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Stochastic Processes and their Applications 125 (2015) 428–457

stochastic processes and their applications

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Central limit theorems for supercritical superprocesses

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Received 18 July 2014; received in revised form 6 September 2014; accepted 18 September 2014
Available online 28 September 2014

Abstract

In this paper, we establish a central limit theorem for a large class of general supercritical superprocesses with spatially dependent branching mechanisms satisfying a second moment condition. This central limit theorem generalizes and unifies all the central limit theorems obtained recently in Miłoś (2012) and Ren et al. (2014) for supercritical super Ornstein–Uhlenbeck processes. The advantage of this central limit theorem is that it allows us to characterize the limit Gaussian field. In the case of supercritical super Ornstein–Uhlenbeck processes with non-spatially dependent branching mechanisms, our central limit theorem reveals more independent structures of the limit Gaussian field.

MSC: primary 60J68; secondary 60F05; 60G57; 60J45

Keywords: Central limit theorem; Supercritical superprocess; Excursion measures of superprocesses

1. Introduction

Central limit theorems for supercritical branching processes were initiated by [13,14]. In these two papers, Kesten and Stigum established central limit theorems for supercritical multitype Galton–Watson processes by using the Jordan canonical form of the expectation matrix

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http://dx.doi.org/10.1016/j.spa.2014.09.014
0304-4149/

M. Then in [4–6], Athreya proved central limit theorems for supercritical multi-type continuous time branching processes, using the Jordan canonical form and the eigenvectors of the matrix M_t, the mean matrix at time t. Asmussen and Keiding [3] used martingale central limit theorems to prove central limit theorems for supercritical multitype branching processes. In [2], Asmussen and Hering established spatial central limit theorems for general supercritical branching Markov processes under a certain condition. However, the condition in [2] is not easy to check and essentially the only examples given in [2] of branching Markov processes satisfying this condition are branching diffusions in bounded smooth domains. In [1], Adamczak and Miłoś proved some central limit theorems for supercritical branching Ornstein–Uhlenbeck processes with binary branching mechanism. We note that branching Ornstein–Uhlenbeck processes do not satisfy the condition in [2]. In [18], Miłoś proved some central limit theorems for supercritical super Ornstein–Uhlenbeck processes with branching mechanisms satisfying a fourth moment condition. In [19], we established central limit theorems for supercritical super Ornstein–Uhlenbeck processes with (non-spatially dependent) branching mechanisms satisfying only a second moment condition. More importantly, the central limit theorems in [19] are more satisfactory since our limit normal random variables are non-degenerate. In the recent paper [20], we obtained central limit theorems for a large class of general supercritical branching Markov processes with spatially dependent branching mechanisms satisfying only a second moment condition. The main results of [20] are the central limit theorems contained in [20, Theorems 1.8, 1.9, 1.10, 1.12]. [20, Theorem 1.8] is the branching Markov process analog of the convergence of the first and fourth components in Theorem 1.4. [20, Theorem 1.9] is the branching Markov process analog of Remark 1.9, while [20, Theorems 1.10, 1.12] are the branching Markov process analogs of the results in Remark 1.10.

It is a natural next step to try to establish counterparts of the central limit theorems of [20] for general supercritical superprocesses with spatially dependent branching mechanisms satisfying only a second moment condition. This is far from trivial. For a branching Markov process $\{Z_t : t \geq 0\}$, to consider the proper scaling limit of $\langle f, Z_t \rangle$ as $t \to \infty$, where f is a test function, it is equivalent to consider the scaling limit of $\langle f, Z_{t+s} \rangle$ as $s \to \infty$ for any $t > 0$. Note that $Z_{t+s} = \sum_{u \in \mathcal{L}_t} Z_s^{u,t}$, where $\mathcal{L}_t$ is the set of particles which are alive at time t, and $Z_s^{u,t}$ is the branching Markov process starting from the particle $u \in \mathcal{L}_t$. So, conditioned on Z_t, Z_{t+s} is the sum of a finite number of independent terms, and then basically we only need to consider central limit theorems of independent random variables. However, a superprocess is an appropriate scaling limit of branching Markov processes, see [8,17], for example. It describes the time evolution of a cloud of uncountable number of particles, where each particle carries mass 0 and moves in space independently according to a Markov process. The particle picture for superprocesses is not very clear. Recently [15] gave a backbone decomposition of superdiffusions, where the backbone is a branching diffusion. One could combine the ideas of [19] with that of [20] to use the backbone decomposition to prove central limit theorems for general supercritical superprocesses with spatial dependent branching mechanisms satisfying only a second moment condition, provided that the backbone decomposition is known. However, up to now, the backbone decomposition has only been established for supercritical superdiffusions with spatial dependent branching mechanisms.

In this paper, our assumption on the spatial process is exactly the same as in [20], while our assumptions on the branching mechanism are similar in spirit to those of [20]. We will use the excursion measures of the superprocess as a tool to replace the backbone decomposition. With this new tool, the general methodology of [20] can be adapted to the present setting of general supercritical superprocesses.

Actually, we will go even further in the present paper. We will prove one central limit theorem which generalizes and unifies all the central limit theorems of [18,19]. See the Corollaries and Remarks after Theorem 1.4. The advantage of this central limit theorem is that it allows us to characterize the limit Gaussian field. In the case of supercritical super Ornstein–Uhlenbeck processes with non-spatially dependent branching mechanisms satisfying a second moment condition, our central limit theorem reveals more independent structures of the limit Gaussian field, see Corollaries 1.5–1.7.

1.1. Spatial process

Our assumptions on the underlying spatial process are the same as in [20]. In this subsection, we recall the assumptions on the spatial process.

E is a locally compact separable metric space and m is a σ-finite Borel measure on E with full support. ∂ is a point not contained in E and will be interpreted as the cemetery point. Every function f on E is automatically extended to $E_\partial := E \cup \{\partial\}$ by setting $f(\partial) = 0$. We will assume that $\xi = \{\xi_t, \Pi_x\}$ is an m-symmetric Hunt process on E and $\zeta := \inf\{t > 0 : \xi_t = \partial\}$ is the lifetime of ξ. The semigroup of ξ will be denoted by $\{P_t : t \geq 0\}$. We will always assume that there exists a family of continuous strictly positive symmetric functions $\{p_t(x, y) : t > 0\}$ on $E \times E$ such that

$$P_t f(x) = \int_E p_t(x, y) f(y)\, m(dy).$$

It is well-known that for $p \geq 1$, $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $L^p(E, m)$.

Define $\widetilde{a}_t(x) := p_t(x, x)$. We will always assume that $\widetilde{a}_t(x)$ satisfies the following two conditions:

(a) For any $t > 0$, we have

$$\int_E \widetilde{a}_t(x)\, m(dx) < \infty.$$

(b) There exists $t_0 > 0$ such that $\widetilde{a}_{t_0}(x) \in L^2(E,\ m)$.

It is easy to check (see [20]) that condition (b) above is equivalent to

(b′) There exists $t_0 > 0$ such that for all $t \geq t_0$, $\widetilde{a}_t(x) \in L^2(E, m)$.

These two conditions are satisfied by a lot of Markov processes. In [20], we gave several classes of examples of Markov processes, including Ornstein–Uhlenbeck processes, satisfying these two conditions.

1.2. Superprocesses

In this subsection, we will spell out our assumptions on the superprocess we are going to work with. Let $\mathcal{B}_b(E)$ ($\mathcal{B}_b^+(E)$) be the set of (positive) bounded Borel measurable functions on E.

The superprocess $X = \{X_t : t \geq 0\}$ we are going to work with is determined by three parameters: a spatial motion $\xi = \{\xi_t, \Pi_x\}$ on E satisfying the assumptions of the previous subsection, a branching rate function $\beta(x)$ on E which is a non-negative bounded measurable function and a branching mechanism ψ of the form

$$\psi(x, \lambda) = -a(x)\lambda + b(x)\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda y} - 1 + \lambda y) n(x, dy), \quad x \in E,\ \lambda > 0, \tag{1.1}$$

where $a \in \mathcal{B}_b(E)$, $b \in \mathcal{B}_b^+(E)$ and n is a kernel from E to $(0, \infty)$ satisfying

$$\sup_{x\in E} \int_0^\infty y^2 n(x, dy) < \infty. \tag{1.2}$$

Let $\mathcal{M}_F(E)$ be the space of finite measures on E equipped with the topology of weak convergence. The existence of such superprocesses is well-known, see, for instance, [10] or [17]. X is a cadlag Markov process taking values in $\mathcal{M}_F(E)$. For any $\mu \in \mathcal{M}_F(E)$, we denote the law of X with initial configuration μ by $\mathbb{P}_\mu$. As usual, $\langle f, \mu\rangle := \int f(x)\mu(dx)$ and $\|\mu\| := \langle 1, \mu\rangle$. Then for every $f \in \mathcal{B}_b^+(E)$ and $\mu \in \mathcal{M}_F(E)$,

$$-\log \mathbb{P}_\mu \left(e^{-\langle f, X_t\rangle}\right) = \langle u_f(\cdot, t), \mu\rangle, \tag{1.3}$$

where $u_f(x, t)$ is the unique positive solution to the equation

$$u_f(x, t) + \Pi_x \int_0^t \psi(\xi_s, u_f(\xi_s, t - s))\beta(\xi_s)ds = \Pi_x f(\xi_t), \tag{1.4}$$

where $\psi(\partial, \lambda) = 0, \lambda > 0$. Define

$$\alpha(x) := \beta(x)a(x) \quad \text{and} \quad A(x) := \beta(x)\left(2b(x) + \int_0^\infty y^2 n(x, dy)\right). \tag{1.5}$$

Then, by our assumptions, $\alpha(x) \in \mathcal{B}_b(E)$ and $A(x) \in \mathcal{B}_b(E)$. Thus there exists $M > 0$ such that

$$\sup_{x\in E} (|\alpha(x)| + A(x)) \le M. \tag{1.6}$$

For any $f \in \mathcal{B}_b(E)$ and $(t, x) \in (0, \infty) \times E$, define

$$T_t f(x) := \Pi_x \left[e^{\int_0^t \alpha(\xi_s)\, ds} f(\xi_t)\right]. \tag{1.7}$$

It is well-known that $T_t f(x) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle$ for every $x \in E$.

It is shown in [20] that there exists a family of continuous strictly positive symmetric functions $\{q_t(x, y), t > 0\}$ on $E \times E$ such that $q_t(x, y) \le e^{Mt} p_t(x, y)$ and for any $f \in \mathcal{B}_b(E)$,

$$T_t f(x) = \int_E q_t(x, y) f(y)\, m(dy).$$

It follows immediately that, for any $p \ge 1$, $\{T_t : t \ge 0\}$ is a strongly continuous semigroup on $L^p(E, m)$ and

$$\|T_t f\|_p^p \le e^{pMt}\|f\|_p^p. \tag{1.8}$$

Define $a_t(x) := q_t(x, x)$. It follows from the assumptions (a) and (b) in the previous subsection that a_t enjoys the following properties:

(i) For any $t > 0$, we have

$$\int_E a_t(x)\, m(dx) < \infty.$$

(ii) There exists $t_0 > 0$ such that for all $t \ge t_0$, $a_t(x) \in L^2(E, m)$.

It follows from (i) above that, for any $t > 0$, T_t is a compact operator. The infinitesimal generator L of $\{T_t : t \ge 0\}$ in $L^2(E, m)$ has purely discrete spectrum with eigenvalues

$-\lambda_1 > -\lambda_2 > -\lambda_3 > \cdots$. It is known that either the number of these eigenvalues is finite, or $\lim_{k\to\infty}\lambda_k = \infty$. The first eigenvalue $-\lambda_1$ is simple and the eigenfunction ϕ_1 associated with $-\lambda_1$ can be chosen to be strictly positive everywhere and continuous. We will assume that $\|\phi_1\|_2 = 1$. ϕ_1 is sometimes denoted as $\phi_1^{(1)}$. For $k > 1$, let $\{\phi_j^{(k)}, j = 1, 2, \ldots, n_k\}$ be an orthonormal basis of the eigenspace (which is finite dimensional) associated with $-\lambda_k$. It is well-known that $\{\phi_j^{(k)}, j = 1, 2, \ldots, n_k; k = 1, 2, \ldots\}$ forms a complete orthonormal basis of $L^2(E, m)$ and all the eigenfunctions are continuous. For any $k \geq 1$, $j = 1, \ldots, n_k$ and $t > 0$, we have $T_t\phi_j^{(k)}(x) = e^{-\lambda_k t}\phi_j^{(k)}(x)$ and

$$e^{-\lambda_k t/2}|\phi_j^{(k)}|(x) \leq a_t(x)^{1/2}, \quad x \in E. \tag{1.9}$$

It follows from the relation above that all the eigenfunctions $\phi_j^{(k)}$ belong to $L^4(E, m)$. For any $x, y \in E$ and $t > 0$, we have

$$q_t(x, y) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\sum_{j=1}^{n_k}\phi_j^{(k)}(x)\phi_j^{(k)}(y), \tag{1.10}$$

where the series is locally uniformly convergent on $E \times E$. The basic facts recalled in this paragraph are well-known, for instance, one can refer to [7, Section 2].

In this paper, we always assume that the superprocess X is supercritical, that is, $\lambda_1 < 0$. Under this assumption, the process X has a strictly positive survival probability, see the next paragraph. Note that the number of negative eigenvalues is infinite except in the case when the total number of eigenvalues is finite.

We will use $\{\mathcal{F}_t : t \geq 0\}$ to denote the filtration of X, that is $\mathcal{F}_t = \sigma(X_s : s \in [0, t])$. Using the expectation formula of $\langle\phi_1, X_t\rangle$ and the Markov property of X, it is easy to show that (see Lemma 1.1), for any nonzero $\mu \in \mathcal{M}_F(E)$, under $\mathbb{P}_\mu$, the process $W_t := e^{\lambda_1 t}\langle\phi_1, X_t\rangle$ is a positive martingale. Therefore it converges:

$$W_t \to W_\infty, \quad \mathbb{P}_\mu\text{-a.s. as } t \to \infty.$$

Using the assumption (1.2) we can show that, as $t \to \infty$, W_t also converges in $L^2(\mathbb{P}_\mu)$, so W_∞ is non-degenerate and the second moment is finite. Moreover, we have $\mathbb{P}_\mu(W_\infty) = \langle\phi_1, \mu\rangle$. Put $\mathcal{E} = \{W_\infty = 0\}$, then $\mathbb{P}_\mu(\mathcal{E}) < 1$. It is clear that $\mathcal{E}^c \subset \{X_t(E) > 0, \forall t \geq 0\}$.

