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Law of Large Numbers for Branching Symmetric Hunt Processes with Measure-Valued Branching Rates

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Abstract We establish weak and strong laws of large numbers for a class of branching symmetric Hunt processes with the branching rate being a smooth measure with respect to the underlying Hunt process, and the branching mechanism being general and state dependent. Our work is motivated by recent work on the strong law of large numbers for branching symmetric Markov processes by Chen and Shiozawa (J Funct Anal 250:374–399, 2007) and for branching diffusions by Engländer et al. (Ann Inst Henri Poincaré Probab Stat 46:279–298, 2010). Our results can be applied to some interesting examples that are covered by neither of these papers.

Keywords Law of large numbers · Branching Hunt processes · Spine approach · h-transform · Spectral gap

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1 Introduction

1.1 Motivation

The law of large numbers (LLN) has been the object of interest for measure-valued Markov processes including branching Markov processes and superprocesses. For branching Markov processes, the earliest work in this field dates back to 1970s when Watanabe [24,25] studied the asymptotic properties of a branching symmetric diffusion, using a suitable Fourier analysis. Later, Asmussen and Hering [3] established an almost-sure limit theorem for a general supercritical branching process under some regularity conditions. Recently, there is a revived interest in this field using modern techniques such as Dirichlet form method, martingales and spine method. Chen and Shiozawa [6] (see also [22]) used a Dirichlet form and spectral theory approach to obtain strong law of large numbers (SLLN) for branching symmetric Hunt processes. Among other assumptions, a spectral gap condition was used to obtain a Poincaré inequality that plays an important role in the proof of SLLN along lattice times. They proved SLLN holds for branching processes under the assumptions that the branching rate is given by a measure in Kato class $\mathbf{K}_\infty(X)$ and the branching mechanism has bounded second moment.

The spine method developed recently for measure-valued Markov processes is a powerful probabilistic tool in studying various properties of these processes; see, e.g., [9–12,14,17,18]. In [11], Engländer, Harris and Kyprianou used spine method to obtain SLLN for branching (possibly non-symmetric) diffusions corresponding to the operator $Lu + \beta(u^2 - u)$ on a domain $D \subset \mathbb{R}^d$ (where $\beta \geq 0$ is non-trivial) under certain spectral conditions. They imposed a condition on how far particles can spread in space (see condition (iii) on page 282 of [11]). That the underlying process is a diffusion plays an important role in their argument and the branching rate there has to be a function rather than a measure. The approach of [11] also involves p-th moment calculation with $p > 1$ which may not be valid for general branching mechanisms. Recently, Eckhoff, Kyprianou and Winkel [9] discussed the strong law of large numbers (SLLN) along lattice times for branching diffusions, which serves as the backbone or skeleton for superdiffusions. It is proved in [9, Theorem 2.14] that if the branching mechanism satisfies a p-th moment condition with $p \in (1, 2]$, the underlying diffusion and the support of the branching diffusion satisfy conditions similar to that presented in [11], then SLLN along lattice times holds.

In this paper, we combine the functional analytic methods used in [6] with spine techniques to study weak and strong laws of large numbers for branching symmetric Hunt processes as well as the $L \log L$ criteria. This approach allows us to obtain new results for a large class of branching Markov processes, for which (i) the underlying spatial motions are general symmetric Hunt processes, which can be discontinuous and may not be intrinsically ultracontractive; (ii) the branching rates are given by general smooth measures rather than functions or Kato class measures; (iii) the offspring distributions are only assumed to have bounded first moments with no assumption on their second moments. In addition, we use L^1-approach instead of L^p-approach for $p \in (1, 2]$. Now we describe the setting and main results of this paper in detail, followed by several examples illuminating the main results.

1.2 Branching Symmetric Hunt Processes and Assumptions

Suppose we are given three initial ingredients: a Hunt process, a smooth measure and a branching mechanism. We introduce them one by one:

- *A Hunt process* X: Suppose E is a locally compact separable metric space and $E_\partial := E \cup \{\partial\}$ is its one point compactification. m is a positive Radon measure on E with full support. Let $X = (\Omega, \mathcal{H}, \mathcal{H}_t, \theta_t, X_t, \Pi_x, \zeta)$ be a m-symmetric Hunt process on E. Here $\{\mathcal{H}_t : t \geq 0\}$ is the minimal admissible filtration, $\{\theta_t : t \geq 0\}$ the time-shift operator of X satisfying $X_t \circ \theta_s = X_{t+s}$ for $s, t \geq 0$, and $\zeta := \inf\{t > 0 : X_t = \partial\}$ the life time of X. Suppose for each $t > 0$, X has a symmetric transition density function $p(t, x, y)$ with respect to the measure m. Let $\{P_t : t \geq 0\}$ be the Markovian transition semigroup of X, i.e.,

$$P_t f(x) := \Pi_x \left[f(X_t)\right] = \int_E f(y) p(t, x, y) m(\mathrm{d}y)$$

for any nonnegative measurable function f. The symmetric Dirichlet form on $L^2(E, m)$ generated by X will be denoted as $(\mathcal{E}, \mathcal{F})$:

$$\mathcal{F} = \left\{u \in L^2(E, m) : \lim_{t \to 0} \frac{1}{t} \int_E (u(x) - P_t u(x))\, u(x) m(\mathrm{d}x) < +\infty\right\},$$

$$\mathcal{E}(u, v) = \lim_{t \to 0} \frac{1}{t} \int_E (u(x) - P_t u(x))\, v(x) m(\mathrm{d}x), \quad u, v \in \mathcal{F}.$$

It is known (cf. [5]) that $(\mathcal{E}, \mathcal{F})$ is quasi-regular and hence is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. In the sequel, we assume that X is m-irreducible in the sense that if $A \in \mathcal{B}(E)$ has positive m-measure, then $\Pi_x(T_A < +\infty) > 0$ for all $x \in E$, where $T_A := \inf\{t > 0 : X_t \in A\}$ is the first hitting time of A.
- *A branching rate* μ: Suppose μ is a positive smooth Radon measure on $(E, \mathcal{B}(E))$. It uniquely determines a positive continuous additive functional (PCAF) A_t^μ by the following Revuz formula:

$$\int_E f(x) \mu(\mathrm{d}x) = \lim_{t \to 0} \frac{1}{t} \Pi_m \left[\int_0^t f(X_s) d A_s^\mu\right], \quad f \in \mathcal{B}^+(E).$$

Here $\Pi_m(\cdot) := \int_E \Pi_x(\cdot) m(\mathrm{d}x)$.
- *Offspring distributions* $\{\{p_n(x) : n \geq 0\}, x \in E\}$: Suppose $\{\{p_n(x) : n \geq 0\}, x \in E\}$ is a family of probability mass functions such that $0 \leq p_n(x) \leq 1$ and $\sum_{n=0}^\infty p_n(x) = 1$. For each $x \in E$, $\{p_n(x) : n \geq 0\}$ serves as the offspring distribution of a particle located at x. Let $\{A(x) : x \in E\}$ be a collection of random variables taking values in $\{0, 1, 2, \ldots\}$ and distributed as $\mathrm{P}(A(x) = n) = p_n(x)$.

Define

$$Q(x) := \sum_{n=0}^{+\infty} n p_n(x), \quad x \in E. \tag{1.1}$$

Throughout this paper we assume that the offspring distribution $\{p_n(x) : n \geq 0\}$ satisfies the following condition:

$$p_0(x) \equiv 0, \quad p_1(x) \not\equiv 1 \quad \text{and} \quad \sup_{x \in E} Q(x) < \infty. \tag{1.2}$$

From these ingredients we can build a branching Markov process according to the following recipe: Under a probability measure $\mathbb{P}_x$, a particle starts from $x \in E$ and moves around in E_∂ like a copy of X. We use $\emptyset$ to denote the original particle, $X_\emptyset(t)$ its position at time t and $\zeta_\emptyset$ its fission time. We say that $\emptyset$ splits at the rate μ in the sense that

$$\mathbb{P}_x\left(\zeta_\emptyset > t | X_\emptyset(s) : s \geq 0\right) = \exp(-A_t^\mu).$$

When $\zeta_\emptyset \geq \zeta$, it dies at time ζ. On the other hand, when $\zeta_\emptyset < \zeta$, it splits into a random number of children, the number being distributed as a copy of $A(X_\emptyset(\zeta_\emptyset-))$. These children, starting from their point of creation, will move and reproduce independently in the same way as their parents. If a particle u is alive at time t, we refer to its location in E as $X_u(t)$. Therefore the time-t configuration is a E-valued point process $\mathbb{X}_t = \{X_u(t) : u \in \mathcal{Z}_t\}$, where $\mathcal{Z}_t$ is the set of particles alive at time t. With abuse of notation, we can also regard $\mathbb{X}_t$ as a random point measure on E defined by $\mathbb{X}_t := \sum_{u \in \mathcal{Z}_t} \delta_{X_u(t)}$. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $\mathbb{X}$ and $\mathcal{F}_\infty = \sigma\{\mathcal{F}_t : t \geq 0\}$. Hence it defines a branching symmetric Hunt process $\mathbb{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, \mathbb{X}_t, \mathbb{P}_x)$ on E with the motion component X, the branching rate measure μ and the branching mechanism function $\{p_n(x) : n \geq 0\}$. When the branching rate measure μ is absolutely continuous with respect to m, i.e., $\mu(\mathrm{d}y) = \beta(y)m(\mathrm{d}y)$ for some nonnegative function β, the corresponding PCAF A_t^μ is equal to $\int_0^t \beta(X_s)\mathrm{d}s$, and given that a particle u is alive at t, its probability of splitting in $(t, t+dt)$ is $\beta(X_u(t))dt + o(dt)$. Since the function β determines the rate at which every particle splits, β is called the branching rate function in literature.

Throughout this paper, we use $\mathcal{B}_b(E)$ (respectively, $\mathcal{B}^+(E)$) to denote the space of bounded (respectively, nonnegative) measurable functions on $(E, \mathcal{B}(E))$. Any function f on E will be automatically extended to E_∂ by setting $f(\partial) = 0$. We use $\langle f, g\rangle$ to denote $\int_E f(x)g(x)m(\mathrm{d}x)$ and ":=" as a way of definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, and $\log^+ a := \log(a \vee 1)$.

For every $f \in \mathcal{B}^+(E)$ and $t \geq 0$, define

$$\mathbb{X}_t(f) := \sum_{u \in \mathcal{Z}_t} f(X_u(t)).$$

We define the Feynman–Kac semigroup $P_t^{(Q-1)\mu}$ by

$$P_t^{(Q-1)\mu} f(x) := \Pi_x\left[\exp(A_t^{(Q-1)\mu}) f(X_t)\right], \quad f \in \mathcal{B}^+(E). \tag{1.3}$$

Since X has a transition density function, it follows that for each $t > 0$, $P_t^{(Q-1)\mu}$ admits an integral kernel with respect to the measure m. We denote this kernel by $p^{(Q-1)\mu}(t,x,y)$. The semigroup $P_t^{(Q-1)\mu}$ associates with a quadratic form $(\mathcal{E}^{(Q-1)\mu}, \mathcal{F}^\mu)$, where $\mathcal{F}^\mu = \mathcal{F} \cap L^2(E;\mu)$ and

$$\mathcal{E}^{(Q-1)\mu}(u,u) = \mathcal{E}(u,u) - \int_E u(x)^2 (Q(x)-1)\mu(\mathrm{d}x), \quad u \in \mathcal{F}^\mu.$$

We say that a signed smooth measure ν belongs to the Kato class $\mathbf{K}(X)$, if

$$\lim_{t\downarrow 0} \sup_{x\in E} \Pi_x[A_t^{|\nu|}] = 0. \tag{1.4}$$

For every nonnegative $\nu \in \mathbf{K}(X)$, we have $\|G_\alpha \nu\|_\infty < \infty$ for every $\alpha > 0$, where G_α is the α-resolvent of X and $G_\alpha\nu$ is the α-potential of ν. Define $\mathcal{E}_\alpha(u,u) := \mathcal{E}(u,u) + \alpha\int_E u^2 dm$. By [23],

$$\int_E u(x)^2 \nu(\mathrm{d}x) \le \|G_\alpha \nu\|_\infty \mathcal{E}_\alpha(u,u), \quad u \in \mathcal{F}. \tag{1.5}$$

Thus when μ is in $\mathbf{K}(X)$, $\mathcal{F}^\mu = \mathcal{F}$, and the quadratic form $(\mathcal{E}^{(Q-1)\mu}, \mathcal{F}^\mu)$ is bounded from below.

For an arbitrary smooth measure μ, we define

$$\lambda_1 := \inf\left\{\mathcal{E}^{(Q-1)\mu}(u,u) : u \in \mathcal{F}^\mu \text{ with } \int_E u(x)^2 m(\mathrm{d}x) = 1\right\}. \tag{1.6}$$

Assumption 1 Let μ be a nonnegative smooth measure on E so that the Schrödinger semigroup $P_t^{(Q-1)\mu}$ admits a symmetric kernel $p^{(Q-1)\mu}(t,x,y)$ with respect to the measure m and is jointly continuous in $(x,y) \in E\times E$ for every $t>0$. Moreover, $\lambda_1 \in (-\infty,0)$ and there is a positive continuous function $h \in \mathcal{F}^\mu$ with $\int_E h(x)^2 m(\mathrm{d}x) = 1$ so that $\mathcal{E}^{(Q-1)\mu}(h,h) = \lambda_1$.

Observe that if u is a minimizer for (1.6), then so is $|u|$. Assumption 1 says that there is a minimizer for (1.6) that can be chosen to be positive and continuous. Clearly the following property holds for h:

$$\mathcal{E}(h,v) = \int_E h(x)v(x)(Q(x)-1)\mu(\mathrm{d}x) + \lambda_1\langle h, v\rangle \quad \text{for every } v \in \mathcal{F}^\mu. \tag{1.7}$$

The finiteness of λ_1 implies that the bilinear form $(\mathcal{E}^{(Q-1)\mu}, \mathcal{F}^\mu)$ is bounded from below, and hence by [2], $\{P_t^{(Q-1)\mu} : t \ge 0\}$ is a strongly continuous semigroup on $L^2(E,m)$. Let $\sigma(\mathcal{E}^{(Q-1)\mu})$ denote the spectrum of the self-adjoint operator associated with $\mathcal{E}^{(Q-1)\mu}$. Let λ_2 be the second bottom of $\sigma(\mathcal{E}^{(Q-1)\mu})$, that is,

$$\lambda_2 := \inf\left\{\mathcal{E}^{(Q-1)\mu}(u,u) : \ u \in \mathcal{F}^\mu, \int_E u(x)h(x)m(\mathrm{d}x) = 0, \int_E u(x)^2 m(\mathrm{d}x) = 1\right\}.$$

Assumption 2 There is a positive spectral gap in $\sigma(\mathcal{E}^{(Q-1)\mu})$: $\lambda_h := \lambda_2 - \lambda_1 > 0$.

