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Small value probabilities for continuous state branching processes with immigration

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Abstract We consider the small value probability of supercritical continuous state branching processes with immigration. From Pinsky (1972) it is known that under regularity condition on the branching mechanism and immigration mechanism, the normalized population size converges to a non-degenerate finite and positive limit $\mathcal{W}$ as t tends to infinity. We provide sharp estimate on asymptotic behavior of $\mathbb{P}(\mathcal{W} \leqslant \varepsilon)$ as $\varepsilon \to 0^+$ by studying the Laplace transform of $\mathcal{W}$. Without immigration, we also give a simpler proof for the small value probability in the non-subordinator case via the prolific backbone decomposition.

Keywords continuous state branching process, small value probability, immigration

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1 Introduction

Let $(\mathcal{Z}, \mathbb{P}_x) = (\mathcal{Z}_t, \mathbb{P}_x : t \geqslant 0)$ be a continuous state branching process with immigration, starting from $x \geqslant 0$, with branching mechanism ψ and immigration mechanism φ (see [18, p. 287] for definition). The Laplace transform of $\mathcal{Z}$ is given by

$$\mathbb{E}_x \mathrm{e}^{-\lambda \mathcal{Z}_t} = \exp\left(-x u_t(\lambda) - \int_0^t \varphi(u_s(\lambda))\mathrm{d}s\right), \quad t \geqslant 0, \quad \lambda > 0, \quad x \geqslant 0, \tag{1.1}$$

where the function $u_t(\lambda)$ satisfies

$$u_0(\lambda) = \lambda, \quad \frac{\partial}{\partial t} u_t(\lambda) + \psi(u_t(\lambda)) = 0. \tag{1.2}$$

Here

$$\psi(\lambda) = \alpha_0 \lambda + \alpha \lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda x} - 1 + \lambda x \mathbb{I}_{\{x<1\}})\Pi(\mathrm{d}x), \tag{1.3}$$

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and

$$\varphi(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda x})\Lambda(\mathrm{d}x) + b\lambda,$$

with $\alpha_0 \in \mathbb{R}$, $\alpha \geqslant 0$, $b \geqslant 0$, Π and Λ being nonnegative measures on $(0, \infty)$ such that

$$\int_0^\infty (1 \wedge x^2)\Pi(\mathrm{d}x) < \infty, \quad \int_0^\infty (1 \wedge x)\Lambda(\mathrm{d}x) < \infty. \tag{1.4}$$

Usually the process $\mathcal{Z} = (\mathcal{Z}_t, t \geqslant 0)$ is called a CBI(ψ, φ). In particular, if $\varphi = 0$, CBI$(\psi, 0)$ is a continuous state branching process (without immigration), which is denoted by $Z = (Z_t, t \geqslant 0)$ in the present paper and is called CB(ψ) (see [18, p. 271] for detailed definition).

Furthermore, for the processes without immigration, under stronger condition

$$\int_0^\infty (x \wedge x^2)\Pi(\mathrm{d}x) < \infty \tag{1.5}$$

than (1.4) for the measure Π, one has $\mathbb{E}_x Z_t < \infty$ and can write ψ in (1.3) in the form

$$\psi(\lambda) = -m\lambda + \alpha\lambda^2 + \int_0^\infty (\mathrm{e}^{-\lambda x} - 1 + \lambda x)\Pi(\mathrm{d}x) \tag{1.6}$$

with the growth parameter $m = -\psi'(0^+)$, i.e., $\mathbb{E}_x Z_t = \mathrm{e}^{mt}$. We assume $m > 0$ in this paper, which corresponds to the supercritical case.

For the process CB(ψ), according to [14, p. 676], the martingale

$$W_t := \mathrm{e}^{-mt} Z_t \to W \quad \text{as } t \to \infty, \ \mathbb{P}_x\text{-a.s. \& } L^1 \tag{1.7}$$

for some finite and non-degenerate random variable W, if and only if

$$\int_0^\infty x^2 \wedge (|x \log x|)\Pi(\mathrm{d}x) < \infty. \tag{1.8}$$

For the process with immigration, from [19],

$$\mathrm{e}^{-mt}\mathcal{Z}_t \to \mathcal{W} \quad \text{as } t \to \infty, \ \mathbb{P}_x\text{-a.s.} \tag{1.9}$$

for some non-degenerate and finite random variable $\mathcal{W}$, if and only if

$$\int_0^\infty (x^2 \wedge |x \log x|)\Pi(\mathrm{d}x) < \infty, \quad \int_0^\infty (x \wedge |\log x|)\Lambda(\mathrm{d}x) < \infty. \tag{1.10}$$

In order to study the small value probability of $\mathcal{W}$ in this paper, we assume throughout the condition (1.10) which is equivalent to the existence of non-trivial limit $\mathcal{W}$. The small value probability of W is studied in [3]. One main result is stated below (others are stated in Theorem 5.1):

Theorem 1.1 (See [3]). *Assume the branching mechanism ψ is not corresponding to the Laplace exponent of a subordinator and $\int^\infty 1/\psi(\lambda)d\lambda < \infty$. Then*

$$\mathbb{P}_1(W = 0) = \mathrm{e}^{-\gamma} \quad \textit{with} \quad \gamma = \inf\{s > 0 : \psi(s) = 0\}.$$

Write $\rho = \psi'(\gamma)/m > 0$, then

$$\mathbb{P}_1(0 < W \leqslant \varepsilon) \sim C\varepsilon^\rho \quad \textit{as } \varepsilon \to 0^+, \tag{1.11}$$

for some constant $C > 0$.

Here and throughout this paper we use $f(x) \sim g(x)$ as $x \to 0^+$ (∞) to represent $f(x)/g(x) \to 1$ as $x \to 0^+$ (∞), and C to represent constant which can be different in different lines.

The proof of Theorem 1.1 in [3] is based on the link between continuous state branching processes and Lévy processes via time change, see, e.g., Theorem 10.2 of [18]. The Laplace exponent $\phi(\lambda) = -\log \mathbb{E}_1 \mathrm{e}^{-\lambda W}$ has inverse function Θ, which can be expressed through the branching mechanism ψ, as seen in Theorem 4.2 of [3]. Through the study of Θ, the asymptotic behavior of W was obtained in [3]. In the next section, we give a simpler proof for Theorem 1.1 without the use of Θ. Our approach is based on the prolific backbone introduced in [2].

