

CENTRAL LIMIT THEOREMS FOR SUPERCRITICAL BRANCHING NONSYMMETRIC MARKOV PROCESSES

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In this paper, we establish a spatial central limit theorem for a large class of supercritical branching, not necessarily symmetric, Markov processes with spatially dependent branching mechanisms satisfying a second moment condition. This central limit theorem generalizes and unifies all the central limit theorems obtained recently in Ren, Song and Zhang [*J. Funct. Anal.* **266** (2014) 1716–1756] for supercritical branching symmetric Markov processes. To prove our central limit theorem, we have to carefully develop the spectral theory of nonsymmetric strongly continuous semigroups, which should be of independent interest.

1. Introduction. Central limit theorems for supercritical branching processes were initiated by Kesten and Stigum in [13, 14]. In these two papers, they established central limit theorems for supercritical multi-type Galton–Watson processes by using the Jordan canonical form of the expectation matrix M . Then in [4–6], Athreya proved central limit theorems for supercritical multi-type continuous time branching processes, using the Jordan canonical form and the eigenvectors of the matrix M_t , the mean matrix at time t . Asmussen and Keiding [3] used martingale central limit theorems to prove central limit theorems for supercritical multi-type branching processes. In [2], Asmussen and Hering established spatial central limit theorems for general supercritical branching Markov processes under a certain condition. However, the condition in [2] is not easy to check and essentially the only examples given in [2] of branching Markov processes satisfying this condition are branching diffusions in bounded smooth domains. We note that the limit normal random variables in [2] may be degenerate.

The recent study of spatial central limit theorems for branching Markov processes started with a paper by Adamczak and Miłoś [1] where they proved some central limit theorems for supercritical branching Ornstein–Uhlenbeck processes with binary branching mechanism. We note that branching Ornstein–Uhlenbeck

Received April 2014; revised November 2014.

¹Supported by NSFC (Grant No. 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education.

²Supported in part by a grant from the Simons Foundation (208236).

³Supported by the China Scholarship Council.

MSC2010 subject classifications. Primary 60F05, 60J80; secondary 60J25, 60J35.

Key words and phrases. Central limit theorem, branching Markov process, supercritical, martingale.

processes do not satisfy the condition in [2]. In [23], Miłoś proved some central limit theorems for supercritical super Ornstein–Uhlenbeck processes with branching mechanisms satisfying a fourth moment condition. Similar to the case of [2], the limit normal random variables in [1, 23] may be degenerate. In [25], we established central limit theorems for supercritical super Ornstein–Uhlenbeck processes with branching mechanisms satisfying only a second moment condition. More importantly, the central limit theorems in [25] are more satisfactory since our limit normal random variables are nondegenerate. In [26], we obtained central limit theorems for a large class of general supercritical branching symmetric Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. In [28], we obtained central limit theorems for a large class of general supercritical superprocesses with symmetric spatial motions and with spatially dependent branching mechanisms satisfying only a second moment condition. Furthermore, we also obtained the covariance structure of the limit Gaussian field in [28].

Compared with [4–6, 13, 14], the spatial processes in [1, 23, 25, 26, 28] are assumed to be symmetric. The reason for this assumption is that one of the main tools in [1, 23, 25, 26, 28] is the well-developed spectral theory of self-adjoint operators.

The main purpose of this paper is to establish central limit theorems for general supercritical branching, not necessarily symmetric, Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. See our main result, Theorem 1.16, for the statement of our central limit theorems. To prove our main result, we need to carefully develop the spectral theory of not necessarily symmetric, strongly continuous semigroups. We believe these spectral results are of independent interest and should be very useful in studying nonsymmetric Markov processes.

In this paper, \mathbb{R} and \mathbb{C} stand for the sets of real and complex numbers, respectively, and all vectors in \mathbb{R}^n or \mathbb{C}^n will be understood as column vectors. For any $z \in \mathbb{C}$, we use $\Re(z)$ and $\Im(z)$ to denote real and imaginary parts of z , respectively. For a matrix A , we use \bar{A} and A^T to denote the conjugate and transpose of A , respectively.

1.1. Spatial process. In this subsection, we spell out our assumptions on the spatial Markov process. Throughout this paper, E stands for a locally compact separable metric space, m is a σ -finite Borel measure on E with full support and ∂ is a separate point not contained in E . ∂ will be interpreted as the cemetery point. We will use E_∂ to denote $E \cup \{\partial\}$. Every function f on E is automatically extended to E_∂ by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is a Hunt process on E , and $\zeta := \inf\{t > 0: \xi_t = \partial\}$ is the lifetime of ξ . We will use $\{P_t: t \geq 0\}$ to denote the semigroup of ξ . Our standing assumption on ξ is that there exists a family of continuous, strictly positive functions $\{p(t, x, y): t > 0\}$ on $E \times E$ such

that, for any $t > 0$ and nonnegative function f on E ,

$$P_t f(x) = \int_E p(t, x, y) f(y) m(dy).$$

We will use $\{\widehat{P}_t : t \geq 0\}$ to denote the dual semigroup of $\{P_t : t \geq 0\}$ defined by

$$\widehat{P}_t f(x) = \int_E p(t, y, x) f(y) m(dy).$$

For $p \geq 1$, we define $L^p(E, m; \mathbb{C}) := \{f : E \rightarrow \mathbb{C} : \int_E |f(x)|^p m(dx) < \infty\}$ and $L^p(E, m) := \{f \in L^p(E, m; \mathbb{C}) : f \text{ is real}\}$. We also define

$$(1.1) \quad a_t(x) := \int_E p(t, x, y)^2 m(dy), \quad \widehat{a}_t(x) := \int_E p(t, y, x)^2 m(dy).$$

In this paper, we assume the following:

ASSUMPTION 1. (a) For all $t > 0$ and $x \in E$, $\int_E p(t, y, x) m(dy) \leq 1$.

(b) For any $t > 0$, a_t and \widehat{a}_t are continuous functions in E , and they belong to $L^1(E, m)$.

(c) There exists $t_0 > 0$ such that $a_{t_0}, \widehat{a}_{t_0} \in L^2(E, m)$.

By the Chapman–Kolmogorov equation, the Cauchy–Schwarz inequality and (1.1), we have

$$(1.2) \quad p(t + s, x, y) = \int_E p(t, x, z) p(s, z, y) m(dz) \leq (a_t(x))^{1/2} (\widehat{a}_s(y))^{1/2},$$

which implies

$$a_{t+s}(x) \leq \int_E \widehat{a}_s(y) m(dy) a_t(x) \quad \text{and} \quad \widehat{a}_{t+s}(x) \leq \int_E a_s(y) m(dy) \widehat{a}_t(x).$$

So Assumption 1(c) above is equivalent to the following:

(c') There exists $t_0 > 0$ such that for all $t \geq t_0$, $a_t, \widehat{a}_t \in L^2(E, m)$.

Using Assumption 1(a), we have that, for $p \in [1, \infty)$, $\{P_t : t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ are contraction semigroups on $L^p(E, m; \mathbb{C})$. In fact, for any $f \in L^p(E, m; \mathbb{C})$, using Hölder’s inequality, Fubini’s theorem and Assumption 1(a), we have

$$\begin{aligned} \|P_t f\|_p^p &= \int_E \left| \int_E p(t, x, y) f(y) m(dy) \right|^p m(dx) \\ &\leq \int_E \int_E p(t, x, y) |f(y)|^p m(dy) m(dx) \\ &= \int_E \left(\int_E p(t, x, y) m(dx) \right) |f(y)|^p m(dy) \\ &\leq \int_E |f(y)|^p m(dy). \end{aligned}$$

\widehat{P}_t can be dealt with similarly.

One can check that the semigroups $\{P_t : t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ are strongly continuous on $L^p(E, m; \mathbb{C})$ for any $p \in [1, \infty)$, even though only the strong continuity on $L^2(E, m; \mathbb{C})$ is needed later. Here we give a sketch of the proof of this fact. Since X is a Hunt process, for any continuous function f on E with compact support, we have by the dominated convergence theorem,

$$\lim_{t \downarrow 0} P_t f(x) = f(x), \quad x \in E.$$

Since the collection of continuous functions of compact support is dense in $L^2(E, m; \mathbb{C})$, it follows from [21], Proposition II.4.3, that $\{P_t : t \geq 0\}$ is strongly continuous on $L^2(E, m; \mathbb{C})$. Now the strong continuity of $\{\widehat{P}_t : t \geq 0\}$ on $L^2(E, m; \mathbb{C})$ follows from general theory; see, for instance, [24], Corollary 1.10.6. If $f \geq 0$ is a bounded function on E which vanishes outside a set $B \subset E$ of finite m -measure, then

$$\begin{aligned} \lim_{t \rightarrow 0} \int_B P_t f(x) m(dx) &= \lim_{t \rightarrow 0} \int_E 1_B(x) P_t f(x) m(dx) \\ &= \int_E 1_B(x) f(x) m(dx) = \|f\|_1 \end{aligned}$$

by the strong continuity of $\{P_t : t \geq 0\}$ on $L^2(E, m; \mathbb{C})$. Using $\|P_t f\|_1 \leq \|f\|_1$, we have

$$\lim_{t \rightarrow 0} \int_E 1_{E \setminus B}(x) P_t f(x) m(dx) = 0.$$

This implies that

$$\begin{aligned} \lim_{t \rightarrow 0} \|P_t f - f\|_1 &\leq \lim_{t \rightarrow 0} \int_B |P_t f(x) - f(x)| m(dx) \\ &= \lim_{t \rightarrow 0} \int_E |1_B(x) P_t f(x) - f(x)| m(dx) \\ &\leq \lim_{t \rightarrow 0} \|P_t f - f\|_2 m(B)^{1/2} = 0. \end{aligned}$$

Combining the conclusion above with the fact that the collection $\{f : E \mapsto [0, \infty) : f \text{ is bounded on } E \text{ and vanishes outside a set of finite } m\text{-measure}\}$ is dense in $L^1_+(E, m)$, we immediately get the strong continuity of $\{P_t : t \geq 0\}$ on $L^1(E, m; \mathbb{C})$. The strong continuity of $\{\widehat{P}_t : t \geq 0\}$ on $L^1(E, m; \mathbb{C})$ can be proved similarly. The strong continuity of $\{P_t : t \geq 0\}$ and $\{\widehat{P}_t : t \geq 0\}$ on $L^1(E, m; \mathbb{C})$ and $L^2(E, m; \mathbb{C})$ implies the same on $L^p(E, m; \mathbb{C})$ for $p \in (1, 2)$ by interpolation. The same follows for $p \in (2, \infty)$ by using the fact that $L^p(E, m; \mathbb{C})$ is reflexive for $p \in (1, \infty)$.

We claim that the function $t \rightarrow \int_E a_t(x)m(dx)$ is decreasing. In fact, by Fubini’s theorem and Hölder’s inequality, we get

$$\begin{aligned} a_{t+s}(x) &= \int_E p(t+s, x, y) \int_E p(t, x, z)p(s, z, y)m(dz)m(dy) \\ &= \int_E p(t, x, z) \int_E p(t+s, x, y)p(s, z, y)m(dy)m(dz) \\ &\leq a_{t+s}(x)^{1/2} \int_E p(t, x, z)a_s(z)^{1/2}m(dz), \end{aligned}$$

which implies

$$(1.3) \quad a_{t+s}(x) \leq \left(\int_E p(t, x, z)a_s(z)^{1/2}m(dz) \right)^2 \leq \int_E p(t, x, z)a_s(z)m(dz).$$

Thus, by Fubini’s theorem and Assumption 1(a), we get

$$\begin{aligned} \int_E a_{t+s}(x)m(dx) &\leq \int_E a_s(z) \int_E p(t, x, z)m(dx)m(dz) \\ &\leq \int_E a_s(z)m(dz). \end{aligned}$$

Therefore, the function $t \rightarrow \int_E a_t(x)m(dx)$ is decreasing.

Now we give some examples of nonsymmetric Markov processes satisfying the above assumptions. The purpose of these examples is to show that the above assumptions are satisfied by many Markov processes. We will not try to give the most general examples possible. For examples of symmetric Markov processes satisfying the above assumptions, see [26].

EXAMPLE 1.1. Suppose that E consists of finitely many points. If $\xi = \{\xi_t : t \geq 0\}$ is an irreducible conservative Markov process in E , then ξ satisfies Assumption 1 for some finite measure m on E with full support.

EXAMPLE 1.2. Suppose that $\alpha \in (0, 2)$ and that $\xi^{(1)} = \{\xi_t^{(1)} : t \geq 0\}$ is a strictly α -stable process in \mathbb{R}^d . Suppose that, in the case $d \geq 2$, the spherical part η of the Lévy measure μ of $\xi^{(1)}$ satisfies the following assumption: there exist a positive function Φ on the unit sphere S in \mathbb{R}^d and $\kappa > 1$ such that

$$\Phi = \frac{d\eta}{d\sigma} \quad \text{and} \quad \kappa^{-1} \leq \Phi(z) \leq \kappa \quad \text{on } S,$$

where σ is the surface measure on S . In the case $d = 1$, we assume that the Lévy measure of $\xi^{(1)}$ is given by

$$\mu(dx) = c_1x^{-1-\alpha}1_{\{x>0\}} + c_2|x|^{-1-\alpha}1_{\{x<0\}}$$

with $c_1, c_2 > 0$. Suppose that D is an open set in \mathbb{R}^d of finite Lebesgue measure. Let ξ be the process in D obtained by killing $\xi^{(1)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the Lebesgue measure. For details, see [18], Example 4.1.

EXAMPLE 1.3. Suppose that $\alpha \in (0, 2)$ and that $\xi^{(2)} = \{\xi_t^{(2)} : t \geq 0\}$ is a truncated strictly α -stable process in \mathbb{R}^d : that is, $\xi^{(2)}$ is a Lévy process with Lévy measure given by

$$\tilde{\mu}(dx) = \mu(dx)1_{\{|x|<1\}},$$

where μ is the Lévy measure of the process $\xi^{(1)}$ in the previous example. Suppose that D is a connected open set in \mathbb{R}^d of finite Lebesgue measure. Let ξ be the process in D obtained by killing $\xi^{(2)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the Lebesgue measure. For details, see [18], Example 4.2 and Proposition 4.4.

EXAMPLE 1.4. Suppose $\alpha \in (0, 2)$, $\xi^{(1)} = \{\xi_t^{(1)} : t \geq 0\}$ is a strictly α -stable process in \mathbb{R}^d satisfying the assumptions in Example 1.2 and that $\xi^{(3)} = \{\xi_t^{(3)} : t \geq 0\}$ is an independent Brownian motion in \mathbb{R}^d . Let $\xi^{(4)}$ be the process defined by $\xi_t^{(4)} = \xi_t^{(1)} + \xi_t^{(3)}$. Suppose that D is an open set in \mathbb{R}^d of finite Lebesgue measure. Let ξ be the process in D obtained by killing $\xi^{(4)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the Lebesgue measure. For details, see [18], Example 4.5 and Lemma 4.6.

EXAMPLE 1.5. Suppose $\alpha \in (0, 2)$, $\xi^{(2)} = \{\xi_t^{(2)} : t \geq 0\}$ is a truncated strictly α -stable process in \mathbb{R}^d satisfying the assumptions in Example 1.3 and that $\xi^{(3)} = \{\xi_t^{(3)} : t \geq 0\}$ is an independent Brownian motion in \mathbb{R}^d . Let $\xi^{(5)}$ be the process defined by $\xi_t^{(5)} = \xi_t^{(2)} + \xi_t^{(3)}$. Suppose that D is a connected open set in \mathbb{R}^d of finite Lebesgue measure. Let ξ be the process in D obtained by killing $\xi^{(5)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the Lebesgue measure. For details, see [18], Example 4.7 and Lemma 4.8.

EXAMPLE 1.6. Suppose $d \geq 3$ and that $\mu = (\mu^1, \dots, \mu^d)$, where each μ^j is a signed measure on \mathbb{R}^d such that

$$\lim_{r \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j|(dy)}{|x - y|^{d-1}} = 0.$$

Let $\xi^{(6)} = \{\xi_t^{(6)} : t \geq 0\}$ be a Brownian motion with drift μ in \mathbb{R}^d ; see [15]. Suppose that D is a bounded, connected open set in \mathbb{R}^d , and suppose $K > 0$ is a constant such that $D \subset B(0, K/2)$. Put $B = B(0, K)$. Let G_B be the Green function of $\xi^{(6)}$ in B , and define $H(x) := \int_B G_B(x, y) dy$. Then H is a strictly positive continuous function on B . Let ξ be the process obtained by killing $\xi^{(6)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the measure defined by $m(dx) = H(x) dx$. For details, see [31], Example 4.6 or [16, 17].

EXAMPLE 1.7. Suppose $d \geq 2$, $\alpha \in (1, 2)$ and that $\mu = (\mu^1, \dots, \mu^d)$, where each μ^j is a signed measure on \mathbb{R}^d such that

$$\limsup_{r \rightarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|\mu^j|(dy)}{|x-y|^{d-\alpha+1}} = 0.$$

Let $\xi^{(7)} = \{\xi_t^{(7)} : t \geq 0\}$ be an α -stable process with drift μ in \mathbb{R}^d ; see [19]. Suppose that D is a bounded open set in \mathbb{R}^d and suppose $K > 0$ is such that $D \subset B(0, K/2)$. Put $B = B(0, K)$. Let G_B be the Green function of $\xi^{(7)}$ in B , and define $H(x) := \int_B G_B(x, y) dy$. Then H is a strictly positive continuous function on B . Let ξ be the process obtained by killing $\xi^{(7)}$ upon exiting D . Then ξ satisfies Assumption 1 with $E = D$ and m being the measure defined by $m(dx) = H(x) dx$. For details, see [31], Example 4.7 or [9].

1.2. *Branching Markov processes.* The branching Markov process $\{X_t : t \geq 0\}$ on E we are going to work with is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E satisfying the assumptions at the beginning of the previous subsection, a branching rate function $\beta(x)$ on E which is a nonnegative bounded measurable function and an offspring distribution $\{p_n(x) : n = 0, 1, 2, \dots\}$ satisfying:

ASSUMPTION 2.

$$(1.4) \quad \sup_{x \in E} \sum_{n=0}^{\infty} n^2 p_n(x) < \infty.$$

We denote the generating function of the offspring distribution by

$$\varphi(x, z) = \sum_{n=0}^{\infty} p_n(x) z^n, \quad x \in E, |z| \leq 1.$$

Consider a branching system on E characterized by the following properties: (i) each individual has a random birth and death time; (ii) given that an individual is born at $x \in E$, the conditional distribution of its path is determined by Π_x ; (iii) given the path ξ of an individual up to time t and given that the particle is alive at time t , its probability of dying in the interval $[t, t + dt)$ is $\beta(\xi_t) dt + o(dt)$; (iv) when an individual dies at $x \in E$, it splits into n individuals all positioned at x , with probability $p_n(x)$; (v) when an individual reaches ∂ , it disappears from the system; (vi) all the individuals, once born, evolve independently.

Let $\mathcal{M}_a(E)$ be the space of finite integer-valued atomic measures on E , and let $\mathcal{B}_b(E)$ be the set of bounded real-valued Borel measurable functions on E . Let $X_t(B)$ be the number of particles alive at time t located in $B \in \mathcal{B}(E)$. Then $X = \{X_t, t \geq 0\}$ is an $\mathcal{M}_a(E)$ -valued Markov process. For any $\nu \in \mathcal{M}_a(E)$, we denote

the law of X with initial configuration ν by \mathbb{P}_ν . As usual, $\langle f, \nu \rangle := \int_E f(x)\nu(dx)$. For $0 \leq f \in \mathcal{B}_b(E)$, let

$$(1.5) \quad \omega(t, x) := \mathbb{P}_{\delta_x} e^{-\langle f, X_t \rangle}.$$

Then $\omega(t, x)$ is the unique positive solution to the equation

$$(1.6) \quad \omega(t, x) = \Pi_x \int_0^t \psi(\xi_s, \omega(t-s, \xi_s)) ds + \Pi_x(e^{-f(\xi_t)}),$$

where $\psi(x, z) = \beta(x)(\varphi(x, z) - z)$, $x \in E, z \in [0, 1]$, while $\psi(\partial, z) = 0, z \in [0, 1]$. By the branching property, we have

$$\mathbb{P}_\nu e^{-\langle f, X_t \rangle} = e^{\langle \log \omega(t, \cdot), \nu \rangle}.$$

For recent developments on measure-valued branching Markov processes, see, for instance, [7, 20]. Define

$$(1.7) \quad \alpha(x) := \frac{\partial \psi}{\partial z}(x, 1) = \beta(x) \left(\sum_{n=1}^\infty n p_n(x) - 1 \right)$$

and

$$(1.8) \quad A(x) := \frac{\partial^2 \psi}{\partial z^2}(x, 1) = \beta(x) \sum_{n=2}^\infty (n-1) n p_n(x).$$

By (1.4), there exists $K > 0$, such that

$$(1.9) \quad \sup_{x \in E} (|\alpha(x)| + A(x)) \leq K.$$

For any $f \in \mathcal{B}_b(E)$ and $(t, x) \in (0, \infty) \times E$, define

$$T_t f(x) := \Pi_x [e^{\int_0^t \alpha(\xi_s) ds} f(\xi_t)].$$

By applying (1.5) and (1.6) to θf and differentiating with respect to θ at $\theta = 0$, we get that $T_t f(x) = \mathbb{P}_{\delta_x} \langle f, X_t \rangle$ for every $x \in E$.

It is elementary to show that (see [27], Lemma 2.1) there exists a function $q(t, x, y)$ on $(0, \infty) \times E \times E$ which is continuous in (x, y) for each $t > 0$ such that

$$(1.10) \quad e^{-Kt} p(t, x, y) \leq q(t, x, y) \leq e^{Kt} p(t, x, y), \quad (t, x, y) \in (0, \infty) \times E \times E$$

and that for any bounded Borel function f on E and $(t, x) \in (0, \infty) \times E$,

$$T_t f(x) = \int_E q(t, x, y) f(y) m(dy).$$

Define

$$(1.11) \quad b_t(x) := \int_E q(t, x, y)^2 m(dy), \quad \widehat{b}_t(x) := \int_E q(t, y, x)^2 m(dy).$$

The functions $x \rightarrow b_t(x)$ and $x \rightarrow \widehat{b}_t(x)$ are continuous. In fact, by (1.2),

$$(1.12) \quad q(t, x, y) \leq e^{Kt} p(t, x, y) \leq e^{Kt} a_{t/2}(x)^{1/2} \widehat{a}_{t/2}(y)^{1/2}.$$

Since $q(t, \cdot, y)$ and $a_{t/2}$ are continuous, by the dominated convergence theorem, we get b_t is continuous. Similarly, \widehat{b}_t is also continuous. Thus, it follows from (1.12) and Assumption 1(b) and (c') that b_t and \widehat{b}_t enjoy the following properties:

- (i) For any $t > 0$, we have $b_t \in L^1(E, m)$. Moreover, $b_t(x)$ and $\widehat{b}_t(x)$ are continuous in $x \in E$.
- (ii) There exists $t_0 > 0$ such that for all $t \geq t_0$, $b_t, \widehat{b}_t \in L^2(E, m)$.

1.3. *Preliminaries.* Note that, by (1.10), we have $|T_t f(x)| \leq e^{Kt} P_t |f|(x)$. Thus, for any $p \geq 1$,

$$(1.13) \quad \|T_t f\|_p \leq e^{Kt} \|P_t |f|\|_p \leq e^{Kt} \|f\|_p.$$

Recall that α is defined in (1.7). By the boundedness of α and Khas'minskii's lemma ([10], Lemma 3.7), one can follow the elementary arguments in the proofs of [10], Propositions 3.8 and 3.9, to show that

$$(1.14) \quad \limsup_{t \rightarrow 0} \sup_{x \in E} \Pi_x |e^{\int_0^t \alpha(\xi_s) ds} - 1|^2 = 0.$$

Thus for any $f \in L^2(E, m; \mathbb{C})$,

$$\begin{aligned} |T_t f(x) - P_t f(x)|^2 &= |\Pi_x (e^{\int_0^t \alpha(\xi_s) ds} - 1) f(\xi_t)|^2 \\ &\leq \Pi_x |e^{\int_0^t \alpha(\xi_s) ds} - 1|^2 \cdot \Pi_x |f(\xi_t)|^2 \end{aligned}$$

by the Cauchy-Schwarz inequality. Hence by Assumption 1(a), we have

$$\begin{aligned} &\int_E |T_t f(x) - P_t f(x)|^2 m(dx) \\ &\leq \sup_{x \in E} \Pi_x |e^{\int_0^t \alpha(\xi_s) ds} - 1|^2 \int_E p(t, x, y) |f(y)|^2 m(dy) m(dx) \\ &\leq \int_E |f(y)|^2 m(dy) \cdot \sup_{x \in E} \Pi_x |e^{\int_0^t \alpha(\xi_s) ds} - 1|^2, \end{aligned}$$

which goes to zero as $t \downarrow 0$ by (1.14). Thus $\{T_t : t \geq 0\}$ is strongly continuous on $L^2(E, m; \mathbb{C})$.