Define h-transformed semigroup $\{P_t^h; t \geq 0\}$ from $\{P_t^{(Q-1)\mu}; t \geq 0\}$ by

$$P_t^h f(x) = \frac{e^{\lambda_1 t}}{h(x)} P_t^{(Q-1)\mu}(hf)(x) \quad \text{for } x \in E \text{ and } f \in \mathcal{B}^+(E). \tag{1.8}$$

Then it is easy to see that $\{P_t^h; t \geq 0\}$ is an $\widetilde{m}$-symmetric semigroup, where $\widetilde{m} := h^2 m$, and 1 is an eigenfunction of P_t^h with eigenvalue 1. Furthermore the spectrum of the infinitesimal generator of $\{P_t^h; t \geq 0\}$ in $L^2(E; \widetilde{m})$ is the spectrum of the infinitesimal generator of $\{P_t^{(Q-1)\mu}; t \geq 0\}$ in $L^2(E; m)$ shifted by λ_1. Hence under Assumption 2, we have the following Poincaré inequality:

$$\| P_t^h \varphi \|_{L^2(E,\widetilde{m})} \leq e^{-\lambda_h t} \|\varphi\|_{L^2(E,\widetilde{m})} \tag{1.9}$$

for all $\varphi \in L^2(E, \widetilde{m})$ with $\int_E \varphi(x)\widetilde{m}(dx) = 0$.

Remark 1.1 If the underlying process X satisfies that for each $t > 0$, the transition density function $p(t, x, y)$ is bounded and is continuous in x for every fixed $y \in E$ and that the branching rate measure μ is in the Kato class $\mathbf{K}(X)$ of X, and then it follows from [1] that the Feymann–Kac semigroup $P_t^{(Q-1)\mu}$ maps bounded functions to continuous functions and is bounded from $L^p(E, m)$ to $L^q(E, m)$ for any $1 \leq p \leq q \leq +\infty$. By Friedrichs theorem, Assumptions 1 and 2 hold if we assume in addition that

the embedding of $(\mathcal{F}, \mathcal{E}_1)$ into $L^2(E, \mu)$ is compact.

Such an assumption is imposed in [6] to ensure the spectral gap condition and to obtain Poincaré inequality (1.9).

1.3 Main Results

Recall that $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $\mathbb{X}$. Observe that (cf. [22, Lemma 3.3]) for every $x \in E$ and every $f \in \mathcal{B}^+(E)$,

$$\mathbb{E}_x [\mathbb{X}_t(f)] = P_t^{(Q-1)\mu} f(x). \tag{1.10}$$

It is easy to see that $M_t := e^{\lambda_1 t}\mathbb{X}_t(h)$ is a positive $\mathbb{P}_x$-martingale with respect to $\mathcal{F}_t$. Let $M_\infty := \lim_{t\to+\infty} M_t$. It is natural to ask when M_∞ is non-degenerate, that is, when $\mathbb{P}_x(M_\infty > 0) > 0$ for $x \in E$? Under the assumptions that (i) $m(E) < \infty$, (ii) the Feymann–Kac semigroup $P_t^{(Q-1)\mu}$ is intrinsically ultracontractive, and (iii) h is bounded, it is proved in [18] that the condition

$$\int_E \int_E \sum_{k=0}^{+\infty} k p_k(y) h(y)^2 \log^+(kh(y))\mu(dy) < +\infty \tag{1.11}$$

is necessary and sufficient for M_∞ to be non-degenerate. Condition (1.11) is usually called the $L \log L$ criteria. The first main result of this paper reveals that, in general, condition (1.12) below is sufficient for M_∞ to be non-degenerate.

Theorem 1.2 *Suppose Assumptions 1–2 hold. If*

$$\int_E h(y)^2 \log^+ h(y) m(dy) + \int_E \sum_{k=0}^{+\infty} k p_k(y) h(y)^2 \log^+(kh(y)) \mu(dy) < +\infty, \tag{1.12}$$

then M_t converges to M_∞ in $L^1(\mathbb{P}_x)$ for every $x \in E$, and, consequently, $\mathbb{P}_x(M_\infty > 0) > 0$.

Thus under condition (1.12), $\mathbb{X}_t(h)$ grows exponentially with rate $-\lambda_1$. Note that when h is bounded, (1.12) is equivalent to (1.11). The next question to ask is that, for a general test function $f \in \mathcal{B}^+(E)$, what is the limiting behavior of $\mathbb{X}_t(f)$ as $t \to \infty$? By (1.8) and (1.10), it is not hard to deduce (see (2.8) below) that for every $f \in \mathcal{B}^+(E)$ with $f \le ch$ for some constant $c > 0$,

$$\mathrm{e}^{\lambda_1 t}\mathbb{E}_x[\mathbb{X}_t(f)] = h(x) P_t^h(f/h)(x) \to h(x)\langle f, h\rangle \quad \text{as } t \to +\infty.$$

So, the mean of $\mathbb{X}_t(f)$ also grows exponentially with rate $-\lambda_1$. Our previous question is related to the question: for $f \in \mathcal{B}^+(E)$ with $f \le ch$ for some constant $c > 0$, does $\mathbb{X}_t(f)$ grow exponentially with the same rate? If so, can one identify its limit? We first answer these questions in Theorem 1.3 and Corollary 1.4 in terms of convergence in $L^1(\mathbb{P}_x)$ and in probability, under an additional condition (1.13).

Note that under Assumption 1, for every $t > 0$, P_t^h has a symmetric continuous transition density function $p^h(t, x, y)$ on $E \times E$ with respect to the measure $\widetilde{m}$, which is related to $p^{(Q-1)\mu}(t, x, y)$ by the following formula:

$$p^h(t, x, y) = \mathrm{e}^{\lambda_1 t}\frac{p^{(Q-1)\mu}(t, x, y)}{h(x)h(y)}, \quad x, y \in E.$$

Theorem 1.3 *(Weak law of large numbers) Suppose Assumptions 1–2 and (1.12) hold. If there exists some $t_0 > 0$ such that*

$$\int_E p^{(Q-1)\mu}(t_0, y, y) m(dy) < +\infty, \quad \textit{or equivalently,}$$
$$\int_E p^h(t_0, y, y)\widetilde{m}(dy) < +\infty, \tag{1.13}$$

then for any $x \in E$ and any $f \in \mathcal{B}^+(E)$ with $f \le ch$ for some $c > 0$, we have

$$\lim_{t\to+\infty} e^{\lambda_1 t}\mathbb{X}_t(f) = M_\infty\langle f, h\rangle \quad \textit{in } L^1(\mathbb{P}_x).$$

Corollary 1.4 *Under the assumptions of Theorem 1.3, it holds that*

$$\lim_{t\to+\infty} \frac{\mathbb{X}_t(f)}{\mathbb{E}_x[\mathbb{X}_t(f)]} = \frac{M_\infty}{h(x)} \quad \textit{in probability with respect to } \mathbb{P}_x,$$

for every $x \in E$ and every $f \in \mathcal{B}^+(E)$ with $f \le ch$ for some $c > 0$.

For almost sure convergence result, we need a stronger condition (1.14) below.

Theorem 1.5 *(Strong law of large numbers) Suppose Assumptions 1–2 and* (1.12) *hold. If there exists* $t_1 > 0$ *such that*

$$\sup_{y\in E} \frac{p^{(Q-1)\mu}(t_1, y, y)}{h(y)^2} < +\infty, \quad \textit{or, equivalently,} \quad \sup_{y\in E} p^h(t_1, y, y) < +\infty, \tag{1.14}$$

then there exists $\Omega_{\mathbf{0}} \subset \Omega$ *of* $\mathbb{P}_x$*-full probability for every* $x \in E$*, such that, for every* $\omega \in \Omega_{\mathbf{0}}$ *and every* $f \in \mathcal{B}_b(E)$ *with compact support whose set of discontinuous points has zero* m*-measure, we have*

$$\lim_{t\to+\infty} e^{\lambda_1 t}\mathbb{X}_t(f)(\omega) = M_\infty(\omega)\langle f, h\rangle. \tag{1.15}$$

Corollary 1.6 *Suppose the assumptions of Theorem 1.5 hold and let* $\Omega_{\mathbf{0}}$ *be defined in Theorem 1.5. Then*

$$\lim_{t\to+\infty} \frac{\mathbb{X}_t(f)(\omega)}{\mathbb{E}_x\left[\mathbb{X}_t(f)\right]} = \frac{M_\infty(\omega)}{h(x)}$$

for every $\omega \in \Omega_{\mathbf{0}}$ *and for every* $f \in \mathcal{B}_b(E)$ *with compact support whose set of discontinuous points has zero* m*-measure.*

Remark 1.7 The condition (1.13) is equivalent to

$$\int_E \int_E p^h(t_0/2, x, y)^2 \widetilde{m}(\mathrm{d}y)\widetilde{m}(\mathrm{d}x) < +\infty.$$

Hence by [7, Page 156], P_t^h is a compact operator on $L^2(E, \widetilde{m})$ for every $t \geq t_0/2$. Consequently Assumption 2 is automatically satisfied if either (1.13) or (1.14) holds.

To understand condition (1.14), we give some equivalent statements of (1.14) under our Assumptions 1–2.

Proposition 1.8 *Suppose Assumptions 1–2 hold. The following are equivalent to* (1.14).

(i) *There exists* $t_1 > 0$ *such that for any* $t > t_1$,

$$\sup_{x,y\in E} |p^h(t, x, y) - 1| \leq c_1 e^{-c_2 t} \tag{1.16}$$

for some $c_1, c_2 > 0$.

(ii)

$$p^h(t, x, y) \to 1, \ \textit{as } t \uparrow +\infty \textit{ uniformly in } (x, y) \in E \times E. \tag{1.17}$$

(iii) *There exist constants* $t, c_t > 0$ *such that*

$$p^{(Q-1)\mu}(t,x,y) \le c_t h(x)h(y) \quad \textit{for every } x, y \in E. \tag{1.18}$$

Property (1.18) is called *asymptotically intrinsically ultracontractive* (AIU) by Kaleta and Lőrinczi in [15]. If the inequality in (1.18) is true for every $t > 0$, and every $x, y \in E$, then $\{P_t^{(Q-1)\mu} : t > 0\}$ is called *intrinsically ultracontractive* (IU). It is shown in [15] that in case of symmetric α-stable processes ($\alpha \in (0,2)$), AIU is a weaker property than IU.

1.4 Examples

In this subsection, we illustrate our main results by several concrete examples. For simplicity, we consider binary branching only, i.e., every particle gives birth to precisely two children, in which case $Q(x) \equiv 2$ on E. Since $\sum_{k=0}^{+\infty} kp_k(x)\log^+ k \equiv 2\log 2$ on E, condition (1.12) is reduced to

$$\int_E h(y)^2 \log^+ h(y)\,(m(\mathrm{d}y) + \mu(\mathrm{d}y)) < +\infty. \tag{1.19}$$

Example 1 (WLLN for branching OU processes with a quadratic branching rate function) Let $E = \mathbb{R}^d$. In Example 10 of [11], (X, Π_x) is an Ornstein–Ulenbeck (OU) process on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\sigma^2\Delta - cx\cdot\nabla \quad \text{on } \mathbb{R}^d,$$

where $\sigma,\ c > 0$. Without loss of generality, we assume $\sigma = 1$. Let $m(\mathrm{d}x) = \left(\frac{c}{\pi}\right)^{d/2} \mathrm{e}^{-c|x|^2}\mathrm{d}x$. Then X is symmetric with respect to the probability measure m, and the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X on $L^2(\mathbb{R}^d, m)$ is given by

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d, m) : \int_{\mathbb{R}^d} |\nabla f(x)|^2 m(\mathrm{d}x) < +\infty \right\},$$
$$\mathcal{E}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla f(x)|^2 m(\mathrm{d}x).$$

Let $\beta(x) = b|x|^2 + a$ with $a, b > 0$ be the branching rate function. Let P_t^β be the corresponding Feynman–Kac semigroup,

$$P_t^\beta f(x) := \Pi_x\left[\exp\left(\int_0^t \beta(X_s)\mathrm{d}s\right) f(X_t)\right].$$

Suppose $c > \sqrt{2b}$ and $\alpha = \sqrt{c^2 - 2b}$. Let

$$\lambda_c := \inf\{\lambda \in \mathbb{R} : \text{ there exists } u > 0 \text{ such that } (\mathcal{L} + \beta - \lambda)u = 0 \text{ in } \mathbb{R}^d\}$$

be the generalized principal eigenvalue. Let ϕ denote the corresponding ground state, i.e., $\phi > 0$ such that $(\mathcal{L}+\beta-\lambda_c)\phi = 0$. As indicated in [11], $\lambda_c = \frac{1}{2}(c-\alpha)+a > 0$ and $\phi(x) = \left(\frac{\alpha}{c}\right)^{d/4}\exp(\frac{1}{2}(c-\alpha)|x|^2)$. Note that $\phi \in \mathcal{F}^\mu$ and $\phi = \mathrm{e}^{-\lambda_c t}P_t^\beta\phi$ on $\mathbb{R}^d$. It is easy to see that in this example $\lambda_1 = -\lambda_c$ and $h(x) = \phi(x)$. The transformed process (X^h, Π_x^h) is also an OU process with infinitesimal generator

$$\mathcal{L} = \frac{1}{2}\Delta - \alpha x\cdot\nabla \text{ on } \mathbb{R}^d.$$

Note that its invariant probability measure is $\widetilde{m}(\mathrm{d}x) = h(x)^2 m(\mathrm{d}x) = \left(\frac{\alpha}{\pi}\right)^{d/2}\mathrm{e}^{-\alpha|x|^2}\mathrm{d}x$. Let $p^h(t,x,y)$ be the transition density of X^h with respect to $\widetilde{m}$. It is known that

$$p^h(t,x,y) = \left(\frac{1}{1-\mathrm{e}^{-2\alpha t}}\right)^{d/2}\exp\left(-\frac{\alpha}{(\mathrm{e}^{2\alpha t}-1)}\left(|x|^2+|y|^2-2x\cdot y\mathrm{e}^{\alpha t}\right)\right)$$

In particular,

$$p^h(t,x,x) = \left(\frac{1}{1-\mathrm{e}^{-2\alpha t}}\right)^{d/2}\exp\left(\frac{2\alpha}{\mathrm{e}^{\alpha t}+1}|x|^2\right).$$

Thus $\int_{\mathbb{R}^d} p^h(t,x,x)\widetilde{m}(\mathrm{d}x) < +\infty$ for $t > 0$. Moreover, we observe that condition (1.19) is satisfied for this example. Therefore Theorem 1.3 holds for this example.