The small value probability for branching related processes has been studied by many authors, starting from the pioneering work of Harris [15]. Especially for the Galton-Watson process, more and more refined estimates are obtained in [4, 7, 8, 10, 12, 13]. For continuous state branching processes, the constancy phenomena happens, see [16, 17] and [3]. For example, in the non-subordinator setting, the limit

$$\lim_{\varepsilon \to 0^+} \varepsilon^{-\rho} \mathbb{P}_1(0 < W \leqslant \varepsilon)$$

exists for some constant $\rho > 0$, which is not necessarily true in discrete time discrete state branching processes, see [9] and the related references therein.

The main goal of this paper is to study the small value probability of $\mathcal{W}$ in (1.9). The precise connections between Laplace transform and small value probability is standard, via Tauberian type theorems, see Lemma 2.3. One can express the Laplace transform of $\mathcal{W}$ in terms of W and the immigration mechanism. Then the small value probability of $\mathcal{W}$ is obtained by using the corresponding asymptotic properties of W studied in [3], as stated in Theorems 1.1 and 5.1. Roughly speaking, there are two distinct behaviors. The first is the polynomial decay rate analogous to Theorem 1.1 in the non-subordinator case. The other behavior is the exponential decay rate analogous to Theorem 5.1 in the subordinator case, which is also studied in [3]. Both are stated in Section 2, and their proofs are given in Sections 4 and 5, respectively. In Section 3, we provide a simpler proof of Theorem 1.1 based on the prolific backbone.

2 Statements of results

Let $\mathcal{W}$ be the non-degenerate and finite limit of $\mathrm{e}^{-mt}\mathcal{Z}_t$ under the necessary and sufficient condition (1.10), where the growth parameter m is defined in (1.6). We first consider the polynomial decay rate.

Theorem 2.1. *Assuming as in Theorem* 1.1, *the branching mechanism* ψ *is not corresponding to the Laplace exponent of a subordinator and* $\int^\infty 1/\psi(\lambda) d\lambda < \infty$. *Then*

$$\mathbb{E}_x \mathrm{e}^{-\lambda \mathcal{W}} \sim C \mathrm{e}^{-\gamma x} \lambda^{-\tau} \quad \text{as } \lambda \to \infty, \tag{2.1}$$

where $\tau = \varphi(\gamma)/m$ *and* $C = \exp(-(bC' + C'')/m)$ *with constants* C' *and* C'' *defined in* (4.16) *and* (4.26), *respectively. Equivalently, we obtain the small value probability of* $\mathcal{W}$,

$$\mathbb{P}_x(\mathcal{W} \leqslant \varepsilon) \sim \frac{C \mathrm{e}^{-\gamma x}}{\Gamma(1+\tau)} \varepsilon^\tau \quad \text{as } \varepsilon \to 0^+. \tag{2.2}$$

Note that the equivalence of (2.1) and (2.2) follows from Lemma 2.3(i) below.

Next, we consider the exponential decay rate analogous to Theorem 5.1 studied in [3] in the subordinator case. In order to get the precise behavior of small value probability of $\mathcal{W}$, we only consider the immigration mechanisms that have the following special form,

$$\varphi(s) = bs + s^\beta, \quad 0 < \beta < 1. \tag{2.3}$$

Also note that in this setting, ψ is corresponding to a subordinator, that is, one has $\alpha = 0$ and $\int_0^\infty x\Pi(\mathrm{d}x) < \infty$ in (1.6). Therefore we can rewrite ψ as

$$\psi(\lambda) = a\lambda + \int_0^\infty (\mathrm{e}^{-\lambda x} - 1)\Pi(\mathrm{d}x), \tag{2.4}$$

where $a = \int_0^\infty x\Pi(\mathrm{d}x) - m$ is the drift of subordinator. Our result in this setting is the following estimates.

Theorem 2.2. *Let immigration mechanism be given in* (2.3). *Assuming as in Theorem* 5.1, *the branching mechanism* ψ *is corresponding to the Laplace exponent of a subordinator.*

(i) *If* ψ *has zero drift and finite Lévy measure* $\Pi(0,\infty)=\alpha m$, *then as* $\varepsilon\to 0^+$,

$$-\log\mathbb{P}_x(\mathcal{W}\leqslant\varepsilon)\sim(2m)^{-1}b\alpha\cdot|\log\varepsilon|^2+\alpha^{\beta}(m(\beta+1))^{-1}\mathbb{I}_{\{b=0\}}\cdot|\log\varepsilon|^{\beta+1}, \tag{2.5}$$

where $\mathbb{I}$ *denotes indicator function.*

(ii) *If* ψ *has zero drift and infinity Lévy measure, then as* $\varepsilon\to 0^+$,

$$-\log\mathbb{P}_x(\mathcal{W}\leqslant\varepsilon)\sim m^{-1}b\cdot R_1^*(1/\varepsilon)+\mathbb{I}_{\{b=0\}}\cdot(xL^*(1/\varepsilon)+m^{-1}\cdot R_2^*(1/\varepsilon)), \tag{2.6}$$

where L^* *is defined in* (5.3), *and* R_1^* *and* R_2^* *are conjugate functions to* R_1 *and* R_2 *respectively, with* R_1 *and* R_2 *being slowly varying functions defined in* (5.9).

(iii) *If* ψ *has drift* $a>0$ *and the initial value* $x>0$, *then as* $\varepsilon\to 0^+$,

$$-\log\mathbb{P}_x(\mathcal{W}\leqslant\varepsilon)\sim(x+b/a)^{m/(m-a)}\cdot\varepsilon^{-a/(m-a)}L\left(1/\varepsilon\right), \tag{2.7}$$

where L *is the slowly varying function defined in Theorem* 5.1(iii).