For $f, g \in L^2(E, m; \mathbb{C})$, define

$$\langle f, g \rangle_m := \int_E f(x) \overline{g(x)} m(dx).$$

Let $\{\widehat{T}_t, t > 0\}$ be the adjoint semigroup of $\{T_t : t \geq 0\}$ on $L^2(E, m; \mathbb{C})$, that is, for $f, g \in L^2(E, m; \mathbb{C})$,

$$\langle T_t f, g \rangle_m = \langle f, \widehat{T}_t g \rangle_m.$$

Thus,

$$\widehat{T}_t g(x) = \int_E q(t, y, x) g(y) m(dy).$$

It is well known (see, e.g., [24], Corollary 1.10.6, Lemma 1.10.1), that $\{\widehat{T}_t : t \geq 0\}$ is a strongly continuous semigroup on $L^2(E, m; \mathbb{C})$ and that

$$(1.15) \quad \|\widehat{T}_t\|_2 = \|T_t\|_2 \leq e^{Kt}.$$

For all $t > 0$ and $f \in L^2(E, m; \mathbb{C})$, $T_t f$ and $\widehat{T}_t f$ are continuous. In fact, since $q(t, x, y)$ is continuous, by (1.12) and Assumption 1(b), using the dominated convergence theorem, we get $T_t f$ and $\widehat{T}_t f$ are continuous. It follows from property (i) at the end of Section 1.2 that, for any $t > 0$, T_t and \widehat{T}_t are compact operators on $L^2(E, m; \mathbb{C})$. Let \mathcal{A} and $\widehat{\mathcal{A}}$ be the infinitesimal generators of $\{T_t : t \geq 0\}$ and $\{\widehat{T}_t : t \geq 0\}$ in $L^2(E, m; \mathbb{C})$, respectively. Let $\sigma(\mathcal{A})$ and $\sigma(\widehat{\mathcal{A}})$ be the spectra of \mathcal{A} and $\widehat{\mathcal{A}}$, respectively. It follows from [24], Theorem 2.2.4 and Corollary 2.3.7, that both $\sigma(\mathcal{A})$ and $\sigma(\widehat{\mathcal{A}})$ consist of eigenvalues only, and that \mathcal{A} and $\widehat{\mathcal{A}}$ have the same number, say N , of eigenvalues. Of course N might be finite or infinite. Let $\mathbb{I} = \{1, 2, \dots, N\}$, when $N < \infty$; otherwise $\mathbb{I} = \{1, 2, \dots\}$. Under the assumptions of Section 1.1, using (1.10) and Jentzsch’s theorem [29], Theorem V.6.6 on page 337, we know that the common value $-\lambda_1 = \sup \Re(\sigma(\mathcal{A})) = \sup \Re(\sigma(\widehat{\mathcal{A}}))$ is an eigenvalue of multiplicity one for both \mathcal{A} and $\widehat{\mathcal{A}}$, and that an eigenfunction ϕ_1 of \mathcal{A} associated with $-\lambda_1$ can be chosen to be strictly positive almost everywhere with $\|\phi_1\|_2 = 1$, and an eigenfunction ψ_1 of $\widehat{\mathcal{A}}$ associated with $-\lambda_1$ can be chosen to be strictly positive almost everywhere with $\langle \phi_1, \psi_1 \rangle_m = 1$. We list the eigenvalues $\{-\lambda_k, k \in \mathbb{I}\}$ of \mathcal{A} in an order so $\lambda_1 < \Re(\lambda_2) \leq \Re(\lambda_3) \leq \dots$. Then $\{-\bar{\lambda}_k, k \in \mathbb{I}\}$ are the eigenvalues of $\widehat{\mathcal{A}}$. For convenience, we define, for any positive integer k not in \mathbb{I} , $\lambda_k = \bar{\lambda}_k = \infty$. For $k \in \mathbb{I}$, we write $\Re_k := \Re(\lambda_k)$ and $\Im_k := \Im(\lambda_k)$. We use the convention $\Re_\infty = \infty$.

Let $\sigma(T_t)$ be the spectrum of T_t in $L^2(E, m; \mathbb{C})$. It follows from [24], Theorem 2.2.4, that $\sigma(T_t) \setminus \{0\} := \{e^{-\lambda_k t} : k \in \mathbb{I}\}$. We claim that there exists $t^* > 0$ such that, for any $k \neq j$, $e^{-\lambda_k t^*} \neq e^{-\lambda_j t^*}$. In fact, if $\Re_k \neq \Re_j$, then for all $t > 0$, $e^{-\lambda_k t} \neq e^{-\lambda_j t}$. If, for $k \neq j$, $\Re_k = \Re_j$, then the set $\{t > 0 : e^{-\lambda_k t} = e^{-\lambda_j t}\} = \{2n\pi / \Im(\lambda_k - \lambda_j) : n \in \mathbb{Z}\}$ is countable. Thus the set $\bigcup_{k \neq j \text{ and } \Re_k = \Re_j} \{t > 0 : e^{-\lambda_k t} = e^{-\lambda_j t}\}$ is countable. Hence the claim is valid. We will fix this t^* throughout this paper.

Now we recall some basic facts about spectral theory; for more details, see [8], Chapter 6. For any $k \in \mathbb{I}$, we define $\mathcal{N}_{k,0} := \{0\}$, and for $n \geq 1$,

$$\mathcal{N}_{k,n} := \mathcal{N}((e^{-\lambda_k t^*} I - T_{t^*})^n) = \{f \in L^2(E, m; \mathbb{C}) : (e^{-\lambda_k t^*} I - T_{t^*})^n f = 0\}$$

and

$$(1.16) \quad \mathcal{R}_{k,n} := \mathcal{R}((e^{-\lambda_k t^*} I - T_{t^*})^n) = (e^{-\lambda_k t^*} I - T_{t^*})^n(L^2(E, m; \mathbb{C})).$$

For each $k \in \mathbb{I}$, there exists an integer $\nu_k \geq 1$ such that

$$\mathcal{N}_{k,n} \subsetneq \mathcal{N}_{k,n+1}, \quad n = 0, 1, \dots, \nu_k - 1; \quad \mathcal{N}_{k,n} = \mathcal{N}_{k,n+1}, \quad n \geq \nu_k$$

and

$$\mathcal{R}_{k,n} \supsetneq \mathcal{R}_{k,n+1}, \quad n = 0, 1, \dots, \nu_k - 1; \quad \mathcal{R}_{k,n} = \mathcal{R}_{k,n+1}, \quad n \geq \nu_k.$$

For all $k \in \mathbb{I}$ and $n \geq 0$, $\mathcal{N}_{k,n}$ is a finite dimensional linear subspace of $L^2(E, m; \mathbb{C})$. $\mathcal{N}_{k,n}$ and $\mathcal{R}_{k,n}$ are invariant subspaces of T_t . In fact, for any $f \in \mathcal{N}_{k,n}$,

$$(e^{-\lambda_k t^*} I - T_{t^*})^n (T_t f) = T_t (e^{-\lambda_k t^*} I - T_{t^*})^n f = 0,$$

which implies that $T_t f \in \mathcal{N}_{k,n}$. If $f = (e^{-\lambda_k t^*} I - T_{t^*})^n g$, then $T_t f = T_t (e^{-\lambda_k t^*} I - T_{t^*})^n g = (e^{-\lambda_k t^*} I - T_{t^*})^n T_t g \in \mathcal{R}_{k,n}$. Thus $\{T_t|_{\mathcal{N}_{k,\nu_k}}, t > 0\}$ is a semigroup on \mathcal{N}_{k,ν_k} . We denote the corresponding infinitesimal generator as \mathcal{A}_k . By [8], Theorem 6.7.4, $\sigma(T_{t^*}|_{\mathcal{N}_{k,\nu_k}}) = \{e^{-\lambda_k t^*}\}$. Since $\sigma(\mathcal{A}_k) \subset \sigma(\mathcal{A})$, we have $\sigma(\mathcal{A}_k) = \{-\lambda_k\}$. Define $n_k := \dim(\mathcal{N}_{k,\nu_k})$ and $r_k := \dim(\mathcal{N}_{k,1})$. Then from linear algebra we know that there exists a basis $\{\phi_j^{(k)}, j = 1, 2, \dots, n_k\}$ of \mathcal{N}_{k,ν_k} such that

$$\begin{aligned} &\mathcal{A}_k(\phi_1^{(k)}, \phi_2^{(k)}, \dots, \phi_{n_k}^{(k)}) \\ &= (\phi_1^{(k)}, \phi_2^{(k)}, \dots, \phi_{n_k}^{(k)}) \begin{pmatrix} J_{k,1} & & & 0 \\ & J_{k,2} & & \\ & & \ddots & \\ 0 & & & J_{k,r_k} \end{pmatrix} \\ &=: (\phi_1^{(k)}, \phi_2^{(k)}, \dots, \phi_{n_k}^{(k)}) D_k, \end{aligned}$$

where

$$J_{k,j} = \begin{pmatrix} -\lambda_k & 1 & & 0 \\ & -\lambda_k & 1 & \\ & & \ddots & \ddots \\ 0 & & & -\lambda_k & 1 \\ & & & & -\lambda_k \end{pmatrix}, \quad \text{a } d_{k,j} \times d_{k,j} \text{ matrix}$$

with $\sum_{j=1}^{r_k} d_{k,j} = n_k$. D_k is uniquely determined by the dimensions of $\mathcal{N}_{k,n}$, $n = 1, 2, \dots, \nu_k$; see [22], Section 7.8, for more details. Here and in the remainder of this paper we use the convention that when an operator, like \mathcal{A} or \mathcal{A}_k or T_t , acts on a vector-valued function, it acts componentwise. For convenience, we define the following \mathbb{C}^{n_k} -valued functions:

$$(1.17) \quad \Phi_k(x) := (\phi_1^{(k)}(x), \phi_2^{(k)}(x), \dots, \phi_{n_k}^{(k)}(x))^T.$$

Put

$$(1.18) \quad D_k(t) := \begin{pmatrix} J_{k,1}(t) & & & 0 \\ & J_{k,2}(t) & & \\ & & \ddots & \\ 0 & & & J_{k,r_k}(t) \end{pmatrix},$$

where $J_{k,j}(t)$ is a $d_{k,j} \times d_{k,j}$ matrix given by

$$J_{k,j}(t) = \begin{pmatrix} 1 & t & t^2/2! & \dots & t^{d_{k,j}-1}/(d_{k,j}-1)! \\ 0 & 1 & t & t^2/2! & \dots \\ & & \ddots & \ddots & \\ & & & 1 & t \\ 0 & & & & 1 \end{pmatrix}.$$

Then we have for a.e. $x \in E$,

$$(1.19) \quad T_t(\Phi_k)^T(x) = e^{-\lambda_k t} (\Phi_k(x))^T D_k(t).$$

More details can be found in [22], page 609. Under our assumptions, $T_t(\Phi_k)^T(x)$ is continuous. Thus, by (1.19), we can choose Φ_k to be continuous, which implies (1.19) holds for all $x \in E$. We note that here the matrix $D_k(t)$ satisfies the semigroup property, that is, for $t, s > 0$, $D_k(t + s) = D_k(t)D_k(s)$ and $D_k(t)$ is invertible with $D_k(t)^{-1} = D_k(-t)$.

For any vector $a = (a_1, \dots, a_n)^T \in \mathbb{C}^n$, we define the L^p norm of a by $|a|_p := (\sum_{j=1}^n |a_j|^p)^{1/p}$ when $1 \leq p < \infty$ and $|a|_\infty := \max_i (|a_i|)$ when $p = \infty$.

By Hölder’s inequality, $|T_t(\phi_j^{(k)})(x)| \leq b_t(x)^{1/2}$. By (1.19), we get $(\Phi_k)^T = e^{\lambda_k t} T_t(\Phi_k)^T (D_k(t))^{-1}$. Thus

$$(1.20) \quad |\Phi_k(x)|_\infty \leq c(t, k) b_t(x)^{1/2},$$

where $c(t, k)$ does not depend on x . When we choose $t = t_0$, by Assumption 1(b) and (c), we get that $\phi_j^{(k)} \in L^2(E, m; \mathbb{C}) \cap L^4(E, m; \mathbb{C})$.

Now we consider the corresponding objects for \widehat{T}_t . We know that $\sigma(\widehat{T}_t) \setminus \{0\} = \{e^{-\bar{\lambda}_k t^*}, k \in \mathbb{I}\}$. Define

$$\widehat{\mathcal{N}}_{k,n} := \mathcal{N}((e^{-\bar{\lambda}_k t^*} I - \widehat{T}_t)^n) = \{f \in L^2(E, m; \mathbb{C}) : (e^{-\bar{\lambda}_k t^*} I - \widehat{T}_t)^n f = 0\}.$$

Note that

$$(1.21) \quad (e^{-\lambda_k t^*} I - T_{t^*})^n = e^{-n\lambda_k t^*} I - \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} e^{-(n-j)\lambda_k t^*} T_{t^*}^j.$$

Since $\sum_{j=1}^n (-1)^{j-1} \binom{n}{j} e^{-(n-j)\lambda_k t^*} T_{t^*}^j$ is a compact operator, by [8], Theorem 6.6.13, $\widehat{\mathcal{N}}_{k,n}$ is of the same dimension as $\mathcal{N}_{k,n}$. In particular, $\dim(\widehat{\mathcal{N}}_{k,\nu_k}) = \dim(\mathcal{N}_{k,\nu_k}) = n_k$. Thus we have

$$\widehat{\mathcal{N}}_{k,n} \subsetneq \widehat{\mathcal{N}}_{k,n+1}, \quad n = 0, 1, \dots, \nu_k - 1; \quad \widehat{\mathcal{N}}_{k,n} = \widehat{\mathcal{N}}_{k,n+1}, \quad n \geq \nu_k.$$

Similarly, we can get, for all $k \in \mathbb{I}$ and $n \geq 0$, $\widehat{\mathcal{N}}_{k,n}$ is an invariant subspace of \widehat{T}_t . Hence, $\{\widehat{T}_t|_{\widehat{\mathcal{N}}_{k,\nu_k}}, t > 0\}$ is a semigroup on $\widehat{\mathcal{N}}_{k,\nu_k}$ with infinitesimal generator \widehat{A}_k .

Let $\{\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)}\}$ be a basis of $\widehat{\mathcal{N}}_{k,\nu_k}$ such that

$$(1.22) \quad \widehat{T}_t(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)}) = (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)}) \widehat{D}_k(t),$$

where $\widehat{D}_k(t)$ is an $n_k \times n_k$ invertible matrix. Since $\widehat{T}_t(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)})(x)$ is continuous, we can choose $(\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)})$ to be continuous. We define an $n_k \times n_k$ matrix \widetilde{A}_k by

$$(1.23) \quad (\widetilde{A}_k)_{j,l} := \langle \phi_j^{(k)}, \widehat{\psi}_l^{(k)} \rangle_m.$$

LEMMA 1.8. For each $k \in \mathbb{I}$,

$$(1.24) \quad L^2(E, m; \mathbb{C}) = \mathcal{N}_{k, \nu_k} \oplus (\widehat{\mathcal{N}}_{k, \nu_k})^\perp = \widehat{\mathcal{N}}_{k, \nu_k} \oplus (\mathcal{N}_{k, \nu_k})^\perp.$$

Moreover, the matrix \widetilde{A}_k defined in (1.23) is invertible.

PROOF. By [8], Theorem 6.6.7, we have $L^2(E, m; \mathbb{C}) = \mathcal{N}_{k, \nu_k} \oplus \mathcal{R}_{k, \nu_k}$. It follows from (1.21) and [8], Theorem 6.6.14, that $\mathcal{R}_{k, \nu_k} = (\widehat{\mathcal{N}}_{k, \nu_k})^\perp$. Thus $L^2(E, m; \mathbb{C}) = \mathcal{N}_{k, \nu_k} \oplus (\widehat{\mathcal{N}}_{k, \nu_k})^\perp$. Similarly, we have $L^2(E, m; \mathbb{C}) = \widehat{\mathcal{N}}_{k, \nu_k} \oplus (\mathcal{N}_{k, \nu_k})^\perp$.

For any vector $a = (a_1, \dots, a_{n_k})^T \in \mathbb{C}^{n_k}$, we have by the definition of \widetilde{A}_k in (1.23),

$$\widetilde{A}_k a = (\langle \phi_1^{(k)}, h \rangle_m, \langle \phi_2^{(k)}, h \rangle_m, \dots, \langle \phi_{n_k}^{(k)}, h \rangle_m)^T,$$

where $h = (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)})\bar{a} \in \widehat{\mathcal{N}}_{k, \nu_k}$.

If $\widetilde{A}_k a = 0$, then $h \in (\mathcal{N}_{k, \nu_k})^\perp$. Since $\widehat{\mathcal{N}}_{k, \nu_k} \cap (\mathcal{N}_{k, \nu_k})^\perp = \{0\}$, we have $h = 0$, which implies $a = 0$. Therefore, \widetilde{A}_k is invertible. \square

LEMMA 1.9. For any $k \in \mathbb{I}$, define

$$(1.25) \quad \begin{aligned} (\Psi_k(x))^T &:= (\psi_1^{(k)}(x), \psi_2^{(k)}(x), \dots, \psi_{n_k}^{(k)}(x)) \\ &:= (\widehat{\psi}_1^{(k)}(x), \widehat{\psi}_2^{(k)}(x), \dots, \widehat{\psi}_{n_k}^{(k)}(x))\overline{\widetilde{A}_k^{-1}}. \end{aligned}$$

Then $\{\psi_1^{(k)}, \psi_2^{(k)}, \dots, \psi_{n_k}^{(k)}\}$ is a basis of $\widehat{\mathcal{N}}_{k, \nu_k}$ such that the $n_k \times n_k$ matrix $A_k := (\langle \phi_j^{(k)}, \psi_l^{(k)} \rangle_m)$ satisfies

$$(1.26) \quad A_k = I$$

and for any $x \in \mathbb{E}$,

$$(1.27) \quad \widehat{T}_t(\Psi_k)(x) = e^{-\bar{\lambda}_k t} D_k(t) \Psi_k(x).$$

Moreover, the basis of $\widehat{\mathcal{N}}_{k, \nu_k}$ satisfying (1.26) is unique.

PROOF. Since $\overline{\widetilde{A}_k^{-1}}$ is invertible, $\{\psi_1^{(k)}, \psi_2^{(k)}, \dots, \psi_{n_k}^{(k)}\}$ is a basis of $\widehat{\mathcal{N}}_{k, \nu_k}$. According to the definition of \widetilde{A}_k given by (1.23), we have

$$\widetilde{A}_k = \int_E \Phi_k(x) \overline{\widehat{\Psi}_k(x)^T} m(dx),$$

where $\widehat{\Psi}_k := (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)})^T$, and the integration of a matrix is understood element-wise.

By (1.19) and (1.22), we get

$$\begin{aligned} e^{-\lambda_k t} (D_k(t))^T \widetilde{A}_k &= \int_E e^{-\lambda_k t} (D_k(t))^T \Phi_k(x) \overline{\widehat{\Psi}_k(x)^T} m(dx) \\ &= \int_E T_t \Phi_k(x) \overline{\widehat{\Psi}_k(x)^T} m(dx) = \int_E \Phi_k(x) \overline{(\widehat{T}_t \widehat{\Psi}_k^T)(x)} m(dx) \\ &= \int_E \Phi_k(x) \overline{\widehat{\Psi}_k^T(x) \widehat{D}_k(t)} m(dx) = \widetilde{A}_k \overline{\widehat{D}_k(t)}. \end{aligned}$$

Since $D_k(t)$ is a real matrix, we have

$$(1.28) \quad e^{-\bar{\lambda}_k t} \overline{\widetilde{A}_k^{-1}} (D_k(t))^T = \widehat{D}_k(t) \overline{\widetilde{A}_k^{-1}}.$$

By (1.22) and (1.28), we have

$$\begin{aligned} \widehat{T}_t(\psi_1^{(k)}, \psi_2^{(k)}, \dots, \psi_{n_k}^{(k)}) &= (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)}) \widehat{D}_k(t) \overline{\widetilde{A}_k^{-1}} \\ &= e^{-\bar{\lambda}_k t} (\widehat{\psi}_1^{(k)}, \widehat{\psi}_2^{(k)}, \dots, \widehat{\psi}_{n_k}^{(k)}) \overline{\widetilde{A}_k^{-1}} (D_k(t))^T \\ &= e^{-\bar{\lambda}_k t} (\psi_1^{(k)}, \psi_2^{(k)}, \dots, \psi_{n_k}^{(k)}) (D_k(t))^T. \end{aligned}$$

Assume that there exists another basis $\widetilde{\Psi}_k(x)$ of $\widehat{\mathcal{N}}_{k, v_k}$ satisfying (1.26). Then there exists a matrix B such that $(\widetilde{\Psi}_k(x))^T = (\Psi_k(x))^T B$. Thus

$$I = \int_E \Phi_k(x) \overline{(\widetilde{\Psi}_k(x))^T} m(dx) = \int_E \Phi_k(x) \overline{(\Psi_k(x))^T} m(dx) \overline{B} = \overline{B},$$

which implies $B = I$. Hence, we get $\widetilde{\Psi}_k(x) = \Psi_k(x)$. The proof is now complete. □

REMARK 1.10. Recall that $\Phi_k(x)$, $D_k(t)$ and $\Psi_k(x)$ are defined in (1.17), (1.18) and (1.25), respectively. We know that $T_t(\Phi_k^T)(x) = e^{-\bar{\lambda}_k t} \overline{\Phi_j^T(x)} D_k(t)$. Thus $e^{-\bar{\lambda}_k t}$ is also an eigenvalue of T_t . Hence there exists a unique k' such that $\lambda_{k'} = \bar{\lambda}_k$. It is obvious that $D_k(t) = D_{k'}(t)$, and we can choose $\Phi_{k'}(x) = \overline{\Phi_k(x)}$. By Lemma 1.9, we have $\Psi_{k'}(x) = \overline{\Psi_k(x)}$. In particular, if λ_k is real, then $k' = k$.

LEMMA 1.11. For $j, k \in \mathbb{I}$ and $j \neq k$, we have

$$(1.29) \quad \mathcal{N}_{j, v_j} \subset \mathcal{R}_{k, v_k} = (\widehat{\mathcal{N}}_{k, v_k})^\perp.$$

In particular, $\mathcal{N}_{j, v_j} \cap \mathcal{N}_{k, v_k} = \{0\}$.

PROOF. Assume $f \in \mathcal{N}_{j, v_j}$, then $(e^{-\lambda_j t^*} I - T_{t^*})^{v_j} f = 0$. Since $v_j \geq 1$, we can define $g = (e^{-\lambda_j t^*} I - T_{t^*})^{v_j - 1} f$. Thus $e^{-\lambda_j t^*} g = T_{t^*} g$. Hence, $(e^{-\lambda_k t^*} I - T_{t^*}) g = (e^{-\lambda_k t^*} - e^{-\lambda_j t^*}) g$, which implies

$$(e^{-\lambda_k t^*} I - T_{t^*})^{v_k} g = (e^{-\lambda_k t^*} - e^{-\lambda_j t^*})^{v_k} g.$$

Therefore $g = (e^{-\lambda_k t^*} - e^{-\lambda_j t^*})^{-\nu_k} (e^{-\lambda_k t^*} I - T_{t^*})^{\nu_k} g \in \mathcal{R}_{k, \nu_k}$.

Assume $f = f_1 + f_2$ with $f_1 \in \mathcal{N}_{k, \nu_k}$ and $f_2 \in \mathcal{R}_{k, \nu_k}$. Then $(e^{-\lambda_j t^*} I - T_{t^*})^{\nu_j - 1} f_1 \in \mathcal{N}_{k, \nu_k}$. On the other hand, $(e^{-\lambda_j t^*} I - T_{t^*})^{\nu_j - 1} f_1 = g - (e^{-\lambda_j t^*} I - T_{t^*})^{\nu_j - 1} f_2 \in \mathcal{R}_{k, \nu_k}$. Thus $(e^{-\lambda_j t^*} I - T_{t^*})^{\nu_j - 1} f_1 = 0$.

