{P}_\mu \left(\int_{t_1}^{\infty} e^{-qu} \sup_{t \geq t_1} \left(\left| M_t^{(u)} \right| \mathbf{1}_{t<u} \right) du < \infty \right) = 1. \tag{2.31}$$

By the L_2-maximum inequality and (2.27), we have

$$\begin{aligned} &\mathbb{P}_\mu \left(\int_{t_1}^{\infty} e^{-qu} \sup_{t \geq t_1} \left(\left| M_t^{(u)} \right| \mathbf{1}_{t<u} \right) du \right) \\ &\leq 2 \int_{t_1}^{\infty} e^{-qu} \sqrt{\mathbb{P}_\mu \left| M_u^{(u)} \right|^2} du \\ &= 2 \int_{t_1}^{\infty} e^{-qu} \sqrt{\int_E \mathbb{V}ar_{\delta_x} \langle f, X_u \rangle \mu(dx)} du. \end{aligned} \tag{2.32}$$

By (2.8) and (2.4), we have, for $u > t_1$,

$$\int_E \mathbb{V}ar_{\delta_x} \langle f, X_u \rangle \mu(dx) \leq e^{Ku} \int_E T_u \left(f^2 \right)(x) \mu(dx) \lesssim e^{Ku} e^{-\lambda_1 u} \int_E a_{t_1}(x)^{1/2} \mu(dx).$$

Since $a_{t_1}(x)$ is continuous in E and μ has compact support, it follows that $\int_E a_{t_1}(z)^{1/2} \times \mu(dz) < \infty$. Thus, by (2.32), we have

$$\mathbb{P}_\mu \left(\int_{t_1}^{\infty} e^{-qu} \sup_{t \geq t_1} \left(\left| M_t^{(u)} \right| \mathbf{1}_{t<u} \right) du \right) \lesssim \int_{t_1}^{\infty} e^{-qu} e^{(K-\lambda_1)u/2} du \sqrt{\int_E a_{t_1}(x)^{1/2} \mu(dx)} < \infty.$$

Now (2.31) follows immediately. Since $t_1 > 0$ are arbitrary, we have

$$\mathbb{P}_\mu \left(J_2^f(t) \text{ is càdlàg in } (0, \infty) \right) = 1. \tag{2.33}$$

□

3 Proof of the Main Result

Suppose that $(X^n)_{n \geq 0}$ and X are all $\mathbb{D}(\mathbb{R}^d)$-valued random variables. If for any $k \geq 1$ and any $t_1, \ldots t_k \in \mathbb{R}_+$,

$$\left(X_{t_1}^n, X_{t_2}^n, \ldots, X_{t_k}^n \right) \xrightarrow{d} (X_{t_1}, \ldots, X_{t_k}), \qquad \text{as } n \to \infty,$$

then we write

$$X^n \xrightarrow{\mathcal{L}(\mathbb{R}_+)} X, \qquad \text{as } n \to \infty.$$

3.1 Finite Dimensional Convergence

The following lemma is a generalization of [34, Remark 1.3]. Recall that W_∞ is the limit of the nonnegative martingale $W_t = H_t^{1,1} = e^{\lambda_1 t}\langle \phi_1, X_t\rangle$ as $t \to \infty$.

Lemma 3.1 *If $f \in L^2(E,m)$ is nonnegative and $\mu \in \mathcal{M}_C(E)$, then*

$$e^{\lambda_1 t}\langle f, X_t\rangle \to (f,\phi_1)_m W_\infty, \ \textit{in } L^1(\mathbb{P}_\mu). \tag{3.1}$$

Proof If f is bounded, then the conclusion follows from [34, Remark 1.3]. So we will assume that f is unbounded. For any $M > 0$, let $f_M(x) := f(x)\mathbf{1}_{f(x)\le M}$ and $\hat{f}_M := f - f_M$. Then $f_M \ge 0$, $f_M \in L^2(E,m)\cap L^4(E,m)$ and $\hat{f}_M \ge 0$ is nontrivial. In [34, Remark 1.3], we have proved that

$$\lim_{t\to\infty} \mathbb{P}_\mu\left|e^{\lambda_1 t}\langle f_M, X_t\rangle - (f_M,\phi_1)_m W_\infty\right| = 0. \tag{3.2}$$

Since $\hat{f}_M \ge 0$ is nontrivial, we have $\gamma(\hat{f}_M) = 1$. For $t > t_0$, by (2.4), we have $e^{\lambda_1 t}T_t\hat{f}_M(x) \le e^{\lambda_1 t_0}(\int_E a_{t_0/2}(x)m(dx))a_{t_0}(x)^{1/2}\|\hat{f}_M\|_2$. Thus, we get

$$\begin{aligned}
&\mathbb{P}_\mu\left|e^{\lambda_1 t}\langle \hat{f}_M, X_t\rangle - (\hat{f}_M,\phi_1)_m W_\infty\right| \\
&\quad \le e^{\lambda_1 t}\mathbb{P}_\mu\langle \hat{f}_M, X_t\rangle + (\hat{f}_M,\phi_1)_m\mathbb{P}_\mu(W_\infty) \\
&\quad = e^{\lambda_1 t}\langle T_t\hat{f}_M, \mu\rangle + (\hat{f}_M,\phi_1)_m\mathbb{P}_\mu(W_\infty) \\
&\quad \le e^{\lambda_1 t_0}\left(\int_E a_{t_0/2}(x)m(dx)\right)\left\langle a_{t_0}^{1/2}, \mu\right\rangle\|\hat{f}_M\|_2 + \mathbb{P}_\mu(W_\infty)\|\hat{f}_M\|_2.
\end{aligned} \tag{3.3}$$

By (3.2) and (3.3), we have

$$\limsup_{t\to\infty} \mathbb{P}_\mu\left|e^{\lambda_1 t}\langle f, X_t\rangle - (f,\phi_1)_m W_\infty\right| \lesssim \|\hat{f}_M\|_2. \tag{3.4}$$

Letting $M \to \infty$, we arrive at (3.1). □

Recall that

$$H_t^{k,j} := e^{\lambda_k t}\left\langle \phi_j^{(k)}, X_t\right\rangle, \quad t \ge 0,$$

and for $g(x) = \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}(x)$, $x \in E$,

$$F_t(g) := \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k t}b_j^k H_\infty^{k,j},$$

where $H_\infty^{k,j}$ is the martingale limit of $H_t^{k,j}$. And recall that

$$I_u g(x) := \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{\lambda_k u}b_j^k\phi_j^{(k)}(x), \quad x \in E.$$

It is easy to see that $I_{s+t}g = I_s(I_t g)$ and $T_u(I_u g) = I_u(T_u g) = g$. Thus, we have, as $u \to \infty$,

$$\langle I_u g, X_{t+u}\rangle \to F_t(g), \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.5}$$

Define

$$\widetilde{H}_t^{k,j}(\omega) := e^{\lambda_k t}\langle \phi_j^{(k)}, \omega_t\rangle, \quad t \geq 0, \omega \in \mathbb{D},$$

and

$$H_\infty(g)(\omega) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \widetilde{H}_\infty^{k,j}(\omega).$$

It follows from [34, Lemma 3.1] that the limit $\widetilde{H}_\infty^{k,j} := \lim_{t\to\infty} \widetilde{H}_t^{k.j}$ exists $\mathbb{N}_x$-a.e., in $L^1(\mathbb{N}_x)$ and in $L^2(\mathbb{N}_x)$. Then, as $u \to \infty$,

$$\langle I_u g, \omega_u\rangle \to H_\infty(g)(\omega), \quad \mathbb{N}_x\text{-a.e.}, \quad \text{in } L^1(\mathbb{N}_x) \text{ and in } L^2(\mathbb{N}_x). \tag{3.6}$$

Since $\mathbb{N}_x\langle I_u g, \omega_u\rangle = \mathbb{P}_{\delta_x}\langle I_u g, X_u\rangle = g(x)$, we get that

$$\mathbb{N}_x\big(H_\infty(g)\big) = g(x). \tag{3.7}$$

By (2.18) and (2.7), we have

$$\mathbb{N}_x\langle I_u g, \omega_u\rangle^2 = \mathbb{V}\mathrm{ar}_{\delta_x}\langle I_u g, X_u\rangle = \int_0^u T_s\big[A(I_s g)^2\big](x)ds, \tag{3.8}$$

which implies that

$$\mathbb{N}_x\big(H_\infty(g)\big)^2 = \int_0^\infty T_s\big[A(I_s g)^2\big](x)ds. \tag{3.9}$$

The following simple fact will be used later:

$$\left|e^{ix} - \sum_{m=0}^n \frac{(ix)^m}{m!}\right| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right). \tag{3.10}$$

Note that, in contrast with (1.19), the following Lemma 3.2 says that (3.11), which is about the convergence of finite dimensional distributions, is valid for any $f \in \mathcal{C}_s$, not just for $U_q f$ with $f \in \mathcal{C}_s$.

Lemma 3.2 *Assume that $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$, $g \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_C(E)$. Suppose that $Y_t^{1,f}$, $Y_t^{2,h}$, and $Y_t^{3,g}$ are defined as in Theorem 1.2. Then, for any $0 \leq \tau_1 \leq \tau_2 \cdots \leq \tau_k$, under $\mathbb{P}_\mu$, as $t \to \infty$,*

$$\begin{aligned}&\big(W_t, Y_t^{1,f}(\tau_1), \ldots, Y_t^{1,f}(\tau_k), Y_t^{2,h}(\tau_1), \ldots, Y_t^{2,h}(\tau_k), Y_t^{3,g}(\tau_1), \ldots, Y_t^{3,g}(\tau_k)\big)\\ &\overset{d}{\to} \big(W_\infty, \sqrt{W_\infty}G^{1,f}(\tau_1), \ldots, \sqrt{W_\infty}G^{1,f}(\tau_k), \sqrt{W_\infty}G^{2,h}, \ldots, \sqrt{W_\infty}G^{2,h},\\ &\quad \sqrt{W_\infty}G^{3,g}(\tau_1), \ldots, \sqrt{W_\infty}G^{3,g}(\tau_k)\big).\end{aligned} \tag{3.11}$$

Here $G^{2,h} \sim \mathcal{N}(0, \rho_h^2)$ is a constant process and $(G^{1,f}(\tau_1), \ldots, G^{1,f}(\tau_k), G^{3,g}(\tau_1), \ldots, G^{3,g}(\tau_k))$ is an $\mathbb{R}^{2k}$-valued Gaussian random variable, on some probability space $(\Omega, \mathcal{F}, P)$, with mean 0 and covariance

$$P\big(G^{1,f}(\tau_j)G^{1,f}(\tau_l)\big) = \sigma_{f,\tau_l-\tau_j}, \quad \textit{for } 1 \leq j \leq l \leq k, \tag{3.12}$$

$$P\big(G^{3,g}(\tau_j)G^{3,g}(\tau_l)\big) = \beta_{g,\tau_l-\tau_j}, \quad \textit{for } 1 \le j \le l \le k, \tag{3.13}$$

and

$$P\big(G^{3,g}(\tau_j)G^{1,f}(\tau_l)\big) = \begin{cases} \eta_{\tau_j,\tau_l}(f,g), & \textit{if } 1 \le j < l \le k, \\ 0, & \textit{if } 1 \le l \le j \le k. \end{cases} \tag{3.14}$$

Moreover, W_∞, $G^{2,h}$ *and* $(G^{1,f}(\tau_1), \dots, G^{1,f}(\tau_k), G^{3,g}(\tau_1), \dots, G^{3,g}(\tau_k))$ *are independent.*

Proof To prove this theorem, we need to find the limit of the following quantity

$$\phi(t) =: \mathbb{P}_\mu \exp\left\{ i\theta W_t + \sum_{j=1}^k i\theta_{1,j} Y_t^{1,f}(\tau_j) + \sum_{j=1}^k i\theta_{2,j} Y_t^{2,h}(\tau_j) + \sum_{j=1}^k i\theta_{3,j} Y_t^{3,g}(\tau_j) \right\}, \tag{3.15}$$

where $\theta, \theta_{l,j} \in \mathbb{R}$, $l = 1, 2, 3,\ j = 1, \dots, k$. This proof is pretty long, so we divide it into several steps.