(iv) *If* ψ *has drift* $a>0$ *and the initial value* $x=0$, *then as* $\varepsilon\to 0^+$,

$$\begin{aligned}-\log\mathbb{P}_0(\mathcal{W}\leqslant\varepsilon)\sim{}&(b/a)^{m/(m-a)}\cdot\varepsilon^{-a/(m-a)}L\left(1/\varepsilon\right)\\&+m^{-m/(m-a\beta)}(m-a\beta)(a\beta)^{-1}\mathbb{I}_{\{b=0\}}\cdot\varepsilon^{-a\beta/(m-a\beta)}L_2^*(\varepsilon^{-m/(m-a\beta)}),\end{aligned} \tag{2.8}$$

where L *is the slowly varying function defined in Theorem* 5.1(iii), L_2^* *is the conjugate slowly varying function to* L_2 *and* $L_2(\lambda)=L_1(\lambda)^{\beta(m-a)/(m-a\beta)}$, *where* L_1 *is the slowly varying function defined in* (5.11).

For more details about the above-mentioned slowly varying functions, see the remark after Theorem 5.1 in Section 5. Note also that all asymptotic expressions involved in slowly varying functions are the simplest possible but still somewhat implicit due to the use of conjugate functions.

As already indicated earlier, various Tauberian type theorems are very useful in this paper. And thus we collect them in the following lemma for the convenience of readers. The case (i) below is exactly Theorem 1.7.1 of [5], and the case (ii) is Theorem 4 and Corollary 1 of [6].

Lemma 2.3. *Assume* V *is a positive random variable.*

(i) (*Karamata Tauberian Theorem*) *For constants* $C>0$ *and* $\alpha>0$ *and a slowly varying function* L,

$$\mathbb{E}\mathrm{e}^{-\lambda V}\sim C\lambda^{-\alpha}L(\lambda)\quad as\ \lambda\to\infty,$$

if and only if

$$\mathbb{P}(V\leqslant t)\sim\frac{C}{\Gamma(1+\alpha)}t^{\alpha}L(1/t)\quad as\ t\to 0^+.$$

(ii) (*de Bruijn's Tauberian Theorem*) *Assume* $0\leqslant\alpha<1$ *is a constant,* L *is a slowly varying function at infinity, and* L^* *is the conjugate slowly varying function to* L *defined in* [6]. *Then*

$$\log\mathbb{E}\mathrm{e}^{-\lambda V}\sim-\lambda^{\alpha}/L(\lambda)^{1-\alpha}\quad as\ \lambda\to\infty,$$

if and only if

$$\log\mathbb{P}(V\leqslant t)\sim-(1-\alpha)\alpha^{\alpha/(1-\alpha)}t^{-\alpha/(1-\alpha)}L^*(t^{-1/(1-\alpha)})\quad as\ t\to 0^+.$$

In particular, when $\alpha=0$, *then*

$$\log\mathbb{E}\mathrm{e}^{-\lambda V}\sim-1/L(\lambda)\quad iff\quad\log\mathbb{P}(V\leqslant t)\sim-L^*(t^{-1}).$$

Note that there are two useful results about the conjugate slowly varying function in our arguments in Section 5. Namely,

(i) For any constant $C > 0$,

$$(CL)^*(x) \sim C^{-1} \cdot L^*(x). \tag{2.9}$$

(ii) For any integer $k > 0$ and constant α,

$$((\log_k x)^\alpha)^* \sim (\log_k x)^{-\alpha}, \tag{2.10}$$

where $\log_k$ is the k-th iterate of log. Both of them can be found in Proposition 1.5.14 and Appendix 5 of [5].

3 The prolific backbone approach

The main goal of this section is to provide a simpler proof of Theorem 1.1 based on the prolific backbone. The prolific backbone, according to [2] and further research in [11], is a decomposition of a continuous state branching process with initial mass $a > 0$ into three types of immigration along a continuous time Galton-Watson process. Denote the backbone as $(P(t,a), t \geqslant 0)$, and assume the branching mechanism in (1.3)

$$\alpha > 0 \quad \text{or} \quad \int_{(0,\infty)} (1 \wedge x)\Pi(\mathrm{d}x) = \infty. \tag{3.1}$$

Then the following property about the backbone holds as described in Theorem 5 of [2].

Theorem 3.1 (See [2]). *For every $a \geqslant 0$, the process $P(\cdot, a)$ is an immortal branching process in continuous time, with initial distribution given by the Poisson law with parameter $a\gamma$. Its reproduction measure μ is characterized in terms of the branching mechanism of Z by*

$$\Phi(s) = \sum_{n=2}^{\infty} (s^n - s)\mu(n) = \gamma^{-1} \cdot \psi(\gamma(1-s)), \quad s \in [0,1],$$

where $\gamma = \inf\{s > 0 : \psi(s) = 0\}$ and μ is given explicitly in [2].

Note that the prolific backbone decomposition for supercritical superprocesses is studied in [1]. The degenerate case of superprocesses without spatial motion gives the prolific backbone for the continuous state branching processes.

We start with a basic property of the continuous time discrete state branching processes.

Lemma 3.2. *There exists a finite and positive random variable $\widetilde{W}$,*

$$\mathrm{e}^{-mt} P(t,a) \to \widetilde{W} \quad \text{as } t \to \infty, \tag{3.2}$$

with

$$\widetilde{W} =^d \gamma W, \tag{3.3}$$

where W is the martingale limit of the continuous state branching process Z defined in (1.7).

Proof. Since $\mathrm{e}^{-mt}P(t,a)$ is a positive martingale, (3.2) naturally holds for some finite and positive random $\widetilde{W}$. In order to prove (3.3), we only need to show

$$\mathbb{E} \exp(-\lambda \mathrm{e}^{-mt} P(t,a)) \to \mathbb{E}_1 \mathrm{e}^{-\lambda\gamma W}.$$

From [2] it is known that

$$\mathbb{E} \exp(-\lambda \mathrm{e}^{-mt} P(t,a)) = \mathbb{E}_a \exp(-(1 - \exp(-\lambda \mathrm{e}^{-mt}))\gamma Z_t)$$

$$= \exp(-au_t((1-\exp(-\lambda \mathrm{e}^{-mt}))\gamma)). \tag{3.4}$$

Define the Laplace exponent of the martingale convergence limit W in (1.7) as

$$\phi(\lambda) = -\log \mathbb{E}_1 \mathrm{e}^{-\lambda W}. \tag{3.5}$$

For the Laplace functional u defined in (1.1), one has

$$u_t(\lambda \mathrm{e}^{-mt}) = -\log \mathbb{E}_1 \exp(-\lambda \mathrm{e}^{-mt} Z_t) \to \phi(\lambda) \quad \text{as } t \to \infty.$$

