Conditional limit theorems for critical continuous-state branching processes

Yan-Xia Ren*, Ting Yang and Guo-Huan Zhao

Abstract

In this paper we study the conditional limit theorems for critical continuous-state branching processes with branching mechanism $\psi(\lambda) = \lambda^{1+\alpha}L(1/\lambda)$ where $\alpha \in [0,1]$ and L is slowly varying at ∞. We prove that if $\alpha \in (0,1]$, there are norming constants $Q_t \to 0$ (as $t \uparrow +\infty$) such that for every $x > 0$, $P_x(Q_t X_t \in \cdot | X_t > 0)$ converges weakly to a non-degenerate limit. The converse assertion is also true provided the regularity of ψ at 0. We give a conditional limit theorem for the case $\alpha = 0$. The limit theorems we obtain in this paper allow infinite variance of the branching process.

1 Introduction

A $[0,+\infty)$-valued strong Markov process $X = \{X_t : t \geq 0\}$ with probabilities $\{P_x : x > 0\}$ is called a (conservative) continuous-state branching process (CB process) if it has paths that are right continuous with left limits, and it employs the following branching property: for any $\lambda \geq 0$ and $x, y > 0$,

$$E_{x+y}(e^{-\lambda X_t}) = E_x(e^{-\lambda X_t})E_y(e^{-\lambda X_t}). \tag{1.1}$$

It can be characterized by the branching mechanism ψ which is also the Laplace exponent of a Lévy process with non-negative jumps. Set $\rho := \psi'(0+)$, then $E_x X_t = xe^{-\rho t}$. We call a CB process *supercritical, critical* or *subcritical* as $\rho < 0$, $= 0$, or > 0.

Let $\tau := \inf\{t \geq 0 : X_t = 0\}$ denote the extinction time of X_t and $q(x) := P_x(\tau < +\infty)$. When $q(x) < 1$ for some (and then for all) $x > 0$, the asymptotic behavior of X_t is studied in [3]. It was proved that there are positive constants η_t such that $\eta_t X_t$ converges almost surely to a non-degenerate random variable

*Research supported in part by NNSF of China (Grant No. 10971003 and No. 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education.

as $t \to +\infty$.Note that $q(x) \equiv 1$ if and only if X is subcritical or critical with ψ satisfying

$$\int_{\theta}^{+\infty} \frac{1}{\psi(\xi)} d\xi < +\infty \tag{1.2}$$

for some $\theta > 0$. In this case, one can study the asymptotic behavior of X by conditioning it on $\{\tau > t\}$ (see [7, 5, 9, 10] and the references therein). In the subcritical case, it was proved that $P_x(X_t \in \cdot | \tau > t)$ converges weakly as $t \to +\infty$ to the so-called Yaglom distribution. However in the critical case, the limiting distribution of X_t conditioned on non-extinction is trivial, converging to the Dirac measure at ∞. To evaluate the asymptotic behavior of X_t more accurately, we therefore have to normalize the process appropriately.

Throughout this paper, we assume ψ satisfies

$$\psi(\lambda) = \lambda^{1+\alpha} L(1/\lambda) \quad \forall \lambda \geq 0 \tag{1.3}$$

where $\alpha \in [0, 1]$ and L is slowly varying at infinity. Our assumption on ψ does not require the finiteness of $E_x X_t^2$.

It is well known that a CB process can be viewed as the analogue of Galton-Watson branching process in continuous time and continuous state space. So it is necessary for us to take a look at the asymptotic behavior of critical G-W branching processes. Let $f(s)$ denote the probability generating function of the offspring law of the critical G-W process Z_n. Let $\bar{F}(n) = P_1(Z_n > 0)$. Slack [13, 14] proved that $P_1(\bar{F}(n) Z_n \leq y | Z_n > 0)$ converges weakly to a non-degenerate limit if and only if

$$f(s) = s + (1-s)^{1+\alpha} L\left(\frac{1}{1-s}\right) \tag{1.4}$$

for some $\alpha \in (0, 1]$ and L slowly varying at $+\infty$. Later Nagaev *et.al.*[6] proved a conditional limit theorem for $f(s)$ satisfying (1.4) with $\alpha = 0$. Recently, Pakes [8] generalized the above results to continuous time Markov branching process. The proofs given in [8], based on Karamata's theory for regular varying functions, are much easier. However, for discrete-state branching process, there leaves open the question of whether (1.4) is implied by the more general conditional convergence of $P_1(b_n Z_n \leq y | Z_n > 0)$ for some positive sequence $\{b_n\}$ with $b_n \to 0$.

This paper is structured as follows: In Section 2, we collect some basic facts about regularly varying functions and CB processes. Section 3 is devoted to the conditional limit theorems for ψ with $\alpha \in (0, 1]$. We prove that there exists positive norming constants $Q_t \to 0$ such that $P_x(Q_t X_t \in \cdot | \tau > t)$ converges weakly to a non-degenerate limit. An admissible norming is $Q_t = P_1(\tau > t)$. This is analogous to the result we mentioned in the above paragraph for discrete-state branching processes. Later we prove that the converse assertion is also true provided some regularity of ψ at 0 (or equivalently, provided some regularity of the Lévy measure of ψ at infinity). In Section 4, we give a conditional limit theorem for the case

$\alpha = 0$. Its discrete state analogue is proved independently in [6] and [8]. The last section provides some concrete examples which satisfy the assumptions in Section 3 or Section 4. The branching mechanisms in these examples are well known and taken from [11].

2 Preliminary

In the rest of this paper, we shall use the notation $f(x) \sim g(x)$ for functions f and g to mean that $f(x)/g(x) \to 1$ as $x \to +\infty$ or 0. Let $x \wedge y := \min\{x, y\}$.