In this paper, we also assume that, for any $t > 0$ and $x \in E$,

$$\mathbb{P}_{\delta_x}\{\|X_t\| = 0\} \in (0, 1). \tag{1.11}$$

Here we give a sufficient condition for (1.11). Suppose that $\Phi(z) = \inf_{x\in E}\psi(x, z)\beta(x)$ can be written in the form:

$$\Phi(z) = \widetilde{a}z + \widetilde{b}z^2 + \int_0^\infty (e^{-zy} - 1 + zy)\widetilde{n}(dy)$$

with $\widetilde{a} \in \mathbb{R}$, $\widetilde{b} \geq 0$ and $\widetilde{n}$ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (y \wedge y^2)\widetilde{n}(dy) < \infty$. If $\widetilde{b} + \widetilde{n}(0, \infty) > 0$ and $\Phi(z)$ satisfies

$$\int^\infty \frac{1}{\Phi(z)}\,dz < \infty, \tag{1.12}$$

then (1.11) holds. For the last claim, see, for instance, [8, Lemma 11.5.1].

1.3. Main result

We will use $\langle \cdot, \cdot \rangle_m$ to denote inner product in $L^2(E, m)$. Any $f \in L^2(E, m)$ admits the following eigen-expansion:

$$f(x) = \sum_{k=1}^{\infty} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x), \tag{1.13}$$

where $a_j^k = \langle f, \phi_j^{(k)} \rangle_m$ and the series converges in $L^2(E, m)$. a_1^1 will sometimes be written as a_1. For $f \in L^2(E, m)$, define

$$\gamma(f) := \inf\{k \geq 1 : \text{ there exists } j \text{ with } 1 \leq j \leq n_k \text{ such that } a_j^k \neq 0\},$$

where we use the usual convention $\inf \varnothing = \infty$. Note that $\gamma(f) = \infty$ if and only if $f = 0$, m-a.e. We put $\lambda_\infty := \infty$.

For any $f \in L^2(E, m)$, we define

$$f^*(x) := \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)} \phi_j^{(\gamma(f))}(x).$$

We note that if $f \in L^2(E, m)$ is nonnegative and $m(x : f(x) > 0) > 0$, then $\langle f, \phi_1 \rangle_m > 0$ which implies $\gamma(f) = 1$ and $f^*(x) = a_1\phi_1(x) = \langle f, \phi_1 \rangle_m \phi_1(x)$. The following three subsets of $L^2(E, m)$ will be needed in the statement of the main result:

$$\mathcal{C}_l := \left\{ g(x) = \sum_{k:\lambda_1 > 2\lambda_k} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : b_j^k \in \mathbb{R} \text{ and } g \neq 0 \right\},$$

$$\mathcal{C}_c := \left\{ g(x) = \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) : 2\lambda_k = \lambda_1, b_j^k \in \mathbb{R} \text{ and } g \neq 0 \right\}$$

and

$$\mathcal{C}_s := \left\{ g(x) \in L^2(E, m) \cap L^4(E, m) : g \neq 0 \text{ and } \lambda_1 < 2\lambda_{\gamma(g)} \right\}.$$

Note that $\mathcal{C}_l$ consists of these functions in $L^2(E, m) \cap L^4(E, m)$ that only have nontrivial projection onto the eigen-spaces corresponding to those "large" eigenvalues $-\lambda_k$ satisfying $\lambda_1 > 2\lambda_k$. The space $\mathcal{C}_l$ is of finite dimension. The space $\mathcal{C}_c$ is the (finite dimensional) eigen-space corresponding to the "critical" eigenvalue $-\lambda_k$ with $\lambda_1 = 2\lambda_k$. Note that there may not be a critical eigenvalue and in this case, $\mathcal{C}_c$ is empty. The space $\mathcal{C}_s$ consists of these functions in $L^2(E, m) \cap L^4(E, m)$ that only have nontrivial projections onto the eigen-spaces corresponding to those "small" eigenvalues $-\lambda_k$ satisfying $\lambda_1 < 2\lambda_k$. The space $\mathcal{C}_s$ is of infinite dimensional in general.

In this subsection we give the main result of this paper. The proof will be given in Section 3. In the remainder of this paper, whenever we deal with an initial configuration $\mu \in \mathcal{M}_F(E)$, we are implicitly assuming that it has compact support.

1.3.1. Some basic convergence results

Define

$$H_t^{k,j} := e^{\lambda_k t} \langle \phi_j^{(k)}, X_t \rangle, \quad t \geq 0.$$

Using the same argument as in the proof of [20, Lemma 3.1], we can show that

Lemma 1.1. *$H_t^{k,j}$ is a martingale under $\mathbb{P}_\mu$. Moreover, if $\lambda_1 > 2\lambda_k$, $\sup_{t>3t_0} \mathbb{P}_\mu (H_t^{k,j})^2 < \infty$. Thus the limit*

$$H_\infty^{k,j} := \lim_{t\to\infty} H_t^{k.j}$$

exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$.

Theorem 1.2. *If $f \in L^2(E,m) \cap L^4(E,m)$ with $\lambda_1 > 2\lambda_{\gamma(f)}$, then, as $t \to \infty$,*

$$e^{\lambda_{\gamma(f)}t}\langle f, X_t\rangle \to \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)} H_\infty^{\gamma(f),j}, \quad \text{in } L^2(\mathbb{P}_\mu).$$

Proof. The proof is similar to that of [20, Theorem 1.6]. We omit the details here. □

Remark 1.3. When $\gamma(f) = 1$, $H_t^{1,1}$ reduces to W_t, and thus $H_\infty^{1,1} = W_\infty$. Therefore by Theorem 1.2 and the fact that $a_1 = \langle f, \phi_1\rangle_m$, we get that, as $t \to \infty$,

$$e^{\lambda_1 t}\langle f, X_t\rangle \to \langle f, \phi_1\rangle_m W_\infty, \quad \text{in } L^2(\mathbb{P}_\mu).$$

In particular, the convergence also holds in $\mathbb{P}_\mu$-probability.

1.3.2. Main result

For $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$, we define

$$\sigma_f^2 := \int_0^\infty e^{\lambda_1 s}\langle A(T_s f)^2, \phi_1\rangle_m \, ds \tag{1.14}$$

and

$$\rho_h^2 := \left\langle Ah^2, \phi_1\right\rangle_m . \tag{1.15}$$

For $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, we define

$$I_s g(x) := \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} e^{\lambda_k s} b_j^k \phi_j^{(k)}(x) \quad \text{and} \quad \beta_g^2 := \int_0^\infty e^{-\lambda_1 s}\left\langle A(I_s g)^2, \phi_1\right\rangle_m ds. \tag{1.16}$$

Theorem 1.4. *If $f \in \mathcal{C}_s$, $h \in \mathcal{C}_c$ and $g(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x) \in \mathcal{C}_l$, then $\sigma_f^2 < \infty$, $\rho_h^2 < \infty$ and $\beta_g^2 < \infty$. Furthermore, it holds that, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, as $t \to \infty$,*

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, \ \frac{\langle g, X_t\rangle - \sum_{k:2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} b_j^k H_\infty^{k,j}}{\sqrt{\langle \phi_1, X_t\rangle}}, \ \frac{\langle h, X_t\rangle}{\sqrt{t\langle \phi_1, X_t\rangle}}, \ \frac{\langle f, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}} \right)$$
$$\xrightarrow{d} (W^*, G_3(g), G_2(h), \ G_1(f)), \tag{1.17}$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$, $G_3(g) \sim \mathcal{N}(0, \beta_g^2)$, $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W^*, $G_3(g)$, $G_2(h)$ and $G_1(f)$ are independent.

This theorem says that, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$, as $t \to \infty$, the limits of the second, third and fourth components on the right hand side of (1.17) are nondegenerate normal random variables. Furthermore, the limit normal random variables are independent. As consequences of this theorem, we could also get the covariance of the limit random variables $G_1(f_1)$ and $G_1(f_2)$ when $f_1, f_2 \in \mathcal{C}_s$, the covariance of the limit random variables $G_2(h_1)$ and $G_2(h_2)$ when $h_1, h_2 \in \mathcal{C}_c$, and the covariance of the limit random variables $G_3(g_1)$ and $G_3(g_2)$ when $g_1, g_2 \in \mathcal{C}_l$.

For $f_1, f_2 \in \mathcal{C}_s$, define

$$\sigma(f_1, f_2) = \int_0^\infty e^{\lambda_1 s}\langle A(T_s f_1)(T_s f_2), \phi_1\rangle_m \, ds.$$

Note that $\sigma(f, f) = \sigma_f^2$.

Corollary 1.5. *If $f_1, f_2 \in \mathcal{C}_s$, then, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,*

$$\left(\frac{\langle f_1, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}}, \frac{\langle f_2, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}}\right) \xrightarrow{d} (G_1(f_1), G_1(f_2)), \quad t \to \infty,$$

where $(G_1(f_1), G_1(f_2))$ is a bivariate normal random variable with covariance

$$\mathrm{Cov}(G_1(f_i), G_1(f_j)) = \sigma(f_i, f_j), \quad i, j = 1, 2. \tag{1.18}$$

Consider the special situation when both the branching mechanism and the branching rate function are non-spatially dependent, and ϕ_1 is a constant function (this is the case of Ornstein–Uhlenbeck processes). If $f_1 = \phi_j^{(k)}$ and $f_2 = \phi_{j'}^{(k')}$ are distinct eigenfunctions satisfying $\lambda_1 < 2\lambda_k$ and $\lambda_1 < 2\lambda_{k'}$, then $G_1(f_1)$ and $G_1(f_2)$ are independent.

Proof. Using the convergence of the fourth component in Theorem 1.4, we get

$$\begin{aligned}
&\mathbf{P}_\mu\left(\exp\left\{i\theta_1\frac{\langle f_1, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}} + i\theta_2\frac{\langle f_2, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}}\right\}\Bigg|\mathcal{E}^c\right)\\
&\quad = \mathbf{P}_\mu\left(\exp\left\{i\frac{\langle \theta_1 f_1 + \theta_2 f_2, X_t\rangle}{\sqrt{\langle \phi_1, X_t\rangle}}\right\}\Bigg|\mathcal{E}^c\right) \to \exp\left\{-\frac{1}{2}\sigma^2_{(\theta_1 f_1 + \theta_2 f_2)}\right\}, \quad \text{as } t \to \infty,
\end{aligned}$$

where

$$\begin{aligned}
\sigma^2_{(\theta_1 f_1 + \theta_2 f_2)} &= \int_0^\infty e^{\lambda_1 s}\langle A(T_s(\theta_1 f_1 + \theta_2 f_2))^2, \phi_1\rangle_m \, ds\\
&= \theta_1^2\sigma_{f_1}^2 + 2\theta_1\theta_2\sigma(f_1, f_2) + \theta_2\sigma_{f_2}^2.
\end{aligned}$$

Note that $\exp\left\{-\frac{1}{2}\left(\theta_1^2\sigma_{f_1}^2 + 2\theta_1\theta_2\sigma(f_1, f_2) + \theta_2\sigma_{f_2}^2\right)\right\}$ is the characteristic function of $(G_1(f_1), G_1(f_2))$, which is a bivariate normal random variable with covariance $\mathrm{Cov}(G_1(f_i), G_1(f_j)) = \sigma(f_i, f_j)$, $i, j = 1, 2$. The desired result now follows immediately.

In particular, if both the branching mechanism and the branching rate function are non-spatially dependent, then $A(x) = A$ is a constant. If ϕ_1 is a constant function, and $f_1 = \phi_j^{(k)}$ and $f_2 = \phi_{j'}^{(k')}$ are distinct eigenfunctions satisfying $\lambda_1 < 2\lambda_k$ and $\lambda_1 < 2\lambda_{k'}$, then

$$\sigma(f_1, f_2) = A\phi_1 \int_0^\infty e^{(\lambda_1 - \lambda_k - \lambda_{k'})s}\langle \phi_j^{(k)}, \phi_{j'}^{(k')}\rangle_m \, ds = 0$$

and thus $G_1(f_1)$ and $G_1(f_2)$ are independent. □

For $h_1, h_2 \in \mathcal{C}_c$, define

$$\rho(h_1, h_2) = \langle A h_1 h_2, \phi_1 \rangle_m.$$

Using the convergence of the third component in Theorem 1.4 and an argument similar to that in the proof of Corollary 1.5, we get

Corollary 1.6. *If $h_1, h_2 \in \mathcal{C}_c$, then we have, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,*

$$\left(\frac{\langle h_1, X_t \rangle}{\sqrt{t \langle \phi_1, X_t \rangle}}, \frac{\langle h_2, X_t \rangle}{\sqrt{t \langle \phi_1, X_t \rangle}} \right) \stackrel{d}{\to} (G_2(h_1), G_2(h_2)), \quad t \to \infty,$$

where $(G_2(h_1), G_2(h_2))$ is a bivariate normal random variable with covariance

$$Cov(G_2(h_i), G_2(h_j)) = \rho(h_i, h_j), \quad i, j = 1, 2.$$

Consider the special situation when both the branching mechanism and the branching rate function are non-spatial dependent and ϕ_1 is a constant function. If $h_1 = \phi_j^{(k)}$ and $h_2 = \phi_{j'}^{(k)}$ are distinct eigenfunctions satisfying $\lambda_1 = 2\lambda_k$, then $G_2(h_1)$ and $G_2(h_2)$ are independent.

For $g_1(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x)$ and $g_2(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} c_j^k \phi_j^{(k)}(x)$, define

$$\beta(g_1, g_2) = \int_0^\infty e^{-\lambda_1 s} \langle A(I_s g_1)(I_s g_2), \phi_1 \rangle_m \, ds.$$

Using the convergence of the second component in Theorem 1.4 and an argument similar to that in the proof of Corollary 1.5, we get

Corollary 1.7. *If $g_1(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} b_j^k \phi_j^{(k)}(x)$ and $g_2(x) = \sum_{k:2\lambda_k<\lambda_1} \sum_{j=1}^{n_k} c_j^k \phi_j^{(k)}(x)$, then we have, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,*

$$\left(\frac{\langle g_1, X_t \rangle - \sum_{k:2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} b_j^k H_\infty^{k,j}}{\sqrt{\langle \phi_1, X_t \rangle}}, \frac{\langle g_2, X_t \rangle - \sum_{k:2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} c_j^k H_\infty^{k,j}}{\sqrt{\langle \phi_1, X_t \rangle}} \right) \stackrel{d}{\to} (G_3(g_1), G_3(g_2)),$$

where $(G_3(g_1), G_3(g_2))$ is a bivariate normal random variable with covariance

$$Cov(G_3(g_i), G_3(g_j)) = \beta(g_i, g_j), \quad i, j = 1, 2.$$

Consider the special situation when both the branching mechanism and the branching rate function are non-spatial dependent and ϕ_1 is a constant function. If $g_1 = \phi_j^{(k)}$ and $g_2 = \phi_{j'}^{(k')}$ are distinct eigenfunctions satisfying $\lambda_1 > 2\lambda_k$ and $\lambda_1 > 2\lambda_{k'}$, then $G_3(g_1)$ and $G_3(g_2)$ are independent.