This example does not satisfy the assumptions in [6]. To be more specific, here the ground state h is unbounded and $\beta(x) = b|x|^2 + a$ is not in the Kato class $\mathbf{K}_\infty(X)$ of X.

Example 2 (WLLN for branching Hunt processes with a bounded branching rate function) Let E be a locally compact separable metric space and m a positive Radon measure on E with full support. Suppose the branching rate function β is a nonnegative bounded function on E. Suppose the underlying Hunt process (X, Π_x) satisfies that for every $t > 0$, there exists a family of jointly continuous, symmetric and positive kernels $p(t,x,y)$ such that $P_t f(x) = \int_E p(t,x,y)f(y)m(\mathrm{d}y)$, and that there exists $s_1 > 0$ so that

$$\int_E p(s_1,x,x)m(\mathrm{d}x) < +\infty. \tag{1.20}$$

In this case, the Feyman–Kac semigroup

$$P_t^\beta f(x) := \Pi_x\left[\exp\left(\int_0^t \beta(X_s)\mathrm{d}s\right)f(X_t)\right]$$

has a jointly continuous and positive kernel $p^\beta(t,x,y)$. It is easy to see that

$$\mathrm{e}^{-\|\beta\|_\infty t}p(t,x,y) \le p^\beta(t,x,y) \le \mathrm{e}^{\|\beta\|_\infty t}p(t,x,y) \quad \text{for every } t>0 \text{ and } x,y\in E. \tag{1.21}$$

Properties (1.20) and (1.21) imply that $\int_E p^\beta(s_1, x, x)m(\mathrm{d}x) < +\infty$. Thus P_t^β is a compact operator on $L^2(E, m)$ for every $t \geq s_1$. By Jentzch's theorem (see, for example, [21, Theorem V.6.6]), $-\lambda_1$ is a simple eigenvalue of $\mathcal{L} + \beta$ where $\mathcal{L}$ is the infinitesimal operator of X, and an eigenfunction h of $\mathcal{L} + \beta$ associated with $-\lambda_1$ can be chosen to be positive and continuous on E. Suppose $\lambda_1 < 0$. We assume in addition that there exists $s_2 > 0$ such that

$$\int_E p(s_2, x, x)^2 m(\mathrm{d}x) < +\infty. \tag{1.22}$$

It follows from (1.21) and Hölder's inequality that for every $t > s_2$, P_t^β is a bounded operator from $L^2(E, m)$ to $L^4(E, m)$. Thus $h = \mathrm{e}^{\lambda_1 t} P_t^\beta h \in L^4(E, m)$, and so condition (1.19) is satisfied. Hence Theorem 1.3 holds.

Conditions (1.20) and (1.22) are satisfied by a large class of Hunt processes, which contains subordinated OU processes as special cases. By "subordinated OU process," we mean the process $X_t = Y_{S_t}$, where Y_t is an OU process on $\mathbb{R}^d$ and S_t is a subordinator on $\mathbb{R}_+$ independent of Y_t. In the special case $S_t \equiv t$, X_t reduces to the OU process. In general, the sample path of X_t is discontinuous. Suppose the infinitesimal generator of Y_t is

$$\widehat{\mathcal{L}} = \frac{1}{2}\sigma^2 \Delta - bx \cdot \nabla \quad \text{on } \mathbb{R}^d$$

where $\sigma, b > 0$ are constants, and S_t is a subordinator with positive drift coefficient $a > 0$. As is indicated in Example 1, Y_t is symmetric with respect to the reference measure $m(\mathrm{d}x) := \left(\frac{b}{\pi\sigma^2}\right)^{d/2} \exp(-b|x|^2/\sigma^2)\mathrm{d}x$. We use $\hat{p}(t, x, y)$ to denote the transition density of Y_t with respect to m. It is known that

$$\hat{p}(t, x, y) = \left(1 - \mathrm{e}^{-2bt}\right)^{-d/2} \exp\left(-\frac{b}{\sigma^2\left(\mathrm{e}^{2bt} - 1\right)}\left(|x|^2 + |y|^2 - 2x \cdot y\mathrm{e}^{bt}\right)\right).$$

By definition, the transition density of X_t with respect to m is given by

$$p(t, x, y) = \mathbb{E}\left[\hat{p}(S_t, x, y)\right].$$

It is proved in [19, Example 1.1] that (1.20) and (1.22) hold for a subordinated OU process. Therefore, Theorem 1.3 holds for branching subordinated OU processes with a bounded branching rate function.

Example 3 (LLN for branching diffusions on bounded domains with branching rate given by a Kato class measure) Suppose $d \geq 3$, $E \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ domain (that is, the boundary of E can be locally characterized by $C^{1,1}$ functions) and m is the Lebesgue measure on E. Let

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} \partial_i (a_{ij} \partial_j)$$

with $a_{ij}(x) \in C^1(\mathbb{R}^d)$ for every $i, j = 1, \ldots, d$. Suppose the matrix $(a_{ij}(x))$ is symmetric and uniformly elliptic. It is known that there exists a symmetric diffusion process Y on $\mathbb{R}^d$ with generator $\mathcal{L}$. Let X be the killed process of Y upon E, i.e.,

$$X_t = \begin{cases} Y_t, & t < \tau_E, \\ \partial, & t \geq \tau_E, \end{cases}$$

where $\tau_E := \inf\{t > 0 : \ Y_t \notin E\}$ and ∂ is a cemetery state. Then X has a transition density function $p_E(t, x, y)$ which is jointly continuous in (x, y) and positive for every $t > 0$. The following two-sided estimates of $p_E(t, x, y)$ are established in [20, Theorem 2.1], extending an earlier result of Q. Zhang. Let $f_E(t, x, y) := \left(1 \wedge \frac{\delta_E(x)}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_E(y)}{\sqrt{t}}\right)$, where $\delta_E(x)$ denotes the distance between x and the boundary of E. There exist positive constants c_i, $i = 1, \ldots, 4$, such that for every $(t, x, y) \in (0, 1] \times E \times E$,

$$c_1 f_E(t, x, y) t^{-d/2} \mathrm{e}^{-c_2 |x-y|^2/t} \leq p_E(t, x, y) \leq c_3 f_E(t, x, y) t^{-d/2} \mathrm{e}^{-c_4 |x-y|^2/t}.$$

We say that a signed smooth Radon measure ν on $\mathbb{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ ($\alpha \in (0, 2]$) if

$$\lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq r} \frac{|\nu|(\mathrm{d}y)}{|x-y|^{d-\alpha}} = 0. \tag{1.23}$$

In fact $\mathbf{K}_{d,\alpha}$ is the Kato class of the rotationally symmetric α-stable processes on $\mathbb{R}^d$. We assume the branching rate measure μ is a nonnegative Radon measure in $\mathbf{K}_{d,2}$. For any $f \in \mathcal{B}^+(E)$, let

$$P_t^\mu f(x) := \mathrm{E}_x \left[\exp(A_t^\mu) f(X_t) \right].$$

Then P_t^μ has a transition density $p_E^\mu(t, x, y)$ which is jointly continuous in (x, y) and positive for every $t > 0$. It is shown in [16, Theorem 4.4] that there exist positive constants c_i, $i = 5, \ldots, 8$, such that for every $(t, x, y) \in (0, 1] \times E \times E$,

$$c_5 f_E(t, x, y) t^{-d/2} \mathrm{e}^{-c_6 |x-y|^2/t} \leq p_E^\mu(t, x, y) \leq c_7 f_E(t, x, y) t^{-d/2} \mathrm{e}^{-c_8 |x-y|^2/t}. \tag{1.24}$$

The infinitesimal generator of P_t^μ is $(\mathcal{L} + \mu)|_E$ with zero Dirichlet boundary condition. It follows from Jentzch's theorem that $-\lambda_1$ is a simple eigenvalue of $(\mathcal{L} + \mu)|_E$ and that an eigenfunction h associated with $-\lambda_1$ can be chosen to be positive with $\|h\|_{L^2(E, \mathrm{d}x)} = 1$. Immediately, h is continuous on E by the dominated convergence theorem. We assume $\lambda_1 < 0$. Recall that E is bounded. Using the equation $h = \mathrm{e}^{\lambda_1} P_1^{\mu} h$ and the estimates in (1.24), we get that for every $x \in E$,

$$c_9(1 \wedge \delta_E(x)) \leq h(x) \leq c_{10}(1 \wedge \delta_E(x)) \tag{1.25}$$

for some positive constants c_9, c_{10}. Let

$$p_E^h(t, x, y) := \frac{\mathrm{e}^{\lambda_1 t} p_E^{\mu}(t, x, y)}{h(x)h(y)} \quad \text{for } x, y \in E.$$

Immediately condition (1.19) holds by the boundedness of h, and condition (1.14) holds by (1.24) and (1.25). Therefore both Theorem 1.3 and Theorem 1.5 hold for this example.

Example 4 (LLN for branching killed α-stable processes with a bounded branching rate function) Suppose $d \geq 1$, $E = \mathbb{R}^d$, m is the Lebesgue measure on $\mathbb{R}^d$ and $\alpha \in (0, 2)$. Suppose Y is a symmetric α-stable process on $\mathbb{R}^d$, and $c(x)$ is a nonnegative function in $\mathbf{K}_{d,\alpha}$ (a function q is said to be in $\mathbf{K}_{d,\alpha}$, if the measure $\nu(\mathrm{d}x) := q(x)\mathrm{d}x$ is in $\mathbf{K}_{d,\alpha}$ where $\mathbf{K}_{d,\alpha}$ is defined in (1.23)). Let X be the subprocess of Y such that for all $f \in \mathcal{B}_b(\mathbb{R}^d)$,

$$P_t f(x) := \mathrm{E}_x[f(X_t)] = \mathrm{E}_x\left[\exp\left(-\int_0^t c(Y_s)\mathrm{d}s\right) f(Y_t)\right].$$

It is known that the infinitesimal generator of X is $\mathcal{L} = \Delta^{\alpha/2} - c(x)$, where $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ is the generator of a symmetric α-stable process. Let the branching rate function β be a nonnegative bounded function on $\mathbb{R}^d$. Let $V(x) := c(x) - \beta(x)$. Clearly, $|V| \in \mathbf{K}_{d,\alpha}$. For any $f \in \mathcal{B}^+(\mathbb{R}^d)$, let

$$P_t^{\beta} f(x) := \mathrm{E}_x\left[\exp\left(\int_0^t \beta(X_s)\mathrm{d}s\right) f(X_t)\right].$$

Note that for every $t > 0$, P_t is bounded from $L^1(\mathbb{R}^d, \mathrm{d}x)$ to $L^{\infty}(\mathbb{R}^d, \mathrm{d}x)$ and satisfies the strong Feller property. It follows from [1] that for very $t > 0$, P_t^{β} is bounded from $L^{p_1}(\mathbb{R}^d, \mathrm{d}x)$ to $L^{p_2}(\mathbb{R}^d, \mathrm{d}x)$ for any $1 \leq p_1 \leq p_2 \leq +\infty$. Thus under our Assumption 1, the ground state h is a positive bounded continuous function on $\mathbb{R}^d$. The semigroup $\{P_t^{\beta} : t \geq 0\}$ is the Feynman–Kac semigroup with infinitesimal generator $\mathcal{L}^{\beta} = \Delta^{\alpha/2} - V$. Assume in addition that $V(x) = c(x) - \beta(x)$ satisfies the following conditions: $\liminf_{|x| \to +\infty} V^+(x)/\log|x| > 0$ and V^- has compact support. Then by [15, Theorem 5.5], the semigroup P_t^{β} is AIU. Hence both Theorem 1.3 and Theorem 1.5 are true for this example. It is known from [15] that in case of symmetric α-stable processes, AIU is weaker than IU. For instance, $V(x) = c(x) - \beta(x)$ with $c(x) = \log|x| 1_{\{|x| \geq R\}}$ and β has compact support in $B(0, R)$ for some $R \geq 1$ is such an example.

The rest of the paper is organized as follows. In Sect. 2, we review some facts on Girsanov transform and h-transforms in the context of symmetric Markov processes, and prove Proposition 1.8. Spine construction of branching processes is recalled in Sect. 3. We then present proof for the $L \log L$ criteria, Theorem 1.2, in Sect. 4. Weak law of large numbers, Theorem 1.3, is proved in Sect. 5, while Theorem 1.5 on the strong law of large numbers will be proved in Sect. 6. The lower case constants $c_1, c_2, \ldots,$ will denote the generic constants used in this article, whose exact values are not important, and can change from one appearance to another.