Suppose X is a CB process with branching mechanism ψ. Generally ψ is specified by the Lévy-Khintschine formula

$$\psi(\lambda) = a\lambda + b\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda x} - 1 + \lambda x)\Lambda(dx), \quad \lambda \geq 0,$$

where $a \in (-\infty, +\infty)$, $b \geq 0$ and Λ is a non-negative measure on $(0, +\infty)$ satisfying $\int_{(0,+\infty)}(x^2 \wedge x)\Lambda(dx) < +\infty$. Λ is called the Lévy measure of ψ. Obviously, ψ is convex and infinitely differentiable on $(0, +\infty)$. Since we aim at conditioning critical CB process on non-extinction, we assume that ψ satisfies (1.2) with $\psi'(0+) = 0$. Under this assumption, ψ is a strictly convex function on $[0, +\infty)$, $\psi(+\infty) = +\infty$, and $\psi(\lambda) = 0$ if and only if $\lambda = 0$. This assumption also implies that $P_x(\tau < +\infty) = 1$ for every $x > 0$.

For $x > 0$ and $\lambda, t \geq 0$, let $E_x(e^{-\lambda X_t}) = e^{-xu_t(\lambda)}$. Then $u_t(\lambda)$ is the unique positive solution to the backward equation

$$\frac{\partial}{\partial t}u_t(\lambda) = -\psi(u_t(\lambda)), \quad u_0(\lambda) = \lambda. \tag{2.1}$$

From (2.1) and the semi-group property $u_t(u_s(\lambda)) = u_{t+s}(\lambda)$, we also get the forward equation

$$\frac{\partial}{\partial t}u_t(\lambda) = -\psi(\lambda)\frac{\partial}{\partial \lambda}u_t(\lambda), \quad u_0(\lambda) = \lambda. \tag{2.2}$$

Note that our moment condition on Λ implies that $E_xX_t = xe^{-\rho t} < +\infty$ for all $x > 0$ and $t \geq 0$.

Next define

$$\phi(z) := \int_z^{+\infty} \frac{1}{\psi(\xi)}\, d\xi, \quad \forall z > 0.$$

The mapping $\phi : (0, +\infty) \to (0, +\infty)$ is bijective with $\phi(0) = +\infty$ and $\phi(+\infty) = 0$. We use φ to denote the inverse function of ϕ. From (2.1), we have

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(\xi)}\, d\xi = t, \quad \lambda, t \geq 0.$$

Hence

$$u_t(\lambda) = \varphi(t + \phi(\lambda)), \quad \lambda, t \geq 0. \tag{2.3}$$

Since $\phi(+\infty) = 0$, we have $u_t(+\infty) = \varphi(t)$, and for any $x > 0$ and $t \geq 0$,

$$P_x(\tau > t) = P_x(X_t > 0) = 1 - \lim_{\lambda \to +\infty} e^{-x u_t(\lambda)} = 1 - e^{-x\varphi(t)}. \tag{2.4}$$

Let $\bar{F}(t) := P_1(\tau > t)$. Obviously, we have $\bar{F}(t) \sim \varphi(t)$ as $t \uparrow +\infty$.

Results about regular varying functions will be used a lot in the remaining paper, so we collect some basic facts here. A positive measurable function L is said to be slowly varying at ∞ if it is defined on $(0, +\infty)$ and $\lim_{x \to +\infty} L(\lambda x)/L(x) = 1$ for all $\lambda > 0$. This convergence holds uniformly with respect to λ on every compact subset of $(0, +\infty)$. Let $\mathcal{S}$ denote the set of all slowly varying functions at ∞. If $L \in \mathcal{S}$, then for any $\delta > 0$, $\lim_{x \to +\infty} x^{\delta} L(x) = +\infty$, and $\lim_{x \to +\infty} x^{-\delta} L(x) = 0$.

If a positive function f defined on $(0, +\infty)$ satisfies that $f(\lambda x)/f(x) \to \lambda^p$ as $x \to +\infty$ (resp. 0) for any $\lambda > 0$, then f is called regularly varying at ∞ (resp. 0) with index $p \in (-\infty, +\infty)$, denoted by $f \in \mathcal{R}_p(\infty)$ (resp. $f \in \mathcal{R}_p(0)$). Obviously, $f(x) \in \mathcal{R}_p(0)$ is equivalent to $f(1/x) \in \mathcal{R}_{-p}(\infty)$. If $f \in \mathcal{R}_p(\infty)$ (resp. $f \in \mathcal{R}_p(0)$), it can be represented by $f(x) = x^p L(x)$ (resp. $f(x) = x^p L(1/x)$) for some $L \in \mathcal{S}$.

3 The case $0 < \alpha \leq 1$

The following technical lemma follows from Theorem 1.5.2 and Theorem 1.5.12 in [1]. We omit the details here.

Lemma 1.

(1) If $p \in (-\infty, +\infty)$, $f \in \mathcal{R}_p(\infty)$ (resp. $\mathcal{R}_p(0)$), $T_1(t), T_2(t) \to +\infty$ (resp. 0) and $T_1(t) \sim T_2(t)$ as $t \uparrow +\infty$, then $f(T_1(t)) \sim f(T_2(t))$.

(2) Suppose $f \in \mathcal{R}_p(\infty)$, $T_1(t), T_2(t) \to +\infty$ as $t \to +\infty$, and $f(T_1(t))/f(T_2(t)) \sim c \in (0, +\infty)$. If $p > 0$, then $T_1(t)/T_2(t) \sim c^{1/p}$; otherwise if $p < 0$ and f has inverse function f^{-1}, then $f^{-1} \in \mathcal{R}_{1/p}(0)$ and $T_1(t)/T_2(t) \sim c^{1/p}$.