Remark 1.8. If $2\lambda_k < \lambda_1$, then, it holds under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$ that, as $t \to \infty$,

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, \frac{\left(\langle \phi_j^{(k)}, X_t \rangle - e^{-\lambda_k t} H_\infty^{k,j} \right)}{\langle \phi_1, X_t \rangle^{1/2}} \right) \stackrel{d}{\to} (W^*, G_3),$$

where $G_3 \sim \mathcal{N}\left(0, \frac{1}{\lambda_1 - 2\lambda_k}\langle A(\phi_j^{(k)})^2, \phi_1\rangle_m\right)$. In particular, for ϕ_1, we have

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, \ \frac{\left(\langle \phi_1, X_t\rangle - e^{-\lambda_1 t} W_\infty\right)}{\langle \phi_1, X_t\rangle^{1/2}}\right) \xrightarrow{d} (W^*, \ G_3), \quad t \to \infty,$$

where $G_3 \sim \mathcal{N}\left(0, -\frac{1}{\lambda_1}\int_E A(x)(\phi_1(x))^3\, m(dx)\right)$.

All the central limit theorems in [19] are consequences of Theorem 1.4. To see this, we recall the following notation from [19]. For $f \in L^2(E, m)$, define

$$f_{(s)}(x) := \sum_{k: 2\lambda_k < \lambda_1} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x),$$

$$f_{(l)}(x) = \sum_{k: 2\lambda_k > \lambda_1} \sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x),$$

$$f_{(c)}(x) := f(x) - f_{(s)}(x) - f_{(l)}(x).$$

Then $f_{(s)} \in \mathcal{C}_l$, $f_{(c)} \in \mathcal{C}_c$ and $f_{(l)} \in \mathcal{C}_s$. Obviously, [19, Theorem 1.4] is an immediate consequence of the convergence of the first and fourth components in Theorem 1.4. Now we explain that Theorems 1.6, 1.10 and 1.13 of [19] also follow easily from Theorem 1.4.

Remark 1.9. If $f \in L^2(E, m) \cap L^4(E, m)$ with $\lambda_1 = 2\lambda_{\gamma(f)}$, then $f = f_{(c)} + f_{(l)}$. Using the convergence of the fourth component in Theorem 1.4 for $f_{(l)}$, it holds under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$ that

$$\frac{\langle f_{(l)}, X_t\rangle}{\sqrt{t\langle \phi_1, X_t\rangle}} \xrightarrow{d} 0, \quad t \to \infty.$$

Thus using the convergence of the first and third components in Theorem 1.4, we get, under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$,

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, \ \frac{\langle f, X_t\rangle}{\sqrt{t\langle \phi_1, X_t\rangle}}\right) \xrightarrow{d} (W^*, \ G_2(f_{(c)})), \quad t \to \infty,$$

where W^* has the same distribution as W_∞ conditioned on $\mathcal{E}^c$ and $G_2(f_{(c)}) \sim \mathcal{N}(0, \rho^2_{f_{(c)}})$. Moreover, W^* and $G_2(f_{(c)})$ are independent. Thus [19, Theorem 1.6] is a consequence of Theorem 1.4.

Remark 1.10. Assume $f \in L^2(E, m) \cap L^4(E, m)$ satisfies $\lambda_1 > 2\lambda_{\gamma(f)}$.

If $f_{(c)} = 0$, then $f = f_{(l)} + f_{(s)}$. Using the convergence of the first, second and fourth components in Theorem 1.4, we get for any nonzero $\mu \in \mathcal{M}_F(E)$, it holds under $\mathbb{P}_\mu(\cdot \mid \mathcal{E}^c)$ that, as $t \to \infty$,

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, \ \frac{\left(\langle f, X_t\rangle - \sum_{2\lambda_k < \lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} a_j^k H_\infty^{k,j}\right)}{\langle \phi_1, X_t\rangle^{1/2}}\right) \xrightarrow{d} (W^*, \ G_1(f_{(l)}) + G_3(f_{(s)})),$$

where W^*, $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are the same as those in Theorem 1.4. Since $G_3(f_{(s)})$ and $G_1(f_{(l)})$ are independent, $G_1(f_{(l)}) + G_3(f_{(s)}) \sim \mathcal{N}\left(0, \sigma^2_{f_{(l)}} + \beta^2_{f_{(s)}}\right)$. Thus [19, Theorem 1.10] is a consequence of Theorem 1.4.

If $f_{(c)} \neq 0$, then as $t \to \infty$,

$$\frac{\left(\langle f_{(l)} + f_{(s)}, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} a_j^k H_\infty^{k,j}\right)}{\sqrt{t\langle \phi_1, X_t\rangle}} \stackrel{d}{\to} 0.$$

Then using the convergence of the first and third components in Theorem 1.4, we get

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle,\ \frac{\left(\langle f, X_t\rangle - \sum_{2\lambda_k<\lambda_1} e^{-\lambda_k t} \sum_{j=1}^{n_k} a_j^k H_\infty^{k,j}\right)}{\sqrt{t\langle \phi_1, X_t\rangle}}\right) \stackrel{d}{\to} (W^*,\ G_2(f_{(c)})),$$

where W^* and $G_2(f_{(c)})$ are the same as those in Remark 1.9. Thus [19, Theorem 1.13] is a consequence of Theorem 1.4.

2. Preliminaries

2.1. Excursion measures of $\{X_t, t \geq 0\}$

We use $\mathbb{D}$ to denote the space of $\mathcal{M}_F(E)$-valued right continuous functions $t \mapsto \omega_t$ on $(0, \infty)$ having zero as a trap. We use $(\mathcal{A}, \mathcal{A}_t)$ to denote the natural σ-algebras on $\mathbb{D}$ generated by the coordinate process.

It is known (see [17, Section 8.4]) that one can associate with $\{\mathbb{P}_{\delta_x} : x \in E\}$ a family of σ-finite measures $\{\mathbb{N}_x : x \in E\}$ defined on $(\mathbb{D}, \mathcal{A})$ such that $\mathbb{N}_x(\{0\}) = 0$,

$$\int_{\mathbb{D}} (1 - e^{-\langle f,\omega_t\rangle})\mathbb{N}_x(d\omega) = -\log \mathbb{P}_{\delta_x}(e^{-\langle f,X_t\rangle}), \quad f \in \mathcal{B}_b^+(E),\ t > 0, \tag{2.1}$$

and, for every $0 < t_1 < \cdots < t_n < \infty$, and nonzero $\mu_1, \ldots, \mu_n \in M_F(E)$,

$$\begin{aligned}\mathbb{N}_x(\omega_{t_1} \in d\mu_1, \ldots, \omega_{t_n} \in d\mu_n) &= \mathbb{N}_x(\omega_{t_1} \in d\mu_1)\mathbb{P}_{\mu_1}(X_{t_2-t_1} \in d\mu_2)\\ &\quad \cdots \mathbb{P}_{\mu_{n-1}}(X_{t_n-t_{n-1}} \in d\mu_n).\end{aligned} \tag{2.2}$$

For earlier work on excursion measures of superprocesses, see [12,16,11].

For any $\mu \in M_F(E)$, let $N(d\omega)$ be a Poisson random measure on the space $\mathbb{D}$ with intensity $\int_E \mathbb{N}_x(d\omega)\mu(dx)$, in a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \mathbf{P}_\mu)$. Define another process $\{\Lambda_t : t \geq 0\}$ by $\Lambda_0 = \mu$ and

$$\Lambda_t := \int_{\mathbb{D}} \omega_t N(d\omega), \quad t > 0.$$

Let $\widetilde{\mathcal{F}}_t$ be the σ-algebra generated by the random variables $\{N(A) : A \in \mathcal{A}_t\}$. Then, $\{\Lambda, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \mathbf{P}_\mu\}$ has the same law as $\{X, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_\mu\}$, see [17, Theorem 8.24] for a proof.

Now we list some properties of $\mathbb{N}_x$. The proofs are similar to those in [11, Corollary 1.2, Proposition 1.1].

Proposition 2.1. *If* $\mathbb{P}_{\delta_x}|\langle f, X_t\rangle| < \infty$, *then*

$$\int_{\mathbb{D}} \langle f, \omega_t\rangle\, \mathbb{N}_x(d\omega) = \mathbb{P}_{\delta_x}\langle f, X_t\rangle. \tag{2.3}$$

If $\mathbb{P}_{\delta_x}\langle f, X_t\rangle^2 < \infty$, *then*

$$\int_{\mathbb{D}} \langle f, \omega_t\rangle^2\, \mathbb{N}_x(d\omega) = \mathbb{V}ar_{\delta_x}\langle f, X_t\rangle. \tag{2.4}$$

Proposition 2.2.

$$\mathbb{N}_x(\|\omega_t\| \neq 0) = -\log \mathbb{P}_{\delta_x}(\|X_t\| = 0). \tag{2.5}$$

Remark 2.3. By (1.11) and Proposition 2.2, for each $t > 0$ and $x \in E$, we have

$$0 < \mathbb{N}_x(\|\omega_t\| \neq 0) < \infty.$$

Thus, we can define another probability measure $\widetilde{\mathbb{N}}_x$ on $\mathbb{D}$ as follows:

$$\widetilde{\mathbb{N}}_x(B) = \frac{\mathbb{N}_x\left(B \cap \{\|\omega_1\| \neq 0\}\right)}{\mathbb{N}_x(\|\omega_1\| \neq 0)}. \tag{2.6}$$

Notice that, for $f \in L^2(E, m)$, $\mathbb{N}_x(\langle |f|, \omega_t\rangle) = T_t|f|(x) < \infty$, which implies that $\mathbb{N}_x(\langle |f|, \omega_t\rangle = \infty) = 0$. Thus, for $f \in L^2(E, m)$,

$$\begin{aligned}\mathbb{P}_\mu\left(e^{i\theta\langle f, X_t\rangle}\right) &= \mathbf{P}_\mu\left(e^{i\theta\langle f, \Lambda_t\rangle}\right) = \mathbf{P}_\mu\left(e^{i\theta\int_{\mathbb{D}}\langle f, \omega_t\rangle\, N(d\omega)}\right)\\ &= \exp\left\{\int_E\int_{\mathbb{D}}\left(e^{i\theta\langle f, \omega_t\rangle} - 1\right)\mathbb{N}_x(d\omega)\mu(dx)\right\}.\end{aligned}$$

Thus, by the Markov property of superprocesses, we have

$$\begin{aligned}\mathbb{P}_\mu\left[\exp\{i\theta\langle f, X_{t+s}\rangle\}\,|X_t\right] &= \mathbb{P}_{X_t}\left(e^{i\theta\langle f, X_s\rangle}\right)\\ &= \exp\left\{\int_E\int_{\mathbb{D}}(e^{i\theta\langle f, \omega_s\rangle} - 1)\mathbb{N}_x(d\omega)X_t(dx)\right\}.\end{aligned} \tag{2.7}$$

2.2. *Estimates on the moments of X*

In the remainder of this paper we will use the following notation: for two positive functions f and g on E, $f(x) \lesssim g(x)$ means that there exists a constant $c > 0$ such that $f(x) \le cg(x)$ for all $x \in E$.

First, we recall some results about the semigroup (T_t), the proofs of which can be found in [20].

Lemma 2.4. *For any* $f \in L^2(E, m)$, $x \in E$ *and* $t > 0$, *we have*

$$T_t f(x) = \sum_{k=\gamma(f)}^{\infty} e^{-\lambda_k t}\sum_{j=1}^{n_k} a_j^k \phi_j^{(k)}(x) \tag{2.8}$$

and

$$\lim_{t\to\infty} e^{\lambda_{\gamma(f)}t}T_t f(x) = \sum_{j=1}^{n_{\gamma(f)}} a_j^{\gamma(f)}\phi_j^{(\gamma(f))}(x), \tag{2.9}$$

where the series in (2.8) converges absolutely and uniformly in any compact subset of E. Moreover, for any $t_1 > 0$,

$$\sup_{t>t_1} e^{\lambda_{\gamma(f)}t}|T_t f(x)| \le e^{\lambda_{\gamma(f)}t_1}\|f\|_2\left(\int_E a_{t_1/2}(x)\,m(dx)\right)a_{t_1}(x)^{1/2}, \tag{2.10}$$

$$\sup_{t>t_1} e^{(\lambda_{\gamma(f)+1}-\lambda_{\gamma(f)})t}\left|e^{\lambda_{\gamma(f)}t}T_t f(x) - f^*(x)\right|$$
$$\le e^{\lambda_{\gamma(f)+1}t_1}\|f\|_2\left(\int_E a_{t_1/2}(x)\,m(dx)\right)(a_{t_1}(x))^{1/2}. \tag{2.11}$$

Lemma 2.5. *Suppose that $\{f_t(x) : t > 0\}$ is a family of functions in $L^2(E, m)$. If $\lim_{t\to\infty}\|f_t\|_2 = 0$, then for any $x \in E$,*

$$\lim_{t\to\infty} e^{\lambda_1 t}T_t f_t(x) = 0.$$

Recall the second moments of the superprocess $\{X_t : t \ge 0\}$ (see, for example, [17, Corollary 2.39]): for $f \in \mathcal{B}_b(E)$, we have for any $t > 0$,

$$\mathbb{P}_\mu\langle f, X_t\rangle^2 = \left(\mathbb{P}_\mu\langle f, X_t\rangle\right)^2 + \int_E\int_0^t T_s[A(T_{t-s}f)^2](x)\,ds\mu(dx). \tag{2.12}$$

Thus,

$$\mathbb{V}\mathrm{ar}_\mu\langle f, X_t\rangle = \langle\mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle f, X_t\rangle, \mu\rangle = \int_E\int_0^t T_s[A(T_{t-s}f)^2](x)\,ds\mu(dx), \tag{2.13}$$

where $\mathbb{V}\mathrm{ar}_\mu$ stands for the variance under $\mathbb{P}_\mu$. Note that the second moment formula (2.12) for superprocesses is different from that of [20, (2.11)] for branching Markov processes.