Step 1. In this step, we reduce the problem of finding the limit above to the limit of $\phi_1(t)$ defined in (3.21) below. We put $\theta_{1,0} = \theta_{2,0} = \theta_{3,0} = 0$, $\tau_0 = 0$ and $s_j := \tau_j - \tau_{j-1}$, $j = 1, \dots, k$. Define, for $l = 0, \dots, k$,

$$\tilde{f}_l(x) := \sum_{j=l}^k \theta_{1,l} e^{\lambda_1(\tau_j - \tau_l)/2} T_{\tau_j - \tau_l} f(x), \qquad \widehat{g}_l(x) := \sum_{j=0}^l \theta_{3,j} e^{\lambda_1(\tau_j - \tau_l)/2} I_{\tau_l - \tau_j} g(x)$$

and

$$B_l(x) := \tilde{f}_l(x) + \theta_{3,l} g(x) - \widehat{g}_l(x). \tag{3.16}$$

For $j = 1, \dots, k$, by (3.5),

$$F_{t+\tau_j}(g) = \lim_{u\to\infty} \langle I_{u+\tau_k-\tau_j} g, X_{u+t+\tau_k} \rangle. \tag{3.17}$$

Using this, we get that

$$\begin{aligned} \phi(t) &= \mathbb{P}_\mu \exp\left\{ i\theta W_t + \sum_{j=1}^k i\theta_{1,j} Y_t^{1,f}(\tau_j) + \sum_{j=1}^k i\theta_{2,j} Y_t^{2,h}(\tau_j) + \sum_{j=1}^k i\theta_{3,j} Y_t^{3,g}(\tau_j) \right\} \\ &= \mathbb{P}_\mu \exp\left\{ i\theta W_t + \sum_{j=1}^k \big[i\theta_{1,j} Y_t^{1,f}(\tau_j) + i\theta_{2,j} Y_t^{2,h}(\tau_j) \right. \\ &\qquad \left. + i\theta_{3,j} e^{\lambda_1(t+\tau_j)/2} \big(\langle g, X_{t+\tau_j} \rangle - F_{t+\tau_j}(g) \big) \big] \right\} \\ &= \lim_{u\to\infty} \mathbb{P}_\mu \exp\left\{ i\theta W_t + \sum_{j=1}^k \big[i\theta_{1,j} Y_t^{1,f}(\tau_j) + i\theta_{2,j} Y_t^{2,h}(\tau_j) + i\theta_{3,j} e^{\lambda_1(t+\tau_j)/2} \langle g, X_{t+\tau_j} \rangle \big] \right. \\ &\qquad \left. - i \left\langle \sum_{j=1}^k \theta_{3,j} e^{\lambda_1(t+\tau_j)/2} I_{u+\tau_k-\tau_j} g, X_{u+t+\tau_k} \right\rangle \right\} \end{aligned}$$

$$= \lim_{u\to\infty} \mathbb{P}_\mu \exp\Big\{ i\theta W_t + \sum_{j=1}^k i e^{\lambda_1(t+\tau_j)/2} \big\langle \theta_{1,j} f + t^{-1/2}\theta_{2,j} h + \theta_{3,j} g, X_{t+\tau_j}\big\rangle$$

$$- i e^{\lambda_1(t+\tau_k)/2} \langle I_u \widehat{g}_k, X_{u+t+\tau_k}\rangle \Big\}$$

$$= \lim_{u\to\infty} \mathbb{P}_\mu \exp\Big\{ i\theta W_t + \sum_{j=1}^k i e^{\lambda_1(t+\tau_j)/2} \big\langle \theta_{1,j} f + t^{-1/2}\theta_{2,j} h + \theta_{3,j} g, X_{t+\tau_j}\big\rangle$$

$$- i e^{\lambda_1(t+\tau_k)/2} \langle \widehat{g}_k, X_{t+\tau_k}\rangle + i \big\langle J_u^{(k)}(t,\cdot), X_{t+\tau_k}\big\rangle \Big\}, \tag{3.18}$$

where

$$J_u^{(k)}(t,x) := \int_{\mathbb{D}} \Big(\exp\big\{ -i e^{\lambda_1(t+\tau_k)/2} \big\langle I_u(\widehat{g}_k), \omega_u\big\rangle\big\} - 1 + i e^{\lambda_1(t+\tau_k)/2} \big\langle I_u(\widehat{g}_k), \omega_u\big\rangle\Big) \mathbb{N}_x(d\omega).$$

The last equality in (3.18) follows from the Markov property of X, (2.16) and the fact that

$$\int_{\mathbb{D}} \langle I_u \widehat{g}_k, \omega_u\rangle \mathbb{N}_x(d\omega) = \mathbb{P}_{\delta_x} \langle I_u \widehat{g}_k, X_u\rangle = \widehat{g}_k(x).$$

By (3.6), we have that as $u \to \infty$,

$$\big\langle I_u(\widehat{g}_k), \omega_u\big\rangle \to H_\infty(\widehat{g}_k)(\omega), \quad \mathbb{N}_x\text{-a.e.}, \quad \text{in } L^1(\mathbb{N}_x) \text{ and in } L^2(\mathbb{N}_x).$$

Then one can prove that

$$\lim_{u\to\infty} \big\langle J_u^{(k)}(t,\cdot), X_{t+\tau_k}\big\rangle = \big\langle J^{(k)}(t,\cdot), X_{t+\tau_k}\big\rangle, \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.19}$$

where

$$J^{(k)}(t,x) := \int_{\mathbb{D}} \Big(\exp\big\{ -i e^{\lambda_1(t+\tau_k)/2} H_\infty(\widehat{g}_k)\big\} - 1 + i e^{\lambda_1(t+\tau_k)/2} H_\infty(\widehat{g}_k)\Big) \mathbb{N}_x(d\omega).$$

For the detailed proof of (3.19), we refer readers to the proof of [34, Theorem 1.4]. Thus, by (3.18) and the dominated convergence theorem, we get that

$$\phi(t) = \mathbb{P}_\mu \exp\Big\{ i\theta W_t + \sum_{j=1}^k i e^{\lambda_1(t+\tau_j)/2} \big\langle \theta_{1,j} f + t^{-1/2}\theta_{2,j} h + \theta_{3,j} g, X_{t+\tau_j}\big\rangle$$

$$- i e^{\lambda_1(t+\tau_k)/2} \langle \widehat{g}_k, X_{t+\tau_k}\rangle + i \big\langle J^{(k)}(t,\cdot), X_{t+\tau_k}\big\rangle \Big\}.$$

It is known (see [34, (3.46)]) that

$$\lim_{t\to\infty} \big\langle J^{(k)}(t,\cdot), X_{t+\tau_k}\big\rangle = \exp\Big\{ -\frac{1}{2} \Big(\mathbb{N}_\cdot \big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\Big)_m W_\infty \Big\} \quad \text{in } \mathbb{P}_\mu\text{-probability}.$$

By the definition in (3.16), $B_k(x) = \theta_{1,k} f(x) + \theta_{3,k} g(x) - \widehat{g}_k(x)$. Thus, as $t \to \infty$,

$$\lim_{t\to\infty} \big|\phi(t) - \phi_1(t)\big| = 0, \tag{3.20}$$

where

$$\begin{aligned}\phi_1(t) := \mathbb{P}_\mu \exp\bigg\{&\Big(i\theta - \frac{1}{2}\big(\mathbb{N}.\big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\big)_m\Big)W_t \\ &+ \sum_{j=1}^{k-1} i e^{\lambda_1(t+\tau_j)/2}\big\langle \theta_{1,j} f_j + t^{-1/2}\theta_{2,j}h + \theta_{3,j}g, X_{t+\tau_j}\big\rangle \\ &+ i e^{\lambda_1(t+\tau_k)/2}\big\langle B_k + t^{-1/2}\theta_{2,k}h, X_{t+\tau_k}\big\rangle\bigg\}. \end{aligned} \tag{3.21}$$

Therefore, to find the limit in (3.15), we only need to find the limit of $\phi_1(t)$.

Step 2. In this step, we reduce the problem of finding the above limit to the problem of finding the limit of $\phi_2(t)$:

$$\begin{aligned}\phi_2(t) := \mathbb{P}_\mu \exp\bigg\{&\Big(i\theta - \frac{1}{2}\big(\mathbb{N}.\big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\big)_m - \frac{1}{2}\sum_{j=1}^{k} C_j\Big)W_t \\ &+ i e^{\lambda_1 t/2}\langle \tilde{f}_0, X_t\rangle + i t^{-1/2} e^{\lambda_1 t/2}\Big\langle \sum_{j=1}^{k}\theta_{2,j}h, X_t\Big\rangle\bigg\}, \end{aligned} \tag{3.22}$$

where C_j, $j = 1, \dots, k$, are the constants defined in (3.25) below. In the following we explain the details of this reduction.

By the Markov property of X, we have

$$\begin{aligned}&\mathbb{P}_\mu\big[\exp\big\{i e^{\lambda_1(t+\tau_k)/2}\big\langle B_k + t^{-1/2}\theta_{2,k}h, X_{t+\tau_k}\big\rangle\big\}\big|\mathcal{F}_{t+\tau_{k-1}}\big] \\ &\quad= \exp\bigg\{\bigg\langle \int_{\mathbb{D}}\big(\exp\big\{i e^{\lambda_1(t+\tau_k)/2}\big\langle B_k + t^{-1/2}\theta_{2,k}h, \omega_{s_k}\big\rangle\big\} - 1\big)\mathbb{N}.(d\omega), X_{t+\tau_{k-1}}\bigg\rangle\bigg\} \\ &\quad= \exp\big\{i e^{\lambda_1(t+\tau_k)/2}\big\langle \mathbb{N}.\big\langle B_k + t^{-1/2}\theta_{2,k}h, \omega_{s_k}\big\rangle, X_{t+\tau_{k-1}}\big\rangle\big\} \\ &\qquad\times \exp\bigg\{-\frac{1}{2}e^{\lambda_1(t+\tau_k)}\big\langle \mathbb{N}.\langle B_k, \omega_{s_k}\rangle^2, X_{t+\tau_{k-1}}\big\rangle\bigg\} \times \exp\big\{\big\langle R(t,\cdot), X_{t+\tau_{k-1}}\big\rangle\big\} \\ &\quad=: (I) \times (II) \times (III), \end{aligned}$$

where

$$\begin{aligned}R(t,x) := \int_{\mathbb{D}}\Big(&\exp\big\{i e^{\lambda_1(t+\tau_k)/2}\big\langle t^{-1/2}\theta_{2,k}h + B_k, \omega_{s_k}\big\rangle\big\} - 1 \\ &- i e^{\lambda_1(t+\tau_k)/2}\big\langle t^{-1/2}\theta_{2,k}h + B_k, \omega_{s_k}\big\rangle + \frac{1}{2}e^{\lambda_1(t+\tau_k)}\langle B_k, \omega_{s_k}\rangle^2\Big)\mathbb{N}_x(d\omega), \quad x \in E. \end{aligned}$$

For part (I), by the definition of $\widehat{g}_k$, we get that

$$\begin{aligned}\theta_{3,k}g(x) - \widehat{g}_k(x) &= -\sum_{j=0}^{k-1}\theta_{3,j}e^{\lambda_1(\tau_j-\tau_k)/2}I_{\tau_k-\tau_j}g(x) \\ &= -e^{-\lambda_1(\tau_k-\tau_{k-1})/2}I_{\tau_k-\tau_{k-1}}\widehat{g}_{k-1}(x), \quad x \in E. \end{aligned} \tag{3.23}$$

Since $h \in \mathcal{C}_c$, we have $T_s h(x) = e^{-\lambda_1 s/2} h(x)$. Thus, for $x \in E$,

$$\begin{aligned}\mathbb{N}_x\big(\langle B_k + t^{-1/2}\theta_{2,k} h, \omega_{s_k}\rangle\big) &= T_{s_k}\big(B_k + t^{-1/2}\theta_{2,k} h\big)(x)\\ &= \theta_{1,k} T_{s_k} f(x) + t^{-1/2}\theta_{2,k} e^{-\lambda_1 s_k/2} h(x) - e^{-\lambda_1 s_k/2}\widehat{g}_{k-1}(x).\end{aligned}$$

Hence, we have

$$(I) = \exp\big\{ i e^{\lambda_1(t+\tau_{k-1})/2} \big\langle \theta_{1,k} e^{\lambda_1 s_k/2} T_{s_k} f + t^{-1/2}\theta_{2,k} h - \widehat{g}_{k-1}, X_{t+\tau_{k-1}} \big\rangle \big\}. \tag{3.24}$$