Theorem 1. *If (1.3) holds with $0 < \alpha \leq 1$, then for all $x > 0$ and $y \geq 0$,*

$$\lim_{t \to +\infty} P_x\left(\bar{F}(t) X_t \leq y | \tau > t\right) = H_\alpha(y), \tag{3.1}$$

where $H_\alpha(y)$ is a probability distribution function, and its Laplace transform is given by

$$h_\alpha(\theta) = \int_{[0,+\infty)} e^{-\theta y} dH_\alpha(y) = 1 - (1 + \theta^{-\alpha})^{-1/\alpha}. \tag{3.2}$$

Moreover, $\bar{F}(t)$ is regularly varying at $+\infty$ with index $-1/\alpha$, and consequently, for any $\delta > 0$,

$$\lim_{t\to+\infty} t^{\frac{1}{\alpha}+\delta}\bar{F}(t) = +\infty, \quad \lim_{t\to+\infty} t^{\frac{1}{\alpha}-\delta}\bar{F}(t) = 0.$$

Proof. For any $z > 0$, set $g(z) := \phi(1/z) = \int_0^z \xi^{\alpha-1}/L(\xi)\,d\xi$. Then by Karamata's theorem (see, for example [1, Theorem 1.5.11]), we have $g \in \mathcal{R}_\alpha(\infty)$, more specifically, $g(z) \sim \alpha^{-1}z^\alpha L(z)^{-1}$ as $z \to +\infty$. Consequently, we get $\phi \in \mathcal{R}_{-\alpha}(0)$, $\phi(z) \sim \alpha^{-1}z^{-\alpha}L(1/z)^{-1}$ as $z \downarrow 0$, and $\varphi \in \mathcal{R}_{-1/\alpha}(\infty)$.

Since $1 - e^{-u} \sim u$ as $u \downarrow 0$, we have for any $x, \theta > 0$,

$$\begin{aligned} \lim_{t\to+\infty} E_x\left(e^{-\theta\bar{F}(t)X_t}|\tau > t\right) &= 1 - \lim_{t\to+\infty} \frac{1 - e^{-x\varphi(t+\phi(\theta\bar{F}(t)))}}{1 - e^{-x\varphi(t)}} \\ &= 1 - \lim_{t\to+\infty} \frac{\varphi(t+\phi(\theta\bar{F}(t)))}{\varphi(t)}. \end{aligned} \tag{3.3}$$

It follows from Lemma 1 and the fact that $\bar{F}(t) \sim \varphi(t)$ as $t \uparrow +\infty$, we have

$$\phi(\theta\bar{F}(t)) \sim \phi(\theta\varphi(t)) \sim \theta^{-\alpha}\phi(\varphi(t)) = \theta^{-\alpha}t.$$

Hence we have $\varphi(t + \phi(\theta\bar{F}(t))) \sim \varphi((1 + \theta^{-\alpha})t)$. By (3.3) and the regularity of φ at ∞, we get

$$\lim_{t\to+\infty} E_x\left(e^{-\theta\bar{F}(t)X_t}|\tau > t\right) = 1 - \lim_{t\to+\infty} \frac{\varphi((1+\theta^{-\alpha})t)}{\varphi(t)} = 1 - (1 + \theta^{-\alpha})^{-1/\alpha}. \tag{3.4}$$

The assertion follows from the continuity theory for Laplace transforms (see, for example, [2, Section 6.6]). □

Remark 1. *The stationary-excess operation on $H_\alpha(y)$ is defined by $\widetilde{H}_\alpha(y) := \int_{(0,y]} \bar{H}_\alpha(x)dx / \int_{(0,+\infty)} \bar{H}_\alpha(x)dx$, where $\bar{H}_\alpha(y) = 1 - H_\alpha(y)$. $\widetilde{H}_\alpha(y)$ is also a probability distribution function, and a simple calculation shows that its Laplace transform is $(1+\theta^{-\alpha})^{-1/\alpha}$. $\widetilde{H}_\alpha(y)$ is often called a generalized positive Linnik law. When $\alpha = 1$, it gives the well-known standard exponential law. For more information on Linnik Law, we refer readers to [8, Section 4] and references therein.*

The remainder of this section is devoted to the converse assertions to Theorem 1. Suppose that X_t is a critical CB process. If there exist $x > 0$ and positive constants $Q_t \to 0$ (as $t \uparrow +\infty$) such that $P_x(Q_tX_t \in \cdot|\tau > t)$ converges weakly to a non-degenerate limit, then $\liminf_{t\to+\infty} Q_t/\bar{F}(t) > 0$. In fact, by Fatou's lemma

$$\begin{aligned} 0 &< \liminf_{t\to+\infty} \int_0^{+\infty} P_x(Q_tX_t > y|\,\tau > t)\,dy \\ &= \liminf_{t\to+\infty} E_x(Q_tX_t\,|\,\tau > t) \\ &= \liminf_{t\to+\infty} Q_t/\bar{F}(t). \end{aligned}$$

Lemma 2. *Suppose ψ is the branching mechanism of a non-trivial critical CB process. If ψ is regularly varying at 0, then $\psi \in \mathcal{R}_{1+\alpha}(0)$ with $\alpha \in [0,1]$.*

Proof. Suppose $\psi(\lambda) = \lambda^p L(1/\lambda)$ for some $p \in (-\infty, +\infty)$ and $L \in \mathcal{S}$. Since

$$0 = \psi'(0+) = \lim_{\lambda \downarrow 0} \frac{\psi(\lambda)}{\lambda} = \lim_{\lambda \downarrow 0} \lambda^{p-1} L(1/\lambda),$$

we have $p \geq 1$. If $p > 2$, then

$$\psi''(0+) = \lim_{\lambda \downarrow 0} \frac{2\psi(\lambda)}{\lambda^2} = \lim_{\lambda \downarrow 0} 2\lambda^{p-2} L(1/\lambda) = 0. \tag{3.5}$$

Recall that $\psi''(\lambda) = 2b + \int_0^{+\infty} x^2 e^{-\lambda x} \Lambda(dx)$ for some $b \geq 0$ and $\int_{(0,+\infty)} (x \wedge x^2) \Lambda(dx) < +\infty$. So (3.5) implies that $b = 0$ and $\Lambda(dx) \equiv 0$, in which case ψ is trivial. Hence $p \leq 2$. We set $\alpha = p - 1$, thus proving the conclusion. □

Theorem 2. *Suppose X_t is a critical CB process with branching mechanism ψ. If for some $x > 0$, $P_x\left(\bar{F}(t) X_t \leq y | \tau > t\right)$ converges weakly to a non-degenerate distribution function $H(y)$, then (1.3) holds with $\alpha \in (0,1]$.*