For any $f \in L^2(E, m) \cap L^4(E, m)$ and $x \in E$, since $(T_{t-s}f)^2(x) \le e^{M(t-s)}T_{t-s}(f^2)(x)$, we have

$$\int_0^t T_s[A(T_{t-s}f)^2](x)\,ds \le e^{Mt}T_t(f^2)(x) < \infty.$$

Thus, using a routine limit argument, one can easily check that (2.12) and (2.13) also hold for $f \in L^2(E, m) \cap L^4(E, m)$.

Lemma 2.6. *Assume that $f \in L^2(E, m) \cap L^4(E, m)$.*

(1) *If $\lambda_1 < 2\lambda_{\gamma(f)}$, then for any $x \in E$,*

$$\lim_{t\to\infty} e^{\lambda_1 t/2}\mathbb{P}_{\delta_x}\langle f, X_t\rangle = 0, \tag{2.14}$$

$$\lim_{t\to\infty} e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle = \sigma_f^2\phi_1(x), \tag{2.15}$$

where $\sigma^2(f)$ is defined by (1.14). Moreover, for $(t, x) \in (3t_0, \infty) \times E$, we have

$$e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle \lesssim a_{t_0}(x)^{1/2}. \tag{2.16}$$

(2) *If $\lambda_1 = 2\lambda_{\gamma(f)}$, then for any $(t, x) \in (3t_0, \infty) \times E$,*

$$\left|t^{-1}e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \rho_{f^*}^2\phi_1(x)\right| \lesssim t^{-1}a_{t_0}(x)^{1/2}, \tag{2.17}$$

where $\rho_{f^*}^2$ is defined by (1.15).

(3) *If* $\lambda_1 > 2\lambda_{\gamma(f)}$, *then for any* $x \in E$,

$$\lim_{t\to\infty} e^{2\lambda_{\gamma(f)}t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle = \eta_f^2(x), \tag{2.18}$$

where

$$\eta_f^2(x) := \int_0^\infty e^{2\lambda_{\gamma(f)}s} T_s(A(f^*)^2)(x)\, ds.$$

Moreover, for any $(t, x) \in (3t_0, \infty) \times E$,

$$e^{2\lambda_{\gamma(f)}t}\mathbb{P}_{\delta_x}\langle f, X_t\rangle^2 \lesssim a_{t_0}(x)^{1/2}. \tag{2.19}$$

Proof. Since the first moment formulas for superprocesses and branching Markov processes are the same, we get (2.14) easily. Although the second moment formula for superprocesses is different from that for branching Markov processes, we can still get all results on the variance of the superprocess X from the proof of [20, Lemma 2.3]. In fact,

$$\mathbb{V}\mathrm{ar}_x\langle f, X_t\rangle = \int_0^t T_s[A(T_{t-s}f)^2](x)\, ds.$$

The limit behavior of the right side of the above equation, as $t \to \infty$, was given in the proof of [20, Lemma 2.3]. □

Lemma 2.7. *Assume that* $f \in L^2(E, m) \cap L^4(E, m)$. *If* $\lambda_1 < 2\lambda_{\gamma(f)}$, *then for any* $(t, x) \in (3t_0, \infty) \times E$,

$$\left|e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \sigma_f^2\phi_1(x)\right| \lesssim \left(e^{(\lambda_1-2\lambda_{\gamma(f)})t} + e^{(\lambda_1-\lambda_2)t}\right) a_{t_0}(x)^{1/2}. \tag{2.20}$$

Proof. Without loss of generality, we assume that $m(x : f(x) \neq 0) > 0$. By (2.13), we get, for $t > 3t_0$,

$$\begin{aligned}
&\left|e^{\lambda_1 t}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_t\rangle - \int_0^\infty e^{\lambda_1 s}\langle A(T_s f)^2, \phi_1\rangle_m\, ds\phi_1(x)\right| \\
&\quad = \left|e^{\lambda_1 t}\int_0^t T_{t-s}[A(T_s f)^2](x)\, ds - \int_0^\infty e^{\lambda_1 s}\langle A(T_s f)^2, \phi_1\rangle_m\, ds\phi_1(x)\right| \\
&\quad \le e^{\lambda_1 t}\int_0^{t-t_0}\left|T_{t-s}[A(T_s f)^2](x) - e^{-\lambda_1(t-s)}\langle A(T_s f)^2, \phi_1\rangle_m\phi_1(x)\right| ds \\
&\qquad + e^{\lambda_1 t}\int_{t-t_0}^t T_{t-s}[A(T_s f)^2](x)\, ds + \int_{t-t_0}^\infty e^{\lambda_1 s}\langle A(T_s f)^2, \phi_1\rangle_m\, ds\phi_1(x) \\
&\quad =: V_1(t, x) + V_2(t, x) + V_3(t, x).
\end{aligned} \tag{2.21}$$

For $V_2(t, x)$, by [20, (2.26)], we have

$$V_2(t, x) \lesssim e^{(\lambda_1-2\lambda_{\gamma(f)})t} a_{t_0}(x)^{1/2}. \tag{2.22}$$

For $V_3(t, x)$, by (2.10), for $s > t - t_0 > t_0$, $|T_s f(x)| \lesssim e^{-\lambda_{\gamma(f)}s} a_{t_0}(x)^{1/2}$. By (1.9), $\phi_1(x) \le e^{\lambda_1 t_0/2} a_{t_0}(x)^{1/2}$. Thus, we get

$$\begin{aligned}
V_3(t, x) &\lesssim \int_{t-t_0}^\infty e^{(\lambda_1-2\lambda_{\gamma(f)})s}\, ds\langle a_{t_0}, \phi_1\rangle_m\phi_1(x) \\
&\lesssim e^{(\lambda_1-2\lambda_{\gamma(f)})t} a_{t_0}(x)^{1/2}.
\end{aligned} \tag{2.23}$$

Finally, we consider $V_1(t,x)$. Let $g := A(T_s f)^2$, noticing that g is nonnegative and non-trivial, we have that $\gamma(g) = 1$ and $g^*(x) = \langle A(T_s f)^2, \phi_1\rangle_m \phi_1(x)$. Using (2.11) with f replaced by g, for $t - s > t_0$, we have

$$\left| T_{t-s}[A(T_s f)^2](x) - e^{-\lambda_1(t-s)}\langle A(T_s f)^2, \phi_1\rangle_m \phi_1(x)\right| \lesssim e^{-\lambda_2(t-s)}\|A(T_s f)^2\|_2 a_{t_0}(x)^{1/2}.$$

For $s > t_0$, by (2.10), $|T_s f(x)| \lesssim e^{-\lambda_{\gamma(f)}s} a_{t_0}(x)^{1/2}$. Thus,

$$\|A(T_s f)^2\|_2 \lesssim e^{-2\lambda_{\gamma(f)}s}\|a_{t_0}\|_2.$$

For $s \le t_0$, by (1.8), it is easy to get

$$\|A(T_s f)^2\|_2 \le M\|T_s f\|_4^2 \le M e^{2Ms}\|f\|_4^2.$$

Therefore, we have

$$\begin{aligned} V_1(t,x) &\lesssim e^{\lambda_1 t}\int_{t_0}^{t-t_0} e^{-\lambda_2(t-s)} e^{-2\lambda_{\gamma(f)}s}\, ds\, a_{t_0}(x)^{1/2} + e^{\lambda_1 t}\int_0^{t_0} e^{-\lambda_2(t-s)}\, ds\, a_{t_0}(x)^{1/2} \\ &\lesssim \left(e^{(\lambda_1 - 2\lambda_{\gamma(f)})t} + e^{(\lambda_1-\lambda_2)t}\right) a_{t_0}(x)^{1/2}. \end{aligned} \tag{2.24}$$

Now (2.20) follows immediately from (2.22)–(2.24). □

Lemma 2.8. *Assume that $f \in L^2(E,m)\cap L^4(E,m)$ with $\lambda_1 < 2\lambda_{\gamma(f)}$ and $h \in L^2(E,m)\cap L^4(E,m)$ with $\lambda_1 = 2\lambda_{\gamma(h)}$. Then, for any $(t,x) \in (3t_0,\infty)\times E$,*

$$\mathbb{C}\mathrm{ov}_{\delta_x}(e^{\lambda_1 t/2}\langle f, X_t\rangle, t^{-1/2}e^{\lambda_1 t/2}\langle h, X_t\rangle) \lesssim t^{-1/2}(a_{t_0}(x))^{1/2}, \tag{2.25}$$

where $\mathbb{C}\mathrm{ov}_{\delta_x}$ is the covariance under $\mathbb{P}_{\delta_x}$.

Proof. By (2.13), we have

$$\begin{aligned} &\left|\mathbb{C}\mathrm{ov}_{\delta_x}(e^{\lambda_1 t/2}\langle f, X_t\rangle, t^{-1/2}e^{\lambda_1 t/2}\langle h, X_t\rangle)\right| \\ &= t^{-1/2}e^{\lambda_1 t}\frac{1}{4}\left|\left(\mathbb{V}\mathrm{ar}_{\delta_x}\langle (f+h), X_t\rangle - \mathbb{V}\mathrm{ar}_{\delta_x}\langle (f-h), X_t\rangle\right)\right| \\ &= t^{-1/2}e^{\lambda_1 t}\left|\int_0^t T_{t-s}\left[A(T_s f)(T_s h)\right](x)\, ds\right| \\ &\le t^{-1/2}e^{\lambda_1 t}\left(\int_0^{t-t_0} T_{t-s}[A\,|(T_s f)(T_s h)|](x)\, ds + \int_{t-t_0}^t T_{t-s}[A\,|(T_s f)(T_s h)|](x)\, ds\right) \\ &=: V_4(t,x) + V_5(t,x). \end{aligned}$$

First, we deal with $V_4(t,x)$. By (2.10), for $t - s > t_0$,

$$T_{t-s}[A\,|(T_s f)(T_s h)|](x) \lesssim e^{-\lambda_1(t-s)}\|A(T_s f)(T_s h)\|_2 (a_{t_0}(x))^{1/2}.$$

If $s > t_0$, then by (2.10), we get

$$\|A(T_s f)(T_s h)\|_2 \lesssim e^{-(\lambda_1/2+\lambda_{\gamma(f)})s}\|a_{t_0}\|_2.$$

If $s \le t_0$, by (1.8), it is easy to get

$$\|A(T_s f)(T_s h)\|_2 \le M\|T_s f\|_4 \|T_s h\|_4 \le M e^{2Ms}\|f\|_4 \|h\|_4.$$

Therefore, we have

$$V_4(t,x) \lesssim t^{-1/2}e^{\lambda_1 t}\left(\int_{t_0}^{t-t_0} e^{-\lambda_1(t-s)}e^{-(\lambda_1/2+\lambda_{\gamma(f)})s}\,ds + \int_0^{t_0} e^{-\lambda_1(t-s)}\,ds\right) a_{t_0}(x)^{1/2}$$
$$= t^{-1/2}\left(\int_{t_0}^{t-t_0} e^{(\lambda_1/2-\lambda_{\gamma(f)})s}\,ds + \int_0^{t_0} e^{\lambda_1 s}\,ds\right) a_{t_0}(x)^{1/2}$$
$$\lesssim t^{-1/2}a_{t_0}(x)^{1/2}. \tag{2.26}$$

For $V_5(t,x)$, if $s > t-t_0 \geq 2t_0$, then by (2.10), we get

$$V_5(t,x) \lesssim t^{-1/2}e^{\lambda_1 t}\int_{t-t_0}^{t} e^{-(\lambda_1/2+\lambda_{\gamma(f)})s}T_{t-s}(a_{2t_0})(x)\,ds$$
$$= t^{-1/2}e^{(\lambda_1/2-\lambda_{\gamma(f)})t}\int_0^{t_0} e^{(\lambda_1/2+\lambda_{\gamma(f)})s}T_s(a_{2t_0})(x)\,ds$$
$$\lesssim t^{-1/2}e^{(\lambda_1/2-\lambda_{\gamma(f)})t}\int_0^{t_0} T_s(a_{2t_0})(x)\,ds$$
$$\lesssim t^{-1/2}(a_{t_0}(x))^{1/2}. \tag{2.27}$$

The last inequality follows from the fact that

$$\int_0^{t_0} T_s(a_{2t_0})(x)\,ds \lesssim a_{t_0}(x)^{1/2}, \tag{2.28}$$

which is [20, (2.25)]. Therefore, by (2.26) and (2.27), we get (2.25) immediately. □

3. Proof of the main theorem

In this section, we will prove the main result of this paper. The general methodology is similar to that of [20], the difference being that we use the excursion measures of the superprocess rather than the backbone decomposition (which is not yet available in the general setup of this paper) of superprocess.

We first recall some facts about weak convergence which will be used later. For $f:\mathbb{R}^n \to \mathbb{R}$, let $\|f\|_L := \sup_{x\neq y}|f(x)-f(y)|/\|x-y\|$ and $\|f\|_{BL} := \|f\|_\infty + \|f\|_L$. For any distributions ν_1 and ν_2 on $\mathbb{R}^n$, define

$$d(\nu_1,\nu_2) := \sup\left\{\left|\int f\,d\nu_1 - \int f\,d\nu_2\right| \;:\; \|f\|_{BL}\leq 1\right\}.$$

Then d is a metric. It follows from [9, Theorem 11.3.3] that the topology generated by d is equivalent to the weak convergence topology. From the definition, we can easily see that, if ν_1 and ν_2 are the distributions of two $\mathbb{R}^n$-valued random variables X and Y respectively, then

$$d(\nu_1,\nu_2) \leq E\|X-Y\| \leq \sqrt{E\|X-Y\|^2}. \tag{3.1}$$

The following simple fact will be used several times later in this section:

$$\left|e^{ix} - \sum_{m=0}^{n}\frac{(ix)^m}{m!}\right| \leq \min\left(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!}\right). \tag{3.2}$$

Before we prove Theorem 1.4, we prove several lemmas first. The first lemma below says that the result in Lemma 1.1 also holds under $\mathbb{N}_x$. Recall the probability measure $\widetilde{\mathbb{N}}_x$ defined in (2.6). On the measurable space $(\mathbb{D}, \mathcal{A})$, define

$$\widetilde{H}_t^{k,j}(\omega) := e^{\lambda_k t}\langle \phi_j^{(k)}, \omega_t\rangle, \quad t \geq 0,\ \omega \in \mathbb{D}.$$

Lemma 3.1. *For $x \in E$, if $\lambda_1 > 2\lambda_k$, then the limit*

$$\widetilde{H}_\infty^{k,j} := \lim_{t\to\infty} \widetilde{H}_t^{k,j}$$

exists $\mathbb{N}_x$-a.e., in $L^1(\mathbb{N}_x)$ and in $L^2(\mathbb{N}_x)$.