2 Preliminary

Recall $h \in \mathcal{F}^{\mu}$ is the minimizer in Assumption 1. Since $h \in \mathcal{F}$, by Fukushima's decomposition, we have for q.e. $x \in E$, Π_x-a.s.

$$h(X_t) - h(X_0) = M_t^h + N_t^h, \quad t \geq 0,$$

where M^h is a martingale additive functional of X having finite energy and N_t^h is a continuous additive functional of X having zero energy. It follows from (1.7) and [13, Theorem 5.4.2] that N_t^h is of bounded variation, and

$$N_t^h = -\lambda_1 \int_0^t h(X_s)\mathrm{d}s - \int_0^t h(X_s)\mathrm{d}A_s^{(Q-1)\mu}, \quad \forall t \geq 0.$$

Following [4, Section 2] (see also [6, Section 2]), we define a local martingale on the random time interval $[0, \zeta_p)$ by

$$M_t := \int_0^t \frac{1}{h(X_{s-})}\mathrm{d}M_s^h, \quad t \in [0, \zeta_p), \tag{2.1}$$

where ζ_p is the predictable part of the life time ζ of X. Then the solution R_t of the stochastic differential equation

$$R_t = 1 + \int_0^t R_{s-}\mathrm{d}M_s, \quad t \in [0, \zeta_p), \tag{2.2}$$

is a positive local martingale on $[0, \zeta_p)$ and hence a supermartingale. As a result, the formula

$$\mathrm{d}\Pi_x^h = R_t \mathrm{d}\Pi_x \quad \text{on } \mathcal{H}_t \cap \{t < \zeta\} \quad \text{for } x \in E$$

uniquely determines a family of subprobability measures $\{\Pi_x^h : x \in E\}$ on $(\Omega, \mathcal{H})$. We denote X under $\{\Pi_x^h : x \in E\}$ by X^h, that is

$$\Pi_x^h\left[f(X_t^h)\right] = \Pi_x\left[R_t f(X_t) : t < \zeta\right]$$

for any $t \geq 0$ and $f \in \mathcal{B}^+(E)$. It follows from [4, Theorem 2.6] that the process X^h is an irreducible recurrent $\widetilde{m}$-symmetric right Markov process, where $\widetilde{m}(\mathrm{d}y) := h(y)^2 m(\mathrm{d}y)$. Note that by (2.1), (2.2) and Doléan-Dade's formula,

$$R_t = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right) \prod_{0<s\leq t} \frac{h(X_s)}{h(X_{s-})} \exp\left(1 - \frac{h(X_s)}{h(X_{s-})}\right), \quad t \in [0, \zeta_p), \quad (2.3)$$

where M^c is the continuous martingale part of M. Applying Ito's formula to $\log h(X_t)$, we obtain that for q.e. $x \in E$, Π_x-a.s. on $[0, \zeta)$,

$$\begin{aligned} \log h(X_t) - \log h(X_0) = M_t - \frac{1}{2}\langle M^c\rangle_t + \sum_{s\leq t}\left(\log \frac{h(X_s)}{h(X_{s-})} - \frac{h(X_s) - h(X_{s-})}{h(X_{s-})}\right) \\ - \lambda_1 t - A_t^{(Q-1)\mu}. \end{aligned} \quad (2.4)$$

By (2.3) and (2.4), we get

$$R_t = \exp\left(\lambda_1 t + A_t^{(Q-1)\mu}\right) \frac{h(X_t)}{h(X_0)} \quad \text{on } [0, \zeta).$$

Therefore for any $f \in \mathcal{B}^+(E)$,

$$\begin{aligned} \Pi_x^h\left[f(X_t^h)\right] &= \frac{\mathrm{e}^{\lambda_1 t}}{h(x)} \Pi_x\left[\mathrm{e}^{A_t^{(Q-1)\mu}} h(X_t) f(X_t)\right] \\ &= \frac{\mathrm{e}^{\lambda_1 t}}{h(x)} P_t^{(Q-1)\mu}(hf)(x) = P_t^h f(x). \end{aligned} \quad (2.5)$$

Let $(\mathcal{E}^h, \mathcal{F}^h)$ be the symmetric Dirichlet form on $L^2(E, \widetilde{m})$ generated by X^h. Then (2.5) says that the transition semigroup of X^h is exactly the semigroup $\{P_t^h : t \geq 0\}$ obtained from $P_t^{(Q-1)\mu}$ through Doob's h-transform. Consequently, $f \in \mathcal{F}^h$ if and only if $fh \in \mathcal{F}^\mu$, and

$$\mathcal{E}^h(f, f) = \mathcal{E}^{(Q-1)\mu}(fh, fh) - \lambda_1 \int_E f(x)^2 h(x)^2 m(\mathrm{d}x).$$

In other words, $\Phi^h : f \mapsto fh$ is an isometry from $(\mathcal{E}^h, \mathcal{F}^h)$ onto $(\mathcal{E}^{(Q-1)\mu+\lambda_1 m}, \mathcal{F}^\mu)$ and from $L^2(E, \widetilde{m})$ onto $L^2(E, m)$. Let $\sigma(\mathcal{E}^h)$ denote the spectrum of the positive definite self-adjoint operator associated with $\mathcal{E}^h$. We know from [4, Theorem 2.6] that the constant function 1 belongs to $\mathcal{F}^h$, and $\mathcal{E}^h(1, 1) = 0$. Hence $0 \in \sigma(\mathcal{E}^h)$ is a simple eigenvalue and 1 is the corresponding eigenfunction. Therefore

$$\lambda_1^h := \inf\left\{\mathcal{E}^h(u, u) : u \in \mathcal{F}^h \text{ with } \int_E u(x)^2 \widetilde{m}(\mathrm{d}x) = 1\right\} = 0.$$

Let λ_2^h be the second bottom of $\sigma(\mathcal{E}^h)$, i.e.,

$$\lambda_2^h := \inf\left\{\mathcal{E}^h(u,u):\ u\in\mathcal{F}^h \text{ with } \int_E u(x)\widetilde{m}(\mathrm{d}x)=0 \text{ and } \int_E u(x)^2\widetilde{m}(\mathrm{d}x)=1\right\}.$$

In view of the isometry Φ^h, we have $\lambda_2^h=\lambda_2-\lambda_1$. So Assumption 2 is equivalent to assuming $\lambda_2^h>0$.

Define

$$\widetilde{a}_t(x) := p^h(t,x,x) \quad \text{for } t>0 \text{ and } x\in E. \tag{2.6}$$

Note that by (1.9) for any $x\in E$ and $t,s\geq 0$,

$$\begin{aligned}
|P^h_{t+s}\varphi(x)| &= |P^h_s P^h_t\varphi(x)| \\
&\leq \int_E p^h(s,x,y)|P^h_t\varphi(y)|\widetilde{m}(\mathrm{d}y) \\
&\leq \left(\int_E p^h(s,x,y)^2\widetilde{m}(\mathrm{d}y)\right)^{1/2} \|P^h_t\varphi\|_{L^2(E,\widetilde{m})} \\
&\leq \widetilde{a}_{2s}(x)^{1/2}\mathrm{e}^{-\lambda_h t}\|\varphi\|_{L^2(E,\widetilde{m})}.
\end{aligned} \tag{2.7}$$

We use $\langle f,g\rangle_{L^2(\widetilde{m})}$ to denote $\int_E f(x)g(x)\widetilde{m}(\mathrm{d}x)$. For every $g\in L^2(E,\widetilde{m})$, we can decompose g as $g(x)=\langle g,1\rangle_{L^2(\widetilde{m})}+\varphi(x)$ with $\varphi(x)$ satisfying $\int_E\varphi(x)\widetilde{m}(\mathrm{d}x)=0$. Then by (2.7), for any $t>s\geq 0$ and any $x\in E$,

$$\begin{aligned}
|P^h_t g(x)-\langle g,1\rangle_{L^2(\widetilde{m})}| = |P^h_t\varphi(x)| &\leq \mathrm{e}^{-\lambda_h(t-s)}\widetilde{a}_{2s}(x)^{1/2}\|\varphi\|_{L^2(E,\widetilde{m})} \\
&\leq \mathrm{e}^{-\lambda_h(t-s)}\widetilde{a}_{2s}(x)^{1/2}\|g\|_{L^2(E,\widetilde{m})}.
\end{aligned} \tag{2.8}$$

Proof of Proposition 1.8 We only need to prove that (1.14) implies (1.16). For any $f\in L^2(E,\widetilde{m})$, define $c_f:=\int_E f(x)\widetilde{m}(\mathrm{d}x)$. Immediately, we have for any $t>0$, $\int_E(f-c_f)\widetilde{m}(\mathrm{d}x)=0$, and $f-c_f\in L^2(E,\widetilde{m})$. By (1.9), we have

$$\|P^h_t f-c_f\|_{L^2(E,\widetilde{m})}\leq \mathrm{e}^{-\lambda_h t}\|f\|_{L^2(E,\widetilde{m})}. \tag{2.9}$$

Let $t_1>0$ be the constant in (1.14). By the semigroup property, for any $t\geq t_1/2$ and $x\in E$,

$$\begin{aligned}
p^h(t,x,y) &= \int_E p^h(t_1/2,x,z)p^h(t-t_1/2,y,z)\widetilde{m}(\mathrm{d}z) \\
&= P^h_{t-t_1/2}f_x(y),
\end{aligned} \tag{2.10}$$

where $f_x(\cdot) := p^h(t_1/2, x, \cdot)$. Note that by Hölder's inequality,

$$\begin{aligned}|p^h(t,x,y)-1| &= \left|\int_E p^h(t/2,x,z)p^h(t/2,z,y)\widetilde{m}(\mathrm{d}z)-1\right| \\ &= \left|\int_E (p^h(t/2,x,z)-1)(p^h(t/2,z,y)-1)\widetilde{m}(\mathrm{d}z)\right| \\ &\le \left(\int_E (p^h(t/2,x,z)-1)^2\widetilde{m}(\mathrm{d}z)\right)^{1/2} \\ &\quad \times \left(\int_E (p^h(t/2,z,y)-1)^2\widetilde{m}(\mathrm{d}z)\right)^{1/2} \end{aligned} \tag{2.11}$$

Note that $c_{f_x} = \int_E p^h(t_1/2, x, y)\widetilde{m}(\mathrm{d}y) = 1$ and $\int_E f_x^2(y)\widetilde{m}(\mathrm{d}y) = \widetilde{a}_{t_1}(x)$. By (2.10) and (2.9), for any $t > t_1$,

$$\begin{aligned}\left(\int_E (p^h(t/2,x,z)-1)^2\widetilde{m}(\mathrm{d}z)\right)^{1/2} &= \|P^h_{(t-t_1)/2}f_x - c_{f_x}\|_{L^2(E,\widetilde{m})} \\ &\le \mathrm{e}^{-\lambda_h(t-t_1)/2}\|f_x\|_{L^2(E,\widetilde{m})} \\ &= \mathrm{e}^{-\lambda_h(t-t_1)/2}\widetilde{a}_{t_1}(x)^{1/2}. \end{aligned} \tag{2.12}$$

Similarly, for any $t > t_1$,

$$\left(\int_E (p^h(t/2,z,y)-1)^2\widetilde{m}(\mathrm{d}z)\right)^{1/2} \le \mathrm{e}^{-\lambda_h(t-t_1)/2}\widetilde{a}_{t_1}(y)^{1/2}. \tag{2.13}$$

Combining (2.11), (2.12) and (2.13), we have for any $t > t_1$,

$$|p^h(t,x,y)-1| \le \mathrm{e}^{-\lambda_h(t-t_1)}\widetilde{a}_{t_1}(x)^{1/2}\widetilde{a}_{t_1}(y)^{1/2}. \tag{2.14}$$

Therefore (1.14) implies that for any $t > t_1$, (1.16) holds. □

3 Spine Construction

To establish the L^1 convergence of the martingale M_t, we apply the "spine" and change of measure techniques presented in [14] for branching diffusions to our branching Hunt processes. The notation used here is closely related to that used in [14]. It is known that the family structure of the particles in a branching Markov process can be well expressed by marked Galton–Watson trees (see, for example, [14,18] and the references therein). Let $\mathcal{T}$ denote the space of all marked G-W trees. For a fixed $\tau \in \mathcal{T}$, all particles in τ are labeled according to the Ulam–Harris convention, for example, Ø231 or 231 is the first child of the third child of the second child of the initial ancestor Ø. Besides, each particle $u \in \tau$ has a mark (X_u, σ_u, A_u) where $X_u : [b_u, \zeta_u) \to E_\partial$ is the spatial location of u during its life span (b_u is its birth time and ζ_u its fission time),

$\sigma_u = \zeta_u - b_u$ is the length of its life span, and A_u denotes the number of its offspring. We use $u \prec v$ to mean that u is an ancestor of v.

Since in this paper every particle is assumed to give birth to at least one child, for each tree τ, we can choose a distinguished genealogical path of descent from the initial ancestor $\emptyset$. Such a line is called a *spine* and denoted as $\xi = \{\xi_0 = \emptyset, \xi_1, \xi_2, \ldots\}$, where ξ_i is the label of the ith spine node. Define $\mathrm{node}_t(\xi) := u$ if $u \in \xi$ and is alive at time t. We shall use $\{\widetilde{X}_t : t \geq 0\}$ and $\{n_t : t \geq 0\}$, respectively, to denote the spatial path and the counting process of fission times along the spine. Let $\widetilde{\mathcal{T}}$ denote the space of G-W trees with a distinguished spine. We introduce some fundamental filtrations that encapsulate different knowledge:

$$\begin{aligned}
&\mathcal{F}_t := \sigma\left\{(u, X_u, \sigma_u, A_u), \zeta_u \leq t;\ (u, X_u(s), s \in [b_u, t]), t \in [b_u, \zeta_u)\right\},\\
&\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t := \sigma\{\mathcal{F}_t; t \geq 0\};\\
&\widetilde{\mathcal{F}}_t := \sigma\left\{\mathcal{F}_t, \mathrm{node}_t(\xi)\right\}, \quad \widetilde{\mathcal{F}}_\infty = \bigvee_{t\geq 0} \widetilde{\mathcal{F}}_t;\\
&\mathcal{G}_t := \sigma\{\widetilde{X}_s : s \leq t\}, \quad \mathcal{G}_\infty := \bigvee_{t\geq 0} \mathcal{G}_t;\\
&\widetilde{\mathcal{G}}_t := \sigma\left\{\mathcal{G}_t, (\mathrm{node}_s(\xi) : s \leq t), (A_u :\ u \prec \mathrm{node}_t(\xi))\right\}, \quad \widetilde{\mathcal{G}}_\infty := \bigvee_{t\geq 0} \widetilde{\mathcal{G}}_t.
\end{aligned}$$

We assume $\mathbb{P}_x$ is the measure on $(\widetilde{\mathcal{T}}, \mathcal{F}_\infty)$ such that the filtered probability space $\left(\widetilde{\mathcal{T}}, \mathcal{F}_\infty, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x\right)$ is the canonical model for the branching Hunt process $\mathbb{X}$ described in the introduction. We know from [14] that the measure $\mathbb{P}_x$ on $(\widetilde{\mathcal{T}}, \mathcal{F}_\infty)$ can be extended to the probability measure $\widetilde{\mathbb{P}}_x$ on $(\widetilde{\mathcal{T}}, \widetilde{\mathcal{F}}_\infty)$ such that the nth spine node is uniformly chosen from the children of the $(n-1)$th spine node.