Proof. Let $H(y,t) := P_x\left(\bar{F}(t) X_t \leq y | \tau > t\right)$. Under the assumption, we have

$$\lim_{t \to +\infty} \int_{[0,+\infty)} g(y) dH(y,t) = \int_{[0,+\infty)} g(y) dH(y) \tag{3.6}$$

for any continuous function g defined on $[0, +\infty)$ such that $\lim_{y \to +\infty} g(y) = 0$. Suppose $\theta > 0$. Using (3.6) with $g(y) = e^{-\theta y}$ we get

$$\begin{aligned} h(\theta) := \int_{[0,+\infty)} e^{-\theta y} dH(y) &= \lim_{t \to +\infty} \int_{[0,+\infty)} e^{-\theta y} dH(y,t) \\ &= \lim_{t \to +\infty} E_x\left(e^{-\theta \bar{F}(t) X_t} | \tau > t\right) \\ &= 1 - \lim_{t \to +\infty} \frac{1 - \exp\{-x u_t(\theta \bar{F}(t))\}}{1 - \exp\{-x \varphi(t)\}} \\ &= 1 - \lim_{t \to +\infty} \frac{u_t(\theta \bar{F}(t))}{\varphi(t)}. \end{aligned} \tag{3.7}$$

So as $t \uparrow +\infty$

$$u_t(\theta \bar{F}(t)) \sim \bar{h}(\theta) \varphi(t) \sim \bar{h}(\theta) \bar{F}(t), \tag{3.8}$$

where $\bar{h}(\theta) = 1 - h(\theta)$. On the other hand,using (3.6) with $g(y) = y\, e^{-\theta y}$, we obtain

$$\begin{aligned} \bar{h}'(\theta) = \int_{[0,+\infty)} y e^{-\theta y} dH(y) &= \lim_{t \to +\infty} \int_{[0,+\infty)} y e^{-\theta y} dH(y,t) \\ &= \lim_{t \to +\infty} E_x\left(\bar{F}(t) X_t e^{-\theta \bar{F}(t) X_t} | \tau > t\right) \\ &= \lim_{t \to +\infty} \frac{\bar{F}(t) E_x(X_t e^{-\theta \bar{F}(t) X_t})}{1 - e^{-x\varphi(t)}}. \end{aligned} \tag{3.9}$$

From (2.1) and (2.2), we have

$$\frac{\partial}{\partial \lambda} u_t(\lambda) = \frac{\psi(u_t(\lambda))}{\psi(\lambda)}, \quad \forall \lambda > 0.$$

Thus

$$E_x(X_t e^{-\lambda X_t}) = -\frac{\partial}{\partial \lambda} e^{-x u_t(\lambda)} = x e^{-x u_t(\lambda)} \frac{\psi(u_t(\lambda))}{\psi(\lambda)}. \tag{3.10}$$

It follows from (3.8), (3.9) and(3.10) that

$$\begin{aligned} \bar{h}'(\theta) &= \lim_{t \to +\infty} \frac{x \bar{F}(t)}{1 - e^{-x\varphi(t)}} e^{-x u_t(\theta \bar{F}(t))} \frac{\psi(u_t(\theta \bar{F}(t)))}{\psi(\theta \bar{F}(t))} \\ &= \lim_{t \to +\infty} \frac{\psi(u_t(\theta \bar{F}(t)))}{\psi(\theta \bar{F}(t))} \\ &= \lim_{t \to +\infty} \frac{\psi(\bar{h}(\theta) \bar{F}(t))}{\psi(\theta \bar{F}(t))}. \end{aligned} \tag{3.11}$$

The last equality follows from a standard argument using the continuity and monotonicity of ψ. Let $\lambda(\theta) := \bar{h}(\theta)/\theta = \int_0^{+\infty} e^{-\theta y} \bar{H}(y) dy$ where $\bar{H}(y) = 1 - H(y)$. $\lambda(\theta)$ is decreasing on $(0, +\infty)$. Since $\bar{F}(t)$ decreases continuously to 0 as $t \uparrow +\infty$ and ψ is monotone on $(0, +\infty)$, (3.11) implies that

$$\lim_{s \downarrow 0} \frac{\psi(\lambda(\theta) s)}{\psi(s)} = \xi(\lambda(\theta)), \quad \forall \theta > 0, \tag{3.12}$$

for some function ξ such that $\xi(\lambda(\theta)) = \bar{h}'(\theta)$. From the continuity and monotonicity of $\lambda(\theta)$, we have for any $\lambda \in (0, \lambda(0+))$,

$$\lim_{s \downarrow 0} \frac{\psi(\lambda s)}{\psi(s)} = \xi(\lambda). \tag{3.13}$$

Characterization theorem (see [1, Theorem 1.4.1]) says that (3.13) holds for all $\lambda > 0$, and there exists $p \in (-\infty, +\infty)$ such that $\xi(\lambda) \equiv \lambda^p$, *i.e.* ψ is regularly varying at 0 with index p. Let $\alpha = p - 1$, then $\alpha \in [0, 1]$ by Lemma 2. If $\alpha = 0$, we have

$$\frac{\bar{h}(\theta)}{\theta} = \lambda(\theta) = \xi(\lambda(\theta)) = \bar{h}'(\theta).$$

This has the solution $h(\theta) = 1 - c\theta$ for some constant c. This is the Laplace transform of a distribution function if and only if $c = 0$, in which case $H(y) \equiv 1$ is the distribution function of Dirac measure at 0. Therefore $\alpha > 0$. □

Suppose μ is a positive measure supported on $(0, +\infty)$. We say μ is regularly varying at $+\infty$ if $u(x) := \mu((0, x])$ is regularly varying at $+\infty$. The following theorem tells us that (1.3) with $\alpha \in (0, 1]$ is implied by the more general limit $P_x(Q_t X_t \leq y | \tau > t) \to H(y)$ where Q_t are positive constants such that $Q_t \to 0$.

Theorem 3. *Let ψ be the branching mechanism of a non-trivial critical CB process with Lévy measure Λ. Suppose $x^2\Lambda(dx)$ is regularly varying at $+\infty$. If there exist $x > 0$ and positive constants $Q_t \to 0$ (as $t \uparrow +\infty$) such that $P_x(Q_t X_t \le y | \tau > t)$ converges weakly to a non-degenerate limit $H(y)$, then (1.3) holds with $\alpha \in (0, 1]$. In this case, $Q_t/\bar{F}(t) \sim c \in (0, +\infty)$, and the Laplace transform of $H(y)$ is given by*

$$h(\theta) = \int_{[0,+\infty)} e^{-\theta y} dH(y) = 1 - (1 + c^{-\alpha}\theta^{-\alpha})^{-1/\alpha}.$$

To proof Theorem 3, we need the following lemma.