For part (*II*), we define for $j = 1, \ldots, k$,

$$C_j := e^{\lambda_1 s_j} \big(\mathbb{N}_\cdot \langle B_j, \omega_{s_j}\rangle^2, \phi_1\big)_m = e^{\lambda_1 s_j} \big(\mathbb{V}ar_{\delta_\cdot} \langle B_j, \omega_{s_j}\rangle, \phi_1\big)_m. \tag{3.25}$$

By Lemma 3.1, we get that, as $t \to \infty$,

$$e^{\lambda_1(t+\tau_k)} \big\langle \mathbb{N}_\cdot \langle B_k, \omega_{s_k}\rangle^2, X_{t+\tau_{k-1}} \big\rangle \to C_k W_\infty$$

in $\mathbb{P}_\mu$-probability. Thus, we get that, as $t \to \infty$,

$$(II) \to \exp\left\{ -\frac{1}{2} C_k W_\infty \right\}, \quad \text{in } \mathbb{P}_\mu\text{-probability.} \tag{3.26}$$

Now, we deal with part (*III*). For $x_1, x_2 \in \mathbb{R}$, by (3.10), we have

$$\begin{aligned}&\left| e^{i(x_1+x_2)} - 1 - i(x_1+x_2) + \frac{1}{2}(x_1)^2 \right|\\ &\quad \le \left| e^{ix_1} - 1 - ix_1 + \frac{1}{2}(x_1)^2 \right| + \left| e^{ix_2} - 1 - ix_2 \right| + \left| e^{ix_1} - 1 \right| \left| e^{ix_2} - 1 \right|\\ &\quad \le |x_1|^2 \left(1 \wedge \frac{|x_1|}{6} \right) + \frac{1}{2}|x_2|^2 + |x_1 x_2|.\end{aligned} \tag{3.27}$$

Using (3.27) with $x_1 = e^{\lambda_1(t+\tau_k)/2}\langle B_k, \omega_{s_k}\rangle$ and $x_2 = \theta_{2,k} t^{-1/2} e^{\lambda_1(t+\tau_k)/2}\langle h, \omega_{s_k}\rangle$, we get

$$\begin{aligned}\big|R(t,x)\big| &\le e^{\lambda_1(t+\tau_k)} \mathbb{N}_x \left[\langle B_k, \omega_{s_k}\rangle^2 \left(1 \bigwedge \frac{e^{\lambda_1(t+\tau_k)/2} |\langle B_k, \omega_{s_k}\rangle|}{6} \right) \right]\\ &\quad + \frac{(\theta_{2,k})^2}{2} t^{-1} e^{\lambda_1(t+\tau_k)} \mathbb{N}_x \langle h, \omega_{s_k}\rangle^2 + |\theta_{2,k}| t^{-1/2} e^{\lambda_1(t+\tau_k)} \mathbb{N}_x \big| \langle h, \omega_{s_k}\rangle \langle B_k, \omega_{s_k}\rangle \big|\\ &= e^{\lambda_1(t+\tau_k)} \Bigg(\mathbb{N}_x \left[\langle B_k, \omega_{s_k}\rangle^2 \left(1 \bigwedge \frac{e^{\lambda_1(t+\tau_k)/2} |\langle B_k, \omega_{s_k}\rangle|}{6} \right) \right]\\ &\quad + \frac{(\theta_{2,k})^2}{2} t^{-1} \mathbb{N}_x \langle h, \omega_{s_k}\rangle^2 + |\theta_{2,k}| t^{-1/2} \mathbb{N}_x \big| \langle h, \omega_{s_k}\rangle \langle B_k, \omega_{s_k}\rangle \big| \Bigg)\\ &=: e^{\lambda_1(t+\tau_k)} U(t,x).\end{aligned}$$

Notice that $U(\cdot, x) \downarrow 0$, as $t \to \infty$. Thus, for $t > u$,

$$\begin{aligned}\limsup_{t\to\infty} e^{\lambda_1(t+\tau_k)} \mathbb{P}_\mu \big\langle U(t,\cdot), X_{t+\tau_{k-1}} \big\rangle &\le \limsup_{t\to\infty} e^{\lambda_1(t+\tau_k)} \big\langle T_{t+\tau_{k-1}} U(u,\cdot), \mu \big\rangle\\ &= e^{\lambda_1 s_k} \big(U(u,\cdot), \phi_1 \big)_m \langle \phi_1, \mu \rangle,\end{aligned}$$

where the last equality follows from (2.3) since $\gamma(U(u,\cdot)) = 1$ for any $u > 0$. Letting $u \to \infty$, we get that

$$\lim_{t\to\infty} e^{\lambda_1(t+\tau_k)}\mathbb{P}_\mu\langle U(t,\cdot), X_{t+\tau_{k-1}}\rangle = 0,$$

which implies that

$$\lim_{t\to\infty} \mathbb{P}_\mu\big|\langle R(t,\cdot), X_{t+\tau_{k-1}}\rangle\big| = 0. \tag{3.28}$$

Thus, by (3.24), (3.26) and (3.28), we have that, as $t \to \infty$,

$$\begin{aligned}&\Big|\mathbb{P}_\mu\big[\exp\{ie^{\lambda_1(t+\tau_k)/2}\langle B_k + t^{-1/2}\theta_{2,k}h, X_{t+\tau_k}\rangle\}\big|\mathcal{F}_{t+\tau_{k-1}}\big]\\ &\quad - \exp\Big\{-\frac{1}{2}C_k W_t + ie^{\lambda_1(t+\tau_{k-1})/2}\langle\theta_{1,k}e^{\lambda_1(s_k)/2}T_{s_k}f + t^{-1/2}\theta_{2,k}h - \widehat{g}_{k-1}, X_{t+\tau_{k-1}}\rangle\Big\}\Big|\\ &\to 0 \quad \text{in } \mathbb{P}_\mu\text{-probability.}\end{aligned}$$

Hence, using the Markov property and the dominated convergence theorem, we get that, as $t \to \infty$,

$$\begin{aligned}&\Bigg|\phi_1(t) - \mathbb{P}_\mu \exp\Bigg\{\bigg(i\theta - \frac{1}{2}\big(\mathbb{N}.\big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\big)_m - \frac{1}{2}C_k\bigg)W_t\\ &\qquad + \sum_{j=1}^{k-2} ie^{\lambda_1(t+\tau_j)/2}\langle\theta_{1,j}f + t^{-1/2}\theta_{2,j}h + \theta_{3,j}g, X_{t+\tau_j}\rangle\\ &\qquad + ie^{\lambda_1(t+\tau_{k-1})/2}\langle B_{k-1} + t^{-1/2}(\theta_{2,k-1}+\theta_{2,k})h, X_{t+\tau_{k-1}}\rangle\Bigg\}\Bigg|\\ &\to 0.\end{aligned}$$

Repeating the above procedure k times, we obtain that, as $t \to \infty$,

$$\begin{aligned}&\Bigg|\phi_1(t) - \mathbb{P}_\mu \exp\Bigg\{\bigg(i\theta - \frac{1}{2}\big(\mathbb{N}.\big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\big)_m - \frac{1}{2}\sum_{j=1}^{k}C_j\bigg)W_t\\ &\qquad + ie^{\lambda_1 t/2}\langle\tilde{f}_0, X_t\rangle + it^{-1/2}e^{\lambda_1 t/2}\Big\langle\sum_{j=1}^{k}\theta_{2,j}h, X_t\Big\rangle\Bigg\}\Bigg| \to 0.\end{aligned} \tag{3.29}$$

Therefore to find the limit in (3.21), we only need to find the limit of $\phi_2(t)$ defined in (3.22).

Step 3. In this step, we try to find the limit in (3.22). By Lemma 1.1 with h replaced by $\sum_{j=1}^{k}\theta_{2,j}h$ and f replaced by $\tilde{f}_0$, we have

$$\begin{aligned}&\bigg(W_t, t^{-1/2}e^{\lambda_1 t/2}\Big\langle\sum_{j=1}^{k}\theta_{2,j}h, X_t\Big\rangle, e^{\lambda_1 t/2}\langle\tilde{f}_0, X_t\rangle\bigg)\\ &\quad \stackrel{d}{\to} \bigg(W_\infty, \sqrt{W_\infty}G_2\Big(\sum_{j=1}^{k}\theta_{2,j}h\Big), \sqrt{W_\infty}G_1(\tilde{f}_0)\bigg),\end{aligned}$$

where $G_2(\sum_{j=1}^k \theta_{2,j} h) \sim \mathcal{N}(0, (\sum_{j=1}^k \theta_{2,j})^2 \rho_h^2)$, $G_1(\tilde{f}_0) \sim \mathcal{N}(0, \sigma_{\tilde{f}_0}^2)$. Moreover, W_∞, $G_2(\sum_{j=1}^k \theta_{2,j} h)$ and $G_1(\tilde{f}_0)$ are independent. Thus, using equivalent definitions of convergence in distribution and noticing that $\mathbb{N}.(H_\infty(\widehat{g}_k))^2 \geq 0$ and $\sum_{j=1}^k C_j \geq 0$, we get that

$$\lim_{t\to\infty} \phi_2(t) = \mathbb{P}_\mu \exp\left\{\left(i\theta - \frac{1}{2}\left(\mathbb{N}.\left(H_\infty(\widehat{g}_k)\right)^2, \phi_1\right)_m - \frac{1}{2}\sum_{j=1}^k C_j - \frac{1}{2}\sigma_{\tilde{f}_0}^2 - \frac{1}{2}\left(\sum_{j=1}^k \theta_{2,j}\right)^2 \rho_h^2\right) W_\infty\right\}. \tag{3.30}$$

Now we calculate the quantity $(\mathbb{N}.(H_\infty(\widehat{g}_k))^2, \phi_1)_m + \sum_{j=1}^k C_j + \sigma_{\tilde{f}_0}^2$. By the definition of C_j in (3.25), we have,

$$\begin{aligned}
&\left(\mathbb{N}.\left(H_\infty(\widehat{g}_k)\right)^2, \phi_1\right)_m + \sum_{j=1}^k C_j + \sigma_{\tilde{f}_0}^2 \\
&= \left[\left(\mathbb{N}.\left(H_\infty(\widehat{g}_k)\right)^2, \phi_1\right)_m + \sum_{j=1}^k e^{\lambda_1 s_j}\left(\mathbb{V}ar_{\delta.}\langle \theta_{3,j} g - \widehat{g}_j, \omega_{s_j}\rangle, \phi_1\right)_m\right] \\
&\quad + \left[\sum_{j=1}^k e^{\lambda_1 s_j}\left(\mathbb{V}ar_{\delta.}\langle \tilde{f}_j, \omega_{s_j}\rangle, \phi_1\right)_m + \sigma_{\tilde{f}_0}^2\right] \\
&\quad + 2\sum_{j=1}^k e^{\lambda_1 s_j}\left(\mathbb{C}ov_{\delta.}\left(\langle \tilde{f}_j, \omega_{s_j}\rangle, \langle \theta_{3,j} g - \widehat{g}_j, \omega_{s_j}\rangle\right), \phi_1\right)_m.
\end{aligned}$$

In the following, we calculate the three parts separately.