Together with $\exp(-\lambda \mathrm{e}^{-mt}) \sim 1 - \lambda \mathrm{e}^{-mt}$ when t is large, one can obtain from (3.4),

$$\mathbb{E} \exp(-\lambda \mathrm{e}^{-mt} P(t,a)) \to \mathrm{e}^{-a\phi(\lambda\gamma)} = \mathbb{E}_a \mathrm{e}^{-\lambda\gamma W},$$

which implies (3.3). □

Proof of Theorem 1.1. From the connection between branching rate r and branching mechanism Φ of a continuous time discrete state branching process, it is easy to check that for $(P(t,a), t \geqslant 0)$ in Theorem 3.1 one has

$$r = -\Phi'(0) = \psi'(\gamma). \tag{3.6}$$

Note also from Theorem 3.1 that each individual of $P(t,a)$ reproduces at least two offsprings at each splitting time. Thus the extinction of $P(t,a)$ happens if and only if $\{P(0,a) = 0\}$. Then one can obtain

$$\mathbb{P}\left(\lim_{t\to\infty} P(t,a) = 0\right) = \mathbb{P}\left(P(0,a) = 0\right) = \mathrm{e}^{-a\gamma},$$

since $P(0,a)$ is Poisson distributed with parameter $a\gamma$. Note that conditioned on $\{P(0,a) = k\}$ for any positive integer k, $(P(t,a), t \geqslant 0)$ is a continuous time discrete state branching process initiated from k individuals, with branching mechanism Φ defined in Theorem 3.1. Then

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\lambda P(t,a)}, P(t,a) > 0] &= \mathbb{E}[\mathrm{e}^{-\lambda P(t,a)}, P(0,a) > 0] \\ &= \sum_{k=1}^{\infty} \frac{(a\gamma)^k}{k!} \mathrm{e}^{-a\gamma} \mathbb{E} \mathrm{e}^{-\lambda X(t,k)}, \end{aligned} \tag{3.7}$$

where $(X(t,k), t \geqslant 0)$ is a copy of $(P(t,a), t \geqslant 0)$ conditioned on $\{P(0,a) = k\}$. Define the generating function

$$F_t(s) = \mathbb{E} s^{X(t,1)}. \tag{3.8}$$

Then continuing with (3.7) and using the branching property of $(X(t,k), t \geqslant 0)$, one has

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\lambda P(t,a)}, P(t,a) > 0] &= \mathrm{e}^{-a\gamma} \sum_{k=1}^{\infty} \frac{(a\gamma)^k}{k!} F_t(\mathrm{e}^{-\lambda})^k \\ &= \mathrm{e}^{-a\gamma}(\exp(a\gamma F_t(\mathrm{e}^{-\lambda})) - 1). \end{aligned} \tag{3.9}$$

By the classic result of continuous time discrete state branching process, $\mathrm{e}^{-mt} X(t,1)$ is a positive martingale and has a positive and finite limit, denoted as $\widehat{W}$. Then one has

$$F_t(\exp(-\lambda \mathrm{e}^{-mt})) = \mathbb{E} \exp(-\lambda \mathrm{e}^{-mt} X(t,1)) \to \mathbb{E} \mathrm{e}^{-\lambda \widehat{W}} := \widehat{\phi}(\lambda) \quad \text{as } t \to \infty.$$

From (3.13) and Lemma 2.3(i), $\widehat{\phi}$ has the following asymptotic behavior,

$$\widehat{\phi}(\lambda) \sim C\lambda^{-\psi'(\gamma)/m} \quad \text{as } \lambda \to \infty, \tag{3.10}$$

for some constant $C > 0$. Combining (3.9) and (3.10), one can obtain

$$\mathbb{E}[\exp(-\lambda \mathrm{e}^{-mt} P(t,a)), P(t,a) > 0] \to \mathrm{e}^{-a\gamma}(\exp(a\gamma \widehat{\phi}(\lambda)) - 1) \quad \text{as } t \to \infty$$

$$\sim e^{-a\gamma} a\gamma C \cdot \lambda^{-\psi'(\gamma)/m} \qquad \text{as } \lambda \to \infty.$$

Note $\{\widetilde{W} = 0\} = \{P(t,a) \to 0\}$ a.s. Therefore the martingale limit $\widetilde{W}$ satisfies

$$\mathbb{E}[e^{-\lambda \widetilde{W}}, \widetilde{W} > 0] \sim C\lambda^{-\psi'(\gamma)/m} \quad \text{as } \lambda \to \infty,$$

for some constant $C > 0$. Then using Lemma 2.3(i) and the connection between W and $\widetilde{W}$ in (3.3), the small value probability of W in (1.11) is obtained. □

Remark 3.3. For the continuous time discrete state branching process $(X(t,1), t \geqslant 0)$ with branching mechanism Φ defined in Theorem 3.1, its embedded chain $(X(n,1), n = 0, 1, \ldots)$ is a Galton-Watson process with generating function $\widehat{f}(s) = \mathbb{E}s^{X(1,1)}$, and also has the same martingale limit $\widehat{W}$. Assume $(\widehat{p}_k, k \geqslant 0)$ is the distribution of $X(1,1)$ and $\widehat{m}$ is its mean. Then one has

$$\begin{aligned} &\widehat{p}_0 = \mathbb{P}(X(1,1) = 0) = 0, \\ &\widehat{p}_1 = \mathbb{P}(X(\cdot,1) \text{ does not split during (0,1)}) = e^{-r} = e^{-\psi'(\gamma)}, \\ &\widehat{m} = \mathbb{E}X(1,1) = e^{\Phi'(1)} = e^{-\psi'(0^+)} = e^m. \end{aligned} \tag{3.11}$$

Thus by classic result about the small value probability of Galton-Watson process we have

$$\mathbb{P}(\widehat{W} \leqslant \varepsilon) \asymp \varepsilon^{|\log \widehat{p}_1|/\log \widehat{m}} = \varepsilon^{\psi'(\gamma)/m}. \tag{3.12}$$

Since there is no oscillation for the continuous time branching process (see [4, 16, 17]), we obtain

$$\mathbb{P}(\widehat{W} \leqslant \varepsilon) \sim C\varepsilon^{\psi'(\gamma)/m} \quad \text{as } \varepsilon \to 0^+, \tag{3.13}$$

for some constant $C > 0$. One may say more about the constant C mentioned above according to the argument of Theorem 5.2 of [3].