If $\nu_j = 1$, then $f = g \in \mathcal{R}_{k, \nu_k}$. If $\nu_j > 1$ and $f_1 \neq 0$, then $e^{-\lambda_j t^*} \in \sigma(T_{t^*}|_{\mathcal{N}_{k, \nu_k}})$. By [8], Theorem 6.7.4, $\sigma(T_{t^*}|_{\mathcal{N}_{k, \nu_k}}) = \{e^{-\lambda_k t^*}\}$. This is a contradiction. Thus, $f_1 = 0$, which implies $f = f_2 \in \mathcal{R}_{k, \nu_k}$. Therefore $\mathcal{N}_{j, \nu_j} \subset \mathcal{R}_{k, \nu_k}$. \square

By Lemma 1.11, for $k \in \mathbb{I}$, we can define

$$(1.30) \quad \begin{aligned} \mathcal{M}_k &:= \mathcal{N}_{1, \nu_1} \oplus \mathcal{N}_{2, \nu_2} \oplus \cdots \oplus \mathcal{N}_{k, \nu_k} \quad \text{and} \\ \widehat{\mathcal{M}}_k &:= \widehat{\mathcal{N}}_{1, \nu_1} \oplus \widehat{\mathcal{N}}_{2, \nu_2} \oplus \cdots \oplus \widehat{\mathcal{N}}_{k, \nu_k}. \end{aligned}$$

COROLLARY 1.12. For any $k \in \mathbb{I}$,

$$(1.31) \quad L^2(E, m; \mathbb{C}) = \mathcal{M}_k \oplus (\widehat{\mathcal{M}}_k)^\perp = \widehat{\mathcal{M}}_k \oplus (\mathcal{M}_k)^\perp.$$

PROOF. By (1.24), (1.31) holds for $k = 1$. Assume that (1.31) holds for $k - 1$. Then

$$L^2(E, m; \mathbb{C}) = \mathcal{M}_{k-1} \oplus (\widehat{\mathcal{M}}_{k-1})^\perp.$$

For any $f \in (\widehat{\mathcal{M}}_{k-1})^\perp$, by (1.24), we have $f = f_3 + f_4$, where $f_3 \in \mathcal{N}_{k, \nu_k}$ and $f_4 \in (\widehat{\mathcal{N}}_{k, \nu_k})^\perp$. By (1.29), $f_3 \in \bigcap_{j=1}^{k-1} (\widehat{\mathcal{N}}_{j, \nu_j})^\perp = (\widehat{\mathcal{M}}_{k-1})^\perp$, which implies $f_4 = f - f_3 \in (\widehat{\mathcal{M}}_{k-1})^\perp$. Thus we obtain

$$f_4 \in (\widehat{\mathcal{N}}_{k, \nu_k})^\perp \cap (\widehat{\mathcal{M}}_{k-1})^\perp = (\widehat{\mathcal{M}}_k)^\perp.$$

Hence

$$(\widehat{\mathcal{M}}_{k-1})^\perp = \mathcal{N}_{k, \nu_k} \oplus (\widehat{\mathcal{M}}_k)^\perp.$$

Therefore, by induction, the first part of (1.31) holds for all $k \in \mathbb{I}$.

The proof of $L^2(E, m; \mathbb{C}) = \widehat{\mathcal{M}}_k \oplus (\mathcal{M}_k)^\perp$ is similar. \square

REMARK 1.13. Recall that $\Phi_k(x)$ and $\Psi_k(x)$ are defined in (1.17) and (1.25), respectively. Since $-\lambda_1$ is simple, which means $n_1 = r_1 = \nu_1 = 1$, we know that $\Phi_1(x) = \phi_1(x)$ and $\Psi_1(x) = \psi_1(x)$. Moreover, since $T_t \phi_1(x) = e^{-\lambda_1 t} \phi_1(x)$ and $\widehat{T}_t \psi_1(x) = e^{-\lambda_1 t} \psi_1(x)$ for every x , ϕ_1 and ψ_1 are continuous and strictly positive. By the definition of $D_k(t)$ in (1.18), we see that $D_1(t) \equiv 1$.

By Lemma 1.11, $\{\phi_l^{(j)}, j = 1, \dots, k, l = 1, \dots, n_j\}$ is a basis of \mathcal{M}_k and $\{\psi_l^{(j)}, j = 1, \dots, k, l = 1, \dots, n_j\}$ is a basis of $\widehat{\mathcal{M}}_k$. By (1.29) and (1.26), we get $\langle \phi_l^{(j)}, \psi_n^{(k)} \rangle_m = 1$, when $j = k$ and $l = n$; otherwise $\langle \phi_l^{(j)}, \psi_n^{(k)} \rangle_m = 0$.

In this paper, we always assume that the branching Markov process X is supercritical, that is:

ASSUMPTION 3. $\lambda_1 < 0$.

We will use $\{\mathcal{F}_t : t \geq 0\}$ to denote the filtration of X , that is, $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$. Using the expectation formula of $\langle \phi_1, X_t \rangle$ and the Markov property of X , one can show that (see Lemma 3.1) for any nonzero $\nu \in \mathcal{M}_a(E)$, under \mathbb{P}_ν , the process $W_t := e^{\lambda_1 t} \langle \phi_1, X_t \rangle$ is a positive martingale. Therefore it converges in the following way:

$$W_t \rightarrow W_\infty, \quad \mathbb{P}_\nu\text{-a.s. as } t \rightarrow \infty.$$

Using Assumption 2, we can show (see Lemma 3.1 below) that as $t \rightarrow \infty$, W_t also converges in $L^2(\mathbb{P}_\nu)$, so W_∞ is nondegenerate, and the second moment is finite. Moreover, we have $\mathbb{P}_\nu(W_\infty) = \langle \phi_1, \nu \rangle$. Put $\mathcal{E} = \{W_\infty = 0\}$, then $\mathbb{P}_\nu(\mathcal{E}) < 1$. It is clear that $\mathcal{E}^c \subset \{X_t(E) > 0, \forall t \geq 0\}$.

1.4. *Main results.* Recall that $\Phi_k(x)$, $D_k(t)$, $\Psi_k(x)$, \mathcal{M}_k and $\widehat{\mathcal{M}}_k$ are defined in (1.17), (1.18), (1.25) and (1.30), respectively. For any $k \in \mathbb{I}$, every function $f \in L^2(E, m; \mathbb{C})$ can be written uniquely as the sum of a function $f_k \in \mathcal{M}_k$ and a function in $(\widehat{\mathcal{M}}_k)^\perp$. Similarly, every function $f \in L^2(E, m; \mathbb{C})$ can be written uniquely as the sum of a function $\widehat{f}_k \in \widehat{\mathcal{M}}_k$ and a function in $(\mathcal{M}_k)^\perp$. Using Lemma 1.9, we get that

$$f_k(x) = \sum_{j=1}^k (\Phi_j(x))^T \langle f, \Psi_j \rangle_m \in \mathcal{M}_k$$

and

$$(1.32) \quad \widehat{f}_k(x) = \sum_{j=1}^k (\Psi_j(x))^T \langle f, \Phi_j \rangle_m \in \widehat{\mathcal{M}}_k,$$

where

$$\langle f, \Psi_j \rangle_m := (\langle f, \psi_1^{(j)} \rangle_m, \langle f, \psi_2^{(j)} \rangle_m, \dots, \langle f, \psi_{n_j}^{(j)} \rangle_m)^T$$

and

$$\langle f, \Phi_j \rangle_m := (\langle f, \phi_1^{(j)} \rangle_m, \langle f, \phi_2^{(j)} \rangle_m, \dots, \langle f, \phi_{n_j}^{(j)} \rangle_m)^T.$$

For any $f \in L^2(E, m; \mathbb{C})$, we define

$$(1.33) \quad \gamma(f) := \inf\{j \in \mathbb{I} : \langle f, \Psi_j \rangle_m \neq 0\},$$

where we use the usual convention that $\inf \emptyset = \infty$. If $\gamma(f) < \infty$, define

$$(1.34) \quad \zeta(f) := \sup\{j \in \mathbb{I} : \mathfrak{R}j = \mathfrak{R}_{\gamma(f)}\}.$$

For each $j \in \mathbb{I}$, every component of the function $t \mapsto D_j(t)\langle f, \Psi_j \rangle_m$ is a polynomial of t . Denote the degree of the l th component of $D_j(t)\langle f, \Psi_j \rangle_m$ by $\tau_{j,l}(f)$. We define

$$(1.35) \quad \tau(f) := \sup\{\tau_{j,l}(f) : \gamma(f) \leq j \leq \zeta(f), 1 \leq l \leq n_j\}.$$

Then for any j with $\Re j = \Re_{\gamma(f)}$,

$$(1.36) \quad F_{f,j} := \lim_{t \rightarrow \infty} t^{-\tau(f)} D_j(t)\langle f, \Psi_j \rangle_m$$

exists and there exists a j such that $F_{f,j} \neq 0$.

Note that if $g \in L^2(E, m)$, then for any $j \in \mathbb{I}$,

$$\overline{\langle g, \Psi_j \rangle_m} = \langle g, \overline{\Psi_j} \rangle_m = \langle g, \Psi_{j'} \rangle_m,$$

where j' is defined in Remark 1.10. For $g(x) = \sum_{k:\lambda_1 \geq 2\Re k} (\Phi_k(x))^T v_k$, we have $v_k = \langle g, \Psi_j \rangle_m$. Thus $g(x)$ is real if and only if $\bar{v}_k = v_{k'}$. The following three subsets of $L^2(E, m)$ will be needed in the statement of our main result:

$$(1.37) \quad \mathcal{C}_l := \left\{ g(x) = \sum_{k \in \mathbb{I}:\lambda_1 > 2\Re k} (\Phi_k(x))^T v_k : v_k \in \mathbb{C}^{n_k} \text{ with } \bar{v}_k = v_{k'} \right\},$$

$$(1.38) \quad \mathcal{C}_c := \left\{ g(x) = \sum_{k \in \mathbb{I}:\lambda_1 = 2\Re k} (\Phi_k(x))^T v_k : v_k \in \mathbb{C}^{n_k} \text{ with } \bar{v}_k = v_{k'} \right\}$$

and

$$(1.39) \quad \mathcal{C}_s := \{g \in L^2(E, m) \cap L^4(E, m) : \lambda_1 < 2\Re_{\gamma(g)}\}.$$

1.4.1. *Some basic laws of large numbers.* Recall that $\Phi_k(x)$ and $D_k(t)$ are defined in (1.17) and (1.18), respectively. Recall also that \mathbb{I} is defined in the paragraph below (1.15). For any $k \in \mathbb{I}$, we define an n_k -dimensional random vector $H_t^{(k)}$ as follows:

$$(1.40) \quad H_t^{(k)} := e^{\lambda_k t} (\langle \phi_1^{(k)}, X_t \rangle, \dots, \langle \phi_{n_k}^{(k)}, X_t \rangle) (D_k(t))^{-1}.$$

One can show (see Lemma 3.1 below) that if $\lambda_1 > 2\Re k$, then, for any $v \in \mathcal{M}_a(E)$ and $v \in \mathbb{C}^{n_k}$, $H_t^{(k)} v$ is a martingale under \mathbb{P}_v and bounded in $L^2(\mathbb{P}_v)$. Thus the limit $H_\infty^{(k)} := \lim_{t \rightarrow \infty} H_t^{(k)}$ exists \mathbb{P}_v -a.s. and in $L^2(\mathbb{P}_v)$.

THEOREM 1.14. *If $f \in L^2(E, m; \mathbb{C}) \cap L^4(E, m; \mathbb{C})$ with $\lambda_1 > 2\Re_{\gamma(f)}$, then for any nonzero $v \in \mathcal{M}_a(E)$, as $t \rightarrow \infty$,*

$$t^{-\tau(f)} e^{\Re_{\gamma(f)} t} \langle f, X_t \rangle - \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\Im_j t} H_\infty^{(j)} F_{f,j} \rightarrow 0 \quad \text{in } L^2(\mathbb{P}_v),$$

where $F_{f,j}$ is defined in (1.36).

REMARK 1.15. Recall that $\gamma(f)$, $\zeta(f)$ and $\tau(f)$ are defined in (1.33), (1.34) and (1.35), respectively. Suppose $f \in L^2(E, m; \mathbb{C}) \cap L^4(E, m; \mathbb{C})$ with $\gamma(f) = 1$. Then $\zeta(f) = 1$. Since $D_1(t) \equiv 1$, $\tau(f) = 0$. Thus $H_t^{(1)}$ reduces to W_t and $H_\infty^{(1)} = W_\infty$. Therefore by Theorem 1.14 and the fact that $F_{f,1} = \langle f, \psi_1 \rangle_m$, we get that for any nonzero $v \in \mathcal{M}_a(E)$,

$$e^{\lambda_1 t} \langle f, X_t \rangle \rightarrow \langle f, \psi_1 \rangle_m W_\infty \quad \text{in } L^2(\mathbb{P}_v),$$

as $t \rightarrow \infty$. It is obvious that the convergence also holds in \mathbb{P}_v -probability.

In particular, if f is nonzero and nonnegative, then $\langle f, \psi_1 \rangle_m \neq 0$ which implies $\gamma(f) = 1$.

1.4.2. *Central limit theorem.* Our aim is to describe the limit behavior of $\langle f, X_t \rangle$ for f belonging to the subsets C_s , C_c and C_l of $L^2(E, m)$. Recall that C_l , C_c and C_s are defined in (1.37), (1.38) and (1.39), respectively. For $f \in C_s$, define

$$(1.41) \quad \sigma_f^2 := \int_0^\infty e^{\lambda_1 s} \langle A|T_s f|^2, \psi_1 \rangle_m ds + \langle |f|^2, \psi_1 \rangle_m.$$

For $h = \sum_{k:\lambda_1=2\mathfrak{N}_k} (\Phi_k(x))^T v_k \in C_c$, define

$$(1.42) \quad \rho_h^2 := (1 + 2\tau(h))^{-1} \langle AF_h, \psi_1 \rangle_m,$$

where $F_h(x) := \sum_{k:\lambda_1=2\mathfrak{N}_k} |(\Phi_k(x))^T F_{h,k}|^2$. For

$$g(x) = \sum_{k:\lambda_1>2\mathfrak{N}_k} (\Phi_k(x))^T v_k \in C_l,$$

define

$$(1.43) \quad \begin{aligned} I_s g(x) &:= \sum_{k:\lambda_1>2\mathfrak{N}_k} e^{\lambda_1 s} \Phi_k(x)^T D_k(s)^{-1} v_k, \\ \beta_g^2 &:= \int_0^\infty e^{-\lambda_1 u} \langle A|I_u g|^2, \psi_1 \rangle_m du - \langle g^2, \psi_1 \rangle_m \end{aligned}$$

and

$$E_t(g) := \sum_{k:\lambda_1>2\mathfrak{N}_k} (e^{-\lambda_1 t} H_\infty^{(k)} D_k(t) v_k).$$

THEOREM 1.16. *If $f \in C_s$, $h \in C_c$ and $g \in C_l$, then σ_f^2 , ρ_h^2 and β_g^2 all belong to $(0, \infty)$. Furthermore, it holds that, under $\mathbb{P}_v(\cdot|\mathcal{E}^c)$, as $t \rightarrow \infty$,*

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{\langle g, X_t \rangle - E_t(g)}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle h, X_t \rangle}{\sqrt{t^{1+2\tau(h)} \langle \phi_1, X_t \rangle}}, \frac{\langle f, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \right)$$

$$\xrightarrow{d} (W^*, G_3(g), G_2(h), G_1(f)),$$

where W^* has the same distribution as W_∞ conditioned on \mathcal{E}^c , $G_3(g) \sim \mathcal{N}(0, \beta_g^2)$, $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W^* , $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent.

The main difference between the setup in the theorem above and the setup in [26] is that now the spatial motion is not assumed to be symmetric. Even in the symmetric case, the theorem above is a unification of all the central limit theorems contained in [26], Theorems 1.8–1.10 and 1.12. As we will explain at the end of this subsection (see Corollaries 1.20–1.21 and the sentence before Corollary 1.20), all the central limit theorems in [26], Theorems 1.8–1.10 and 1.12, are consequences of the theorem above. Furthermore, as we will explain in the three corollaries below, we can also get the covariance structure of the limiting Gaussian field.

Whenever $f \in \mathcal{C}_s$, we will use $G_1(f)$ to denote a normal random variable $\mathcal{N}(0, \sigma_f^2)$. For $f_1, f_2 \in \mathcal{C}_s$, define

$$\sigma(f_1, f_2) := \int_0^\infty e^{\lambda_1 s} \langle A(T_s f_1)(T_s f_2), \psi_1 \rangle_m ds + \langle f_1 f_2, \psi_1 \rangle_m.$$

COROLLARY 1.17. *If $f_1, f_2 \in \mathcal{C}_s$, then, under $\mathbb{P}_v(\cdot | \mathcal{E}^c)$,*

$$\left(\frac{\langle f_1, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle f_2, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (G_1(f_1), G_1(f_2)), \quad t \rightarrow \infty,$$

and $(G_1(f_1), G_1(f_2))$ is a bivariate normal random variable with covariance

$$(1.44) \quad \text{Cov}(G_1(f_1), G_1(f_2)) = \sigma(f_1, f_2).$$

PROOF. Using the convergence of the fourth component in Theorem 1.16, we get

$$\begin{aligned} & \mathbb{P}_v \left(\exp \left\{ i\theta_1 \frac{\langle f_1, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} + i\theta_2 \frac{\langle f_2, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \right\} \middle| \mathcal{E}^c \right) \\ &= \mathbb{P}_v \left(\exp \left\{ i \frac{\langle \theta_1 f_1 + \theta_2 f_2, X_t \rangle}{\sqrt{\langle \phi_1, X_t \rangle}} \right\} \middle| \mathcal{E}^c \right) \\ &\rightarrow \exp \left\{ -\frac{1}{2} \sigma_{(\theta_1 f_1 + \theta_2 f_2)}^2 \right\} \quad \text{as } t \rightarrow \infty, \end{aligned}$$

where

$$\begin{aligned} \sigma_{(\theta_1 f_1 + \theta_2 f_2)}^2 &= \int_0^\infty e^{\lambda_1 s} \langle A(T_s(\theta_1 f_1 + \theta_2 f_2))^2, \psi_1 \rangle_m ds \\ &\quad + \langle (\theta_1 f_1 + \theta_2 f_2)^2, \psi_1 \rangle_m \\ &= \theta_1^2 \sigma_{f_1}^2 + 2\theta_1 \theta_2 \sigma(f_1, f_2) + \theta_2^2 \sigma_{f_2}^2. \end{aligned}$$

Now (1.44) follows immediately. \square

Whenever $h \in \mathcal{C}_c$, we will use $G_2(h)$ to denote a normal random variable $\mathcal{N}(0, \rho_h^2)$, where ρ_h^2 is defined in (1.42). For $h_1, h_2 \in \mathcal{C}_c$, define

$$(1.45) \quad \rho(h_1, h_2) := (1 + \tau(h_1) + \tau(h_2))^{-1} \langle AF_{h_1, h_2}, \psi_1 \rangle_m,$$

where

$$(1.46) \quad \begin{aligned} F_{h_1, h_2}(x) &:= \sum_{j: \lambda_1 = 2\Re_j} \Phi_j(x)^T F_{h_1, j} \Phi_{j'}(x)^T F_{h_2, j'} \\ &= \sum_{j: \lambda_1 = 2\Re_j} \Phi_j(x)^T F_{h_1, j} \overline{\Phi_j(x)^T F_{h_2, j}}. \end{aligned}$$

COROLLARY 1.18. *If $h_1, h_2 \in \mathcal{C}_c$, then we have, under $\mathbb{P}_v(\cdot | \mathcal{E}^c)$,*

$$\left(\frac{\langle h_1, X_t \rangle}{\sqrt{t^{1+2\tau(h_1)} \langle \phi_1, X_t \rangle}}, \frac{\langle h_2, X_t \rangle}{\sqrt{t^{1+2\tau(h_2)} \langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (G_2(h_1), G_2(h_2)), \quad t \rightarrow \infty,$$

and $(G_2(h_1), G_2(h_2))$ is a bivariate normal random variable with covariance

$$\text{Cov}(G_2(h_1), G_2(h_2)) = \rho(h_1, h_2).$$

Whenever $g \in \mathcal{C}_l$, we will use $G_3(g)$ to denote a normal random variable $\mathcal{N}(0, \beta_g^2)$, where β_g^2 is defined in (1.43). For $g_1(x), g_2(x) \in \mathcal{C}_l$, define

$$\beta(g_1, g_2) := \int_0^\infty e^{-\lambda_1 s} \langle A(I_s g_1)(I_s g_2), \psi_1 \rangle_m ds - \langle g_1 g_2, \psi_1 \rangle_m.$$

Using the convergence of the second component in Theorem 1.16 and an argument similar to that in the proof of Corollary 1.17, we get the following:

COROLLARY 1.19. *If $g_1(x), g_2(x) \in \mathcal{C}_l$, then we have, under $\mathbb{P}_v(\cdot | \mathcal{E}^c)$,*

$$\left(\frac{\langle g_1, X_t \rangle - E_t(g_1)}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle g_2, X_t \rangle - E_t(g_2)}{\sqrt{\langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (G_3(g_1), G_3(g_2)), \quad t \rightarrow \infty,$$

and $(G_3(g_1), G_3(g_2))$ is a bivariate normal random variable with covariance

$$\text{Cov}(G_3(g_1), G_3(g_2)) = \beta(g_1, g_2).$$

For any $f \in L^2(E, m) \cap L^4(E, m)$, define

$$f_{(s)}(x) := \sum_{j: 2\Re_j < \lambda_1} (\Phi_j(x))^T \langle f, \Psi_j \rangle_m,$$

$$f_{(c)}(x) := \sum_{j: 2\Re_j = \lambda_1} (\Phi_j(x))^T \langle f, \Psi_j \rangle_m,$$

$$f_{(l)}(x) := f(x) - f_{(s)}(x) - f_{(l)}(x).$$

Then $f_{(s)} \in \mathcal{C}_l$, $f_{(c)} \in \mathcal{C}_c$ and $f_{(l)} \in \mathcal{C}_s$. Obviously, [26], Theorem 1.8, is an immediate consequence of the convergence of the first and fourth components in Theorem 1.16.

REMARK 1.20. If $f \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 = 2\mathfrak{N}_\gamma(f)$, then $f = f_{(c)} + f_{(l)}$. Using the convergence of the fourth component in Theorem 1.16 for $f_{(l)}$, it holds under $\mathbb{P}_\nu(\cdot|\mathcal{E}^c)$ that

$$\frac{\langle f_{(l)}, X_t \rangle}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \xrightarrow{d} 0, \quad t \rightarrow \infty.$$

Thus using the convergence of the first and third components in Theorem 1.16, we get, under $\mathbb{P}_\nu(\cdot|\mathcal{E}^c)$,

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{\langle f, X_t \rangle}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (W^*, G_2(f_{(c)})), \quad t \rightarrow \infty,$$

where W^* has the same distribution as W_∞ conditioned on \mathcal{E}^c and $G_2(f_{(c)}) \sim \mathcal{N}(0, \rho_{f_{(c)}}^2)$. Moreover, W^* and $G_2(f_{(c)})$ are independent. Thus [26], Theorem 1.9, is a consequence of Theorem 1.16.

REMARK 1.21. Assume $f \in L^2(E, m) \cap L^4(E, m)$ satisfies $\lambda_1 > 2\mathfrak{N}_\gamma(f)$.

If $f_{(c)} = 0$, then $f = f_{(l)} + f_{(s)}$. Using the convergence of the first, second and fourth components in Theorem 1.16, we get for any nonzero $\nu \in \mathcal{M}_a(E)$, it holds under $\mathbb{P}_\nu(\cdot|\mathcal{E}^c)$ that, as $t \rightarrow \infty$,

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{(\langle f, X_t \rangle - \sum_{k:2\mathfrak{N}_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t) \langle f, \Psi_k \rangle_m)}{\langle \phi_1, X_t \rangle^{1/2}} \right) \xrightarrow{d} (W^*, G_1(f_{(l)}) + G_3(f_{(s)})),$$

where W^* , $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are the same as those in Theorem 1.16. Since $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are independent,

$$G_1(f_{(l)}) + G_3(f_{(s)}) \sim \mathcal{N}(0, \sigma_{f_{(l)}}^2 + \beta_{f_{(s)}}^2).$$

Thus [26], Theorem 1.10, is a consequence of Theorem 1.16.

If $f_{(c)} \neq 0$, then as $t \rightarrow \infty$,

$$\frac{(\langle f_{(l)} + f_{(s)}, X_t \rangle - \sum_{k:2\mathfrak{N}_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t) \langle f, \Psi_k \rangle_m)}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \xrightarrow{d} 0.$$

Then using the convergence of the first and third components in Theorem 1.16, we get

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{(\langle f, X_t \rangle - \sum_{k:2\mathfrak{N}_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t) \langle f, \Psi_k \rangle_m)}{\sqrt{t^{1+2\tau(f)} \langle \phi_1, X_t \rangle}} \right) \xrightarrow{d} (W^*, G_2(f_{(c)})),$$

where W^* and $G_2(f_{(c)})$ are the same as those in Remark 1.20. Thus [26], Theorem 1.12, is a consequence of Theorem 1.16.

2. Estimates on the moments of X . In the remainder of this paper we will use the following notation: for two positive functions $f(t, x)$ and $g(t, x)$, $f(t, x) \lesssim g(t, x)$ means that there exists a constant $c > 0$ such that $f(t, x) \leq cg(t, x)$ for all t, x .