1. By (3.9) and (3.23), we have that, for $j = 1, \ldots, k$,

$$\begin{aligned}
\left(\mathbb{N}.\left(H_\infty(\widehat{g}_j)\right)^2, \phi_1\right)_m &= \int_0^\infty e^{-\lambda_1 s}\left(A(I_s\widehat{g}_j)^2, \phi_1\right)_m ds \\
&= \int_0^\infty e^{-\lambda_1 s}\left(A\left(I_s\left(\theta_{3,j} g + e^{-\lambda_1(\tau_j - \tau_{j-1})/2} I_{\tau_j - \tau_{j-1}}\widehat{g}_{j-1}\right)\right)^2, \phi_1\right)_m ds \\
&= \theta_{3,j}^2 \beta_g^2 + 2\theta_{3,j}\sum_{l=0}^{j-1} \theta_{3,l}\beta_{g,\tau_j - \tau_l} \\
&\quad + \int_{\tau_j - \tau_{j-1}}^\infty e^{-\lambda_1 s}\left(A(I_s\widehat{g}_{j-1})^2, \phi_1\right)_m ds.
\end{aligned}$$

By (3.8) and (3.23), we get that

$$\begin{aligned}
\left(\mathbb{V}ar_{\delta.}\langle \theta_{3,j} g - \widehat{g}_j, \omega_{s_j}\rangle, \phi_1\right)_m &= \left(\mathbb{V}ar_{\delta.}\langle I_{\tau_j - \tau_{j-1}}\widehat{g}_{j-1}, \omega_{\tau_j - \tau_{j-1}}\rangle, \phi_1\right)_m \\
&= \int_0^{\tau_j - \tau_{j-1}} e^{-\lambda_1 s}\left(A\left[I_s(\widehat{g}_{j-1})\right]^2, \phi_1\right)_m ds.
\end{aligned}$$

Thus, we have, for $j = 1, \ldots, k$,

$$\big(\mathbb{N}_\cdot\big(H_\infty(\widehat{g}_j)\big)^2, \phi_1\big)_m + \big(\mathbb{V}ar_{\delta_\cdot}\langle \theta_{3,j} g - \widehat{g}_j, \omega_{s_j}\rangle, \phi_1\big)_m$$
$$= \theta_{3,j}^2 \beta_g^2 + 2\theta_{3,j} \sum_{l=0}^{j-1} \theta_{3,l} \beta_{g,\tau_j-\tau_l} + \big(\mathbb{N}_\cdot\big(H_\infty(\widehat{g}_{j-1})\big)^2, \phi_1\big)_m.$$

Summing over j and using the fact that $\widehat{g}_0 = 0$, we get

$$\big(\mathbb{N}_\cdot\big(H_\infty(\widehat{g}_k)\big)^2, \phi_1\big)_m + \sum_{j=1}^{k} \big(\mathbb{V}ar_{\delta_\cdot}\langle \theta_{3,j} g - \widehat{g}_j, \omega_{s_j}\rangle, \phi_1\big)_m$$
$$= \sum_{j=1}^{k} \theta_{3,j}^2 \beta_g^2 + 2\sum_{j=1}^{k}\sum_{l=0}^{j-1} \theta_{3,j}\theta_{3,l} \beta_{g,\tau_j-\tau_l}. \tag{3.31}$$

2. Since $\tilde{f}_j = \theta_{1,j} f + e^{\lambda_1(\tau_{j+1}-\tau_j)/2} T_{\tau_{j+1}-\tau_j} \tilde{f}_{j+1} = \theta_{1,j} f + \sum_{l=j+1}^{k} \theta_{1,l} e^{\lambda_1(\tau_l-\tau_j)/2} T_{\tau_l-\tau_j} f$, we have

$$\sigma_{\tilde{f}_j}^2 = \int_0^\infty e^{\lambda_1 u}\big(A[T_u \tilde{f}_j]^2, \phi_1\big)_m du$$
$$= \int_0^\infty e^{\lambda_1 u}\big(A\big[T_u\big(\theta_{1,j} f + e^{\lambda_1(\tau_{j+1}-\tau_j)/2} T_{\tau_{j+1}-\tau_j} \tilde{f}_{j+1}\big)\big]^2, \phi_1\big)_m du$$
$$= \theta_{1,j}^2 \sigma_f^2 + 2\sum_{l=j+1}^{k} \theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j}$$
$$+ e^{\lambda_1(\tau_{j+1}-\tau_j)} \int_0^\infty e^{\lambda_1 u}\big(A[T_{u+\tau_{j+1}-\tau_j} \tilde{f}_{j+1}]^2, \phi_1\big)_m du$$
$$= \theta_{1,j}^2 \sigma_f^2 + 2\sum_{l=j+1}^{k} \theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j} + \int_{\tau_{j+1}-\tau_j}^\infty e^{\lambda_1 u}\big(A[T_u \tilde{f}_{j+1}]^2, \phi_1\big)_m du.$$

By (2.7), we have

$$e^{\lambda_1 s_{j+1}}\big(\mathbb{V}ar_{\delta_\cdot}\langle \tilde{f}_{j+1}, \omega_{s_{j+1}}\rangle, \phi_1\big)_m = \int_0^{\tau_{j+1}-\tau_j} e^{\lambda_1 u}\big(A[T_u \tilde{f}_{j+1}]^2, \phi_1\big)_m du.$$

Thus, we get, for $j = 0, \ldots, k-1$,

$$\sigma_{\tilde{f}_j}^2 + e^{\lambda_1 s_{j+1}}\big(\mathbb{V}ar_{\delta_\cdot}\langle \tilde{f}_{j+1}, \omega_{s_{j+1}}\rangle, \phi_1\big)_m = \theta_{1,j}^2\sigma_f^2 + 2\sum_{l=j+1}^{k} \theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j} + \sigma_{\tilde{f}_{j+1}}^2.$$

Therefore, summing over j on both sides of the above equality, we get

$$\sum_{j=1}^{k} e^{\lambda_1 s_j}\big(\mathbb{V}ar_{\delta_\cdot}\langle \tilde{f}_j, \omega_{s_j}\rangle, \phi_1\big)_m + \sigma_{\tilde{f}_0}^2 = \sum_{j=0}^{k-1}\theta_{1,j}^2\sigma_f^2 + 2\sum_{j=0}^{k-1}\sum_{l=j+1}^{k} \theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j} + \sigma_{\tilde{f}_k}^2$$
$$= \sum_{j=1}^{k}\theta_{1,j}^2\sigma_f^2 + 2\sum_{j=1}^{k-1}\sum_{l=j+1}^{k} \theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j}, \tag{3.32}$$

where the last equality follows from the fact that $\theta_{1,0} = 0$ and $\tilde{f}_k = \theta_{1,k} f$.

3. Since $\tilde{f}_j=\sum_{l=j}^{k}\theta_{1,l}e^{\lambda_1(\tau_l-\tau_j)/2}T_{\tau_l-\tau_j}f$ and $\theta_{3,j}g-\widehat{g}_j=-\sum_{r=0}^{j-1}\theta_{3,r}e^{-\lambda_1(\tau_j-\tau_r)/2}I_{\tau_j-\tau_r}g$, we have

$$
\begin{aligned}
&e^{\lambda_1 s_j}\big(\mathbb{C}ov_{\delta_\cdot}\big(\langle\tilde{f}_j,\omega_{s_j}\rangle,\langle\theta_{3,j}g-\widehat{g}_j,\omega_{s_j}\rangle\big),\phi_1\big)_m\\
&=\int_0^{\tau_j-\tau_{j-1}}e^{\lambda_1 u}\big(AT_u(\tilde{f}_j)T_u(\theta_{3,j}g-\widehat{g}_j),\phi_1\big)_m du\\
&=-\sum_{l=j}^{k}\sum_{r=0}^{j-1}\theta_{1,l}\theta_{3,r}e^{\lambda_1(\tau_l+\tau_r-2\tau_j)/2}\int_0^{\tau_j-\tau_{j-1}}e^{\lambda_1 u}(AT_{u+\tau_l-\tau_j}fI_{\tau_j-\tau_r-u}g,\phi_1)_m du\\
&=-\sum_{l=j}^{k}\sum_{r=0}^{j-1}\theta_{1,l}\theta_{3,r}e^{\lambda_1(\tau_l+\tau_r)/2}\int_{\tau_{j-1}}^{\tau_j}e^{-\lambda_1 u}(AT_{\tau_l-u}fI_{u-\tau_r}g,\phi_1)_m du.
\end{aligned}
$$

Thus, we get that

$$
\begin{aligned}
&2\sum_{j=1}^{k}e^{\lambda_1 s_j}\big(\mathbb{C}ov_{\delta_\cdot}\big(\langle\tilde{f}_j,\omega_{s_j}\rangle,\langle\theta_{3,j}g-\widehat{g}_j,\omega_{s_j}\rangle\big),\phi_1\big)_m\\
&\quad=-2\sum_{l=1}^{k}\sum_{r=0}^{l-1}\sum_{j=r+1}^{l}\theta_{1,l}\theta_{3,r}e^{\lambda_1(\tau_l+\tau_r)/2}\int_{\tau_{j-1}}^{\tau_j}e^{-\lambda_1 u}(AT_{\tau_l-u}fI_{u-\tau_r}g,\phi_1)_m du\\
&\quad=-2\sum_{l=1}^{k}\sum_{r=0}^{l-1}\theta_{1,l}\theta_{3,r}e^{\lambda_1(\tau_l+\tau_r)/2}\int_{\tau_r}^{\tau_l}e^{-\lambda_1 u}(AT_{\tau_l-u}fI_{u-\tau_r}g,\phi_1)_m du.
\end{aligned}
\tag{3.33}
$$

Now combining (3.31)–(3.33), we obtain that

$$
\begin{aligned}
&\big(\mathbb{N}_\cdot\big(H_\infty(\widehat{g}_k)\big)^2,\phi_1\big)_m+\sum_{j=1}^{k}C_j+\sigma_{\tilde{f}_0}^2\\
&=\sum_{j=1}^{k}\theta_{3,j}^2\beta_g^2+2\sum_{j=1}^{k}\sum_{l=0}^{j-1}\theta_{3,j}\theta_{3,l}\beta_{g,\tau_j-\tau_l}+\sum_{j=1}^{k}\theta_{1,j}^2\sigma_f^2+2\sum_{j=1}^{k-1}\sum_{l=j+1}^{k}\theta_{1,j}\theta_{1,l}\sigma_{f,\tau_l-\tau_j}\\
&\quad-2\sum_{l=1}^{k}\sum_{r=0}^{l-1}\theta_{1,l}\theta_{3,r}e^{\lambda_1(\tau_l+\tau_r)/2}\int_{\tau_r}^{\tau_l}e^{-\lambda_1 u}(AT_{\tau_l-u}fI_{u-\tau_r}g,\phi_1)_m du.
\end{aligned}
\tag{3.34}
$$

Step 4. Combining Steps 1 and 2 with (3.30) and (3.34), we get (3.11) immediately. The proof is now complete. □

Remark 3.3 By Lemma 3.2, we know that, for any $f\in\mathcal{C}_s$ and $g\in\mathcal{C}_l$, there exists a Gaussian process $(G^{1,U_qf},G^{3,g})$ with mean 0 and covariance function defined as in Theorem 1.2. Furthermore, the next lemma shows that, this Gaussian process has a continuous version. Thus, the Gaussian process $(G^{1,U_qf},G^{3,g})$ defined in Theorem 1.2 exists.

Lemma 3.4 *Assume that $f\in\mathcal{C}_s$ and $g\in\mathcal{C}_l$. If $(G^{1,U_qf}(\tau),G^{3,g}(\tau))_{\tau\geq 0}$ is a Gaussian process, on some probability space $(\Omega,\mathcal{F},P)$, with mean 0 and covariance function defined as in Theorem 1.2, then, $(G^{1,U_qf},G^{3,g})$ has a continuous version.*

Proof By Kolmogorov's continuity criterion, it suffices to show that, for any $\tau_2 > \tau_1 \ge 0$,

$$P\left|G^{1,U_q f}(\tau_2) - G^{1,U_q f}(\tau_1)\right|^4 + P\left|G^{3,g}(\tau_2) - G^{3,g}(\tau_1)\right|^4 \le C|\tau_2 - \tau_1|^2, \tag{3.35}$$

where C is a constant.