Lemma 3. *Suppose ψ is the branching mechanism of a non-trivial critical CB process. Then ψ is regularly varying at 0 if and only if $x^2\Lambda(dx)$ is regularly varying at $+\infty$.*

Proof. We may and do assume that

$$\psi(\lambda) = b\lambda^2 + \int_{(0,+\infty)} (e^{-\lambda x} - 1 + \lambda x)\Lambda(dx)$$

where $b \ge 0$ and $\int_{(0,+\infty)}(x \wedge x^2)\Lambda(dx) < +\infty$. Let $U(z) := \int_{(0,z]} x^2\Lambda(dx)$ and $\hat{U}(\theta) := \int_{(0,+\infty)} e^{-\theta x} dU(x)$. If $\psi''(0+) < +\infty$, then $\psi \in \mathcal{R}_2(0)$ and $\int_{[1,+\infty)} x^2\Lambda(dx) < +\infty$. Obviously $\lim_{z\to+\infty} U(z) = \int_{(0,+\infty)} x^2\Lambda(dx) < +\infty$, which implies that $x^2\Lambda(dx)$ is slowly varying at $+\infty$.

Now we suppose $\psi''(0+) = +\infty$, in which case $\int_{[1,+\infty)} x^2\Lambda(dx) = +\infty$. If ψ is regularly varying at 0 with index $p \in [1, 2]$, then for any $A > 0$, using L'Hospital rule, we have

$$\begin{aligned} A^p &= \lim_{\lambda\to0+} \frac{\psi(A\lambda)}{\psi(\lambda)} = \lim_{\lambda\to0+} A^2 \frac{\psi''(A\lambda)}{\psi''(\lambda)} \\ &= \lim_{\lambda\to0+} A^2 \frac{2b + \hat{U}(A\lambda)}{2b + \hat{U}(\lambda)} = \lim_{\lambda\to0+} A^2 \frac{\hat{U}(A\lambda)}{\hat{U}(\lambda)}. \end{aligned} \tag{3.14}$$

The last equality is because $\lim_{\theta\to0+} \hat{U}(\theta) = \lim_{\theta\to0+} \int_{(0,+\infty)} e^{-\theta x} x^2\Lambda(dx) = +\infty$. Thus $\hat{U}$ is regularly varying at 0 with index $p-2 \in [-1, 0]$. By Tauberian theorem (see, for example [1, Theorem 1.7.1]), $x^2\Lambda(dx)$ is regularly varying at $+\infty$ with index $2 - p \in [0, 1]$. The converse assertion is clear through the equalities in (3.14). □

Proof of Theorem 3. The proof is similar to that of Theorem 2. We provide details here for the reader's convenience. Let $H(y, t) := P_x(Q_t X_t \le y | \tau > t)$, $h(\theta) := \int_{[0,+\infty)} e^{-\theta y} dH(y, t)$ and $\bar{h}(\theta) := 1 - h(\theta)$. Similarly we can get the analogues to (3.8) and (3.11):

$$u_t(\theta Q_t) \sim \bar{h}(\theta)\bar{F}(t) \ \text{ as } t \to +\infty, \tag{3.15}$$

and

$$\lim_{t\to+\infty}\frac{Q_t}{\bar{F}(t)}\frac{\psi(u_t(\theta Q_t))}{\psi(\theta Q_t)}=\bar{h}'(\theta). \tag{3.16}$$

It follows from Lemma 3 that ψ is regularly varying at 0. Using Lemma 1, (3.15) and (3.16), we have

$$\lim_{t\to+\infty}\frac{Q_t}{\bar{F}(t)}\frac{\psi(\bar{h}(\theta)\bar{F}(t))}{\psi(\theta Q_t)}=\bar{h}'(\theta). \tag{3.17}$$

In view of Lemma 2, we may and do assume $\psi\in\mathcal{R}_{1+\alpha}(0)$ with $\alpha\in[0,1]$. We first consider the case $\alpha>0$. Put $g(z):=(z\psi(1/z))^{-1}$, $z>0$. Then $g\in\mathcal{R}_\alpha(+\infty)$. (3.17) implies that

$$\lim_{t\to+\infty}\frac{g(1/\theta Q_t)}{g(1/\bar{h}(\theta)\bar{F}(t))}=\lim_{t\to+\infty}\frac{\psi(\bar{h}(\theta)\bar{F}(t))}{\psi(\theta Q_t)}\frac{\theta Q_t}{\bar{h}(\theta)\bar{F}(t)}=\frac{\theta}{\bar{h}(\theta)}\bar{h}'(\theta),\quad\forall\theta>0. \tag{3.18}$$

By Lemma 1, we have for all $\theta>0$,

$$\frac{\theta Q_t}{\bar{h}(\theta)\bar{F}(t)}\sim\left(\frac{\theta}{\bar{h}(\theta)}\bar{h}'(\theta)\right)^{-1/\alpha},\quad\text{as } t\uparrow+\infty,$$

or equivalently,

$$\frac{Q_t}{\bar{F}(t)}\sim\left(\frac{\theta}{\bar{h}(\theta)}\right)^{-1/\alpha-1}\bar{h}'(\theta)^{-1/\alpha},\quad\text{as } t\uparrow+\infty.$$

Hence we have $Q_t/\bar{F}(t)\sim c$ for some constant $c\in(0,+\infty)$, and

$$\left(\frac{\theta}{\bar{h}(\theta)}\right)^{-1/\alpha-1}\bar{h}'(\theta)^{-1/\alpha}\equiv c,\quad\theta\in(0,\infty).$$

In view of the initial condition $\bar{h}(0)=1$, the above equation has the unique solution $h(\theta)=1-(1+c^{-\alpha}\theta^{-\alpha})^{-1/\alpha}$.