2.1. *Estimates on the first moment of X .* Recall that \mathcal{M}_k and $\widehat{\mathcal{M}}_k$ are defined in (1.30), and \mathbb{I} is defined in the paragraph below (1.15).

LEMMA 2.1. *For each $k \in \mathbb{I}$, if $a < \mathfrak{R}_{k+1}$, there exists a constant $c(k, a) > 0$ such that for all $t > 0$,*

$$\|T_t|_{(\widehat{\mathcal{M}}_k)^\perp}\|_2 \leq c(k, a)e^{-at} \quad \text{and} \quad \|\widehat{T}_t|_{(\mathcal{M}_k)^\perp}\|_2 \leq c(k, a)e^{-at}.$$

PROOF. Since $(\mathcal{M}_k)^\perp$ is invariant for \widehat{T}_t , $\{\widehat{T}_t|_{(\mathcal{M}_k)^\perp} : t > 0\}$ is a semigroup on $(\mathcal{M}_k)^\perp$. By [8], Theorem 6.7.5, we have

$$\sigma(\widehat{T}_t^*|_{(\mathcal{M}_k)^\perp}) \setminus \{0\} = \{e^{-\bar{\lambda}_j t^*}, k + 1 \leq j \in \mathbb{I}\}.$$

Thus if $k + 1 \in \mathbb{I}$, the spectral radius of $\widehat{T}_t^*|_{(\mathcal{M}_k)^\perp}$ is

$$r(\widehat{T}_t^*|_{(\mathcal{M}_k)^\perp}) = e^{-\mathfrak{R}_{k+1} t^*} < e^{-at^*}.$$

If $k + 1$ does not belong to \mathbb{I} , then $r(\widehat{T}_t^*|_{(\mathcal{M}_k)^\perp}) = 0 < e^{-at^*}$.

By [8], Theorem 6.3.10, $r(\widehat{T}_t^*|_{(\mathcal{M}_k)^\perp}) = \lim_{n \rightarrow \infty} (\|\widehat{T}_{nt^*}^*|_{(\mathcal{M}_k)^\perp}\|_2)^{1/n}$; thus there exists a constant n_1 , such that

$$(2.1) \quad \|\widehat{T}_{n_1 t^*}^*|_{(\mathcal{M}_k)^\perp}\|_2 \leq e^{-an_1 t^*}.$$

By (1.15), we have

$$(2.2) \quad \sup_{0 \leq t \leq n_1 t^*} \|\widehat{T}_t|_{(\mathcal{M}_k)^\perp}\|_2 \leq \sup_{0 \leq t \leq n_1 t^*} \|\widehat{T}_t\|_2 \leq e^{Kn_1 t^*}.$$

For any $t > 0$, there exist $l \in \mathbb{N}$ and $r \in [0, n_1)$, such that $t = n_1 l t^* + r t^*$. By (2.1) and (2.2), we have

$$\begin{aligned} \|\widehat{T}_t|_{(\mathcal{M}_k)^\perp}\|_2 &\leq \|\widehat{T}_{n_1 l t^*}|_{(\mathcal{M}_k)^\perp}\|_2^l \|\widehat{T}_{r t^*}|_{(\mathcal{M}_k)^\perp}\|_2 \leq e^{-an_1 l t^*} e^{Kn_1 t^*} \\ &\leq e^{Kn_1 t^*} \left(\sup_{0 \leq r \leq n_1} e^{ar t^*} \right) e^{-at}. \end{aligned}$$

Thus we can find $c(k, a) > 1$ such that $\|\widehat{T}_t|_{(\mathcal{M}_k)^\perp}\|_2 \leq c(k, a)e^{-at}$. Similarly, we can show that $\|T_t|_{(\widehat{\mathcal{M}}_k)^\perp}\|_2 \leq c(k, a)e^{-at}$. \square

Recall that $\Phi_k(x)$, $D_k(t)$, $\Psi_k(x)$, $b_t(x)$ and $\widehat{b}_t(x)$ are defined in (1.17), (1.18) (1.25) and (1.11), respectively.

LEMMA 2.2. For each $k \in \mathbb{I}$ and $t_1 > 0$, if $a < \mathfrak{R}_{k+1}$, there exists a constant $c(k, a, t_1) > 0$ such that for all $(t, x, y) \in (2t_1, \infty) \times E \times E$,

$$(2.3) \quad \left| q(t, x, y) - \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \right| \leq c e^{-at} b_{t_1}(x)^{1/2} \widehat{b}_{t_1}(y)^{1/2}.$$

PROOF. Recall that for any $f \in L^2(E, m; \mathbb{C})$ and $k \in \mathbb{I}$, \widehat{f}_k is defined in (1.32). Since $|\langle f, \phi_l^{(j)} \rangle_m| \leq \|f\|_2$, we have

$$|\widehat{f}_k(x)| \leq \|f\|_2 \sum_{j=1}^k \sum_{l=1}^{n_j} |\psi_l^{(j)}(x)|.$$

Thus, we get $\|\widehat{f}_k\|_2 \leq c_1(k) \|f\|_2$. By Lemma 2.1, for any $a < \mathfrak{R}_{k+1}$, there exists a constant $c_2 = c_2(k, a) > 0$ such that for all $t > 0$,

$$(2.4) \quad \|\widehat{T}_t(f - \widehat{f}_k)\|_2 \leq c_2 e^{-at} \|f - \widehat{f}_k\|_2 \leq c_3 e^{-at} \|f\|_2,$$

where $c_3 = c_2(1 + c_1(k))$. For $t > t_1$, we have

$$q(t, x, y) = \int_E q(t_1, x, z) q(t - t_1, z, y) m(dz) = \widehat{T}_{t-t_1}(h_x)(y),$$

where $h_x(z) = q(t_1, x, z) \in L^2(E, m)$. Note that

$$\langle h_x, \phi_l^{(j)} \rangle_m = \int_E q(t_1, x, z) \overline{\phi_l^{(j)}(z)} m(dz) = \overline{T_{t_1}(\phi_l^{(j)})(x)}.$$

Let

$$h_{x,k}(z) := \sum_{j=1}^k \sum_{l=1}^{n_j} \langle h_x, \phi_l^{(j)} \rangle_m \psi_l^{(j)}(z) = \sum_{j=1}^k \overline{T_{t_1}((\Phi_j)^T)(x)} \Psi_j(z).$$

By (1.19) and (1.27), we have

$$\begin{aligned} \widehat{T}_{t-t_1}(h_{x,k})(y) &= \sum_{j=1}^k \overline{T_{t_1}(\Phi_j)^T(x)} \widehat{T}_{t-t_1}(\Psi_j)(y) \\ &= \sum_{j=1}^k e^{-\bar{\lambda}_j t} (\overline{\Phi_j(x)})^T D_j(t_1) D_j(t - t_1) \Psi_j(y) \\ &= \sum_{j=1}^k e^{-\bar{\lambda}_j t} (\overline{\Phi_j(x)})^T D_j(t) \Psi_j(y). \end{aligned}$$

Thus, by (2.4), we have

$$\begin{aligned} \int_E \left| q(t, x, y) - \sum_{j=1}^k e^{-\bar{\lambda}_j t} (\overline{\Phi_j(x)})^T D_j(t) \Psi_j(y) \right|^2 m(dy) \\ \leq (c_3)^2 e^{-2a(t-t_1)} \|h_x\|_2^2 = c_4 e^{-2at} b_{t_1}(x), \end{aligned}$$

where $c_4 = c_4(k, a, t_1) = c_3^2 e^{-2at_1}$. Since $q(t, x, y)$ is a real-valued function, we have, for $t > t_1$,

$$(2.5) \quad \int_E \left| q(t, x, y) - \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \right|^2 m(dy) \leq c_4 e^{-2at} b_{t_1}(x).$$

Repeating the above argument with T_t , we get that there exists $c_5 = c_5(k, a, t_1) > 0$ such that for $t > t_1$,

$$(2.6) \quad \int_E \left| q(t, z, y) - \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(z))^T D_j(t) \overline{\Psi_j(y)} \right|^2 m(dz) \leq c_5 e^{-2at} \widehat{b}_{t_1}(y).$$

Since $D_j(t) = D_j(t/2)D_j(t/2)$, we get

$$(2.7) \quad \begin{aligned} & e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \\ &= e^{-\lambda_j t/2} \int_E q(t/2, x, z) (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} m(dz) \end{aligned}$$

and

$$(2.8) \quad \begin{aligned} & e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \\ &= e^{-\lambda_j t/2} \int_E q(t/2, z, y) (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} m(dz). \end{aligned}$$

Thus by (1.26), we have

$$(2.9) \quad \begin{aligned} & \int_E \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} \right) \\ & \quad \times \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} \right) m(dz) \\ &= \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t/2) D_j(t/2) \overline{\Psi_j(y)} \\ &= \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)}. \end{aligned}$$

Thus, by the semigroup property of T_t and (2.7)–(2.9), we obtain

$$\begin{aligned} & q(t, x, y) - \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \\ &= \int_E q(t/2, x, z) q(t/2, z, y) m(dz) \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=1}^k e^{-\lambda_j t/2} \int_E q(t/2, x, z) (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} m(dz) \\
 & - \sum_{j=1}^k e^{-\lambda_j t/2} \int_E q(t/2, z, y) (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} m(dz) \\
 & + \int_E \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} \right) \\
 & \times \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} \right) m(dz) \\
 & = \int_E \left(q(t/2, x, z) - \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(x))^T D_j(t/2) \overline{\Psi_j(z)} \right) \right) \\
 & \times \left(q(t/2, z, y) - \left(\sum_{j=1}^k e^{-\lambda_j t/2} (\Phi_j(z))^T D_j(t/2) \overline{\Psi_j(y)} \right) \right) m(dz).
 \end{aligned}$$

Therefore, by Hölder’s inequality, (2.5) and (2.6), we get, for $t > 2t_1$,

$$\begin{aligned}
 & \left| q(t, x, y) - \sum_{j=1}^k e^{-\lambda_j t} (\Phi_j(x))^T D_j(t) \overline{\Psi_j(y)} \right| \\
 & \leq \sqrt{c_4 c_5} e^{-at} b_{t_1}(x)^{1/2} \widehat{b}_{t_1}(y)^{1/2}. \quad \square
 \end{aligned}$$

Recall that $\gamma(f)$, $\zeta(f)$, $\tau(f)$, $F_{f,j}$ and $b_t(x)$ are defined in (1.33), (1.34), (1.35), (1.36) and (1.11), respectively.

COROLLARY 2.3. *Assume $f \in L^2(E, m; \mathbb{C})$. If $\gamma(f) < \infty$, then, for any $t_1 > 0$, there exists a constant $c(f, t_1) > 0$ such that for all $(t, x) \in (2t_1, \infty) \times E$,*

$$\begin{aligned}
 (2.10) \quad & \left| t^{-\tau(f)} e^{\Re \gamma(f)t} T_t f(x) - \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i \Im \gamma_j t} (\Phi_j(x))^T F_{f,j} \right| \\
 & \leq c(f, t_1) t^{-1} b_{t_1}(x)^{1/2}.
 \end{aligned}$$

Moreover, we have, for $(t, x) \in (2t_1, \infty) \times E$,

$$(2.11) \quad |T_t f(x)| \lesssim t^{\tau(f)} e^{-\Re \gamma(f)t} b_{t_1}(x)^{1/2}.$$

If $\gamma(f) = \infty$, for any $t_1 > 0$, we have, for $(t, x) \in (2t_1, \infty) \times E$,

$$(2.12) \quad |T_t f(x)| \lesssim b_{t_1}(x)^{1/2}.$$

PROOF. First, we consider the case $\gamma(f) < \infty$, which implies $\gamma(f) \in \mathbb{I}$. By the definition of $\zeta(f)$, we have $\mathfrak{R}_{\gamma(f)} < \mathfrak{R}_{\zeta(f)+1}$. Since $\langle f, \widehat{b}_{t_1}^{1/2} \rangle_m \leq \|\widehat{b}_{t_1}^{1/2}\|_2 \|f\|_2$, applying Lemma 2.2 with $k = \zeta(f)$ and a fixed a with $\mathfrak{R}_{\gamma(f)} < a < \mathfrak{R}_{\zeta(f)+1}$, we get that there exists $c_1 = c_1(f, t_1) > 0$ such that for $(t, x) \in (2t_1, \infty) \times E$,

$$(2.13) \quad \left| T_t f(x) - e^{-\mathfrak{R}_{\gamma(f)}t} \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\mathfrak{S}_j t} (\Phi_j(x))^T D_j(t) \langle f, \Psi_j \rangle_m \right| \leq c_1 e^{-at} b_{t_1}(x)^{1/2}.$$

Recall that $\Phi_k(x)$, $D_k(t)$ and $\Psi_k(x)$ are defined in (1.17), (1.18) and (1.25), respectively. If $\tau(f) \geq 1$, the degree of each component of $D_j(t) \langle f, \Psi_j \rangle_m - t^{\tau(f)} F_{f,j}$ is no larger than $\tau(f) - 1$. Thus, for $t > 2t_1$,

$$|D_j(t) \langle f, \Psi_j \rangle_m - t^{\tau(f)} F_{f,j}|_\infty \lesssim t^{\tau(f)-1},$$

which implies that

$$(2.14) \quad |t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle_m - F_{f,j}|_\infty \lesssim t^{-1}.$$

If $\tau(f) = 0$, $D_j(t) \langle f, \Psi_j \rangle_m - t^{\tau(f)} F_{f,j} = 0$. By (1.20), we get, for $(t, x) \in (2t_1, \infty) \times E$,

$$(2.15) \quad \left| \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\mathfrak{S}_j t} (\Phi_j(x))^T D_j(t) \langle f, \Psi_j \rangle_m - t^{\tau(f)} \sum_{j=\gamma(f)}^{\zeta(f)} e^{-i\mathfrak{S}_j t} (\Phi_j(x))^T F_{f,j} \right| \lesssim t^{\tau(f)-1} |\Phi_j(x)|_\infty \lesssim t^{\tau(f)-1} b_{t_1}(x)^{1/2}.$$

Now (2.10) follows from (2.13) and (2.15). By (2.10) and (1.20), we get (2.11) immediately.

Now we deal with the case $\gamma(f) = \infty$. Let $k_0 := \sup\{j : \mathfrak{R}_j \leq 0\}$. Thus we have $k_0 \in \mathbb{I}$ and $\mathfrak{R}_{k_0+1} > 0$. Since $\gamma(f) = \infty$, we have $\langle f, \Psi_k \rangle_m = 0$ for any $k \in \mathbb{I}$. Now, applying Lemma 2.2 with $k = k_0$ and $a = 0$, we get (2.12) immediately. \square

REMARK 2.4. Since $D_1(t) \equiv 1$, using (2.3) with $k = 1$ and $\lambda_1 < a < \mathfrak{R}_2$, we get that, for any $t_1 > 0$, there exists $c_1(t_1, a) > 0$ such that for any $f \in L^2(E, m)$ and $(t, x) \in (2t_1, \infty) \times E$,

$$(2.16) \quad |e^{\lambda_1 t} T_t f(x) - \langle f, \psi_1 \rangle_m \phi_1(x)| \leq c_1(t_1, a) e^{-(a-\lambda_1)t} \|f\|_2 b_{t_1}(x)^{1/2},$$

and hence there exists $c_2(t_1, a) > 0$ such that

$$(2.17) \quad e^{\lambda_1 t} |T_t f(x)| \leq c_2 \|f\|_2 b_{t_1}(x)^{1/2}.$$

2.2. *Estimates on the second moment of X.* We first recall the formula for the second moment of the branching Markov process $\{X_t : t \geq 0\}$ for $f \in \mathcal{B}_b(E)$, we have for any $(t, x) \in (0, \infty) \times E$,

$$(2.18) \quad \mathbb{P}_{\delta_x} \langle f, X_t \rangle^2 = \int_0^t T_s [A |T_{t-s} f|^2](x) ds + T_t (f^2)(x),$$

where the function $A(\cdot)$ is defined in (1.8). The second moment formula above was proved in [30], Lemma 3.3, for branching symmetric stable processes, but the argument there works for general branching nonsymmetric Markov processes. For any $f \in L^2(E, m) \cap L^4(E, m)$ and $x \in E$, since $(T_{t-s} f)^2(x) \leq e^{K(t-s)} T_{t-s} (f^2)(x)$, we have by (1.9),

$$\int_0^t T_s [A(T_{t-s} f)^2](x) ds \leq e^{Kt} T_t (f^2)(x) < \infty,$$

which implies

$$(2.19) \quad \int_0^t T_s [A(T_{t-s} f)^2](x) ds + T_t (f^2)(x) \leq (1 + e^{Kt}) T_t (f^2)(x) < \infty.$$

Thus, using a routine limit argument, one can check that (2.18) also holds for $f \in L^2(E, m) \cap L^4(E, m)$. Thus, for $f \in L^2(E, m; \mathbb{C}) \cap L^4(E, m; \mathbb{C})$, we have

$$(2.20) \quad \begin{aligned} \mathbb{P}_{\delta_x} |\langle f, X_t \rangle|^2 &= \mathbb{P}_{\delta_x} \langle \Re(f), X_t \rangle^2 + \mathbb{P}_{\delta_x} \langle \Im(f), X_t \rangle^2 \\ &= \int_0^t T_s [A |T_{t-s} f|^2](x) ds + T_t (|f|^2)(x). \end{aligned}$$

Let Var_ν be the variance under \mathbb{P}_ν . Then by the branching property, we have $\text{Var}_\nu \langle f, X_t \rangle = \langle \text{Var}_\delta \langle f, X_t \rangle, \nu \rangle$. By (2.19), (2.17) and properties (i) and (ii) at the end of Section 1.2, we get that there exists a constant $c = c(t_0)$ such that for $t > 2t_0$,

$$\begin{aligned} \text{Var}_{\delta_x} \langle f, X_t \rangle &\leq \mathbb{P}_{\delta_x} |\langle f, X_t \rangle|^2 \leq (1 + e^{Kt}) T_t (|f|^2)(x) \\ &\leq c(1 + e^{Kt}) e^{-\lambda_1 t} b_t(x)^{1/2} \| |f|^2 \|_2 \in L^2(E, m) \cap L^4(E, m). \end{aligned}$$

Recall that t_0 is the constant in Assumption 1(c), and that b_t and $\gamma(f)$ are defined in (1.11) and (1.33), respectively.

LEMMA 2.5. *Assume that $f \in L^2(E, m; \mathbb{C}) \cap L^4(E, m; \mathbb{C})$. If $\lambda_1 > 2\Re \gamma(f)$, then for any $(t, x) \in (10t_0, \infty) \times E$, we have*

$$(2.21) \quad \sup_{t > 10t_0} t^{-2\tau(f)} e^{2\Re \gamma(f)t} \mathbb{P}_{\delta_x} |\langle f, X_t \rangle|^2 \lesssim b_{t_0}(x)^{1/2}.$$

PROOF. In this proof, we always assume $t > 10t_0$. For $s \leq 2t_0$, we have $T_{t-s} [A |T_s f|^2](x) \leq K e^{Ks} T_t (|f|^2)(x) \lesssim T_t (|f|^2)(x)$. Thus, by (2.11), we have for $t > 10t_0$,

$$(2.22) \quad \int_0^{2t_0} T_{t-s} [A |T_s f|^2](x) ds \lesssim T_t (|f|^2)(x) \lesssim e^{-\lambda_1 t} b_{t_0}(x)^{1/2},$$

where the function $A(\cdot)$ is defined in (1.8). It follows from (2.11) again that for $(s, x) \in (8t_0, \infty) \times E$, $|T_s f(x)| \lesssim s^{\tau(f)} e^{-2\Re_\gamma(f)s} b_{4t_0}(x)^{1/2}$. Thus, for $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned}
 \int_{t-2t_0}^t T_{t-s}[A|T_s f|^2](x) ds &\lesssim t^{2\tau(f)} \int_{t-2t_0}^t e^{-2\Re_\gamma(f)s} T_{t-s}(b_{4t_0})(x) ds \\
 (2.23) \qquad \qquad \qquad &= t^{2\tau(f)} e^{-2\Re_\gamma(f)t} \int_0^{2t_0} e^{2\Re_\gamma(f)s} T_s(b_{4t_0})(x) ds \\
 &\lesssim t^{2\tau(f)} e^{-2\Re_\gamma(f)t} \int_0^{2t_0} T_s(b_{4t_0})(x) ds.
 \end{aligned}$$

We now show that for any $x \in E$, $\int_0^{2t_0} T_s(b_{4t_0})(x) ds < \infty$. By (1.3), we get

$$b_{4t_0}(x) \leq e^{8Kt_0} a_{4t_0}(x) \leq e^{10Kt_0} T_{2t_0}(a_{2t_0})(x).$$

Thus, by (2.17), we have

$$\begin{aligned}
 \int_0^{2t_0} T_s(b_{4t_0})(x) ds &\leq e^{10Kt_0} \int_0^{2t_0} T_{s+2t_0}(a_{2t_0})(x) ds \\
 (2.24) \qquad \qquad \qquad &\lesssim \int_0^{2t_0} e^{-\lambda_1(s+2t_0)} ds b_{t_0}(x)^{1/2} \lesssim b_{t_0}(x)^{1/2}.
 \end{aligned}$$

By (2.23)–(2.24), we get

$$(2.25) \qquad \int_{t-2t_0}^t T_{t-s}[A|T_s f|^2](x) ds \lesssim t^{2\tau(f)} e^{-2\Re_\gamma(f)t} b_{t_0}(x)^{1/2}.$$

For $s \in [2t_0, t - 2t_0]$, by (2.11), we have $|T_s f(x)|^2 \lesssim s^{2\tau(f)} e^{-2\Re_\gamma(f)s} b_{t_0}(x)$. By (2.17), we get $T_{t-s}[A(T_s f)^2](x) \lesssim s^{2\tau(f)} e^{-2\Re_\gamma(f)s} e^{-\lambda_1(t-s)} b_{t_0}(x)^{1/2}$. So, for $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned}
 \int_{2t_0}^{t-2t_0} T_{t-s}[A|T_s f|^2](x) ds \\
 (2.26) \qquad \qquad \qquad &\lesssim t^{2\tau(f)} e^{-\lambda_1 t} \int_0^t e^{(\lambda_1 - 2\Re_\gamma(f))s} ds b_{t_0}(x)^{1/2} \\
 &\lesssim t^{2\tau(f)} e^{-2\Re_\gamma(f)t} b_{t_0}(x)^{1/2}.
 \end{aligned}$$

Combining (2.22), (2.25) and (2.26), when $\lambda_1 > 2\Re_\gamma(f)$, we get

$$\int_0^t T_{t-s}[A|T_s f|^2](x) ds \lesssim t^{2\tau(f)} e^{-2\Re_\gamma(f)t} b_{t_0}(x)^{1/2}.$$

Since $\lambda_1 > 2\Re_\gamma(f)$, by (2.17), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$T_t(|f|^2)(x) \lesssim e^{-\lambda_1 t} b_{t_0}(x)^{1/2} \lesssim t^{2\tau(f)} e^{-2\Re_\gamma(f)t} b_{t_0}(x)^{1/2}.$$

Now combining the two displays above with (2.18), we arrive at (2.21). \square

Recall that t_0 is the constant in Assumption 1(c), that ϕ_1 and ψ_1 are the eigenfunctions associated with $-\lambda_1$ of \mathcal{A} and $\widehat{\mathcal{A}}$, respectively, and that b_t and σ_f^2 are defined in (1.11) and (1.41), respectively.

LEMMA 2.6. *Assume that $f \in L^2(E, m) \cap L^4(E, m)$. If $\lambda_1 < 2\Re_\gamma(f)$, then for $(t, x) \in (10t_0, \infty) \times E$,*

$$(2.27) \quad |e^{\lambda_1 t} \mathbb{V}ar_{\delta_x} \langle f, X_t \rangle - \sigma_f^2 \phi_1(x)| \lesssim c_t (b_{t_0}(x))^{1/2} + b_{t_0}(x),$$

where c_t is independent of x with $\lim_{t \rightarrow \infty} c_t = 0$ and σ_f^2 is defined in (1.41).