Proof. On the set $\{\omega \in \mathbb{D} : \|\omega_1\| = 0\}$, we have $\omega_t = 0, t > 1$, thus, $\widetilde{H}_\infty^{k,j}(\omega) = 0$. Thus, we only need to show $\widetilde{H}_\infty^{k,j}$ exists $\widetilde{\mathbb{N}}_x$-a.s. and in $L^2(\widetilde{\mathbb{N}}_x)$.

For $t > s \geq 1$, since $\{\|\omega_1\| = 0\} \subset \{\|\omega_s\| = 0\} \subset \{\|\omega_t\| = 0\}$, we have

$$\begin{aligned}\mathbb{N}_x\left(\langle \phi_j^{(k)}, \omega_t\rangle; \|\omega_1\| \neq 0 | \mathcal{A}_s\right) &= \mathbb{N}_x\left(\langle \phi_j^{(k)}, \omega_t\rangle | \mathcal{A}_s\right) = \mathbb{P}_{\omega_s}\left(\langle \phi_j^{(k)}, X_{t-s}\rangle\right)\\ &= e^{-\lambda_k(t-s)}\langle \phi_j^{(k)}, \omega_s\rangle,\end{aligned}$$

which implies $\{\widetilde{H}_t^{k,j}, t \geq 1\}$ is a martingale under $\widetilde{\mathbb{N}}_x$. By (2.4), we have

$$\mathbb{N}_x\left(\langle \phi_j^{(k)}, \omega_t\rangle^2; \|\omega_1\| \neq 0\right) = \mathbb{N}_x\left(\langle \phi_j^{(k)}, \omega_t\rangle^2\right) = \mathbb{V}ar_{\delta_x}\langle \phi_j^{(k)}, X_t\rangle.$$

Then by Lemma 1.1, we easily get $\limsup_{t\to\infty} \widetilde{\mathbb{N}}_x(\widetilde{H}_t^{k,j})^2 < \infty$, which implies $\widetilde{H}_\infty^{k,j}$ exists $\widetilde{\mathbb{N}}_x$-a.s. and in $L^2(\widetilde{\mathbb{N}}_x)$. $\square$

Lemma 3.2. *If $f \in \mathcal{C}_s$, then $\sigma_f^2 < \infty$ and, for any nonzero $\mu \in \mathcal{M}_F(E)$, it holds under $\mathbb{P}_\mu$ that*

$$\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle,\ e^{\lambda_1 t/2}\langle f, X_t\rangle\right) \xrightarrow{d} \left(W_\infty,\ G_1(f)\sqrt{W_\infty}\right), \quad t \to \infty,$$

where $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, W_∞ and $G_1(f)$ are independent.

Proof. We need to consider the limit of the $\mathbb{R}^2$-valued random variable $U_1(t)$ defined by

$$U_1(t) := \left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle,\ e^{\lambda_1 t/2}\langle f, X_t\rangle\right), \tag{3.3}$$

or equivalently, we need to consider the limit of $U_1(t+s)$ as $t \to \infty$ for any $s > 0$. The main idea is as follows. For $s, t > t_0$,

$$\begin{aligned}U_1(s+t) = &\left(e^{\lambda_1(t+s)}\langle \phi_1, X_{t+s}\rangle,\ e^{\lambda_1(t+s)/2}\langle f, X_{t+s}\rangle - e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle\right)\\ &+ \left(0,\ e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle\right).\end{aligned} \tag{3.4}$$

The double limit, first as $t \to \infty$ and then $s \to \infty$, of the first term of the right side of (3.4) is equal to the double limit, first as $t \to \infty$ and then $s \to \infty$, of another $\mathbb{R}^2$-valued random variable $U_2(s, t)$ where

$$U_2(s,t) := \left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, e^{\lambda_1(t+s)/2}\langle f, X_{t+s}\rangle - e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle\right).$$

We will prove that the second term on the right hand side of (3.4) has no contribution to the double limit, first as $t \to \infty$ and then $s \to \infty$, of the left hand side (see (3.12)).

We claim that, under $\mathbb{P}_\mu$,

$$U_2(s,t) \xrightarrow{d} \left(W_\infty, \sqrt{W_\infty} G_1(s)\right), \quad \text{as } t \to \infty, \tag{3.5}$$

where $G_1(s) \sim \mathcal{N}(0, \sigma_f^2(s))$ with $\sigma_f^2(s)$ to be given later. In fact, denote the characteristic function of $U_2(s,t)$ under $\mathbb{P}_\mu$ by $\kappa(\theta_1, \theta_2, s, t)$:

$$\begin{aligned}
\kappa(\theta_1,\theta_2,s,t) &= \mathbb{P}_\mu\Big(\exp\Big\{i\theta_1 e^{\lambda_1 t}\langle\phi_1, X_t\rangle \\
&\quad + i\theta_2 e^{\lambda_1(t+s)/2}\langle f, X_{t+s}\rangle - i\theta_2 e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle\Big\}\Big) \\
&= \mathbb{P}_\mu\bigg(\exp\bigg\{i\theta_1 e^{\lambda_1 t}\langle\phi_1, X_t\rangle + \int_E\int_{\mathbb{D}}\bigg(\exp\left\{i\theta_2 e^{\lambda_1(t+s)/2}\langle f, \omega_s\rangle\right\} \\
&\quad - 1 - i\theta_2 e^{\lambda_1(t+s)/2}\langle f, \omega_s\rangle\bigg)\mathbb{N}_x(d\omega)X_t(dx)\bigg\}\bigg),
\end{aligned} \tag{3.6}$$

where in the last equality we used the Markov property of X, (2.3) and (2.7). Define

$$R_s(\theta, x) = \int_{\mathbb{D}}\left(\exp\{\langle i\theta f, \omega_s\rangle\} - 1 - i\theta\langle f, \omega_s\rangle + \frac{1}{2}\theta^2\langle f, \omega_s\rangle^2\right)\mathbb{N}_x(d\omega).$$

Then, by (2.4), we get

$$\begin{aligned}
\kappa(\theta_1,\theta_2,s,t) &= \mathbb{P}_\mu\bigg(\exp\bigg\{i\theta_1 e^{\lambda_1 t}\langle\phi_1, X_t\rangle + \int_E\int_{\mathbb{D}}\left(-\frac{1}{2}e^{\lambda_1(t+s)}\theta_2^2\langle f, \omega_s\rangle^2\right) \\
&\quad \times \mathbb{N}_x(d\omega)X_t(dx) + \langle R_s(e^{\lambda_1(t+s)/2}\theta_2, \cdot), X_t\rangle\bigg\}\bigg) \\
&= \mathbb{P}_\mu\bigg(\exp\bigg\{i\theta_1 e^{\lambda_1 t}\langle\phi_1, X_t\rangle - \frac{1}{2}\theta_2^2 e^{\lambda_1 t}\langle V_s, X_t\rangle \\
&\quad + \langle R_s(e^{\lambda_1(t+s)/2}\theta_2, \cdot), X_t\rangle\bigg\}\bigg),
\end{aligned} \tag{3.7}$$

where $V_s(x) := e^{\lambda_1 s}\mathbb{V}\mathrm{ar}_{\delta_x}\langle f, X_s\rangle$. By (3.2), we have

$$\begin{aligned}
\left|R_s(e^{\lambda_1(t+s)/2}\theta_2, x)\right| &\le \theta_2^2 e^{\lambda_1(t+s)}\mathbb{N}_x\left(\langle f, \omega_s\rangle^2\left(\frac{e^{\lambda_1(t+s)/2}\theta_2\langle f, \omega_s\rangle}{6}\wedge 1\right)\right) \\
&= \theta_2^2 e^{\lambda_1 t}\mathbb{N}_x\left(Y_s^2\left(\frac{\theta_2 e^{\lambda_1 t/2}Y_s}{6}\wedge 1\right)\right),
\end{aligned} \tag{3.8}$$

where $Y_s := e^{\lambda_1 s/2}\langle f, \omega_s\rangle$. Let

$$h(x,s,t) := \mathbb{N}_x\left(Y_s^2\left(\frac{\theta_2 e^{\lambda_1 t/2}Y_s}{6}\wedge 1\right)\right).$$

We note that $h(x,s,t) \downarrow 0$ as $t \uparrow \infty$ and by (2.16), we get

$$h(x,s,t) \le \mathbb{N}_x(Y_s^2) = e^{\lambda_1 s}\mathbb{V}\mathrm{ar}_{\delta_x}(\langle f, X_s\rangle) \lesssim a_{t_0}(x)^{1/2} \in L^2(E,m).$$

Thus, by (2.9), we have, for any $u < t$,

$$\limsup_{t\to\infty} e^{\lambda_1 t}T_t(h(\cdot,s,t)) \le \limsup_{t\to\infty} e^{\lambda_1 t}T_t(h(\cdot,s,u)) = \langle h(\cdot,s,u), \phi_1\rangle_m \phi_1(x).$$

Letting $u \to \infty$, we get $\lim_{t\to\infty} e^{\lambda_1 t} T_t(h(\cdot, s, t)) = 0$. Therefore we have

$$\mathbb{P}_\mu \left| \langle R_s(e^{\lambda_1(t+s)/2}\theta_2, \cdot), X_t\rangle \right| \le \theta_2^2 e^{\lambda_1 t} T_t(h(\cdot, s, t)) \to 0, \quad \text{as } t \to \infty,$$

which implies

$$\lim_{t\to\infty} \langle R_s(e^{\lambda_1(t+s)/2}\theta_2, \cdot), X_t\rangle = 0, \quad \text{in probability.}$$

Furthermore, by Remark 1.3 and the fact $V_s(x) \lesssim a_{t_0}(x)^{1/2} \in L^2(E, m) \cap L^4(E, m)$, we have

$$\lim_{t\to\infty} e^{\lambda_1 t} \langle V_s, X_t\rangle = \sigma_f^2(s) W_\infty, \quad \text{in probability,}$$

where $\sigma_f^2(s) := \langle V_s,\ \phi_1\rangle_m$. Hence by the dominated convergence theorem, we get

$$\lim_{t\to\infty} \kappa(\theta_1, \theta_2, s, t) = \mathbb{P}_\mu \left(\exp\{i\theta_1 W_\infty\} \exp\left\{ -\frac{1}{2}\theta_2^2 \sigma_f^2(s) W_\infty \right\} \right), \tag{3.9}$$

which implies our claim (3.5).

Since $e^{\lambda_1(t+s)}\langle \phi_1, X_{t+s}\rangle - e^{\lambda_1 t}\langle \phi_1, X_t\rangle \to 0$ in probability, as $t \to \infty$, we easily get that under $\mathbb{P}_\mu$,

$$\begin{aligned} U_3(s,t) &:= \left(e^{\lambda_1(t+s)}\langle \phi_1, X_{t+s}\rangle, e^{\lambda_1(t+s)/2}(\langle f, X_{t+s}\rangle - \langle T_s f, X_t\rangle) \right) \\ &\overset{d}{\to} \left(W_\infty, \sqrt{W_\infty} G_1(s) \right), \end{aligned}$$

as $t \to \infty$. By (2.15), we have $\lim_{s\to\infty} V_s(x) = \sigma_f^2 \phi_1(x)$, thus $\lim_{s\to\infty} \sigma_f^2(s) = \sigma_f^2$. So

$$\lim_{s\to\infty} d(G_1(s), G_1(f)) = 0. \tag{3.10}$$

Let $\mathcal{D}(s+t)$ and $\widetilde{\mathcal{D}}(s,t)$ be the distributions of $U_1(s+t)$ and $U_3(s,t)$ respectively, and let $\mathcal{D}(s)$ and $\mathcal{D}$ be the distributions of $(W_\infty, \sqrt{W_\infty} G_1(s))$ and $(W_\infty, \sqrt{W_\infty} G_1(f))$ respectively. Then, using (3.1), we have

$$\begin{aligned} \limsup_{t\to\infty} d(\mathcal{D}(s+t), \mathcal{D}) &\le \limsup_{t\to\infty} [d(\mathcal{D}(s+t), \widetilde{\mathcal{D}}(s,t)) + d(\widetilde{\mathcal{D}}(s,t), \mathcal{D}(s)) \\ &\quad + d(\mathcal{D}(s), \mathcal{D})] \\ &\le \limsup_{t\to\infty} (\mathbb{P}_\mu (e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle)^2)^{1/2} + 0 + d(\mathcal{D}(s), \mathcal{D}). \end{aligned} \tag{3.11}$$

Using this and the definition of $\limsup_{t\to\infty}$, we easily get that

$$\begin{aligned} \limsup_{t\to\infty} d(\mathcal{D}(t), \mathcal{D}) &= \limsup_{t\to\infty} d(\mathcal{D}(s+t), \mathcal{D}) \\ &\le \limsup_{t\to\infty} (\mathbb{P}_\mu (e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle)^2)^{1/2} + d(\mathcal{D}(s), \mathcal{D}). \end{aligned}$$

Letting $s \to \infty$, we get

$$\limsup_{t\to\infty} d(\mathcal{D}(t), \mathcal{D}) \le \limsup_{s\to\infty} \limsup_{t\to\infty} (\mathbb{P}_\mu (e^{\lambda_1(t+s)/2}\langle T_s f, X_t\rangle)^2)^{1/2}.$$

Therefore, we are left to prove that

$$\limsup_{s\to\infty} \limsup_{t\to\infty} e^{\lambda_1(t+s)} \mathbb{P}_\mu (\langle T_s f, X_t\rangle)^2 = 0. \tag{3.12}$$

By (2.13) and (2.10), we have for any $x \in E$,

$$
\begin{aligned}
e^{\lambda_1(t+s)}\mathbb{V}\mathrm{ar}_{\delta_x}\langle T_s f, X_t\rangle \quad &= e^{\lambda_1(s+t)}\int_0^t T_{t-u}[A(T_{s+u}f)^2](x)\,du \\
&= e^{(\lambda_1-2\lambda_{\gamma(f)})s}\int_0^t e^{(\lambda_1-2\lambda_{\gamma(f)})u}e^{\lambda_1(t-u)}T_{t-u}[A(e^{\lambda_{\gamma(f)}(s+u)}T_{s+u}f)^2](x)\,du \\
&\lesssim e^{(\lambda_1-2\lambda_{\gamma(f)})s}\left(\int_0^t e^{(\lambda_1-2\lambda_{\gamma(f)})u}e^{\lambda_1(t-u)}T_{t-u}[a_{2t_0}](x)\,du\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^t e^{(\lambda_1-2\lambda_{\gamma(f)})u}e^{\lambda_1(t-u)}T_{t-u}(a_{2t_0})(x)\,du \\
&\quad = \left(\int_0^{t-t_0}+\int_{t-t_0}^t\right)e^{(\lambda_1-2\lambda_{\gamma(f)})u}e^{\lambda_1(t-u)}T_{t-u}(a_{2t_0})(x)\,du \\
&\quad \lesssim \int_0^{t-t_0}e^{(\lambda_1-2\lambda_{\gamma(f)})u}\,du\,a_{t_0}(x)^{1/2}+\int_0^{t_0}e^{(\lambda_1-2\lambda_{\gamma(f)})(t-u)}e^{\lambda_1 u}T_u(a_{2t_0})(x)\,du \\
&\quad \lesssim a_{t_0}(x)^{1/2}+\int_0^{t_0}T_u(a_{2t_0})(x)\,du \lesssim a_{t_0}(x)^{1/2}.
\end{aligned}
$$