Otherwise if $\alpha=0$, we assume $\psi(\lambda)=\lambda l(\lambda)$ where l is slowing varying at 0. From (3.17), we get

$$\lim_{t\to+\infty}\frac{l(\bar{F}(t))}{l(Q_t)}=\frac{\theta}{\bar{h}(\theta)}\bar{h}'(\theta),\quad\forall\theta>0.$$

Thus there exists a constant c_1 independent of θ such that

$$\frac{\theta}{\bar{h}(\theta)}\bar{h}'(\theta)\equiv c_1,\quad\theta\in(0,\infty).$$

This has the solution $h(\theta)=1-c_2\theta^{c_1}$ for some constant c_2. $h(\theta)$ is the Laplace transform of a distribution function only if $c_2=0$, in which case $H(y)\equiv 1, y\in[0,\infty)$ is the distribution function of the Dirac measure at 0. This contradicts our assumption that H is the distribution function of a non-degenerate random variable. Hence $\alpha>0$. We complete the proof. □

Remark 2. *Through the above proof we see that for ψ satisfying* (1.3) *with $\alpha = 0$, the limit distribution of $P_x\left(Q_t X_t \in \cdot \,|\, \tau > t\right)$, if exists, must be the Dirac measure at* 0.

4 The case $\alpha = 0$

In this section, we stay in the regime $\alpha = 0$. Suppose $\psi(\lambda) = \lambda L(1/\lambda)$ satisfies our assumption (1.2) and $\psi'(0+) = 0$. From Remark 2 we know that for $\alpha = 0$, any possible positive sequence $Q_t \to 0$ overnormalizes X_t. So we need to find an alternative way to normalize X_t. [8] considers the analogous conditional limit theorem for critical Markov branching processes with the offspring generating function $f(s) = s + (1-s)L(1/(1-s))$ where $L \in \mathcal{S}$. The proof in [8] can be adapted here to get the convergence result for a CB process.

Set

$$V(x) := \phi(1/x) = \int_{1/x}^{+\infty} \frac{1}{\psi(\xi)}\, d\xi = \int_0^x \frac{1}{\xi L(\xi)}\, d\xi, \quad x > 0.$$

Obviously, V is differentiable, strictly increasing on $(0, +\infty)$, $V'(x) = x^{-1}L(x)^{-1}$, $V(0) = 0$ and $V(+\infty) = \int_0^{+\infty} 1/\psi(\xi)d\xi = +\infty$. By Karamata's theorem, we have $V \in \mathcal{S}$, and $V(x)L(x) \to +\infty$ as $x \to +\infty$.

Let R denote the inverse function of V. It is easy to see that $R(x) = 1/\varphi(x)$, R is continuous, strictly increasing on $(0, +\infty)$ with $R(+\infty) = +\infty$ and $R(0) = 0$. By [1, Theorem 2.4.7], R belongs to the class of Karamata rapidly varying functions denoted by KR_∞. We refer readers to [1, Section 2.4] for more information about KR_∞. Since $y = V(R(y))$, we have

$$1 = V'(R(y))R'(y) = \frac{R'(y)}{R(y)L(R(y))}, \quad \forall y > 0,$$

or equivalently

$$\frac{R'(y)}{R(y)} = L(R(y)), \quad \forall y > 0.$$

Thus there exist $c, A > 0$ such that

$$R(y) = c \exp\left\{\int_A^y L(R(z))dz\right\}, \quad y \in [A, +\infty). \tag{4.1}$$

Lemma 4 ([8] Lemma 5.2)**.** *As $t \uparrow +\infty$, $I(y,t) := \int_t^{t+y/L(R(t))} L(R(z))dz \to y$, and this convergence holds locally uniformly with respect to $y \in (-\infty, +\infty)$.*

Theorem 4. *If* (1.3) *holds with* $\alpha = 0$*, then*

$$V(\bar{F}(t)^{-1}) \sim t, \quad \textit{as } t \uparrow +\infty, \tag{4.2}$$

and

$$\lim_{t\to+\infty} P_x\left(L(\bar{F}(t)^{-1})V(X_t) \le y | \tau > t\right) = 1 - e^{-y} \tag{4.3}$$

for any $x > 0$ *and* $y \ge 0$*.*

Proof. (4.2) follows from the fact that $V(\bar{F}(t)^{-1}) \sim V(R(t)) = t$ as $t \uparrow +\infty$. Henceforth we only need to prove (4.3). By the monotonicity of V, we have

$$P_x\left(L(\bar{F}(t)^{-1})V(X_t) \le y | \tau > t\right) = P_x\left(X_t \le R\left(y/L(\bar{F}(t)^{-1})\right) | \tau > t\right). \tag{4.4}$$

For any $\theta > 0$, using the argument of (3.3), we have

$$\begin{aligned} & \lim_{t\to+\infty} P_x\left(\exp\left\{-\theta \frac{X_t}{R\left(y/L(\bar{F}(t)^{-1})\right)}\right\} | \tau > t\right) \\ = \; & 1 - \lim_{t\to+\infty} \frac{\varphi\left(t + \phi\left(\theta/R\left(y/L(\bar{F}(t)^{-1})\right)\right)\right)}{\varphi(t)} \\ = \; & 1 - \lim_{t\to+\infty} \frac{R(t)}{R\left(t + \phi\left(\theta/R\left(y/L(\bar{F}(t)^{-1})\right)\right)\right)}, \end{aligned} \tag{4.5}$$

where in the last equality we used the fact that $R(t) = 1/\varphi(t), t > 0$.