PROOF. First, we consider the case $\gamma(f) < \infty$. In this proof, we always assume $t > 10t_0$ and $f \in L^2(E, m) \cap L^4(E, m)$. By (2.11), we have

$$(2.28) \quad e^{\lambda_1 t/2} |\mathbb{P}_{\delta_x} \langle f, X_t \rangle| \lesssim t^{\tau(f)} e^{-(2\Re_\gamma(f) - \lambda_1)t/2} b_{t_0}(x)^{1/2}.$$

We first show that $\sigma_f^2 < \infty$. For $s \leq 2t_0$, by (1.13), we have

$$(2.29) \quad \|A|T_s f|^2\|_2 \leq K \|T_s f\|_4^2 \leq K e^{2Ks} \|f\|_4^2,$$

where the function $A(\cdot)$ is defined in (1.8). For $s > 2t_0$, by (2.11), $|T_s f(x)| \lesssim e^{-\Re_\gamma(f)s} s^{\tau(f)} b_{t_0}(x)^{1/2}$. Thus, we have

$$(2.30) \quad \begin{aligned} \int_0^\infty e^{\lambda_1 s} \langle A|T_s f|^2, \psi_1 \rangle_m ds &\leq K \|\psi_1\|_2 \int_0^\infty e^{\lambda_1 s} \| |T_s f|^2 \|_2 ds \\ &\lesssim \int_0^{2t_0} e^{\lambda_1 s} ds + \int_{2t_0}^\infty e^{(\lambda_1 - 2\Re_\gamma(f))s} s^{2\tau(f)} ds \\ &< \infty. \end{aligned}$$

Combining this with (1.41) we get that $\sigma_f^2 < \infty$. By (2.20), we have

$$\begin{aligned} &|e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle f, X_t \rangle^2 - \sigma_f^2 \phi_1(x)| \\ &\leq e^{\lambda_1 t} \int_0^{t-2t_0} |T_{t-s}[A|T_s f|^2](x) - e^{-\lambda_1(t-s)} \langle A|T_s f|^2, \psi_1 \rangle_m \phi_1(x)| ds \\ &\quad + e^{\lambda_1 t} \int_{t-2t_0}^t T_{t-s}[A|T_s f|^2](x) ds + \int_{t-2t_0}^\infty e^{\lambda_1 s} \langle A|T_s f|^2, \psi_1 \rangle_m ds \phi_1(x) \\ &\quad + |e^{\lambda_1 t} T_t(|f|^2)(x) - \langle |f|^2, \psi_1 \rangle_m \phi_1(x)| \\ &=: V_1(t, x) + V_2(t, x) + V_3(t, x) + V_4(t, x). \end{aligned}$$

First, we consider $V_1(t, x)$. By (2.16), for $t - s > 2t_0$, there exists $a \in (\lambda_1, \Re_2)$ such that

$$\begin{aligned} &|T_{t-s}[A|T_s f|^2](x) - e^{-\lambda_1(t-s)} \langle A|T_s f|^2, \psi_1 \rangle_m \phi_1(x)| \\ &\lesssim e^{-a(t-s)} \|A(T_s f)^2\|_2 b_{t_0}(x)^{1/2}. \end{aligned}$$

Therefore, by (2.11) and (2.29), we have

$$\begin{aligned}
 V_1(t, x) &\lesssim e^{\lambda_1 t} t^{2\tau(f)} \int_{2t_0}^{t-2t_0} e^{-a(t-s)} e^{-2\Re_{\gamma(f)} s} ds b_{t_0}(x)^{1/2} \\
 &\quad + e^{\lambda_1 t} \int_0^{2t_0} e^{-a(t-s)} ds b_{t_0}(x)^{1/2} \\
 (2.31) \quad &\lesssim e^{-(a-\lambda_1)t} t^{2\tau(f)} \int_0^t e^{(a-2\Re_{\gamma(f)})s} ds b_{t_0}(x)^{1/2} + e^{-(a-\lambda_1)t} b_{t_0}(x)^{1/2} \\
 &\lesssim t^{2\tau(f)} (e^{(\lambda_1-2\Re_{\gamma(f)})t} + e^{-(a-\lambda_1)t}) b_{t_0}(x)^{1/2}.
 \end{aligned}$$

Now we deal with $V_2(t, x)$. By (2.25), we have

$$(2.32) \quad V_2(t, x) \lesssim t^{2\tau(f)} e^{(\lambda_1-2\Re_{\gamma(f)})t} b_{t_0}(x)^{1/2}.$$

For $V_3(t, x)$, by (2.30), we get $\int_{t-2t_0}^\infty e^{\lambda_1 s} \langle A|T_s f|^2, \psi_1 \rangle_m ds \rightarrow 0$, as $t \rightarrow \infty$. By (1.20), we have $\phi_1(x) \lesssim b_{t_0}(x)^{1/2}$.

Finally, we consider $V_4(t, x)$. By (2.16), we have

$$(2.33) \quad V_4(t, x) \lesssim e^{-(a-\lambda_1)t} b_{t_0}(x)^{1/2}.$$

Thus, by (2.31)–(2.33), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$(2.34) \quad |e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle f, X_t \rangle^2 - \sigma_f^2 \phi_1(x)| \lesssim c_t b_{t_0}(x)^{1/2},$$

with $\lim_{t \rightarrow \infty} c_t = 0$. Now (2.27) follows immediately from (2.28) and (2.34).

Now we consider the case $\gamma(f) = \infty$. The proof is similar to that of the case $\gamma(f) < \infty$, the only difference being that we now use (2.12) instead of (2.11). \square

Recall that t_0 is the constant in Assumption 1(c), that ϕ_1 and ψ_1 are the eigenfunctions associated with $-\lambda_1$ of \mathcal{A} and $\widehat{\mathcal{A}}$, respectively, and that b_t , $\gamma(f)$ and $\tau(f)$ are defined in (1.11), (1.33) and (1.35), respectively.

LEMMA 2.7. Assume that $f, h \in L^2(E, m) \cap L^4(E, m)$. If $\lambda_1 = 2\Re_{\gamma(f)} = 2\Re_{\gamma(h)}$, then for $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned}
 (2.35) \quad &|t^{-(1+\tau(f)+\tau(h))} e^{\lambda_1 t} \text{Cov}_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) - \rho(f, h) \phi_1(x)| \\
 &\lesssim t^{-1} (b_{t_0}(x)^{1/2} + b_{t_0}(x)),
 \end{aligned}$$

where Cov_{δ_x} is the covariance under \mathbb{P}_{δ_x} , and $\rho(f, h)$ is defined by (1.45) with f and h in place of h_1 and h_2 , respectively. In particular, we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$(2.36) \quad |t^{-(1+2\tau(f))} e^{\lambda_1 t} \text{Var}_{\delta_x} \langle f, X_t \rangle - \rho_f^2 \phi_1(x)| \lesssim t^{-1} (b_{t_0}(x)^{1/2} + b_{t_0}(x)),$$

where ρ_f^2 is defined by (1.42). Moreover, we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$(2.37) \quad t^{-(1+2\tau(f))} e^{\lambda_1 t} \text{Var}_{\delta_x} \langle f, X_t \rangle \lesssim (b_{t_0}(x)^{1/2} + b_{t_0}(x)).$$

PROOF. In this proof we always assume $t > 10t_0$ and $f, h \in L^2(E, m) \cap L^4(E, m)$. By (2.20), we have

$$\begin{aligned}
 & \mathbb{C}ov_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) \\
 (2.38) \quad &= \frac{1}{4}(\mathbb{V}ar_{\delta_x}(\langle f+h, X_t \rangle) - \mathbb{V}ar_{\delta_x}(\langle f-h, X_t \rangle)) \\
 &= \int_0^t T_{t-s}[A(T_s f)(T_s h)](x) ds + T_t(fh)(x) - T_t(f)(x)T_t(h)(x).
 \end{aligned}$$

Let

$$C_f(s, x) := \sum_{j:\lambda_1=2\mathfrak{N}_j} (e^{-i\mathfrak{S}_j s}(\Phi_j(x)))^T F_{f,j}$$

and

$$C_h(s, x) := \sum_{j:\lambda_1=2\mathfrak{N}_j} (e^{-i\mathfrak{S}_j s}(\Phi_j(x)))^T F_{h,j},$$

where $\Phi_j(x)$ and $F_{f,j}$ are defined in (1.17) and (1.36), respectively. Define

$$\begin{aligned}
 V_5(t, x) &:= e^{\lambda_1 t} \int_{2t_0}^{t-2t_0} T_{t-s}[A(T_s f)(T_s h)](x) ds, \\
 V_6(t, x) &:= e^{\lambda_1 t} \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} e^{-\lambda_1 s} T_{t-s}[AC_f(s, \cdot)C_h(s, \cdot)](x) ds, \\
 V_7(t, x) &:= \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} \langle AC_f(s, \cdot)C_h(s, \cdot), \psi_1 \rangle_m ds \phi_1(x)
 \end{aligned}$$

and

$$V_8(t, x) := \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} \langle AF_{f,h}, \psi_1 \rangle_m ds \phi_1(x),$$

where A is defined in (1.8), and $F_{f,h}$ is defined in (1.46) with f and h in place of h_1 and h_2 , respectively. By the definition of $\rho(f, h)$ we have that

$$\rho(f, h) = t^{-(1+\tau(f)+\tau(h))} \int_0^t s^{\tau(f)+\tau(h)} \langle AF_{f,h}, \psi_1 \rangle_m ds.$$

Thus we have

$$\begin{aligned}
 & \left| e^{\lambda_1 t} \int_0^t T_{t-s}[A(T_s f)(T_s h)](x) ds - t^{1+\tau(f)+\tau(h)} \rho(f, h) \phi_1(x) \right| \\
 & \leq e^{\lambda_1 t} \left(\int_0^{2t_0} + \int_{t-2t_0}^t \right) T_{t-s}[A|T_s f||T_s h|](x) ds \\
 & \quad + |V_5(t, x) - V_6(t, x)| + |V_6(t, x) - V_7(t, x)| + |V_7(t, x) - V_8(t, x)| \\
 & \quad + \left(\int_0^{2t_0} + \int_{t-2t_0}^t \right) s^{\tau(f)+\tau(h)} ds \langle AF_{f,h}, \psi_1 \rangle_m \phi_1(x).
 \end{aligned}$$

By (2.17), for $s \leq t - 2t_0$, we have

$$T_{t-s}[A|T_s f||T_s h|](x) \lesssim e^{-\lambda_1(t-s)} \|A|T_s f||T_s h|\|_2(b_{t_0}(x))^{1/2}.$$

It follows from (1.13) and the Cauchy–Schwarz inequality that

$$\|A|T_s f||T_s h|\|_2 \leq K \|T_s f\|_4 \|T_s h\|_4 \leq K e^{2Ks} \|f\|_4 \|h\|_4.$$

Thus

$$(2.39) \quad \begin{aligned} e^{\lambda_1 t} \int_0^{2t_0} T_{t-s}[A|T_s f||T_s h|](x) ds &\lesssim \int_0^{2t_0} e^{\lambda_1 s} ds (b_{t_0}(x))^{1/2} \\ &\lesssim (b_{t_0}(x))^{1/2}. \end{aligned}$$

For $s > t - 2t_0$, using arguments similar to those leading to (2.25), we get

$$(2.40) \quad e^{\lambda_1 t} \int_{t-2t_0}^t T_{t-s}[A|T_s f||T_s h|](x) ds$$

$$(2.41) \quad \begin{aligned} &\lesssim t^{\tau(f)+\tau(h)} e^{\lambda_1 t} e^{-(\Re_\gamma(h)+\Re_\gamma(f))t} (b_{t_0}(x))^{1/2} \\ &= t^{\tau(f)+\tau(h)} (b_{t_0}(x))^{1/2}. \end{aligned}$$

It follows from (1.20) that

$$(2.42) \quad \begin{aligned} &\left(\int_0^{2t_0} + \int_{t-2t_0}^t \right) s^{\tau(f)+\tau(h)} ds \langle AF_{f,h}, \psi_1 \rangle_m \phi_1(x) \\ &\lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}. \end{aligned}$$

Next we consider $|V_5(t, x) - V_6(t, x)|$. By (2.10), we have, for $(s, x) \in (2t_0, \infty) \times E$,

$$\begin{aligned} &|T_s f(x) - s^{\tau(f)} e^{-\lambda_1 s/2} C_f(s, x)| \\ &\lesssim s^{\tau(f)-1} e^{-\lambda_1 s/2} b_{t_0}(x)^{1/2}. \end{aligned}$$

The same is also true for h . Thus by (2.11) and (1.20), we get that, for $(s, x) \in (2t_0, \infty) \times E$,

$$\begin{aligned} &||T_s f(x)T_s h(x)| - s^{\tau(f)+\tau(h)} e^{-\lambda_1 s} C_f(s, x)C_h(s, x)| \\ &\lesssim |T_s f(x) - s^{\tau(f)} e^{-\lambda_1 s/2} C_f(s, x)| |T_s h(x) - s^{\tau(h)} e^{-\lambda_1 s/2} C_h(s, x)| \\ &\quad + s^{\tau(h)} e^{-\lambda_1 s/2} |T_s f(x) - s^{\tau(f)} e^{-\lambda_1 s/2} C_f(s, x)| |C_h(s, x)| \\ &\quad + s^{\tau(f)} e^{-\lambda_1 s/2} |T_s h(x) - s^{\tau(h)} e^{-\lambda_1 s/2} C_h(s, x)| |C_f(s, x)| \\ &\lesssim s^{\tau(f)+\tau(h)-1} e^{-\lambda_1 s} b_{t_0}(x). \end{aligned}$$

Therefore, by (2.17), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned}
 & |V_5(t, x) - V_6(t, x)| \\
 (2.43) \quad & \lesssim \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)-1} e^{\lambda_1(t-s)} T_{t-s}(b_{t_0})(x) ds \\
 & \lesssim \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)-1} ds b_{t_0}(x)^{1/2} \lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}.
 \end{aligned}$$

For $|V_6(t, x) - V_7(t, x)|$, by (2.16), there exists $\lambda_1 < a < \Re_2$, such that, for $t - s > 2t_0$,

$$\begin{aligned}
 & |e^{\lambda_1(t-s)} T_{t-s}[AC_f(s, \cdot)C_h(s, \cdot)](x) - \langle AC_f(s, \cdot)C_h(s, \cdot), \psi_1 \rangle_m \phi_1(x)| \\
 & \lesssim e^{-(a-\lambda_1)(t-s)} \|C_f(s, \cdot)C_h(s, \cdot)\|_2 b_{t_0}(x)^{1/2}.
 \end{aligned}$$

By (1.20), we get, for $s > 2t_0$, $|C_f(s, x)C_h(s, x)| \lesssim b_{t_0}(x)$. Thus, we get

$$\begin{aligned}
 & |V_6(t, x) - V_7(t, x)| \lesssim \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} e^{-(a-\lambda_1)(t-s)} ds b_{t_0}(x)^{1/2} \\
 (2.44) \quad & \lesssim t^{\tau(f)+\tau(h)} \int_{2t_0}^{t-2t_0} e^{-(a-\lambda_1)(t-s)} ds b_{t_0}(x)^{1/2} \\
 & \lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}.
 \end{aligned}$$

Now we deal with $|V_7(t, x) - V_8(t, x)|$. We can check that $C_h(s, x)$ is real. In fact, for each j with $\lambda_1 = 2\Re_j$, we also have $\lambda_1 = 2\Re_j$ and

$$e^{-i\Im_j s} (\Phi_{j'}(x))^T F_{h,j'} = \overline{e^{-i\Im_j s} (\Phi_j(x))^T F_{h,j}}.$$

Thus we have

$$C_h(s, x) = \overline{C_h(s, x)} = \sum_{j:\lambda_1=2\Re_j} (e^{i\Im_j s} (\Phi_j(x))^T F_{h,j}).$$

Therefore,

$$\begin{aligned}
 C_f(s, x)C_h(s, x) &= \sum_{j:\lambda_1=2\Re_j} (\Phi_j(x))^T F_{f,j} \overline{(\Phi_j(x))^T F_{h,j}} \\
 &+ \sum_{\gamma(f) \leq j \neq l \leq \zeta(f)} (e^{-i(\Im_j - \Im_l)s} (\Phi_j(x))^T F_{f,j} \overline{(\Phi_l(x))^T F_{h,l}}).
 \end{aligned}$$

When $j \neq l$, since $\lambda_j \neq \lambda_l$ and $\Re_j = \Re_l$, we have $\Im_j \neq \Im_l$.

We claim that for any nonzero $\theta \in \mathbb{R}$ and $n \geq 0$, we have for $t > 2t_0$,

$$(2.45) \quad \left| \int_{2t_0}^{t-2t_0} s^n e^{i\theta s} ds \right| \lesssim t^n.$$

Then, using (1.46), we get

$$\begin{aligned}
 &|V_7(t, x) - V_8(t, x)| \\
 &\lesssim \sum_{\gamma(f) \leq j \neq l \leq \zeta(f)} \left| \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} e^{-i(\mathfrak{S}_j - \mathfrak{S}_l)s} ds \right| \\
 &\quad \times \left| \langle (\Phi_j(x))^T F_{f,j} (\overline{\Phi_l(x)})^T \overline{F}_{h,l}, \psi_1 \rangle_m \right| \phi_1(x) \\
 &\lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}.
 \end{aligned}$$

Now we prove (2.45). Using integration by parts, for $n \geq 1$, we get

$$\begin{aligned}
 \left| \int_{2t_0}^{t-2t_0} s^n e^{i\theta s} ds \right| &= \left| \frac{s^n e^{i\theta s} \Big|_{2t_0}^{t-2t_0} - \int_{2t_0}^{t-2t_0} n s^{n-1} e^{i\theta s} ds}{i\theta} \right| \\
 &\lesssim t^n + \int_{2t_0}^{t-2t_0} s^{n-1} ds \lesssim t^n.
 \end{aligned}$$

For $n = 0$, we have

$$\left| \int_{2t_0}^{t-2t_0} e^{i\theta s} ds \right| = \left| \frac{e^{i\theta(t-2t_0)} - e^{i2\theta t_0}}{i\theta} \right| \leq 2/|\theta|.$$

Therefore, (2.45) follows immediately.

Combining (2.39), (2.40), (2.42), (2.43), (2.44) and (2.46), we get $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned}
 &\left| e^{\lambda_1 t} \int_0^t T_{t-s} [A(T_s f)(T_s h)](x) ds - t^{1+\tau(f)+\tau(h)} \rho(f, h) \phi_1(x) \right| \\
 &\lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x)^{1/2}.
 \end{aligned}$$

By (2.17), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$e^{\lambda_1 t} T_t(|fh|)(x) \lesssim b_{t_0}(x)^{1/2}.$$

And by (2.16) and $\lambda_1 = 2\Re_{\gamma(f)}$,

$$e^{\lambda_1 t} |T_t f(x)| |T_t h(x)| \lesssim t^{\tau(f)+\tau(h)} b_{t_0}(x).$$

Now (2.35) follows immediately. \square

Recall that t_0 is the constant in Assumption 1(c), and that b_t and $\gamma(f)$ are defined in (1.11) and (1.33), respectively.

LEMMA 2.8. Assume that $f \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 < 2\Re_{\gamma(f)}$ and $h \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 = 2\Re_{\gamma(h)}$. Then, for any $(t, x) \in (10t_0, \infty) \times E$,

$$(2.46) \quad e^{\lambda_1 t} \text{Cov}_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) \lesssim ((b_{t_0}(x))^{1/2} + b_{t_0}(x)).$$

PROOF. In this proof, we always assume that $t > 10t_0$, $f \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 < 2\mathfrak{R}\gamma(f)$ and $h \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 = 2\mathfrak{R}\gamma(h)$. First, we assume $\gamma(f) < \infty$. By (2.38), we have

$$\begin{aligned} & \text{Cov}_{\delta_x}(\langle f, X_t \rangle, \langle h, X_t \rangle) \\ &= \int_0^t T_{t-s}[A(T_s f)(T_s h)](x) ds + T_t(fh)(x) - T_t(f)(x)T_t(h)(x), \end{aligned}$$

where the function $A(\cdot)$ is defined in (1.8). By (2.39) and (2.41), we have, for $(t, x) \in (10t_0, \infty) \times E$,

$$\begin{aligned} & e^{\lambda_1 t} \left(\int_0^{2t_0} + \int_{t-2t_0}^t \right) T_{t-s}[A|T_s f||T_s h|](x) ds \\ & \lesssim b_{t_0}(x)^{1/2} + t^{\tau(f)+\tau(h)} e^{(\lambda_1/2-\mathfrak{R}\gamma(f))t} (b_{t_0}(x))^{1/2} \lesssim (b_{t_0}(x))^{1/2}. \end{aligned}$$

By (2.11), we have

$$\begin{aligned} & e^{\lambda_1 t} \int_{2t_0}^{t-2t_0} T_{t-s}[A|T_s f||T_s h|](x) ds \\ & \lesssim e^{\lambda_1 t} \int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} e^{-(\lambda_1/2+\mathfrak{R}\gamma(f))s} T_{t-s}(b_{t_0})(x) ds \\ & \lesssim \left(\int_{2t_0}^{t-2t_0} s^{\tau(f)+\tau(h)} e^{(\lambda_1/2-\mathfrak{R}\gamma(f))s} ds \right) b_{t_0}(x)^{1/2} \lesssim b_{t_0}(x)^{1/2}. \end{aligned}$$

Thus we have

$$e^{\lambda_1 t} \left| \int_0^t T_{t-s}[A(T_s f)(T_s h)](x) ds \right| \lesssim (b_{t_0}(x))^{1/2}.$$

By (2.17), we get

$$e^{\lambda_1 t} |T_t(fh)(x)| \leq e^{\lambda_1 t} T_t(|fh|)(x) \lesssim b_{t_0}(x)^{1/2}.$$

By (2.11), for $(t, x) \in (10t_0, \infty) \times E$, we have

$$e^{\lambda_1 t} |T_t f(x)T_t h(x)| \lesssim t^{\tau(f)+\tau(h)} e^{(\lambda_1/2-\mathfrak{R}\gamma(f))t} b_{t_0}(x) \lesssim b_{t_0}(x).$$

Now (2.46) follows immediately.

Repeating the proof above by using (2.12) instead of (2.11), we get that (2.46) also holds when $\gamma(f) = \infty$. \square

3. Proof of the main result. In this section, we will prove the main result of this paper. When referring to individuals in X , we will use the classical Ulam–Harris notation so that every individual in X has a unique label; see [12]. For each individual $u \in \mathcal{T}$ we shall write b_u and d_u for its birth and death times, respectively, and $\{z_u(r) : r \in [b_u, d_u]\}$ for its spatial trajectory. Define

$$\mathcal{L}_t = \{u \in \mathcal{T}, b_u \leq t < d_u\}, \quad t \geq 0.$$

Thus X_{s+t} has the following decomposition:

$$(3.1) \quad X_{s+t} = \sum_{u \in \mathcal{L}_t} X_s^{u,t},$$

where given $\mathcal{F}_t, X_s^{u,t}, u \in \mathcal{L}_t$, are independent, and $X_s^{u,t}$ has the same law as X_s under $\mathbb{P}_{\delta_{zu(t)}}$.

3.1. *A basic law of large numbers.* Recall that $H_t^{(k)}$ is defined in (1.40).

LEMMA 3.1. *Assume that v is an n_k -dimensional vector. If $\lambda_1 > 2\mathfrak{R}_k$, then, for any $v \in \mathcal{M}_a(E), H_t^{(k)}v$ is a martingale under \mathbb{P}_v . Moreover, the limit*

$$(3.2) \quad H_\infty^{(k)} := \lim_{t \rightarrow \infty} H_t^{(k)}$$

exists \mathbb{P}_v -a.s. and in $L^2(\mathbb{P}_v)$.

PROOF. By the branching property, it suffices to prove the lemma for $v = \delta_x$ with $x \in E$. By (1.19), we have

$$\mathbb{P}_{\delta_x} H_t^{(k)}v = e^{\lambda_k t} T_t((\Phi_k)^T)(x)(D_k(t))^{-1}v = (\Phi_k(x))^T v.$$

Recall that $\Phi_k(x)$ and $D_k(t)$ are defined in (1.17) and (1.18), respectively, and that $|v|_\infty$ is defined in the paragraph above (1.20). Thus, by the Markov property, we get that $H_t^{(k)}v$ is a martingale under \mathbb{P}_{δ_x} . Recall that t_0 is the constant in Assumption 1(c) and that b_t is defined in (1.11). We claim that, for $(t, x) \in (2t_0, \infty) \times E$,

$$(3.3) \quad \mathbb{P}_{\delta_x} |H_t^{(k)}v|^2 \lesssim |v|_\infty^2 b_{t_0}(x)^{1/2},$$

from which (3.2) follows immediately.