4 Small value probabilities of $\mathcal{W}$

In this section, we consider the small value probability of $\mathcal{W}$, which is the limit of $e^{-mt}\mathcal{Z}_t$. Recalling that $u_t(\cdot)$ in (1.2) is the Laplace functional of Z_t, and ϕ in (3.5) is the Laplace exponent of W in (1.7) as the martingale limit of $e^{-mt}Z_t$, we have

$$\mathbb{E}_x \exp(-\lambda e^{-mt} Z_t) = \exp(-x u_t(\lambda e^{-mt})) \to \exp(-x\phi(\lambda)) \quad \text{as } t \to \infty. \tag{4.1}$$

In order to get the expression of the Laplace transform of $\mathcal{W}$, we start with the Laplace transform of $e^{-mt}\mathcal{Z}_t$. Using (1.1), one has

$$\begin{aligned} \mathbb{E}_x \exp\left(-\lambda e^{-mt}\mathcal{Z}_t\right) &= \exp\left(-x u_t(\lambda e^{-mt}) - \int_0^t \varphi(u_s(\lambda e^{-mt}))\mathrm{d}s\right) \\ &= \exp\left(-x u_t(\lambda e^{-mt}) - \int_0^t \varphi(u_{t-r}(\lambda e^{-m(t-r)} e^{-mr}))\mathrm{d}r\right) \\ &= \exp\left(-x u_t(\lambda e^{-mt}) - \int_0^\infty \varphi(u_{t-r}(\lambda e^{-m(t-r)} e^{-mr}))\mathbb{I}_{\{r \leqslant t\}}\mathrm{d}r\right), \end{aligned} \tag{4.2}$$

where we used the variable substitution $r = t - s$. Recalling $e^{-mt}\mathcal{Z}_t \to \mathcal{W}$ in (1.9) and letting t converge to infinity in (4.2), one can get

$$\mathbb{E}_x e^{-\lambda \mathcal{W}} = \exp\left(-x\phi(\lambda) - \int_0^\infty \varphi(\phi(\lambda e^{-mr}))\mathrm{d}r\right). \tag{4.3}$$

In fact, the convergence in (4.3) can be assured by the dominated convergence theorem. To be more precise, using Jensen's inequality and $\mathbb{E}_1 Z_t = e^{mt}$ we have

$$u_t(\lambda) = -\log \mathbb{E}_1 e^{-\lambda Z_t} \leqslant \lambda e^{mt}.$$

Therefore, by the increasing property of φ we have

$$\varphi(u_{t-r}(\lambda \mathrm{e}^{-mt})) \leqslant \varphi(\lambda \mathrm{e}^{-mr}).$$

Using variable substitution $s = \lambda \mathrm{e}^{-mr}$, we have for any $\lambda > 0$,

$$\int_0^\infty \varphi(\lambda \mathrm{e}^{-mr})\mathrm{d}r = \int_0^\lambda (ms)^{-1}\varphi(s)\mathrm{d}s < \infty, \tag{4.4}$$

where the finiteness can be seen from (4.15) and (4.25). This ensures the dominated convergence theorem.

For convenience in the following argument, we define

$$\varphi_1(\lambda) = \int_0^\infty (1 - \mathrm{e}^{-\lambda x})\Lambda(\mathrm{d}x). \tag{4.5}$$

Then (4.3) can be rewritten as

$$\mathbb{E}_x \mathrm{e}^{-\lambda \mathcal{W}} = \exp\bigg(- x\phi(\lambda) - \int_0^\lambda (mt)^{-1} \cdot (\varphi_1(\phi(t)) + b\phi(t))\,\mathrm{d}t\bigg). \tag{4.6}$$

We start with the asymptotic behavior of the Laplace exponent ϕ defined in (3.5). Using Lemma 2.3(i) and the small value probability of W in (1.11), we get for some constant $C > 0$,

$$\mathbb{E}_1 \mathrm{e}^{-tW}\mathbb{I}_{\{W>0\}} \sim Ct^{-\rho} \quad \text{as } t \to \infty.$$

Furthermore, using $\mathbb{E}_1\mathrm{e}^{-tW} = \mathrm{e}^{-\gamma} + \mathbb{E}_1\mathrm{e}^{-tW}\mathbb{I}_{\{W>0\}}$, one can get

$$\phi(t) = -\log \mathbb{E}_1 \mathrm{e}^{-tW} = \gamma - \log(1 + \mathrm{e}^{\gamma}\mathbb{E}_1\mathrm{e}^{-tW}\mathbb{I}_{\{W>0\}}).$$

Thus it is easy to check that

$$0 \leqslant \gamma - \phi(t) \sim \mathrm{e}^{\gamma}\mathbb{E}_1\mathrm{e}^{-tW}\mathbb{I}_{\{W>0\}} \sim Ct^{-\rho} \quad \text{as } t \to \infty, \tag{4.7}$$

and $0 < \phi(1) < \gamma$, which is used in the following.

Lemma 4.1. *For φ_1 defined in* (4.5)*, we have*

$$0 < \varphi_1'(t) \leqslant C \tag{4.8}$$

uniformly on $[\phi(1), \infty)$, for some constant $C > 0$.