(1) Since $G^{1,U_q f}(\tau_2) - G^{1,U_q f}(\tau_1) \sim \mathcal{N}(0, \Sigma(\tau_1, \tau_2))$ with $\Sigma(\tau_1, \tau_2) = P|G^{1,U_q f}(\tau_2) - G^{1,U_q f}(\tau_1)|^2$, we have

$$P\left|G^{1,U_q f}(\tau_2) - G^{1,U_q f}(\tau_1)\right|^4 = 3\Sigma(\tau_1, \tau_2)^2. \tag{3.36}$$

In the following, we write $U_q f$ as $f^{(q)}$. By (3.12), we have

$$\begin{aligned}
\Sigma(\tau_1, \tau_2) &= P\left|G^{1,U_q f}(\tau_2) - G^{1,U_q f}(\tau_1)\right|^2 \\
&= 2\int_0^\infty e^{\lambda_1 s}\left(A\left(T_s f^{(q)}\right)^2, \phi_1\right)_m ds \\
&\qquad - 2e^{\lambda_1(\tau_2-\tau_1)/2}\int_0^\infty e^{\lambda_1 s}\left(A\left(T_s f^{(q)}\right)\left(T_{s+\tau_2-\tau_1} f^{(q)}\right), \phi_1\right)_m ds \\
&= 2\int_0^\infty e^{\lambda_1 s}\left(A\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right), \phi_1\right)_m ds \\
&\le 2K\int_0^\infty e^{\lambda_1 s}\left\|\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right)\right\|_2 ds.
\end{aligned}$$

We rewrite the last integral above as the sum of integrals over $(0, t_0)$ and (t_0, ∞). For $s > t_0$,

$$\begin{aligned}
&\left\|\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right)\right\|_2 \\
&\quad\lesssim e^{-2\lambda_{\gamma(f)}s}\|a_{t_0}\|_2\left\|f^{(q)}\right\|_2\left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_2.
\end{aligned} \tag{3.37}$$

Thus,

$$\int_{t_0}^\infty e^{\lambda_1 s}\left\|\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right)\right\|_2 ds \lesssim \left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_2. \tag{3.38}$$

For $s \le t_0$, since $\|T_s\|_4 \le e^{Ks}$, we have

$$\begin{aligned}
&\left\|\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right)\right\|_2 \\
&\quad\le \left\|T_s f^{(q)}\right\|_4\left\|T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right\|_4 \\
&\quad\le e^{2Ks}\left\|f^{(q)}\right\|_4\left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_4.
\end{aligned}$$

Thus,

$$\int_0^{t_0} e^{\lambda_1 s}\left\|\left(T_s f^{(q)}\right)\left(T_s\left(f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right)\right)\right\|_2 ds \lesssim \left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_4. \tag{3.39}$$

Combining (3.38) and (3.39) we get that

$$\Sigma(\tau_1, \tau_2) \lesssim \left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_2 + \left\|f^{(q)} - e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1} f^{(q)}\right\|_4. \tag{3.40}$$

It follows from Fubini's theorem that, for $p=2,4$,

$$\begin{aligned}&\left\|U_q f-e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1}U_q f\right\|_p\\&=\left\|\int_0^\infty e^{-qu}T_u f\,du-e^{(\lambda_1/2+q)(\tau_2-\tau_1)}\int_{\tau_2-\tau_1}^\infty e^{-qu}T_u f\,du\right\|_p\\&\le\left\|\int_0^{\tau_2-\tau_1} e^{-qu}T_u f\,du\right\|_p+\left(e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\right)\left\|\int_{\tau_2-\tau_1}^\infty e^{-qu}T_u f\,du\right\|_p\\&\le\int_0^{\tau_2-\tau_1} e^{-qu}\|T_u f\|_p du+\left(e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\right)\int_{\tau_2-\tau_1}^\infty e^{-qu}\|T_u f\|_p du.\end{aligned}$$

Since $\|T_u f\|_p\le e^{Ku}\|f\|_p$ and $q>K$, we have

$$\int_0^{\tau_2-\tau_1} e^{-qu}\|T_u f\|_p du\le\int_0^{\tau_2-\tau_1} e^{-qu}e^{Ku}du\|f\|_p\le(\tau_2-\tau_1)\|f\|_p. \tag{3.41}$$

If $\tau_2-\tau_1>t_0$, by (2.4), for $u>\tau_2-\tau_1$, we have $\|T_u f\|_p\lesssim e^{-\lambda_{\gamma(f)}u}\|f\|_2\|a_{t_0}^{1/2}\|_p$. Thus,

$$\begin{aligned}&\left(e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\right)\int_{\tau_2-\tau_1}^\infty e^{-qu}\|T_u f\|_p du\\&\quad\lesssim e^{(\lambda_1/2+q)(\tau_2-\tau_1)}\int_{\tau_2-\tau_1}^\infty e^{-qu}e^{-\lambda_{\gamma(f)}u}du\|f\|_2\\&\quad\lesssim e^{(\lambda_1/2-\lambda_{\gamma(f)})(\tau_2-\tau_1)}\lesssim\tau_2-\tau_1.\end{aligned} \tag{3.42}$$

If $\tau_2-\tau_1\le t_0$, then $e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\lesssim\tau_2-\tau_1$. Thus,

$$\begin{aligned}\left(e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\right)\int_{\tau_2-\tau_1}^\infty e^{-qu}\|T_u f\|_p du&\le\left(e^{(\lambda_1/2+q)(\tau_2-\tau_1)}-1\right)\|f\|_p\int_0^\infty e^{-qu}e^{Ku}du\\&\lesssim\tau_2-\tau_1.\end{aligned} \tag{3.43}$$

Now, combining (3.41)–(3.43), we obtain that, for $p=2,4$,

$$\left\|U_q f-e^{\lambda_1(\tau_2-\tau_1)/2}T_{\tau_2-\tau_1}U_q f\right\|_p\lesssim\tau_2-\tau_1.$$

Now, by (3.40), we have

$$\Sigma(\tau_1,\tau_2)\le C(\tau_2-\tau_1). \tag{3.44}$$

Thus, by (3.36) and (3.44), we get

$$P\left|G^{1,U_q f}(\tau_2)-G^{1,U_q f}(\tau_1)\right|^4\le C(\tau_2-\tau_1)^2. \tag{3.45}$$

(2) We claim that

$$P\left|G^{3,g}(\tau_2)-G^{3,g}(\tau_1)\right|^4\le C(\tau_2-\tau_1)^2, \tag{3.46}$$

where C is a constant. To prove (3.46), using the same argument as that of leading to (3.36), it suffices to show that, for $0\le\tau_1\le\tau_2$,

$$P\left(G^{3,g}(\tau_2)-G^{3,g}(\tau_1)\right)^2\le C(\tau_2-\tau_1). \tag{3.47}$$

Note that

$$\begin{aligned}P\big(G^{3,g}(\tau_2)-G^{3,g}(\tau_1)\big)^2 &= 2\beta_{g,0}-2\beta_{g,\tau_2-\tau_1}\\ &= 2\int_0^\infty e^{-\lambda_1 s}\big(A(I_s g)^2,\phi_1\big)_m ds\\ &\quad -2e^{-\lambda_1(\tau_2-\tau_1)/2}\int_0^\infty e^{-\lambda_1 s}\big(A(I_s g)(I_{s+\tau_2-\tau_1}g),\phi_1\big)_m ds\\ &= 2\int_0^\infty e^{-\lambda_1 s}\big(A(I_s g)\big(I_s g-e^{-\lambda_1(\tau_2-\tau_1)/2}I_{s+\tau_2-\tau_1}g\big),\phi_1\big)_m ds.\end{aligned}$$

Since $g\in\mathcal{C}_l$, $g(x)=\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}(x)$, where $b_j^k=(g,\phi_j^{(k)})_m$. By (1.9), we have that for any $x\in E$,

$$|I_s g(x)|\le\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}e^{\lambda_k s}|b_j^k||\phi_j^k(x)|\lesssim e^{\lambda_{k_0}s}a_{2t_0}(x)^{1/2},$$

where $k_0=\sup\{k:2\lambda_k<\lambda_1\}$. By the definition of $I_u g$,

$$\begin{aligned}&\big|I_s g-e^{-\lambda_1(\tau_2-\tau_1)/2}I_{s+\tau_2-\tau_1}g\big|\\ &=\left|\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}e^{\lambda_k s}\big(1-e^{(\lambda_k-\lambda_1/2)(\tau_2-\tau_1)}\big)b_j^k\phi_j^{(k)}(x)\right|\\ &\le(-\lambda_1/2)(\tau_2-\tau_1)\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k}e^{\lambda_k s}|b_j^k||\phi_j^{(k)}(x)|\lesssim(-\lambda_1/2)(\tau_2-\tau_1)e^{\lambda_{k_0}s}a_{2t_0}(x)^{1/2}.\end{aligned}$$

It follows that

$$\begin{aligned}P\big(G^{3,g}(\tau_2)-G^{3,g}(\tau_1)\big)^2&\lesssim(-\lambda_1)K(\tau_2-\tau_1)\int_0^\infty e^{-\lambda_1 s}e^{2\lambda_{k_0}s}(a_{2t_0},\phi_1)_m ds\\ &=(-\lambda_1)K(\lambda_1-2\lambda_{k_0})^{-1}(a_{2t_0},\phi_1)_m(\tau_2-\tau_1).\end{aligned}$$

Now the proof is complete. □

By Lemma 3.2, we get the following Corollary immediately.

Corollary 3.5 *Let $f\in\mathcal{C}_s$, $h\in\mathcal{C}_c$, $g\in\mathcal{C}_l$ and $\mu\in\mathcal{M}_C(E)$. Suppose that $Y_t^{1,U_q f}$, $Y_t^{2,h}$, $Y_t^{3,g}$, $G^{1,U_q f}$, $G^{2,h}$ and $G^{3,g}$ are defined as in Theorem 1.2. Then, under $\mathbb{P}_\mu$, as $t\to\infty$,*

$$\big(W_t,Y_t^{1,U_q f},Y_t^{2,h},Y_t^{3,g}\big)\xrightarrow{\mathcal{L}(\mathbb{R}_+^4)}\big(W_\infty,\sqrt{W_\infty}G^{1,U_q f},\sqrt{W_\infty}G^{2,h}\sqrt{W_\infty}G^{3,g}\big).\tag{3.48}$$

3.2 The Tightness of $(W_t, Y_t^{1,U_q f}, Y_t^{2,h}, Y_t^{3,g})_{t>0}$ in $\mathbb{D}(\mathbb{R}^4)$

Recall that a sequence (X^n) of càdlàg processes is called C-tight if it is tight, and if all its weakly convergent limit points are continuous processes. In this subsection, we will show that $(W_t,Y_t^{1,U_q f},Y_t^{2,h},Y_t^{3,g})_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R}^4)$ (with W_t, for each $t>0$, being considered as a constant process). By [20, Chapter VI, Corollary 3.33], it suffices to show that $(Y_t^{1,U_q f})_{t>0}$, $(Y_t^{2,h})_{t>0}$ and $(Y_t^{3,g})_{t>0}$ are C-tight in $\mathbb{D}(\mathbb{R})$.

3.2.1 The Tightness of $(Y_t^{1,U_q f})_{t>0}$ in $\mathbb{D}(\mathbb{R})$

The main purpose of this subsection is to prove that $(Y_t^{1,U_q f}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$. The next lemma gives a sufficient condition for the tightness of a sequence $(X^n)_{n\geq 1}$ in $\mathbb{D}(\mathbb{R}^d)$.

Lemma 3.6 *Assume $(X^n)_{n\geq 1}$ is a sequence of $\mathbb{D}(\mathbb{R}^d)$-valued random variables, each X^n being defined on the space $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t\geq 0}, P^n)$. If (X^n) satisfies the following two conditions:*

(1) *For all $N > 0$,*

$$\limsup_{n\to\infty} P^n\Big(\sup_{t\leq N}|X_t^n|\Big) < \infty. \tag{3.49}$$

(2) *For all $N > 0$,*

$$\lim_{\theta\to 0}\limsup_{n}\sup_{S,T\in\mathcal{T}_N^n: S\leq T\leq S+\theta} P^n\big(|X_T^n - X_S^n|\big) = 0, \tag{3.50}$$

where $\mathcal{T}_N^n$ denotes the set of all $\{\mathcal{F}_t^n\}$-stopping times that are bounded by N.

Then, the sequence (X^n) is tight in $\mathbb{D}(\mathbb{R}^d)$.

Proof This follows immediately from Theorem 4.5 in [20, Chapter VI]. □

To prove the tightness of $(Y_t^{1,U_q f}(\cdot))_{t>0}$ in $\mathbb{D}(\mathbb{R})$, we will check that $Y_t^{1,U_q f}$ satisfies the two conditions above. Recall that t_0 is the constant in the condition (b) in Sect. 1.1.