For every $t \geq 0$, as in [14], we define

$$\begin{aligned}
&\eta(t) := \exp\left(\lambda_1 t + A_t^{(Q-1)\mu}\right)\frac{h(\widetilde{X}_t)}{h(x)}, \quad Z(t) := \mathrm{e}^{\lambda_1 t}\frac{\mathbb{X}_t(h)}{h(x)}, \quad \text{and}\\
&\widetilde{\eta}(t) := \mathrm{e}^{\lambda_1 t}\frac{h(\widetilde{X}_t)}{h(x)}\prod_{v\prec \mathrm{node}_t(\xi)} A_v.
\end{aligned}$$

Then $\eta(t)$, $Z(t)$ and $\widetilde{\eta}(t)$ are positive $\widetilde{\mathbb{P}}_x$-martingales with respect to the filtrations $\mathcal{G}_t$, $\mathcal{F}_t$ and $\widetilde{\mathcal{F}}_t$, respectively. Moreover, both $\eta(t)$ and $Z(t)$ are projections of $\widetilde{\eta}(t)$ onto their filtrations:

$$Z(t) = \widetilde{\mathbb{P}}_x\left[\widetilde{\eta}(t)|\mathcal{F}_t\right], \quad \eta(t) = \widetilde{\mathbb{P}}_x\left[\widetilde{\eta}(t)|\mathcal{G}_t\right] \quad \text{for } t \geq 0.$$

We call $\eta(t)$ the *single-particle martingale* and $Z(t)$ the *branching-particle martingale*. As in [14], we define a new probability measure $\widetilde{\mathbb{Q}}_x$ by setting

$$\mathrm{d}\widetilde{\mathbb{Q}}_x|_{\widetilde{\mathcal{F}}_t} = \widetilde{\eta}(t)\mathrm{d}\widetilde{\mathbb{P}}_x|_{\widetilde{\mathcal{F}}_t} \quad \text{for } t \geq 0. \tag{3.1}$$

This implies that

$$\mathrm{d}\widetilde{\mathbb{Q}}_x|_{\mathcal{F}_t} = Z(t)\mathrm{d}\widetilde{\mathbb{P}}_x|_{\mathcal{F}_t} \quad \text{for } t \geq 0. \tag{3.2}$$

The influence of the measure change (3.1) lies in the following three aspects: Firstly, under $\widetilde{\mathbb{Q}}_x$ the motion of the spine is biased by the martingale $\eta(t)$. Secondly, the branching events along the spine occur at an accelerated rate. Finally, the number of children of the spine nodes is size-biased distributed, that is, for every spine node v located at x, A_v is distributed according to the probability mass function $\{\widehat{p}_k(x) := kp_k(x)/Q(x) : k = 0, 1, \ldots\}$, while other (non-spine) nodes, once born, remain unaffected. More specifically, under measure $\widetilde{\mathbb{Q}}_x$,

(i) The spine's spatial process $\widetilde{X}$ moves in E as a copy of (X^h, Π^h_x);
(ii) The branching events along the spine occur at an accelerated rate $\widetilde{\mu}(\mathrm{d}x) := Q(x)h(x)^2\mu(\mathrm{d}x)$. This implies that given $\mathcal{G}_\infty$, for every $t > 0$, the number of fission times n_t is a Poisson random variable with parameter $\widetilde{A}_t$, where $\widetilde{A}_t$ is a PCAF of $\widetilde{X}$ having Revuz measure $\widetilde{\mu}$. To emphasize this dependence, we also write $A^{\widetilde{\mu}}_t$ for $\widetilde{A}_t$.
(iii) At the fission time of node v in the spine, the single spine particle is replaced by A_v children, with A_v being distributed according to the size-biased distribution $\{\widehat{p}_k(\widetilde{X}_{\zeta_v-}) : k = 0, 1, \ldots\}$. Each child is selected to be the next spine node with equal probabilities.
(iv) Each of the remaining $A_v - 1$ children gives rise to independent subtrees, which are not part of the spine and evolve as independent processes determined by the measure $\mathbb{P}$ shifted to their point and time of creation.

For more details on martingale changes of measures for branching Hunt processes, see [14] and [18].

4 $L \log L$ Criteria

In this section, we prove Theorem 1.2. It follows from [5, Theorem 4.1.1] that if ν is a smooth measure of X^h, then for every $g \in \mathcal{B}^+(E)$ and $t \geq 0$,

$$\Pi^h_{\widetilde{m}}\left[\int_0^t g(X^h_s)\mathrm{d}A^\nu_s\right] = \int_0^t \int_E g(y)\nu(\mathrm{d}y)\mathrm{d}s. \tag{4.1}$$

Since t is arbitrary, the monotone class theorem implies that for any $f(s, x) = l(s)g(x)$ with $l \in \mathcal{B}^+[0, +\infty)$ and $g \in \mathcal{B}^+(E)$,

$$\Pi^h_{\widetilde{m}}\left[\int_0^t f(s, X^h_s)\mathrm{d}A^\nu_s\right] = \int_0^t \int_E f(s, y)\nu(\mathrm{d}y)\mathrm{d}s. \tag{4.2}$$

Note that for every $f \in \mathcal{B}^+([0, +\infty) \times E)$, there exists a sequence of functions $f_n(s, x) = l_n(s)g_n(x)$ with $l_n \in \mathcal{B}^+[0, +\infty)$ and $g_n \in \mathcal{B}^+(E)$ such that $f_n(s, x)$ converges increasingly to $f(s, x)$ for all s and x. Thus by the monotone convergence theorem, (4.2) holds for all $f \in \mathcal{B}^+([0, +\infty) \times E)$.

Proof of Theorem 1.2 Since M_t is a positive martingale, it suffices to prove that $\mathbb{E}_x M_\infty = h(x)$ for all $x \in E$. By [8, Theorem 5.3.3] (or [18, Lemma 3.1]), it suffices to show that if (1.12) holds, then

$$\widetilde{\mathbb{Q}}_x\left(\limsup_{\mathbb{N}\ni n\to+\infty} M_{n\sigma} < +\infty\right) = 1 \quad \text{for every } \sigma > 0 \quad \text{and } x \in E. \tag{4.3}$$

Recall that $\widetilde{\mathcal{G}}_\infty$ contains all the information about the spine. We have the following *spine decomposition* for M_t,

$$\begin{aligned}
\widetilde{\mathbb{Q}}_x\left[M_t \,\middle|\, \widetilde{\mathcal{G}}_\infty\right] &= \widetilde{\mathbb{Q}}_x\left[\mathrm{e}^{\lambda_1 t}\sum_{u\in\mathcal{Z}_t} h(X_u(t)) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right] \\
&= \mathrm{e}^{\lambda_1 t} h(\widetilde{X}_t) + \sum_{u\prec\xi_{n_t}} (A_u - 1)\mathrm{e}^{\lambda_1\zeta_u} E_{\widetilde{X}_{\zeta_u}}\left[\mathrm{e}^{\lambda_1(t-\zeta_u)}\mathbb{X}_{t-\zeta_u}(h)\right] \\
&= \mathrm{e}^{\lambda_1 t} h(\widetilde{X}_t) + \sum_{u\prec\xi_{n_t}} (A_u - 1)\mathrm{e}^{\lambda_1\zeta_u} h(\widetilde{X}_{\zeta_u}).
\end{aligned} \tag{4.4}$$

Let $\mathcal{G}_t := \sigma\{\widetilde{X}_s : s \le t\}$ and $\mathcal{G}_0$ be the trivial σ-field. It follows from the second Borel–Cantelli lemma (see, for example, [8, Section 5.4]) and the Markov property that for any $\sigma > 0$ and $M \ge 1$,

$$\begin{aligned}
\widetilde{\mathbb{Q}}_{\widetilde{m}}\left(\mathrm{e}^{\lambda_1 n\sigma} h(\widetilde{X}_{n\sigma}) > M \text{ i.o.}\right) &= \widetilde{\mathbb{Q}}_{\widetilde{m}}\left(\sum_{n=1}^{+\infty} \widetilde{\mathbb{Q}}_{\widetilde{m}}\left(\mathrm{e}^{\lambda_1 n\sigma} h(\widetilde{X}_{n\sigma}) > M \,\middle|\, \mathcal{G}_{(n-1)\sigma}\right) = +\infty\right) \\
&= \Pi^h_{\widetilde{m}}\left(\sum_{n=1}^{+\infty} \Pi^h_{\widetilde{X}_{(n-1)\sigma}}\left(\mathrm{e}^{\lambda_1 n\sigma} h(\widetilde{X}_\sigma) > M\right) = +\infty\right).
\end{aligned} \tag{4.5}$$

Recall that $\widetilde{m}$ is the invariant probability measure of $(\widetilde{X}, \Pi^h_x)$. Thus by Fubini's theorem and Markov property, we have

$$\begin{aligned}
\Pi^h_{\widetilde{m}}\left[\sum_{n=1}^{+\infty} \Pi^h_{\widetilde{X}_{(n-1)\sigma}}\left(\mathrm{e}^{\lambda_1 n\sigma} h(\widetilde{X}_\sigma) > M\right)\right] &= \sum_{n=1}^{+\infty} \Pi^h_{\widetilde{m}}\left(\mathrm{e}^{\lambda_1 n\sigma} h(\widetilde{X}_{n\sigma}) > M\right) \\
&= \sum_{n=1}^{+\infty} \int_E 1_{\{\mathrm{e}^{\lambda_1 n\sigma} h(y) > M\}} \widetilde{m}(\mathrm{d}y) \\
&= \sum_{n=1}^{+\infty} \widetilde{m}\left(\frac{\log^+ h(y) - \log M}{-\lambda_1} > n\sigma\right) \\
&\le (-\lambda_1)^{-1} \int_E \log^+ h(y)\widetilde{m}(\mathrm{d}y) < \infty.
\end{aligned}$$

Therefore by (4.5) we have $\widetilde{\mathbb{Q}}_{\widetilde{m}}\left(\mathrm{e}^{\lambda_1 n\sigma}h(\widetilde{X}_{n\sigma}) > M \text{ i.o.}\right) = 0$. Consequently,

$$\widetilde{\mathbb{Q}}_m\left(\limsup_{\mathbb{N}\ni n\to+\infty} \mathrm{e}^{\lambda_1 n\sigma}h(\widetilde{X}_{n\sigma}) < +\infty\right) = 1.$$

It is easy to check that the function $x \mapsto \widetilde{\mathbb{Q}}_x\left(\limsup_{\mathbb{N}\ni n\to+\infty} \mathrm{e}^{\lambda_1 n\sigma}h(\widetilde{X}_{n\sigma}) < +\infty\right)$ is an invariant function for $(\widetilde{X}, \Pi^h_{\widetilde{m}})$. Recall that $\widetilde{X}$ has a transition density function with respect to $\widetilde{m}$. By [5, Theorem A.2.17],

$$\widetilde{\mathbb{Q}}_x\left(\limsup_{\mathbb{N}\ni n\to+\infty} \mathrm{e}^{\lambda_1 n\sigma}h(\widetilde{X}_{n\sigma}) < +\infty\right) = 1 \quad \text{for every } x \in E.$$

Suppose $\varepsilon \in (0, -\lambda_1)$. For simplicity, we use ζ_i and A_i to denote respectively the fission time and offspring number of the ith spine node.

$$\begin{aligned}\sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) &= \sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) 1_{\{A_i h(\widetilde{X}_{\zeta_i})\le \mathrm{e}^{\varepsilon\zeta_i}\}} \\ &\quad + \sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) 1_{\{A_i h(\widetilde{X}_{\zeta_i}) > \mathrm{e}^{\varepsilon\zeta_i}\}} \\ &=: I + II. \end{aligned} \tag{4.6}$$

Recall that $\mathcal{G}_\infty$ contains all the information of spine's motion $\{\widetilde{X}_t : t \ge 0\}$ in E. For any set $B \in \mathcal{B}[0, +\infty) \times \mathcal{B}(\mathbb{Z}_+)$, define $N(B) := \sharp\{i \ge 1 : (\zeta_i, A_i) \in B\}$. Then conditioned on $\mathcal{G}_\infty$, N is a Poisson random measure on $[0, +\infty) \times \mathbb{Z}_+$ with intensity $dA^{\widetilde{\mu}}_s \sum_{k\in\mathbb{Z}_+} \hat{p}_k(\widetilde{X}_s)\delta_k(\mathrm{d}y)$, where $\widetilde{\mu}(\mathrm{d}x) = Q(x)h(x)^2\mu(\mathrm{d}x)$. We have

$$\widetilde{\mathbb{Q}}_x\left[\sum_{i=1}^{+\infty} 1_{\{A_i h(\widetilde{X}_{\zeta_i}) > \mathrm{e}^{\varepsilon\zeta_i}\}}\right] = \Pi^h_x\left[\int_0^{+\infty} \sum_{k=0}^{+\infty} \hat{p}_k(\xi_s) 1_{\{kh(\widetilde{X}_s) > \mathrm{e}^{\varepsilon s}\}} dA^{\widetilde{\mu}}_s\right].$$

Note that by (4.2) and our assumption (1.12),

$$\begin{aligned}\widetilde{\mathbb{Q}}_{\widetilde{m}}\left[\sum_{i=1}^{+\infty} 1_{\{A_i h(\widetilde{X}_{\zeta_i}) > \mathrm{e}^{\varepsilon\zeta_i}\}}\right] &= \Pi^h_{\widetilde{m}}\left[\int_0^{+\infty} \sum_{k=0}^{+\infty} \hat{p}_k(\xi_s) 1_{\{kh(\widetilde{X}_s) > \mathrm{e}^{\varepsilon s}\}} dA^{\widetilde{\mu}}_s\right] \\ &= \int_0^{+\infty} \int_E \sum_{k=0}^{+\infty} \hat{p}_k(y) 1_{\{\log^+(kh(y)) > \varepsilon s\}} \widetilde{\mu}(\mathrm{d}y)\mathrm{d}s \\ &= \varepsilon^{-1} \int_E \sum_{k=0}^{+\infty} k p_k(y) \log^+(kh(y)) h(y)^2 \mu(\mathrm{d}y) < +\infty. \end{aligned}$$

Thus

$$\widetilde{\mathbb{Q}}_x\left(\sum_{i=1}^{+\infty} 1_{\{A_i h(\widetilde{X}_{\zeta_i})>\mathrm{e}^{\varepsilon\zeta_i}\}} < +\infty\right) = 1 \quad \text{for } \widetilde{m}\text{-a.e. } x \in E.$$

This implies that II is the sum of a finite many terms and so $\widetilde{\mathbb{Q}}_x(II < +\infty) = 1$ for $\widetilde{m}$-a.e. $x \in E$. On the other hand, since $Q(x)$ is bounded on E, we have

$$\begin{aligned}
\widetilde{\mathbb{Q}}_{\widetilde{m}}[I] &= \Pi^h_{\widetilde{m}}\left[\int_0^{+\infty} \mathrm{e}^{\lambda_1 s}\sum_{k=0}^{+\infty} \hat{p}_k(\widetilde{X}_s)kh(\widetilde{X}_s)1_{\{kh(\xi_s)\le \mathrm{e}^{\varepsilon s}\}}dA^{\widetilde{\mu}}_s\right]\\
&= \int_0^{+\infty}\int_E \mathrm{e}^{\lambda_1 s}\sum_{k=0}^{+\infty} \hat{p}_k(y)kh(y)1_{\{kh(y)\le \mathrm{e}^{\varepsilon s}\}}\widetilde{\mu}(\mathrm{d}y)\mathrm{d}s\\
&\le \int_0^{+\infty}\int_E \mathrm{e}^{(\lambda_1+\varepsilon)s}\sum_{k=0}^{+\infty} \hat{p}_k(y)Q(y)h(y)^2\mu(\mathrm{d}y)\mathrm{d}s\\
&\le \|Q\|_\infty \int_E h(y)^2\mu(\mathrm{d}y) < +\infty.
\end{aligned}$$