Proof. For any $t > \phi(1)$, we have

$$\begin{aligned}\varphi_1'(t) &= \int_0^\infty x\mathrm{e}^{-tx}\Lambda(\mathrm{d}x)\\ &\leqslant \int_0^{\mathrm{e}} x\Lambda(\mathrm{d}x) + \int_{\mathrm{e}}^\infty xe^{-\phi(1)x}\Lambda(\mathrm{d}x).\end{aligned}$$

By the property of the measure Λ in (1.4), we have

$$\int_0^{\mathrm{e}} x\Lambda(\mathrm{d}x) := C_1, \tag{4.9}$$

where C_1 is a positive constant. Moreover, there is a constant $B > 0$ such that $x\mathrm{e}^{-\phi(1)x} \leqslant B$ on $[\mathrm{e}, \infty)$. Therefore we obtain

$$\int_{\mathrm{e}}^\infty x\mathrm{e}^{-tx}\Lambda(\mathrm{d}x) \leqslant \int_{\mathrm{e}}^\infty B\Lambda(\mathrm{d}x) := C_2, \tag{4.10}$$

where C_2 is a positive constant since (1.10). Combining (4.9) and (4.10) we obtain the result in the lemma with $C = C_1 + C_2$. □

Proof of Theorem 2.1. In order to obtain the small value probability of $\mathcal{W}$, we only need to estimate the asymptotic behavior of its Laplace transform near the infinity, which is expressed as in (4.6). Firstly. we consider $\int_0^\lambda t^{-1}\phi(t)\mathrm{d}t$, which can be written as

$$\int_0^\lambda t^{-1}\phi(t)\mathrm{d}t = \int_0^1 t^{-1}\phi(t)\mathrm{d}t + \int_1^\lambda t^{-1}\phi(t)\mathrm{d}t. \tag{4.11}$$

By Jensen's inequality, for any $t > 0$,

$$0 < \phi(t) = -\log \mathbb{E}_1 \mathrm{e}^{-tW} \leqslant t\mathbb{E}_1 W = t.$$

Thus the first term of (4.11) is a positive constant determined by ϕ and bounded by 1. For the second term of (4.11), it can be written as

$$\gamma \log \lambda - \int_1^\lambda t^{-1}(\gamma - \phi(t))\mathrm{d}t = \gamma \log \lambda - \int_1^\infty t^{-1}(\gamma - \phi(t))\mathrm{d}t + \int_\lambda^\infty t^{-1}(\gamma - \phi(t))\mathrm{d}t. \tag{4.12}$$

From the asymptotic behavior of ϕ described in (4.7), we obtain that

$$\int_\lambda^\infty t^{-1}(\gamma - \phi(t))\mathrm{d}t \sim \int_\lambda^\infty Ct^{-\rho-1}\mathrm{d}t \sim \rho^{-1}C\lambda^{-\rho} \quad \text{as } \lambda \to \infty. \tag{4.13}$$

Therefore, it is easy to obtain that

$$\int_1^\infty t^{-1}(\gamma - \phi(t))\mathrm{d}t \tag{4.14}$$

is a finite and positive constant, determined by γ and ϕ.

Combining (4.11)–(4.14), we obtain

$$\int_0^\lambda t^{-1}\phi(t)\mathrm{d}t = \gamma \log \lambda + C' \quad \text{as } \lambda \to \infty, \tag{4.15}$$

with

$$C' = \int_0^1 t^{-1}\phi(t)\mathrm{d}t - \int_1^\infty t^{-1}\big(\gamma - \phi(t)\big)\mathrm{d}t. \tag{4.16}$$

For the rest in (4.6), it is similar to check that

$$\begin{aligned}\int_0^\lambda t^{-1}\varphi_1(\phi(t))\mathrm{d}t &= \varphi_1(\gamma)\log\lambda + \int_0^1 t^{-1}\varphi_1(\phi(t))\mathrm{d}t \\ &\quad - \int_1^\infty t^{-1}(\varphi_1(\gamma) - \varphi_1(\phi(t)))\mathrm{d}t + \int_\lambda^\infty t^{-1}(\varphi_1(\gamma) - \varphi_1(\phi(t)))\mathrm{d}t. \end{aligned} \tag{4.17}$$

Firstly, we need to prove that the second term of (4.17) is a finite constant determined by ϕ and φ_1. Note $\varphi_1(x)$ is increasing with respect to x and $0 < \phi(t) \leqslant t$, hence

$$\int_0^1 t^{-1}\varphi_1(\phi(t))\mathrm{d}t \leqslant \int_0^1 t^{-1}\varphi_1(t)\mathrm{d}t = \int_0^1\int_0^\infty t^{-1}(1 - \mathrm{e}^{-tx})\Lambda(\mathrm{d}x)\mathrm{d}t. \tag{4.18}$$

Then using Fubini's theorem and variable substitution $xt = s$, we get

$$\begin{aligned}\int_0^1 t^{-1}\varphi_1(t)\mathrm{d}t &= \int_0^\infty \int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s\Lambda(\mathrm{d}x) \\ &= \int_0^1\int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s\Lambda(\mathrm{d}x) + \int_1^\infty\int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s\Lambda(\mathrm{d}x). \end{aligned} \tag{4.19}$$

Since $\mathrm{e}^{-s} \geqslant 1 - s$ on $[0, 1]$, the first integral of (4.19) satisfies

$$\int_0^1 \int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s\Lambda(\mathrm{d}x) \leqslant \int_0^1 x\Lambda(\mathrm{d}x) < \infty. \tag{4.20}$$

The last inequality follows from (1.4). For the second integral of (4.19), when $x > 1$,

$$\begin{aligned} \int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s &= \int_0^1 s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s + \int_1^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s \\ &\leqslant \int_0^1 1\mathrm{d}s + \int_1^x s^{-1}\mathrm{d}s = 1 + \log x. \end{aligned}$$

Then for the second integral of (4.19) we have

$$\int_1^\infty \int_0^x s^{-1}(1 - \mathrm{e}^{-s})\mathrm{d}s\Lambda(\mathrm{d}x) \leqslant \int_1^\infty (1 + \log x)\Lambda(\mathrm{d}x) < \infty. \tag{4.21}$$

For the last inequality, (1.10) is used. Combining (4.18)–(4.21), we obtain that

$$\int_0^1 t^{-1}\varphi_1(\phi(t))\mathrm{d}t \tag{4.22}$$

is a finite and positive constant determined by ϕ and φ_1.