Now we prove the claim. Let $f_t(x) = e^{\lambda_k t} v^T (D_k(t))^{-1} \Phi_k(x)$. Then we have $H_t^{(k)}v = \langle f_t, X_t \rangle$, and by (1.19), for $s < t$, we have

$$T_s(f_t)(x) = e^{\lambda_k(t-s)} v^T (D_k(t-s))^{-1} \Phi_k(x) = f_{t-s}(x).$$

By (2.20), we have

$$\mathbb{P}_{\delta_x} |H_t^{(k)}v|^2 = \mathbb{P}_{\delta_x} |\langle f_t, X_t \rangle|^2 = \int_0^t T_s[A|f_s|^2](x) ds + T_t(|f_t|^2)(x),$$

where $A(\cdot)$ is defined in (1.8). Since each component of $D_k(s)^{-1} = D_k(-s)$ is a polynomial of s with degree no larger than ν_k , we get $|D_k(s)^{-1}|_\infty \lesssim (1 + s^{\nu_k})$. Thus, for all $s > 0$, we have

$$(3.4) \quad |f_s(x)| \lesssim e^{\mathfrak{R}_k s} |v|_\infty |D_k(s)|_\infty |\Phi_k(x)|_\infty \lesssim |v|_\infty (1 + s^{\nu_k}) e^{\mathfrak{R}_k s} b_{4t_0}(x)^{1/2}.$$

By (2.17), we have, for $(s, x) \in (2t_0, \infty) \times E$,

$$(3.5) \quad \begin{aligned} T_s(|f_s|^2)(x) &\lesssim e^{-\lambda_1 s} \| |f_s|^2 \|_2 b_{t_0}(x)^{1/2} \\ &\lesssim |v|_\infty^2 (1 + s^{2\nu_k}) e^{-(\lambda_1 - 2\mathfrak{N}_k)s} b_{t_0}(x)^{1/2}. \end{aligned}$$

Thus we have

$$(3.6) \quad \int_{2t_0}^t T_s[A|f_s|^2](x) ds \lesssim |v|_\infty^2 b_{t_0}(x)^{1/2}.$$

By (3.4) and (2.24), we get

$$(3.7) \quad \int_0^{2t_0} T_s[A|f_s|^2](x) ds \lesssim |v|_\infty^2 \int_0^{2t_0} T_s b_{4t_0}(x) ds \lesssim |v|_\infty^2 b_{t_0}(x)^{1/2}.$$

Thus, by (3.6)–(3.7), we have

$$(3.8) \quad \int_0^t T_s[A|f_s|^2](x) ds \lesssim |v|_\infty^2 b_{t_0}(x)^{1/2}.$$

Since $\lambda_1 > 2\mathfrak{N}_k$, we have $\sup_{s>2t_0} (1 + s^{2\nu_k}) e^{-(\lambda_1 - 2\mathfrak{N}_k)s} < \infty$. Thus, by (3.5), we get

$$T_t(|f_t|^2)(x) \lesssim |v|_\infty^2 b_{t_0}(x)^{1/2},$$

from which (3.3) follows immediately. \square

Now, we present the proof of Theorem 1.14. Recall that $\Phi_k(x)$, $D_k(t)$ and $\Psi_k(x)$ are defined in (1.17), (1.18), (1.25), $\gamma(f)$, $\zeta(f)$, $\tau(f)$, $F_{f,j}$ and $b_t(x)$ are defined in (1.33), (1.34), (1.35), (1.36) and (1.11), respectively.

PROOF OF THEOREM 1.14. By the branching property, it suffices to prove the theorem for $v = \delta_x$ with $x \in E$. Put

$$f^*(x) := \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T \langle f, \Psi_j \rangle, \quad \tilde{f}(x) := f(x) - f^*(x)$$

and $f_t(x) := \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T D_j(t)^{-1} F_{f,j}$. Then

$$t^{-\tau(f)} f^*(x) - f_t(x) = \sum_{j=\gamma(f)}^{\zeta(f)} \Phi_j(x)^T D_j(t)^{-1} (t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle - F_{f,j}).$$

By the definition of $\gamma(f)$ and $\zeta(f)$ in (1.33) and (1.34), we have for any $j = \gamma(f), \dots, \zeta(f)$,

$$\begin{aligned} &|e^{\mathfrak{N}_\nu(f)t} \langle \Phi_j^T, X_t \rangle D_j(t)^{-1} (t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle - F_{f,j})| \\ &= |e^{\lambda_j t} \langle \Phi_j^T, X_t \rangle D_j(t)^{-1} (t^{-\tau(f)} D_j(t) \langle f, \Psi_j \rangle - F_{f,j})|. \end{aligned}$$

Thus by (3.3) with $v = t^{-\tau(f)}D_j(t)\langle f, \Psi_j \rangle - F_{f,j}$, we get that, for $(t, x) \in (2t_0, \infty) \times E$,

$$\begin{aligned} & \mathbb{P}_{\delta_x} |e^{\Re_{\gamma(f)}t} \langle \Phi_j^T, X_t \rangle D_j(t)^{-1} (t^{-\tau(f)}D_j(t)\langle f, \Psi_j \rangle - F_{f,j})|^2 \\ & \lesssim |t^{-\tau(f)}D_j(t)\langle f, \Psi_j \rangle - F_{f,j}|_{\infty}^2 b_{t_0}(x)^{1/2}. \end{aligned}$$

Combining this with (2.14), we get that, for $(t, x) \in (2t_0, \infty) \times E$,

$$\begin{aligned} (3.9) \quad & \mathbb{P}_{\delta_x} |t^{-\tau(f)}e^{\Re_{\gamma(f)}t} \langle f^*, X_t \rangle - e^{\Re_{\gamma(f)}t} \langle f_t, X_t \rangle|^2 \\ & \lesssim \sum_{j=\gamma(f)}^{\xi(f)} |t^{-\tau(f)}D_j(t)\langle f, \Psi_j \rangle - F_{f,j}|_{\infty}^2 b_{t_0}(x)^{1/2} \lesssim t^{-2}b_{t_0}(x)^{1/2}. \end{aligned}$$

By the definition of $H_t^{(j)}$ and (3.2), we have, as $t \rightarrow \infty$,

$$\begin{aligned} (3.10) \quad & e^{\Re_{\gamma(f)}t} \langle f_t, X_t \rangle - \sum_{j=\gamma(f)}^{\xi(f)} (e^{-i\Im_j t} H_{\infty}^{(j)} F_{f,j}) \\ & = \sum_{j=\gamma(f)}^{\xi(f)} (e^{-i\Im_j t} (H_t^{(j)} - H_{\infty}^{(j)}) F_{f,j}) \rightarrow 0, \end{aligned}$$

in $L^2(\mathbb{P}_{\delta_x})$. Thus, by (3.9)–(3.10), we obtain that, as $t \rightarrow \infty$,

$$t^{-\tau(f)}e^{\Re_{\gamma(f)}t} \langle f^*, X_t \rangle - \sum_{j=\gamma(f)}^{\xi(f)} (e^{-i\Im_j t} H_{\infty}^{(j)} F_{f,j}) \rightarrow 0 \quad \text{in } L^2(\mathbb{P}_{\delta_x}).$$

Now, to complete the proof, we only need to show that, as $t \rightarrow \infty$,

$$(3.11) \quad t^{-2\tau(f)}e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 \rightarrow 0.$$

(1) If $\lambda_1 > 2\Re_{\gamma}(\tilde{f})$, then by (2.21), we get, for $(t, x) \in (2t_0, \infty) \times E$, as $t \rightarrow \infty$,

$$t^{-2\tau(f)}e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 \lesssim t^{-2\tau(f)}t^{2\tau(\tilde{f})}e^{2(\Re_{\gamma(f)} - \Re_{\gamma}(\tilde{f}))t} b_{t_0}(x)^{1/2} \rightarrow 0.$$

(2) If $\lambda_1 = 2\Re_{\gamma}(\tilde{f})$, then by (2.36), we get, as $t \rightarrow \infty$,

$$\begin{aligned} & t^{-2\tau(f)}e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 \\ & = t^{-2\tau(f)}t^{(1+2\tau(\tilde{f}))t} e^{(2\Re_{\gamma(f)} - \lambda_1)t} t^{-(1+2\tau(\tilde{f}))t} e^{\lambda_1 t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 \rightarrow 0. \end{aligned}$$

(3) If $\lambda_1 < 2\Re_{\gamma}(\tilde{f})$, then by (2.27), we get, as $t \rightarrow \infty$,

$$t^{-2\tau(f)}e^{2\Re_{\gamma(f)}t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 = t^{-2\tau(f)}e^{(2\Re_{\gamma(f)} - \lambda_1)t} e^{\lambda_1 t} \mathbb{P}_{\delta_x} |\langle \tilde{f}, X_t \rangle|^2 \rightarrow 0.$$

Combining the three cases above, we get (3.11). The proof is now complete. \square

3.2. *Proof of the main theorem.* First, we recall a metric on the space of distributions on \mathbb{R}^n . For $f : \mathbb{R}^n \rightarrow \mathbb{R}$, define

$$\|f\|_{\text{BL}} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

For any distributions ν_1 and ν_2 on \mathbb{R}^n , define

$$d(\nu_1, \nu_2) := \sup \left\{ \left| \int f d\nu_1 - \int f d\nu_2 \right| : \|f\|_{\text{BL}} \leq 1 \right\}.$$

Then d is a metric. It follows from [11], Theorem 11.3.3, that the topology generated by this metric is equivalent to the weak convergence topology. From the definition, we can see that if ν_1 and ν_2 are the distributions of two \mathbb{R}^n -valued random variables X and Y , respectively, then

$$(3.12) \quad d(\nu_1, \nu_2) \leq \mathbb{E}\|X - Y\| \leq \sqrt{\mathbb{E}\|X - Y\|^2}.$$

Recall that \mathcal{C}_s and σ_f^2 are defined in (1.39) and (1.41), respectively.

LEMMA 3.2. *If $f \in \mathcal{C}_s$, then $\sigma_f^2 \in (0, \infty)$, and for any nonzero $\nu \in \mathcal{M}_a(E)$, it holds under \mathbb{P}_ν that*

$$(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle) \xrightarrow{d} (W_\infty, G_1(f)\sqrt{W_\infty}), \quad t \rightarrow \infty,$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W_∞ and $G_1(f)$ are independent.

PROOF. The proof is similar that of [26], Theorem 1.8. We define an \mathbb{R}^2 -valued random variable $U_1(t)$ by

$$U_1(t) := (e^{\lambda_1 t} \langle \phi_1, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle).$$

By the branching property (see the argument of the beginning of the proof of [26], Theorem 1.8), to prove this lemma, it suffices to show that for any $x \in E$, under \mathbb{P}_{δ_x} ,

$$(3.13) \quad U_1(t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_1(f)),$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$ is independent of W_∞ .

Now we show that (3.13) is valid. In the remainder of this proof, we assume $s, t > 10t_0$ and write

$$U_1(s + t) = (e^{\lambda_1(s+t)} \langle \phi_1, X_{t+s} \rangle, e^{(\lambda_1/2)(s+t)} \langle f, X_{s+t} \rangle).$$

Recall the decomposition of X_{s+t} in (3.1). Define

$$Y_1^{u,t}(s) := e^{\lambda_1 s/2} \langle f, X_s^{u,t} \rangle \quad \text{and} \quad y_1^{u,t}(s) := \mathbb{P}_{\delta_x}(Y_1^{u,t}(s) | \mathcal{F}_t).$$

Given \mathcal{F}_t , $Y_1^{u,t}(s)$ has the same law as $Y_1(s) := e^{\lambda_1 s/2} \langle f, X_s \rangle$ under $\mathbb{P}_{\delta_{z_u(t)}}$. Then we have

$$\begin{aligned} & e^{(\lambda_1/2)(s+t)} \langle f, X_{s+t} \rangle \\ &= e^{(\lambda_1/2)t} \sum_{u \in \mathcal{L}_t} Y_1^{u,t}(s) \\ &= e^{(\lambda_1/2)t} \sum_{u \in \mathcal{L}_t} (Y_1^{u,t}(s) - y_s^{u,t}) + e^{(\lambda_1/2)(t+s)} \mathbb{P}_{\delta_x}(\langle f, X_{s+t} \rangle | \mathcal{F}_t) \\ &=: J_1(s, t) + J_2(s, t). \end{aligned}$$

We first consider $J_2(s, t)$. By the Markov property, we have

$$J_2(s, t) = e^{(\lambda_1/2)(s+t)} \langle T_s f, X_t \rangle.$$

We claim that

$$(3.14) \quad \limsup_{s \rightarrow \infty} \limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x} J_2(s, t)^2 = 0.$$

By (2.20), we get

$$\mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 = \int_0^t T_{t-u} [A(T_{u+s}(f))]^2(x) du + T_t (T_s f)^2(x),$$

where the function $A(\cdot)$ is defined in (1.8). For the case $\gamma(f) < \infty$, using the arguments leading to [26], (3.10), in the proof of [26], Theorem 1.8, we can show that

$$(3.15) \quad \begin{aligned} \limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x} J_2(s, t)^2 &= \limsup_{t \rightarrow \infty} e^{\lambda_1(t+s)} \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \\ &\lesssim s^{2\tau(f)} e^{(\lambda_1 - 2\Re_{\gamma(f)})s} b_{t_0}(x)^{1/2}. \end{aligned}$$

Here we only give a sketch of proving (3.15). Since $u + s \geq s > 10t_0$, by (2.11), we get

$$(3.16) \quad |T_{u+s} f(x)|^2 \lesssim (u + s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(u+s)} b_{4t_0}(x).$$

Thus, for $t > 10t_0$, using (2.17), we have

$$\begin{aligned} & \int_0^{t-2t_0} T_{t-u} [A(T_{s+u} f)]^2(x) du \\ & \lesssim e^{-2s\Re_{\gamma(f)}} \int_0^{t-2t_0} (u + s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}u} e^{-\lambda_1(t-u)} du b_{t_0}(x)^{1/2} \\ & \lesssim e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)^{1/2} \\ & \quad \times \left(\int_0^{t-2t_0} u^{2\tau(f)} e^{(\lambda_1 - 2\Re_{\gamma(f)})u} du + s^{2\tau(f)} \int_0^{t-2t_0} e^{(\lambda_1 - 2\Re_{\gamma(f)})u} du \right) \\ & \lesssim s^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)^{1/2}. \end{aligned}$$

And by (3.16) and (2.24), we have

$$(3.17) \quad \int_{t-2t_0}^t T_{t-u} [A(T_{s+u}f)^2](x) du \lesssim (t+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(t+s)} b_{t_0}(x)^{1/2}.$$

By (2.11), we get that $|T_s f(x)|^2 \lesssim s^{2\tau(f)} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)$. Thus we have

$$(3.18) \quad T_t(T_s f)^2(x) \lesssim s^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)^{1/2}.$$

Consequently, we have

$$(3.19) \quad \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim (t+s)^{2\tau(f)} e^{-2\Re_{\gamma(f)}(t+s)} b_{t_0}(x)^{1/2} + s^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re_{\gamma(f)}s} b_{t_0}(x)^{1/2},$$

which implies (3.15). Similarly, for the case $\gamma(f) = \infty$, we have

$$(3.20) \quad \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim b_{t_0}(x)^{1/2} + e^{-\lambda_1 t} b_{t_0}(x)^{1/2}.$$

Thus

$$(3.21) \quad \limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x} J_2(s, t)^2 = \limsup_{t \rightarrow \infty} e^{\lambda_1(t+s)} \mathbb{P}_{\delta_x} \langle T_s f, X_t \rangle^2 \lesssim e^{\lambda_1 s} b_{t_0}(x)^{1/2}.$$

Combining (3.15) and (3.21), we get (3.14).

Next we consider $J_1(s, t)$. We define an \mathbb{R}^2 -valued random variable $U_2(s, t)$ by

$$U_2(s, t) := (e^{\lambda_1 t} \langle \phi_1, X_t \rangle, J_1(s, t)).$$

Let $V_s(x) := \mathbb{V}ar_{\delta_x} Y_1(s)$. We claim that, for any $x \in E$, under \mathbb{P}_{δ_x} ,

$$(3.22) \quad U_2(s, t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_1(s)) \quad \text{as } t \rightarrow \infty,$$

where $G_1(s) \sim \mathcal{N}(0, \sigma_f^2(s))$ is independent of W_∞ and $\sigma_f^2(s) = \langle V_s, \phi_1 \rangle$. The proof of (3.22) is similar to that of [26], (3.11). We omit the details here. Thus we get that, for any $x \in E$, under \mathbb{P}_{δ_x} , as $t \rightarrow \infty$,

$$U_3(s, t) := (e^{\lambda_1(t+s)} \langle \phi_1, X_{t+s} \rangle, J_1(s, t)) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_1(s)).$$

By (2.27), we have $\lim_{s \rightarrow \infty} \langle V_s, \psi_1 \rangle_m = \sigma_f^2$. Let $G_1(f)$ be a $\mathcal{N}(0, \sigma_f^2)$ random variable independent of W_∞ . Then

$$\lim_{s \rightarrow \infty} d(G_1(s), G_1(f)) = 0.$$

Let $\mathcal{D}(s+t)$ and $\tilde{\mathcal{D}}(s, t)$ be the distributions of $U_1(s+t)$ and $U_3(s, t)$, respectively, and let $\mathcal{D}(s)$ and \mathcal{D} be the distributions of $(W_\infty, \sqrt{W_\infty} G_1(s))$ and $(W_\infty, \sqrt{W_\infty} G_1(f))$, respectively. Then, using (3.12), we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} d(\mathcal{D}(s+t), \mathcal{D}) \\ & \leq \limsup_{t \rightarrow \infty} [d(\mathcal{D}(s+t), \tilde{\mathcal{D}}(s, t)) + d(\tilde{\mathcal{D}}(s, t), \mathcal{D}(s)) + d(\mathcal{D}(s), \mathcal{D})] \\ & \leq \limsup_{t \rightarrow \infty} (\mathbb{P}_{\delta_x} J_2(s, t)^2)^{1/2} + 0 + d(\mathcal{D}(s), \mathcal{D}). \end{aligned}$$

Using this and the definition of $\limsup_{t \rightarrow \infty}$, we get that

$$\begin{aligned} \limsup_{t \rightarrow \infty} d(\mathcal{D}(t), \mathcal{D}) &= \limsup_{t \rightarrow \infty} d(\mathcal{D}(s+t), \mathcal{D}) \\ &\leq \limsup_{t \rightarrow \infty} (\mathbb{P}_{\delta_x}(J_2(s, t)^2)^{1/2} + d(\mathcal{D}(s), \mathcal{D})). \end{aligned}$$

Letting $s \rightarrow \infty$, we get $\limsup_{t \rightarrow \infty} d(\mathcal{D}(t), \mathcal{D}) = 0$. The proof is now complete. \square

Recall that $\Phi_k(x)$, $D_k(t)$ and $\tau(f)$ are defined in (1.17), (1.18) and (1.35), respectively.

LEMMA 3.3. Assume $f(x) = \sum_{j:\lambda_1=2\mathfrak{N}_j} (\Phi_j(x))^T v_j \in \mathcal{C}_c$, where $v_j \in \mathbb{C}^{n_j}$. Define

$$S_t f(x) := t^{-(1+2\tau(f))/2} e^{(\lambda_1/2)t} (\langle f, X_t \rangle - T_t f(x)), \quad (t, x) \in (0, \infty) \times E.$$

Then for any $c > 0$, $\delta > 0$ and $x \in E$, we have

$$(3.23) \quad \lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} (|S_t f(x)|^2; |S_t f(x)| > ce^{\delta t}) = 0.$$

PROOF. In this proof, we always assume $t > 10t_0$. For each j , define

$$S_{j,t}(x) := t^{-(1+2\tau(f))/2} e^{\lambda_1 t/2} (\langle \Phi_j^T, X_t \rangle - e^{-\lambda_j t} (\Phi_j(x))^T D_j(t)).$$

Thus $S_t f(x) = \sum_{j:\lambda_1=2\mathfrak{N}_j} S_{j,t}(x) v_j$. Using the fact that for every $n \geq 1$,

$$(3.24) \quad \left| \sum_{l=1}^n x_l \right|^2 1_{|\sum_{l=1}^n x_l|^2 > M} \leq n \sum_{l=1}^n |x_l|^2 1_{|x_l|^2 > M/n},$$

we see that, to prove (3.23), it suffices to show that, as $t \rightarrow \infty$,

$$F(t, x, v_j) := \mathbb{P}_{\delta_x} (|S_{j,t}(x) v_j|^2; |S_{j,t}(x) v_j| > ce^{\delta t}) \rightarrow 0.$$

Choose an integer $n_0 > 2t_0$. We write $t = l_t n_0 + \varepsilon_t$, where $l_t \in \mathbb{N}$ and $0 \leq \varepsilon_t < n_0$. By (1.19), we get that $T_u(\Phi_j^T)(x) = e^{-\lambda_j u} \Phi_j(x)^T D_j(t)$. Since $\lambda_1 = 2\mathfrak{N}_j$, for any $(t, x) \in (0, \infty) \times E$, we have

$$\begin{aligned} &S_{j,t+n_0}(x) \\ &= \left(\frac{1}{t+n_0} \right)^{1/2+\tau(f)} e^{\lambda_1(t+n_0)/2} \\ &\quad \times (\langle \Phi_j^T, X_{t+n_0} \rangle - \langle e^{-\lambda_j n_0} \Phi_j^T, X_t \rangle D_j(n_0)) \\ &\quad + \left(\frac{1}{t+n_0} \right)^{1/2+\tau(f)} e^{-i\mathfrak{N}_j n_0} e^{\lambda_1 t/2} \end{aligned} \tag{3.25}$$

$$\begin{aligned} & \times (\langle \Phi_j^T, X_t \rangle - e^{-\lambda_j t} (\Phi_j(x))^T D_j(t)) D_j(n_0) \\ &= \left(\frac{1}{t+n_0} \right)^{1/2+\tau(f)} R_j(t) \\ & \quad + e^{-i\Im_j n_0} \left(\frac{t}{t+n_0} \right)^{1/2+\tau(f)} S_{j,t}(x) D_j(n_0), \end{aligned}$$

where

$$R_j(t) := e^{(\lambda_1/2)(t+n_0)} (\langle \Phi_j^T, X_{t+n_0} \rangle - \langle e^{-\lambda_j} \Phi_j^T, X_t \rangle D_j(n_0)).$$

Put

$$\begin{aligned} A_1(t, x, v_j) &:= \{|S_{j,t}(x) D_j(n_0) v_j| > c e^{\delta t}\}, \\ A_2(t, x, v_j) &:= \{|S_{j,t}(x) D_j(n_0) v_j| \leq c e^{\delta t}, |S_{j,t+n_0}(x) v_j| > c e^{\delta(t+n_0)}\} \end{aligned}$$

and

$$A(t, x, v_j) := A_1(t, x, v_j) \cup A_2(t, x, v_j).$$

Then, for any $(t, x) \in (0, \infty) \times E$, we have

$$\begin{aligned} & F(t+n_0, x, v_j) \\ & \leq \mathbb{P}_{\delta_x}(|S_{j,t+n_0}(x) v_j|^2; A_1(t, x, v_j)) + \mathbb{P}_{\delta_x}(|S_{j,t+n_0}(x) v_j|^2; A_2(t, x, v_j)) \\ & =: M_1(t, x) + M_2(t, x). \end{aligned}$$

Since $A_1(t, x, v_j) \in \mathcal{F}_t$ and $\mathbb{P}_{\delta_x}(R_j(t) | \mathcal{F}_t) = 0$ for any $(t, x) \in (0, \infty) \times E$, we have by (3.25) that

$$\begin{aligned} M_1(t, x) &= \left(\frac{1}{t+n_0} \right)^{1+2\tau(f)} \mathbb{P}_{\delta_x}(|R_j(t) v_j|^2; A_1(t, x, v_j)) \\ & \quad + \left(\frac{t}{t+n_0} \right)^{1+2\tau(f)} F(t, x, D_j(n_0) v_j) \end{aligned}$$

and

$$\begin{aligned} M_2(t, x) &\leq 2 \left(\frac{1}{t+n_0} \right)^{1+2\tau(f)} \mathbb{P}_{\delta_x}(|R_j(t) v_j|^2; A_2(t, x, v_j)) \\ & \quad + 2 \left(\frac{t}{t+n_0} \right)^{1+2\tau(f)} \mathbb{P}_{\delta_x}(|S_{j,t}(x) D_j(n_0) v_j|^2; A_2(t, x, v_j)). \end{aligned}$$

Thus, for any $(t, x) \in (0, \infty) \times E$, we have

$$\begin{aligned} (3.26) \quad F(t+n_0, x, v_j) &\leq \left(\frac{t}{t+n_0} \right)^{1+2\tau(f)} F(t, x, D_j(n_0) v_j) \\ & \quad + \left(\frac{1}{t+n_0} \right)^{1+2\tau(f)} (F_1(t, x, v_j) + F_2(t, x, v_j)), \end{aligned}$$

where

$$F_1(t, x, v_j) := 2\mathbb{P}_{\delta_x}(|R_j(t)v_j|^2; A_1(t, x, v_j) \cup A_2(t, x, v_j)),$$

$$F_2(t, x, v_j) := 2t^{1+2\tau(f)}\mathbb{P}_{\delta_x}(|S_{j,t}(x)D_j(n_0)v_j|^2; A_2(t, x, v_j)).$$