Lemma 3.7 *If $f \in \mathcal{C}_s$ and $\mu \in \mathcal{M}_C(E)$, then for any $N > 0$,*

$$\sup_{t>3t_0}\mathbb{P}_\mu\Big(\sup_{\tau\leq N}\big|Y_t^{1,U_q f}(\tau)\big|\Big) < \infty. \tag{3.51}$$

Proof In this proof, we always assume that $t > 3t_0$. By (2.28), for any $t > 0$,

$$\mathbb{P}_\mu\big(Y_t^{1,U_q f}(\tau) = e^{\lambda_1(t+\tau)/2}J_1^f(t+\tau) + e^{(q+\lambda_1/2)(t+\tau)}J_2^f(t+\tau), \forall\tau\geq 0\big) = 1.$$

First, we consider $J_1^f(t+\tau)$. Recall that $J_1^f(t) = \langle T_t(U_q f), \mu\rangle$, $t\geq 0$. By (2.4), we have

$$\begin{aligned}
\sup_{\tau\leq N} e^{\lambda_1(t+\tau)/2}\big|J_1^f(t+\tau)\big| &\leq \sup_{\tau\leq N} e^{\lambda_1(t+\tau)/2}\big\langle\big|T_{t+\tau}(U_q f)\big|, \mu\big\rangle \\
&\lesssim \sup_{\tau\leq N} e^{\lambda_1(t+\tau)/2}e^{-\lambda_{\gamma(f)}(t+\tau)}\|U_q f\|_2\big\langle a_{t_0}^{1/2}, \mu\big\rangle \\
&\lesssim e^{(\lambda_1/2-\lambda_{\gamma(f)})t}\|f\|_2.
\end{aligned} \tag{3.52}$$

Next, we deal with $J_2^f(t+\tau)$. Recall that

$$J_2^f(t+\tau) = \int_{t+\tau}^{\infty} e^{-qu}M_{t+\tau}^{(u)}du.$$

Using (2.32) with $t_1 = t$, we have, for $t > 3t_0$,

$$\begin{aligned}\mathbb{P}_\mu\Big(\sup_{\tau\le N}\big|J_2^f(t+\tau)\big|\Big) &\le \mathbb{P}_\mu\int_t^\infty e^{-qu}\sup_{\tau\le N}\big(\big|M_{t+\tau}^{(u)}\big|\mathbf{1}_{t+\tau<u}\big)du\\ &\le 2\int_t^\infty e^{-qu}\sqrt{\int_E \mathbb{V}ar_{\delta_x}\langle f, X_u\rangle \mu(dx)}du\\ &\lesssim \int_t^\infty e^{-qu}e^{-\lambda_1 u/2}du\sqrt{\int_E a_{t_0}(x)^{1/2}\mu(dx)}\\ &= (q+\lambda_1/2)^{-1}e^{-(q+\lambda_1/2)t}\sqrt{\int_E a_{t_0}(x)^{1/2}\mu(dx)},\end{aligned} \tag{3.53}$$

where in the third inequality we use (2.10). It follows that,

$$\sup_{t>3t_0}\mathbb{P}_\mu\Big(\sup_{\tau\le N}e^{(q+\lambda_1/2)(t+\tau)}\big|J_2^f(t+\tau)\big|\Big) \le \sup_{t>3t_0}e^{(q+\lambda_1/2)(t+N)}\mathbb{P}_\mu\Big(\sup_{\tau\le N}\big|J_2^f(t+\tau)\big|\Big) < \infty.$$

The proof is now complete. □

Next, we prove that

Lemma 3.8 *If $f \in \mathcal{C}_s$ and $\mu \in \mathcal{M}_C(E)$, then*

$$\lim_{\theta\to 0}\limsup_{t\to\infty}\sup_{S,T\in\mathcal{T}_N^t:S<T<S+\theta}\mathbb{P}_\mu\big(\big|Y_t^{1,U_q f}(T) - Y_t^{1,U_q f}(S)\big|\big) = 0, \tag{3.54}$$

where $\mathcal{T}_N^t$ is the set of all $\{\mathcal{G}_{t+\tau} : \tau \ge 0\}$-stopping times that are bounded by N.

Proof In this proof, we always assume that $t > 3t_0$. By (2.28), we have, $\mathbb{P}_\mu$-a.s.,

$$\begin{aligned}\big|Y_t^{1,U_q f}(T) - Y_t^{1,U_q f}(S)\big| \le{}& \big|e^{\lambda_1(t+T)/2}J_1^f(t+T) - e^{\lambda_1(t+S)/2}J_1^f(t+S)\big|\\ &+ \big|e^{(q+\lambda_1/2)(t+T)}J_2^f(t+T) - e^{(q+\lambda_1/2)(t+S)}J_2^f(t+S)\big|\\ =:{}& J_{3,1}(t,T,S) + J_{3,2}(t,T,S).\end{aligned}$$

For $J_{3,1}(t,T,S)$, by (3.52), we have that, as $t\to\infty$,

$$\mathbb{P}_\mu J_{3,1}(t,T,S) \le 2\mathbb{P}_\mu\Big(\sup_{\tau\le N}e^{\lambda_1(t+\tau)/2}\big|J_1^f(t+\tau)\big|\Big) \lesssim e^{(\lambda_1/2-\lambda_{\gamma(f)})t}\|f\|_2 \to 0. \tag{3.55}$$

Note that

$$\begin{aligned}&J_{3,2}(t,T,S)\\ &\le e^{(q+\lambda_1/2)(t+S)}\big|J_2^f(t+T) - J_2^f(t+S)\big| + \big|e^{(q+\lambda_1/2)(t+T)} - e^{(q+\lambda_1/2)(t+S)}\big|\big|J_2^f(t+T)\big|\\ &\le e^{(q+\lambda_1/2)(t+N)}\big|J_2^f(t+T) - J_2^f(t+S)\big| + e^{(q+\lambda_1/2)(t+N)}\big|e^{(q+\lambda_1/2)\theta} - 1\big|\big|J_2^f(t+T)\big|.\end{aligned}$$

By (3.53), we get that, for $t > 3t_0$,

$$\begin{aligned}
&\sup_{S,T\in\mathcal{T}_N^t:S<T<S+\theta} e^{(q+\lambda_1/2)(t+N)}\left|e^{(q+\lambda_1/2)\theta}-1\right|\mathbb{P}_\mu\left|J_2^f(t+T)\right|\\
&\quad\lesssim e^{(q+\lambda_1/2)(t+N)}\left|e^{(q+\lambda_1/2)\theta}-1\right|\mathbb{P}_\mu\Big(\sup_{\tau\le N}\left|J_2^f(t+\tau)\right|\Big)\\
&\quad\lesssim \left|e^{(q+\lambda_1/2)\theta}-1\right|\to 0,\quad \text{as } \theta\to 0.
\end{aligned}\tag{3.56}$$

By (3.55) and (3.56), to prove (3.54), it suffices to show that

$$\lim_{\theta\to 0}\limsup_{t\to\infty}\sup_{S,T\in\mathcal{T}_N^t:S<T<S+\theta} e^{(q+\lambda_1/2)t}\mathbb{P}_\mu\left|J_2^f(t+T)-J_2^f(t+S)\right|=0.\tag{3.57}$$

By the definition of J_2^f, we have

$$\begin{aligned}
\left|J_2^f(t+T)-J_2^f(t+S)\right| &= \left|\int_{t+T}^\infty e^{-qu}M_{t+T}^{(u)}du-\int_{t+S}^\infty e^{-qu}M_{t+S}^{(u)}du\right|\\
&\le \int_{t+T}^\infty e^{-qu}\left|M_{t+T}^{(u)}-M_{t+S}^{(u)}\right|du+\int_{t+S}^{t+T} e^{-qu}\left|M_{t+S}^{(u)}\right|du\\
&\le \int_t^\infty e^{-qu}\left|M_{(t+T)\wedge u}^{(u)}-M_{(t+S)\wedge u}^{(u)}\right|du+\int_{t+S}^{t+T} e^{-qu}\left|M_{t+S}^{(u)}\right|du\\
&=: J_4(t,T,S)+J_5(t,T,S).
\end{aligned}$$

First, we deal with J_4. Since $T, S\in\mathcal{T}_N^t$, $(t+T)\wedge u$ and $(t+S)\wedge u$ are both $\{\mathcal{G}_\tau:\tau\ge 0\}$-stopping times. Thus, by (2.26), we have

$$\begin{aligned}
&\mathbb{P}_\mu J_4(t,T,S)\\
&\quad\le \int_t^\infty e^{-qu}\sqrt{\mathbb{P}_\mu\left|M_{(t+T)\wedge u}^{(u)}-M_{(t+S)\wedge u}^{(u)}\right|^2}du\\
&\quad= \int_t^\infty e^{-qu}\sqrt{\mathbb{P}_\mu\left(\langle M^{(u)}\rangle_{(t+T)\wedge u}-\langle M^{(u)}\rangle_{(t+S)\wedge u}\right)}du\\
&\quad= \int_t^\infty e^{-qu}\sqrt{\mathbb{P}_\mu\int_{(t+S)\wedge u}^{(t+T)\wedge u}\langle A(T_{u-s}f)^2, X_s\rangle ds}du\\
&\quad= \int_0^\infty e^{-q(u+t)}\sqrt{\mathbb{P}_\mu\int_{S\wedge u}^{T\wedge u}\langle A(T_{u-s}f)^2, X_{s+t}\rangle ds}du\\
&\quad\le \int_0^\infty e^{-q(u+t)}\\
&\qquad\times\sqrt{\int_0^{N\wedge u} e^{-\lambda_1(t+s)}\mathbb{P}_\mu\left|e^{\lambda_1(t+s)}\langle A(T_{u-s}f)^2, X_{s+t}\rangle-\left(A(T_{u-s}f)^2,\phi_1\right)_m W_\infty\right|ds}du\\
&\qquad+\int_0^\infty e^{-q(u+t)}\sqrt{\mathbb{P}_\mu\int_{S\wedge u}^{T\wedge u} e^{-\lambda_1(t+s)}\left(A(T_{u-s}f)^2,\phi_1\right)_m W_\infty ds}du\\
&\quad=: J_{4,1}(t)+J_{4,2}(t,T,S).
\end{aligned}$$

Now we consider $J_{4,1}$. Let $V(u-s,t+s) := \mathbb{P}_\mu|e^{\lambda_1(t+s)}\langle A(T_{u-s}f)^2, X_{s+t}\rangle - (A(T_{u-s}f)^2, \phi_1)_m W_\infty|$. Then,

$$J_{4,1}(t) \le e^{-(q+\lambda_1/2)t}e^{-\lambda_1 N/2}\int_0^\infty e^{-qu}\sqrt{\int_0^{N\wedge u} V(u-s,t+s)dsdu}. \tag{3.58}$$

Since $(T_{u-s}f)^2(x) \le e^{K(u-s)}T_{u-s}(f^2)(x)$, we get that, for $t > 3t_0$ and $s \in (0, N\wedge u)$,

$$\begin{aligned}
V(u-s,t+s) &\le e^{\lambda_1(t+s)}\int_E T_{t+s}\big[A(T_{u-s}f)^2\big](x)\mu(dx) + K\big\|(T_{u-s}f)^2\big\|_2\mathbb{P}_\mu(W_\infty)\\
&\le e^{\lambda_1(t+s)}e^{K(u-s)}K\int_E T_{t+u}\big(f^2\big)(x)\mu(dx) + K\|T_{u-s}f\|_4^2\mathbb{P}_\mu(W_\infty)\\
&\lesssim e^{\lambda_1(t+s)}e^{K(u-s)}e^{-\lambda_1(t+u)}K\int_E a_{t_0}(x)^{1/2}\mu(dx) + Ke^{2K(u-s)}\|f\|_4^2\mathbb{P}_\mu(W_\infty)\\
&\lesssim e^{(K-\lambda_1)(u-s)} + e^{2K(u-s)} \le e^{(K-\lambda_1)u} + e^{2Ku},
\end{aligned}$$

where in the third inequality we used (2.4) and the fact that $\|T_{u-s}\|_4 \le e^{K(u-s)}$. Note that

$$\int_0^\infty e^{-qu}\sqrt{\int_0^N e^{(K-\lambda_1)u} + e^{2Ku}dsdu} \le N^{1/2}\int_0^\infty e^{-(q-K/2+\lambda_1/2)u} + e^{-(q-K)u}du < \infty.$$

By Lemma 3.1, we get that $V(u-s,t+s) \to 0$ as $t\to\infty$. By the dominated convergence theorem, we get that

$$\lim_{t\to\infty}\int_0^\infty e^{-qu}\sqrt{\int_0^N V(u-s,t+s)dsdu} = 0.$$

It follows from (3.58) that

$$\lim_{t\to\infty} e^{(q+\lambda_1/2)t}J_{4,1}(t) = 0. \tag{3.59}$$

For $J_{4,2}(t,T,S)$, since $(A(T_{u-s}f)^2, \phi_1)_m \le \|A(T_{u-s}f)^2\|_2 \le Ke^{2K(u-s)}\|f\|_4^2 \le Ke^{2Ku}\|f\|_4^2$, we have

$$\begin{aligned}
J_{4,2}(t,T,S) &\le \|f\|_4 e^{-(q+\lambda_1/2)t}e^{-\lambda_1 N/2}\int_0^\infty e^{-(q-K)u}\sqrt{\mathbb{P}_\mu\big(K(T\wedge u - S\wedge u)W_\infty\big)}du\\
&\lesssim \theta^{1/2}e^{-(q+\lambda_1/2)t}\int_0^\infty e^{-(q-K)u}du = (q-K)^{-1}\theta^{1/2}e^{-(q+\lambda_1/2)t},
\end{aligned}$$

where in the second inequality we used the fact that $T\wedge u - S\wedge u < \theta$. Thus, we get

$$\lim_{\theta\to 0}\limsup_{t\to\infty}\sup_{S,T\in\mathcal{T}_N^t: S<T<S+\theta} e^{(q+\lambda_1/2)t}J_{4,2}(t,T,S) = 0. \tag{3.60}$$