For the last term of (4.17), using $\phi(t) \uparrow \gamma$ as $t \uparrow \infty$, Lemma 4.1 and the asymptotic behavior of ϕ in (4.7), we obtain

$$\begin{aligned} &\int_\lambda^\infty t^{-1}(\varphi_1(\gamma) - \varphi_1(\phi(t)))\mathrm{d}t \\ &\quad \leqslant \int_\lambda^\infty C \cdot t^{-1}(\gamma - \phi(t))\mathrm{d}t \sim \text{constant} \cdot \lambda^{-\rho} \quad \text{as } \lambda \to \infty. \end{aligned} \tag{4.23}$$

Furthermore, from (4.23) we obtain

$$\int_1^\infty t^{-1}(\varphi_1(\gamma) - \varphi_1(\phi(t)))\mathrm{d}t \tag{4.24}$$

is a finite and positive constant determined by γ, ϕ and φ_1.

Combining (4.17) and (4.22)–(4.24), we obtain

$$\int_0^\lambda t^{-1}\varphi_1(\phi(t))\mathrm{d}t = \varphi_1(\gamma)\log\lambda + C'' \quad \text{as } \lambda \to \infty, \tag{4.25}$$

with

$$C'' = \int_0^1 t^{-1}\varphi_1(\phi(t))\mathrm{d}t - \int_1^\infty t^{-1}(\varphi_1(\gamma) - \varphi_1(\phi(t)))\mathrm{d}t. \tag{4.26}$$

Combining (4.15) and (4.25), together with (4.6) and (4.7), we obtain (2.1). □

5 The subordinator case

In this section, we investigate the small value probability of $\mathcal{W}$ when the branching mechanism ψ is corresponding to a subordinator. Firstly we introduce the small value probability of W in this setting, which are due to [3, Section 5].

Theorem 5.1. *Assume ψ is corresponding to a subordinator.*

(i) *If ψ has zero drift and finite Lévy measure $\Pi(0, \infty) = \alpha m$, then*

$$\mathbb{P}_1(W \leqslant \varepsilon) \sim \varepsilon^\alpha L(1/\varepsilon) \quad \text{as } \varepsilon \to 0^+ \tag{5.1}$$

for some function L varying slowly at infinity. One can take L constant if and only if

$$\int_0^1 x^{-1}\Pi(\mathrm{d}x) < \infty. \tag{5.2}$$

(ii) *If ψ has zero drift and infinity Lévy measure, then*

$$-\log \mathbb{P}_1(W \leqslant \varepsilon) \sim L^*(1/\varepsilon) \quad \text{as } \varepsilon \to 0^+, \tag{5.3}$$

where L^* is a slowly varying function of the form

$$L^*(x) = \int_1^x u^{-1}L(u)\mathrm{d}u$$

with L slowly varying.

(iii) *If ψ has drift $a > 0$, then*

$$-\log \mathbb{P}_1(W \leqslant \varepsilon) \sim \varepsilon^{-a/(m-a)}L(1/\varepsilon) \quad \text{as } \varepsilon \to 0^+ \tag{5.4}$$

for some function L slowly varying at infinity.

Remark 5.2. In [3], these slowly varying functions mentioned in Theorem 5.1 are described in detail. For example, the slowly varying function L in Theorem 5.1(i) was defined in the following way. Note ϕ is the Laplace exponent of W defined in (3.5). Suppose Θ is its inverse function. Then according to Theorem 4.2 of [3], Θ satisfies

$$\Theta'(s)/\Theta(s) = -m/\psi(s), \qquad 0 \leqslant s < \infty,$$
$$\Theta(s) = s\exp\left(-\int_0^s (m/\psi(u) + 1/u)\mathrm{d}u\right), \quad 0 \leqslant s < \infty,$$

from which one can deduce that $\mathrm{e}^{-\phi(s)}$ is $-\alpha$-varying at infinity, and L is defined as $s^\alpha \mathrm{e}^{-\phi(s)}$.

In the remaining part of this paper, we assume that the immigration mechanism satisfies the special form in (2.3). Next, we will give the proof of Theorem 2.2, i.e., the small value probability of $\mathcal{W}$ in the subordinator setting.

Proof of Theorem 2.2. We use Tauberian type theorems in Lemma 2.3 to prove this theorem. To this purpose we need to get the asymptotic behavior of the Laplace transform of $\mathcal{W}$ in (4.6). These require, in each case, the limit behaviors of the following functions as $\lambda \to \infty$:

$$\phi(\lambda), \quad \int_0^\lambda t^{-1}\varphi_1(\phi(t))\mathrm{d}t \quad \text{and} \quad \int_0^\lambda t^{-1}\phi(t)\mathrm{d}t,$$

where $\varphi_1(t) = t^\beta, t > 0$ with $\beta \in (0,1)$.

(i) In this case, the small value probability of W is α-varying as expressed in (5.1). In particular when the condition (5.2) holds, one obtains by Lemma 2.3(i),

$$\mathbb{E}_1\mathrm{e}^{-tW} \sim Ct^{-\alpha} \quad \text{as } t \to \infty, \tag{5.5}$$

for some constant C. Thus the Laplace exponent of W satisfies

$$\phi(t) = -\log \mathbb{E}_1\mathrm{e}^{-tW} \sim \alpha \log t \quad \text{as } t \to \infty. \tag{5.6}$$

Therefore we have

$$\int_0^\lambda t^{-1}\phi(t)\mathrm{d}t \sim 2^{-1}\alpha \cdot (\log \lambda)^2 \quad \text{as } \lambda \to \infty. \tag{5.7}$$

Meanwhile, if (5.2) does not hold, one can use Proposition 1.3.6(i) of [5], i.e., for any slowly varying function L,

$$(\log L(t))/\log t \to 0 \quad \text{as } t \to \infty.$$

Then according to (5.1), we still have (5.6) and thus (5.7) holds.

Similarly, we can obtain

$$\int_0^\lambda t^{-1}\varphi_1(\phi(t))\mathrm{d}t \sim \alpha^\beta(\beta+1)^{-1}\cdot(\log\lambda)^{\beta+1} \quad \text{as } \lambda\to\infty.$$

Therefore, using the identity (4.6), we obtain as $\lambda\to\infty$,

$$-\log\mathbb{E}_x\mathrm{e}^{-\lambda\mathcal{W}} \sim (2m)^{-1}b\alpha\cdot(\log\lambda)^2+\alpha^\beta(m(\beta+1))^{-1}\mathbb{I}_{\{b=0\}}\cdot(\log\lambda)^{\beta+1}.$$

Finally, one can obtain (2.5) by Lemma 2.3(ii) and the property of slowly varying function (2.10).