The last inequality follows from (2.28). Thus,

$$
\begin{aligned}
\limsup_{t\to\infty} e^{\lambda_1(t+s)}\mathbb{V}\mathrm{ar}_{\mu}\langle T_s f, X_t\rangle &= \limsup_{t\to\infty} e^{\lambda_1(t+s)}\langle \mathbb{V}\mathrm{ar}_{\delta_\cdot}\langle T_s f, X_t\rangle, \mu\rangle \\
&\lesssim e^{(\lambda_1-2\lambda_{\gamma(f)})s}\langle a_{t_0}(x)^{1/2}, \mu\rangle. \qquad (3.13)
\end{aligned}
$$

By (2.14), we get

$$
\lim_{t\to\infty} e^{\lambda_1(t+s)/2}\mathbb{P}_\mu\langle T_s f, X_t\rangle = \lim_{t\to\infty} e^{\lambda_1(t+s)/2}\langle T_{(t+s)}f, \mu\rangle = 0. \qquad (3.14)
$$

Now (3.12) follows easily from (3.13) and (3.14). The proof is now complete. □

Lemma 3.3. *Assume that $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. Define*

$$
Y_1(t) := t^{-1/2}e^{\lambda_1 t/2}\langle h, X_t\rangle, \qquad Y_2(t) := e^{\lambda_1 t/2}\langle f, X_t\rangle, \quad t > 0,
$$

and

$$
Y_t := Y_1(t) + Y_2(t).
$$

Then for any $c > 0$, $\delta > 0$ and $x \in E$, we have

$$
\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(|Y_t|^2; |Y_t| > ce^{\delta t}\right) = 0. \qquad (3.15)
$$

Proof. For any $\epsilon > 0$ and $\eta > 0$, we have

$$
\begin{aligned}
\mathbb{P}_{\delta_x}\left(|Y_t|^2; |Y_t| > ce^{\delta t}\right) &\le 2\mathbb{P}_{\delta_x}\left(|Y_1(t)|^2; |Y_t| > ce^{\delta t}\right) + 2\mathbb{P}_{\delta_x}\left(|Y_2(t)|^2; |Y_t| > ce^{\delta t}\right) \\
&\le 2\mathbb{P}_{\delta_x}\left(|Y_1(t)|^2; |Y_1(t)| > \epsilon e^{\delta t}\right) + 2\epsilon^2 e^{2\delta t}\mathbb{P}_{\delta_x}\left(|Y_t| > ce^{\delta t}\right) \\
&\quad + 2\mathbb{P}_{\delta_x}\left(|Y_2(t)|^2; |Y_2(t)|^2 > \eta\right) + 2\eta\mathbb{P}_{\delta_x}\left(|Y_t| > ce^{\delta t}\right) \\
&=: J_1(t,\epsilon) + J_2(t,\epsilon) + J_3(t,\eta) + J_4(t,\eta).
\end{aligned}
$$

Repeating the proof of [20, Lemma 3.2] (with the $S_t f$ there replaced by $Y_1(t)$), we can get

$$\lim_{t\to\infty} J_1(t,\epsilon) = 2\lim_{t\to\infty} \mathbb{P}_{\delta_x}\left(|Y_1(t)|^2; |Y_1(t)| > \epsilon e^{\delta t}\right) = 0. \tag{3.16}$$

By (2.14) and (2.15), we easily get

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(|Y_2(t)|^2) = \sigma_f^2 \phi_1(x). \tag{3.17}$$

By (2.17) and the fact $\mathbb{P}_{\delta_x}(Y_1(t)) = t^{-1/2}h(x)$, we get

$$\lim_{t\to\infty} \mathbb{P}_{\delta_x}(|Y_1(t)|^2) = \lim_{t\to\infty}\left(\mathbb{V}\mathrm{ar}_{\delta_x}(Y_1(t)) + t^{-1}h^2(x)\right) = \rho_h^2\phi_1(x).$$

Thus,

$$\limsup_{t\to\infty} \mathbb{P}_{\delta_x}(|Y_t|^2) \le 2\lim_{t\to\infty} \mathbb{P}_{\delta_x}(|Y_1(t)|^2 + |Y_2(t)|^2) = 2(\sigma_f^2 + \rho_h^2)\phi_1(x). \tag{3.18}$$

Thus by Chebyshev's inequality, we have

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty} J_2(t,\epsilon) \le 2\lim_{\epsilon\to 0}\epsilon^2 c^{-2}\limsup_{t\to\infty}\mathbb{P}_{\delta_x}(|Y_t|^2) = 0. \tag{3.19}$$

For $J_3(t,\eta)$, by Lemma 3.2, $Y_2(t) \xrightarrow{d} G_1(f)\sqrt{W_\infty}$. Let $\Psi_\eta(r) = r$ on $[0, \eta-1]$, $\Psi_\eta(r) = 0$ on $[\eta, \infty]$, and let Ψ_η be linear on $[\eta-1, \eta]$. Then, by (3.17),

$$\begin{aligned}
\limsup_{t\to\infty}\mathbb{P}_{\delta_x}\left(|Y_2(t)|^2; |Y_2(t)|^2 > \eta\right) &= \limsup_{t\to\infty}\left(\mathbb{P}_{\delta_x}\left(|Y_2(t)|^2\right)\right.\\
&\qquad \left.-\ \mathbb{P}_{\delta_x}\left(|Y_2(t)|^2; |Y_2(t)|^2 \le \eta\right)\right)\\
&\le \limsup_{t\to\infty}\left(\mathbb{P}_{\delta_x}\left(|Y_2(t)|^2\right) - \mathbb{P}_{\delta_x}\left(\Psi_\eta(|Y_2(t)|^2)\right)\right)\\
&= \sigma_f^2\phi_1(x) - \mathbb{P}_{\delta_x}\left(\Psi_\eta(G_1(f)^2 W_\infty)\right).
\end{aligned}$$

By the monotone convergence theorem and the fact that $G_1(f)$ and W_∞ are independent, we have

$$\lim_{\eta\to\infty}\mathbb{P}_{\delta_x}\left(\Psi_\eta(G_1(f)^2W_\infty)\right) = \mathbb{P}_{\delta_x}\left(G_1(f)^2 W_\infty\right) = \mathbb{P}_{\delta_x}\left(G_1(f)^2\right)\mathbb{P}_{\delta_x}W_\infty = \sigma_f^2\phi_1(x).$$

Thus,

$$\lim_{\eta\to\infty}\limsup_{t\to\infty} J_3(t,\eta) = 0. \tag{3.20}$$

By Chebyshev's inequality and (3.18),

$$\limsup_{t\to\infty} J_4(t,\eta) \le 2\eta c^{-2}\limsup_{t\to\infty} e^{-2\delta t}\mathbb{P}_{\delta_x}(|Y_t|^2) = 0. \tag{3.21}$$

Thus, (3.15) follows easily from (3.16) and (3.19)–(3.21). □

Lemma 3.4. *Assume that $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. Define*

$$\widetilde{Y}_1(t)(\omega) := t^{-1/2}e^{\lambda_1 t/2}\langle h, \omega_t\rangle, \qquad \widetilde{Y}_2(t)(\omega) := e^{\lambda_1 t/2}\langle f, \omega_t\rangle, \quad t > 0, \omega \in \mathbb{D},$$

and

$$\widetilde{Y}_t := \widetilde{Y}_1(t) + \widetilde{Y}_2(t).$$

For any $c > 0$ and $\delta > 0$, we have

$$\lim_{t\to\infty} \mathbb{N}_x \left(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t} \right) = 0. \tag{3.22}$$

Proof. For $t > 1$,

$$\mathbb{N}_x \left(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t} \right) = \mathbb{N}_x \left(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}, \|\omega_1\| \neq 0 \right).$$

Thus, we only need to prove

$$\lim_{t\to\infty} \widetilde{\mathbb{N}}_x \left(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t} \right) = 0.$$

For any $x \in E$, let $N(d\omega)$ be a Poisson random measure with intensity $\mathbb{N}_x(d\omega)$ defined on the probability space $\{\tilde{\Omega}, \tilde{\mathcal{F}}, \mathbf{P}_{\delta_x}\}$ and

$$\Lambda_t = \int_{\mathbb{D}} \omega_t N(d\omega).$$

We know that, under $\mathbf{P}_{\delta_x}$, $\{\Lambda_t, t \geq 0\}$ has the same law as $\{X_t, t \geq 0\}$ under $\mathbb{P}_{\delta_x}$. Define

$$\Lambda_t^* := \int_{\widetilde{\mathbb{D}}} \omega_t N(d\omega) \quad \text{and} \quad Y_t(\Lambda^*) := t^{-1/2} e^{\lambda_1 t/2} \langle h, \Lambda_t^* \rangle + e^{\lambda_1 t/2} \langle f, \Lambda_t^* \rangle,$$

where $\widetilde{\mathbb{D}} := \{\omega \in \mathbb{D} : \|\omega_1\| \neq 0\}$. It is clear that for $t > 1$, $\Lambda_t^* = \Lambda_t$ and $Y_t(\Lambda^*) \overset{d}{=} Y_t$. Since $\mathbb{N}_x(\widetilde{\mathbb{D}}) < \infty$, Λ_t^* is a compound Poisson process and can be written as

$$\Lambda_t^* = \sum_{j=1}^{K} \widetilde{X}_t^j,$$

where $\widetilde{X}_t^j$, $j = 1, 2, \ldots$ are i.i.d. with the same law as ω_t under $\tilde{\mathbb{N}}_x$ and K is a Poisson random variable with parameter $\mathbb{N}_x(\widetilde{\mathbb{D}})$ which is independent of $\widetilde{X}_t^j$, $j = 1, 2, \ldots$. Let

$$Y_t(\widetilde{X}^j) := t^{-1/2} e^{\lambda_1 t/2} \langle h, \widetilde{X}_t^j \rangle + e^{\lambda_1 t/2} \langle f, \widetilde{X}_t^j \rangle.$$

Then, $Y_t(\widetilde{X}^j)$ is independent of K and has the same law as $\widetilde{Y}_t$ under $\widetilde{\mathbb{N}}_x$. Therefore, for $t > 1$,

$$\begin{aligned}
\mathbb{P}_{\delta_x}(|Y_t|^2; |Y_t| > ce^{\delta t}) &= \mathbf{P}_{\delta_x}(|Y_t(\Lambda^*)|^2; |Y_t(\Lambda^*)| > ce^{\delta t}) \\
&\geq \mathbf{P}_{\delta_x}(|Y_t(\widetilde{X}^1)|^2; |Y_t(\widetilde{X}^1)| > ce^{\delta t}, K = 1) \\
&= \mathbf{P}_{\delta_x}(K = 1) \mathbf{P}_{\delta_x}(|Y_t(\widetilde{X}^1)|^2; |Y_t(\widetilde{X}^1)| > ce^{\delta t}) \\
&= \mathbb{N}_x(\widetilde{\mathbb{D}}) e^{-\mathbb{N}_x(\widetilde{\mathbb{D}})} \widetilde{\mathbb{N}}_x(|\widetilde{Y}_t|^2; |\widetilde{Y}_t| > ce^{\delta t}).
\end{aligned}$$

Now (3.22) follows easily from Lemma 3.3. □

Lemma 3.5. *Assume that $f \in \mathcal{C}_s$ and $h \in \mathcal{C}_c$. Then*

$$\left(e^{\lambda_1 t} \langle \phi_1, X_t \rangle, t^{-1/2} e^{\lambda_1 t/2} \langle h, X_t \rangle, e^{\lambda_1 t/2} \langle f, X_t \rangle \right)$$
$$\overset{d}{\to} \left(W_\infty, \sqrt{W_\infty} G_2(h), \sqrt{W_\infty} G_1(f) \right), \tag{3.23}$$

where $G_2(h) \sim \mathcal{N}(0, \rho_h^2)$ and $G_1(f) \sim \mathcal{N}(0, \sigma_f^2)$. Moreover, $W_\infty, G_2(h)$ and $G_1(f)$ are independent.

Proof. In the proof, we always assume $t > 3t_0$. We define an $\mathbb{R}^3$-valued random variable by

$$U_1(t) := \left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle, t^{-1/2}e^{\lambda_1 t/2}\langle h, X_t\rangle, e^{\lambda_1 t/2}\langle f, X_t\rangle\right).$$

Let $n > 2$ and write

$$U_1(nt) = \left(e^{\lambda_1 nt}\langle \phi_1, X_{nt}\rangle,\ (nt)^{-1/2}e^{\lambda_1 nt/2}\langle h, X_{nt}\rangle, e^{\lambda_1 nt/2}\langle f, X_{nt}\rangle\right).$$

To consider the limit of $U_1(t)$ as $t \to \infty$, it is equivalent to consider the limit of $U_1(nt)$ for any $n > 2$. The main idea is as follows. For $t > t_0, n > 2$,

$$\begin{aligned} U_1(nt) = &\left(e^{\lambda_1 nt}\langle \phi_1, X_{nt}\rangle,\ \frac{e^{\lambda_1 nt/2}(\langle h, X_{nt}\rangle - \langle T_{(n-1)t}h, X_t\rangle)}{((n)t)^{1/2}},\right.\\ &\left. e^{\lambda_1 nt/2}(\langle f, X_{nt}\rangle - \langle T_{(n-1)t}f, X_t\rangle)\right)\\ &+ \left(0,\ (nt)^{-1/2}e^{\lambda_1 nt/2}\langle T_{(n-1)t}h, X_t\rangle,\ e^{\lambda_1 nt/2}\langle T_{(n-1)t}f, X_t\rangle\right). \end{aligned} \tag{3.24}$$

The double limit, first as $t \to \infty$ and then $n \to \infty$, of the first term of the right side of (3.24) is equal to the double limit, first as $t \to \infty$ and then $n \to \infty$, of another $\mathbb{R}^2$-valued random variable $U_2(n, t)$ where

$$\begin{aligned} U_2(n, t) := &\left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle,\ \frac{e^{\lambda_1 nt/2}(\langle h, X_{nt}\rangle - \langle T_{(n-1)t}h, X_t\rangle)}{((n-1)t)^{1/2}},\right.\\ &\left. e^{\lambda_1 nt/2}(\langle f, X_{nt}\rangle - \langle T_{(n-1)t}f, X_t\rangle)\right). \end{aligned}$$

We will prove that the second term on the right hand side of (3.24) has no contribution to the double limit, first as $t \to \infty$ and then $n \to \infty$, of the left hand side of (3.24).