Iterating (3.26), we get for t large enough,

$$\begin{aligned} &F(t + n_0, x, v_j) \\ &\leq \left(\frac{1}{t + n_0}\right)^{1+2\tau(f)} \sum_{m=5}^{l_t} (F_1(mn_0 + \varepsilon_t, x, D_j((l_t - m)n_0)v_j)) \\ &\quad + \left(\frac{1}{t + n_0}\right)^{1+2\tau(f)} \sum_{m=5}^{l_t} (F_2(mn_0 + \varepsilon_t, x, D_j((l_t - m)n_0)v_j)) \\ &\quad + \left(\frac{5n_0 + \varepsilon_t}{t + n_0}\right)^{1+2\tau(f)} F(5n_0 + \varepsilon_t, x, D_j((l_t - 4)n_0)v_j) \\ &=: L_1(t, x) + L_2(t, x) \\ &\quad + \left(\frac{5n_0 + \varepsilon_t}{t + 1}\right)^{1+2\tau(f)} F(5n_0 + \varepsilon_t, x, D_j((l_t - 4)n_0)v_j). \end{aligned}$$

First, we consider $L_1(t, x)$. By the definition of $\tau(f)$ in (1.35), we have for $s > 0$,

$$(3.27) \quad |D_j(s)v_j|_2 \lesssim |D_j(s)v_j|_\infty \lesssim 1 + s^{\tau(f)}.$$

Thus, we have for $0 \leq s \leq t$ and $t \geq 2t_0$,

$$(3.28) \quad |R_j(s)D_j(t - s)v_j|^2 \leq |R_j(s)|_2^2 |D_j(t - s)v_j|_2^2 \lesssim t^{2\tau(f)} |R_j(s)|_2^2.$$

It follows that for any $\varepsilon \in (0, 1)$,

$$\begin{aligned} L_1(t, x) &\leq \frac{2}{t + n_0} \sum_{5 \leq m \leq \varepsilon l_t} \mathbb{P}_{\delta_x}(|R_j(mn_0 + \varepsilon_t)|_2^2) \\ &\quad + \frac{2}{t + n_0} \sum_{l_t \varepsilon \leq m \leq l_t} \mathbb{P}_{\delta_x}(|R_j(mn_0 + \varepsilon_t)|_2^2 1_{A(mn_0 + \varepsilon_t, x, D_j((l_t - m)n_0)v_j)}) \\ &=: L_{1,1}(t, x) + L_{1,2}(t, x). \end{aligned}$$

By the definition of $R_j(s)$, we have

$$(3.29) \quad |R_j(s)|_2^2 = e^{\lambda_1(s+n_0)} \sum_{l=1}^{n_j} |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2.$$

Note that

$$\begin{aligned} & |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2 \\ &= |\langle \mathfrak{R}(\phi_l^{(j)}), X_{s+n_0} \rangle - \langle T_{n_0}(\mathfrak{R}(\phi_l^{(j)})), X_s \rangle|^2 \\ &\quad + |\langle \mathfrak{S}(\phi_l^{(j)}), X_{s+n_0} \rangle - \langle T_{n_0}(\mathfrak{S}(\phi_l^{(j)})), X_s \rangle|^2. \end{aligned}$$

Thus we have

$$\begin{aligned} & \mathbb{P}_{\delta_x} |\langle \phi_l^{(j)}, X_{s+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_s \rangle|^2 \\ &= T_s(\text{Var}_{\delta} \langle \mathfrak{R}(\phi_l^{(j)}), X_{n_0} \rangle)(x) + T_s(\text{Var}_{\delta} \langle \mathfrak{S}(\phi_l^{(j)}), X_{n_0} \rangle)(x). \end{aligned}$$

Hence, by (2.17), we get, for $s \geq 5n_0 > 2t_0$,

$$\begin{aligned} & \mathbb{P}_{\delta_x} |R_j(s)|_2^2 \\ (3.30) \quad &= e^{\lambda_1(s+n_0)} \sum_{l=1}^{n_j} \mathbb{P}_{\delta_x} |\langle \phi_l^{(j)}, X_{t+n_0} \rangle - \langle T_{n_0}(\phi_l^{(j)}), X_t \rangle|^2 \\ &\lesssim b_{t_0}(x)^{1/2}. \end{aligned}$$

Therefore, we have, for $(t, x) \in (5n_0, \infty) \times E$,

$$(3.31) \quad L_{1,1}(t, x) \lesssim \varepsilon b_{t_0}(x)^{1/2}.$$

We claim that, for any $x \in E$:

(i)

$$(3.32) \quad \lim_{M \rightarrow \infty} \limsup_{s \rightarrow \infty} \mathbb{P}_{\delta_x} (|R_j(s)|_2^2; |R_j(s)|_2^2 > M) = 0,$$

(ii) and, as $t \rightarrow \infty$,

$$\sup_{t \varepsilon \leq s \leq t} \mathbb{P}_{\delta_x} (A_1(s, x, D_j(t-s)v_j) \cup A_2(s, x, D_j(t-s)v_j)) \rightarrow 0.$$

Using these two claims we get that, as $t \rightarrow \infty$,

$$\begin{aligned} & L_{1,2}(t, x) \\ &\leq \frac{2}{t+n_0} \sum_{\varepsilon_{l_t} \leq m \leq l_t} (\mathbb{P}_{\delta_x} (|R_j(mn_0 + \varepsilon_t)|_2^2; |R_j(mn_0 + \varepsilon_t)|_2^2 > M) \\ &\quad + M \mathbb{P}_{\delta_x} (A(mn_0 + \varepsilon_t, x, D_j((l_t - m)n_0)v_j))) \\ &\lesssim \sup_{s \geq t \varepsilon} \mathbb{P}_{\delta_x} (|R_j(s)|_2^2; |R_j(s)|_2^2 > M) \\ &\quad + M \sup_{t \varepsilon \leq s \leq t} \mathbb{P}_{\delta_x} (A(s, x, D_j(t-s)v_j)) \\ &\rightarrow \limsup_{s \rightarrow \infty} \mathbb{P}_{\delta_x} (|R_j(s)|_2^2; |R_j(s)|_2^2 > M). \end{aligned}$$

Letting $M \rightarrow \infty$, we get

$$(3.33) \quad \lim_{t \rightarrow \infty} L_{1,2}(t, x) = 0.$$

Now we prove the two claims.

(i) For $l = 1, 2, \dots, n_j$, define

$$R_{j,l,1}(s) := e^{\lambda_1(s+n_0)/2} \langle \mathfrak{R}(\phi_l^j), X_{s+n_0} \rangle - \langle T_{n_0}(\mathfrak{R}(\phi_l^j)), X_s \rangle$$

and

$$R_{j,l,2}(s) := e^{\lambda_1(s+n_0)/2} \langle \mathfrak{S}(\phi_l^j), X_{s+n_0} \rangle - \langle T_{n_0}(\mathfrak{S}(\phi_l^j)), X_s \rangle.$$

Using (3.24) and (3.29), we see that, to prove (3.32), we only need to show that, for $k = 1, 2$,

$$(3.34) \quad \lim_{M \rightarrow \infty} \limsup_{s \rightarrow \infty} \mathbb{P}_{\delta_x}(|R_{j,l,k}(s)|^2, |R_{j,l,k}(s)|^2 > M) = 0.$$

Repeating the proof of (3.22) with $s = n_0$, we see that (3.22) is valid for $f \in L^2(E, m) \cap L^4(E, m)$. Thus, for $l = 1, 2, \dots, n_j$, as $s \rightarrow \infty$,

$$R_{j,l,1}(s) \xrightarrow{d} \sqrt{W_\infty} G,$$

where $G \sim \mathcal{N}(0, e^{\lambda_1 n_0} \langle \text{Var}_\delta \langle \mathfrak{R}(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m)$. And by (2.16), we get, as $s \rightarrow \infty$,

$$(3.35) \quad \begin{aligned} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2) &= e^{\lambda_1(s+n_0)} T_s(\text{Var}_\delta \langle \mathfrak{R}(\phi_l^j), X_{n_0} \rangle)(x) \\ &\rightarrow e^{\lambda_1 n_0} \langle \text{Var}_\delta \langle \mathfrak{R}(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x). \end{aligned}$$

Let $h_M(r) = r$ on $[0, M - 1]$, $h_M(r) = 0$ on $[M, \infty]$, and let h_M be linear on $[M - 1, M]$. By (3.35), we have that for any $x \in E$,

$$\begin{aligned} &\limsup_{s \rightarrow \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2, |R_{j,l,1}(s)|^2 > M) \\ &\leq \limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2) - \mathbb{P}_{\delta_x}(h_M(|R_{j,l,1}(s)|^2)) \\ &= e^{\lambda_1 n_0} \langle \text{Var}_\delta \langle \mathfrak{R}(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x) - \mathbb{P}_{\delta_x}(h_M(W_\infty G^2)). \end{aligned}$$

By the monotone convergence theorem, we have that for any $x \in E$,

$$\begin{aligned} &\lim_{M \rightarrow \infty} \mathbb{P}_{\delta_x}(h_M(W_\infty G^2)) = \mathbb{P}_{\delta_x}(W_\infty G^2) \\ &= \mathbb{P}_{\delta_x}(W_\infty) \mathbb{P}_{\delta_x}(G^2) = e^{\lambda_1 n_0} \langle \text{Var}_\delta \langle \mathfrak{R}(\phi_l^j), X_{n_0} \rangle, \psi_1 \rangle_m \phi_1(x), \end{aligned}$$

which implies

$$\lim_{M \rightarrow \infty} \limsup_{s \rightarrow \infty} \mathbb{P}_{\delta_x}(|R_{j,l,1}(s)|^2, |R_{j,l,1}(s)|^2 > M) = 0,$$

which says (3.34) holds for $k = 1$. Using similar arguments, we get (3.34) holds for $k = 2$.

(ii) Recall that v_j is defined below (1.16). Since $\tau(\phi_l^j) \leq v_j$, by (2.37), we get for $10t_0 \leq s$,

$$(3.36) \quad \mathbb{P}_{\delta_x} |S_{j,s}(x)|_2^2 \lesssim s^{1+2v_j} s^{-(1+2\tau(f))} \leq s^{2v_j}.$$

By (3.27), we get, for $10t_0 \leq s \leq t$,

$$(3.37) \quad \mathbb{P}_{\delta_x} |S_{j,s}(x)D_j(t+1-s)v_j|^2 \lesssim s^{2v_j} (1+t^{2\tau(f)}).$$

By Chebyshev's inequality and (3.37), we have that, for any $x \in E$, as $t \rightarrow \infty$

$$\begin{aligned} & \sup_{t\varepsilon \leq s \leq t} \mathbb{P}_{\delta_x} (A_1(s, x, D_j(t-s))) \\ & \leq \sup_{t\varepsilon \leq s \leq t} c^{-2} e^{-2\delta s} \mathbb{P}_{\delta_x} |S_{j,s}(x)D_j(t+1-s)v_j|^2 \\ & \lesssim e^{-2\delta\varepsilon t} t^{2v_j} (1+t^{2\tau(f)}) \rightarrow 0. \end{aligned}$$

Note that, under \mathbb{P}_{δ_x} , for any $t > 0$,

$$(3.38) \quad \begin{aligned} & A_2(s, x, D_j(t-s)v_j) \\ & \subset \{|R_j(s)D_j(t-s)v_j| > ce^{\delta s} (e^{\delta n_0} - 1) s^{(2\tau(f)+1)/2}\}. \end{aligned}$$

By (3.28) and (3.30), we get

$$\mathbb{P}_{\delta_x} |R_j(s)D_j(t-s)v_j|^2 \lesssim t^{2\tau(f)} b_{t_0}(x)^{1/2}.$$

Similarly, by Chebyshev's inequality, we have that, for any $x \in E$, as $t \rightarrow \infty$,

$$\begin{aligned} & \sup_{t\varepsilon \leq s \leq t} \mathbb{P}_{\delta_x} A_2(s, x, D_j(t-s)v_j) \\ & \leq \sup_{t\varepsilon \leq s \leq t} c^{-2} (e^{\delta n_0} - 1)^{-2} e^{-2\delta s} s^{-(1+2\tau(f))} \mathbb{P}_{\delta_x} |R_j(s)D_j(t-s)v_j|^2 \\ & \lesssim e^{-2\delta\varepsilon t} (t\varepsilon)^{-(1+2\tau(f))} t^{2\tau(f)} \rightarrow 0. \end{aligned}$$

Thus we have finished proving the two claims. Therefore, by (3.31) and (3.33), we get

$$\limsup_{t \rightarrow \infty} L_1(t, x) \lesssim \varepsilon b_{t_0}(x)^{1/2}.$$

Letting $\varepsilon \rightarrow 0$, we get

$$\lim_{t \rightarrow \infty} L_1(t, x) = 0.$$

Now we consider $L_2(t, x)$. By (3.38), we have that for any $x \in E$,

$$\begin{aligned}
 &F_2(s, x, D_j(t-s)v_j) \\
 &= 2s^{(1+2\tau(f))} \mathbb{P}_{\delta_x}(|S_{j,s}(x)D_j(t+n_0-s)v_j|^2; A_2(s, x, D_j(t-s)v_j)) \\
 &\leq 2s^{(1+2\tau(f))} ce^{\delta s} \mathbb{P}_{\delta_x}(|S_{j,s}(x)D_j(t+n_0-s)v_j| \\
 &\quad \times \mathbf{1}_{\{|R_j(s)D_j(t-s)v_j| > ce^{\delta s}(e^{\delta n_0}-1)s^{(2\tau(f)+1)/2}\}}) \\
 &\leq 2c^{-1}(e^{\delta n_0}-1)e^{-\delta s} \mathbb{P}_{\delta_x}(|S_{j,s}(x)D_j(t+n_0-s)v_j||R_j(s)D_j(t-s)v_j|^2) \\
 &\lesssim e^{-\delta s} e^{\lambda_1(s+n_0)} t^{\tau(f)} \\
 &\quad \times \mathbb{P}_{\delta_x}(|S_{j,s}(x)|_2 \sqrt{\text{Var}_{\delta}((\Phi_j^T D_j(t-s)v_j, X_{n_0}))}, X_s)) \\
 &\lesssim e^{-\delta s} t^{\tau(f)} \sqrt{\mathbb{P}_{\delta_x}|S_{j,s}(x)|_2^2} \\
 &\quad \times \sqrt{e^{2\lambda_1 s} \mathbb{P}_{\delta_x}(\text{Var}_{\delta}((\Phi_j^T D_j(t-s)v_j, X_{n_0}))}, X_s)^2}.
 \end{aligned}$$

By (2.37) and (1.20), we get for $s \leq t$,

$$\begin{aligned}
 \text{Var}_{\delta_x}((\Phi_j^T D_j(t-s)v_j, X_{n_0})) &\leq \mathbb{P}_{\delta_x} |(\Phi_j^T D_j(t-s)v_j, X_{n_0})|^2 \\
 &\lesssim t^{2\tau(f)} \mathbb{P}_{\delta_x}(b_{t_0}^{1/2}, X_{n_0})^2.
 \end{aligned}$$

Thus by (3.36) and (2.21), we have for $5n_0 \leq s \leq t$,

$$\begin{aligned}
 F_2(s, x, D_j(t-s)v_j) &\lesssim e^{-\delta s} t^{2\tau(f)} s^{v_j} \sqrt{e^{2\lambda_1 s} \mathbb{P}_{\delta_x}(b_{t_0}^{1/2}, X_s)^2} \\
 &\lesssim e^{-\delta s} t^{2\tau(f)} s^{v_j}.
 \end{aligned}$$

Thus we get, as $t \rightarrow \infty$,

$$\begin{aligned}
 L_2(t, x) &\lesssim \frac{1}{t+n_0} \sum_{m=5}^{l_t} e^{-\delta(mn_0+\varepsilon_t)} (mn_0+\varepsilon_t)^{(1+2v_j)/2} \\
 &\leq \frac{1}{t+n_0} \sum_{m=5}^{l_t} e^{-\delta mn_0} ((m+1)n_0)^{(1+2v_j)/2} \rightarrow 0.
 \end{aligned}$$

To complete the proof, we only need to show that for any $x \in E$,

$$(3.39) \quad \lim_{t \rightarrow \infty} \left(\frac{5n_0 + \varepsilon_t}{t+n_0} \right)^{1+2\tau(f)} F(5n_0 + \varepsilon_t, x, D_j((l_t-4)n_0)v_j) = 0.$$

By (3.27) and (3.36), we get that for any $x \in E$,

$$\begin{aligned}
 &(5n_0 + \varepsilon_t)^{1+2\tau(f)} F(5n_0 + \varepsilon_t, x, D_j((l_t-4)n_0)v_j) \\
 &\leq (6n_0)^{1+2\tau(f)} \sup_{5n_0 \leq s \leq 6n_0} \mathbb{P}_{\delta_x} |S_{j,s}(x)D_j((l_t-4)n_0)v_j|^2 \lesssim t^{2\tau(f)} (6n_0)^{2v_j},
 \end{aligned}$$

which implies (3.39).

The proof is now complete. \square

Recall that $\mathcal{C}_s, \mathcal{C}_c$ and $\tau(f)$ are defined in (1.39), (1.38) and (1.35), respectively.

LEMMA 3.4. *Assume that $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. Define for any $t > 0$,*

$$Y_1(t) := e^{\lambda_1 t/2} (\langle f, X_t \rangle - T_t f(x)),$$

$$Y_2(t) := t^{-(1+\tau(h)/2)} e^{\lambda_1 t/2} (\langle h, X_t \rangle - T_t h(x))$$

and $Y_t := Y_1(t) + Y_2(t)$. Then for any $c > 0, \delta > 0$ and $x \in E$, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} (|Y_t|^2; |Y_t| > ce^{\delta t}) = 0.$$

PROOF. By (3.24) and Lemma 3.3, it suffices to show that

$$(3.40) \quad \lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} (|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t}) = 0.$$

If $\gamma(f) < \infty$, by (2.11), we get, as $t \rightarrow \infty$,

$$e^{\lambda_1 t/2} |T_t f(x)| \lesssim t^{\tau(f)} e^{(\lambda_1/2 - \Re_{\gamma(f)})t} b_{t_0}(x)^{1/2} \rightarrow 0.$$

If $\gamma(f) = \infty$, by (2.12), we get, as $t \rightarrow \infty$,

$$e^{\lambda_1 t/2} |T_t f(x)| \lesssim e^{\lambda_1 t/2} b_{t_0}(x)^{1/2} \rightarrow 0.$$

Thus by Lemma 3.2, $Y_1(t) \xrightarrow{d} \sqrt{W_\infty} G_1(f)$. By Lemma 2.6, we have

$$\lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} (|Y_1(t)|^2) = \sigma_f^2 \phi_1(x).$$

Thus for any $M > 0$, we have

$$\begin{aligned} & \mathbb{P}_{\delta_x} (|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t}) \\ & \leq \mathbb{P}_{\delta_x} (|Y_1(t)|^2; |Y_1(t)| > M) + M^2 \mathbb{P}_{\delta_x} (|Y_1(t)| > ce^{\delta t}) \\ & =: I_1(t, x, M) + I_2(t, x, M). \end{aligned}$$

Let $h_M(r) = r$ on $[0, M - 1]$, $h_M(r) = 0$ on $[M, \infty]$, and let h_M be linear on $[M - 1, M]$. Then

$$\begin{aligned} \limsup_{t \rightarrow \infty} I_1(t, x, M) & \leq \limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x} (|Y_1(t)|^2) - \mathbb{P}_{\delta_x} (h_M(|Y_1(t)|)^2) \\ & = \sigma_f^2 \phi_1(x) - \mathbb{P}_{\delta_x} (h_M(|G_1(f)|\sqrt{W_\infty})^2). \end{aligned}$$

By Chebyshev's inequality, we have, as $t \rightarrow \infty$,

$$I_2(t, x, M) \leq M^2 c^{-2} e^{-2\delta t} \mathbb{P}_{\delta_x} (|Y_1(t)|^2) \rightarrow 0.$$

Thus we have

$$\limsup_{t \rightarrow \infty} \mathbb{P}_{\delta_x}(|Y_1(t)|^2; |Y_1(t)| > ce^{\delta t}) \leq \sigma_f^2 \phi_1(x) - \mathbb{P}_{\delta_x}(h_M(|G_1(f)\sqrt{W_\infty}|)^2).$$

Letting $M \rightarrow \infty$, by the monotone convergence theorem, we have that for any $x \in E$,

$$\lim_{M \rightarrow \infty} \mathbb{P}_{\delta_x}(h_M(|G_1(f)\sqrt{W_\infty}|)^2) = \mathbb{P}_{\delta_x}(G_1(f)^2 W_\infty) = \sigma_f^2 \phi_1(x),$$

which implies (3.40). The proof is now complete. \square

Recall that $\mathcal{C}_s, \mathcal{C}_c, \tau(f), \rho_h^2$ and σ_f^2 are defined in (1.39), (1.38), (1.35), (1.42) and (1.41), respectively.

LEMMA 3.5. *Assume that $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. Then*

$$(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f)),$$

where $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, $W_\infty, G_2(h)$ and $G_1(f)$ are independent.

PROOF. In this proof, we always assume $t > 10t_0, f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. We define an \mathbb{R}^3 -valued random variable by

$$U_1(t) := (e^{\lambda_1 t} \langle \phi_1, X_t \rangle, t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle).$$

For $n > 2$, we define

$$U_1(nt) = (e^{\lambda_1 nt} \langle \phi_1, X_{nt} \rangle, (nt)^{-(1+2\tau(h))/2} e^{\lambda_1 nt/2} \langle h, X_{nt} \rangle, e^{\lambda_1 nt/2} \langle f, X_{nt} \rangle).$$

Now we define another \mathbb{R}^3 -valued random variable $U_2(n, t)$ by

$$U_2(n, t) := \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{e^{\lambda_1 nt/2} (\langle h, X_{nt} \rangle - \langle T_{(n-1)t} h, X_t \rangle)}{((n-1)t)^{(1+2\tau(h))/2}}, e^{\lambda_1 nt/2} (\langle f, X_{nt} \rangle - \langle T_{(n-1)t} f, X_t \rangle) \right).$$

We claim that

$$(3.41) \quad U_2(n, t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f)) \quad \text{as } t \rightarrow \infty.$$

Denote the characteristic function of $U_2(n, t)$ under \mathbb{P}_μ by $\kappa_2(\theta_1, \theta_2, \theta_3, n, t)$. Define, for $s, t > 0$,

$$Y_1^{u,t}(s) := e^{\lambda_1 s/2} \langle f, X_s^{u,t} \rangle, \quad Y_2^{u,t}(s) := s^{-(1+2\tau(h))/2} e^{\lambda_1 s/2} \langle h, X_s^{u,t} \rangle.$$

We also define

$$Y_1(s) := e^{\lambda_1 s/2} \langle f, X_s \rangle, \quad Y_2(s) := s^{-(1+2\tau(h))/2} e^{\lambda_1 s/2} \langle h, X_s \rangle$$

and

$$Y_s(\theta_2, \theta_3) := \theta_2 Y_2(s) + \theta_3 Y_1(s).$$

Given \mathcal{F}_t , for $k = 1, 2$, $Y_k^{u,t}(s)$ has the same distribution as $Y_k(s)$ under $\mathbb{P}_{\delta_{z_u(t)}}$. Thus, for $k = 1, 2$,

$$y_k^{u,t}(s) := \mathbb{P}_{\delta_x}(Y_k^{u,t}(s) | \mathcal{F}_t) = \mathbb{P}_{\delta_{z_u(t)}} Y_k(s).$$

Thus, by (3.1), we have

$$U_2(n, t) = \left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, e^{\lambda_1 t/2} \sum_{u \in \mathcal{L}_t} (Y_2^{u,t}((n-1)t) - y_2^{u,t}((n-1)t)), \right. \\ \left. e^{\lambda_1 t/2} \sum_{u \in \mathcal{L}_t} (Y_1^{u,t}((n-1)t) - y_1^{u,t}((n-1)t)) \right).$$

Let $h(s, x, \theta, \theta_2, \theta_3) = \mathbb{P}_{\delta_x}(\exp\{i\theta(Y_s(\theta_2, \theta_3) - \mathbb{P}_{\delta_x} Y_s(\theta_2, \theta_3))\})$. Thus we get

$$\kappa_2(\theta_1, \theta_2, \theta_3, n, t) \\ = \mathbb{P}_{\delta_x} \left(\exp\{i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle\} \prod_{u \in \mathcal{L}_t} h((n-1)t, z_u(t), e^{\lambda_1 t/2}, \theta_2, \theta_3) \right).$$

Let $t_k, m_k \rightarrow \infty$, as $k \rightarrow \infty$. Now we consider

$$S_k := e^{\lambda_1 t_k/2} \sum_{j=1}^{m_k} (Y_{k,j} - y_{k,j}),$$

where $Y_{k,j}$ has the same law as $Y_{(n-1)t_k}(\theta_2, \theta_3)$ under $\mathbb{P}_{\delta_{a_{k,j}}}$ and $y_{k,j} = \mathbb{P}_{\delta_{a_{k,j}}} Y_{(n-1)t_k}(\theta_2, \theta_3)$ with $a_{k,j} \in E$. Further, for each positive integer k , $Y_{k,j}, j = 1, 2, \dots$ are independent. Denote $V_{t_k}^n(x) := \mathbb{V}\text{ar}_{\delta_x} Y_{(n-1)t_k}(\theta_2, \theta_3)$. Suppose the following Lindeberg conditions hold:

(i) as $k \rightarrow \infty$,

$$e^{\lambda_1 t_k} \sum_{j=1}^{m_k} \mathbb{E}(Y_{k,j} - y_{k,j})^2 = e^{\lambda_1 t_k} \sum_{j=1}^{m_k} V_{t_k}^n(a_{k,j}) \rightarrow \sigma^2;$$

(ii) for every $c > 0$,

$$e^{\lambda_1 t_k} \sum_{j=1}^{m_k} \mathbb{E}(|Y_{k,j} - y_{k,j}|^2, |Y_{k,j} - y_{k,j}| > c e^{-\lambda_1 t_k/2}) \\ = e^{\lambda_1 t_k} \sum_{j=1}^{m_k} g_{(n-1)t_k}(a_{k,j}, \theta_2, \theta_3) \rightarrow 0, \quad k \rightarrow \infty,$$

where

$$g_s(x, \theta_2, \theta_3) = \mathbb{P}_{\delta_x} \left(\left| Y_s(\theta_2, \theta_3) - \mathbb{P}_{\delta_x} Y_s(\theta_2, \theta_3) \right|^2 \right. \\ \left. \times \mathbb{1}_{\{|Y_s(\theta_2, \theta_3) - \mathbb{P}_{\delta_x} Y_s(\theta_2, \theta_3)| > ce^{-\lambda_1 s / (2(n-1))}\}} \right).$$

Then $S_k \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, which implies

$$(3.42) \quad \prod_{j=1}^{m_k} h((n-1)t_k, a_{k,j}, e^{\lambda_1 t_k / 2}, \theta_2, \theta_3) \rightarrow e^{-(1/2)\sigma^2 \theta^2} \quad \text{as } k \rightarrow \infty.$$

By the definition of Y_s , we get

$$V_t^n(x) := \text{Var}_{\delta_x} Y_{(n-1)t}(\theta_2, \theta_3) \\ = \theta_2^2 \text{Var}_{\delta_x} Y_2((n-1)t) + \theta_3^2 \text{Var}_{\delta_x} Y_1((n-1)t) \\ + 2\theta_2\theta_3((n-1)t)^{-(1+2\tau(h))/2} e^{\lambda_1(n-1)t} \\ \times \text{Cov}_{\delta_x}(\langle f, X_{(n-1)t} \rangle, \langle h, X_{(n-1)t} \rangle).$$

Thus, by (2.27), (2.36) and (2.46), we get that

$$|V_t^n(x) - (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) \phi_1(x)| \\ \lesssim (c_{(n-1)t} + t^{-1} + t^{-(1+2\tau(h))/2}) (b_{t_0}(x)^{1/2} + b_{t_0}(x)),$$

where $c_t \rightarrow 0$ as $t \rightarrow \infty$. By (2.17), we get, as $t \rightarrow \infty$,

$$e^{\lambda_1 t} T_t |V_t^n(x) - (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) \phi_1(x)|(x) \\ \lesssim (c_{(n-1)t} + t^{-1} + t^{-(1+2\tau(h))/2}) e^{\lambda_1 t} T_t (\sqrt{b_{t_0}} + b_{t_0})(x) \rightarrow 0,$$

which implies

$$\lim_{t \rightarrow \infty} e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} V_t^n(z_u(t)) = \lim_{t \rightarrow \infty} e^{\lambda_1 t} (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) \langle \phi_1, X_t \rangle \\ = (\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) W_\infty,$$

in probability.