Combining (3.59) and (3.60), we get

$$\lim_{\theta\to 0}\limsup_{t\to\infty}\sup_{S,T\in\mathcal{T}_N^t: S<T<S+\theta} e^{(q+\lambda_1/2)t}\mathbb{P}_\mu J_4(t,T,S) = 0. \tag{3.61}$$

Finally, we consider $J_5(t,T,S)$. By Hölder's inequality, we get

$$\begin{aligned}
\mathbb{P}_\mu J_5(t,T,S) &= \mathbb{P}_\mu \int_{t+S}^{t+T} e^{-qu}\big|M^{(u)}_{t+S}\big|du \le \sqrt{\mathbb{P}_\mu \int_{t+S}^{t+T} e^{-2qu}\big|M^{(u)}_{t+S}\big|^2 du}\sqrt{\mathbb{P}_\mu(T-S)} \\
&\le \theta^{1/2}\sqrt{\int_t^{t+N} e^{-2qu}\mathbb{P}_\mu\big|M^{(u)}_{(t+S)\wedge u}\big|^2 du} \\
&= \theta^{1/2}\sqrt{\int_t^{t+N} e^{-2qu}\mathbb{P}_\mu\big\langle M^{(u)}\big\rangle_{(t+S)\wedge u} du} \\
&\le \theta^{1/2}\sqrt{\int_t^{t+N} e^{-2qu}\mathbb{P}_\mu\big\langle M^{(u)}\big\rangle_{u} du} \\
&= \theta^{1/2}\sqrt{\int_t^{t+N} e^{-2qu}\int_E \mathbb{V}ar_{\delta_x}\langle f, X_u\rangle \mu(dx) du} \\
&\lesssim \theta^{1/2}\sqrt{\int_t^{t+N} e^{-2qu}e^{-\lambda_1 u}du \int_E a_{t_0}(x)^{1/2}\mu(dx)} \lesssim \theta^{1/2}e^{-(q+\lambda_1/2)t},
\end{aligned}$$

where in the second to the last inequality we used (2.10). Thus, we get that

$$\lim_{\theta\to 0}\limsup_{t\to\infty}\sup_{S,T\in\mathcal{T}_N^t: S<T<S+\theta} e^{(q+\lambda_1/2)t}\mathbb{P}_\mu J_5(t,T,S) = 0. \tag{3.62}$$

Combining (3.61) and (3.62), we get (3.57) immediately. The proof is now complete. □

Lemma 3.9 *If $f \in \mathcal{C}_s$ and $\mu \in \mathcal{M}_C(E)$, then, under $\mathbb{P}_\mu$, the family of processes $(Y_t^{1,U_q f}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$.*

Proof It follows from Lemmas 3.7 and 3.8 that $(Y_t^{1,U_q f}(\cdot))_{t>0}$ is tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. By Corollary 3.5 and the fact that $\sqrt{W_\infty}G^{1,U_q f}$ is a continuous process, we obtain that $(Y_t^{1,U_q f}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. □

3.2.2 The Tightness of $(Y_t^{2,h})_{t>0}$ in $\mathbb{D}(\mathbb{R})$

The next lemma will be used to prove the tightness of $(Y_t^{2,h}(\cdot))_{t>0}$.

Lemma 3.10 *Suppose that $\{C(\tau),\tau\ge 0\}$ and, for each $t>0$, $\{C_t(\tau),\tau\ge 0\}$ are non-decreasing càdlàg processes defined on the space $(\Omega,\mathcal{F},P)$ such that $C_t(0)=C(0)=0$ and for all $\tau\ge 0$,*

$$\lim_{t\to\infty} C_t(\tau) = C(\tau) \quad \textit{in probability}. \tag{3.63}$$

If C is a continuous process, then

$$\lim_{t\to\infty} \delta(C_t, C) = 0 \quad \textit{in probability}, \tag{3.64}$$

where δ is the Skorohod metric defined in [20, *Chapter VI*, 1.26]. *Moreover, as $t \to \infty$,*

$$C_t - C \xrightarrow{d} 0,$$

which implies that $(C_t)_{t\geq 0}$ is C-tight in $\mathbb{D}(\mathbb{R})$.

Proof Let D be the subset of all the positive rational numbers. For any subsequence (n_k), by a diagonal argument, we can find a further subsequence (n'_k) and a set $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for all $\tau \in D$ and $\omega \in \Omega_0$,

$$\lim_{k\to\infty} C_{n'_k}(\tau)(\omega) = C(\tau)(\omega). \tag{3.65}$$

Thus, by [20, Chapter VI, Theorem 2.15(c)], we have, for $\omega \in \Omega_0$,

$$\lim_{k\to\infty} \delta\big(C_{n'_k}(\omega), C(\omega)\big) = 0,$$

which implies (3.64). The remaining assertion follows immediately from (3.64). □

Lemma 3.11 *If $h \in \mathcal{C}_c$ and $\mu \in \mathcal{M}_C(E)$, then the family of processes $(Y_t^{2,h}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$.*

Proof For $h \in \mathcal{C}_c$, we have $T_t h = e^{-\lambda_1 t/2} h$. Thus, by (2.22), we get that, for $t \geq 0$, $\mathbb{P}_\mu$-a.s.

$$\langle h, X_t\rangle = e^{-\lambda_1 t/2}\langle h, X_0\rangle + e^{-\lambda_1 t/2}\int_0^t \int_E e^{\lambda_1 s/2} h(x) M(ds, dx).$$

Since both sides of the above equation are càdlàg, we have

$$\mathbb{P}_\mu\left(\langle h, X_t\rangle = e^{-\lambda_1 t/2}\langle h, X_0\rangle + e^{-\lambda_1 t/2}\int_0^t \int_E e^{\lambda_1 s/2} h(x) M(ds, dx), \forall t > 0\right) = 1.$$

Thus, we have

$$\begin{aligned} Y_t^{2,h}(\tau) &= t^{-1/2}\langle h, X_0\rangle + t^{-1/2}\int_0^{t+\tau} \int_E e^{\lambda_1 s/2} h(x) M(ds, dx) \\ &= Y_t^{2,h}(0) + t^{-1/2}\int_t^{t+\tau} \int_E e^{\lambda_1 s/2} h(x) M(ds, dx). \end{aligned}$$

Therefore, $\{Y_t^{2,h}(\tau), \tau \geq 0\}$ is a square-integrable martingale with

$$\big\langle Y_t^{2,h}\big\rangle(\tau) = t^{-1}\int_t^{t+\tau} e^{\lambda_1 s}\big\langle Ah^2, X_s\big\rangle ds. \tag{3.66}$$

By (2.4), we have for $t > t_0$,

$$t^{-1}\mathbb{P}_\mu\left(\int_t^{t+\tau} e^{\lambda_1 s}\big\langle Ah^2, X_s\big\rangle ds\right) = t^{-1}\int_E \int_t^{t+\tau} e^{\lambda_1 s} T_s\big(Ah^2\big)(x) ds\,\mu(dx) \lesssim t^{-1}\tau.$$

Thus, for any $\tau \geq 0$, as $t \to \infty$,

$$\big\langle Y_t^{2,h}\big\rangle(\tau) \to 0 \quad \text{in } \mathbb{P}_\mu\text{-probability}. \tag{3.67}$$

Hence by Lemma 3.10 we have that $(\langle Y_t^{2,h}\rangle)_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. Since $Y_t^{2,g}(0) = t^{-1/2}e^{-\lambda_1 t/2}\langle g, X_t\rangle \to \mathcal{N}(0, \rho_g^2)$ in distribution as $t \to \infty$, we know that $\{Y_t^{2,h}(0), t \geq 0\}$ is tight in $\mathbb{R}$ under $\mathbb{P}_\mu$. Therefore, by [20, Chapter VI, Theorem 4.13], we get that $(Y_t^{2,h}(\cdot))_{t>0}$ is tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. By Corollary 3.5 and the fact that $\sqrt{W_\infty}G^{2,h}$ is a continuous process, we obtain that $(Y_t^{2,h}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. The proof is now complete. □

3.2.3 The Tightness of $(Y_t^{3,g})_{t>0}$ in $\mathbb{D}(\mathbb{R})$

Lemma 3.12 *If $g = \sum_{k:\lambda_1>2\lambda_k}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)} \in \mathcal{C}_l$ and $\mu \in \mathcal{M}_C(E)$, then the family of processes $(Y_t^{3,g}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$.*

Proof Note that

$$\begin{aligned} Y_t^{3,g}(\tau) &= \sum_{k:\lambda_1>2\lambda_k}\sum_{j=1}^{n_k} e^{(\lambda_1/2-\lambda_k)(t+\tau)} b_j^k \big(H_{t+\tau}^{k,j} - H_t^{k,j}\big) \\ &\quad + \sum_{k:\lambda_1>2\lambda_k}\sum_{j=1}^{n_k} e^{(\lambda_1/2-\lambda_k)(t+\tau)} b_j^k \big(H_t^{k,j} - H_\infty^{k,j}\big) \\ &=: Z_t^1(\tau) + Z_t^2(\tau). \end{aligned}$$

For $Z_t^2(\tau)$, it is known (see [34, Remark 1.8]) that under $\mathbb{P}_\mu$

$$e^{(\lambda_1/2-\lambda_k)t}\big(H_t^{k,j} - H_\infty^{k,j}\big) \xrightarrow{d} G\sqrt{W_\infty},$$

where G is a normal random variable. It follows that under $\mathbb{P}_\mu$, as $t \to \infty$,

$$e^{(\lambda_1/2-\lambda_k)(t+\cdot)} b_j^k \big(H_t^{k,j} - H_\infty^{k,j}\big) \xrightarrow{d} b_j^k G\sqrt{W_\infty} e^{(\lambda_1/2-\lambda_k)\cdot}.$$

Thus, $e^{(\lambda_1/2-\lambda_k)(t+\cdot)} b_j^k (H_t^{k,j} - H_\infty^{k,j})$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. By [20, Corollary 3.33], $(Z_t^2)_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. Thus, to prove $(Y_t^{3,g})_{t>0}$ is tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$, it suffices to show that $(Z_t^1)_{t>0}$ is tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$.