We claim that

$$U_2(n, t) \overset{d}{\to} \left(W_\infty, \sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)\right), \quad \text{as } t \to \infty. \tag{3.25}$$

Denote the characteristic function of $U_2(n, t)$ under $\mathbb{P}_\mu$ by $\kappa_2(\theta_1, \theta_2, \theta_3, n, t)$. Define

$$Y_1(t, \theta_2) := \theta_2 t^{-1/2}e^{\lambda_1 t/2}\langle h, X_t\rangle, \qquad Y_2(t, \theta_3) := \theta_3 e^{\lambda_1 t/2}\langle f, X_t\rangle, \quad t > 0,$$

and

$$Y_t(\theta_2, \theta_3) = Y_1(t, \theta_2) + Y_2(t, \theta_3).$$

We define the corresponding random variables on $\mathbb{D}$ as $\tilde{Y}_1(t, \theta_2)$, $\tilde{Y}_2(t, \theta_3)$ and $\tilde{Y}_t(\theta_2, \theta_3)$. Using an argument similar to that leading to (3.6), we get

$$\begin{aligned} \kappa_2(\theta_1, \theta_2, \theta_3, n, t) = \mathbb{P}_\mu\Big(\exp\Big\{i\theta_1 e^{\lambda_1 t}\langle \phi_1, X_t\rangle + \int_E\int_{\mathbb{D}}\Big(\exp\Big\{ie^{\lambda_1 t/2}\tilde{Y}_{(n-1)t}(\theta_2, \theta_3)(\omega)\Big\}\\ - 1 - ie^{\lambda_1 t/2}\tilde{Y}_{(n-1)t}(\theta_2, \theta_3)(\omega)\Big)\mathbb{N}_x(d\omega)X_t(dx)\Big\}\Big). \end{aligned}$$

Define

$$\begin{aligned} R'_t(x, \theta) := \int_{\mathbb{D}}\Big(&\exp\{i\theta\tilde{Y}_t(\theta_2, \theta_3)(\omega)\} - 1 - i\theta\tilde{Y}_t(\theta_2, \theta_3)(\omega)\\ &+ \frac{1}{2}\theta^2(\tilde{Y}_t(\theta_2, \theta_3)(\omega))^2\Big)\mathbb{N}_x(d\omega) \end{aligned}$$

and

$$J(n,t,x) := \int_{\mathbb{D}} \left(\exp\{ie^{\lambda_1 t/2}\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)(\omega)\} - 1 - ie^{\lambda_1 t/2}\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)(\omega)\right)\mathbb{N}_x(d\omega).$$

Then

$$J(n,t,x) = -\frac{1}{2}e^{\lambda_1 t}\mathbb{N}_x(\tilde{Y}_{(n-1)t}(\theta_2,\theta_3))^2 + R'_{(n-1)t}(x, e^{\lambda_1 t/2}),$$

and

$$\kappa_2(\theta_1,\theta_2,\theta_3,n,t) = \mathbb{P}_\mu\left(\exp\left\{i\theta_1 e^{\lambda_1 t}\langle\phi_1, X_t\rangle + \langle J(n,t,\cdot), X_t\rangle\right\}\right).$$

Let $V_t^n(x) := \mathbb{N}_x(\tilde{Y}_{(n-1)t}(\theta_2,\theta_3))^2$. Then

$$\begin{aligned}\langle J(n,t,\cdot), X_t\rangle &= -\frac{1}{2}e^{\lambda_1 t}\langle V_t^n, X_t\rangle + \langle R'_{(n-1)t}(\cdot, e^{\lambda_1 t/2}), X_t\rangle \\ &:= J_1(n,t) + J_2(n,t).\end{aligned}$$

We first consider $J_1(n,t)$. By (2.4),

$$\begin{aligned}V_t^n(x) &= \mathbb{V}\mathrm{ar}_{\delta_x}(Y_{(n-1)t}(\theta_2,\theta_3)) \\ &= \mathbb{V}\mathrm{ar}_{\delta_x}(Y_1((n-1)t,\theta_2)) + \mathbb{V}\mathrm{ar}_{\delta_x}(Y_2((n-1)t,\theta_3)) \\ &\quad + \mathbb{C}\mathrm{ov}_{\delta_x}(Y_1((n-1)t,\theta_2), Y_2((n-1)t,\theta_3)).\end{aligned}$$

So by (2.17), (2.20) and (2.25), we have, for $t > 3t_0$,

$$\begin{aligned}&\left|V_t^n(x) - (\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)\phi_1(x)\right| \\ &\le \left|\mathbb{V}\mathrm{ar}_{\delta_x}(Y_1((n-1)t,\theta_2)) - \theta_2^2\rho_h^2\phi_1(x)\right| + \left|\mathbb{V}\mathrm{ar}_{\delta_x}(Y_2((n-1)t,\theta_3)) - \theta_3^2\sigma_f^2\phi_1(x)\right| \\ &\quad + \left|\mathbb{C}\mathrm{ov}_{\delta_x}(Y_1((n-1)t,\theta_2), Y_2((n-1)t,\theta_3))\right| \\ &\lesssim \left(e^{(\lambda_1 - 2\lambda_{\gamma(f)})(n-1)t} + e^{(\lambda_1-\lambda_2)(n-1)t} + ((n-1)t)^{-1/2} + ((n-1)t)^{-1}\right)a_{t_0}(x)^{1/2}.\end{aligned} \tag{3.26}$$

Thus, we have that as $t \to \infty$,

$$\begin{aligned}&e^{\lambda_1 t}\left\langle\left|V_t^n(x) - (\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)\phi_1(x)\right|, X_t\right\rangle \\ &\lesssim \left(e^{(\lambda_1 - 2\lambda_{\gamma(f)})(n-1)t} + e^{(\lambda_1-\lambda_2)(n-1)t} + ((n-1)t)^{-1/2} + ((n-1)t)^{-1}\right) \\ &\quad \times e^{\lambda_1 t}\langle (a_{t_0})^{1/2}, X_t\rangle \to 0,\end{aligned}$$

in probability. It follows that

$$\lim_{t\to\infty} J_1(n,t) = \lim_{t\to\infty} -\frac{1}{2}e^{\lambda_1 t}(\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)\langle\phi_1, X_t\rangle = -\frac{1}{2}(\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)W_\infty \quad \text{in probability.} \tag{3.27}$$

For $J_2(n,t)$, by (3.2), we have, for any $\epsilon > 0$,

$$\begin{aligned}|R'_{(n-1)t}(x, e^{\lambda_1 t/2})| &\le \frac{1}{6}e^{\frac{3}{2}\lambda_1 t}\mathbb{N}_x\left(|\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|^3; |\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)| < \epsilon e^{-\lambda_1 t/2}\right) \\ &\quad + e^{\lambda_1 t}\mathbb{N}_x\left(|\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|^2; |\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)| \ge \epsilon e^{-\lambda_1 t/2}\right)\end{aligned}$$

$$\begin{aligned}&\le \frac{\epsilon}{6}e^{\lambda_1 t}\mathbb{N}_x\left(|\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|^2\right)\\&\quad + e^{\lambda_1 t}\mathbb{N}_x\left(|\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|^2;\, |\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|\ge \epsilon e^{-\lambda_1 t/2}\right)\\&= \frac{\epsilon}{6}e^{\lambda_1 t}V_t^n(x) + e^{\lambda_1 t}F_t^n(x),\end{aligned}$$

where $F_t^n(x) = \mathbb{N}_x\left(|\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|^2;\, |\tilde{Y}_{(n-1)t}(\theta_2,\theta_3)|\ge \epsilon e^{-\lambda_1 t/2}\right)$. Note that

$$e^{\lambda_1 t}\mathbb{P}_\mu\langle F_t^n(x), X_t\rangle = e^{\lambda_1 t}\langle T_t(F_t^n), \mu\rangle. \tag{3.28}$$

It follows from Lemma 3.4 that $\lim_{t\to\infty} F_t^n(x) = 0$. By (3.26), we also have

$$F_t^n(x) \le V_t^n(x) \lesssim a_{t_0}(x)^{1/2},$$

which implies that $\|F_t^n\|_2 \to 0$ as $t\to\infty$. By Lemma 2.5,

$$\lim_{t\to\infty} e^{\lambda_1 t}T_t(F_t^n)(x) = 0.$$

Note that, by (2.10), $e^{\lambda_1 t}T_t(F_t^n) \lesssim e^{\lambda_1 t}T_t(a_{t_0}^{1/2}) \lesssim a_{t_0}^{1/2}$. Since μ has compact support and a_{t_0} is continuous, we have $\langle a_{t_0}, \mu\rangle < \infty$. By (3.28) and the dominated convergence theorem, we obtain $\lim_{t\to\infty} e^{\lambda_1 t}\mathbb{P}_\mu\langle F_t^n(x), X_t\rangle = 0$, which implies that $e^{\lambda_1 t}\langle F_t^n(x), X_t\rangle \to 0$ in probability. Furthermore, by (3.27), we have that as $t\to\infty$,

$$\frac{\epsilon}{6}e^{\lambda_1 t}\langle V_t^n, X_t\rangle \to \frac{\epsilon}{6}(\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)W_\infty \quad \text{in probability.}$$

Thus, letting $\epsilon\to 0$, we get that as $t\to\infty$,

$$J_2(n,t)\to 0 \quad \text{in probability.} \tag{3.29}$$

Thus, when $t\to\infty$,

$$\exp\{\langle J(n,t,\cdot), X_t\rangle\} \to \exp\left\{-\frac{1}{2}(\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)W_\infty\right\} \tag{3.30}$$

in probability. Since the real part of $J(n,t,x)$ is less than 0, we have

$$|\exp\{\langle J(n,t,\cdot), X_t\rangle\}| \le 1.$$

So by the dominated convergence theorem, we get that

$$\lim_{t\to\infty}\kappa_2(\theta_1,\theta_2,\theta_3,n,t) = \mathbb{P}_\mu\left[\exp\{i\theta_1 W_\infty\}\exp\left\{-\frac{1}{2}(\theta_2^2\rho_h^2 + \theta_3^2\sigma_f^2)W_\infty\right\}\right], \tag{3.31}$$

which implies our claim (3.25).

By (3.25) and the fact $e^{\lambda_1 nt}\langle\phi_1, X_{nt}\rangle - e^{\lambda_1 t}\langle\phi_1, X_t\rangle \to 0$, in probability, as $t\to\infty$, we easily get

$$\begin{aligned}U_3(n,t) &:= \Bigg(e^{\lambda_1 nt}\langle\phi_1, X_{nt}\rangle, \frac{e^{\lambda_1 nt/2}(\langle h, X_{nt}\rangle - \langle T_{(n-1)t}h, X_t\rangle)}{(nt)^{1/2}},\\&\qquad e^{\lambda_1 nt/2}(\langle f, X_{nt}\rangle - \langle T_{(n-1)t}f, X_t\rangle)\Bigg)\\&\xrightarrow{d} \left(W_\infty, \sqrt{\frac{n-1}{n}}\sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)\right).\end{aligned}$$

Using (2.17) and the fact $\mathbb{P}_\mu\langle h, X_t\rangle = \langle T_t h, \mu\rangle = e^{-\lambda_1 t/2}\langle h, \mu\rangle$, we can get

$$(nt)^{-1}e^{\lambda_1 nt}\mathbb{P}_\mu(\langle T_{(n-1)t}h, X_t\rangle)^2 = (nt)^{-1}e^{\lambda_1 t}\mathbb{V}ar_\mu\langle h, X_t\rangle + (nt)^{-1}e^{\lambda_1 t}(\mathbb{P}_\mu\langle h, X_t\rangle)^2 \lesssim n^{-1}(1+t^{-1}). \tag{3.32}$$

Using (3.13) with $s = (n-1)t$, and then letting $t \to \infty$, by (2.14) we get

$$e^{\lambda_1 nt}\mathbb{P}_\mu(\langle T_{(n-1)t} f, X_t\rangle)^2 \lesssim e^{(\lambda_1 - 2\lambda_{\gamma(f)})(n-1)t}\langle a_{t_0}(x)^{1/2}, \mu\rangle + e^{\lambda_1 nt}\langle T_{nt} f, \mu\rangle^2 \to 0. \tag{3.33}$$

Let $\mathcal{D}(nt)$ and $\widetilde{\mathcal{D}}^n(t)$ be the distributions of $U_1(nt)$ and $U_3(n,t)$ respectively, and let $\mathcal{D}^n$ and $\mathcal{D}$ be the distributions of $\left(W_\infty, \sqrt{\frac{n-1}{n}}\sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)\right)$ and $\left(W_\infty, \sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)\right)$ respectively. Then, using (3.1), we have

$$\begin{aligned}\limsup_{t\to\infty} d(\mathcal{D}(nt), \mathcal{D}) &\le \limsup_{t\to\infty}[d(\mathcal{D}(nt), \widetilde{\mathcal{D}}^n(t)) + d(\widetilde{\mathcal{D}}^n(t), \mathcal{D}^n) + d(\mathcal{D}^n, \mathcal{D})] \\ &\le \limsup_{t\to\infty}\Big((nt)^{-1}e^{\lambda_1 nt}\mathbb{P}_\mu\langle T_{(n-1)t}h, X_t\rangle^2 \\ &\quad + e^{\lambda_1 nt}\mathbb{P}_\mu\langle T_{(n-1)t} f, X_t\rangle^2\Big)^{1/2} + 0 + d(\mathcal{D}^n, \mathcal{D}).\end{aligned} \tag{3.34}$$

Using the definition of $\limsup_{t\to\infty}$, (3.32) and (3.33), we easily get that

$$\limsup_{t\to\infty} d(\mathcal{D}(t), \mathcal{D}) = \limsup_{t\to\infty} d(\mathcal{D}(nt), \mathcal{D}) \le c/\sqrt{n} + d(\mathcal{D}^n, \mathcal{D}),$$

where c is a constant. Letting $n \to \infty$, we get $\limsup_{t\to\infty} d(\mathcal{D}(t), \mathcal{D}) = 0$. The proof is now complete. □

Recall that

$$g(x) = \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} b_j^k\phi_j^{(k)}(x) \quad \text{and} \quad I_u g(x) = \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{\lambda_k u}b_j^k\phi_j^{(k)}(x).$$