(ii) Using Lemma 2.3(ii) and (5.3), we have

$$\phi(t)\sim\widehat{L}(t)^{-1} \quad \text{as } t\to\infty, \tag{5.8}$$

where $\widehat{L}(t)$ is the conjugate slowly varying function to L^* defined in (5.3). Then

$$\begin{aligned}-\log\mathbb{E}_x\mathrm{e}^{-\lambda\mathcal{W}} &\sim x\widehat{L}(\lambda)^{-1}+m^{-1}b\cdot\int_1^\lambda(t\widehat{L}(t))^{-1}\mathrm{d}t+m^{-1}\cdot\int_1^\lambda t^{-1}\widehat{L}(t)^{-\beta}\mathrm{d}t\\ &:= x\widehat{L}(\lambda)^{-1}+m^{-1}b\cdot R_1(\lambda)^{-1}+m^{-1}\cdot R_2(\lambda)^{-1}.\end{aligned} \tag{5.9}$$

Note that as $\lambda\to\infty$, $\widehat{L}(\lambda)\to0$ and $\widehat{L}(\lambda)/R_1(\lambda)\to\infty$. Therefore one can get

$$-\log\mathbb{E}_x\mathrm{e}^{-\lambda\mathcal{W}}\sim m^{-1}b\cdot R_1(\lambda)^{-1}+\mathbb{I}_{\{b=0\}}(x\widehat{L}(\lambda)^{-1}+m^{-1}\cdot R_2(\lambda)^{-1}) \quad \text{as } \lambda\to\infty,$$

which implies (2.6) by Lemma 2.3(ii).

(iii) & (iv) In this case, using the small value probability of W expressed as in (5.4) and Lemma 2.3(ii), one has

$$\phi(\lambda)\sim\lambda^{a/m}L_1(\lambda)^{a/m-1} \quad \text{as } \lambda\to\infty. \tag{5.10}$$

Here $L_1(\lambda)$ is a slowly varying function at infinity and its conjugate varying function satisfies

$$L_1^*(\lambda)\sim m(m-a)^{-1}(m/a)^{a/(m-a)}\cdot L(\lambda^{1-a/m}) \tag{5.11}$$

with L defined in Theorem 5.1(iii). Then for any $\varepsilon>0$, there exists $\lambda_\varepsilon>0$ such that

$$1-\varepsilon\leqslant\phi(t)\cdot t^{-a/m}L_1(t)^{1-a/m}\leqslant1+\varepsilon, \quad t>\lambda_\varepsilon. \tag{5.12}$$

Furthermore, using variable substitution $t=\lambda s$, one has

$$\begin{aligned}\int_{\lambda_\varepsilon}^\lambda t^{a/m-1}L_1(t)^{a/m-1}\mathrm{d}t &= \lambda^{a/m}\int_{\lambda_\varepsilon/\lambda}^1 s^{a/m-1}L_1(\lambda s)^{a/m-1}\mathrm{d}s\\ &= \lambda^{a/m}(L_1(\lambda))^{a/m-1}\int_{\lambda_\varepsilon/\lambda}^1 s^{a/m-1}\left(L_1(\lambda s)/L_1(\lambda)\right)^{a/m-1}\mathrm{d}s.\end{aligned}$$

Recall that in this case $a>0$ is the drift of ψ and $0<a<m$. Using Potter's theorem (Theorem 1.5.6 of [5]), there is a constant $\lambda_\varepsilon>0$ such that $L_1(\lambda s)/L_1(\lambda)\leqslant 2s^{(1-a/m)/2}$ for all $\lambda>\lambda_\varepsilon$. Using the dominated convergence theorem, one can obtain

$$\int_{\lambda_\varepsilon/\lambda}^1 s^{a/m-1}\left(L_1(\lambda s)/L_1(\lambda)\right)^{a/m-1}\mathrm{d}s\to\int_0^1 s^{a/m-1}\mathrm{d}s=m/a \quad \text{as } \lambda\to\infty, \tag{5.13}$$

since $L_1(\lambda s)/L_1(\lambda)\to1$. Note that

$$\int_0^{\lambda_\varepsilon}t^{-1}\phi(t)\mathrm{d}t\cdot[\lambda^{a/m}(L_1(\lambda))^{a/m-1}]^{-1}\to0 \quad \text{as } \lambda\to\infty.$$

Together with (5.12)–(5.13), we obtain that

$$\int_0^\lambda t^{-1}\phi(t)\mathrm{d}t \sim a^{-1}m\cdot\lambda^{a/m}(L_1(\lambda))^{a/m-1} \quad \text{as } \lambda\to\infty. \tag{5.14}$$

Using similar arguments, we can also obtain

$$\int_0^\lambda t^{-1}(\phi(t))^\beta\mathrm{d}t \sim (a\beta)^{-1}m\cdot\lambda^{a\beta/m}L_1(\lambda)^{(a/m-1)\beta} \quad \text{as } \lambda\to\infty. \tag{5.15}$$

Combining the Laplace transform of $\mathcal{W}$ in (4.6), the asymptotic behaviors in (5.10), (5.14) and (5.15), we obtain as $\lambda\to\infty$,

$$-\log\mathbb{E}_x\mathrm{e}^{-\lambda\mathcal{W}} \sim (x+b/a)\cdot\lambda^{a/m}L_1(\lambda)^{a/m-1} \quad \text{when } x>0,$$

and

$$-\log\mathbb{E}_0\mathrm{e}^{-\lambda\mathcal{W}} \sim a^{-1}b\cdot\lambda^{a/m}L_1(\lambda)^{a/m-1} + (a\beta)^{-1}\mathbb{I}_{\{b=0\}}\cdot\lambda^{a\beta/m}L_1(\lambda)^{(a/m-1)\beta}.$$

Thus using Lemma 2.3(ii) and the properties of slowly varying functions (5.11) and (2.9), we obtain (2.7) and (2.8) respectively. □

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