Since $V \in \mathcal{S}$ and $\bar{F}(t) \sim \varphi(t) = R(t)^{-1}$ as $t \uparrow +\infty$, we get

$$\begin{aligned} \phi\left(\theta/R\left(y/L(\bar{F}(t)^{-1})\right)\right) &= V\left(\frac{1}{\theta} R\left(y/L(\bar{F}(t)^{-1})\right)\right) \\ &\sim V\left(R\left(y/L(\bar{F}(t)^{-1})\right)\right) \\ &= \frac{y}{L(\bar{F}(t)^{-1})} \\ &\sim \frac{y}{L(R(t))}. \end{aligned} \tag{4.6}$$

Thus by (4.1), (4.6) and Lemma 4, we have

$$\begin{aligned} & \lim_{t\to+\infty} \frac{R(t)}{R\left(t + \phi\left(\theta/R\left(y/L(\bar{F}(t)^{-1})\right)\right)\right)} \\ = \; & \lim_{t\to+\infty} \exp\left\{-\int_t^{t+\phi\left(\theta/R\left(y/L(\bar{F}(t)^{-1})\right)\right)} L(R(z))dz\right\} \\ = \; & \lim_{t\to+\infty} \exp\left\{-\int_t^{t+y/L(R(t))} L(R(z))dz\right\} \\ = \; & e^{-y}, \end{aligned}$$

and consequently,

$$\lim_{t\to+\infty} P_x\left(\exp\left\{-\theta\frac{X_t}{R\left(y/L(\bar{F}(t)^{-1})\right)}\right\}|\tau>t\right)=1-e^{-y}.$$

Note that $1-e^{-y}$ is the Laplace transform of the defective law which assigns mass $1-e^{-y}$ at 0 and no mass in $(0,+\infty)$. It follows from the continuity theory for Laplace transform (see, for example [2, Section 6.6]) that

$$\lim_{t\to+\infty} P_x\left(X_t\le R\left(y/L(\bar{F}(t)^{-1})\right)|\tau>t\right)=1-e^{-y},$$

or equivalently by (4.4)

$$\lim_{t\to+\infty} P_x\left(L(\bar{F}(t)^{-1})V(X_t)\le y|\tau>t\right)=1-e^{-y}.$$

□

5 Examples

In this section we collect a few examples of branching mechanisms that satisfy the assumptions in Section 3 or Section 4. Branching mechanisms in Examples 1, 2 and 4 are well-known. It follows from [11, Proposition 5.2] that $\psi(\lambda)=\lambda f(\lambda)$ is a critical branching mechanism if and only if f is a Bernstein function and there exists $b\ge 0$ such that $f(\lambda)=b\lambda+\int_0^\infty(1-e^{-x\lambda})g(x)dx$ with $g\ge 0$ decreasing and $\int_0^\infty(x\wedge 1)g(x)dx<\infty$. Branching mechanisms in Examples 3 and 5 are in given in this from. We refer the reader to [11] for more information on the connections between branching mechanisms and Bernstein functions, and [12] for more examples of Bernstein functions.

Example 1. Let $\psi(\lambda)=c\lambda^{1+\alpha}$ where $c>0$ and $\alpha\in(0,1]$. In this case $\phi(t)=(c\alpha)^{-1}\lambda^{-\alpha}$, $\varphi(t)=(c\alpha t)^{-1/\alpha}$. Thus we have

$$\bar{F}(t)=1-\exp\{-(c\alpha t)^{-1/\alpha}\}\sim(c\alpha t)^{-1/\alpha}\quad\text{as } t\uparrow+\infty.$$

Similarly to (3.4), we get

$$\lim_{t\to+\infty} E_x\left(e^{-\theta t^{-1/\alpha}X_t}|\tau>t\right)=1-\lim_{t\to+\infty}\frac{\varphi(t+\phi(\theta t^{-1/\alpha}))}{\varphi(t)}=1-(1+(c\alpha)^{-1}\theta^{-\alpha})^{-1/\alpha}.$$

Therefore for any $y\ge 0$,

$$\lim_{t\to+\infty} P_x\left(t^{-1/\alpha}X_t\le y|\tau>t\right)=H_\alpha(y),$$

where $H_\alpha(y)$ is uniquely determined by its Laplace transform

$$h(\theta)=\int_0^{+\infty}e^{-\theta y}dH_\alpha(y)=1-(1+(c\alpha)^{-1}\theta^{-\alpha})^{-1/\alpha}.$$

Remark 3. *This case was excluded in Pakes et. al. [9, 10], and was studied independently in Haas et.al. [4] and Zhang [15]. More specifically, [4] discussed Example 1 as a special case of self-similar Markov process, while [15] viewed the corresponding CB process as the scaling limit of a special sequence of Markov branching processes and exploited limit theorems for some general conditioning events.*

Example 2. If $\psi''(0+) = \sigma < +\infty$, then (1.3) holds with $\alpha = 1$ and $\lim_{s\downarrow 0} L(1/s) = \sigma/2$. By Karamata's theorem, we have $\phi(z) \sim z^{-1}L(1/z)^{-1} \sim 2/\sigma z$ as $z \downarrow 0$, and $\varphi \in \mathcal{R}_{-1}(\infty)$. Thus we have

$$\lim_{t\to+\infty} E_x\left(e^{-\theta X_t/t}|\tau > t\right) = 1 - \lim_{t\to+\infty} \frac{\varphi((1+\frac{2}{\sigma}\theta^{-1})t)}{\varphi(t)} = 1 - (1 + \frac{2}{\sigma}\theta^{-1})^{-1}.$$

Therefore

$$\bar{F}(t) \sim \frac{2}{\sigma t} \quad \text{as } t \uparrow +\infty,$$

and for any $y \geq 0$,

$$\lim_{t\to+\infty} P_x\left(X_t/t > y|\tau > t\right) = e^{-\frac{2}{\sigma}y}.$$

This conditional convergence was proved independently in Li [7] and Lambert [5].