Thus we have $\widetilde{\mathbb{Q}}_x(I < +\infty) = 1$ for $\widetilde{m}$-a.e. $x \in E$. Now we have proved that

$$\widetilde{\mathbb{Q}}_x\left(\sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) < +\infty\right) = 1 \quad \text{for } \widetilde{m}\text{-a.e. } x \in E.$$

It is easy to check that the function $x \mapsto \widetilde{\mathbb{Q}}_x\left(\sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) < +\infty\right)$ is an invariant function. Thus by [5, TheoremA.2.17], we get $\widetilde{\mathbb{Q}}_x\left(\sum_{i=1}^{+\infty} \mathrm{e}^{\lambda_1\zeta_i} A_i h(\widetilde{X}_{\zeta_i}) < +\infty\right) = 1$ for every $x \in E$. By (4.4) we have

$$\widetilde{\mathbb{Q}}_x\left(\limsup_{\mathbb{N}\ni n\to+\infty} \widetilde{\mathbb{Q}}_x\left[M_{n\sigma}\,\middle|\,\widetilde{\mathcal{G}}_\infty\right] < +\infty\right) = 1 \quad \text{for every } x \in E.$$

By Fatou's lemma, we get $\widetilde{\mathbb{Q}}_x\left(\liminf_{\mathbb{N}\ni n\to+\infty} M_{n\sigma} < +\infty\right) = 1$. Note that $M^{-1}_{n\sigma}$ is a positive $\widetilde{\mathbb{Q}}_x$-martingale with respect to $\mathcal{F}_{n\sigma}$, so it converges almost surely as $n \to \infty$. It follows then

$$\widetilde{\mathbb{Q}}_x\left(\limsup_{\mathbb{N}\ni n\to+\infty} M_{n\sigma} = \liminf_{\mathbb{N}\ni n\to+\infty} M_{n\sigma} < +\infty\right) = 1.$$

□

5 Weak Law of Large Numbers

In this section, we present a proof for Theorem 1.3.

Lemma 5.1 *If Assumption 1 and (1.12) hold, then for every $t \geq 0$ and $\phi \in \mathcal{B}_b^+(E)$,*

$$\int_E \mathbb{E}_x \left[\mathbb{X}_t(\phi h) \log^+ \mathbb{X}_t(\phi h)\right] h(x) m(dx) < +\infty. \tag{5.1}$$

Proof First we note that for every $x \in E$ and $\phi \in \mathcal{B}_b^+(E)$,

$$\begin{aligned}\mathbb{E}_x \left[\mathbb{X}_t(\phi h) \log^+ \mathbb{X}_t(\phi h)\right] &\leq \mathrm{e}^{-\lambda_1 t} \|\phi\|_\infty h(x) \mathbb{E}_x \left[\mathrm{e}^{\lambda_1 t} h(x)^{-1} \mathbb{X}_t(h) \log^+ \mathbb{X}_t(\phi h)\right] \\ &= \mathrm{e}^{-\lambda_1 t} \|\phi\|_\infty h(x) \widetilde{\mathbb{Q}}_x \left[\log^+ \mathbb{X}_t(\phi h)\right]. \end{aligned} \tag{5.2}$$

Recall that $\widetilde{\mathcal{G}}_\infty$ contains all the information about the spine. Since for any $a, b \geq 0$

$$\log^+(a+b) \leq \log^+ a + \log^+ b + \log 2, \tag{5.3}$$

we have by Jensen's inequality

$$\begin{aligned}\widetilde{\mathbb{Q}}_x \left[\log^+ \mathbb{X}_t(\phi h)\right] &= \widetilde{\mathbb{Q}}_x \left[\widetilde{\mathbb{Q}}_x \left[\log^+ \mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right]\right] \\ &\leq \widetilde{\mathbb{Q}}_x \left[\log \widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \vee 1 \,\middle|\, \widetilde{\mathcal{G}}_\infty\right]\right] \\ &\leq \widetilde{\mathbb{Q}}_x \left[\log^+ \left(\widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right] + 1\right)\right] \\ &\leq \widetilde{\mathbb{Q}}_x \left[\log^+ \widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right]\right] + \log 2. \end{aligned} \tag{5.4}$$

By (5.2) and (5.4), it suffices to show that

$$\int_E \widetilde{\mathbb{Q}}_x \left[\log^+ \widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right]\right] h(x)^2 m(\mathrm{d}x) < +\infty. \tag{5.5}$$

We get the spine decomposition for $\widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right]$ as follows:

$$\begin{aligned}\widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right] &= (\phi h)(\widetilde{X}_t) + \sum_{u \prec \xi_t} (A_u - 1) \mathbb{E}_{\widetilde{X}_{\zeta_u}} \left[\mathbb{X}_{t-\zeta_u}(\phi h)\right] \\ &\leq (\phi h)(\widetilde{X}_t) + \sum_{u \prec \xi_t} A_u \|\phi\|_\infty \mathrm{e}^{-\lambda_1 (t-\zeta_u)} h(\widetilde{X}_{\zeta_u}). \end{aligned} \tag{5.6}$$

Note that $\log^+(ab) \leq \log^+ a + \log^+ b$ for any $a, b > 0$. Using this and an analogy of (5.3) as well as the assumption that $\lambda_1 < 0$, we have

$$\begin{aligned}&\log^+ \widetilde{\mathbb{Q}}_x \left[\mathbb{X}_t(\phi h) \,\middle|\, \widetilde{\mathcal{G}}_\infty\right] \\ &\leq \log^+ \left(\phi h(\widetilde{X}_t) + \sum_{u \prec \xi_t} A_u \|\phi\|_\infty \mathrm{e}^{-\lambda_1 (t-\zeta_u)} h(\widetilde{X}_{\zeta_u})\right) \\ &\leq \log^+ (\phi h(\widetilde{X}_t)) + \sum_{u \prec \xi_t} \log^+ \left(A_u \|\phi\|_\infty \mathrm{e}^{-\lambda_1 (t-\zeta_u)} h(\widetilde{X}_{\zeta_u})\right) + \log^+ n_t \end{aligned}$$

$$\le \log^+(\phi h(\widetilde{X}_t)) + \sum_{u \prec \xi_t} \left(\log^+ \|\phi\|_\infty - \lambda_1(t-\zeta_u) + \log^+\left(A_u h(\widetilde{X}_{\zeta_u})\right)\right) + n_t$$

$$\le \log^+(\phi h(\widetilde{X}_t)) + \sum_{u \prec \xi_t} \log^+\left(A_u h(\widetilde{X}_{\zeta_u})\right) + (1+\log^+\|\phi\|_\infty - \lambda_1 t)n_t. \quad (5.7)$$

By (1.12) and using the fact that $\widetilde{m}(\mathrm{d}y)$ is an invariant distribution,

$$\int_E \widetilde{\mathbb{Q}}_x\left[\log^+(\phi h(\widetilde{X}_t))\right] h(x)^2 m(\mathrm{d}x) = \int_E \log^+(\phi h)(x)\widetilde{m}(\mathrm{d}x) < +\infty.$$

Hence (5.5) is implied by

$$\int_E \widetilde{\mathbb{Q}}_x\left[\sum_{u\prec\xi_t} \log^+(A_u h(\widetilde{X}_{\zeta_u})) + n_t\right] h(x)^2 m(\mathrm{d}x) < +\infty. \quad (5.8)$$

Recall that conditioned on $\mathcal{G}_\infty$, $N(\cdot) = \sharp\{i \ge 1 : (\zeta_i, A_i) \in \cdot\}$ is a Poisson random measure on $[0,+\infty) \times \mathbb{Z}_+$ with intensity $dA_s^{\widetilde{\mu}} \sum_{k\in\mathbb{Z}_+} \widehat{p}_k(\widetilde{X}_s)\delta_k(\mathrm{d}y)$. We have

$$\begin{aligned}
&\int_E \widetilde{\mathbb{Q}}_x\left[\sum_{u\prec\xi_t} \log^+(A_u h(\widetilde{X}_{\zeta_u})) + n_t\right] h(x)^2 m(\mathrm{d}x)\\
&= \int_E \widetilde{\mathbb{Q}}_x\left[\widetilde{\mathbb{Q}}_x\left[\sum_{u\prec\xi_t} \log^+(A_u h(\xi_{\zeta_u})) + n_t \,\Big|\, \mathcal{G}_\infty\right]\right] h(x)^2 m(\mathrm{d}x)\\
&= \int_E \Pi_x^h\left[\int_0^t \sum_{k=0}^{+\infty} \widehat{p}_k(\widetilde{X}_s)\left(\log^+(kh(\widetilde{X}_s)) + 1\right)\mathrm{d}A_s^{\widetilde{\mu}}\right]\widetilde{m}(\mathrm{d}x)\\
&= \int_0^t \mathrm{d}s \int_E \sum_{k=0}^{+\infty} \widehat{p}_k(y)\left(\log^+(kh(y)) + 1\right) Q(y)h(y)^2\mu(\mathrm{d}y)\\
&\le t\left(\int_E \sum_{k=0}^{+\infty} kp_k(y)\log^+(kh(y))h(y)^2\mu(\mathrm{d}y) + \|Q\|_\infty \int_E h(y)^2\mu(\mathrm{d}y)\right).
\end{aligned}$$

Immediately the last term is finite by (1.12). Hence we complete the proof. □

Lemma 5.2 *If Assumptions 1–2 and* (1.12) *hold, then for any* $s,\sigma > 0$, $m \in \mathbb{N}$, *and any* $x \in E$,

$$\lim_{t\to+\infty} e^{\lambda_1(s+t)}\mathbb{X}_{s+t}(\phi h) - \mathbb{E}_x\left[e^{\lambda_1(s+t)}\mathbb{X}_{s+t}(\phi h)\,\Big|\,\mathcal{F}_t\right] = 0 \quad \text{in } L^1(\mathbb{P}_x), \quad (5.9)$$

$$\lim_{\mathbb{N}\ni n\to+\infty} e^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(\phi h) - \mathbb{E}_x\left[e^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(\phi h)\,\Big|\,\mathcal{F}_{n\sigma}\right] = 0 \quad \mathbb{P}_x\text{-a.s.} \quad (5.10)$$

for every $\phi \in \mathcal{B}_b^+(E)$.

Proof For any particle $u \in \mathcal{Z}_s$, let $\{\mathbb{X}^{u,s}_t : t \geq s\}$ denote the branching Markov process initiated by u at time s. It is known that conditioned on $\mathcal{F}_s$, $\mathbb{X}^{u,s}$ and $\mathbb{X}^{v,s}$ are independent for every $u, v \in \mathcal{Z}_s$ with $u \neq v$. For every $s, t \geq 0$, define

$$S_{s,t} := \mathrm{e}^{\lambda_1 t}\mathbb{X}_{s+t}(\phi h) = \mathrm{e}^{\lambda_1 t}\sum_{u\in\mathcal{Z}_t}\mathbb{X}^{u,t}_{s+t}(\phi h),$$

and

$$\widetilde{S}_{s,t} := \mathrm{e}^{\lambda_1 t}\sum_{u\in\mathcal{Z}_t}\mathbb{X}^{u,t}_{s+t}(\phi h)1_{\{\mathbb{X}^{u,t}_{s+t}(\phi h)\leq \mathrm{e}^{-\lambda_1 t}\}}.$$

Obviously $S_{s,t} \geq \widetilde{S}_{s,t}$.

First by the conditional independence and the Markov property, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\left(\widetilde{S}_{m\sigma,n\sigma} - \mathbb{E}_x[\widetilde{S}_{m\sigma,n\sigma} \,\middle|\, \mathcal{F}_{n\sigma}]\right)^2\right]\\
&\quad= \mathbb{E}_x\left[\mathrm{Var}\left[\widetilde{S}_{m\sigma,n\sigma} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right]\\
&\quad= \mathrm{e}^{2\lambda_1 n\sigma}\mathbb{E}_x\left[\sum_{u\in\mathcal{Z}_{n\sigma}}\mathrm{Var}\left[\mathbb{X}^{u,n\sigma}_{(n+m)\sigma}(\phi h)1_{\{\mathbb{X}^{u,n\sigma}_{(n+m)\sigma}(\phi h)\leq \mathrm{e}^{-\lambda_1 n\sigma}\}} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right]\\
&\quad\leq \mathrm{e}^{2\lambda_1 n\sigma}\mathbb{E}_x\left[\sum_{u\in\mathcal{Z}_{n\sigma}}\mathbb{E}_x\left[\left(\mathbb{X}^{u,n\sigma}_{(n+m)\sigma}(\phi h)\right)^2 1_{\{\mathbb{X}^{u,n\sigma}_{(n+m)\sigma}(\phi h)\leq \mathrm{e}^{-\lambda_1 n\sigma}\}} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right]\\
&\quad= \mathrm{e}^{2\lambda_1 n\sigma}\mathbb{E}_x\left[\sum_{u\in\mathcal{Z}_{n\sigma}}\mathbb{E}_{X_u(n\sigma)}\left[(\mathbb{X}_{m\sigma}(\phi h))^2\, 1_{\{\mathbb{X}_{m\sigma}(\phi h)\leq \mathrm{e}^{-\lambda_1 n\sigma}\}}\right]\right].
\end{aligned} \tag{5.11}$$

Let $g_{m\sigma,n\sigma}(y) := \mathbb{E}_y\left[(\mathbb{X}_{m\sigma}(\phi h))^2\, 1_{\{\mathbb{X}_{m\sigma}(\phi h)\leq \mathrm{e}^{-\lambda_1 n\sigma}\}}\right]$. Immediately we have

$$g_{m\sigma,n\sigma}(y) \leq \mathrm{e}^{-\lambda_1 n\sigma}\mathbb{E}_y[\mathbb{X}_{m\sigma}(\phi h)] \leq \mathrm{e}^{-\lambda_1(n+m)\sigma}\|\phi\|_\infty h(y).$$