Since $\{H_{t+\tau}^{k,j} - H_t^{k,j} : \tau \geq 0\}$ is a martingale under $\mathbb{P}_\mu$, using L_p maximum inequality, we get for $\lambda_1 > 2\lambda_k$,

$$\mathbb{P}_\mu\Big(\sup_{\tau\leq N} e^{(\lambda_1/2-\lambda_k)(t+\tau)}\big|H_{t+\tau}^{k,j} - H_t^{k,j}\big|\Big) \leq 2e^{(\lambda_1/2-\lambda_k)(t+N)}\sqrt{\mathbb{P}_\mu\big(H_{t+N}^{k,j} - H_t^{k,j}\big)^2}.$$

By (2.22), we have

$$H_t^{k,j} = \langle\phi_j^{(k)}, \mu\rangle + \int_0^t\int_E e^{\lambda_k s}\phi_j^{(k)}(x)M(ds, dx). \tag{3.68}$$

Thus,

$$\langle H^{k,j}\rangle_t = \int_0^t e^{2\lambda_k s}\langle A(\phi_j^{(k)})^2, X_s\rangle ds. \tag{3.69}$$

Therefore, by (2.4), we get that, for $t > t_0$,

$$\mathbb{P}_\mu\big(H^{k,j}_{t+N} - H^{k,j}_t\big)^2 = \int_E \int_t^{t+N} e^{2\lambda_k s} T_s\big(A\big(\phi_j^{(k)}\big)^2\big)(x) ds \mu(dx)$$
$$\lesssim \int_t^{t+N} e^{2\lambda_k s} e^{-\lambda_1 s} ds \lesssim e^{(2\lambda_k - \lambda_1)t}.$$

Hence,

$$\sup_{t>t_0} \mathbb{P}_\mu\Big(\sup_{\tau\le N} e^{(\lambda_1/2-\lambda_k)(t+\tau)}\big|H^{k,j}_{t+\tau} - H^{k,j}_t\big|\Big) < \infty. \tag{3.70}$$

It follows that

$$\sup_{t>t_0} \mathbb{P}_\mu\Big(\sup_{\tau\le N}\big|Z^1_t(\tau)\big|\Big) \le \sum_{k:\lambda_1>2\lambda_k} \sum_{j=1}^{n_k} \big|b^k_j\big| \sup_{t>t_0} \mathbb{P}_\mu\Big(\sup_{\tau<N} e^{(\lambda_1/2-\lambda_k)(t+\tau)}\big|H^{k,j}_{t+\tau} - H^{k,j}_t\big|\Big) < \infty. \tag{3.71}$$

Next we prove that

$$\lim_{\theta\to 0} \limsup_{t\to\infty} \sup_{T,S\in\mathcal{T}^t_N:0\le T-S\le\theta} \mathbb{P}_\mu\big(\big|Z^1_t(T) - Z^1_t(S)\big|\big) = 0, \tag{3.72}$$

where $\mathcal{T}^t_N$ is the set of all $\{\mathcal{G}_{t+\tau} : \tau \ge 0\}$-stopping times that are bounded by N. It suffices to show that, for $\lambda_1 > 2\lambda_k$,

$$\lim_{\theta\to 0} \limsup_{t\to\infty} \sup_{T,S\in\mathcal{T}^t_N:0\le T-S\le\theta} \mathbb{P}_\mu\big(\big|e^{(\lambda_1/2-\lambda_k)(t+T)}\big(H^{k,j}_{t+T} - H^{k,j}_t\big)$$
$$- e^{(\lambda_1/2-\lambda_k)(t+S)}\big(H^{k,j}_{t+S} - H^{k,j}_t\big)\big|\big) = 0. \tag{3.73}$$

We note that

$$\big|e^{(\lambda_1/2-\lambda_k)(t+T)}\big(H^{k,j}_{t+T} - H^{k,j}_t\big) - e^{(\lambda_1/2-\lambda_k)(t+S)}\big(H^{k,j}_{t+S} - H^{k,j}_t\big)\big|$$
$$\le e^{(\lambda_1/2-\lambda_k)(t+S)}\big|H^{k,j}_{t+T} - H^{k,j}_{t+S}\big| + e^{(\lambda_1/2-\lambda_k)(t+S)}\big(e^{(\lambda_1/2-\lambda_k)\theta} - 1\big)\big|H^{k,j}_{t+T} - H^{k,j}_t\big|$$
$$\le e^{(\lambda_1/2-\lambda_k)(t+N)}\big|H^{k,j}_{t+T} - H^{k,j}_{t+S}\big| + e^{(\lambda_1/2-\lambda_k)(t+N)}\big(e^{(\lambda_1/2-\lambda_k)\theta} - 1\big) \sup_{\tau<N}\big|H^{k,j}_{t+\tau} - H^{k,j}_t\big|.$$

By (3.70), we get that, for $t > t_0$,

$$e^{(\lambda_1/2-\lambda_k)(t+N)}\big(e^{(\lambda_1/2-\lambda_k)\theta} - 1\big)\mathbb{P}_\mu\Big(\sup_{\tau<N}\big|H^{k,j}_{t+\tau} - H^{k,j}_t\big|\Big) \lesssim e^{(\lambda_1/2-\lambda_k)\theta} - 1 \to 0, \tag{3.74}$$

as $\theta \to 0$. By (3.69), we have

$$e^{(\lambda_1/2-\lambda_k)(t+N)}\mathbb{P}_\mu\big|H^{k,j}_{t+T} - H^{k,j}_{t+S}\big|$$
$$\le e^{(\lambda_1/2-\lambda_k)(t+N)}\sqrt{\mathbb{P}_\mu\big|H^{k,j}_{t+T} - H^{k,j}_{t+S}\big|^2}$$
$$= e^{(\lambda_1/2-\lambda_k)(t+N)}\sqrt{\mathbb{P}_\mu\big(\big\langle H^{k,j}\big\rangle_{t+T} - \big\langle H^{k,j}\big\rangle_{t+S}\big)}$$
$$= e^{(\lambda_1/2-\lambda_k)(t+N)}\sqrt{\mathbb{P}_\mu \int_{t+S}^{t+T} e^{2\lambda_k s}\big\langle A\big(\phi^{(k)}_j\big)^2, X_s\big\rangle ds}$$

$$\lesssim \sqrt{\mathbb{P}_\mu \int_{t+S}^{t+T} e^{\lambda_1 s}\langle A(\phi_j^{(k)})^2, X_s\rangle ds}$$

$$\leq \sqrt{\int_t^{t+N} \mathbb{P}_\mu \left|e^{\lambda_1 s}\langle A(\phi_j^{(k)})^2, X_s\rangle - \left(A(\phi_j^{(k)})^2, \phi_1\right)_m W_\infty\right| ds + \theta\left(A(\phi_j^{(k)})^2, \phi_1\right)_m \mathbb{P}_\mu(W_\infty)}.$$

By Lemma 3.1,

$$\lim_{t\to\infty} \int_t^{t+N} \mathbb{P}_\mu \left|e^{\lambda_1 s}\langle A(\phi_j^{(k)})^2, X_s\rangle - \left(A(\phi_j^{(k)})^2, \phi_1\right)_m W_\infty\right| ds = 0.$$

Thus,

$$\lim_{\theta\to 0}\limsup_{t\to\infty} \sup_{T,S\in\mathcal{T}_N^t: 0\leq T-S\leq\theta} e^{(\lambda_1/2-\lambda_k)(t+N)}\mathbb{P}_\mu\left|H_{t+T}^{k,j} - H_{t+S}^{k,j}\right|$$
$$\lesssim \lim_{\theta\to 0}\sqrt{\theta\left(A(\phi_j^{(k)})^2, \phi_1\right)_m \mathbb{P}_\mu(W_\infty)} = 0. \tag{3.75}$$

Combining (3.74) and (3.75), we get (3.73).

By Corollary 3.5 and the fact that $\sqrt{W_\infty}G^{3,g}$ is a continuous process, we obtain that $(Y_t^{3,g}(\cdot))_{t>0}$ is C-tight in $\mathbb{D}(\mathbb{R})$ under $\mathbb{P}_\mu$. The proof is now complete. □

Acknowledgements We thank the two referees for their helpful comments on the first version of this paper.

References

1. Adamczak, R., Miłoś, P.: CLT for Ornstein-Uhlenbeck branching particle system. Electron. J. Probab. **20**(42), 1–35 (2015)
2. Adamczak, R., Miłoś, P.: U-statistics of Ornstein-Uhlenbeck branching particle system. J. Theor. Probab. **27**(4), 1071–1111 (2014)
3. Asmussen, S., Hering, H.: Branching Processes. Birkhäuser, Boston (1983)
4. Asmussen, S., Keiding, N.: Martingale central limit theorems and asymptotic estimation theory for multitype branching processes. Adv. Appl. Probab. **10**, 109–129 (1978)
5. Athreya, K.B.: Limit theorems for multitype continuous time Markov branching processes I: the case of an eigenvector linear functional. Z. Wahrscheinlichkeitstheor. Verw. Geb. **12**, 320–332 (1969)
6. Athreya, K.B.: Limit theorems for multitype continuous time Markov branching processes II: the case of an arbitrary linear functional. Z. Wahrscheinlichkeitstheor. Verw. Geb. **13**, 204–214 (1969)
7. Athreya, K.B.: Some refinements in the theory of supercritical multitype Markov branching processes. Z. Wahrscheinlichkeitstheor. Verw. Geb. **20**, 47–57 (1971)
8. Bojdecki, T., Gorostiza, L.G., Talarczyk, A.: Limit theorems for occupation time fluctuations of branching systems I: long-range dependence. Stoch. Process. Appl. **116**, 1–18 (2006)
9. Bojdecki, T., Gorostiza, L.G., Talarczyk, A.: Limit theorems for occupation time fluctuations of branching systems II: critical and lage dimensions. Stoch. Process. Appl. **116**, 19–35 (2006)
10. Bojdecki, T., Gorostiza, L.G., Talarczyk, A.: Occupation time limits of inhomogeneous Poisson systems of independent particles. Stoch. Process. Appl. **118**, 28–52 (2008)
11. Bojdecki, T., Gorostiza, L.G., Talarczyk, A.: Occupation times of branching systems with initial inhomogeneous Poisson states and related superprocesses. Electron. J. Probab. **14**(46), 1328–1371 (2009)
12. Chung, K.L., Walsh, J.B.: Markov Processes, Brownian Motion, and Time Symmetry. Springer, Berlin (2005)
13. Davies, E.B., Simon, B.: Ultracontractivity and the kernel for Schrödinger operators and Dirichlet Laplacians. J. Funct. Anal. **59**, 335–395 (1984)
14. Dawson, D.A.: Measure-Valued Markov Processes. Springer, Berlin (1993)
15. Dynkin, E.B.: Superprocesses and partial differential equations. Ann. Probab. **21**, 1185–1262 (1993)

16. Dynkin, E.B., Kuznetsov, S.E.: ℕ-measure for branching exit Markov system and their applications to differential equations. Probab. Theory Relat. Fields **130**, 135–150 (2004)
17. El Karoui, N., Roelly, S.: Propriétés de martingales, explosion et représentation de Lévy-Khintchine d'une classe de processus de branchment à valeurs mesures. Stoch. Process. Appl. **38**, 239–266 (1991)
18. Hong, W.: Functional central limit theorem for super α-stable processes. Sci. China Ser. A **47**, 874–881 (2004)
19. Iscoe, I.: A weighted occupation time for a class of measure-valued branching processes. Probab. Theory Relat. Fields **71**, 85–116 (1986)
20. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes. Springer, Berlin (2003)
21. Janson, S.: Functional limit theorems for multitype branching processes and generalized Pólya urns. Stoch. Process. Appl. **110**, 177–245 (2004)
22. Kesten, H., Stigum, B.P.: A limit theorem for multidimensional Galton-Watson processes. Ann. Math. Stat. **37**, 1211–1223 (1966)
23. Kesten, H., Stigum, B.P.: Additional limit theorems for indecomposable multidimensional Galton-Watson processes. Ann. Math. Stat. **37**, 1463–1481 (1966)
24. Li, Z.: Skew convolution semigroups and related immigration processes. Theory Probab. Appl. **46**, 274–296 (2003)
25. Li, Z.: Measure-Valued Branching Markov Processes. Springer, Heidelberg (2011)
26. Li, Z.H., Shiga, T.: Measure-valued branching diffusions: immigrations, excursions and limit theorems. J. Math. Kyoto Univ. **35**, 233–274 (1995)
27. Miłoś, P.: Occupation time fluctuations of Poisson and equilibrium finite variance branching systems. Probab. Math. Stat. **27**, 181–203 (2007)
28. Miłoś, P.: Occupation time fluctuations of Poisson and equilibrium branching systems in critical and large dimensions. Probab. Math. Stat. **28**, 235–256 (2008)
29. Miłoś, P.: Occupation times of subcritical branching immigration systems with Markov motions. Stoch. Process. Appl. **119**, 3211–3237 (2009)
30. Miłoś, P.: Occupation times of subcritical branching immigration systems with Markov motion, CLT and deviation principles. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15**, 1250002 (2012). 28 pp.
31. Miłoś, P.: Spatial CLT for the supercritical Ornstein-Uhlenbeck superprocess (2012). Preprint. arXiv:1203.6661
32. Ren, Y.-X., Song, R., Zhang, R.: Central limit theorems for super Ornstein-Uhlenbeck processes. Acta Appl. Math. **130**, 9–49 (2014)
33. Ren, Y.-X., Song, R., Zhang, R.: Central limit theorems for supercritical branching Markov processes. J. Funct. Anal. **266**, 1716–1756 (2014)
34. Ren, Y.-X., Song, R., Zhang, R.: Central limit theorems for supercritical superprocesses. Stoch. Process. Appl. **125**, 428–457 (2015)
35. Ren, Y.-X., Song, R., Zhang, R.: Central limit theorems for supercritical branching nonsymmetric Markov processes. Ann. Probab. (2015, to appear). Available at: arXiv:1404.0116
36. Zhang, M.: Functional central limit theorem for the super-Brownian motion with super-Brownian immigration. J. Theor. Probab. **18**, 665–685 (2005)