Example 3. Let $\psi(\lambda) = \lambda(\lambda^{-\alpha} + \lambda^{-\beta})^{-1}$ where $0 < \beta < \alpha \leq 1$. By [12] $(\lambda^{-\alpha}+\lambda^{-\beta})^{-1}$ is a Bernstein function, and then ψ is a branching mechanism. Note that $\psi(\lambda) = \lambda^{1+\alpha}L(1/\lambda)$ with $L(z) = (1 + z^{-\alpha+\beta})^{-1}$. By Karamata's theorem, we have $g(z) := \phi(1/z) = \int_0^z \xi^{\alpha-1}/L(\xi)d\xi \in \mathcal{R}_\alpha(\infty)$, and

$$g(z) \sim \alpha^{-1}z^\alpha L(z)^{-1} \sim \alpha^{-1}z^\alpha =: h(z) \quad \text{as } z \uparrow +\infty.$$

Both g and h are strictly increasing on $(0, +\infty)$. Let g^{-1} and h^{-1} respectively denote the inverse functions of g and h. Since

$$1 = g(g^{-1}(z))/h(h^{-1}(z)) \sim g(g^{-1}(z))/g(h^{-1}(z)),$$

by Lemma 1 we have $g^{-1}(z) \sim h^{-1}(z) = (\alpha z)^{1/\alpha}$ as $z \uparrow +\infty$. Consequently, $\varphi(t) = 1/g^{-1}(t) \sim (\alpha t)^{-1/\alpha}$ as $t \uparrow +\infty$. Therefore, we have

$$\bar{F}(t) \sim (\alpha t)^{-1/\alpha} \quad \text{as } t \to +\infty,$$

and for any $y \geq 0$,

$$\lim_{t\to+\infty} P_x\left(t^{-1/\alpha}X_t \leq y|\tau > t\right) = H_\alpha(y),$$

where $H_\alpha(y)$ has the Laplace transform

$$h_\alpha(\theta) = 1 - (1 + \alpha^{-1}\theta^{-\alpha})^{-1/\alpha}.$$

Example 4. Let $\psi(\lambda) = \lambda^{1+\beta} + \lambda^{1+\gamma}$, $0 < \gamma < \beta \leq 1$. Then $\psi(\lambda) = \lambda^{1+\gamma}L(1/\lambda)$ with $L(z) = 1 + z^{\gamma-\beta} \in \mathcal{S}$. Using similar arguments as that in Example 3, we have

$$\bar{F}(t) \sim (\gamma t)^{-1/\gamma} \quad \text{as } t \to +\infty,$$

and for any $y \geq 0$,

$$\lim_{t\to+\infty} P_x\left(t^{-1/\gamma}X_t \leq y|\tau > t\right) = H_\gamma(y),$$

where $H_\gamma(y)$ has the Laplace transform:

$$h_\gamma(\theta) = 1 - (1 + \gamma^{-1}\theta^{-\gamma})^{-1/\gamma}.$$

Example 5. Let $\psi(\lambda) = \lambda \log^{-\beta}(1+\lambda^{-1})$, $\beta \in (0, 1]$ and where $\log^{-\beta}(1+\lambda^{-1})$ is a Bernstein function (see [11, P.133]). Then ψ satisfies (1.3) with $\alpha = 0$ and $L(z) = \log^{-\beta}(1 + z)$. Immediately we have $V(z) \sim (\beta + 1)^{-1} \log^{\beta+1} z$ and $L(z) \sim \log^{-\beta} z$ as $z \uparrow +\infty$. Inserting the asymptotic equivalents of V and L into Theorem 4, we get

$$-\log \bar{F}(t) \sim [(\beta + 1)t]^{\frac{1}{\beta+1}}, \quad \text{as } t \uparrow +\infty,$$

and

$$\lim_{t\to+\infty} P_x\left(\frac{\log^{\beta+1} X_t}{(\beta + 1)\log^{\beta}(\bar{F}(t)^{-1})} \leq y \,|\, \tau > t\right) = 1 - e^{-y}$$

for any $x > 0$ and $y \geq 0$.

Acknowledgement We would like to thank Professor Mei Zhang and her student Xin Zhang from Beijing Normal University for sending us their related work on this topic.

References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels (1987): *Regular Variation*, Cambridge University Press.

[2] K. L. Chung (2001): *A Course in Probability Theory*, Third Ed., Academic Press.

[3] D. R. Grey (1974): Asymptotic behavior of continuous time, continuous state-space branching processes, *J. Appl. Probab.* 11, 669-677.

[4] B. Haas and V. Rivero (2011): Quasi-stationary distribution and Yaglom limits of self-similar Markov processes, *Preprint, arXiv*:1110.4795.

[5] A. Lambert (2007): Quasi-stationary distributions and the continuous-state branching process conditioned to be never extinct, *Elec. J. Prob.* , 12, 420-446.

[6] S. V. Nagaev and V. Wachtel (2007): The critical Galton-Watson process without further power moments, *J. Appl. Probab.* 44, 753-769.

[7] Z.-H. Li (2000): Asymptotic behavior of continuous time and state branching process, *J. Aus. Math. Soc. Seriess A*, 68, 68-84.

[8] A. G. Pakes (2010): Critical Markov branching process limit theorems allowing infinite variance, *Adv. Appl. Prob.*, 42, 460-488.

[9] A. G. Pakes (1988): Some limit theorems for continuous-state branching processes, *J. Aus. Math. Soc. Series A*, 44, 71-87.

[10] A. G. Pakes and A. C. Trajstman (1985): Some properties of continuous-state branching processes, with application to Bartoszynski's virus model, *Adv. Appl. Prob.*, 17, 23-41.

[11] Y.-X. Ren and H. Wang (2008): On states of total weighted occupation times of a class of infinitely divisible superprocesses on a bounded domain, *Potential Anal.*, 28, 105-137.

[12] R. L. Schilling, R. Song and Z. Vondracek (2012): *Bernstein Functions: Theory and Applications*, Second edition. de Gruyter Studies in Mathematics, 37. Walter de Gruyter & Co., Berlin.

[13] R. S. Slack (1968): A branching process with mean one and possibly infinite variance, *Z. Wahrscheinlichkeitsth.* 9,139-145.

[14] R. S. Slack (1972): Further notes on branching processes with mean 1, *Z.Wahrscheinlichkeitsth.* 25, 31-38.

[15] X. Zhang (2012): Asymptotic behavior of continuous-state branching process allowing infinite variance, *unpublished Master dissertation*, Beijing Normal University.

Yan-Xia Ren: LMAM School of Mathematical Sciences & Center for Statistical Science, Peking University, Beijing 100871, P. R. China,
E-mail: `yxren@math.pku.edu.cn`

Ting Yang: Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, P.R.China, E-mail: yangt@amss.ac.cn

Guo-Huan Zhao: LMAM School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China,
E-mail: 1101110040@math.pku.edu.cn