Thus $g_{m\sigma,n\sigma} \in L^2(E, m)$, and

$$\|g_{m\sigma,n\sigma}\|_{L^2(E,m)} \leq \mathrm{e}^{-\lambda_1(n+m)\sigma}\|\phi\|_\infty. \tag{5.12}$$

Using (2.8), we continue the estimates in (5.11) to get that for $n \in \mathbb{N}$ with $n\sigma > 1$,

$$\begin{aligned}
&\mathbb{E}_x\left[\left(\widetilde{S}_{m\sigma,n\sigma} - \mathbb{E}_x\left[\widetilde{S}_{m\sigma,n\sigma} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right)^2\right]\\
&\quad\leq \mathrm{e}^{2\lambda_1 n\sigma}\mathbb{E}_x\left[\mathbb{X}_{n\sigma}(g_{m\sigma,n\sigma})\right]\\
&\quad= \mathrm{e}^{\lambda_1 n\sigma}h(x)P^h_{n\sigma}(g_{m\sigma,n\sigma}/h)(x)
\end{aligned}$$

$$\le e^{\lambda_1 n\sigma} h(x) \left(\langle g_{m\sigma,n\sigma}, h\rangle + e^{\lambda_h/2} \widetilde{a}_1(x)^{1/2} e^{-\lambda_h n\sigma} \| g_{m\sigma,n\sigma} \|_{L^2(E,m)} \right)$$
$$\le e^{\lambda_1 n\sigma} h(x) \langle g_{m\sigma,n\sigma}, h\rangle + e^{\lambda_h/2} h(x) \widetilde{a}_1(x)^{1/2} e^{-\lambda_h n\sigma - \lambda_1 m\sigma} \|\phi\|_\infty. \tag{5.13}$$

Note that

$$\begin{aligned}
&\sum_{n=1}^{+\infty} e^{\lambda_1 n\sigma} \langle g_{m\sigma,n\sigma}, h\rangle \\
&= \sum_{n=1}^{+\infty} e^{\lambda_1 n\sigma} \int_E \mathbb{E}_y \left[(\mathbb{X}_{m\sigma}(\phi h))^2 \, 1_{\{\mathbb{X}_{m\sigma}(\phi h) \le e^{-\lambda_1 n\sigma}\}} \right] h(y) m(dy) \\
&\le \int_E h(y) m(dy) \int_1^{+\infty} e^{\lambda_1 (s-1)\sigma} \mathbb{E}_y \left[(\mathbb{X}_{m\sigma}(\phi h))^2 \, 1_{\{\mathbb{X}_{m\sigma}(\phi h) \le e^{-\lambda_1 s\sigma}\}} \right] ds \\
&= \int_E h(y) m(dy) \int_1^{+\infty} \frac{1}{-\lambda_1 \sigma x^2} \mathbb{E}_y \left[(\mathbb{X}_{m\sigma}(\phi h))^2 \, 1_{\{\mathbb{X}_{m\sigma}(\phi h) \le x e^{-\lambda_1 \sigma}\}} \right] dx \\
&= \int_E h(y) m(dy) \int_0^{+\infty} z^2 \mathbb{P}_y \left(\mathbb{X}_{m\sigma}(\phi h) \in dz \right) \int_{1 \vee z e^{\lambda_1 \sigma}}^{+\infty} \frac{1}{-\lambda_1 \sigma x^2} dx \\
&\le \frac{1}{-\lambda_1 \sigma} e^{-\lambda_1 \sigma} \int_E h(y) m(dy) \int_0^{+\infty} z \mathbb{P}_y \left(\mathbb{X}_{m\sigma}(\phi h) \in dz \right) \\
&= \frac{1}{-\lambda_1 \sigma} e^{-\lambda_1 \sigma} \int_E h(y) \mathbb{E}_y \left[\mathbb{X}_{m\sigma}(\phi h) \right] m(dy) \\
&\le \frac{1}{-\lambda_1 \sigma} e^{-\lambda_1 \sigma - \lambda_1 m\sigma} \|\phi\|_\infty \int_E h(y)^2 m(dy) < +\infty,
\end{aligned} \tag{5.14}$$

where in the second equality above we used the change of variables $x = e^{-\lambda_1 (s-1)\sigma}$. It follows from (5.13) and (5.14) that $\sum_{n=1}^{+\infty} \mathbb{E}_x \left[\left(\widetilde{S}_{m\sigma,n\sigma} - \mathbb{E}_x [\widetilde{S}_{m\sigma,n\sigma} \mid \mathcal{F}_{n\sigma}] \right)^2 \right] < +\infty$. Thus by the Borel–Cantelli lemma,

$$\lim_{n\to+\infty} \widetilde{S}_{m\sigma,n\sigma} - \mathbb{E}_x \left[\widetilde{S}_{m\sigma,n\sigma} \mid \mathcal{F}_{n\sigma} \right] = 0 \quad \mathbb{P}_x\text{-a.s.} \tag{5.15}$$

Note that for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{E}_x \left[S_{m\sigma,n\sigma} - \widetilde{S}_{m\sigma,n\sigma} \right] &= \mathbb{E}_x \left[\mathbb{E}_x [S_{m\sigma,n\sigma} \mid \mathcal{F}_{n\sigma}] - \mathbb{E}_x [\widetilde{S}_{m\sigma,n\sigma} \mid \mathcal{F}_{n\sigma}] \right] \\
&= e^{\lambda_1 n\sigma} \mathbb{E}_x \left[\sum_{u \in \mathcal{Z}_{n\sigma}} \mathbb{E}_{X_u(n\sigma)} \left[\mathbb{X}_{m\sigma}(\phi h) 1_{\{\mathbb{X}_{m\sigma}(\phi h) > e^{-\lambda_1 n\sigma}\}} \right] \right].
\end{aligned} \tag{5.16}$$

Let $f_{m\sigma,n\sigma}(y) := \mathbb{E}_y \left[\mathbb{X}_{m\sigma}(\phi h) 1_{\{\mathbb{X}_{m\sigma}(\phi h) > e^{-\lambda_1 n\sigma}\}} \right]$. Obviously,

$$f_{m\sigma,n\sigma}(y) \le \|\phi\|_\infty \mathbb{E}_y [\mathbb{X}_{m\sigma}(h)] \le e^{-\lambda_1 m\sigma} \|\phi\|_\infty h(y) \tag{5.17}$$

and so $f_{m\sigma,n\sigma} \in L^2(E,m)$. Then by (2.8), we have for $n \in \mathbb{N}$ with $n\sigma > 1$,

$$\begin{aligned}\text{RHS of (5.16)} &= \mathrm{e}^{\lambda_1 n\sigma}\mathbb{E}_x[\mathbb{X}_{n\sigma}(f_{m\sigma,n\sigma})] = h(x)P^h_{n\sigma}(f_{m\sigma,n\sigma}/h)(x)\\ &\le h(x)\langle f_{m\sigma,n\sigma}, h\rangle + \mathrm{e}^{\lambda_h/2}\widetilde{a}_1(x)^{1/2}h(x)\mathrm{e}^{-\lambda_h n\sigma}\|f_{m\sigma,n\sigma}\|_{L^2(E,m)}\\ &\le h(x)\langle f_{m\sigma,n\sigma}, h\rangle + \mathrm{e}^{\lambda_h/2}\widetilde{a}_1(x)^{1/2}h(x)\mathrm{e}^{-\lambda_h n\sigma-\lambda_1 m\sigma}\|\phi\|_\infty,\end{aligned} \tag{5.18}$$

where in the last inequality we used (5.17). Note that

$$\begin{aligned}\sum_{n=1}^{+\infty}\langle f_{m\sigma,n\sigma}, h\rangle &= \sum_{n=1}^{+\infty}\int_E \mathbb{E}_y\left[\mathbb{X}_{m\sigma}(\phi h)1_{\{\mathbb{X}_{m\sigma}(\phi h)>\mathrm{e}^{-\lambda_1 n\sigma}\}}\right]h(y)m(\mathrm{d}y)\\ &\le \int_0^{+\infty}\mathrm{d}s\int_E \mathbb{E}_y\left[\mathbb{X}_{m\sigma}(\phi h)1_{\{\mathbb{X}_{m\sigma}(\phi h)>\mathrm{e}^{-\lambda_1 \sigma s}\}}\right]h(y)m(\mathrm{d}y)\\ &= \int_E h(y)m(\mathrm{d}y)\int_0^{+\infty}\mathrm{d}s\int_{\mathrm{e}^{-\lambda_1\sigma s}}^{+\infty} z\mathbb{P}_y\left(\mathbb{X}_{m\sigma}(\phi h)\in \mathrm{d}z\right)\\ &= \int_E h(y)m(\mathrm{d}y)\int_1^{+\infty}\frac{1}{-\lambda_1\sigma t}dt\int_t^{+\infty} z\mathbb{P}_y\left(\mathbb{X}_{m\sigma}(\phi h)\in \mathrm{d}z\right)\\ &= \frac{1}{-\lambda_1\sigma}\int_E h(y)m(\mathrm{d}y)\int_1^{+\infty} z\mathbb{P}_y\left(\mathbb{X}_{m\sigma}(\phi h)\in \mathrm{d}z\right)\int_1^z\frac{1}{t}dt\\ &= \frac{1}{-\lambda_1\sigma}\int_E h(y)m(\mathrm{d}y)\int_1^{+\infty} z\log^+ z\mathbb{P}_y\left(\mathbb{X}_{m\sigma}(\phi h)\in \mathrm{d}z\right)\\ &= \frac{1}{-\lambda_1\sigma}\int_E \mathbb{E}_y\left[\mathbb{X}_{m\sigma}(\phi h)\log^+\mathbb{X}_{m\sigma}(\phi h)\right]h(y)m(\mathrm{d}y).\end{aligned} \tag{5.19}$$

The last term is finite by Lemma 5.1. Thus we get

$$\sum_{n=1}^{+\infty}\mathbb{E}_x[S_{m\sigma,n\sigma} - \widetilde{S}_{m\sigma,n\sigma}] = \sum_{n=1}^{+\infty}\mathbb{E}_x\left[\mathbb{E}_x[S_{m\sigma,n\sigma}\,\middle|\,\mathcal{F}_{n\sigma}] - \mathbb{E}_x[\widetilde{S}_{m\sigma,n\sigma}\,\middle|\,\mathcal{F}_{n\sigma}]\right] < +\infty.$$

Recall that $S_{m\sigma,n\sigma} \ge \widetilde{S}_{m\sigma,n\sigma}$ for every $n \ge 0$. Again by the Borel–Cantelli lemma, we have

$$\lim_{n\to+\infty} S_{m\sigma,n\sigma} - \widetilde{S}_{m\sigma,n\sigma} = \lim_{n\to+\infty}\mathbb{E}_x[S_{m\sigma,n\sigma}\,|\,\mathcal{F}_{n\sigma}] - \mathbb{E}_x[\widetilde{S}_{m\sigma,n\sigma}\,|\,\mathcal{F}_{n\sigma}] = 0 \quad \mathbb{P}_x\text{-a.s.} \tag{5.20}$$

Now (5.10) follows from (5.15) and (5.20).

Substituting $n\sigma$ by t and $m\sigma$ by s in (5.13), we get for any $t > 1$,

$$\begin{aligned}&\mathbb{E}_x\left[\left(\widetilde{S}_{s,t} - \mathbb{E}_x\left[\widetilde{S}_{s,t}\,\middle|\,\mathcal{F}_t\right]\right)^2\right]\\ &\quad\le \mathrm{e}^{\lambda_1 t}h(x)\langle g_{s,t}, h\rangle + \mathrm{e}^{\lambda_h/2}h(x)\widetilde{a}_1(x)^{1/2}\mathrm{e}^{-\lambda_h t-\lambda_1 s}\|\phi\|_\infty.\end{aligned}$$

Using a similar calculation as in (5.14), we get for $s>0$, $\int_1^{+\infty} \mathrm{e}^{\lambda_1 t}\langle g_{s,t}, h\rangle dt < +\infty$. Consequently $\lim_{t\to+\infty} \mathrm{e}^{\lambda_1 t}\langle g_{s,t}, h\rangle = 0$. Hence $\mathbb{E}_x\left[\left(\widetilde{S}_{s,t} - \mathbb{E}_x\left[\widetilde{S}_{s,t} \,\middle|\, \mathcal{F}_t\right]\right)^2\right] \to 0$ as $t\to+\infty$, or equivalently

$$\lim_{t\to+\infty} \left(\widetilde{S}_{s,t} - \mathbb{E}_x[\widetilde{S}_{s,t} \,|\, \mathcal{F}_t]\right) = 0 \quad \text{in } L^2(\mathbb{P}_x). \tag{5.21}$$

Substituting $n\sigma$ by t and $m\sigma$ by s in (5.18), we get for any $t>1$,

$$\begin{aligned}\mathbb{E}_x\left[S_{s,t} - \widetilde{S}_{s,t}\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[S_{s,t} \,|\, \mathcal{F}_t\right] - \mathbb{E}_x\left[\widetilde{S}_{s,t} \,|\, \mathcal{F}_t\right]\right]\\ &\le h(x)\langle f_{s,t}, h\rangle + \mathrm{e}^{\lambda_h/2}\widetilde{a}_1(x)^{1/2}h(x)\mathrm{e}^{-\lambda_h t-\lambda_1 s}\|\phi\|_\infty.\end{aligned}$$

By similar calculation as in (5.19), we get $\int_0^{+\infty}\langle f_{s,t}, h\rangle dt < +\infty$ for all $s>0$. Hence $\lim_{t\to+\infty}\langle f_{s,t}, h\rangle = 0$. Thus we have

$$\mathbb{E}_x\left[S_{s,t} - \widetilde{S}_{s,t}\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[S_{s,t} \,|\, \mathcal{F}_t\right] - \mathbb{E}_x\left[\widetilde{S}_{s,t} \,|\, \mathcal{F}_t\right]\right] \to 0, \text{ as } t\to+\infty. \tag{5.22}$$

Therefore (5.9) follows from (5.21) and (5.22). □

Lemma 5.3 *Under the conditions of Theorem 1.3, we have*

$$\lim_{s\to+\infty}\lim_{t\to+\infty} e^{\lambda_1(s+t)}\mathbb{E}_x\left[\mathbb{X}_{s+t}(\phi h) \,\middle|\, \mathcal{F}_t\right] = M_\infty\langle\phi h, h\rangle \quad \text{in } L^1(\mathbb{P}_x), \tag{5.23}$$

for every $\phi\in\mathcal{B}_b^+(E)$ and every $x\in E$.