Note that the sum over k is a sum over a finite number of elements. Define

$$H_\infty(\omega) := \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} b_j^k\widetilde{H}_\infty^{k,j}(\omega), \quad \omega \in \mathbb{D}.$$

By Lemma 3.1, we have, as $u \to \infty$

$$\langle I_u g, \omega_u\rangle \to H_\infty, \quad \mathbb{N}_x\text{-a.e., in } L^1(\mathbb{N}_x) \text{ and in } L^2(\mathbb{N}_x).$$

Since $\mathbb{N}_x\langle I_u g, \omega_u\rangle = \mathbb{P}_{\delta_x}\langle I_u g, X_u\rangle = g(x)$, we get

$$\mathbb{N}_x(H_\infty) = g(x). \tag{3.35}$$

By (2.4) and (2.13), we have

$$\mathbb{N}_x\langle I_u g, \omega_u\rangle^2 = \mathbb{V}\mathrm{ar}_{\delta_x}\langle I_u g, X_u\rangle = \int_0^u T_s\left[A\left(\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{\lambda_k s}b_j^k\phi_j^k\right)^2\right](x)\,ds, \tag{3.36}$$

which implies

$$\mathbb{N}_x(H_\infty)^2 = \int_0^\infty T_s\left[A\left(\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{\lambda_k s} b_j^k \phi_j^k\right)^2\right](x)\,ds. \tag{3.37}$$

By (1.9), we have that for any $x \in E$,

$$\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{\lambda_k s}|b_j^k||\phi_j^k(x)| \lesssim e^{\lambda_K s} a_{2t_0}(x)^{1/2},$$

where $K = \sup\{k : 2\lambda_k < \lambda_1\}$. So by (3.37), (2.10) and (2.28), we have that for any $x \in E$,

$$\begin{aligned}\mathbb{N}_x(H_\infty)^2 &\lesssim \int_0^\infty e^{(2\lambda_K-\lambda_1)s} e^{\lambda_1 s} T_s(a_{2t_0})(x)\,ds \\ &= \left(\int_0^{t_0} + \int_{t_0}^\infty\right) e^{(2\lambda_K-\lambda_1)s} e^{\lambda_1 s} T_s(a_{2t_0})(x)\,ds \\ &\lesssim \int_0^{t_0} T_s(a_{2t_0})(x)\,ds + \int_{t_0}^\infty e^{(2\lambda_K-\lambda_1)s}\,ds\; a_{t_0}(x)^{1/2} \\ &\lesssim a_{t_0}(x)^{1/2} \in L^2(E,m)\cap L^4(E,m). \end{aligned} \tag{3.38}$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Consider an $\mathbb{R}^4$-valued random variable $U_4(t)$ defined by:

$$U_4(t) := \left(e^{\lambda_1 t}\langle \phi_1, X_t\rangle,\ e^{\lambda_1 t/2}\left(\langle g, X_t\rangle - \sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k t} b_j^k H_\infty^{k,j}\right), \frac{e^{\lambda_1 t/2}\langle h, X_t\rangle}{t^{1/2}}, e^{\lambda_1 t/2}\langle f, X_t\rangle\right).$$

To get the conclusion of Theorem 1.4, it suffices to show that, under $\mathbb{P}_\mu$,

$$U_4(t) \xrightarrow{d} \left(W_\infty, \sqrt{W_\infty}G_3(g), \sqrt{W_\infty}G_2(h), \sqrt{W_\infty}G_1(f)\right), \tag{3.39}$$

where $W_\infty, G_3(g), G_2(h)$ and $G_1(f)$ are independent. Denote the characteristic function of $U_4(t)$ under $\mathbb{P}_\mu$ by $\kappa_1(\theta_1,\theta_2,\theta_3,\theta_4,t)$. Then, we only need to prove

$$\lim_{t\to\infty}\kappa_1(\theta_1,\theta_2,\theta_3,\theta_4,t) = \mathbb{P}_\mu\left(\exp\{i\theta_1 W_\infty\}\exp\left\{-\frac{1}{2}(\theta_2^2\beta_g^2 + \theta_3^2\rho_h^2 + \theta_4^2\sigma_f^2)W_\infty\right\}\right). \tag{3.40}$$

Note that, by Lemma 1.1, $\sum_{k:2\lambda_k<\lambda_1}\sum_{j=1}^{n_k} e^{-\lambda_k t} b_j^k H_\infty^{k,j} = \lim_{u\to\infty}\langle I_u g, X_{t+u}\rangle$, $\mathbb{P}_\mu$-a.s. We have

$$\begin{aligned}\kappa_1(\theta_1,\theta_2,\theta_3,\theta_4,t) &= \lim_{u\to\infty}\mathbb{P}_\mu\left(\exp\left\{i\theta_1 e^{\lambda_1 t}\langle\phi_1,X_t\rangle + i\theta_2 e^{\lambda_1 t/2}(\langle g,X_t\rangle - \langle I_u g, X_{t+u}\rangle)\right.\right. \\ &\qquad \left.\left. + i\theta_3 t^{-1/2} e^{\lambda_1 t/2}\langle h,X_t\rangle + i\theta_4 e^{\lambda_1 t/2}\langle f,X_t\rangle\right\}\right) \\ &= \lim_{u\to\infty}\mathbb{P}_\mu\Big(\exp\Big\{i\theta_1 e^{\lambda_1 t}\langle\phi_1,X_t\rangle + i\theta_3 t^{-1/2}e^{\lambda_1 t/2}\langle h,X_t\rangle \\ &\qquad + i\theta_4 e^{\lambda_1 t/2}\langle f,X_t\rangle + \langle J_u(t,\cdot),X_t\rangle\Big\}\Big), \end{aligned} \tag{3.41}$$

where

$$J_u(t,x) = \int_{\mathbb{D}} \left(\exp\left\{-i\theta_2 e^{\lambda_1 t/2}\langle I_u g, \omega_u\rangle\right\} - 1 + i\theta_2 e^{\lambda_1 t/2}\langle I_u g, \omega_u\rangle\right)\mathbb{N}_x(d\omega).$$

The last equality above follows from the Markov property of X, (2.7) and the fact

$$\int_{\mathbb{D}} \langle I_u g, \omega_u\rangle \mathbb{N}_x(d\omega) = \mathbb{P}_{\delta_x}\langle I_u g, X_u\rangle = g(x).$$

We will show that

$$\lim_{u\to\infty} J_u(t,x) = \mathbb{N}_x\left(\exp\left\{-i\theta_2 e^{\lambda_1 t/2} H_\infty\right\} - 1 + i\theta_2 e^{\lambda_1 t/2} H_\infty\right) =: J(t,x). \tag{3.42}$$

For $u > 1$, $|e^{-i\theta_2 e^{\lambda_1 t/2}\langle I_u g, \omega_u\rangle} - 1| \le 2\mathbf{1}_{\{\|\omega_1\|\neq 0\}}(\omega)$. By Remark 2.3, $\mathbb{N}_x(\|\omega_1\| \neq 0) < \infty$. Thus, by Lemma 3.1 and the dominated convergence theorem, we get

$$\lim_{u\to\infty} \int_{\mathbb{D}} \left(\exp\left\{-i\theta_2 e^{\lambda_1 t/2}\langle I_u g, \omega_u\rangle\right\} - 1\right)\mathbb{N}_x(d\omega) = \mathbb{N}_x\left(\exp\left\{-i\theta_2 e^{\lambda_1 t/2} H_\infty\right\} - 1\right).$$

By (3.35), we get $\mathbb{N}_x H_\infty = \mathbb{N}_x\langle I_u g, \omega_u\rangle = g(x)$. Then, (3.42) follows immediately.

By (3.2), we get

$$\sup_{u\ge 0} |J_u(t,x)| \le \frac{1}{2}\theta_2^2 e^{\lambda_1 t} \sup_{u\ge 0} \mathbb{N}_x\langle I_u g, \omega_u\rangle^2 < \frac{1}{2}\theta_2^2 e^{\lambda_1 t}\mathbb{N}_x H_\infty^2 < \infty.$$

Note that, by (3.38),

$$\mathbb{P}_\mu\langle \mathbb{N}.H_\infty^2, X_t\rangle \lesssim \mathbb{P}_\mu\langle a_{t_0}^{1/2}, X_t\rangle = \langle T_t a_{t_0}^{1/2}, \mu\rangle < \infty,$$

which implies that $\langle \mathbb{N}.H_\infty^2, X_t\rangle < \infty, \mathbb{P}_\mu$-a.s. So, by the dominated convergence theorem, we get

$$\lim_{u\to\infty}\langle J_u(t,\cdot), X_t\rangle = \langle J(t,\cdot), X_t\rangle, \quad \mathbb{P}_\mu\text{-a.s.}$$

Using the dominated convergence theorem again, we obtain

$$\begin{aligned}\kappa_1(\theta_1,\theta_2,\theta_3,\theta_4,t) = \mathbb{P}_\mu\Big(\exp\Big\{& i\theta_1 e^{\lambda_1 t}\langle \phi_1, X_t\rangle + i\theta_3 t^{-1/2} e^{\lambda_1 t/2}\langle h, X_t\rangle \\ &+ i\theta_4 e^{\lambda_1 t/2}\langle f, X_t\rangle + \langle J(t,\cdot), X_t\rangle\Big\}\Big).\end{aligned}$$

Let

$$R(\theta,x) := \mathbb{N}_x\left(\exp\{i\theta H_\infty\} - 1 - i\theta H_\infty + \frac{1}{2}\theta^2 H_\infty^2\right).$$

Thus,

$$\langle J(t,\cdot), X_t\rangle = -\frac{1}{2}\theta_2^2 e^{\lambda_1 t}\langle V, X_t\rangle + \langle R(-e^{\lambda_1 t/2}\theta_2, \cdot), X_t\rangle,$$

where $V(x) := \mathbb{N}_x(H_\infty)^2$. By (3.2), we have

$$|R(-e^{\lambda_1 t/2}\theta_2, x)| \le e^{\lambda_1 t}\theta_2^2 \mathbb{N}_x\left(|H_\infty|^2\left(\frac{e^{\lambda_1 t/2}\theta_2|H_\infty|}{6} \wedge 1\right)\right), \tag{3.43}$$

which implies that

$$\mathbb{P}_\mu \left| \langle R(-e^{\lambda_1 t/2}\theta_2, \cdot), X_t\rangle \right| \le \theta_2^2 \, e^{\lambda_1 t} \langle \, T_t(k(\cdot, t)), \; \mu \, \rangle,$$

where

$$k(x,t) := \mathbb{N}_x \left(|H_\infty|^2 \left(\frac{e^{\lambda_1 t/2}\theta_2 |H_\infty|}{6} \wedge 1 \right) \right).$$

It is clear that $k(x,t) \downarrow 0$ as $t \uparrow \infty$. Thus as $t \to \infty$, $e^{\lambda_1 t} T_t(k(\cdot, t))(x) \to 0$, which implies

$$\lim_{t\to\infty} \langle R(-e^{\lambda_1 t/2}\theta_2, \cdot), X_t\rangle = 0 \quad \text{in probability.} \tag{3.44}$$

Since $V \in L^2(E,m) \cap L^4(E,m)$, by Remark 1.3, we have

$$\lim_{t\to\infty} e^{\lambda_1 t} \langle V, X_t\rangle = \langle V, \; \phi_1\rangle_m W_\infty \quad \text{in probability.} \tag{3.45}$$

Therefore, combining (3.44) and (3.45), we get

$$\lim_{t\to\infty} \exp\{\langle J(t,\cdot), X_t\rangle\} = \exp\left\{ -\frac{1}{2}\theta_2^2 \langle V, \; \phi_1\rangle_m W_\infty \right\} \quad \text{in probability.} \tag{3.46}$$

Since the real part of $J(t,x)$ is less than 0,

$$|\exp\{\langle J(t,\cdot), X_t\rangle\}| \le 1. \tag{3.47}$$

Recall that $\lim_{t\to\infty} e^{\lambda_1 t}\langle \phi_1, X_t\rangle = W_\infty, \mathbb{P}_\mu$-a.s. Thus by (3.46), (3.47) and the dominated convergence theorem, we get that as $t \to \infty$,

$$\begin{aligned}
&\left| \mathbb{P}_\mu \left(\exp\left\{ \left(i\theta_1 - \frac{1}{2}\theta_2^2 \langle V, \; \phi_1\rangle_m \right) e^{\lambda_1 t}\langle \phi_1, X_t\rangle + i\theta_3 t^{-1/2} e^{\lambda_1 t/2}\langle h, X_t\rangle \right.\right.\right. \\
&\qquad \left.\left.\left. + i\theta_4 e^{\lambda_1 t/2}\langle f, X_t\rangle \right\} \right) - \kappa_1(\theta_1, \theta_2, \theta_3, \theta_4, t) \right| \\
&\quad \le \mathbb{P}_\mu \left| \exp\{\langle J(t,\cdot), X_t\rangle\} - \exp\left\{ -\frac{1}{2}\theta_2^2 \langle V, \; \phi_1\rangle_m e^{\lambda_1 t}\langle \phi_1, X_t\rangle \right\} \right| \to 0.
\end{aligned} \tag{3.48}$$

By Lemma 3.5,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{P}_\mu \left(\exp\left\{ \left(i\theta_1 - \frac{1}{2}\theta_2^2 \langle V, \; \phi_1\rangle_m \right) e^{\lambda_1 t}\langle \phi_1, X_t\rangle + i\theta_3 t^{-1/2} e^{\lambda_1 t/2}\langle h, X_t\rangle \right.\right. \\
&\qquad \left.\left. + \; i\theta_4 e^{\lambda_1 t/2}\langle f, X_t\rangle \right\} \right) \\
&\quad = \mathbb{P}_\mu \left(\exp\{i\theta_1 W_\infty\} \exp\left\{ -\frac{1}{2}(\theta_2^2 \langle V, \; \phi_1\rangle_m + \theta_3^2 \rho_f^2 + \theta_4^2 \sigma_f^2) W_\infty \right\} \right).
\end{aligned} \tag{3.49}$$

By (3.37), we get

$$\langle V, \; \phi_1\rangle_m = \int_0^\infty e^{-\lambda_1 s} \left\langle A(I_s g)^2, \phi_1 \right\rangle_m ds.$$

The proof is now complete. □

Acknowledgments

We thank the referee for helpful comments on the first version of this paper. The research of the first author is supported by NSFC (Grant No. 11271030 and 11128101) and LMEQF. Renming Song's research was supported in part by a grant from the Simons Foundation (208236). Rui Zhang was supported by the China Scholarship Council.

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