By Lemma 3.4, we get, as $s \rightarrow \infty$, $g_s(x, \theta_2, \theta_3) \rightarrow 0$. Since

$$g_{(n-1)t}(x, \theta_2, \theta_3) \leq V_t^n(x) \lesssim b_{t_0}(x)^{1/2} + b_{t_0}(x) \in L^2(E, m),$$

by the dominated convergence theorem, we have that for any $x \in E$,

$$\lim_{t \rightarrow \infty} \|g_{(n-1)t}(x, \theta_2, \theta_3)\|_2 = 0.$$

By Lemma 2.17, we have that, for any $x \in E$, as $t \rightarrow \infty$,

$$e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle g_{(n-1)t}(\cdot, \theta_2, \theta_3), X_t \rangle \lesssim \|g_{(n-1)t}(\cdot, \theta_2, \theta_3)\|_2 b_{t_0}(x)^{1/2} \rightarrow 0,$$

which implies

$$e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} g_{(n-1)t}(z_u(t), \theta_2, \theta_3) = e^{\lambda_1 t} \langle g_{(n-1)t}(x, \theta_2, \theta_3), X_t \rangle \rightarrow 0,$$

in probability. Thus, for any sequence $s_k \rightarrow \infty$, there exists a subsequence s'_k such that, if we let $t_k = s'_k$, $m_k = |X_{t_k}|$ and $\{a_{k,j}, j = 1, \dots, m_k\} = \{z_u(t_k), u \in \mathcal{L}_{t_k}\}$, then the Lindeberg conditions hold \mathbb{P}_{δ_x} -a.s. Therefore, by (3.42), we have

$$\lim_{t \rightarrow \infty} \prod_{u \in \mathcal{L}_t} h((n-1)t, z_u(t), e^{\lambda_1 t/2}, \theta_2, \theta_3) = \exp\left\{-\frac{1}{2}(\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) W_\infty\right\}$$

in probability. Hence by the dominated convergence theorem, we get

$$\begin{aligned} & \lim_{t \rightarrow \infty} \kappa_2(\theta_1, \theta_2, \theta_3, n, t) \\ &= \mathbb{P}_{\delta_x} \left(\exp\{i\theta_1 W_\infty\} \exp\left\{-\frac{1}{2}(\theta_2^2 \rho_h^2 + \theta_3^2 \sigma_f^2) W_\infty\right\} \right), \end{aligned}$$

which implies our claim (3.41).

By (3.41) and the fact that $e^{\lambda_1 n t} \langle \phi_1, X_{nt} \rangle - e^{\lambda_1 t} \langle \phi_1, X_t \rangle \rightarrow 0$, in probability, as $t \rightarrow \infty$, we get that

$$\begin{aligned} & U_3(n, t) \\ &:= \left(e^{\lambda_1 n t} \langle \phi_1, X_{nt} \rangle, \frac{e^{\lambda_1 n t/2} (\langle h, X_{nt} \rangle - \langle T_{(n-1)t} h, X_t \rangle)}{(nt)^{(1+2\tau(h))/2}}, \right. \\ & \quad \left. e^{\lambda_1 n t/2} (\langle f, X_{nt} \rangle - \langle T_{(n-1)t} f, X_t \rangle) \right) \\ & \xrightarrow{d} \left(W_\infty, \left(\frac{n-1}{n} \right)^{(1+2\tau(h))/2} \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f) \right). \end{aligned}$$

Using (3.19) with $s = (n-1)t$, we get that, if $\gamma(f) < \infty$,

$$\begin{aligned} \mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 & \lesssim (nt)^{2\tau(f)} e^{-2nt\Re\gamma(f)} b_{t_0}(x)^{1/2} \\ & \quad + ((n-1)t)^{2\tau(f)} e^{-\lambda_1 t} e^{-2\Re\gamma(f)(n-1)t} b_{t_0}(x)^{1/2}. \end{aligned}$$

If $\gamma(f) = \infty$, using (3.20) with $s = (n-1)t$, we get

$$\mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 \lesssim b_{t_0}(x)^{1/2} + e^{-\lambda_1 t} b_{t_0}(x)^{1/2}.$$

Therefore, we have

$$(3.43) \quad \lim_{t \rightarrow \infty} e^{\lambda_1 n t} \mathbb{P}_{\delta_x} \langle T_{(n-1)t} f, X_t \rangle^2 = 0.$$

By (3.17), when $\lambda_1 = 2\mathfrak{R}_\gamma(h)$, we get

$$\begin{aligned} & \int_0^{t-2t_0} T_{t-u} [A(T_{u+(n-1)t}h)^2](x) du \\ & \lesssim e^{-\lambda_1 nt} \int_0^{t-2t_0} (u + (n-1)t)^{2\tau(h)} du b_{t_0}(x)^{1/2} \\ & \lesssim n^{2\tau(h)} t^{1+2\tau(h)} e^{-\lambda_1 nt} b_{t_0}(x)^{1/2}. \end{aligned}$$

By (3.44), (3.17) and (3.18), when $\lambda_1 = 2\mathfrak{R}_\gamma(h)$, we have

$$\mathbb{P}_{\delta_x} \langle T_{(n-1)t}h, X_t \rangle^2 \lesssim n^{2\tau(h)} t^{1+2\tau(h)} e^{-\lambda_1 nt} b_{t_0}(x)^{1/2} + (nt)^{2\tau(h)} e^{-\lambda_1 nt} b_{t_0}(x)^{1/2}.$$

Therefore, we have

$$(3.44) \quad \lim_{n \rightarrow \infty} \limsup_{t \rightarrow \infty} (nt)^{-(1+2\tau(h))} e^{\lambda_1 nt} \mathbb{P}_{\delta_x} \langle T_{(n-1)t}h, X_t \rangle^2 = 0.$$

Let $\mathcal{D}(nt)$ and $\tilde{\mathcal{D}}^n(t)$ be the distributions of $U_1(nt)$ and $U_3(n, t)$, respectively, and let \mathcal{D}^n and \mathcal{D} be those of

$$\left(W_\infty, \left(\frac{n-1}{n} \right)^{(1+2\tau(h))/2} \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f) \right)$$

and $(W_\infty, \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f))$, respectively. Then, using (3.12), we have

$$\begin{aligned} & \limsup_{t \rightarrow \infty} d(\mathcal{D}(nt), \mathcal{D}) \\ & \leq \limsup_{t \rightarrow \infty} [d(\mathcal{D}(nt), \tilde{\mathcal{D}}^n(t)) + d(\tilde{\mathcal{D}}^n(t), \mathcal{D}^n) + d(\mathcal{D}^n, \mathcal{D})] \\ & \leq \limsup_{t \rightarrow \infty} ((nt)^{-(1+2\tau(h))} e^{\lambda_1 nt} \mathbb{P}_\mu \langle T_{(n-1)t}h, X_t \rangle^2 + e^{\lambda_1 nt} \mathbb{P}_\mu \langle T_{(n-1)t}f, X_t \rangle^2)^{1/2} \\ & \quad + 0 + d(\mathcal{D}^n, \mathcal{D}). \end{aligned}$$

Using the definition of $\limsup_{t \rightarrow \infty}$, (3.43) and (3.44), we get that

$$\begin{aligned} \limsup_{t \rightarrow \infty} d(\mathcal{D}(t), \mathcal{D}) &= \limsup_{t \rightarrow \infty} d(\mathcal{D}(nt), \mathcal{D}) \\ &\leq \limsup_{t \rightarrow \infty} (nt)^{-(1+2\tau(h))} e^{nt\lambda_1 t} \mathbb{P}_{\delta_x} \langle T_{(n-1)t}h, X_t \rangle^2 + d(\mathcal{D}^n, \mathcal{D}). \end{aligned}$$

Letting $n \rightarrow \infty$, we get $\limsup_{t \rightarrow \infty} d(\mathcal{D}(t), \mathcal{D}) = 0$. The proof is now complete. \square

PROOF OF COROLLARY 1.18. Define

$$Y_1(s) := s^{-(1+2\tau(h_1))/2} e^{\lambda_1 s/2} \langle h_1, X_s \rangle, \quad Y_2(s) := s^{-(1+2\tau(h_2))/2} e^{\lambda_1 s/2} \langle h_2, X_s \rangle$$

and

$$Y_s(\theta_2, \theta_3) := \theta_2 Y_1(s) + \theta_3 Y_2(s).$$

Thus we have

$$\begin{aligned} \text{Var}_{\delta_x} Y_{(n-1)t}(\theta_2, \theta_3) &= \theta_2^2 \text{Var}_{\delta_x} Y_1((n-1)t) + \theta_3^2 \text{Var}_{\delta_x} Y_2((n-1)t) \\ &\quad + 2\theta_2\theta_3 \text{Cov}_{\delta_x}(Y_1((n-1)t), Y_2((n-1)t)). \end{aligned}$$

By (2.35) and (2.36), we get

$$\begin{aligned} &|\text{Var}_{\delta_x} Y_{(n-1)t}(\theta_2, \theta_3) - (\theta_2^2 \rho_{h_1}^2 + \theta_3^2 \rho_{h_2}^2 + 2\theta_2\theta_3 \rho(h_1, h_2)) \phi_1(x)| \\ &\lesssim t^{-1} (b_{t_0}(x)^{1/2} + b_{t_0}(x)). \end{aligned}$$

Using arguments similar to those leading to Lemma 3.5, we get

$$\begin{aligned} &\lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} \exp\{i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_2 Y_1(t) + i\theta_3 Y_2(t)\} \\ &= \mathbb{P}_{\delta_x} \exp\left\{i\theta_1 W_\infty - \frac{1}{2}(\theta_2^2 \rho_{h_1}^2 + \theta_3^2 \rho_{h_2}^2 + 2\theta_2\theta_3 \rho(h_1, h_2)) W_\infty\right\}. \end{aligned}$$

The proof of Corollary 1.18 is now complete. \square

Recall that

$$g(x) = \sum_{k: \lambda_1 > 2\mathfrak{R}_k} \Phi_k(x)^T v_k \in \mathcal{C}_c$$

and

$$I_s g(x) = \sum_{k: \lambda_1 > 2\mathfrak{R}_k} e^{\lambda_k s} \Phi_k(x)^T D_k(s)^{-1} v_k.$$

We can show that $I_s g$ is real. In fact, for k with $\lambda_1 > 2\mathfrak{R}_k$, we have $\lambda_1 > 2\mathfrak{R}_{k'}$ and

$$e^{\lambda_{k'} s} \Phi_{k'}(x)^T D_{k'}(s)^{-1} v_{k'} = e^{\bar{\lambda}_k s} \overline{\Phi_k(x)^T D_k(s)^{-1} v_k} = \overline{e^{\lambda_k s} \Phi_k(x)^T D_k(s)^{-1} v_k},$$

which implies that $I_s g(x)$ is real. Define

$$H_\infty := \sum_{k: \lambda_1 > 2\mathfrak{R}_k} H_\infty^{(k)} v_k.$$

By Lemma 3.1, we have, as $s \rightarrow \infty$,

$$\langle I_s g, X_s \rangle \rightarrow H_\infty, \quad \mathbb{P}_{\delta_x}\text{-a.s. and in } L^2(\mathbb{P}_{\delta_x}).$$

Since $\mathbb{P}_{\delta_x} \langle I_s g, X_s \rangle = g(x)$, we get

$$\mathbb{P}_{\delta_x}(H_\infty) = g(x).$$

By (2.20), we have

$$(3.45) \quad \mathbb{P}_{\delta_x} \langle I_s g, X_s \rangle^2 = \int_0^s T_u[A|I_u g|^2](x) du + T_s[(I_s g)^2](x).$$

By (1.20) and the fact that $|D_k(s)^{-1}|_\infty = |D_k(-s)|_\infty \lesssim (1 + s^{\nu_k})$, we get that

$$|I_s g(x)|^2 \lesssim \sum_{k:\lambda_1 > 2\mathfrak{N}_k} e^{2\mathfrak{N}_k s} (1 + s^{2\nu_k}) b_{4t_0}(x).$$

Thus by (2.17), we have, for $s > 2t_0$,

$$\begin{aligned} T_s |I_s g|^2(x) &\lesssim \sum_{k:\lambda_1 > 2\mathfrak{N}_k} e^{2\mathfrak{N}_k s} (1 + s^{2\nu_k}) T_s(b_{4t_0})(x) \\ (3.46) \qquad &\lesssim \sum_{k:2\mathfrak{N}_k < \lambda_1} (1 + s^{2\nu_k}) e^{(2\mathfrak{N}_k - \lambda_1)s} b_{t_0}(x)^{1/2}. \end{aligned}$$

By (2.24), we get

$$\begin{aligned} &\int_0^\infty T_u[A|I_u g|^2](x) du \\ &\lesssim \sum_{k:\lambda_1 > 2\mathfrak{N}_k} \left(\int_0^{2t_0} e^{2\mathfrak{N}_k u} (1 + u^{2\nu_k}) T_u(b_{4t_0})(x) du \right. \\ &\qquad \left. + \int_{2t_0}^\infty (1 + u^{2\nu_k}) e^{(2\mathfrak{N}_k - \lambda_1)u} du b_{t_0}(x)^{1/2} \right) \\ &\lesssim b_{t_0}(x)^{1/2} \in L^2(E, m) \cap L^4(E, m). \end{aligned}$$

Therefore, by (3.45) and (3.46), we get

$$\begin{aligned} \mathbb{P}_{\delta_x}(H_\infty)^2 &= \lim_{s \rightarrow \infty} \mathbb{P}_{\delta_x} |\langle I_s g, X_s \rangle|^2 \\ (3.47) \qquad &= \int_0^\infty T_u[A|I_u g|^2](x) du \in L^2(E, m) \cap L^4(E, m). \end{aligned}$$

Hence we have

$$\begin{aligned} \text{Var}_{\delta_x} H_\infty &= \mathbb{P}_{\delta_x}(H_\infty)^2 - (\mathbb{P}_{\delta_x} H_\infty)^2 \\ (3.48) \qquad &= \int_0^\infty T_u(A|I_u g|^2)(x) du - g(x)^2. \end{aligned}$$

PROOF OF THEOREM 1.16. Recall that

$$E_t(g) = \left(\sum_{k:2\lambda_k < \lambda_1} e^{-\lambda_k t} H_\infty^{(k)} D_k(t) v_k \right)$$

and

$$Y_1(t) := e^{\lambda_1 t/2} \langle f, X_t \rangle, \qquad Y_2(t) := t^{-(1+2\tau(h))/2} e^{\lambda_1 t/2} \langle h, X_t \rangle.$$

Consider an \mathbb{R}^4 -valued random variable $U_4(t)$ defined by

$$U_4(t) := (e^{\lambda_1 t} \langle \phi_1, X_t \rangle, e^{\lambda_1 t/2} (\langle g, X_t \rangle - E_t(g)), Y_2(t), Y_1(t)).$$

To get the conclusion of Theorem 1.16, it suffices to show that, under \mathbb{P}_{δ_x} ,

$$U_4(t) \xrightarrow{d} (W_\infty, \sqrt{W_\infty}G_3(g), \sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)),$$

where $W_\infty, G_3(g), G_2(h)$ and $G_1(f)$ are independent. Denote the characteristic function of $U_4(t)$ under \mathbb{P}_{δ_x} by $\kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t)$. Then, we only need to prove

$$\begin{aligned} &\lim_{t \rightarrow \infty} \kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t) \\ &= \mathbb{P}_\mu \left(\exp\{i\theta_1 W_\infty\} \exp \left\{ -\frac{1}{2}(\theta_2^2 \beta_g^2 + \theta_3^2 \rho_h^2 + \theta_4^2 \sigma_f^2) W_\infty \right\} \right). \end{aligned}$$

Note that, by Lemma 3.1, we get

$$E_t(g) = \lim_{s \rightarrow \infty} \langle I_s g, X_{t+s} \rangle = \sum_{u \in \mathcal{L}_t} \lim_{s \rightarrow \infty} \langle I_s g, X_s^{u,t} \rangle.$$

Since $X_s^{u,t}$ has the same law as X_s under $\mathbb{P}_{\delta_{z_u(t)}}$, $H_\infty^{u,t} := \lim_{s \rightarrow \infty} \langle I_s g, X_s^{u,t} \rangle$ exists and has the same law as H_∞ under $\mathbb{P}_{\delta_{z_u(t)}}$. Thus we get $E_t(g) = \sum_{u \in \mathcal{L}_t} H_\infty^{u,t}$. Let $h(x, \theta) = \mathbb{P}_{\delta_x} \exp\{i\theta(H_\infty - g(x))\}$. Therefore, we obtain that

$$\begin{aligned} &\kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t) \\ &= \mathbb{P}_{\delta_x} \left(\exp\{i\theta_1 e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_3 Y_2(t) + i\theta_4 Y_1(t)\} \prod_{u \in \mathcal{L}_t} h(z_u(t), -\theta_2 e^{\lambda_1 t/2}) \right). \end{aligned}$$

Let $V(x) = \text{Var}_{\delta_x} H_\infty$. We claim that:

(i) as $t \rightarrow \infty$,

$$\begin{aligned} &e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} \mathbb{P}_{\delta_x} |H_\infty^{u,t} - g(z_u(t))|^2 = e^{\lambda_1 t} \langle V, X_t \rangle \\ (3.49) \quad &\hspace{15em} \rightarrow \langle V, \psi_1 \rangle_m W_\infty \quad \text{in probability;} \end{aligned}$$

(ii) for any $\varepsilon > 0$, as $t \rightarrow \infty$,

$$\begin{aligned} &e^{\lambda_1 t} \sum_{u \in \mathcal{L}_t} \mathbb{P}_{\delta_x} (|H_\infty^{u,t} - g(z_u(t))|^2, |H_\infty^{u,t} - g(z_u(t))| > \varepsilon e^{-\lambda_1 t/2}) \\ (3.50) \quad &= e^{\lambda_1 t} \langle k(\cdot, t), X_t \rangle \rightarrow 0 \quad \text{in probability,} \end{aligned}$$

where $k(x, t) := \mathbb{P}_{\delta_x} (|H_\infty - g(x)|^2, |H_\infty - g(x)| > \varepsilon e^{-\lambda_1 t/2})$.

Then using arguments similar to those in the proof Lemma 3.5, we have

$$(3.51) \quad \prod_{u \in \mathcal{L}_t} h(z_u(t), -\theta_2 e^{(\lambda_1/2)t}) \rightarrow \exp \left\{ -\frac{1}{2} \theta_2^2 \langle V, \psi_1 \rangle_m W_\infty \right\},$$

in probability.

Now we prove the claims:

(i) By (3.47), we have $V(x) \in L^2(E, m) \cap L^4(E, m)$. By Remark 1.15, (3.49) follows immediately.

(ii) Note that $k(x, t) \downarrow 0$ as $t \uparrow \infty$ and $k(x, t) \leq V(x) \in L^2(E, m)$ for any $x \in E$. Thus $\lim_{t \rightarrow \infty} \|k(\cdot, t)\|_2 = 0$. So by (2.17), we have that for any $x \in E$,

$$e^{\lambda_1 t} \mathbb{P}_{\delta_x} \langle k(\cdot, t), X_t \rangle \lesssim \|k(\cdot, t)\|_2 b_{t_0}(x)^{1/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

which implies (3.50).

By (3.49), (3.51) and the dominated convergence theorem, we get that as $t \rightarrow \infty$,

$$\begin{aligned} & \left| \kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t) \right. \\ & \left. - \mathbb{P}_{\delta_x} \left(\exp \left\{ \left(i\theta_1 - \frac{1}{2} \theta_2^2 \langle V, \psi_1 \rangle_m \right) e^{\lambda_1 t} \langle \phi_1, X_t \rangle + i\theta_3 Y_2(t) + i\theta_4 Y_1(t) \right\} \right) \right| \\ & \leq \mathbb{P}_{\delta_x} \left| \prod_{u \in \mathcal{L}_t} h(z_u(t), -\theta_2 e^{(\lambda_1/2)t}) - \exp \left\{ -\frac{1}{2} \theta_2^2 \langle V, \psi_1 \rangle_m e^{\lambda_1 t} \langle \phi_1, X_t \rangle \right\} \right| \\ & \rightarrow 0. \end{aligned}$$

By Lemma 3.5, we get

$$\begin{aligned} & \lim_{t \rightarrow \infty} \kappa_3(\theta_1, \theta_2, \theta_3, \theta_4, t) \\ & = \lim_{t \rightarrow \infty} \mathbb{P}_{\delta_x} \left(\exp \left\{ \left(i\theta_1 - \frac{1}{2} \theta_2^2 \langle V, \psi_1 \rangle_m \right) e^{\lambda_1 t} \langle \phi_1, X_t \rangle \right. \right. \\ & \qquad \qquad \qquad \left. \left. + i\theta_3 Y_1(t) + i\theta_4 Y_2(t) \right\} \right) \\ & = \mathbb{P}_{\delta_x} \left(\exp \{ i\theta_1 W_\infty \} \exp \left\{ -\frac{1}{2} (\theta_2^2 \langle V, \psi_1 \rangle_m + \theta_3^2 \rho_h^2 + \theta_4^2 \sigma_f^2) W_\infty \right\} \right). \end{aligned}$$

By (3.48), we get

$$\langle V, \psi_1 \rangle_m = \int_0^\infty e^{-\lambda_1 u} \langle A | I_u g|^2, \psi_1 \rangle_m du - \langle g^2, \psi_1 \rangle_m.$$

The proof is now complete. \square

Acknowledgments. We thank the referee for helpful comments on the first version of this paper.

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