Proof Recall that M_t converges to M_∞ in $L^1(\mathbb{P}_x)$ by Theorem 1.2. It suffices to prove that

$$\lim_{s\to+\infty}\lim_{t\to+\infty}\mathbb{E}_x\left[\left|\mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x[\mathbb{X}_{s+t}(\phi h) \,|\, \mathcal{F}_t] - M_t\langle\phi h, h\rangle\right|\right] = 0. \tag{5.24}$$

Note that by the Markov property,

$$\begin{aligned}\mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x\left[\mathbb{X}_{s+t}(\phi h) \,|\, \mathcal{F}_t\right] &= \mathrm{e}^{\lambda_1(s+t)}\sum_{u\in\mathcal{Z}_t}\mathbb{E}_{X_u(t)}\left[\mathbb{X}_s(\phi h)\right]\\ &= \mathrm{e}^{\lambda_1 t}\mathbb{X}_t\left(hP_s^h(\phi)\right).\end{aligned} \tag{5.25}$$

Thus we have for any $s,t>t_0$,

$$\begin{aligned}&\mathbb{E}_x\left[\left|\mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x[\mathbb{X}_{s+t}(\phi h) \,|\, \mathcal{F}_t] - M_t\langle\phi h, h\rangle\right|\right]\\ &\le \mathrm{e}^{\lambda_1 t}\mathbb{E}_x\left[\left|\mathbb{X}_t\left(hP_s^h\phi - h\langle\phi h, h\rangle\right)\right|\right]\\ &\le \mathrm{e}^{\lambda_1 t}\mathbb{E}_x\left[\mathbb{X}_t\left(\left|hP_s^h\phi - h\langle\phi h, h\rangle\right|\right)\right]\\ &= h(x)P_t^h\left(\left|P_s^h\phi - \langle\phi h, h\rangle\right|\right)(x).\end{aligned}$$

Using (2.8), we continue the estimation above to get:

$$\begin{aligned}
&\mathbb{E}_x\left[\left|\mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x[\mathbb{X}_{s+t}(\phi h)\,|\,\mathcal{F}_t] - M_t\langle \phi h, h\rangle\right|\right]\\
&\quad\le \mathrm{e}^{-\lambda_h(s-t_0/2)}\|\phi\|_\infty h(x) P^h_t(\widetilde{a}^{1/2}_{t_0})(x)\\
&\quad\le \mathrm{e}^{-\lambda_h(s-t_0/2)}\|\phi\|_\infty h(x)\left[\int_E \widetilde{a}_{t_0}(y)^{1/2}\widetilde{m}(\mathrm{d}y)\right.\\
&\qquad\left. + \widetilde{a}_{t_0}(x)^{1/2}\mathrm{e}^{-\lambda_h(t-t_0/2)}\left(\int_E \widetilde{a}_{t_0}(y)\widetilde{m}(\mathrm{d}y)\right)^{1/2}\right]\\
&\quad\to 0 \quad \text{as } t\to+\infty, \text{ and then } s\to+\infty.
\end{aligned}$$

□

Proof of Theorem 1.3 Let $\phi = f/h$, then $\phi\in\mathcal{B}^+_b(E)$. Note that for any $s, t > 0$

$$\begin{aligned}
&\left|\mathrm{e}^{\lambda_1(t+s)}\mathbb{X}_{t+s}(\phi h) - M_\infty\langle\phi h, h\rangle\right|\\
&\quad\le \left|\mathrm{e}^{\lambda_1(t+s)}\mathbb{X}_{t+s}(\phi h) - \mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x\left[\mathbb{X}_{s+t}(\phi h)\,\middle|\,\mathcal{F}_t\right]\right|\\
&\qquad + \left|\mathrm{e}^{\lambda_1(s+t)}\mathbb{E}_x\left[\mathbb{X}_{s+t}(\phi h)\,\middle|\,\mathcal{F}_t\right] - M_\infty\langle\phi h, h\rangle\right|.
\end{aligned}$$

Therefore Theorem 1.3 follows immediately from Lemma 5.2 and Lemma 5.3. □

6 Strong Law of Large Numbers

In this section, we prove Theorem 1.5.

6.1 SLLN Along Lattice Times

Lemma 6.1 *Suppose the assumptions of Theorem 1.5 hold. Then for any $\sigma > 0$ and any $x\in E$,*

$$\lim_{n\to+\infty} e^{\lambda_1 n\sigma}\mathbb{X}_{n\sigma}(\phi h) = M_\infty\langle\phi h, h\rangle \quad \mathbb{P}_x\text{-a.s.} \tag{6.1}$$

for every $\phi\in\mathcal{B}^+_b(E)$

Proof Recall that $M_\infty = \lim_{n\to+\infty}\mathrm{e}^{\lambda_1 n\sigma}\mathbb{X}_{n\sigma}(h)$. Let $g(y) := (\phi h)(y) - h(y)\langle\phi h, h\rangle$. The convergence in (6.1) is equivalent to

$$\lim_{n\to+\infty}\mathrm{e}^{\lambda_1 n\sigma}\mathbb{X}_{n\sigma}(g) = 0 \quad \mathbb{P}_x\text{-a.s.} \tag{6.2}$$

For an arbitrary $m \in \mathbb{N}$,

$$
\begin{aligned}
&\mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(g) \\
&\quad = \left(\mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(g) - \mathbb{E}_x\left[\mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(g) \,\middle|\, \mathcal{F}_{n\sigma}\right]\right) \\
&\qquad + \mathbb{E}_x\left[\mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(g) \,\middle|\, \mathcal{F}_{n\sigma}\right] \\
&\quad =: I_n + II_n.
\end{aligned} \tag{6.3}
$$

Note that by Lemma 5.2, we have

$$
\begin{aligned}
I_n &= \mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(\phi h) \\
&\quad - \mathbb{E}_x\left[\mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{(n+m)\sigma}(\phi h) \,\middle|\, \mathcal{F}_{n\sigma}\right] \to 0 \quad \text{as } n \uparrow +\infty \quad \mathbb{P}_x\text{-a.s.}
\end{aligned} \tag{6.4}
$$

On the other hand, by Markov property, we have

$$
\begin{aligned}
II_n &= \sum_{u \in \mathcal{Z}_{n\sigma}} \mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{E}_{X_u(n\sigma)}[\mathbb{X}_{m\sigma}(g)] \\
&= \sum_{u \in \mathcal{Z}_{n\sigma}} \mathrm{e}^{\lambda_1(n+m)\sigma} P^{(Q-1)\mu}_{m\sigma} g(X_u(n\sigma)) \\
&= \mathrm{e}^{\lambda_1(n+m)\sigma}\mathbb{X}_{n\sigma}(P^{(Q-1)\mu}_{m\sigma} g).
\end{aligned} \tag{6.5}
$$

Note that our assumption (1.14) implies (1.17). Then for any fixed $\varepsilon > 0$, there exist $m > 0$ sufficiently large such that

$$
\sup_{x,y \in E} |p^h(m\sigma, x, y) - 1| \le \varepsilon.
$$

Then for any $x \in E$,

$$
\begin{aligned}
|\mathrm{e}^{\lambda_1 m\sigma} P^{(Q-1)\mu}_{m\sigma} g(x)| &= |h(x) P^h_{m\sigma}(g/h)(x)| \\
&= h(x)\left|\int_E \left(p^h(m\sigma, x, y) - 1\right)\phi(y)\widetilde{m}(\mathrm{d}y)\right| \\
&\le h(x)\int_E \left|p^h(m\sigma, x, y) - 1\right|\phi(y)h(y)^2 m(\mathrm{d}y) \\
&\le \varepsilon h(x)\langle \phi h, h\rangle.
\end{aligned}
$$

Consequently by (6.5), we have

$$
|II_n| \le \varepsilon\langle \phi h, h\rangle M_{n\sigma}. \tag{6.6}
$$

It follows from (6.3), (6.4) and (6.6) that

$$\limsup_{n\to+\infty} |\mathrm{e}^{\lambda_1 n\sigma}\mathbb{X}_{n\sigma}(g)| \le \varepsilon\langle \phi h, h\rangle M_\infty \quad \mathbb{P}_x\text{-a.s.}$$

Hence we get (6.2) by letting $\varepsilon \to 0$. □

6.2 Transition from Lattice Times to Continuous Time

In this subsection, we extend the convergence along lattice times in Lemma 6.1 to convergence along continuous time and give a sketch of proof of Theorem 1.5. The main approach in this subsection is similar to that of [6, Theorem 3.7] (see also [3, Theorem1']). According to the proof of [6, Theorem3.7], to prove Theorem 1.5, it suffices to prove the following lemma:

Lemma 6.2 *Under the conditions of Theorem 1.5, for every open subset U in E and every $x \in E$,*

$$\liminf_{t\to+\infty} e^{\lambda_1 t}\mathbb{X}_t(1_U h) \ge M_\infty \int_U h(y)^2 m(dy) \quad \mathbb{P}_x\text{-a.s.,} \tag{6.7}$$

Note that if (6.7) is true for any bounded open set U in E with $\overline{U} \subset E$, then (6.7) is true for any open set U. In fact, for an arbitrary open set U in E, there exists a sequence of bounded open sets $\{U_n : n \ge 0\}$ such that $\overline{U}_n \subset E$ and $U_n \uparrow U$. Hence if U_n satisfies (6.7), we can deduce that U satisfies (6.7) by monotone convergence theorem.

In the following, we assume that U is an arbitrary bounded open set in E with $\overline{U} \subset E$. For any $\varepsilon, \sigma > 0$, $n \in \mathbb{N}$ and $x \in E$, define

$$U^\varepsilon(x) := \{y \in U : \ h(y) \ge \frac{1}{1+\varepsilon} h(x)\},$$
$$Y^{\sigma,\varepsilon}_{n,u} := \frac{1}{1+\varepsilon}(h1_U)(X_u(n\sigma))1_{\{X_v(t)\in U^\varepsilon(X_u(n\sigma)) \text{ for all } v\in\mathcal{L}^{u,n\sigma}_t \text{ and } t\in[n\sigma,(n+1)\sigma]\}},$$

and

$$S^{\sigma,\varepsilon}_n := \sum_{u\in\mathcal{Z}_{n\sigma}} \mathrm{e}^{\lambda_1 n\sigma} Y^{\sigma,\varepsilon}_{n,u}.$$

Here for each $t \ge n\sigma$, $\mathcal{L}^{u,n\sigma}_t$ denotes the set of particles which are alive at t and are descendants of the particle $u \in \mathcal{L}_{n\sigma}$.

To prove Lemma 6.2, we need the following lemma.

Lemma 6.3 *Suppose that U is an arbitrary bounded open set U in E with $\overline{U} \subset E$. Under the conditions of Theorem 1.5, we have*

$$\lim_{n\to+\infty} S_n^{\sigma,\varepsilon} - \mathbb{E}_x\left[S_n^{\sigma,\varepsilon} \,\middle|\, \mathcal{F}_{n\sigma}\right] = 0 \quad \mathbb{P}_x\text{-}a.s. \tag{6.8}$$

for every $x \in E$.

Proof Note that for $n \in \mathbb{N}$ and $\{Y_i : i = 1, \ldots, n\}$ independent real-valued centered random variables,

$$\mathrm{E}|\sum_{i=1}^{n} Y_i|^2 = \mathrm{Var}\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \mathrm{E}|Y_i|^2.$$

Thus by the conditional independence between subtrees and the Markov property of a branching Markov process, we have

$$\begin{aligned}
&\mathbb{E}_x\left[\left(S_n^{\sigma,\varepsilon} - \mathbb{E}_x\left[S_n^{\sigma,\varepsilon} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right)^2 \middle| \mathcal{F}_{n\sigma}\right]\\
&= \mathrm{e}^{2\lambda_1 n\sigma} \sum_{u\in\mathcal{Z}_{n\sigma}} \mathbb{E}_x\left[\left|Y_{n,u}^{\sigma,\varepsilon} - \mathbb{E}_x\left[Y_{n,u}^{\sigma,\varepsilon} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right|^2 \middle| \mathcal{F}_{n\sigma}\right]\\
&\le 2\mathrm{e}^{2\lambda_1 n\sigma} \sum_{u\in\mathcal{Z}_{n\sigma}} \mathbb{E}_x\left[|Y_{n,u}^{\sigma,\varepsilon}|^2 \,\middle|\, \mathcal{F}_{n\sigma}\right]\\
&= 2\mathrm{e}^{2\lambda_1 n\sigma} \sum_{u\in\mathcal{Z}_{n\sigma}} \mathbb{E}_{X_u(n\sigma)}\left[|Y_{0,\emptyset}^{\sigma,\varepsilon}|^2\right]\\
&\le 2(1+\varepsilon)^{-2}\mathrm{e}^{2\lambda_1 n\sigma} \sum_{u\in\mathcal{Z}_{n\sigma}} (h^2 1_U)(X_u(n\sigma)).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \mathbb{E}_x\left[\left(S_n^{\sigma,\varepsilon} - \mathbb{E}_x\left[S_n^{\sigma,\varepsilon} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right)^2\right]\\
&\le 2(1+\varepsilon)^{-2} \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_1 2n\sigma} \mathbb{E}_x\left[\sum_{u\in\mathcal{Z}_{n\sigma}} (h^2 1_U)(X_u(n\sigma))\right]\\
&\le 2(1+\varepsilon)^{-2} \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_1 2n\sigma} P_{n\sigma}^{(Q-1)\mu}(h^2 1_U)(x)\\
&\le 2(1+\varepsilon)^{-2} h(x) \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_1 2n\sigma} P_{n\sigma}^{h}(h 1_U)(x).
\end{aligned}$$

By Proposition 1.8, for any $\varepsilon > 0$, $|p^h(n\sigma, x, y) - 1| \le \varepsilon$ for every $x, y \in E$ when n is sufficiently large. It follows that $P_{n\sigma}^h(h1_U)(x) \le (1+\varepsilon)\int_U h(y)\widetilde{m}(\mathrm{d}y) < +\infty$, and so

$$\sum_{n=1}^{+\infty} \mathbb{E}_x\left[\left(S_n^{\sigma,\varepsilon} - \mathbb{E}_x\left[S_n^{\sigma,\varepsilon} \,\middle|\, \mathcal{F}_{n\sigma}\right]\right)^2\right] \le ch(x) \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_1 n\sigma} < \infty,$$

where c is a positive constant. This together with the Borel–Cantelli lemma yields (6.8). □

Proof of Lemma 6.2 If U is an arbitrary bounded open set in E with $\overline{U} \subset E$, then (6.7) follows from Lemma 6.3 in the same way as [6, Theorem3.9]. We omit the details here. By the argument after (6.7), we know that (6.7) holds for any open subset U of E.

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